Moscow Institute of Physics and Technology

Russian Physics Olympiads
2005 – 2017

Executive Editors
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Preface

Russian National Olympiad is the major event for high school students participating in subject Olympiads in Russia.

Russian Physics Olympiad (RPhO) includes four stages.

The first (*school*) and the second (*local district*) stages take place in November-December. About 200 thousand high school students participate in the contest at the first stage.

The third *regional* stage is held in the regions of Russian Federation at the end of January. About six thousand students participate in the contest at this stage.

The fourth stage is *the final*. It takes place in the middle of April. The best three hundred physics students from all over Russia gather for the contest every year. The final stage is held separately for 9th-, 10th-, and 11th-graders.

Before 2008 RPhO included five stages, the fourth *federal* stage was held in seven Federal Districts of Russia in the middle of March. The federal stage was canceled after 2008. Since then, the selection of contesters for the final is made after the regional stage.

The regional, federal, and final stages consist of experimental and theoretical contests. During the theoretical contest, five problems are given to be solved in five hours. The experimental contest includes two assignements, a student has two and a half hours to finish each of them.

The previous set of problems given at RPhO was published in 2004 (in Russian only). The present set is its follow-up containing the problems of theoretical contest given to 11th-graders at the federal and final stages in 2005-2017.

We want to express our sincere gratitude to all authors of the problems, to the members of jury, and to the students who prepared assignments and experimental equipment for the Physics Olympiad over the course of years.
We must especially remember two persons whose work shaped the course of the Physics Olympiad in our country for years to come.
Alexander Rafailovich Zilberman, who passed away in 2010, directed Soviet Union National Physics Olympiad and trained the USSR team for the International Physics Olympiad (IPhO) for many years.
Stanislav Mironovich Kozel passed away in 2015. He directed RPhO and trained the national team for more than 20 years.
Among their students there are several absolute winners of IPhO and dozens of gold medalists.
We dedicate this book to their memory.

Executive Editors

A. M. Kiselev
V. P. Slobodyanin

Problems

Federal stage 2005

♦ 1. A Fly
Page 48

A fly is flying between a lens and a plane mirror parallel to the mirror. The lens is located at a distance $L = 20$ cm from the mirror, its optical axis is perpendicular to the mirror. When the fly is crossing the optical axis, the velocities of its images formed by the lens and by the mirror-lens system are equal in magnitude. Determine the focal length F of the lens and the distance a between the lens and the fly.

♦ 2. Boat
Page 48

A round rubber boat was pushed off from a lake shore at a speed v_0 and traveled a distance S_0 until stopped. An identical boat was pushed off from a river bank so that its initial velocity was equal to v_0 and orthogonal to the current. This boat traveled a distance $S_1 = \alpha S_0$ in the reference frame of water until it stopped relative to the water. What was a boat velocity V relative to the bank when it reached the middle of the river which width is $H = \alpha S_0$? Assume that $\alpha = 5/4$, the water drag is directly proportional to boat velocity, and the river current is uniform.

♦ 3. Variable Equilibrium
Page 50

A mixture of gases X_2, Y_2, and X_2Y are confined in a cylinder under a piston. A chemical reaction $2X_2 + Y_2 \leftrightarrow 2X_2Y$ proceeds in the cylinder. In equilibrium (when the reaction proceeds at the same rate in both directions) the system occupied a volume V under a pressure p

while the amount of substances X_2, Y_2, and X_2Y was ν_1, ν_2, and ν_3, respectively. Then the pressure was changed by a small amount Δp. Determine the volume increment ΔV and the increments $\Delta\nu_1$, $\Delta\nu_2$, and $\Delta\nu_3$ when new equilibrium had been reached. The temperature was maintained constant during the process.

Note. It is known that a chemical reaction rate is proportional to the product of concentrations ν_i/V of participating substances. Therefore, the rates of the forward and reverse reactions are proportional to

$$\left(\frac{\nu_1}{V}\right)^2\left(\frac{\nu_2}{V}\right) \quad \text{and} \quad \left(\frac{\nu_3}{V}\right)^2.$$

The proportionality factors can be different but depend on temperature only. A gas is considered to be ideal.

♦ 4. Electric Charge, Hollow Sphere, and Dielectric Page 51

A small sphere, which carries a charge Q, is located at the centre of a fixed uncharged conducting hollow sphere with outer and inner radii R_1 and R_2 ($R_2 < R_1$). The sphere is enclosed by a concentric dielectric layer of permittivity ε and the outer radius R_3 (fig. 1).

What is the minimum work required to move away a small sphere to a distance much greater than R_3 through a narrow channel inside the conductor and the dielectric?

♦ 5. Three Batteries Page 51

Experimentalist Glitch assembled an electric circuit (fig. 2) by mistakenly connecting a battery in parallel, rather than in series, to the rest two ones. Determine the currents via the resistors of the assembled circuit. Each resistor has a resistance R. All three batteries are identical with an emf $\mathscr{E}$ and an internal resistance which is small compared to R.

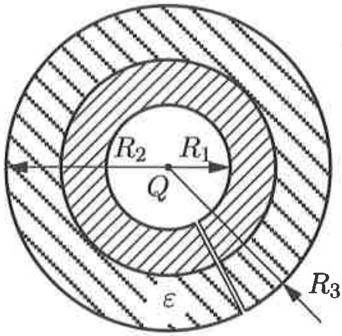

Figure 1

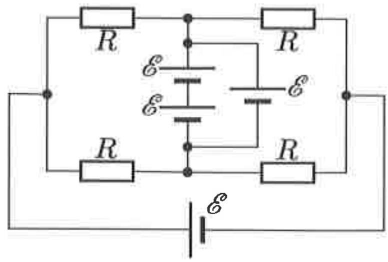

Figure 2

Final stage 2005

♦ 6. Something Fictional

Page **53**

Suppose that the Moon suddenly stopped orbiting the Earth because of a cosmic cataclysm. Determine the time τ in which the Moon will fall on the Earth and a relative velocity v of the planets just before the impact. The distance between the Earth and the Moon is $L = 3.84 \cdot 10^5$ km, the Earth radius is $R = 6370$ km. Both the mass and the radius of the Moon can be neglected compared to those of the Earth.

♦ 7. Not Quasistatic Cyclic Processes

Page **53**

An ideal diatomic gas being in thermal equilibrium occupies precisely one half of an adiabatically isolated container under a massive piston which does not conduct heat. A weight was placed on the piston (fig. 3). When the system reached a new equilibrium state, the gas pressure turned out to increase by 25 %. Then the weight was quickly removed and new equilibrium state had settled. How many such cycles n will be performed until the piston is pushed out of the cylinder during a next weight removal? A friction between the piston and the cylinder is negligible. There is no external pressure.

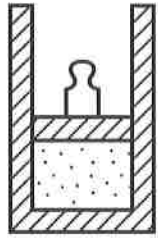

Figure 3

♦ 8. Clausius' Gas

Page 54

When developing the kinetic theory of gases, Clausius corrected the ideal gas law by introducing a parameter b, the proper volume of gas molecules (per mole):

$$p(V - b) = RT.$$

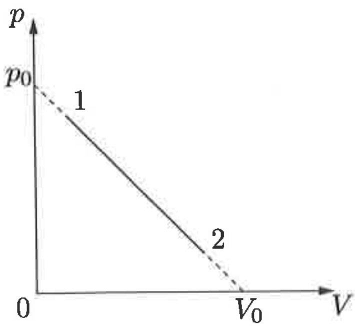

Figure 4

Process 1–2 (fig. 4) is performed over one mole of ideal gas and then over one mole of Clausius' gas. Determine a difference ΔT of the maximum gas temperatures in these experiments and indicate the highest one. It is known that $p_0 = 1.51$ MPa, $b = 44\ \text{cm}^3/\text{mol} \ll V_0$, and $R = 8.314$ J/(mol · K).

♦ 9. Superconducting Solenoid and Power Source

Page 55

There is a superconducting solenoid $l = 10$ cm long with $N = 1000$ turns and a cross-sectional area $S = 1.6\ \text{cm}^2$. At some moment the solenoid is connected to a power source with an emf $\mathscr{E} = 24$ V and an internal resistance $r = 0.2\ \Omega$. It is known that a magnetic induction exceeding $B_0 = 1.26$ T destroys the solenoid superconducting state.

Determine whether the solenoid coil switches from the superconducting state to the normal one in this experiment and, if so, at which time t_0 after connecting to the power source this happens. Otherwise, find out for which emf $\mathscr{E}$ of the source the transition would occur. The magnetic constant is $\mu_0 = 4\pi \cdot 10^{-7}$ SI units.

♦ 10. Photoelectric Effect

Page 56

A zinc ball of a radius $R = 1$ *cm* is located in vacuum far away from other bodies and is charged to a potential $\varphi_0 = -0.5$ *V* (assuming $\varphi = 0$ at infinity). The ball is being illuminated by a monochromatic ultraviolet light with a wavelength $\lambda = 290$ nm.

1. What is the maximum velocity v_1 of photoelectrons flying out of the ball?

2. What is the maximum velocity v_2 of a photoelectron, which left the ball at the beginning, far away from the ball?

3. Determine a ball potential φ_1 after a prolonged exposure to the UV.

4. Determine the net number N of photoelectrons escaped from the ball after a prolonged exposure to the UV light.

The photoelectric threshold for zinc is $\lambda_0 = 332$ nm.
The speed of light is $c = 3.0 \cdot 10^8$ m/s, the Planck constant is $h = 6.63 \cdot 10^{-34}$ J/s, the electric constant is $\varepsilon_0 = 8.85 \cdot 10^{-12}$ F/m, the electron charge is $e = -1.6 \cdot 10^{-19}$ C, and the electron mass is $m = 9.1 \cdot 10^{-31}$ kg.

Federal stage 2006

♦ 11. A Fly in Web

Page 57

A spider made a web shaped as a regular hexagon with a side $l = 45$ cm (fig. 5) and fixed the endpoints of radial threads of radius $r = 0.01$ mm so that their tension turned out to be $F_0 = 6$ mN. Assume a thread deformation to be elastic and its Young's modulus to be $E = 2 \cdot 10^8$ Pa. A thread breaks when its strain exceeds $\varepsilon_{max} = 0.2$.

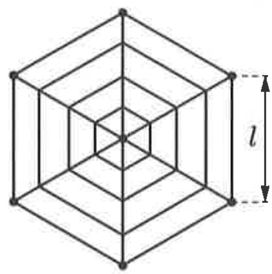

Figure 5

1. Determine the maximum mass M of a fly which does not break the web by hitting it at a speed $v = 2$ m/s. The fly hits at the web centre pependicular to the web plane.

2. A fly of a mass $m = 0.1$ g gets stuck at the web centre. Determine the period T of its small oscillation perpendicular to the web plane. Once in the web, a fly cannot wave its wings.

♦ 12. Work Done by Gas Mixing

Page 58

There are ν_X moles of an ideal gas X and ν_Y moles of an ideal gas Y in a cylinder which temperature T is maintained constant. There are two semitransparent pistons inserted in the cylinder (fig. 6).

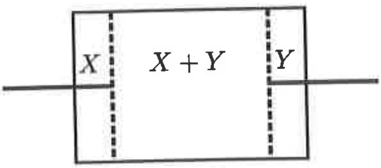

Figure 6

The first piston lets through only molecules of gas X and the second one only the molecules of gas Y. Initially the pistons are in contact and the pure gases X and Y occupy volumes V_{X0} and V_{Y0}, respectively. Then the pistons are slowly moved apart until the mixture of gases X and Y occupies the volume $V_{X0}+V_{Y0}$. What is the net work A done by the gases in this process?

Note. The area of the curvilinear trapezoid bounded by the curve $y = 1/x$ and straight lines $y = 0, x = x_1$ and $x = x_2$ is $S(x_1, x_2) = \ln \frac{x_2}{x_1}$.

♦ 13. Soap Bubble

Page 59

Determine a rate u of the radius R of a soap bubble decreasing when being deflated via a tube of a radius $r \ll R$. The tube volume is negligible compared to the bubble volume, the air inside the bubble can be regarded as being still. The surface tension of the soap solution is σ. The air flowing out of the bubble can be considered as an ideal incompressible liquid of a density ρ.

♦ 14. Conducting Sphere

Page 59

A point charge is located at a distance $R/3$ from the centre O of a conducting sphere of radius R (fig. 8). A point charge Q_2 is located outside the sphere at a distance $2R$. The sphere is connected to the ground via a switch K and a battery of an emf $\mathscr{E}$. A separation between the sphere and the ground is much greater than R. Assume the Earth potential to be zero.

1. Determine the potential φ at the sphere centre when the switch K is open.
2. Determine the charge Q accumulated on the sphere after the switch had been closed and the system attained equilibrium.

♦ 15. Circuit and Solenoid

Page 60

An electric circuit consists of two resistors R_1 and R_2 and a capacitor C (fig. 7). The conducting wire AB passes along a diameter of a loop of a long solenoid in which the current linearly increases with time. Determine the charge q stored by the capacitor in a stationary regime if the current flowing through R_1 in this regime equals I_1.

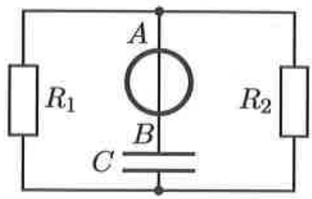

Figure 7

Final stage 2006

♦ 16. Capacitor Parameters

Page 61

The Wheatstone bridge circuit (fig. 9) is used to determine a capacitance and a leakage resistance (C_2, r_2) of a capacitor. The bridge is balanced when a harmonic alternating voltage is applied. It turns out, the balance persists even under variations of the voltage frequency. Determine C_2 and r_2 providing $r_1 = 2500\ \Omega$, $r_3 = 10\ \Omega$, $L_3 = 1$ H, and $r_4 = 800\ \Omega$.

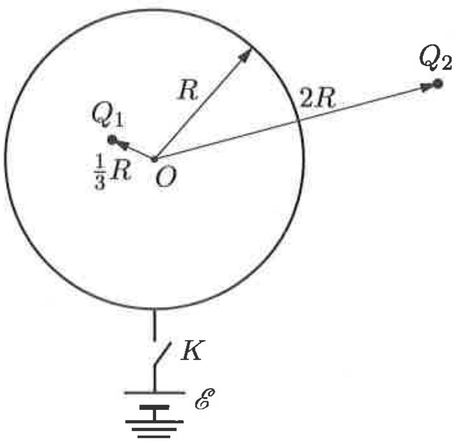

Figure 8

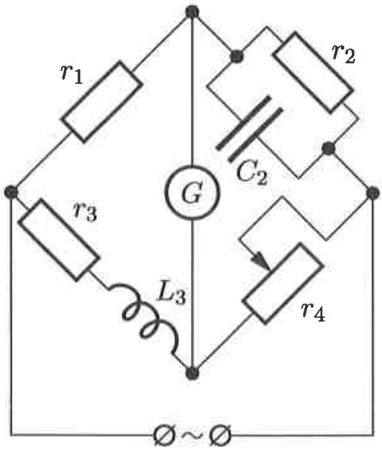

Figure 9

♦ 17. Exploded Projectile

Page 61

A projectile of mass $M = 100$ kg was fired from a cannon. At some point of its trajectory the projectile exploded and split in two fragments with initial momenta $p_1 = 36 \cdot 10^3$ kg $\cdot$ m/s and $p_2 = 24 \cdot 10^3$ kg $\cdot$ m/s. The angle between the momenta was $\alpha = 60°$. Determine the ratio of the fragment masses which minimises a change ΔE of the kinetic energy due to the explosion. Determine ΔE_{min}.

♦ 18. Tethered Puck

Page 62

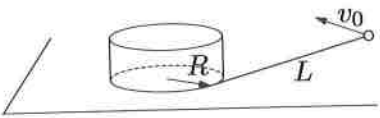

Figure 10

A round vertical cylinder of a radius R is fixed on a horizontal plane (fig. 10). An inextensible thread of a length L is attached at the cylinder side near the bottom, initially the thread is tangent to the side. A small puck (of negligible size) is attached to the other end of the thread. The puck is given an initial velocity v_0 perpendicular to the thread, so the puck starts sliding on the plane.

1. How long will the puck motion (winding the thread around the cylinder) last if there is no friction?
2. How long will the puck motion last if there is a friction between the puck and the plane? The coefficient of friction is μ.

♦ 19. Two Thermodynamic Processes

Page 64

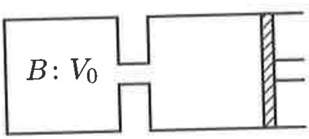

Figure 11

A diagram (fig. 11) shows a container B connected to a cylinder under a piston. The container volume is $V_0 = 0.2$ m^3, the initial net volume of the cylinder and the container is $V_1 = kV_0$ where $k = 2.72$. The system contains air (a diatomic gas) under a pressure $p_0 = 10^5$ Pa and at the ambient temperature. The air from the cylinder is then displaced to the container by moving the piston. Determine a heat transferred to the system surroundings during the process. Consider two possibilities:

1. The piston is moving slowly (quasistatically), so at any moment the system remains in equilibrium with its surroundings.
2. The piston is moving fast enough, so that a heat exchange between the system and the surroundings can be neglected while the air inside the system can be considered as being in thermal equilibrium at any moment. After the process is completed the temperature in the container gradually equilibrates with the ambient temperature.

Note. An adiabatic process is described by the equation

$$pV^\gamma = \text{const},$$

where $\gamma = C_p/C_V$.

♦ 20. At Solenoid Butt

Page 65

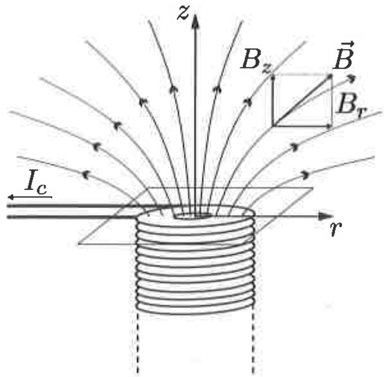

Figure 12

A thin round superconductor ring lies on a thin non-magnetic sheet upon a butt of a long vertical solenoid (fig. 12). The ring is coaxial with the solenoid. Initially, a current I_c through the solenoid and a current I in the ring are zero. A non-uniform magnetic field is produced near the solenoid butt for a nonzero I_c. The vertical B_z and the radial B_r components of the magnetic induction $\vec{B}$ near the butt can be approximated by

$$B_z = B_0(1 - \alpha z),$$
$$B_r = B_0\beta r,$$

where α and β are constants and B_0 is determined by I_c. At some moment, the current I_c starts flowing and gradually increases. Determine:

1. the critical value I_0 of the current I_c at which the superconductor ring starts levitating above the sheet;
2. the ring elevation above the sheet when $I_c = 2I_0$;
3. the frequency of small oscillations of the ring at $I_c = 2I_0$ (assuming the ring is displaced along the solenoid axis by a small distance Δz from the equilibrium position).

Numerical data: $\alpha = 36\ \mathrm{m}^{-1}$, $\beta = 18\ \mathrm{m}^{-1}$, the ring mass is $m = 100$ mg, the ring coefficient of self-inductance is $L = 1.8 \cdot 10^{-8}$ H, the number of turns of the solenoid is $n = 10^3$ per meter, the ring area is $S = 1\ \mathrm{cm}^2$, and the magnetic constant is $\mu_0 = 1.257 \cdot 10^{-6}$ H/m.

Federal stage 2007

♦ 21. Slipping Rope

Page 66

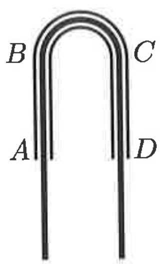

Figure 13

A firmly fixed symmetric tube consists of three parts: two straight vertical segments AB and CD and a semicircle BC (fig. 13). A heavy uniform rope is threaded through the tube, the rope can slide within the tube without friction. Initially, both ends of the rope are at the same height and the rope is at rest. Then the rope has been slightly pushed and started sliding.

Determine the rope acceleration a and a fraction k of the rope length traveled by the lower end to the moment when the vertical component of the force exerted on the rope by the tube vanishes. At any time the length of the bent part BC is negligible compared to the length of a vertical part of the rope.

♦ 22. Pair of Unequal Lenses

Page 68

Two thin lenses L_1 and L_2 with focal lengths F_1 and F_2 are separated by a distance L. A thin lens L_3 is placed between L_1 and L_2, so that any beam coming to the optical system at a small angle to the optical axis remains parallel to itself when leaving the system. Determine a focal length F_3 of L_3, the distance l_1 between L_3 and L_1, and the distance l_2 between L_3 and L_2. Optical axes of all three lenses coincide.

♦ 23. «Running Away» Liquid

Page 69

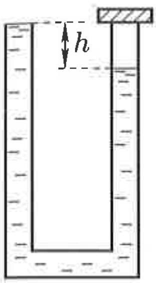

Figure 14

One side of a tall symmetric U-shaped tube with a cross-sectional area S is open to the atmosphere, the other side is sealed. The tube is filled with a liquid of a density ρ, so that the open side is filled to the brim while the liquid level in the sealed side is lower by h because of air left under the lid (fig. 14). The tube is being heated from the initial room temperature T_1 to the liquid boiling temperature T_2 at an atmospheric pressure P_0.

Determine the volume ΔV of the liquid spilled out of the open side before the boiling, provided the liquid level in the sealed side remained above the horizontal part of the tube. Evaporation from the open side and the pressure of saturated vapour at room temperature can be neglected.

♦ 24. Particle and Variable Capacitor

Page 70

One plate of a plane capacitor is fixed while the other one can move. A distance between the plates can be varied between 0 and d. A power source maintains a constant voltage U across the capacitor regardless of the distance between the plates. The task is to accelerate a particle with a charge $q > 0$, which initially stays at rest between the plates, to the maximum kinetic energy possible. In so doing, the particle should not approach a plate to a distance less than a. Determine this energy, the polarity of the power source, the initial particle position, and how one should move the plate in order to attain the maximum energy. The force of gravity and edge effects can be ignored.

Note. The indefinite integral of $1/(x + k)$ is $\ln(x + k)$.

♦ 25. Electromagnetic Gun

Page 71

A long solenoid of a radius r produces a uniform magnetic field B_0 along its axis O (fig. 15). A straight tube AM made of a dielectric is fixed in a plane perpendicular to the axis at a distance R_0 from it. The angle AOM equals $\alpha = \pi/3$. The tube is much shorter than the solenoid. A small sphere of a mass m, which carries a positive charge q, is placed inside the tube. Determine the sphere velocity at the moment of departure from the tube. Consider the following cases:

1. The magnetic field quickly vanishes, so the sphere travels a distance much less than R_0 during this time.
2. The magnetic field decreases at a constant rate $dB/dt = -k < 0$ during the time of sphere motion inside the tube.

A friction and an electromagnetic force exerted by the tube on the sphere are negligible.

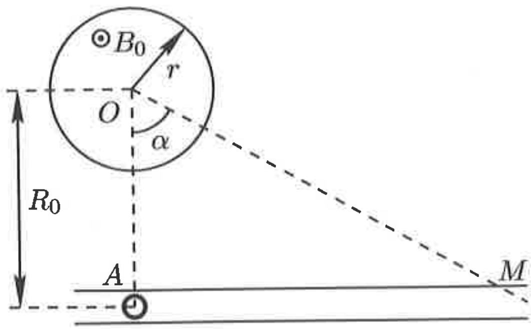

Figure 15

Final stage 2007

♦ 26. Particle Annihilation Page 72

A proton p^+ and an antiproton p^- are separated by a distance $L = 10$ cm in vacuum. Both particles have the same mass $m = 1.67 \cdot 10^{-27}$ kg and the same absolute value of electric charge $e = 1.602 \cdot 10^{-19}$ C. Initially, the particles are at rest. When a distance between the particles becomes less than $l = 10^{-13}$ m, they annihilate and produce γ-quanta.

1. What velocities will the particles have at this separation?
2. What time it will take the particles to approach?
3. Is it necessary to take into account the gravitational force between the particles? Justify your answer with a calculation.

Electric constant is $\varepsilon_0 = 0.885 \cdot 10^{-11}$ $C^2/(N \cdot m^2)$.
Gravitational constant is $G = 6.67 \cdot 10^{-11}$ $N \cdot m^2/kg^2$.

♦ 27. Colliding Disks Page 74

Two small disks with smooth lateral sides lie on horizontal plane with a coefficient of friction μ. Initially, the first disk was at rest and the second one collided with it at a velocity $\vec{v}$. Determine the distance between disks when they stopped moving, providing the first disk has traveled the distance x_1. Assume the collision to be elastic but not necessarily central. What is the maximum and minimum finite distance between the disks for a given absolute value of velocity v and the coefficient of friction μ? Neglect the disk size. The free fall acceleration is g.

♦ 28. Heating Main Page 74

Heating plant supplies a residential district by water at high pressure and an output temperature $T_0 = 120°\mathrm{C}$. Water flows inside a steel pipe of a radius $R = 20$ cm insulated with a $h = 4$ cm layer of mineral wool, the pipe is located in the open air. The water flow rate is $\mu = 100$ kg/s. The ambient temperature is $T_a = -20°\mathrm{C}$. The coefficient of thermal conductivity of the wool is $\chi = 0.08\ \mathrm{W/(m \cdot K)}$. The coefficient of thermal conductivity of steel is several orders of magnitude higher than that of the wool.

Determine the temperature T_f at the pipe other end in two cases:

1. The heating pipe length is $L_1 = 10$ km.
2. The heating pipe length is $L_2 = 100$ km.

The specific heat capacity of water is $c = 4200\ \mathrm{J/(kg \cdot K)}$.

Note. The amount of heat Δq passing through a layer with an area S and a thickness h per a time Δt for a temperature gradient ΔT is given by the equation $\Delta q = \chi(S/h)\Delta T \Delta t$, where χ is a coefficient of thermal conductivity.

♦ 29. Parametric Oscillations Page 76

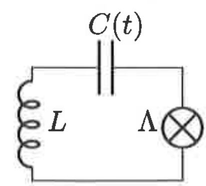

Figure 16

A diagram (fig. 16) shows a circuit with a capacitor C which capacitance is varied by moving its plates. Suppose that a minor disturbance initiated small oscillations with a capacitor voltage amplitude of several millivolts. At the moment of maximum voltage the capacitance is sharply reduced by a factor $\varepsilon = |\Delta C|/C$. In a quarter of the period $\frac{\pi}{2}\sqrt{LC}$ the capacitance is

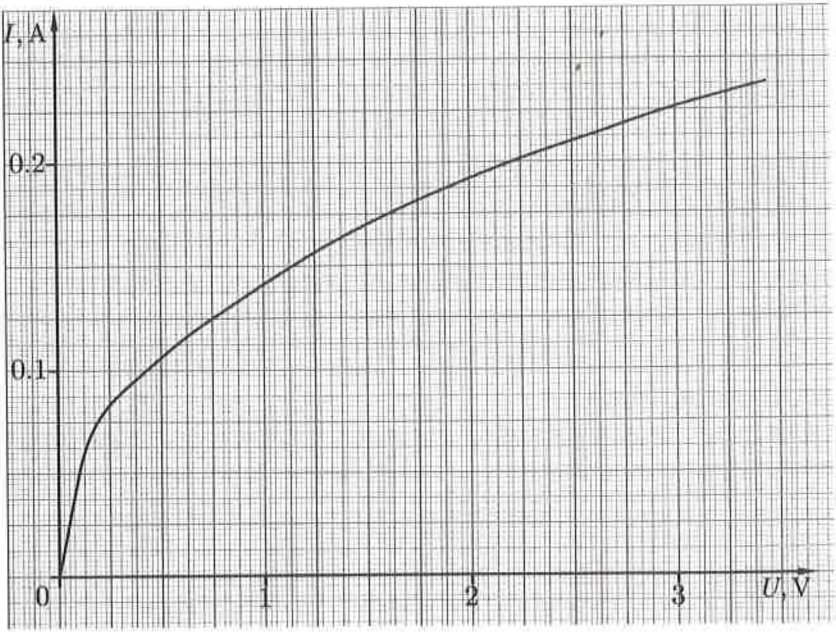

Figure 17

sharply increased to the original value; in the next quarter the capacitance is again reduced by ε etc. Under certain conditions this process can induce undamped electric oscillation in the circuit.

The circuit contains a nonlinear element (incandescent light bulb Λ) which I–V curve is shown in the diagram (fig. 17).

1. Determine the minimum value $\varepsilon_{\min}$ required to initiate nondamped oscillations in the circuit provided $L = 0.1$ H and $C = 10^{-7}$ F.

2. Determine the amplitude of stationary oscillations across the bulb, if $\varepsilon = 3\,\%$.

♦ 30. Light Emitting Diodes Page 78

A metal rod is traveling at a constant velocity between poles of a large round electromagnet of a radius $R = 5$ cm. The magnetic induction between the poles is uniform and equals $B = 1$ T, the rod velocity is perpendicular to the field lines (fig. 18).

The rod length exceeds $2R$, its ends are connected via flexible wires to a circuit containing a battery with an emf $\mathscr{E}_0 = 0.5$ V and two LEDs C_1 and C_2. An LED emits light if a voltage across it exceeds $U \geq 0.25$ V for the polarity shown in the diagram. Suppose the rod initially touches the magnet circumference, i.e. begins to cross the magnetic induction lines. Determine the voltage $U(t)$ across the LEDs and the moments of their turning on and off during the rod field crossing ($0 \leq t \leq 2R/v$). Sketch the dependence $U(t)$ and indicate the intervals of LED C_1 and C_2 lighting.

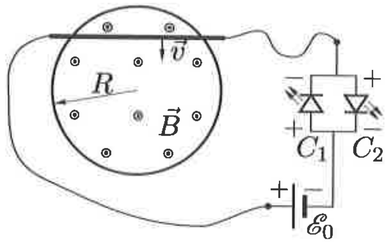

Figure 18

Federal stage 2008

♦ 31. Insulated System

Page 80

A cylinder with a metal bottom and dielectric walls is under a thin massive metal piston located at a height h which is much smaller than the cylinder diameter. A resistor, which is much smaller than the cylinder, is placed inside; the resistor is connected to an electric circuit (fig. 19). The latter is connected with the piston and the cylinder bottom with light flexible wires. Initially, the cylinder is filled with helium at a pressure $p \gg \varepsilon_0 \mathscr{E}^2/h^2$. The system is thermally insulated, placed in vacuum, and stays in thermal equilibrium.

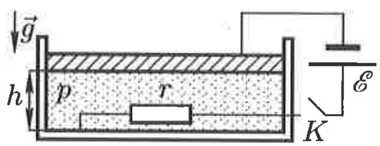

Figure 19

Then the switch K is closed. Determine the maximum height H the piston is able to reach when the system attains an equilibrium.

Heat capacity of the cylinder and piston is negligible. The resistance r can be considered constant. A friction between the piston and the cylinder can be neglected. Helium is considered as an ideal gas. The electric permittivity of helium is taken to be $\varepsilon_{He} = 1$.

♦ 32. Which Efficiency is Greater?

Page 81

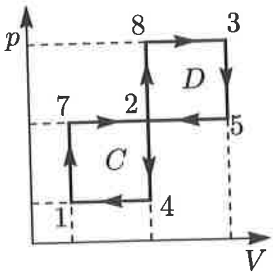

Figure 20

An ideal monoatomic gas undergoes a cyclic process C which consists of two isochors and two isobars. The same gas undergoes a similar process D (fig. 20). Which process has a greater efficiency? Evaluate η_D providing the efficiency of C equals η_C. For both processes $\Delta p_{21} = \Delta p_{32} = \Delta p$ and $\Delta V_{21} = \Delta V_{32} = \Delta V$, although their numerical values are unknown.

♦ 33. Very Thick Lens

Page 82

A transparent plate with a refractive index n is bounded by two spherical surfaces of curvature radii R and $r < R$.

1. What should a plate thickness L be in order to transform a parallel beam incident on the surface with the curvature radius R into a parallel one?
2. By which factor does the beam intensity increase (the energy transferred per unit area per unit time) after it passes through the plate? Neglect a loss of the beam energy inside the plate.
3. What is the angular magnification of a distant object by the plate?

♦ 34. Athlet Ant

Page 84

An ant is sliding on a smooth straw starting from a point A without an initial velocity; the sloping straight segment AB is smoothly joined to an arc BC with a curvature radius R, and the arc is smoothly joined to a horizontal straight segment CD (fig. 22).

It is given that $AB : BC : CD = 1 : 2 : 3$ and the total length of the path is much smaller than R. Evaluate the time of ant sliding on the straw from the point A to the point D.

♦ 35. Massive Rope

Page 85

A weight is held at rest by means of a massive uniform rope, a moving pulley of radius R, and a pulley attached to a fixed mounting

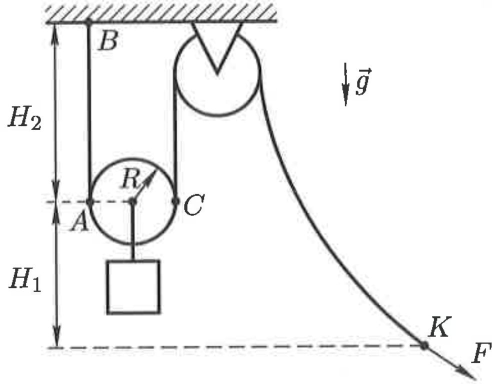

Figure 21

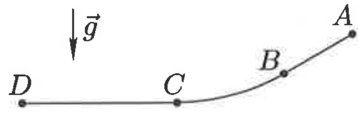

Figure 22

point (fig. 21). The rope mass is m, its length is l, and a net mass of the weight and the moving pulley is M. Vertical distances H_1 and H_2 are known.

1. Determine the rope tension at a point B.
2. Determine the force F applied to the rope at a point K.
   Neglect a friction in the pulley axes.

Final stage 2008

♦ 36. Rotation of Charged Cylinder

Page 86

A long dielectric thin-wall cylinder of a radius R, a length $L \gg R$, and a mass M carries an electric charge of a uniform surface density σ (C/m^2). The cylinder can rotate without friction around its axis under a force exerted by a weight of a mass m suspended by a light thread wound around the cylinder (fig. 23).

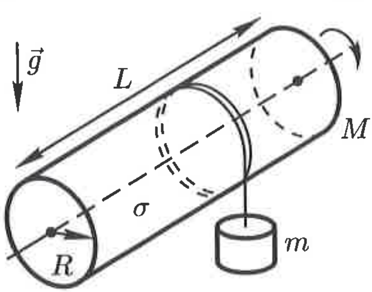

Figure 23

Determine the weight acceleration. The magnetic constant μ_0 is known.

♦ 37. Weight and Springs

Page 87

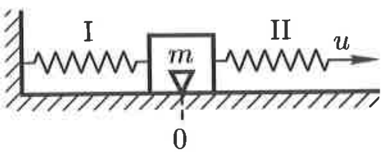

Figure 24

A weight of a mass m lies at rest on a smooth horizontal table; two identical springs with the same spring constant k are attached to it (fig. 24). The left end of the spring I is attached to a wall. The right end of the spring II starts to be slowly pulled at $t = 0$ at a constant velocity u.

1. In what time will the weight have the velocity u at the first time?
2. At what distance from the initial position will the weight be at this moment?

Directive. Consider the motion in the frame moving at a velocity $u/2$.

♦ 38. Utilisation of Wave Power

Page 88

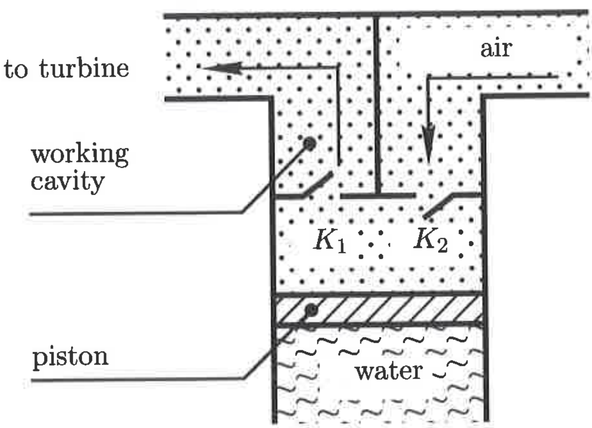

Figure 25

The first device utilising the power of sea waves was engineered in 1964 to produce electricity for a beacon. The design is shown in the figure 25. When the piston goes down, a portion of air is pumped in through a valve K_2, then it is compressed and enters a working cavity through a valve K_1. When the water level goes down, the valve K_1 closes and K_2 opens. The volume of air pumped in for a single cycle is $V_1 = 0.233\ \text{m}^3$, the pressure $p_1 = 1.0 \cdot 10^5$ Pa, and the temperature $t_1 = 7°\text{C}$.

When the water level begins to rise, the valve K_2 closes and the piston adiabatically compresses the air to a pressure $p_2 = 6.0 \cdot 10^5$ Pa. Then the valve K_1 opens and the piston goes upward until all air is expelled into the working cavity. On the way, the air rotates a turbine and an electric generator. After the valve K_1 opens, the air pressure under the piston remains approximately constant.

Determine the work done by water per a cycle. Neglect the piston mass and a friction between the piston and the walls. Air can be considered as an ideal diatomic gas for which $\gamma = C_p/C_V = 7/5$. The universal gas constant is $R = 8.31$ J/(mol · K).

♦ 39. Charged Soap Bubble

Page 89

A soap bubble of a mass $m = 0.01$ g has a coefficient of surface tension $\sigma = 0.01$ N/m, the bubble has been inflated through a short thin tube (fig. 26).

The bubble is then charged with an electric charge $Q = 5.4 \cdot 10^{-8}$ C. The tube remains open.

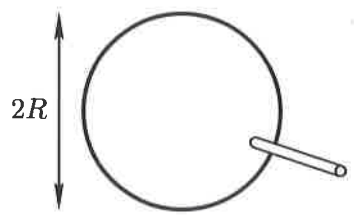

Figure 26

1. Determine an equilibrium radius R_0 of the bubble.
2. Determine a period of small spherical oscillations of the bubble.
3. The bubble suddenly receives a charge $Q_1 = 10\,Q$, estimate a speed of the spray.

Electric constant is $\varepsilon_0 = 8.85 \cdot 10^{-12}$ C^2/(J · m).

♦ 40. Optical System

Page 90

There is a rumour that Snell's archive contains a drawing of some optical system. The ink has faded with time and the only sketches remained are those of a converging lens, an object, and its real image; all of them are being parallel (fig. 27). From the comments to the drawing, it is clear that there was a plane mirror behind the lens. Using the drawing, restore the mirror position and locate the lens focal points.

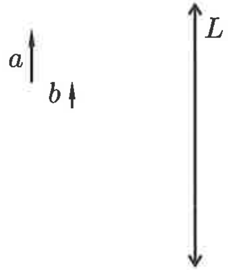

Figure 27

Final stage 2009

♦41. Bungee Jumping

Page 92

In a dizzy sport, a person of a mass $m = 70\ kg$ jumps from a platform to a lake†. A rubber cord of a length L and with a spring constant k is fastened to person's legs. The other end of the cord is attached to the platform. Near the water surface the person must have zero velocity and an acceleration $a_0 = 2g$. Assume $g = 10\ m/s^2$ and that the cord obeys Hooke's law. Neglect person's dimensions, a cord mass, air drag, and energy losses.

Determine:

1. the length L of unstretched cord and its spring constant k;
2. the cord length at the equilibrium position;
3. the person maximum speed $v_{\max}$;
4. in what time τ the person reaches the water surface.

Attention! The accuracy of your calculations could possibly be crucial for a person safety!

♦42. Electric Circuit with Inductor

Page 93

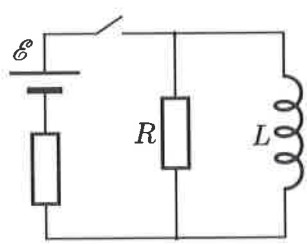

Figure 28

Parameters $\mathscr{E}, R, L$ of the circuit shown in the diagram (fig. 28) are known. Initially the switch is open and there is no current in the circuit containing the inductor. Then the switch was closed for some time and opened again.

It is known that a charge q_0 passed through the inductor when the switch was closed. The net heat released in the circuit after the switch was opened is Q_0.

Assuming the circuit elements to be ideal, determine

1. a current I_0 flowing through the inductor just before the switch was opened;
2. a charge q_1 passed through the resistor R when the switch was closed;

† www.youtube.com/watch?v=uIB7u-Cbq2Q

3. a charge q_2 passed through the resistor R after the switch was opened;
4. a net work A done by a DC power source during the whole process;
5. a net heat Q released in the circuit when the switch was closed.

Directive. Determine a relation between a charge passed through the resistor R and a change of magnetic flux in the inductor.

♦ 43. Kelvin's Problem

Page 94

There is a rumour that Kelvin's archive contains a plot of a thermodynamic cycle performed over one mole of an ideal monoatomic gas (fig. 29). The ink has faded with time, so there are no traces of T-axis (temperature) and V-axis (volume) left. Comments to the plot say that a temperature at the point A is equal to 400 K, the volume is 4 l, the pressure is minimal, and the coordinate origin is in a plot lower part. The plot scale is shown.

Figure 29

1. Reconstruct the T- and V-axes.
2. Determine the maximum gas pressure in the cycle.

♦ 44. Heat Exchange with Environment

Page 95

A heater, with a power output $P_0 = 400$ W, is placed into a container with a water-ice mixture at $\tau = 0$ min. The mixture is being vigorously stirred. A plot in fig. 30 shows a mixture temperature t versus time τ. A heat exchange Q with an environment is proportional to a temperature difference $\Delta t = t - t_0$, where t_0 is the ambient temperature. Assume that $t_0 = 0$ °C, so $Q = \alpha t$, where α is a temperature independent factor. Using the plot of $t(\tau)$, determine:

1. the initial mass m_i of ice in the mixture;
2. the net mass M of the mixture;
3. the factor α;
4. the maximum power output $P_{\max}$ of the heater, such that water would never boil;

5. a time τ_1 from the beginning of ice melting to the beginning of water boiling for the heater power output $P_1 = 300$ W.

The specific heat capacity of water is $c_w = 4200$ J/(kg · K); the specific heat of fusion of ice is $\lambda = 3.2 \cdot 10^5$ J/kg.

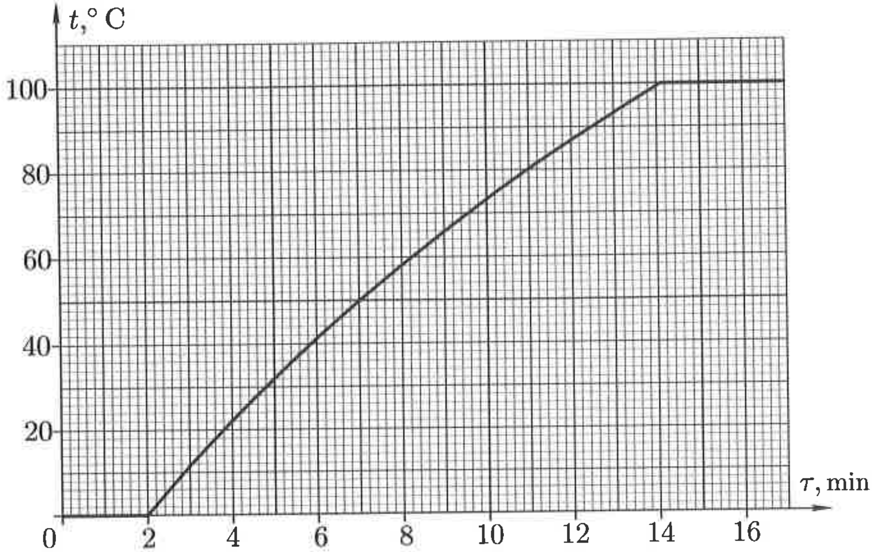

Figure 30

♦ 45. Two Lens Problem

Page 96

During an olympiad the contesters were given a task to determine focal lengths of two thin converging lenses placed at the butts of an empty cylinder which length was $L = 20.0$ cm (fig. 31).

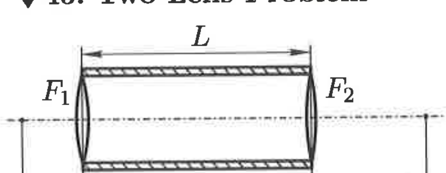

Figure 31

A contester Basil Haughtykin did the experiment accurately and obtained the following results:

1. If a point-like source of light is placed at the cylinder left at a distance $l_1 = 5.0$ cm, the light passing through the system comes out of the right end as a parallel beam.

2. If a parallel beam is incident on the left butt, the light coming out of the right butt converges to a point on the cylinder axis at a distance $l_2 = 10.0$ cm from the right butt.

However, Haughtykin could not determine the focal lengths F_1 and F_2 of these lenses using his data. Please, help poor Basil.

Final stage 2010

♦ 46. Chain on Sphere Page 97

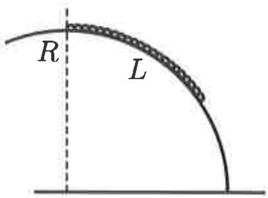

Figure 32

A uniform chain of a length L is being held by its end at the top of a smooth spherical surface of a radius R , where $L < \pi R/2$ (fig. 32). Then the chain end is released.

1. What is an acceleration a (the absolute value) of a chain element right after the release?
2. At which chain point is a tension T maximal right after the release?

Consider a special case when the chain length L equals $\dfrac{2\pi R}{6}$.

♦ 47. Motion of Charged Particles Page 99

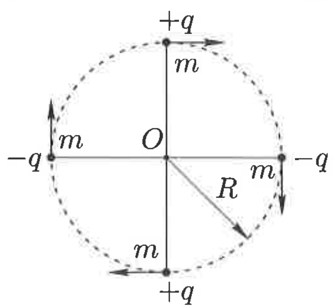

Figure 33

Four point masses m freely float in space being initially at the corners of a square inscribed in a circle of radius R_0. Two particles have electric charge $+q$, the other two have a charge $-q$ (fig. 33). Initially the masses were given the same clockwise velocities tangent to the circle. It is known that the minimum distance between a mass and the circle centre O during the motion equals R_1 ($R_1 < R_0$). Suppose that the system remains symmetrical with respect to the axis O during the time of observation. Gravity force is negligible.

1. Determine the mass trajectories;
2. Determine a period of their motion.

⧫ 48. Unipolar Inductor

Page 100

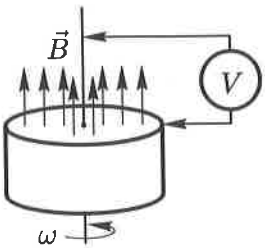

Figure 34

A unipolar inductor is a rapidly rotating disk-shaped permanent magnet. The disk is made of an electrically conducting magnetic alloy, which is able to produce a strong magnetic field, and it is covered by a thin conducting layer of nickel.

Disk rotation generates an electric potential across the axis of rotation and the lateral side, the potential can be measured by a voltmeter (fig. 34). If, on the other hand, a battery is connected to the axis of rotation and to the lateral side, the magnet starts rapidly spinning, thereby becoming an electric motor. Similarly, if one rapidly rotates a shaft of a regular electric motor, it becomes an electric generator; if a voltage is applied to a generator, it becomes a motor.

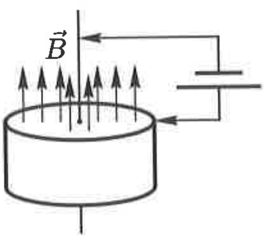

Figure 35

A diagram (fig. 35) shows a scheme of a working *unipolar electric motor*, which rotor is a disk of radius $r_0 = 2$ cm made of a strong permanent magnet fitted on a shaft. The disk starts rapidly spinning when a battery of an emf $\mathscr{E} = 1.5$ V is connected to it via sliding contacts.

Note. To simplify calculations assume that the vector of magnetic induction $\vec{B}$ in the conducting nickel layer is perpendicular to the disk upper surface, its magnitude is constant and equal to $B = 1$ T. Assume that a current in the conducting layer flows along radii.

1. What are the readings of a voltmeter (fig. 34) at a frequency of disk rotation equal to $\nu = 3000$ rev/min ? What is the voltage polarity? Indicate the polarity in the diagram (fig. 34).

2. Assume a friction to be negligible and estimate the maximum frequency of rotation (rev/min) of a magnetised disk (the rotor of unipolar motor in figure 35). Indicate the direction of rotation for the battery polarity and the magnetic induction $\vec{B}$ shown (fig. 35).

♦ 49. Heat Engine

Page 101

A heat engine operates according to a Carnot cycle with a hot source temperature $T_1 = 800$ K. Heat exchange between a working substance and a cold sink with a temperature $T_2 = 300$ K proceeds at a temperature T via a massive body, which is thermally insulated from the environment. The body transfers per a time Δt a net energy Q_2 received from the engine to the sink by means of thermal conductivity according to a law $Q_2 = \alpha(T - T_2)\Delta t$, where $\alpha = 1$ kW/K (fig. 36).
The temperature T of the body-mediator depends on the work $A = P\Delta t$ done by the engine, where P is the engine power.

1. Express P in terms of T_1, T_2 and T.
2. Determine a temperature T of the mediator which maximises the engine output.
3. Determine the maximum power $P_{\max}$ of the engine.
4. What is the heat engine efficiency at the maximum power output?

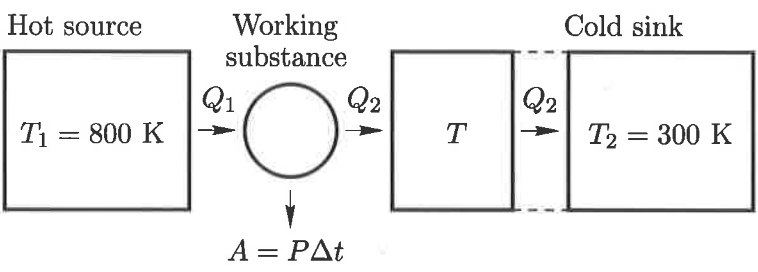

Figure 36

♦ 50. Motion without Sliding

Page 102

A long board of a mass m_1 lies at rest on a smooth horizontal surface of a table; a slab of a mass m_2 is put on the right end of the board. The slab is tethered to a wall by a light uncompressed spring with a spring coefficient k. A weight of a mass M is attached to the board via a light inextensible cord threaded through a pulley (fig. 37). Initially the system is at rest. There is a dry friction between the slab and the board. The board length and the initial distance between the pulley and the board are large enough.

1. Which path L will the slab travel before it starts sliding on the board?
2. Study how this result depends on the coefficient of friction μ.
3. Evaluate a time t the slab will travel the distance L.

Final stage 2011

♦ 51. Trifilar Pendulum Page 103

A massive ring is suspended on three identical thin vertical threads of a length L (fig. 38).

1. Determine a period of small torsional oscillations of the ring around the axis OO'.
2. How much will the period of the torsional oscillations change if a small body of the same mass as that of the ring is placed at the ring centre (the point O) by means of light spokes?

Hint: For $\alpha \ll 1$ the cosine approximation can be used: $\cos\alpha \approx 1-\alpha^2/2$.

Figure 37

Figure 38

♦ 52. Charged Particle Inside Solenoid Page 104

A diagram (fig. 39) shows a cross-section of a long straight coil (a solenoid) with a coil radius r = 10 cm. The number of turns per 1 meter of the solenoid length is n = 500 m^{-1}. A direct current $I = 1.0$ A flows (clockwise) in the solenoid coils.

A charged particle accelerated by a potential $U = 10^3$ V flies into the solenoid via a gap between the coils at a point A. The particle velocity at A is pointing along a solenoid radius. The particle is traveling inside the solenoid in a plane perpendicular to its axis and exits the solenoid at a point C at an angle $\alpha = 60°$ with respect to its initial direction. Determine:

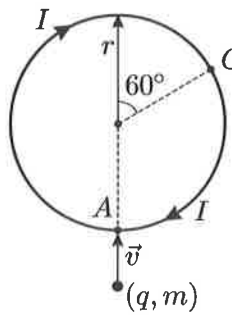

Figure 39

1. a sign of the particle charge;
2. a curvature radius of the particle trajectory inside the solenoid;
3. a specific charge of the particle (i.e. the charge-to-mass ratio).

Magnetic constant is $\mu_0 = 4\pi \cdot 10^{-7}$ (of SI units).

♦ 53. Capacitor Leak

Page 106

A plane capacitor of a capacitance C_0 is filled with a weakly conducting layered medium of a dielectric permittivity $\varepsilon = 1$, which specific conductivity depends on a distance x to the lower capacitor plate as $\rho = \rho_0(1 + \frac{2x}{d})$, where d is the plate separation. The capacitor is connected to a battery of a voltage U_0 (fig. 40). Determine:

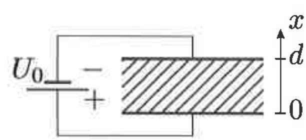

Figure 40

1. a current flowing through the capacitor;
2. a charge on the lower (q_1) and upper (q_2) capacitor plates;
3. a volume charge q within the capacitor (i.e. in the medium between the plates);
4. an electric energy W_e stored by the capacitor.

♦ 54. Piston Stability

Page 107

A vertical cylinder sealed at the bottom has a length $L = 1.50$ m. Its upper end is opened into another cylinder of a much larger diameter (fig. 41). A thin light piston is located in the lower cylinder at a distance $h = h_1 = 380$ mm from its upper end. There are a layer of mercury of a height $h + \Delta h$, where $\Delta h \ll h$,

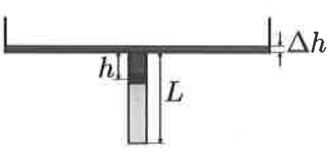

Figure 41

above the piston and helium at a pressure $p_1 = p_0 + \rho_{Hg} g h_1$, where $p_0 = 760$ mmHg is the atmospheric pressure and $\rho_{Hg} = 13.6\ \mathrm{g/cm^3}$ is a density of mercury, below the piston. A change in Δh resulting from a displacement of a piston inside the lower cylinder is negligible because the diameters of the cylinders differ significantly.

The piston is in equilibrium. Is its equilibrium position stable? Are there other equilibrium positions? If so, at which distances h_i of the piston from the upper end are they located? Are these equilibrium positions stable? One can assume that the temperature of helium under the piston does not vary under a small change of the volume.

♦ 55. Planar Waveguide

Page **108**

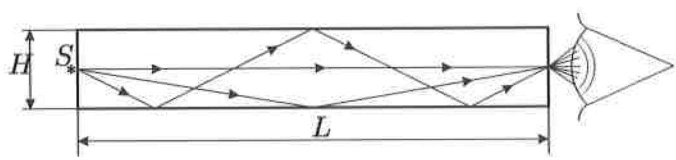

Figure 42

A point-like source S is placed near the left butt of a finely polished transparent plate with a refractive index n (fig. 42). A plate thickness is $H = 1$ cm and its length is $L = 100$ cm. Light from the source is incident on the left butt of the plate at all possible angles $(0-90°)$. An observer's eye catches both direct rays from the source and the rays which underwent multiple internal reflections on the plate lateral sides.

1. What is the maximum number of reflections undergone by a ray going out the right butt of the plate? Solve the problem for two values of the refractive index: $n_1 = 1.73$ and $n_2 = 1.3$.

2. In which of these two cases the light partially escapes the plate through its lateral sides?

Final stage 2012

♦ 56. Bag of Flour

Page 110

A paper bag of flour falls from a height $h = 4$ cm without initial velocity on the measuring pan of a spring scale. The scale pointer had initially deviated to $m_1 = 6$ kg and eventually settled at $m_0 = 2$ kg when oscillations decayed. The spring constant is $k = 1.5$ kN/m. Determine a mass M of the measuring pan.
Note. Assume the free fall acceleration to be $g = 10$ m/s^2.

♦ 57. Magnetism

Page 111

Two metal rods AB and CD can move without friction along two horizontal parallel rails separated by a distance l (fig. 43). A rod has a mass m and a resistance R. A uniform magnetic field B is perpendicular to the plane of the rails. Initially the rods are perpendicular to the rails and separated by a distance d. The rod CD is at rest and the rod AB is given an initial velocity v_0 which is parallel to the rails away from CD.

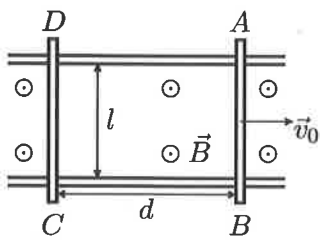

Figure 43

1. How far away will the rods be after a long time?
2. How much heat will be produced by the system?

Neglect a resistance of the rails.

♦ 58. Linear Process

Page 113

One mole of an ideal polyatomic gas proceeds from a state B, in which its temperature is $T_B = 217\,°$C, to a state D. In the process, there is a linear relation between gas pressure and volume, its temperature monotonously decreases, and a heat is being transferred to the gas all the way (fig. 44).

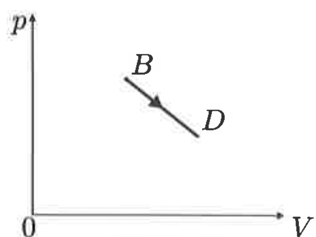

Figure 44

Determine the maximum work the gas can perform in such a process.

♦ 59. Circle of Lenses
Page 114

It is rumoured about a manuscript in Snell' archive describing how a ray can propagate through a system of N identical lenses which optical centres lie on a circle and which planes are perpendicular to this circle and aligned with the circle radii. The ink has faded, so only the traces of the planes of two adjacent lenses and the focal point of a lens remained (fig. 45). According to the text, a ray refracted by a lens follows along a side of a regular N-gon.

The lens and its diameter are shown in the diagram 46. Which type of lens can they possibly be, converging or diverging?

By using a ruler and a compass, restore:

1. positions of two more lenses (to the left and to the right of the lens planes shown);
2. possible positions of optical centres of these four lenses;
3. a possible ray path through the lenses.

Justify your answer.

♦ 60. Dipole in Electric Field
Page 115

A dipole consists of two point charges $+q$ and $-q$ separated by a fixed distance l. The dipole mass equals m. The dipole is aligned with an x-axis and flies at a speed v_0 into a region of electric field which length satisfies $2L \gg l$ (fig. 47). In this region the electric field vector $\vec{E}$

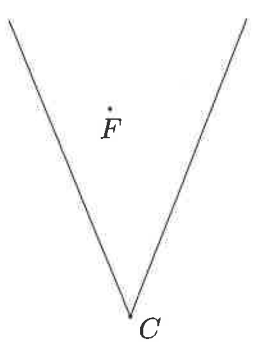

Figure 45

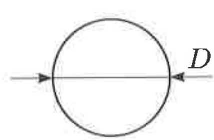

Figure 46

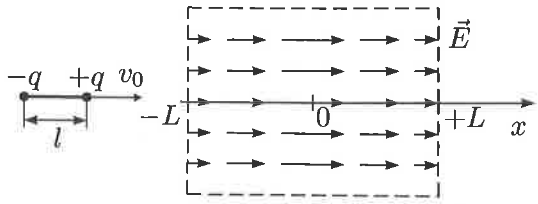

Figure 47

points along the x-axis and its magnitude varies as

$$E(x) = E_0 \left(1 - \frac{x^2}{L^2}\right).$$

Determine the force exerted on the dipole as a function of its coordinate x, the maximum dipole speed, and the time of flight through the region. Assume the dipole alignment with x-axis to persist.

Note. Such an electric field between the plates of a plane capacitor can be produced by an appropriate volume charge.

Final stage 2013

♦ 61. Three in A Field, not Counting Capacitor Page 116

A flat shape consisting of three squares of a side a is made of the same wire (fig. 48). A small sized capacitor of a capacitance C is soldered into a wire segment. The whole system is placed in a uniform magnetic field $\vec{B}$ perpendicular to the shape plane.

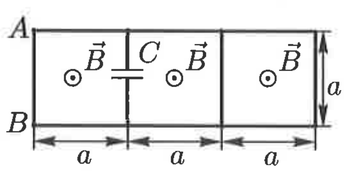

Figure 48

The magnetic field slowly increases at a constant rate $dB/dt = k > 0$. A resistance of a wire segment of the length a equals r. Determine for the stationary regime

1. a magnitude and direction of the current in the segment AB;
2. a charge Q stored by the capacitor and the signs of plate charges;
3. a heat W generated in the circuit in a time τ.

♦ 62. Spoke Oscillation

Page 117

Two identical small metal spheres are attached to the ends of a light spoke of a length L (fig. 49). The spoke is placed on a stand of a width $l \ll L$, so that the spoke middle point is above the stand middle and then it is deflected by a small angle $\varphi_0 \ll 1$. Determine the period of small oscillations of the spoke provided it does not slide over the stand and the energy losses, when the spoke switches from one edge to the other, are negligible.

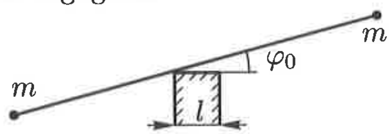

Figure 49

♦ 63. Soda Pressure

Page 118

An industrial process of making bottled soda consists in saturating water with carbon dioxide at a temperature $t_1 = 4\,°\mathrm{C}$ and a pressure $p_1 = 150$ kPa. Then the bottles are sealed and transported to a storage facility where the temperature must not exceed $t_2 = 35\,°\mathrm{C}$ according to regulations.

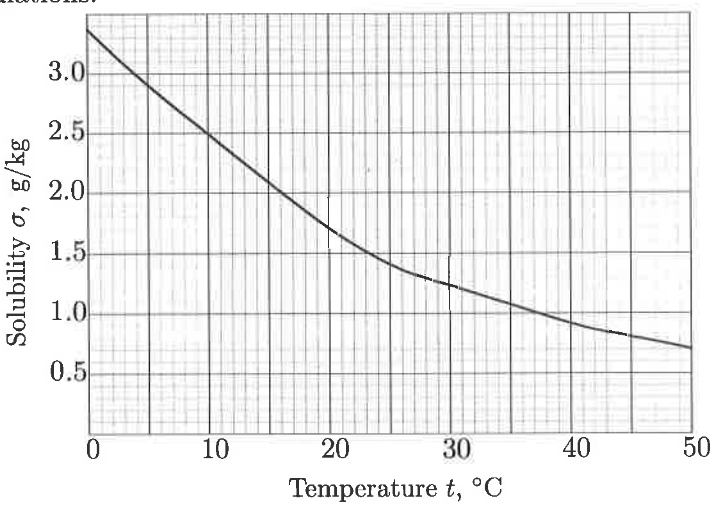

Figure 50

Neglect a change of the volumes of liquid and bottle and determine

1. the minimum empty volume V_0 above the soda level if the maximum pressure in a stored bottle is $p_2 = 370$ kPa;
2. a level x, to which a bottle is filled at the production facility, corresponding to this volume.

A mass of water in the bottle is $m_w = 2$ kg, the molar mass of carbon dioxide is $\mu = 44$ g/mol.

Geometric dimensions of a bottle:
$d = 3$ cm, $b = 10$ cm, $h = 27$ cm, $H = 30$ cm, and $D = 13$ cm.

Note. Assume that a gas solubility σ at constant temperature is proportional to its partial pressure above a liquid (Henry's law). The carbon dioxide solubility in water versus temperature at a partial pressure $p_0 = 100$ kPa is plotted in the diagram. A partial pressure of water vapour in this temperature range is negligible. Solubility is defined as a mass of a gas (in grams) dissolved in 1 kg of a liquid.

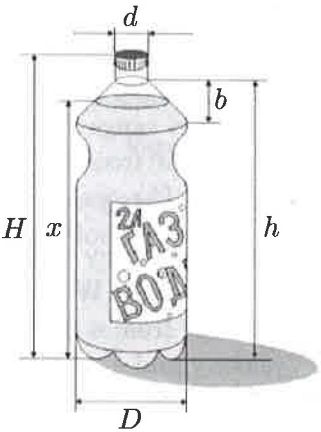

Figure 51

♦ 64. Lens Recovery

Page 120

It is rumoured that Snell's archive contains an optical scheme depicting an ideal thin lens, an object, and its image. The text describes the object as a rod of a length l with two point light sources at the ends. The rod and the main axis lie in the diagram plane and the rod does not intersect the lens plane. The ink has faded, so only the light sources and their images remained and it is not clear which points correspond to the images and which to the sources. It is interesting that the points are located at vertices and at the centre of an equilateral triangle (fig. 52).

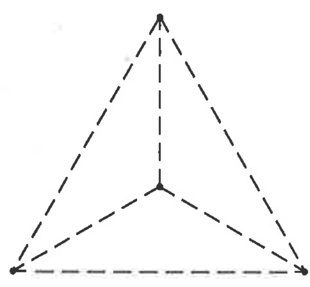
Figure 52

1. Find out whether the point at the triangle centre corresponds to the object or to its image.

2. Recover the optical scheme (the object, the image, the lens, its main axis and the focal points) up to a 120° rotation and a reflection of the diagram.

3. Determine the lens focal length.

Note. A lens is ideal if any parallel beam is focused in its focal plane.

♦ 65. Faulty Rocket

Page **121**

Lieutenant-experimenter Glitch did his research with new flare rockets on a military range. A rocket flying at a constant speed v yields a sound at a constant frequency f_0. Glitch used frequency sensors to register the sound. The speed of sound on the range is $c = 330$ m/s.

1. What frequency will be registered by a sensor if a rocket is coming directly to it? What frequency will be registered by a sensor located far away from a flying rocket if there is an angle φ between a rocket velocity and a direction to the sensor?

2. When doing his research, lieutenant-experimenter Glitch accidentally fired a faulty rocket which began flying in a circle of a radius r at a low height above the ground and at the same speed v. The rocket was successfully brought under control and the lieutenant noticed a time dependence of the sound frequency received by sensors 1 and 2 which was recorded by a plotter. Use the plots (fig. 53) to help the lieutenant to determine a distance L between the sensors.

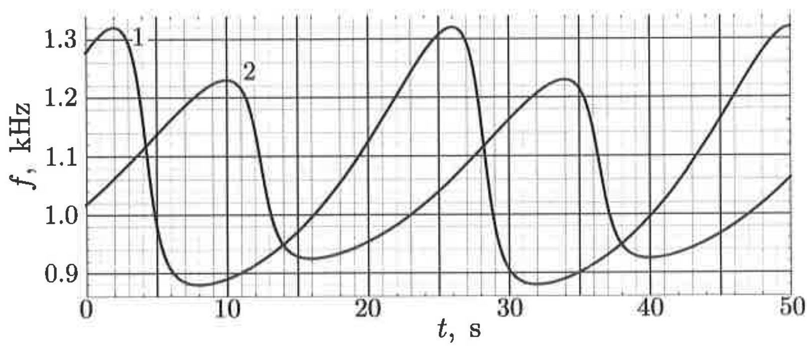

Figure 53

Final stage 2014

♦ 66. Spring Stretching

Page **124**

A thin light spring is extended by Δl_1 and fixed on a smooth horizontal table at points A and B. A ratio of the periods of small transverse (fig. 54) and longitudinal (fig. 55) oscillations of a small weight at the spring midpoint equals $n_1 = 4$. After the extension has been increased by $\Delta x = 3.5$ cm, the ratio became $n_2 = 3$.

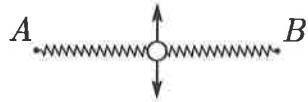

Figure 54

Figure 55

Determine the length l_0 of the loose spring and the extensions Δl_1 and Δl_2 in both cases. Assume that the spring obeys Hooke's law.

♦ 67. Nanomelting

Page **125**

The melting point of a large tin sample is $t_0 = 232\,°\mathrm{C}$. The melting point of a tiny tin sphere of a diameter $d = 20$ nm is 25 degrees less and equals $t_d = 207\,°\mathrm{C}$. This is a so-called dimensional effect; it has been experimentally verified that melting point depends both on a sample size and shape. What is the melting point of a tin foil of a thickness $h = d$?

Assume that tin atoms in a surface layer 2–3 atoms thick possess a certain additional energy compared to atoms in the bulk and the heat of fusion λ per atom is proportional to an average binding energy U of atom while the latter is proportional to the temperature T of the melting point (phase transition): $\lambda \sim U \sim T$.

♦ 68. Figure-Eight of Lord Kelvin

Page **126**

A plot of a cyclic process performed with an unknown amount ν of nitrogen was found in lord Kelvin's archive. The cycle was plotted in coordinates $(T,\ C)$, where C is a heat capacity and T is temperature, and was represented by four straight lines: $a \to b$, $e \to f$, $c \to b$, and $e \to d$ (fig. 56).

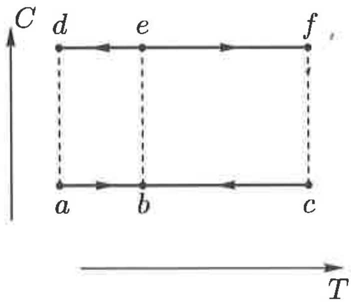

Figure 56

Unfortunately, the position of the origin was lost. According to the manuscript, $C_d = 1.000$ J/K, $C_a = 0.715$ J/K, also

$$T_c - T_b = 2(T_b - T_a) = 200 \text{ K}, \qquad \text{and} \quad \frac{p_c}{p_a} = \frac{V_c}{V_a}.$$

1. Determine the work A done per the cycle and the cycle efficiency η.
2. Determine the temperatures T_a, T_b and T_c.
3. Draw the cycle in coordinates (p, V) and determine ν.

Note. A process at a constant heat capacity C is called polytropic, it obeys the relation

$$pV^n = \text{const},$$

where n is a constant called the polytropic index.

♦ 69. From Snell's Archive Page 129

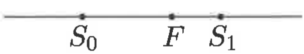

Figure 57

There is a drawing of an optical scheme found in Snell's archive. The drawing depicts a lens, a point light source S_0, and its image S_1. The ink has faded with time, so one can see only the lens optical axis, the source S_0, the image S_1, and one of the focal points F (fig. 57). Restore possible lens positions by using a ruler (without markings on it) and a compass.

♦ 70. Electroshock

Page 132

Two identical small steel spheres of a radius $r = 5$ mm and a mass $m = 4$ g are suspended from a non-conducting ceiling on thin metal wires of a length $l = 1$ m at a distance $d = 10$ cm apart (fig. 58). Initially the spheres are at rest and have no electric charge.

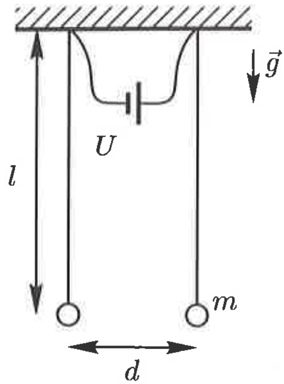

Figure 58

1. Determine the period T of small free oscillations of the spheres.
2. A source of a constant voltage U with a large internal resistance $R = 10^{15}$ Ω is connected to the points of wire suspension. At what minimum voltage $U = U_{\min}$ the spheres will collide after some time?
3. Determine a time t_0, in which the voltage between the spheres reaches the value $U_{\min}$, if $U = U_0 = 1.0 \cdot 10^6$ V.

Final stage 2015

♦ 71. Oscillations

Page 133

A positive electric charge is uniformly distributed over a fixed dielectric semi-sphere. The semi-sphere symmetry axis is vertical. A pendulum is suspended from a pivot O coinciding with the semi-sphere centre of curvature. The pendulum consists of a small sphere with a charge q_1 and a thread which length is less than the semi-sphere radius (fig. 59). A period of harmonic oscillations of the pendulum near the vertical equals T.

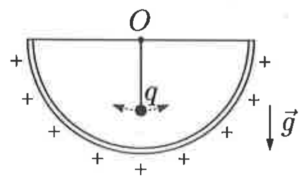

Figure 59

After the sphere charge has been changed to q_2, and $|q_2/q_1| = 2$, the period of oscillations near a new position of equilibrium, in which the thread is also vertical, turned out to be T again.

Evaluate T, if a period of simple harmonic oscillations of the pendulum in the non-charged semi-sphere is $T_0 = 1.0$ s. Ignore an electric field due to polarisation charges.

♦ 72. Little Conducting Cube

Page 135

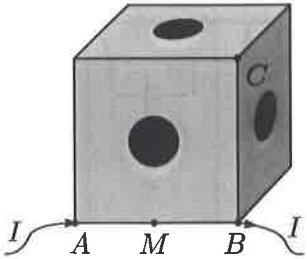

Figure 60

A little cube is made of six identical conducting plates with round holes drilled at their centres. Small identical conducting spheres are inserted into the cube vertices, so that wires can be connected to them. The hole diameters are such that an electric resistance of the cube between two adjacent vertices A and B turns out to be $R_{AB} = r = 32\ k\Omega$.

If a current $I = 1$ mA flows through these vertices in the direction shown in the figure, a voltage across the point M (the midpoint of the edge AB) and the vertex C will be equal to $U_{MC} = \varphi_M - \varphi_C = U = 2.0$ V. Determine a resistance R_{AC} between A and C. By how much will the resistances R_{AB} and R_{AC} change if the plate size and the hole diameter are doubled while the thickness remains the same?

♦ 73. Cosmic Object

Page 136

A cosmic object travels at a constant speed along a straight line and transmits periodic radio pulses. An astronomer has determined that during an observation period a direction to the object turned by a small angle $\Delta\varphi$ while a period of radio pulses arrival changed from T to $T + \Delta T$ ($\Delta T \ll T$). Determine a distance between the astronomer and the object. The speed of radio pulses equals the speed of light c.

♦ 74. «Millicar»

Page 138

A tiny, about the size of ant, car travels on an even horizontal surface along the main optical axis of a converging lens with a focal length f. A point light source S moving along the optical axis is fixed on the car roof. The car velocity varies so that the velocity of an image S_1 of the source S remains constant and equal to v_0. Find out at which distance from the lens such a motion of the «car» is possible. The coefficient of friction between the car wheels and the road is μ.

Final stage 2016

♦ 75. Friction Drive

Page 139

A long cylinder of a radius R_0, which is rotating around its axis at an angular speed ω_0, has been freely pressed (without applying friction to its axis) against a disk of a radius R rotating around its axis. The line of contact between the disk and the cylinder is along a disk radius as shown on figure 61.

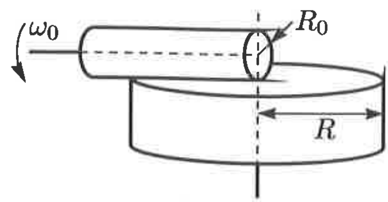

Figure 61

1. Determine a stationary angular velocity ω_μ of disk rotation providing there is a dry friction between the cylinder and the disk.

2. Determine a stationary angular velocity ω_μ of disk rotation in the case of lubricated friction, and a ratio $k = \omega_\eta/\omega_\mu$. Assume that a force of lubricated friction exerted per a unit length of contact is proportional to a relative velocity of the contacting surfaces.

♦ 76. Solar Sail

Page 140

A solar sail is a flat mirror of a mass $m = 1.660$ g and an area $S = 1.000$ m^2. The sail is perpendicular to solar rays and is moving along a straight line passing through the Sun and sail centres. Initially the sail is at a distance $R_0 = 1$ a. u. from the Sun. Determine the distance R_1 to the Sun after $t_1 = 1$ hour of flight if a sail velocity v remained constant although unknown and $v \ll c$?

One astronomical unit is the distance from the Earth to the Sun: 1 a. u. $= 150.0 \cdot 10^6$ km. The photon momentum p and its energy E are related by $pc = E$, where $c = 2.998 \cdot 10^8$ m/s is the speed of light. Do not take into account protons, neutrons and other particles, coming from the Sun. The total flux of solar radiation passing per unit time through a unit area oriented perpendicular to solar rays at a distance 1 a. u. from the Sun is equal to $W_0 = 1.367$ kW/m^2.

Note. Did you know that 1 year $\simeq \pi \cdot 10^7$ s with a half-percent accuracy?

♦ 77. Circular Process
Page 141

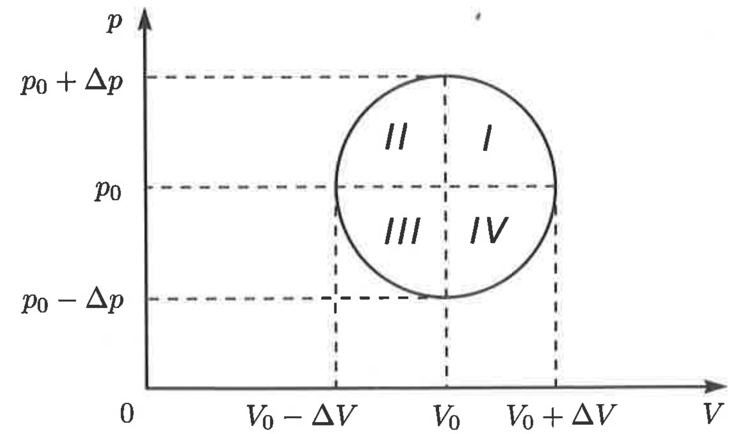

Figure 62

A cyclic process with one mole of an ideal polyatomic gas appears as a circle on the p, V diagram. Coordinates of the circle centre are (p_0, V_0), the diameter along the pressure axis is $2\Delta p$, and the diameter along the volume axis is $2\Delta V$.

1. Determine all pairs of diametrically opposite points of the circle with equal heat capacities. Calculate these heat capacities.
2. Compare heat capacities of two arbitrary diametrically opposite points lying in quadrants 2 and 4 of the circle (fig. 62). In other words, find out at which of these points the heat capacity is greater and why.

Note. Assume that the heat capacity of gas at constant volume is independent of T.

♦ 78. AC Y-circuit
Page 143

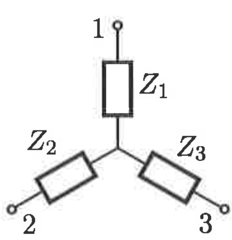

Figure 63

Three elements, which can be resistors, capacitors, and inductors, are connected in a Y-circuit (fig. 63). If an AC source is connected to terminals 1 and 2, a voltmeter connected to terminals 1 and 3 displays 80 V. If connected to terminals 2 and 3, the voltmeter displays 45 V. If the same source is connected to terminals 1 and 3, the voltmeter measures 21 V across 2 and 3 and 28 V across 1 and 2. If the source is connected to 2 and 3, the voltmeter readings are 21 V across 1 and 2 and 28 V across 1 and 3.

1. Determine the source output voltage.
2. Identify the elements corresponding to the legs of Y-circuit. Is it possible to determine element types unambiguously?
3. Determine the ratio of currents $I_{12} : I_{13} : I_{23}$ through the AC source when it is connected to terminals 1 and 2, 1 and 3, and 2 and 3.

The source, the voltmeter, and all circuit elements are assumed to be ideal.

Final stage 2017

♦ 79. I'm Little Rain Cloud... Page 144

Using the model of adiabatic atmosphere, evaluate
1. a height H of the Earth atmosphere;
2. an elevation h_0 of the lower cloud level.

A temperature at the ground level is $t_0 = 27\,°\mathrm{C}$ and a relative humidity of air is $\varphi = 80\%$. Assume $h_0 \ll H$. A pressure P_H (in mmHg) of saturated water vapour versus temperature t is shown in the table below. Air can be considered as an ideal diatomic gas with a molar mass $\mu = 29$ g/mol.

Directive. In adiabatic atmosphere a parcel of gas moving vertically without exchanging heat with the environment remains in mechanical equilibrium.

t, °C	6	8	10	12	14	16
P_H, mmHg	7.01	8.05	9.21	10.5	12.0	13.6
t, °C	18	20	22	24	26	28
P_H, mmHg	15.5	17.5	19.8	22.4	25.2	28.4

Figure 64

♦ 80. Slinky

Page 146

A «slinky» coil (fig. 64) is being held by the upper turn so that its lowest turn is at $h = 1$ m above the ground. The length of the coil being stretched by its own weight is $l = 1.5$ m. The coil is then released. In what time τ will the coil reach the ground? When unstretched, the coil is $l_0 = 6$ cm long, its thin turns fit tightly without exerting pressure on each other. Collapsing turns collide inelastically and the whole coil has already collapsed when it touched the ground. The answer should be calculated with an accuracy of 0.02 s.

♦ 81. Small Bead

Page 147

A charge Q is uniformly distributed over a surface of a dielectric thin fixed tube of a radius R and a length H. A bead carrying a charge of the same sign can freely slide along a thin insulator spoke coinciding with a diameter of the middle (equidistant from the ends) cross-section. Determine a period T of small oscillations of the bead near a point of equilibrium. A bead specific charge $\gamma = q/m$ is known. Neglect the gravity.

♦ 82. Lunar Eclipse

Page 148

The Sun is not a point-like source of light; when viewed from Earth, its angular diameter is $2\delta = 0.52°$. Because of this, the region completely shadowed by Earth turns out to be finite.

1. Suppose a refraction of solar light by Earth atmosphere is negligible. At what distance L_1 from Earth could the complete shadow be still observed? Determine a duration of the total lunar eclipse in this case.

2. Actually the refraction significantly affects the size of the region of total shadow. Assume the height of Earth atmosphere to be $h = 8$ km with an average refraction index $n = 1.00028$.

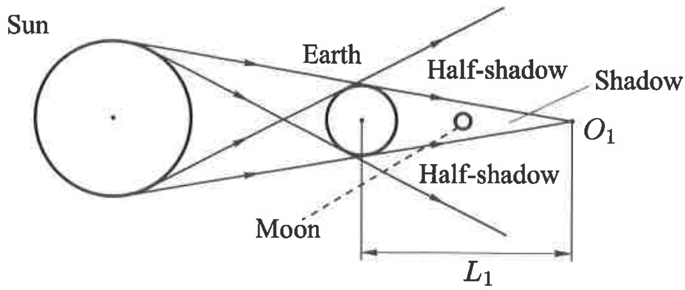

Figure 65

3. Assume that the boundary of the shadow is formed by rays tangent to the Earth surface and determine at what maximum distance L_2 would the complete shadow be observed in this case. What fraction of the Moon disk will be shadowed?

Earth radius is $R = 6400$ km, the free fall acceleration is $g = 9.8$ m/s^2, the angular diameter of the Moon equals to the angular diameter of the Sun 2δ, and the Moon orbital period around Earth is $T_0 = 27.3$ days.

Solutions

1. A Fly *A. Voronov*

The mirror-lens system can be considered as a lens which object is a virtual image M_1 of the fly M formed by the mirror. Let M' and M_1' be the images of M and M_1 formed by the lens. Since velocities of the «flies» M and M_1 are equal, the velocities of their images can be equal in magnitude only if the lens is converging, the image M' is virtual, and M_1' is real. In this case the transverse magnification of the objects M and M_1 is the same:

$$\frac{F}{F-a} = \frac{F}{a_1 - F}.$$

Here a is the distance between the lens and the fly and a_1 is the distance between the lens and the fly image in the mirror. It follows from this equation that $a + a_1 = 2F$. Therefore, the mirror is located at the lens focal plane, i.e. $L = F = 20$ cm. The result is independent of a, so the latter can be any value in the interval of $(0; L)$.

2. Boat *A. Sheronov*

Let Δv be an increment of the boat velocity v in a small time Δt. Let us write the second Newton's law for the boat motion in the lake:

$$\frac{m\Delta v}{\Delta t} = -kv,$$

where m is a boat mass and k is a drag coefficient. Notice that $v\Delta t = \Delta S$ is the boat displacement in Δt, so the above equation can be written as $m\Delta v = -k\Delta S$. Now consider the boat motion in the river.

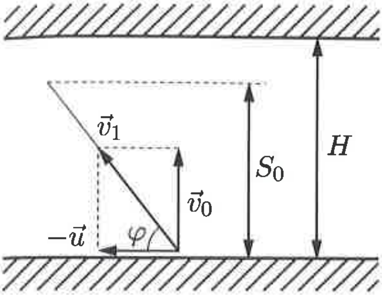

Figure 66

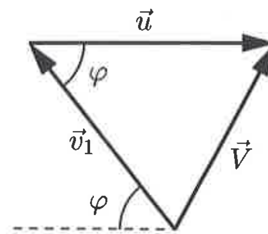

Figure 67

In the water frame the boat is traveling along a straight line at the initial speed

$$v_1 = \sqrt{v_0^2 + u^2}, \tag{1}$$

where u is the current velocity (fig. 66).
Similar to the boat motion in the lake, there is an equation $mv_1 = kS_1$ from which the equation $v_1 = \alpha v_0$ can be easily derived. Taking into account (1) one finds $v_0\sqrt{\alpha^2 - 1}$. The angle φ between vectors $\vec{v}_1$ and $(-\vec{u})$ is determined from the equations:

$$\sin\varphi = \frac{S_0}{S_1} = \frac{1}{\alpha},$$

$$\cos\varphi = \sqrt{1 - \frac{1}{\alpha^2}}.$$

Summing up the increments on both sides of $m\Delta v = -k\Delta S$ up to the moment the boat reached the middle of the river, one obtains:

$$m(v_1 - v_1') = k\frac{H}{2\sin\varphi}.$$

It follows that the boat velocity in the middle of the river is

$$v_1' = v_1 - \frac{kH}{2m\sin\varphi} = \alpha\left(1 - \frac{\alpha}{2}\right)v_0.$$

The boat velocity relative to the bank at that moment can be found from the law of cosines applied to the triangle shown (fig. 67):

$$V = \sqrt{u^2 + v_1'^2 - 2uv_1'\cos\varphi} = \sqrt{1 - \alpha + \frac{\alpha^4}{4}}v_0 = \frac{3}{32}\sqrt{41}v_0 \approx 0.6v_0.$$

3. Variable Equilibrium

O. Shvedov

Suppose that before new equilibrium had been reached, a rate of forward reactions exceeded a rate of reverse ones by $N_A x$. Then

$$\Delta\nu_1 = -2x, \qquad \Delta\nu_2 = -x, \qquad \Delta\nu_3 = 2x, \tag{2}$$

$$\Delta\nu = \Delta\nu_1 + \Delta\nu_2 + \Delta\nu_3 = -x.$$

Let us write the ideal gas law for the equilibrium states:

$$pV = (\nu_1 + \nu_2 + \nu_3)\, RT, \qquad (p + \Delta p)(V + \Delta V) = (\nu_1 + \nu_2 + \nu_3 - x)\, RT.$$

For small increments the solution of this equations is

$$\frac{\Delta p}{p} + \frac{\Delta V}{V} = -\frac{x}{\nu_1 + \nu_2 + \nu_3}. \tag{3}$$

Now let us derive an equation to determine $\Delta V/V$ and x. The quantity

$$\frac{(\nu_1/V)^2\,(\nu_2/V)}{(\nu_3/V)^2}$$

is proportional to the reaction rates and it depends only on temperature, hence, it is the same for the two equilibrium states:

$$\frac{\nu_1^2\nu_2}{\nu_3^2 V} = \frac{(\nu_1 + \Delta\nu_1)^2\,(\nu_2 + \Delta\nu_2)}{(\nu_3 + \Delta\nu_3)^2\,(V + \Delta V)}$$

whence $$\frac{\Delta V}{V} = 2\frac{\Delta\nu_1}{\nu_1} + \frac{\Delta\nu_2}{\nu_2} - 2\frac{\Delta\nu_3}{\nu_3}.$$

Using (2), one obtains

$$\frac{\Delta V}{V} = -x\left(\frac{4}{\nu_1} + \frac{1}{\nu_2} + \frac{4}{\nu_3}\right). \tag{4}$$

Solving together (3) and (4), one finds:

$$x = \frac{\Delta p}{p}\left(\frac{4}{\nu_1} + \frac{1}{\nu_2} + \frac{4}{\nu_3} - \frac{1}{\nu_1 + \nu_2 + \nu_3}\right)^{-1},$$

$$\frac{\Delta V}{V} = -\frac{\Delta p}{p}\,\frac{\frac{4}{\nu_1} + \frac{1}{\nu_2} + \frac{4}{\nu_3}}{\frac{4}{\nu_1} + \frac{1}{\nu_2} + \frac{4}{\nu_3} - \frac{1}{\nu_1+\nu_2+\nu_3}}.$$

Since x is known, $\Delta\nu_1$, $\Delta\nu_2$, and $\Delta\nu_3$ follow from (2).

4. Electric Charge, Hollow Sphere, and Dielectric *V. Chivilev*

The minimum work equals a change in the energy of electric field. By comparing the electric field of the initial and final configurations, one can conclude that the energy increment is equal to $W_2 - W_1$, where W_1 is the energy of the field in the dielectric sphere with inner and outer radii R_2 and R_3 (the field is due to the charge Q located at the center) and W_2 is the field energy in the «empty» space between the spheres with radii R_1 and R_3 (the field is due to the charge Q located at the common centre of the spheres). It is convenient to find W_1 and W_2 as the energies of the corresponding spherical capacitors C_1 and C_2 with the charges equal to Q on their plates.

Let us determine C_2 and W_2. A voltage across a spherical capacitor with plate radii R_1 and R_3 is

$$U = \left(k\frac{Q}{R_1} - k\frac{Q}{R_3}\right) - 0 = kQ\frac{R_3 - R_1}{R_1 R_3}, \qquad \text{where} \quad k = \frac{1}{4\pi\varepsilon_0}.$$

The capacitance is

$$C_2 = \frac{Q}{U} = \frac{R_1 R_3}{k(R_3 - R_1)}$$

and the energy is

$$W_2 = \frac{Q^2}{2C_2} = \frac{kQ^2(R_3 - R_1)}{2R_1 R_3}.$$

Similarly,

$$C_1 = \frac{\varepsilon R_2 R_3}{k(R_3 - R_2)}, \qquad W_1 = \frac{kQ^2(R_3 - R_2)}{2\varepsilon R_2 R_3}.$$

The required work is

$$A = W_2 - W_1 = \frac{kQ^2}{2R_3}\left(\frac{R_3 - R_1}{R_1} - \frac{R_3 - R_2}{\varepsilon R_2}\right).$$

5. Three Batteries *O. Shvedov*

First, let us study the system of identical batteries in the circuit centre (fig. 68). Let the internal resistance of a battery be r. Recall that

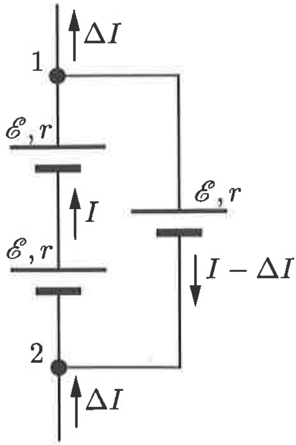

Figure 68

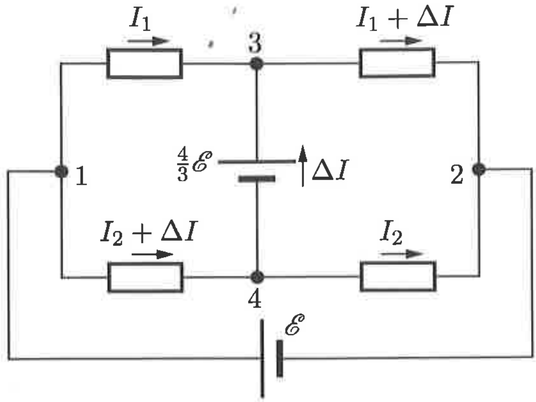

Figure 69

$r \ll R$. A voltage U_{34} across points 1 and 2 can be evaluated from equations

$$U_{12} = 2\mathscr{E} - 2Ir, \qquad U_{12} = \mathscr{E} + (I - \Delta I)\, r,$$

hence $U_{12} = \frac{4}{3}\mathscr{E} - \frac{2}{3}\Delta I r$.
This means that the system is equivalent to a single battery with an emf $\frac{4}{3}\mathscr{E}$ and an internal resistance $\frac{2}{3}r$ which is neglected from now on. Let us replace the circuit with an equivalent one (fig. 69). From symmetry, $I_1 = I_2$. Therefore,

$$\mathscr{E} = U_{14} = I_1 R + (I_1 + \Delta I)\, R,$$

$$I_1 R = U_{13} = U_{12} + U_{23} = (I_2 + \Delta I)\, R - \frac{4}{3}\mathscr{E},$$

and

$$I_1 = -\frac{1}{6}\frac{\mathscr{E}}{R}, \qquad \Delta I = \frac{4}{3}\frac{\mathscr{E}}{R}.$$

Thus the currents via the resistors are

$$I_1 = I_2 = -\frac{1}{6}\frac{\mathscr{E}}{R}, \qquad I_1 + \Delta I = I_2 + \Delta I = \frac{7}{6}\frac{\mathscr{E}}{R}.$$

6. Something Fictional *Folklore*

Since the Earth mass M is much larger than the Moon mass m, the Earth can be considered as being at rest. The falling Moon can be considered as moving along a degenerate ellipse with a large semi-major axis $a = L/2$ and a period of rotation $T = 2\tau$. According to the third Kepler's law

$$\frac{a^3}{T^2} = \frac{GM}{4\pi^2} = \text{const.} \tag{5}$$

The gravitational constant G and the Earth mass can be expressed in terms of the free fall acceleration on the Earth surface:

$$g = \frac{GM}{R^2}, \qquad \text{hence} \quad GM = gR^2. \tag{6}$$

From (5) and (6) one finds

$$\tau = \frac{T}{2} = \pi\sqrt{\frac{a^3}{gR^2}} = \frac{\pi L}{2R}\sqrt{\frac{L}{2g}} \approx 4.19 \cdot 10^5 \text{ s} \approx 4.85 \text{ days.}$$

Using the law of conservation of energy

$$-\frac{GMm}{L} = \frac{mv^2}{2} - \frac{GMm}{R}$$

one finds the relative velocity of the planets just before the impact:

$$v = \sqrt{2gR\left(1 - \frac{R}{L}\right)} \approx \sqrt{2gR} \approx 11.2 \text{ km/s.}$$

Note. There is no need to memorise the constant in the Kepler's law (5) because it can be easily derived for the special case of circular orbit.

7. Not Quasistatic Cyclic Processes *A. Chudnovsky*

A process taking place in the cylinder is not quasi-static, therefore one cannot use the Poisson adiabatic equation. Let us apply the law of conservation of energy in the «gas-piston-weight» system to the process of gas compression by the installed weight. The initial and final states are (P_1, V_1) and (P_2, V_2):

$$\Delta U_{12} = \frac{5}{2}(p_2 V_2 - p_1 V_1) + (4m + m)g(h_2 - h_1) = 0. \qquad (7)$$

Here $4m$ is the piston mass, m is the weight mass, and h_1 and h_2 are the initial and final vertical coordinates of the piston. Let S be the piston cross-sectional area, then

$$\begin{aligned} p_1 &= \frac{4mg}{S}, \quad p_2 = 1.25 p_1 = \frac{5mg}{S}, \\ V_1 &= Sh_1, \quad V_2 = Sh_2. \end{aligned} \qquad (8)$$

Combining (7) and (8) one finds

$$V_2 = \frac{6}{7} V_1.$$

Similarly, consider the process of gas expansion to a state (p_3, V_3) after the weight removal:

$$\Delta U_{23} = \frac{5}{2}(p_3 V_3 - p_2 V_2) + 4mg(h_3 - h_2) = 0. \qquad (9)$$

Since $4mgh_3 = (4mg/S)Sh_3 = p_3 V_3$, it follows from (9) that

$$V_3 = \frac{33}{28} V_2 = \frac{99}{98} V_1.$$

The volume increases by the factor 99/98 after each cycle, therefore the required number of cycles is

$$n = \left[\frac{\ln 2}{\ln 99/98}\right] \approx [68.3] \approx 68.$$

8. Clausius' Gas *V. Slobodyanin*

In the process under study,

$$p = p_0 \left(1 - \frac{V}{V_0}\right).$$

The Clausius' gas temperature

$$T = \frac{p(V-b)}{R} = -\frac{p_0}{RV_0}\left(V^2 - (V_0+b)V + V_0 b\right).$$

The temperature reaches its maximum when the derivative $dT/dV = 0$. This happens when $V = (V_0 + b)/2$. Therefore the maximum temperature T_K of Clausius' gas is

$$T_K = \frac{p_0 V_0}{4R}\left(1 - \frac{b}{V_0}\right)^2 \approx \frac{p_0 V_0}{4R}\left(1 - \frac{2b}{V_0}\right).$$

The maximum temperature T_{Id} of an ideal gas can be obtained by setting $b \to 0$:

$$T_{\text{Id}} = \frac{p_0 V_0}{4R}.$$

Thus, the maximum temperature of ideal gas exceeds by

$$\Delta T = T_{\text{Id}} - T_K \approx \frac{p_0 b}{2R} \approx 4.0\ \text{K}$$

the maximum temperature of Clausius' gas in this process.

9. Superconducting Solenoid and Power Source *S. Kozel*

Let us find the current I_0 for which the magnetic induction reaches its critical value:

$$B_0 = \mu_0 I_0 \frac{N}{l}, \qquad \text{hence} \quad I_0 = \frac{B_0 l}{\mu_0 N} \approx 100.3\ \text{A}.$$

The maximum current provided by the source is

$$I_{\max} = \frac{\mathscr{E}}{r} = 120\ \text{A} > 100\ \text{A} \approx I_0,$$

therefore the superconducting state will be destroyed.
Using Kirchhoff' law

$$L\frac{dI}{dt} + rI = \mathscr{E},$$

one can find a current through the solenoid as a function of time:

$$I = \frac{\mathscr{E}}{r}\left(1 - e^{-t/\tau}\right),$$

where the time constant $\tau = L/r$ and the solenoid inductance $L = \mu_0 N^2 S/l$. The critical current is attained at

$$t_0 = -\tau \ln\left(1 - \frac{I_0 r}{\mathscr{E}}\right) = -\frac{\mu_0 S N^2}{lr} \ln\left(1 - \frac{B_0 l r}{\mu_0 N \mathscr{E}}\right) \approx 18 \text{ ms}.$$

10. Photoelectric Effect *S. Kozel*

1. The energy of an incident quantum with a frequency ν is

$$E = h\nu = \frac{hc}{\lambda}.$$

The work function is determined by the photoelectric threshold:

$$A = \frac{hc}{\lambda_0}.$$

Using Einstein' equation, one finds the maximum kinetic energy of photoelectrons:

$$K_{\text{max}} = E - A = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right) \approx 8.677 \cdot 10^{-20} \text{ J}.$$

Since $K_{\text{max}} = \dfrac{mv_1^2}{2}$,

$$v_1 = \sqrt{\frac{2K_{\text{max}}}{m}} \approx 4.37 \cdot 10^5 \text{ m/s}.$$

2. Applying the conservation of energy to the first electrons, one obtains:

$$K_{\text{max}} + e\varphi_0 = \frac{mv_2^2}{2};$$

$$v_2 = \sqrt{\frac{2(K_{\text{max}} + e\varphi_0)}{m}} \approx 6.05 \cdot 10^5 \text{ m/s}.$$

3. The ball potential after a prolonged exposure is found from the condition that photoelectrons cannot go far away from the ball anymore:

$$K_{\text{max}} + e\varphi_1 = 0, \qquad \text{hence} \quad \varphi_1 = -\frac{K_{\text{max}}}{e} \approx +0.54 \text{ V}.$$

4. The ball potential $\varphi = q/(4\pi\varepsilon_0 R)$ had changed by $\Delta\varphi = \varphi_1 - \varphi_0$ during the time of exposure, therefore its charge changed by

$$\Delta q = 4\pi\varepsilon_0 R\Delta\varphi,$$

so

$$N = \frac{\Delta q}{-e} = \frac{4\pi\varepsilon_0}{-e}R(\varphi_1 - \varphi_0) \approx 7.2 \cdot 10^6.$$

11. A Fly in Web *G. Tarnopolsky, A. Chudnovsky*

1. When a fly strikes the web centre perpendicular to the web plane, only the radial threads are stretched. Using Hooke's law, one determines their initial strain:

$$\varepsilon_0 = \frac{F_0}{ES},$$

where $S = \pi r^2$ is the cross-sectional area of a thread. The maximum mass M of a fly is determined by the condition that the fly stops when the web strain reaches its critical value. The energy of elastic deformation of 6 various threads at a strain ε is:

$$W = 6 \cdot \frac{E\varepsilon^2}{2} \cdot Sl.$$

Recall that $E\varepsilon^2/2$ is the energy density of elastic deformation. Using the conservation of energy

$$\frac{Mv^2}{2} + 6 \cdot \frac{E\varepsilon_0^2}{2} \cdot Sl = 6 \cdot \frac{E\varepsilon_{\max}^2}{2} \cdot Sl,$$

one obtains

$$M = \frac{6ESl}{v^2}\left(\varepsilon_{\max}^2 - \varepsilon_0^2\right) = \frac{6\pi r^2 lE}{v^2}\left(\varepsilon_{\max}^2 - \frac{F_0^2}{\pi^2 r^4 E^2}\right) \approx 1.3 \text{ g}.$$

2. A small variable strain of radial threads can be neglected compared to the initial strain ε_0, since, according to the Pythagorean theorem, the former strain is of the second order in a small displacement x (perpendicular to the web plane) of the fly. The restoring force F is due to six radial forces projected on the direction of oscillation:

$$F \approx -6 \cdot F_0 \cdot \frac{x}{\sqrt{l^2 + x^2}} \approx -\frac{6F_0}{l} \cdot x.$$

Thus the effective spring constant is $k = 6F_0/l$.
The period of oscillation is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{ml}{6F_0}} \approx 0.22 \text{ s}.$$

12. Work Done by Gas Mixing *O. Shvedov*

Let V_X be the volume occupied by the pure gas X (to the pistons left), V_Y be the volume occupied by the pure gas Y (to the pistons right), and V_{XY} be the volume occupied by a mixture of X and Y. The gas X is uniformly distributed over the volume $V_1 = V_X + V_{XY}$ and the gas Y over the volume $V_2 = V_{XY} + V_Y$. The pressure exerted on the left piston by the gas Y is $p_2 = \nu_Y RT/V_2$ and the pressure exerted by the gas X on the right piston is $p_1 = \nu_X RT/V_1$.
Therefore, a small separation of the pistons makes the gases do a work

$$\Delta A = p_1\Delta V_1 + p_2\Delta V_2 = \nu_X RT\frac{\Delta V_1}{V_1} + \nu_Y RT\frac{\Delta V_2}{V_2}.$$

The net work done by the gases is given by a sum

$$A = \sum \Delta A = \nu_X RT \sum \frac{\Delta V_1}{V_1} + \nu_Y RT \sum \frac{\Delta V_2}{V_2}.$$

Each sum equals the area of the corresponding curvilinear trapezoid:

$$\sum \frac{\Delta V_1}{V_1} = S(V_{01}, V_1'), \qquad \sum \frac{\Delta V_2}{V_2} = S(V_{02}, V_2'),$$

where V_{01}, V_{02} are the initial values of V_1, V_2, and V_1', V_2' are their final values. Therefore,

$$A = \nu_X RT \ln\frac{V_1'}{V_{01}} + \nu_Y RT \ln\frac{V_2'}{V_{02}}.$$

Now recall that

$$V_{10} = V_{X0}, \quad V_{20} = V_{Y0},$$
$$V_1' = V_2' = V_{X0} + V_{Y0}.$$

Hence,

$$A = \nu_X RT \ln\frac{V_{X0} + V_{Y0}}{V_{X0}} + \nu_Y RT \ln\frac{V_{X0} + V_{Y0}}{V_{Y0}}.$$

13. Soap Bubble

Y. Lesnichii

Surface tension of the soap film produces an extra pressure inside the bubble equal to

$$\Delta p = 2 \cdot \frac{2\sigma}{R} = \frac{4\sigma}{R}, \tag{10}$$

since the film has two surfaces. The work done by the film approximately equals the kinetic energy of the air flowing out of the bubble:

$$\Delta p \Delta V = \frac{\Delta m v^2}{2},$$

where ΔV is a bubble volume decrement, Δm is a mass of the displaced air, and Δv is an air velocity inside the tube. Therefore,

$$\frac{\rho v^2}{2} = \Delta p, \qquad v = \sqrt{\frac{2\Delta p}{\rho}}, \tag{11}$$

where $\rho = \Delta m/\Delta V$ is the air density. The rate at which a bubble volume V decreases is

$$\frac{dV}{dt} = vS, \tag{12}$$

where $S = \pi r^2$ is a tube cross-sectional area. Substituting $V = \frac{4}{3}\pi R^3$ into (12) and using (10) and (11), one finds

$$4\pi R^2 \frac{dR}{dt} = \sqrt{\frac{8\sigma}{\rho R}} \cdot \pi r^2, \qquad \text{whence} \quad u = \frac{dR}{dt} = \frac{r^2}{R^{5/2}} \sqrt{\frac{\sigma}{2\rho}}.$$

14. Conducting Sphere

V. Chivilev

1. When the switch is open, a charge distribution on the inner and outer surfaces of the sphere is not uniform while the net charge is zero. The potential at the sphere centre is

$$\varphi = k\frac{Q_1}{R/3} + k\frac{Q_2}{2R} = \frac{k}{R}(3Q_1 + \frac{Q_2}{2}).$$

2. After closing the switch, extra charges q_1 and q_2 accumulate on the inner and outer surface, respectively. Obviously, $q_1 = -Q_1$. Let us determine q_2. The sphere potential $\mathscr{E}$ is due to the charges Q_2 and q_2

because there is no electric field due to Q_1 and q_1 outside the sphere. Let us temporarily remove Q_1 and q_1. This does not alter the distribution of q_2. In the electric field of the charges Q_2 and q_2 the sphere potential $\mathscr{E}$ equals the potential at its centre:

$$k\frac{q_2}{R} + k\frac{Q_2}{2R} = \mathscr{E}.$$

Therefore,

$$q_2 = \frac{\mathscr{E}R}{k} - \frac{Q_2}{2} = 4\pi\varepsilon_0\mathscr{E}R - \frac{Q_2}{2},$$

$$Q = q_1 + q_2 = -Q_1 - \frac{Q_2}{2} + 4\pi\varepsilon_0\mathscr{E}R.$$

15. Circuit and Solenoid

D. Aleksandrov

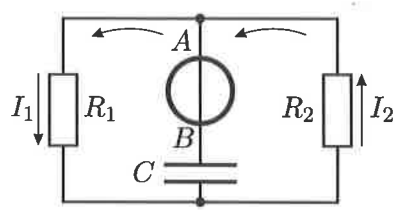

Figure 70

There is an emf induced in any closed circuit by a varying magnetic field, the emf equals the time rate of change of a magnetic flux enclosed by the circuit. Let $\mathscr{E}$ be an emf in the circuit containing R_1 and C (fig. 70), then the emf in the circuit containing R_1 and R_2 equals $2\mathscr{E}$. The second Kirchhoff's law for these circuits reads:

$$\mathscr{E} = I_1R_1 + \frac{q}{C},$$
$$2\mathscr{E} = I_1R_1 + I_2R_2.$$

In the stationary regime the capacitor charge is constant and there is no current via the capacitor, therefore, $I_1 = I_2$. By solving the equations, one obtains:

$$q = \frac{1}{2}CI_1(R_2 - R_1).$$

16. Capacitor Parameters

M. Ogarkov

At the point of balance:

$$r_1 r_4 = Z_3 Z_2, \tag{13}$$

where

$$Z_3 = \sqrt{r_3^2 + (L_3\omega)^2}, \tag{14}$$

$$\frac{1}{Z_2} = \sqrt{\left(\frac{1}{r_2}\right)^2 + (C_2\omega)^2}, \tag{15}$$

and ω is the angular frequency of alternating current. Substituting (14) and (15) into (13), one obtains after some algebra:

$$\left(\frac{r_1 r_4}{r_2}\right)^2 - r_3^2 = \omega^2 \left(L_3^2 - (C_2 r_1 r_4)^2\right). \tag{16}$$

According to the problem statement, the equation (16) is valid at any frequency ω. Therefore, (16) is equivalent to the set of equations:

$$\begin{cases} \left(\dfrac{r_1 r_4}{r_2}\right)^2 - r_3^2 = 0, \\ L_3^2 - (C_2 r_1 r_4)^2 = 0. \end{cases}$$

Solving the set, one obtains:

$$r_2 = \frac{r_1 r_4}{r_3} = 200\ \text{k}\Omega,$$

$$C_2 = \frac{L_3}{r_1 r_4} = 0.5\ \mu\text{F}.$$

17. Exploded Projectile

A. Chudnovsky

Let m_1 and m_2 be the fragment masses, $M = m_1 + m_2$ be the projectile initial mass, and $\vec{p}_0$ be its initial momentum. According to the conservation of momentum,

$$\vec{p}_0 = \vec{p}_1 + \vec{p}_2 \quad \text{or} \quad p_0^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos\alpha.$$

The kinetic energy before and after the explosion:

$$E_{\text{in}} = \frac{p_0^2}{2M} = \frac{p_1^2 + p_2^2 + 2p_1 p_2 \cos\alpha}{2(m_1 + m_2)}, \qquad E_{\text{fin}} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}.$$

A change ΔE of the kinetic energy due to the explosion is

$$\Delta E = E_{fin} - E_{in} = \left(\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}\right) - \frac{p_1^2 + p_2^2 + 2p_1p_2\cos\alpha}{2(m_1+m_2)}.$$

After some algebra one obtains

$$\Delta E = \frac{1}{2M}(p_1^2 k + \frac{p_2^2}{k} - 2p_1p_2\cos\alpha), \qquad \text{where} \quad k = \frac{m_2}{m_1}. \qquad (17)$$

To determine $\Delta E_{\min}$ the derivative of ΔE with respect to k must be set to zero:

$$\frac{d}{dk}\Delta E = \frac{1}{2M}\left(p_1^2 - \frac{p_2^2}{k^2}\right) = 0.$$

Hence,

$$k = \frac{p_2}{p_1} = \frac{2}{3}.$$

Substituting this k into (17), one finds:

$$\Delta E_{\min} = \frac{2p_1p_2 - 2p_1p_2\cos\alpha}{2M} = \frac{p_1p_2(1-\cos\alpha)}{M} = 432 \cdot 10^4 \text{ J}.$$

18. Tethered Puck

E. Butikov

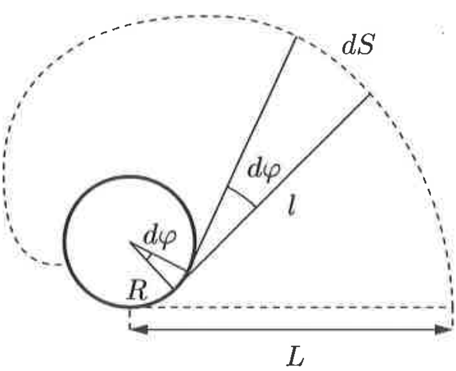

Figure 71

1. Suppose the thread has turned by a small angle $d\varphi$ (fig. 71). Then the puck has traveled a distance $dS = l d\varphi$ while the free part of the thread has shortened by $dl = R\,d\varphi$. Therefore, $dS = l\,dl/R$ and the path traveled by the puck before it collides with the cylinder is

$$S = \int_0^L \frac{l\,dl}{R} = \frac{L^2}{2R}.$$

Since there is no friction, a kinetic energy of the puck and therefore its velocity remains constant. The required time is

$$t = \frac{L^2}{2Rv_0}.$$

2. If there is a friction, the puck goes along an arc with a tangential acceleration $a = -\mu g$. A free puck moving at this acceleration would have traveled the distance $S_1 = \frac{v_0^2}{\mu g}$. Therefore, there are two possibilities:

a) $S \geq S_1$ or $v_0 \leq L\sqrt{\frac{\mu g}{R}}$. In this case, the puck either does not reach the cylinder or just touches it; the travel time is

$$t_a = \frac{v_0}{\mu g};$$

b) If $S \leq S_1$, the puck travels the distance S and either collides with the cylinder or just touches it. The travel time can be found as a time of uniformly accelerated motion:

$$S = \frac{L^2}{2R} = v_0 t - \frac{\mu g t^2}{2}, \qquad \text{or} \quad t^2 - 2\frac{v_0}{\mu g}t + \frac{L^2}{\mu g R} = 0.$$

The quadratic equation has the following roots:

$$t_{1.2} = \frac{v_0}{\mu g} \pm \sqrt{\left(\frac{v_0}{\mu g}\right)^2 - \frac{L^2}{\mu g R}}.$$

The required time t_b must be less than $t_a = \frac{v_0}{\mu g}$ needed for the puck to come at full stop. Hence, the following root should be taken:

$$t_b = \frac{v_0}{\mu g} - \sqrt{\left(\frac{v_0}{\mu g}\right)^2 - \frac{L^2}{\mu g R}} = \frac{v_0}{\mu g}\left(1 - \sqrt{1 - \frac{\mu g L^2}{v_0 R}}\right).$$

c) If $S = S_1$, the times t_a and t_b are equal.

19. Two Thermodynamic Processes *S. Kozel*

In both cases the final temperature in the container equals T_0.

1. Consider the quasistatic compression. At any moment the air temperature equals the ambient temperature, therefore $pV = p_0V_1$. The compression proceeds from the initial volume $V_1 = kV_0$ to the final volume V_0. Since the internal energy of the air remains constant, the amount of heat Q_1 transferred to the air is equal to

$$Q_1 = \Delta A_1 = \int\limits_{V_1}^{V_0} p\,dV = p_0V_1 \int\limits_{V_1}^{V_0} \frac{dV}{V} = p_0V_1 \ln\frac{V_0}{V_1} = -p_0V_1 \ln k.$$

The heat transferred to the surroundings is

$$Q = -Q_1 = kp_0V_0 \ln k = 5.44 \cdot 10^4 \text{ J}.$$

2. Now consider the fast compression. According to the problem statement, a heat exchange between the system and its surroundings is negligible, so the compression is adiabatic. For a diatomic gas $C_V = 5R/2$, $\gamma = 1.4$. Let T_2 be the air temperature at the end of the adiabatic process. Using the equation of adiabatic process,

$$pV^\gamma = \text{const}, \quad \text{or} \quad TV^{\gamma-1} = \text{const},$$

one obtains

$$T_2 = T_0 \frac{V_1}{V_0}^{\gamma-1} = T_0 k^{\gamma-1}. \tag{18}$$

After the compression, the air slowly cools down from T_2 to T_0, a heat is transferred to the surroundings only at this stage. The work done by the air is zero, so

$$Q_1 = \Delta U = (\nu C_V T_0 - \nu C_V T_2) = \frac{5}{2}\nu R(T_0 - T_2).$$

Using the ideal gas law $\nu RT_0 = p_0V_1$ and (18), one obtains the heat transferred to the surroundings:

$$Q = -Q_1 = \frac{5}{2}p_0V_0k(k^{\gamma-1} - 1) = 6.69 \cdot 10^4 \text{ J}.$$

20. At Solenoid Butt *S. Kozel*

The magnetic induction B_0 at the butt of a long solenoid ($z = 0$) equals one half of the magnetic induction inside the solenoid far away from its ends:

$$B_0 = \frac{1}{2}\mu_0 I_c n.$$

Let the ring be located at some distance z from the butt. The net magnetic flux is $\Phi = B_z S + LI = B_0(1-\alpha z)S + LI$ where I is the current through the ring. The superconductor ring preserves the magnetic flux. Initially, $\Phi = 0$. Therefore,

$$I(z) = -\frac{B_0(1-\alpha z)S}{L}.$$

The minus sign indicates that the current in the ring flows in the opposite direction to the current through solenoid coils. Therefore, the ring is repelled by the solenoid. The Ampere's force exerted on the ring is directed upward:

$$F_z = F_A - mg = |I(z)|B_r 2\pi r_0 - mg = \frac{B_0^2(1-\alpha z)S^2}{L}2\beta - mg.$$

At the equilibrium $F_z = 0$, i.e.

$$B_0^2(1-\alpha z) = \frac{mgL}{2\beta S^2}, \qquad \text{or} \qquad \left(\frac{1}{2}\mu_0 I_c n\right)^2 (1-\alpha z) = \frac{mgL}{2\beta S^2}.$$

1. At $z = 0$, the critical value of $B_0^2 = \dfrac{mgL}{2\beta S^2}$, so

$$I_c = I_0 = \sqrt{\frac{mgL}{2\beta}}\frac{2}{S\mu_0 n} = 11.1\ \text{A}.$$

At this current the ring starts levitating.

2. If $I_c > I_0$, the ring is levitating above the sheet at some height $z = z_0$ (and remains coaxial with the solenoid). In this case

$$(1-\alpha z_0) = \left(\frac{I_0}{I_c}\right)^2,$$

hence

$$z_0 = \frac{1}{\alpha}\left[1 - \left(\frac{I_0}{I_c}\right)^2\right].$$

If $I_c = 2I_0$, the distance

$$z_0 = \frac{3}{4\alpha} = 2.08 \text{ cm}.$$

3. In this case, $I_c = 2I_0 = \text{const}$ and the magnetic induction is $B_0 = \mu_0 I_0 n$. Under a small displacement Δz from the equilibrium,

$$F_z = \frac{B_0^2(1-\alpha z_0)S^2}{L}2\beta - \frac{B_0^2\alpha\Delta z S^2}{L}2\beta - mg = -\frac{2\alpha\beta B_0^2 S^2}{L}\Delta z.$$

Thus, there is a quasi-elastic force exerted on the ring with the spring coefficient

$$k = \frac{2\alpha\beta B_0^2 S^2}{L} = \frac{2\alpha\beta(\mu_0 I_0 n)^2 S^2}{L} = 0.14 \text{ N/m}.$$

The frequency of small oscillations of the ring near the equilibrium position is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 6.0 \text{ Hz}.$$

21. Slipping Rope *A. Chudnovsky*

Let ρ be the rope linear density (kg/m), l be the rope length, l_1 and l_2 be the lengths of the rope vertical segments at some moment, T_1 and T_2 be the rope tensions at the points B and C, and F be the vertical component of the force exerted on the rope by the tube (fig. 72). Let us write the second Newton's law for the rope vertical segments:

$$\begin{cases} \rho l_1 a = \rho l_1 g - T_1, \\ \rho l_2 a = T_2 - \rho l_2 g. \end{cases} \tag{19}$$

According to the problem statement, the length of the rope inside the semicircle BC is negligible compared to the length of vertical segments, therefore, $l_1 + l_2 = l$. Besides, a difference between T_1 and T_2

is due to the tangential acceleration a of this short (and, therefore, lightweight) piece of the rope, so let $T_1 = T_2 = T$. Solving the set (19), one obtains

$$a = g\frac{l_1 - l_2}{l_1 + l_2} = 2kg,$$

$$T = 2\rho g\frac{l_1 l_2}{l_1 + l_2} = \frac{1}{2}\rho l g(1 - 4k^2),$$

where the following equations have been used:

$$l_1 - l_2 = 2kl, \quad l_1 = \left(\frac{1}{2} + k\right) l, \quad l_2 = \left(\frac{1}{2} - k\right) l.$$

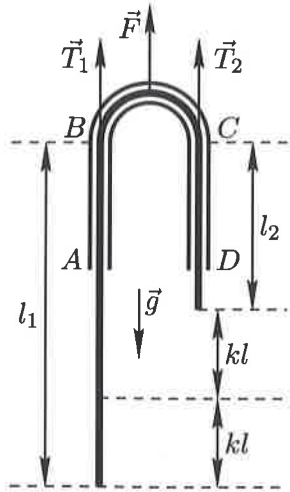

Figure 72

The rope piece inside the semicircle BC is moving along the circle under the forces of rope tension and tube reaction. The net force responsible for the circular motion must be taken into account because for a given rope velocity v the centripetal acceleration is inversely proportional to the diameter of the semicircle BC.

Instead of a time consuming method of summing up the forces over all points of the BC segment, let us use the method of virtual displacements: moving a piece of the rope inside BC by a distance $\Delta l = v\Delta t$ is equivalent to transferring a small rope piece Δl from the point C to B and switching the direction of its velocity. The ensuing change of momentum is due to the impulse of the net force

$$\Delta p = 2(\rho\Delta l)v = (2T - F)\Delta t, \qquad \text{hence} \quad F = 2T - 2\rho v^2. \qquad (20)$$

Velocity v can be determined from the conservation of energy:

$$\frac{\rho l v^2}{2} = E_{kin} = -\Delta E_{pot} = (\rho k l)g(kl), \qquad \text{therefore,} \quad 2\rho v^2 = 4\rho l g k^2.$$

By substituting this formula in (20), one obtains $F = \rho l g(1 - 8k^2)$. At the required moment $F = 0$, so $k = 1/(2\sqrt{2})$ and $a = g/\sqrt{2}$.

22. Pair of Unequal Lenses

O. Shvedov

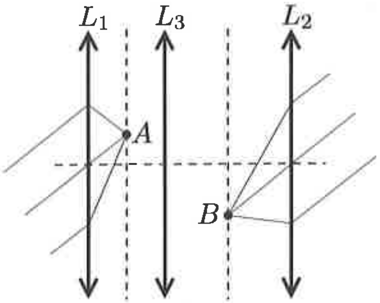

Figure 73

Thin lenses L_1 and L_2 make parallel beams to pass through the points A and B, respectively (fig. 73). Therefore, the lens L_3 must make the beam passing through A to pass through B. Since the beam AB is not refracted, it passes through the lens centre O_3 and this allows one to determine unambiguously the lens location on the optical axis, the lens plane being perpendicular to the main optical axis. The focal length of L_3 is determined from the thin lens formula.

To obtain a quantitative answer, the following coordinates are introduced: the origin is chosen at the centre of L_1, the axis Ox is aligned with the main axis, and the perpendicular axis Oy is chosen so that a parallel beam propagates in the plane Oxy. Let α be a small angle between the parallel beam and the main axis.

Points A and B have coordinates $A(F_1, F_1\alpha)$ and $B(L - F_2, -F_2\alpha)$. The centre O_3 of the lens L_3, which coordinates are $O_3(l_1, 0)$, lies on the line AB; then

$$\frac{l_1 - F_1}{-F_1\alpha} = \frac{L - l_1 - F_2}{-F_2\alpha},$$

$$l_1 = \frac{LF_2^{-1}}{F_1^{-1} + F_2^{-1}} = \frac{LF_1}{F_1 + F_2}, \qquad l_2 = \frac{LF_2}{F_1 + F_2}.$$

Since L_3 must be placed between L_1 and L_2, the following inequalities must hold: $0 < l_1 < L$ and $0 < l_2 < L$. Therefore, the problem has a solution provided both lenses (L_1 and L_2) are either converging or diverging; if one lens is converging and the other one is diverging, the problem has no solution.

To determine the focal length F_3 of L_3, one then uses the thin lens formula:

$$\frac{1}{l_1 - F_1} + \frac{1}{l_2 - F_2} = \frac{1}{F_3}.$$

Hence,

$$F_3 = \frac{F_1 F_2 (L - F_1 - F_2)}{(F_1 + F_2)^2}.$$

The problem has no solution if $L = F_1 + F_2$.

23. «Running Away» Liquid

A. Chudnovsky

Let H be the difference of liquid levels at the beginning of boiling (fig. 74), then gas pressures in the sealed side at the initial and final states are, respectively:

$$\begin{aligned} P_1 &= P_0 + \rho g h, \\ P_2 &= P_0 + \rho g H. \end{aligned} \tag{21}$$

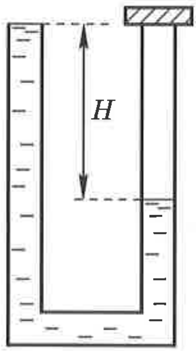

Figure 74

On the other hand, the pressure in the sealed side equals the sum of partial pressures P_v of air and the saturated vapour P_n of the liquid:

$$\begin{aligned} P_1 &= P_{v1} + P_{n1} \approx P_{v1}, \\ P_2 &= P_{v2} + P_{n2} = P_{v2} + P_0. \end{aligned} \tag{22}$$

Each partial pressure increases with temperature, which results in expelling water from the open side. Using the ideal gas law for the air in the sealed side,

$$\frac{P_{v1} \cdot Sh}{T_1} = \frac{P_{v2} \cdot SH}{T_2},$$

one obtains

$$P_{v2} = \frac{hT_2}{HT_1} P_{v1} \approx \frac{hT_2}{HT_1}(P_0 + \rho g h) \tag{23}$$

Equating P_2 from (21) and (22) and substituting P_{v2} from (23), one obtains

$$P_0 + \rho g H = \frac{hT_2}{HT_1}(P_0 + \rho g h) + P_0.$$

Hence,

$$H = h\sqrt{\frac{T_2}{T_1}\left(\frac{P_0}{\rho g h} + 1\right)}.$$

The volume of spilled liquid is

$$\Delta V = S(H-h) = Sh\left[\sqrt{\frac{T_2}{T_1}\left(\frac{P_0}{\rho g h}+1\right)}-1\right].$$

24. Particle and Variable Capacitor *I. Vargin*

The particle energy will be maximal when it is accelerated on the maximum distance by the maximum force possible. The distance is bounded by an allowed distance between the particle and a plate. The electric force exerted on the particle is inversely proportional to the distance between the plates, therefore the plates must be as close as possible at any time.

Let us connect the capacitor to the source so that the signs of the particle charge and the plate coincided. Set the distance between plates to $2a$. Put the particle precisely in the middle between the plates. When the particle starts moving, move the plate so that the distance between the plate and the particle remains equal to a. If $d < 2a$, the particle cannot be accelerated.

Let us choose x-axis along the particle path and the origin at the particle initial position. The equation of particle motion is:

$$m\frac{dv}{dt} = |qE|, \tag{24}$$

where m and v are the particle mass and velocity and $|E| = U/(2a+x)$ is the electric field in the capacitor. Substitute $|E|$ and multiply (24) by dx:

$$m\frac{dv}{dt}dx = |q|\frac{U}{2a+x}dx.$$

Now, using $v = dx/dt$, one obtains from the above:

$$mv\,dv = |q|U\frac{dx}{2a+x}.$$

Integrating this formula between the initial and final states, one obtains:

$$K_{\max} = \frac{mv_{\text{fin}}^2}{2} = |q|U\ln(2a+x)\Big|_0^{d-2a} = |q|U\ln\frac{d}{2a}.$$

25. Electromagnetic Gun

V. Chivilev

1. Let us divide a time of switching off the magnetic field into infinitesimal intervals Δt. Suppose the magnetic flux through a solenoid cross-section changed by $\Delta\Phi$ during such an interval. Then the electromotive force at the point A is directed along the tube and equals

$$E = \frac{-\Delta\Phi}{2\pi R_0 \Delta t}.$$

The force acting on the sphere is $F = qE$. Since $F\Delta t = m\Delta v$, where Δv is a velocity increment,

$$-\frac{q}{2\pi R_0}\Delta\Phi = m\Delta v.$$

Let us sum up both sides of the equation over the time of switching off

$$-\frac{q}{2\pi R_0}\sum \Delta\Phi = m\sum \Delta v.$$

Here

$$\sum \Delta\Phi = 0 - B_0\pi r^2 = -B_0\pi r^2, \qquad \sum \Delta v = v_1.$$

The sphere departs from the tube at a speed v_1 received at the point A:

$$v_1 = \frac{r^2 q B_0}{2mR_0}.$$

2. Let the axis Ox be directed along the tube (fig. 75). Let the sphere be at a point C at a distance R from the solenoid axis. At this moment the sphere velocity equals v and its position x is specified by the angle β. The electromotive force at C equals

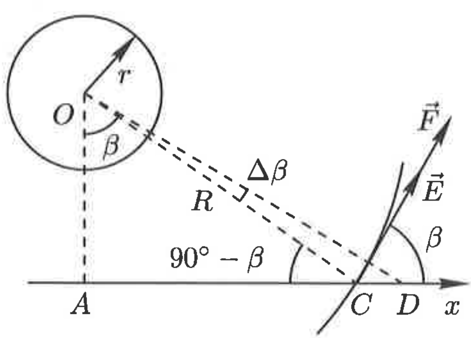

Figure 75

$$E = \frac{\pi r^2 |dB/dt|}{2\pi R} = \frac{kr^2}{2R}.$$

The force exerted on the sphere is $F = qE$ and its x-component is $F_x = F\cos\beta$. The sphere travels a small distance $CD = v\Delta t$

in a time Δt, its velocity receives an increment Δv and $F_x \Delta t = m\Delta v$. Then using the formulae for F_x, F, and E, one derives:

$$\frac{kr^2 q\Delta t}{2R}\cos\beta = m\Delta v \tag{25}$$

Applying the law of sines to the triangle OCD, one obtains:

$$v\Delta t \approx \frac{R\Delta\beta}{\sin(90^\circ - \beta).}$$

Hence, $\Delta t\cos\beta/R = \Delta\beta/v$. This equation allows one to cast (25) in the form:

$$kr^2 q\Delta\beta = 2mv\Delta v.$$

Since $2v\Delta v = \Delta(v^2)$, $kr^2 q\Delta\beta = m\Delta(v^2)$. We have

$$\sum \Delta\beta = \frac{\pi}{3} - 0 = \frac{\pi}{3},$$
$$\sum \Delta(v^2) = v_2^2 - 0 = v_2^2.$$

The sphere departs the tube at a speed

$$v_2 = \sqrt{\frac{\pi kr^2 q}{3m}}.$$

26. Particle Annihilation *S. Kozel*

1. It is convenient to solve the problem by placing the coordinate origin at the centre of mass (the point O, figure 76). The attraction force between the proton and antiproton, which are at the same distance from O, equals

$$F = \frac{e^2}{4\pi\varepsilon_0(2r)^2}.$$

Thus, the force exerted on each particle is proportional to $1/r^2$, where r is a distance from the fixed point O. Therefore, the potential energy E_p of a particle is

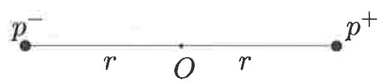

Figure 76

$$E_p = \frac{-e^2}{16\pi\varepsilon_0 r}.$$

The minus sign means that the particle is attracted to O. Now, let us write the law of conservation of energy for a particle:

$$-\frac{e^2}{16\pi\varepsilon_0(L/2)} = -\frac{e^2}{16\pi\varepsilon_0(x/2)} + \frac{mv^2}{2}.$$

Taking into account that $L \gg x$, one obtains

$$v = \sqrt{\frac{e^2}{4\pi\varepsilon_0 xm}} = 1.17 \cdot 10^6 \text{ m/s} \ll c,$$

where c is the speed of light.

2. A particle trajectory can be considered as a degenerate ellipse with a major axis $L/4$. A time t passed before the particle collide (and annihilate) equals one half of the period of rotation around this ellipse. According to the third Kepler's law, the period of rotation equals to a period of rotation around a circle with a radius $L/4$.

The law of motion for rotation around O in a circular orbit of the radius $L/4$ is

$$\frac{mv_0^2}{L/4} = \frac{e^2}{4\pi\varepsilon_0(2\cdot L/4)^2},$$

$$T^2 = \left(\frac{2\pi L/4}{v_0}\right)^2 = \frac{\varepsilon_0 m(\pi L)^3}{e^2},$$

$$t = \frac{T}{2} = \frac{1}{2}\sqrt{\frac{\varepsilon_0 m(\pi L)^3}{e^2}} \approx 67 \text{ ms}.$$

3. The ratio of gravitational and electric forces between the particles is

$$\frac{F_{\text{gr}}}{F_{\text{el}}} = \frac{4\pi\varepsilon_0 Gm^2}{e^2} \approx 10^{-36} \ll 1.$$

Thus, the force of gravity should be discarded.

27. Colliding Disks

I. Vorob'ev, A. Ershov

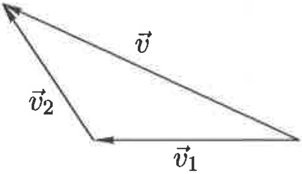

Figure 77

1. The conservation of momentum for disks of equal mass yields: $\vec{v_1} + \vec{v_2} = \vec{v}$, where $\vec{v_1}$ and $\vec{v_2}$ are disk velocities after the collision and $\vec{v}$ is a velocity of the incident disk before the impact. Thus the vectors form a triangle (fig. 77).

The conservation of energy in elastic collision gives $v_1^2 + v_2^2 = v^2$ (the sum of squares of the triangle sides equals to the square of the third side). Therefore the angle between the final velocities is right. The angle between paths x_1 and x_2 traveled by the disks after the collision is right as well.

The first disk travels a distance x_1 before it stops, hence $v_1^2 = 2\mu g x_1$. For the second disk, one has $v_2^2 = 2\mu g x_2$. From the conservation of energy, $x_1 + x_2 = L = v^2/(2\mu g)$. According to the Pythagorean theorem, the square of the distance between the disks after they stop is

$$R^2 = x_1^2 + x_2^2 = 2x_1^2 - 2Lx_1 + L^2 = 2x_1^2 - \frac{x_1 v^2}{\mu g} + \frac{v^4}{4\mu^2 g^2}.$$

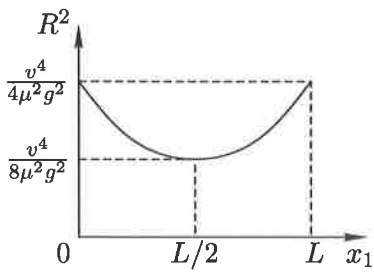

Figure 78

2. To answer the second question, let us study this expression as a function of x_1 (fig. 78). Obviously, $L \geq x_1 \geq 0$. The quantity R^2 is maximal at the ends of the interval, so $R = L = v^2/(2\mu g)$. The minimum is at $x_1 = L/2$, then

$$R^2 = \frac{L^2}{2}, \qquad R = \frac{L}{\sqrt{2}} = \frac{v^2}{2\sqrt{2}\mu g}.$$

The case of $x_1 = 0$ corresponds to a sliding collision while the case of $x_1 = L$ to a head-on collision.

28. Heating Main

V. Deltsov

Consider a small section of the pipe Δx (fig. 79). Water flows into the section at a temperature T and flows out at a temperature $T + \Delta T$ ($\Delta T < 0$). Thus, in this section water looses a heat $\Delta Q = -c\mu \Delta T \Delta t$, where c is the specific heat capacity of water and Δt is a time interval.

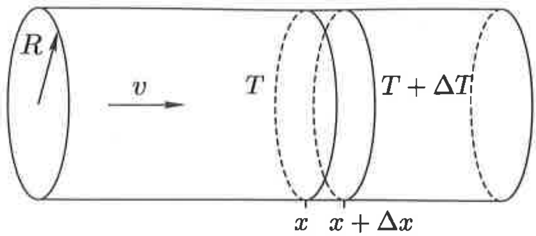

Figure 79

The same amount of heat must be transferred to the ambient air in the stationary regime, i.e.:

$$\Delta Q = \chi \frac{2\pi R \Delta x}{h} (T - T_a) \Delta t.$$

Equating these expressions and replacing ΔT and Δx by dT and dx, one writes

$$\frac{dT}{T - T_a} = -A\, dx, \tag{26}$$

where $A = \chi \frac{2\pi R}{c\mu h}$ is a constant factor. When calculating the numerical value of A one should take into account the thickness of the insulating layer and substitute $(R + h/2) = 22$ cm instead of $R = 20$ cm. Then

$$A = 6.54 \cdot 10^{-6}\ \mathrm{m}^{-1}.$$

Integrating the left hand side of (26) from T_0 to T_f, and the right side from 0 to L, one obtains:

$$\ln \frac{T_f - T_a}{T_0 - T_a} = -AL,$$
$$T_f = T_a + (T_0 - T_a) e^{-AL}.$$

1. For a heating pipe of length $L_1 = 10$ km

$$T_{f1} = -20 + (120 + 20) e^{-0.0654} \approx 111\,^{\circ}\mathrm{C}.$$

2. For a heating pipe of length $L_2 = 100$ km

$$T_{f2} = -20 + (120 + 20) e^{-0.654} \approx 53\,^{\circ}\mathrm{C}.$$

Note. The temperature drop is small for the length $L_1 = 10$ km, therefore, the problem can be solved without integration by simply assuming that the heat exchange proceeds at the temperature difference $(T_0 - T_a)$ like at the initial pipe section. This leads to the equation:

$$-c\mu(T_{f1} - T_0) = \chi \frac{2\pi R L_1}{h}(T_0 - T_a).$$

Solving for T_{f1}, one gets

$$T_{f1} = T_0 - AL_1(T_0 - T_a) \approx 111\,^\circ\text{C}.$$

Similar calculation of T_{f2} results in a wrong value: $T_{f2} \approx 28°\text{C}$. However, let us assume that the heat exchange along the whole length of the pipe proceeds at some average water temperature $(T_0 + T_f)/2$. In this case

$$T_{f2} = \frac{T_0 - AL_2(T_0/2 - T_a)}{1 + AL_2/2} \approx 50\,^\circ\text{C},$$

which is a good approximation.

29. Parametric Oscillations *S. Zhak*

1. An external force, which moves the capacitor plates, does a positive work at the maximum voltage U_m, i.e. twice during the oscillation period $T = 2\pi\sqrt{LC}$. This work increases the energy stored by the capacitor:

$$A = \Delta W = 2\Delta\left(\frac{q^2}{2C}\right) = -\frac{q^2}{C^2}\Delta C = \varepsilon C U_m^2.$$

Here $q = CU_m$ is a capacitor charge, $\Delta C < 0$ is a decrement of its capacitance, and $\varepsilon = -\Delta C/C$ is the absolute value of a relative capacitance increment. The external force does not do a work under a sharp increase of the capacitance to its original value since there is no voltage across the capacitor at this time. During oscillations in the circuit the energy is partially consumed by the light bulb, which is an inertial nonlinear device, as Joule heating. For small oscillations the bulb behaves like a resistor with a resistance determined by the slope of the I–V curve at the origin.

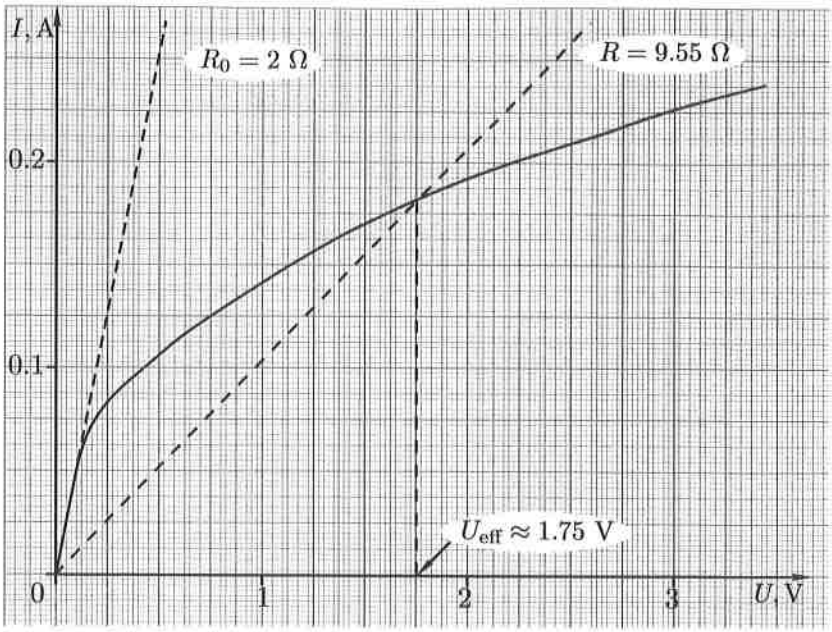

Figure 80

According to the plot (fig. 80), $R_0 = 2\ \Omega$. Thus, the energy consumed by the bulb during an oscillation period T is

$$\Delta Q = \frac{R_0 I_m^2}{2} T = \frac{R_0 T}{2} \omega^2 C^2 U_m^2.$$

Here $I_m = \omega C U_m$ is an amplitude of the current through the bulb. Undamped parametric oscillations will be initiated provided

$$\Delta W \geq \Delta Q, \qquad \text{or} \quad \varepsilon C U_m^2 \geq \frac{R_0 T}{2} \omega^2 C^2 U_m^2.$$

Taking into account that $T = 2\pi/\omega = 2\pi\sqrt{LC}$, one obtains for $\varepsilon_{\min}$:

$$\varepsilon_{\min} = \frac{R_0 T}{2} \omega^2 C = \frac{R_0}{2} \cdot 2\pi\sqrt{LC} \cdot \frac{1}{LC} \cdot C = \pi R_0 \sqrt{\frac{C}{L}}.$$

Substitution of numerical values yields $\varepsilon_{\min} = 6.3 \cdot 10^{-3} = 0.63\,\%$.

2. The oscillation period is $T = 2\pi\sqrt{LC} = 6.3 \cdot 10^{-4}$ s. A temperature of the bulb filament does not have time to change during T, so the bulb

behaves like a resistor. According to **1**, its resistance in the stationary regime at $\varepsilon = 3\%$ equals $R = (\varepsilon/\pi)\sqrt{L/C} = 9.55\ \Omega$. Let us draw the straight line corresponding to this value on the plot. The point of intersection with the I–V curve determines an effective voltage across the bulb, $U_{\text{eff}} = 1.75$ V. Therefore, $U_0 = \sqrt{2}U_{\text{eff}} \approx 2.5$ V.

Note. Because of a graphical method uncertainty, the numerical values of R_0 and U_{eff} may differ approximately by 5%.

30. Light Emitting Diodes *A. Malcev*

When the rod crosses the magnetic field lines, a variable emf of $|\mathscr{E}^{\text{ind}}| = \Delta\Phi/\Delta t$ is induced in it. The sign of $\mathscr{E}^{\text{ind}}$ can be determined either from the Lorentz force or from Faraday's law of induction. This results in the equivalent closed circuit shown in the diagram (fig. 81). Notice that $\mathscr{E}^{\text{ind}}$ and $\mathscr{E}_0$ are oppositely connected.

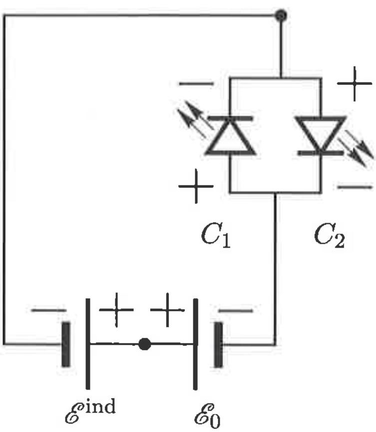

Figure 81

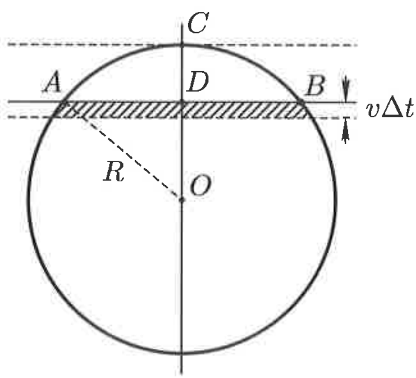

Figure 82

Let us determine a time dependence of $\mathscr{E}^{\text{ind}}$. Initially, the rod was touching the domain of the magnetic field. At t the rod traveled the distance $|CD| = vt$ (fig. 82). According to simple geometry, the length of the chord $|AB|$ equals

$$|AB| = 2|AD| = 2\sqrt{R^2 - (R - vt)^2} = 2\sqrt{vt(2R - vt)}.$$

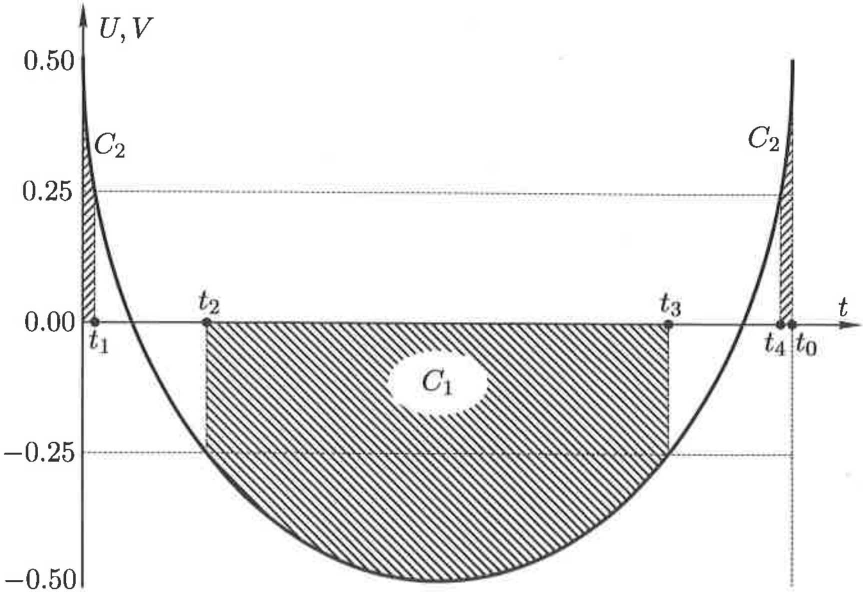

Figure 83

A magnetic flux increment per a small time interval Δt is

$$\Delta\Phi = B \cdot |AB| \cdot v\Delta t.$$

Hence,

$$|\mathscr{E}^{\text{ind}}| = 2Bv\sqrt{vt(2R - vt)}.$$

The maximum of $\mathscr{E}^{\text{ind}}$ is attained at $t = R/v$:

$$|\mathscr{E}^{\text{ind}}|_{\text{max}} = 2BvR = 1 \text{ V}.$$

The voltage across LEDs equals $U = \mathscr{E}_0 - \mathscr{E}^{\text{ind}}$. A qualitative time dependence $U(t)$ is plotted on the diagram (fig. 83). Using the plot, one can see that the LED C_2 turns on when $U \geq 0.25$ V, i.e. in the intervals $[0, t_1]$ and $[t_4, t_0]$, where $t_0 = 2R/v$. The LED C_1 is on for $U < -0.25$ V, i.e. in the interval $[t_2, t_3]$.
Let us determine t_1 and t_2. The times t_3 and t_4 follow from the plot symmetry: $t_3 = t_0 - t_2$, $t_4 = t_0 - t_1$.

For t_1 and t_4

$$\mathscr{E}_0 - 2Bv\sqrt{vt_{1,4}(2R - vt_{1,4})} = 0.25\ \text{V} \doteq \frac{1}{4}|\mathscr{E}^{\text{ind}}|_{\max} = \frac{1}{2}BvR.$$

Taking into account that $\mathscr{E}_0 = \frac{1}{2}|\mathscr{E}^{\text{ind}}|_{\max} = BvR$, one obtains a quadratic equation

$$t_{1,4}^2 - \frac{2R}{v}t_{1,4} + \frac{1}{16}\frac{R^2}{v^2} = 0,$$

whence

$$t_{1,4} = \frac{R}{v}\left(1 \pm \sqrt{\frac{15}{16}}\right) = 5 \cdot 10^{-3} \cdot (1 \pm 0.97)\ \text{s}.$$

Thus, $t_1 = 150\ \mu\text{s}$, $t_4 = 9.85$ ms.
Similarly, one derives t_2 and t_3:

$$t_{2,3}^2 + \frac{2R}{v}t_{2.3} + \frac{9}{16}\frac{R^2}{v^2} = 0,$$

$$t_{2,3} = \frac{R}{v}\left(1 \pm \sqrt{\frac{7}{16}}\right) = 5 \cdot 10^{-3} \cdot (1 \pm 0.66)\ \text{s}.$$

$$t_2 = 1.8\ \text{ms}, \qquad t_3 = 8.3\ \text{ms}.$$

31. Insulated System *I. Erofeev*

Since the system is thermally insulated, its energy is conserved:

$$E_1 + A = E_2,$$

where E_1 and E_2 are the energies of the system before and after the switch is closed and A is the work done by external forces. In our case, it is the work done by electromotive force. Since the system was initially in equilibrium, the piston mass equals $m = pS/g$. Suppose a charge q has passed through the battery. Then the cylinder bottom and the piston will be charged similar to the plates of a capacitor. In the final state, the capacitance is $C = \varepsilon_0 S/H$, where S is the bottom area. Since the voltage across the capacitor equals $\mathscr{E}$, the charge passed through the battery is

$$q = C\mathscr{E} = \frac{\varepsilon_0 S\mathscr{E}}{H}.$$

The charges on the plates are of opposite sign and, therefore, attract each other with a force

$$F = \frac{q^2/S}{2\varepsilon_0} = \frac{\varepsilon_0 S \mathscr{E}^2}{2H^2}.$$

The pressure in the final state will be $p + F/S$.
Internal energy of a monoatomic ideal gas can be expressed via its pressure and volume as $U = 3PV/2$. Let us use the conservation of energy:

$$\frac{3}{2}p_1V_1 + mgh + \mathscr{E}q = \frac{3}{2}p_2V_2 + mgH + \frac{q^2}{2C};$$

$$\frac{5}{2}pS(H-h) = -\frac{1}{4}\frac{\varepsilon_0 S\mathscr{E}^2}{H}, \qquad \left(\frac{H}{h}\right)^2 - \left(\frac{H}{h}\right) = -\frac{\varepsilon_0\mathscr{E}^2}{10h^2p}.$$

Solving the quadratic equation, one obtains

$$H = h\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\varepsilon_0\mathscr{E}^2}{10h^2p}}\right),$$

because the greater root is necessary. Using $\dfrac{\varepsilon_0\mathscr{E}^2}{h^2p} \ll 1$, one gets finally

$$H \approx h\left(1 - \frac{\varepsilon_0\mathscr{E}^2}{10h^2p}\right).$$

32. Which Efficiency is Greater? *V. Slobodyanin*

A work done per cycle $A_0 = \Delta p\Delta V$ is the same for both C and D cycles. The efficiency of a cyclic process is $\eta = A_0/Q$, where Q is a received heat. For the cycle C, $Q_C = \Delta U_{172} + A_{172}$.
Internal energy of an ideal monoatomic gas, $U = (3/2)\nu RT$, can be expressed via its pressure and volume: $U = (3/2)pV$. The gas does not do the work in the isochoric process 17, while its internal energy changes by $\Delta U_{17} = (3/2)\Delta pV_1$. In the isobaric process 72,

$$A_{72} = p_2\Delta V,$$
$$\Delta U_{72} = (3/2)p_2\Delta V.$$

It follows from these equations that

$$Q_C = \Delta U_{17} + \Delta U_{72} + A_{72} = (3/2)\Delta p V_1 + (5/2)p_2\Delta V.$$

Then,

$$\frac{1}{\eta_C} = \frac{Q_C}{A} = \frac{3}{2}\frac{V_1}{\Delta V} + \frac{5}{2}\frac{p_2}{\Delta p}.$$

Similarly, for the cycle D:

$$\frac{1}{\eta_D} = \frac{Q_D}{A} = \frac{3}{2}\frac{V_2}{\Delta V} + \frac{5}{2}\frac{p_3}{\Delta p}.$$

Note that $p_3 = p_2 + \Delta p$ and $V_2 = V_1 + \Delta V$, hence

$$\frac{1}{\eta_D} = \frac{Q_D}{A} = \frac{3}{2}(\frac{V_2}{\Delta V} + 1) + \frac{5}{2}(\frac{p_3}{\Delta p} + 1) = \frac{1}{\eta_C} + 4.$$

Thus, $\eta_D < \eta_C$, and, finally, $\eta_D = \dfrac{\eta_C}{1 + 4\eta_C}$.

33. Very Thick Lens

O. Shvedov

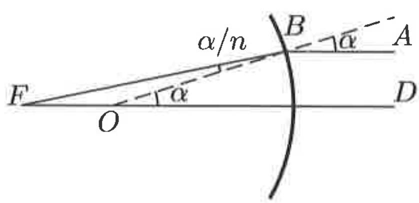

Figure 84

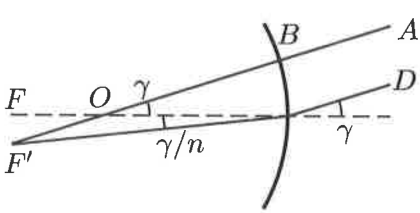

Figure 85

Consider first an auxiliary problem of a parallel beam incident on a spherical surface of radius R which separates a medium with a refraction index n from the air. Consider two rays incident on the surface at small angles. Let the ray DOF (fig. 84) pass through the sphere centre. Another ray ABF is refracted at a point B at the surface, its incidence angle is α. Then, $\angle OBF = \alpha/n$, as the angle of refraction, and $\angle BOD = \alpha$; using $\triangle BOF$, one finds that $\angle BFO = \alpha - \alpha/n$.

Let us apply the law of sines to $\triangle BOF$ and approximate the sine of a small angle by the angle in radians:

$$\frac{\alpha/n}{|OF|} \approx \frac{\alpha(1 - 1/n)}{|BO|} \approx \frac{\alpha}{|BF|},$$

therefore, $|BF| = |BO|/(1 - 1/n) = Rn/(n-1)$.
Thus, a narrow parallel beam after refraction on a spherical interface of two media converges at a focal point located on the ray passing through the centre at a distance $Rn/(n-1)$. Similarly, a parallel beam incident on a concave spherical interface of two media (of a radius R) is transformed into a diverging beam, the rays can be continued to a point of intersection at a distance $Rn/(n-1)$ from the interface.

1. Let us address the questions of the problem. A parallel beam will be transformed by an optical system into a parallel one if the focal points of two spherical surfaces coincide. It is not specified whether the surfaces are concave or convex, so all possibilities should be considered. Suppose the surface with the large radius is concave, the resulting divergent beam can be transformed to a parallel one only by refraction on the surface with the radius $r > R$, which contradicts to the problem statement. On the contrary, the surface with radius r can be both convex or concave (fig. 86). In both cases, $|AF| = Rn/(n-1)$ and $|BF| = rn/(n-1)$.
Therefore, in the first case

$$L = |AB| = (R - r)\frac{n}{n-1}$$

and in the second case

$$L = |AB| = (R + r)\frac{n}{n-1}.$$

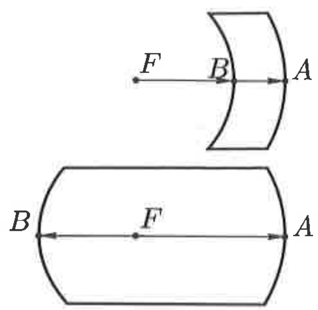

Figure 86

2. The beam decreases in size by a factor R/r, therefore, its intensity increases by the factor $(R/r)^2$.

3. Let us determine the angular magnification of the plate. To do this, consider an auxiliary problem: let us determine an angle γ at which a parallel beam must be incident on the symmetry axis in order to converge at a point F' at a distance y from the focal point F after refraction on the spherical surface with radius R (fig. 85). Due to rotational symmetry,

$$\gamma = \frac{y}{|OF|} = \frac{y(n-1)}{R}.$$

Therefore, a parallel beam incident on the surface at the angle γ converges after the refraction into a point on the focal plane located at a distance $\gamma R/(n-1)$ from the focal point F.
Thus, a parallel beam incident on the plate at an angle γ_1 to the symmetry axis comes out of the plate at an angle γ_2 determined by the equation

$$\frac{\gamma_1 R}{n-1} = \frac{\gamma_2 r}{n-1}.$$

Hence, the required angular magnification is $\dfrac{\gamma_1}{\gamma_2} = \dfrac{r}{R}$.

34. Athlet Ant

V. Efimov

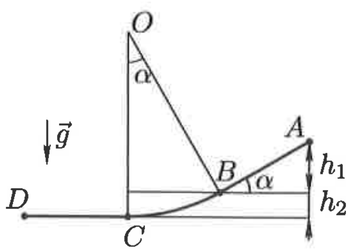

Figure 87

The net time of sliding is a sum of t_1, a time of sliding on the segment AB, a time t_2 of sliding on the arc BC, and a time t_3 of sliding on the segment CD. Let $AB = l$.

1. Let us draw perpendiculars to the ant path from points B and C, they are the radii of the arc BC (fig. 87). Let $\angle BOC = \alpha = 2l/R$. Since $l \ll R$, $\sin\alpha \approx \alpha = 2l/R$. The acceleration on AB equals $a_1 = g\sin\alpha = 2gl/R$.

$$\text{Since} \quad l = \frac{a_1 t_1^2}{2}, \qquad t_1 = \sqrt{\frac{2l}{a_1}} = \sqrt{\frac{R}{g}}.$$

A velocity at the end of this segment is

$$v_B = \sqrt{2gh_1} = \sqrt{2gl\sin\alpha} = 2l\sqrt{\frac{g}{R}}.$$

2. Now, calculate t_2. Using the conservation of energy, one determines an ant velocity v_C at the point C. This is the maximum velocity.

$$mg(h_1 + h_2) = \frac{mv_C^2}{2}, \qquad \text{therefore,} \quad v_C = \sqrt{2g(h_1 + h_2)}.$$

Now, determine h_1 and h_2:

$$h_1 = l\sin\alpha = \frac{2l^2}{R},$$

$$h_2 = R(1-\cos\alpha) = 2R\sin^2\frac{\alpha}{2} = 2R\left(\frac{\alpha}{2}\right)^2 = \frac{2l^2}{R}.$$

$$h_1 + h_2 = 4\frac{l^2}{R}, \qquad \text{so} \quad v_C = 2l\sqrt{\frac{2g}{R}}.$$

Sliding on BC is equivalent to harmonic motion of a simple gravity pendulum of a length R. Its angular frequency is $\omega = \sqrt{g/R}$. A time t_2 of ant sliding from B to C equals the time of its reverse motion from C to B which can be found from the law of simple harmonic motion

$$v_B = v_C\cos\left(\sqrt{\frac{g}{R}}t_2\right), \qquad \text{hence} \quad t_2 = \sqrt{\frac{R}{g}}\arccos\frac{v_B}{v_C} = \frac{\pi}{4}\sqrt{\frac{R}{g}}.$$

3. Ant slides on CD at the constant speed v_C, thus $t_3 = \dfrac{3l}{v_C} = \dfrac{3}{2}\sqrt{\dfrac{R}{2g}}$.

The total time of ant sliding is

$$t = t_1 + t_2 + t_3 = \sqrt{\frac{R}{g}}\left(1 + \frac{\pi}{4} + \frac{3}{2\sqrt{2}}\right) \approx 2.85\sqrt{\frac{R}{g}}.$$

35. Massive Rope *V. Chivilev*

For the moving pulley with the load and the rope section AC to remain at rest, this condition must be met:

$$2T_A = Mg + \frac{m}{l}\pi Rg.$$

Hence, the tension at a point A is

$$T_A = \frac{1}{2}Mg + \frac{\pi mR}{2l}g.$$

Besides, $T_C = T_A$. The tension at B is

$$T_B = T_A + \frac{m}{l}H_2 g = \frac{1}{2}Mg + \frac{\pi m R}{2l}g + \frac{m}{l}H_2 g.$$

To determine F, let us imagine moving a piece KC of the rope by displacing C downwards at a small distance x. The overall work done by all forces exerted on KC equals to a change of potential energy of the piece:

$$T_C x - Fx = \frac{m}{l}xgH_1.$$

Finally,

$$F = T_C - \frac{m}{l}gH_1 = \frac{1}{2}Mg + \frac{\pi m R}{2l}g - \frac{m}{l}H_1 g.$$

36. Rotation of Charged Cylinder *M. Osin*

A cylinder rotation is equivalent to a circular current which generates magnetic field inside the cylinder. The total current flowing on the cylinder surface equals $I = \sigma v L$, where v is the linear velocity of the charge. The current surface density is $i = I/L = \sigma v$. The magnetic induction B inside the cylinder is that one of a long solenoid:

$$B = \frac{\mu_0 I}{L} = \mu_0 \sigma v.$$

The magnetic energy density is

$$w_M = \frac{B^2}{2\mu_0} = \frac{1}{2}\mu_0 \sigma^2 v^2.$$

The total energy of magnetic field is

$$W_M = w_M \cdot \pi R^2 L = \frac{1}{2}kv^2, \qquad \text{where} \quad k = \pi \mu_0 \sigma^2 R^2 L.$$

The kinetic energy of the cylinder and the weight equals

$$W_K = \frac{(m+M)v^2}{2}.$$

Let x-axis be pointing downwards, then the potential energy of the weight can be written as

$$W_P = -mgx + \text{const}.$$

Now, let us apply the conservation of energy to the mechanical energy of the cylinder and the weight and the energy of magnetic field inside the cylinder:

$$W_K + W_P + W_M = \text{const}, \qquad \text{or} \quad \left(m + M + k\right)\frac{v^2}{2} - mgx = \text{const}.$$

Taking into account that $v = dx/dt$ and $a = dv/dt$ and differentiating this equation with respect to time, one obtains:

$$a = \frac{mg}{m + M + k} = \frac{mg}{m + M + \pi\mu_0\sigma^2 R^2 L}.$$

37. Weight and Springs *A. Gudenko*

Let us introduce the following notations: x is a weight displacement from the initial position and $y = ut$ is a displacement of the right end of the spring II (fig. 88).

The elastic forces exerted on the weight are

$$F_1 = kx, \qquad F_2 = (y - x)k = (ut - x)k;$$

the net force:

$$F = F_2 - F_1 = kut - kx - kx = k(ut - 2x).$$

The second Newton's law (the equation of motion of the weight) is

$$ma_x = k(ut - 2x).$$

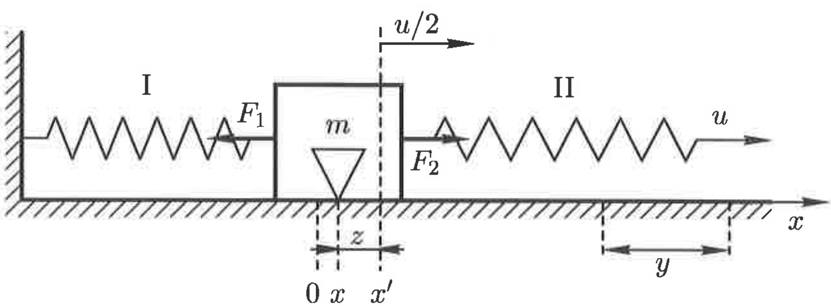

Figure 88

Now let us consider the motion in the frame moving at a constant speed $u/2$ with respect to the frame at «rest». These frames are inertial, therefore, the weight acceleration is the same in both frames: $a_x = a_z$. The origin of the moving frame at t is at $x' = t \cdot u/2$. At this moment the weight coordinate equals $z = -(x' - x) = -x' + x$. The second Newton's law in the moving frame becomes

$$ma_z = k(ut - 2z - ut) = -2kz.$$

This equation describes simple harmonic motion with an angular frequency

$$\omega_0 = \sqrt{\frac{2k}{m}}.$$

At $t = 0$ the weight is at the origin of the moving frame $z_0 = 0$ and has a velocity $v_0 = -u/2$. After one half of the period

$$\Delta t = \frac{T}{2} = \frac{\pi}{\omega_0} = \pi\sqrt{\frac{m}{2k}},$$

the weight velocity in this frame becomes $+u/2$ and the weight is at $z = 0$. Therefore, the weight velocity in the «rest» frame at this moment will be u and its coordinate

$$x = \frac{u}{2}\Delta t = \frac{\pi}{2}u\sqrt{\frac{m}{2k}}.$$

38. Utilisation of Wave Power *Folklore*

The work done by water can be represented by a sum of the following two terms.

1. First, the air is adiabatically compressed:

$$A_1 = \Delta U = \nu C_V \Delta T = \frac{5}{2}\nu R(T_2 - T_1), \tag{27}$$

where ν is an amount of air above the piston. Equation of adiabatic process reads:

$$\frac{p_1}{p_2} = \left(\frac{V_2}{V_1}\right)^{7/5}, \qquad \text{whence} \quad \frac{T_2}{T_1} = \frac{p_2 V_2}{p_1 V_1} = \frac{p_2}{p_1}\left(\frac{p_1}{p_2}\right)^{5/7}.$$

Thus, $T_2 = T_1 \left(\frac{p_2}{p_1}\right)^{2/7} \approx 280 \cdot 6^{2/7} \approx 468$ K.

The number of moles $\nu = \dfrac{p_1 V_1}{RT_1} \approx 10$ mol.

Substituting these values in (27), one obtains $A_1 \approx 39$ kJ.

2. Since the pressure remains constant after the valve K_1 opens, the second process is isobaric:

$$A_2 = p_2 \Delta V = p_2 V_2 = p_1 V_1 \left(\frac{p_1}{p_2}\right)^{5/7} \approx 39 \text{ kJ}.$$

Finally, $A = A_1 + A_2 \approx 78$ kJ.

39. Charged Soap Bubble *A. Olkhovetz*

1. Let us determine the pressure exerted on the bubble due to electrostatic force. Consider a small element ΔS of the surface. The electrostatic field E_0 acting on the bubble is equal to the field E_1 produced by the bubble itself near its surface (this is the case, since an electric field inside the bubble must vanish).

Thus, the force exerted on the bubble is

$$F_e = E_0 \cdot \frac{Q \Delta S}{4\pi R^2}, \qquad \text{where} \quad E_0 = E_1 = \frac{1}{2\varepsilon_0} \cdot \frac{Q}{4\pi R^2}.$$

Therefore, the electrostatic pressure acting on the bubble equals

$$p_e = \frac{F_e}{\Delta S} = \frac{Q^2}{32\pi^2 \varepsilon_0 R^4}.$$

A pressure due to the surface tension is $p_\sigma = -4\sigma/R$. The net pressure is $p = p_e + p_\sigma$. In equilibrium, $p = 0$:

$$\frac{Q^2}{32\pi^2 \varepsilon_0 R_0^4} - \frac{4\sigma}{R_0} = 0.$$

Therefore, the equilibrium radius is

$$R_0 = \sqrt[3]{\frac{Q^2}{128\pi^2 \varepsilon_0 \sigma}} \approx 3.0 \text{ cm}.$$

2. Suppose the bubble radius deviates from the equilibrium value R_0, then the force exerted on a small element ΔS of the surface becomes:

$$F = p\Delta S = 4\sigma\left(\frac{R_0^3}{R^4} - \frac{1}{R}\right)\Delta S.$$

For a small radial increment ($\Delta R \ll R_0$), the force can be written as

$$F = \left.\frac{dp}{dR}\right|_{R=R_0} \cdot \Delta R \cdot \Delta S = 4\sigma\Delta R\Delta S\left.\left(-\frac{4R_0^3}{R^5} + \frac{1}{R^2}\right)\right|_{R=R_0} =$$
$$= -\frac{12\sigma}{R_0^2}\Delta R\Delta S.$$

The minus sign means that the bubble equilibrium is stable.
Let us apply the second Newton's law to the surface element ΔS of mass Δm:

$$\Delta m\Delta\ddot{R} = -\frac{12\sigma}{R_0^2}\Delta R\Delta S,$$

$$\Delta m = \frac{m\Delta S}{4\pi R_0^2}, \qquad \text{hence} \quad \Delta\ddot{R} + 48\frac{\pi\sigma}{m}\Delta R = 0.$$

This is equation of harmonic oscillations with an angular frequency $\omega = \sqrt{48\pi\sigma/m}$. Thus,

$$T = \frac{2\pi}{\omega} = \sqrt{\frac{\pi m}{12\sigma}} \approx 16 \text{ ms}.$$

3. The speed of spray can be estimated from the conservation of energy. Neglecting the surface energy, one obtains:

$$\frac{1}{2}\frac{Q_1^2}{4\pi\varepsilon_0 R_0} = \frac{mv^2}{2}, \qquad \text{therefore,} \quad v = \sqrt{\frac{100Q^2}{4\pi\varepsilon_0 R_0 m}} \approx 94 \text{ m/s}.$$

40. Optical System *G. Tarnopolski*

Reversibility of light propagation implies that a solution should be independent of the assumption of which arrow, a or b, is the object and which is its image formed by the mirror-lens system. The arrows a and b

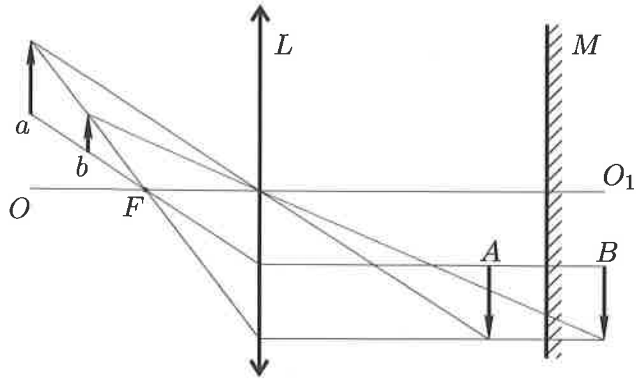

Figure 89

and the lens are parallel, therefore their images A and B are also parallel to the lens. Now, since A and B are parallel, they are also parallel to the mirror, so that a straight line passing through their ends is parallel to the system optical axis. Therefore, a beam propagating along this straight line passes through a focal point and the ends of A and B after being refracted by the lens.

Thus, the drawing could be done as follows (fig. 89):

1. Plot the main optical axis of the lens OO_1 which passes through the lens optical centre perpendicular to the lens plane.

2. Draw straight lines through the endpoints of the object and its image. These lines must pass through the lens focal point on the optical axis.

3. The rear focal point is plotted using a standard construction.

4. One of the arrows, either a or b, is an object and the other one is its image formed by the lens-mirror system. Let us plot the images of these arrows using reversibility of a light ray. To this end, the standard rays can be used, i.e. those ones passing through a lens focal point and its centre. This gives the images A and B.

5. One of the arrows, A or B, is an object with respect to the plane mirror and the other one is its image. Therefore, the plane mirror must be placed in the middle between the arrows A and B.

41. Bungee Jumping

K. Zakharchenko

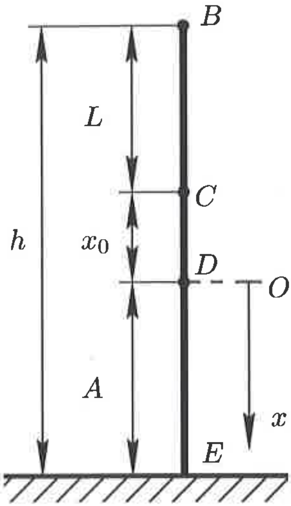

Figure 90

1. Using the conservation of energy and the condition of zero velocity at the water surface, one has:

$$mgh = \frac{k(h-L)^2}{2}.$$

According to Hooke's law, $F_{el} = k(h-L)$. Let us write the second Newton's law for the person at the lowest point:

$$ma_0 = F_{el} - mg = 2mg.$$

Then $F_{el} = 3mg = k(h-L) = \dfrac{2mgh}{h-L}$, whence it follows that

$$L = h/3 = 30 \text{ m},$$

$$k = \frac{2mgh}{(h-L)^2} = \frac{9}{2}\frac{mg}{h} = 35 \text{ N/m}.$$

2. After oscillations decay (at the equilibrium), the cord length increases by $x_0 = mg/k = 2h/9 = 20$ m.

3. The maximum velocity is attained at a point D corresponding to the point of equilibrium (fig. 90). According to the conservation of energy,

$$\frac{mv_{\max}^2}{2} = mg(L + x_0) - \frac{kx_0^2}{2},$$

hence, $v_{\max} = (2/3)\sqrt{2gh} = 28.3$ m/s. Notice that $\sqrt{2gh}$ is the speed of free fall from a height h.

4. To evaluate the time required to reach the water surface let us direct x-axis downward and choose the origin at D. Assume that the person passes the point D ($x = 0$) at $t = 0$. Then its subsequent motion to the point E and back to C is described by an equation $x = A\sin\omega t$. Let us determine the time it takes to pass BC, CD, and DE.

a) The part BC is a free fall. The corresponding time is

$$\tau_{BC} = \sqrt{\frac{2L}{g}} = \sqrt{\frac{2h}{3g}} = \frac{\tau_0}{\sqrt{3}},$$

where $\tau_0 = \sqrt{2h/g}$ is a time of free fall from a height h.

b) When the person passes the point C, the cord starts stretching and the subsequent motion is a simple harmonic motion.

Let the amplitude of oscillations be A. The angular frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{9g}{2h}} = 0.71\ \mathrm{s}^{-1}.$$

The time it takes to reach D can be found from an equation:

$$-x_0 = A\sin(-\tau_{CD}\omega), \qquad \text{or} \quad x_0 = A\sin\omega\tau_{CD}.$$

Since $x_0 = A/2$, $\omega\tau_{CD} = \pi/6$, and $\tau_{CD} = \dfrac{\pi}{6\omega} = \dfrac{\pi}{18}\sqrt{\dfrac{2h}{g}} = \dfrac{\pi}{18}\tau_0$.

c) A time of passing DE equals a quarter of the oscillation period T:

$$\tau_{DE} = \frac{T}{4} = \frac{1}{4}\frac{2\pi}{\omega} = \frac{\pi}{6}\sqrt{\frac{2h}{g}} = \frac{\pi}{6}\tau_0.$$

Thus, the total time τ required to reach the water surface equals

$$\tau = \tau_{BC} + \tau_{CD} + \tau_{DE} = \left(\frac{1}{\sqrt{3}} + \frac{2\pi}{9}\right)\tau_0 = 5.41\ \mathrm{s}.$$

42. Electric Circuit with Inductor

A. Sheronov

1. The heat Q_0 released in the circuit after the switch was opened equals the magnetic energy stored by the inductor at the moment of opening: $Q_0 = LI_0^2/2$, hence, $I_0 = \sqrt{2Q_0/L}$.

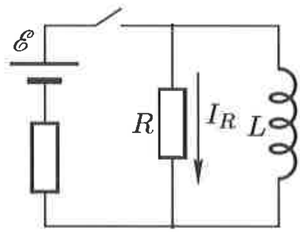

Figure 91

2. Let us determine a relation between a charge passed through the resistor R and a change of magnetic flux $\Delta\Phi$. Regardless of the switch status, a voltage across the resistor R equals the voltage across the inductor. Therefore,

$$L\frac{dI_L}{dt} = RI_R.$$

Multiplying both sides of this equation by dt, and integrating, one obtains:

$$L\Delta I_L = \Delta\Phi = R\Delta q, \tag{28}$$

where $\Delta\Phi$ is a change of magnetic flux and Δq is a charge passed through the resistor R. This relation holds true for any time interval. Let us apply (28) to the period when the switch was closed, this gives:

$$\Delta\Phi_1 = LI_0, \qquad \Delta q = q_1 = \frac{\Delta\Phi_1}{R} = \frac{LI_0}{R} = \frac{\sqrt{2Q_0L}}{R}.$$

3. Equation (28) can be also used after the switch was opened. In this case $\Delta\Phi_2 = -LI_0$, since the flux Φ changes from LI_0 to zero when the current completely decays. Thus, $\Delta q = q_2 = \Delta\Phi_2/R = -LI_0/R$. Therefore, $q_2 = -q_1$, i.e. the net charge passed through the resistor R during the whole process vanishes. When the switch is closed, the charge flows through R in the positive direction (fig. 91); when it is opened, in the negative.

4. The work of the power source equals to the product of its emf and the charge passed through the source. When the switch was closed, the passed charge was $q = q_1 + q_0$, where q_1 is the charge passed through the resistor R and q_0 is the charge passed through the inductor according to the problem statement. Finally, $A = \mathscr{E}q = \mathscr{E}q_0 + (\mathscr{E}/R)\sqrt{2Q_0L}$. Eventually, all the work was converted into heat.

5. According to the problem statement, the heat released in the circuit after the switch was opened was Q_0, therefore the heat released in the circuit when the switch was closed equals

$$Q = A - Q_0 = \mathscr{E}\left(q_0 + \frac{\sqrt{2Q_0L}}{R}\right) - Q_0.$$

43. Kelvin's Problem *G. Tarnopolski*

According to the ideal gas law, the pressure $p = \nu RT/V$. Therefore, the point of minimum pressure on a plot in coordinates (T, V) corresponds to the minimum slope of a straight line passing through this point and the origin. Therefore, the origin must lie somewhere on the tangent to the curve at A (fig. 92).

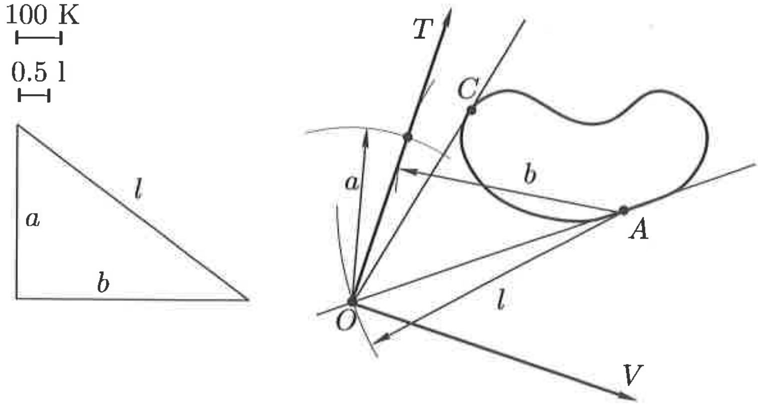

Figure 92

Now, since the scales of both axes are known, one can find a distance l from the origin to A as the hypotenuse of a right-angled triangle which other sides are the segments corresponding to 400 K (of a length a) and 4 l (of a length b). Thus, the origin O lies on the tangent at the distance l from A in the lower part of the diagram.

Note that the temperature axis must point upwards, otherwise the pressure at A would not be minimal. The point corresponding to 400 K is at the distance a from the origin and at the distance b from the point A. This point can be determined as an intersection of the circles centred at O and A with radii a and b. Now one can plot the temperature axis and the volume axis (perpendicular).

The point of maximum pressure corresponds to the maximum slope k_C of a straight line passing through this point and the origin: $p_{\max} = R \cdot k_C \approx 4.75$ MPa.

44. Heat Exchange with Environment *S. Kozel*

1. The first horizontal segment of the plot corresponds to the period τ_0 of ice fusion:

$$m_i \lambda = P_0 \tau_0, \qquad \text{hence} \quad m_i = \frac{P_0 \tau_0}{\lambda} \approx 0.15 \text{ kg}.$$

2. The net mass of the mixture can be determined from the slope at $t_0 = 0$ °C:

$$c_w M \Delta t = P_0 \Delta\tau, \qquad \text{hence} \quad M = \frac{P_0}{c_w}\frac{\Delta\tau}{\Delta t} = \frac{P_0}{k_1 c_w}.$$

The slope $k_1 = (\Delta t/\Delta\tau)_{t=t_0} \approx 0.20$ °C/s, therefore, the mass is $M \approx 0.48$ kg.

3. The factor α can be determined from the slope k_2 at $t = t_1 = 100$ °C:

$$c_w M \Delta t = (P_0 - \alpha t_1)\Delta\tau, \qquad \text{hence,} \quad \alpha = \frac{P_0 - c_w M k_2}{t_1},$$

where $k_2 = (\Delta t/\Delta\tau)_{t=t_1} \approx 0.10$ °C/s. Then $\alpha \approx 2.0$ W/°C.

4. The maximum output at which water does not boil is determined by the condition: $P_{\max} - \alpha \cdot t_1 = 0$, so $P_{\max} \approx 200$ W.

5. To answer the last question, one has to solve an equation with separating variables: $(P - Q)d\tau = (P - \alpha t)d\tau = c_w M\, dt$,

$$\frac{d\tau}{c_w M} = \frac{dt}{P - \alpha t}, \qquad \text{hence} \quad \tau = \frac{c_w M}{\alpha} \ln \frac{P}{P - \alpha t},$$

this is the time dependence of water temperature rising from 0 °C to 100 °C. The time τ_1 from the beginning of ice melting to the boiling at $P_1 = 300$ W equals

$$\tau_1 = \frac{\lambda m_i}{P_1} + \frac{c_w M}{\alpha} \ln \frac{P_1}{P_1 - \alpha t_1} \approx 21 \text{ min.}$$

45. Two Lens Problem *S. Kozel*

1. In the first experiment, light beams (or their extensions) must intersect at the front focal point of the second lens after passing through the first lens. Applying the thin lens formula to L_1, one obtains:

$$\frac{1}{l_1} + \frac{1}{L - F_2} = \frac{1}{F_1}. \tag{29}$$

2. In the second experiment, the beams necessarily pass through the rear focal point of L_1. Applying the thin lens formula to L_2, one obtains:

$$\frac{1}{l_2} + \frac{1}{L - F_1} = \frac{1}{F_2}. \tag{30}$$

Note that the formulae (29) and (30) are identical up to a swap of indices 1 and 2.

3. Now one should exclude a focal length from (29) and (30). Let it be F_2. According to (30),

$$F_2 = \frac{l_2(L - F_1)}{L - F_1 + l_2}.$$

Substituting this formula in (29), one obtains a quadratic equation for F_1:

$$F_1^2(L + l_1 - l_2) - F_1(L^2 + 2l_1 L) + L^2 l_1 = 0.$$

4. Its solutions are:

$$(F_1)_{1,2} = \frac{L^2 + 2l_1 L \pm L\sqrt{L^2 + 4l_1 l_2}}{2(L + l_1 - l_2)} = 20.0 \pm 16.3 \text{ cm}.$$

Similarly, for F_2:

$$(F_2)_{1,2} = \frac{L^2 + 2l_2 L \pm L\sqrt{L^2 + 4l_1 l_2}}{2(L + l_2 - l_1)} = 16.0 \pm 9.8 \text{ cm}.$$

Both solutions are possible. In the first case both lenses are «long-focus»: $F_1 = 36.3$ cm and $F_2 = 25.8$ cm (F_1, $F_2 > L$); in the second case, on the contrary, they are «short-focus»: $F_1 = 3.7$ cm and $F_2 = 6.2$ cm (F_1, $F_2 < L$).

46. Chain on Sphere

V. Plis

1. Consider a small chain element of a length $\Delta L = R\Delta\varphi$. Its mass is $\Delta m = \rho \cdot \Delta L = \rho \cdot R\Delta\varphi$. The forces acting on this element are tensions $\vec{T}(\varphi+\Delta\varphi)$ and $\vec{T}(\varphi)$, a reaction force $\vec{N}$, and a gravitational force $\vec{g}\Delta m$ (fig. 93).

Consider the tangential component of the second Newton's law:

$$\Delta m a_\tau = T(\varphi + \Delta\varphi) - T(\varphi) + \Delta m g \sin\varphi. \quad (31)$$

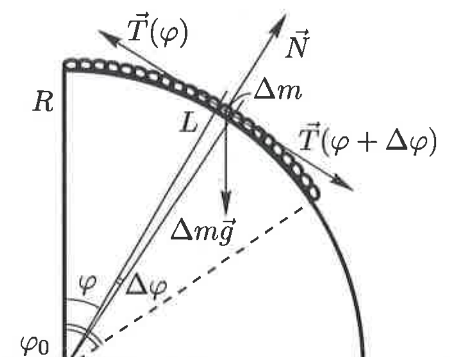

Figure 93

The tangential acceleration of any chain element is the same. A normal acceleration vanishes since all elements have zero velocity right after the release. By summing up the left- and right-hand sides of (31) over the chain length and taking into account that a tension vanishes at the chain free ends, one obtains:

$$\rho R a_\tau \sum \Delta\varphi = \rho R g \sum \sin\varphi \Delta\varphi.$$

The tension is an internal force, so it cancels out according to the third Newton's law. Now, sending $\Delta\varphi \to 0$, one obtains:

$$a_\tau \frac{L}{R} = g \int_0^{\varphi_0} \sin\varphi d\varphi = g(1 - \cos\varphi_0),$$

where $\varphi_0 = L/R$. Thus,

$$a_\tau = g\frac{R}{L}(1 - \cos\varphi_0) = g\frac{R}{L}\left(1 - \cos\frac{L}{R}\right).$$

For $L/R = 2\pi/6 = \pi/3$ this gives

$$a_\tau = \frac{3}{2\pi} g.$$

2. To answer the second question, one should take into account that $\Delta T = T(\varphi + \Delta\varphi) - T(\varphi) = 0$ at the chain cross-section where the tension T is maximal. Let a position of the small element at this point be $\varphi_{\max}$. The acceleration of this element is due to a tangent component of gravitational force only:

$$a_\tau = g\sin\varphi_{\max} = g\frac{R}{L}\left(1 - \cos\frac{L}{R}\right).$$

Therefore, $\sin\varphi_{\max} = \dfrac{R}{L}\left(1 - \cos\dfrac{L}{R}\right)$.

For $L/R = \pi/3$, this gives $\sin\varphi_{\max} = 3/(2\pi) \approx 0.48 \approx 0.5$, so $\alpha_{\max} \approx 30°$. Thus, the point of maximum tension is located approximately in the chain middle.

47. Motion of Charged Particles

H. Matveev, M. Proskurin

1. Due to symmetry, all masses travel along identical trajectories; at any moment they remain at the corners of a square with a side $a = \sqrt{2}\, r$ inscribed in a circle of radius $r(t)$.

Each mass is being acted on by three forces: $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ (fig. 94). According to Coulomb's law, the absolute values of the forces are:

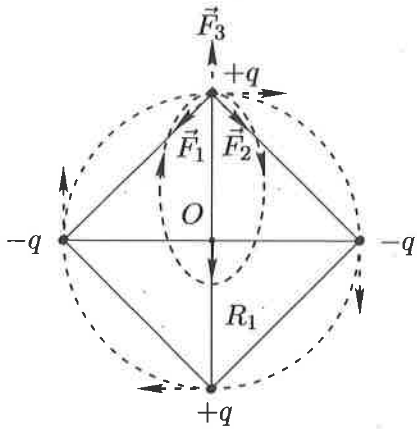

Figure 94

$$F_1 = F_2 = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2r^2},$$

$$F_3 = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4r^2}.$$

A net force always points to the centre (point O), its absolute value is

$$F(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q^2}{2r^2}\sqrt{2} - \frac{q^2}{4r^2}\right) = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r^2}\left(\frac{2\sqrt{2}-1}{4}\right).$$

Therefore, each mass is moving as if it were attracted to the centre by a charge of opposite sign with an absolute value equal to

$$Q = q\left(\frac{2\sqrt{2}-1}{4}\right).$$

Equation for $F(r)$ is similar to Coulomb's law (or Newton's law of universal gravitation), i.e. $F(r) \sim 1/r^2$. Therefore, a mass trajectory is an ellipse which major axis equals $R_0 + R_1$. The point O is at a focal point of the ellipse.

2. A period of rotation T around an elliptical orbit can be found from the third Kepler's law.

First, let us determine a period T_0 of a point mass m going around a circular orbit of radius $r = R_0$ under a force $F(r)$:

$$T_0 = \frac{2\pi R_0}{v_0}, \qquad \frac{mv_0^2}{R_0} = F(R_0) = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{R_0^2}\left(\frac{2\sqrt{2}-1}{4}\right).$$

Solving these equations for v_0, one finds:

$$v_0 = q\sqrt{\frac{1}{4\pi\varepsilon_0} \cdot \frac{2\sqrt{2}-1}{4mR_0}}.$$

This speed should be given to the masses to make them to rotate around a circle of a radius R_0. Then:

$$T_0 = \frac{2\pi}{q}\sqrt{4\pi\varepsilon_0 \frac{4mR_0^3}{2\sqrt{2}-1}}. \tag{32}$$

According to the third Kepler's law,

$$\left(\frac{T}{T_0}\right)^2 = \left(\frac{R_0+R_1}{2R_0}\right)^3,$$

so

$$T = T_0\left(\frac{R_0+R_1}{2R_0}\right)^{\frac{3}{2}}.$$

Using equation (32), one derives the period of rotation:

$$T = \frac{2\pi}{q}\sqrt{4\pi\varepsilon_0 \frac{m(R_0+R_1)^3}{2(2\sqrt{2}-1)}}.$$

48. Unipolar Inductor *A. Gudenko*

1. Free electrons in the conducting layer of rotating disk are subjected to Lorentz force $F_L = evB$, where $e < 0$ is the electron charge and $v = \omega r$ is a linear velocity. The Lorentz force is pointing to the centre for the directions of magnetic induction and disk rotation shown (fig. 34). As a result, electric charges are redistributed: electrons will be driven to the centre while a positive charge will be accumulated on the lateral side. This generates an electric field $\vec{E}$ pointing to the centre. The system reaches equilibrium when Coulomb's force at any point is balanced by the Lorentz force:

$$eE = evB = e\omega rB, \qquad \text{i.e.} \quad E = vB.$$

A potential $\Delta\varphi$ between a lateral side of the disk ($r = r_0$) and its centre ($r = 0$) equals

$$\Delta\varphi = \int_0^r E dr = \frac{1}{2}\omega r_0^2 V.$$

It is this voltage, which is measured by the voltmeter:

$$V = \Delta\varphi = \frac{1}{2}\omega r_0^2 B = \pi\nu r_0^2 B = 62.8 \cdot 10^{-3} \text{ V},$$

here the negative pole is at the centre and the positive one is at the lateral side.

2. The disk is accelerated by a torque arising due to interaction between the magnet and the electric current in the external circuit operating as a stator. As the rotor accelerates, the induced emf $\mathscr{E}_{\text{ind}}$ which, according to Lenz's law is opposite to the batter emf, also increases. The acceleration vanishes when $\mathscr{E}_{\text{ind}}$ becomes equal to the battery emf and the current through the circuit vanishes as well. For this to happen, the disk must rotate in the same way as in the first diagram, i.e. counterclockwise when viewed from the above. The maximum rotation frequency is found from the condition:

$$\mathscr{E}_{\text{ind}} = \Delta\varphi = \mathscr{E}, \qquad \text{or} \quad \pi\nu_{\max} r_0^2 V = \mathscr{E},$$

hence

$$\nu_{\max} = \frac{\mathscr{E}}{\pi r_0^2 V} = 71.7 \cdot 10^3 \text{ rev/min}.$$

49. Heat Engine *S. Kozel*

Consider the work done by the engine per $\Delta t = 1$ s. The efficiency of the Carnot cycle is

$$\eta = \frac{Q_1 - Q_2}{Q_1} = \frac{P}{P + P_2} = \frac{T_1 - T}{T_1}. \tag{33}$$

Here P is the engine power output and P_2 is the heat power transferred to the cold sink. It follows from (33) that

$$P = \frac{P_2}{T}(T_1 - T).$$

According to the problem statement, $P_2 = Q_2/\Delta t = \alpha(T - T_2)$, then

$$P = \alpha\frac{(T - T_2)(T_1 - T)}{T} = \alpha\left[T_1 + T_2 - \left(T + \frac{T_1T_2}{T}\right)\right].$$

The maximum of $(T + T_1T_2/T)$ is at $T = \sqrt{T_1T_2} = 489.9$ K ≈ 490 K. Therefore,

$$P_{\max} = \alpha\left[T_1 + T_2 - 2\sqrt{T_1T_2}\right] = \alpha\left[\sqrt{T_1} - \sqrt{T_2}\right]^2 = 120.1 \text{ kW},$$

$$\eta = \frac{T_1 - T}{T} = 38.7\,\%.$$

50. Motion without Sliding *D. Aleksandrov*

According to the second Newton's law, the equations of motion of the weight, the board, and the slab are:

$$T - F_{\text{fr}} = m_1a_1,$$
$$Mg - T = Ma_1,$$
$$F_{\text{fr}} - kx = m_2a_2.$$

Here T is a cord tension, F_{fr} is a friction force, and x is a spring extension. There is no sliding while $F_{\text{fr}} < \mu m_2 g$. The desired length L can be found from the condition $a_1 = a_2$, i.e.

$$\frac{Mg - \mu m_2 g}{m_1 + M} = \frac{\mu m_2 g - kx}{m_2}.$$

This gives:

$$L = \frac{g}{k}\frac{m_2}{m_1 + M}\cdot[\mu(m_1 + m_2 + M) - M].$$

If

$$\mu < \frac{M}{m_1 + m_2 + M} = \mu_{\min},$$

the sliding starts right away, i.e. $L = 0$.

If μ is large enough, the system, being left to itself, starts oscillatory motion with an amplitude $A = Mg/k$. The maximum spring extension is $2A$. Using the condition $L_{\max} = 2A$, one can determine the minimum coefficient of friction μ_0 at which there will be no sliding:

$$\mu_0 = \frac{M}{m_2}\frac{2m_1 + m_2 + 2M}{m_1 + m_2 + M}.$$

Thus, if $\mu > \mu_0$, then $L \longrightarrow \infty$. If

$$\frac{M}{m_1 + m_2 + M} < \mu < \mu_0 = \frac{M}{m_2}\frac{2m_1 + m_2 + 2M}{m_1 + m_2 + M},$$

then

$$L = \frac{g}{k}\frac{m_2}{m_1 + M} \cdot [\mu(m_1 + m_2 + M) - M].$$

If

$$\mu < \mu_{\min} = \frac{M}{m_1 + m_2 + M}, \qquad \text{then} \quad L = 0.$$

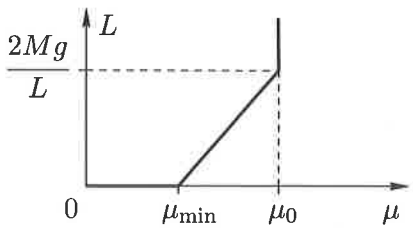

Figure 95

The dependence of L on μ is shown in a diagram (fig. 95).

Now let us calculate a time of slab motion before the onset of sliding. During this time the system executes a simple harmonic motion:

$$L = \frac{Mg}{k}(1 - \cos\omega t),$$

where $\omega^2 = k/(m_1 + m_2 + M)$. Using this equation, one finds that for $\mu \leqslant \mu_0$:

$$t = \left(\frac{m_1 + m_2 + M}{k}\right)^{1/2} \arccos\left(1 - \frac{kL}{Mg}\right).$$

If $\mu > \mu_0$, the sliding never occurs.

51. Trifilar Pendulum *S. Kozel*

1. Let us rotate the ring around the axis OO' by a small angle φ (fig. 96). Then all threads will deviate from their equilibrium positions by a small angle α. According to the diagram,

$$L \cdot \alpha = R \cdot \varphi,$$

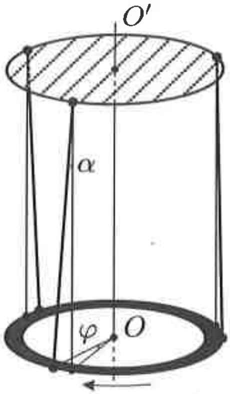

Figure 96

where R is a ring radius. The ring will ascend by a distance

$$x = L(1-\cos\alpha) \approx L\frac{\alpha^2}{2} = \frac{R^2}{2L}\varphi^2.$$

Suppose that all ring points have a speed $v = R\dot{\varphi}$ at this position. Then the net energy of the ring is

$$E = Mgx + \frac{Mv^2}{2} = M\left(\frac{R^2 g}{2L}\varphi^2 + \frac{R^2\dot{\varphi}^2}{2}\right). \quad (34)$$

In the absence of friction the net energy is conserved. By differentiating (34) with respect to time, one obtains

$$\ddot{\varphi} + \frac{g}{L}\cdot\varphi = 0.$$

This is an equation of motion of a simple gravity pendulum of the length L. Therefore, the angular frequency is $\omega_0 = \sqrt{g/L}$, and

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{L}{g}}.$$

2. Placing a point-like mass at the ring centre does not change the kinetic energy of the system while its potential energy must include a sum of masses $(M+m)$. The equation of torsional oscillations now becomes:

$$\ddot{\varphi} + \frac{(M+m)}{m}\frac{g}{L}\cdot\varphi = 0, \qquad \text{therefore,} \quad \omega_0' = \sqrt{\frac{(M+m)}{m}\frac{g}{L}}.$$

For $m = M$ the frequency increases by a factor of $\sqrt{2}$ and, therefore, the period decreases by the same factor of $\sqrt{2}$.

52. Charged Particle Inside Solenoid

S. Kozel

1. A magnetic induction B inside the solenoid is

$$B = \mu_0 I \cdot n = 6.28 \cdot 10^{-4}\ \text{T}.$$

A direction of the magnetic induction vector can be found from the right-hand rule. In this case, $\vec{B}$ points into the diagram as viewed by the reader. A charged particle traveling in a magnetic field is subjected to the Lorentz force which direction can be found from the right-hand rule. Since the particle is deflected to the right, its charge $q < 0$.

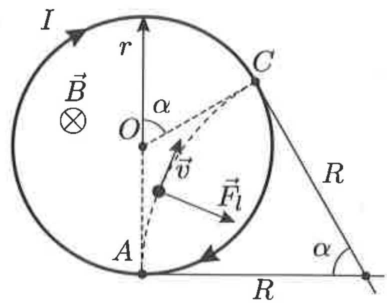

Figure 97

2. A charged particle in a uniform magnetic field travels along a circular arc (fig. 97). The absolute value of the particle velocity remains constant:

$$\frac{mv^2}{R} = |q|vB, \qquad \text{or} \quad R = R_{\text{curvature}} = \frac{mv}{|q|B}. \tag{35}$$

The curvature radius can be found from geometry. The points A and C are at the intersections of two circles which radii are r and R. It follows from symmetry that the particle velocity at C points along a solenoid radius (similar to the point A). Therefore, a centre of the particle circular path (the centre of path curvature) lies at the intersection of the tangents at A and C.
According to the diagram,

$$R = r\frac{1+\cos\alpha}{\sin\alpha} = \sqrt{3}\,r = 17.3 \text{ cm}.$$

Also,

$$2R\sin\frac{\alpha}{2} = |AC| = 2r\sin\frac{\pi-\alpha}{2},$$

which gives $R = r\cot\dfrac{\alpha}{2} = \sqrt{3}\,r = 17.3$ cm.

3. The absolute value of the particle velocity v is determined from the conservation of energy:

$$\frac{mv^2}{2} = |q|U; \qquad v^2 = \frac{2|q|U}{m} = \frac{|q|}{m^2}R^2B^2.$$

Here we also used (35). Hence,

$$\frac{|q|}{m} = \frac{2U}{R^2B^2} \approx 1.7 \cdot 10^{11}\ \frac{\text{C}}{\text{kg}}.$$

Note. For electron $\frac{e}{m} = 1.76 \cdot 10^{11}\ \frac{\text{C}}{\text{kg}}$.

53. Capacitor Leak *M. Proskurin*

1. In the stationary regime, a current $I = \text{const}$ at any x. Consider a layer of a thickness dx at a distance x from the lower plate. According to Ohm's law,

$$dU = \rho\frac{dx}{S}I = I\rho_0\left(1 + \frac{2x}{d}\right)\frac{dx}{S}. \tag{36}$$

Here S is an area of capacitor plates. By integration, one obtains

$$U_0 = \int_0^d I\rho_0\left(1 + \frac{2x}{d}\right)\frac{dx}{S} = \frac{2I\rho_0 d}{S} = \frac{2I\rho_0\varepsilon_0}{C_0}$$

because $C_0 = \frac{\varepsilon_0 S}{d}$. Therefore,

$$I = \frac{U_0C_0}{2\rho_0\varepsilon_0}. \tag{37}$$

2. Let us determine an electric field near the lower (E_1) and the upper (E_2) plates. Using (36) and (37), one finds:

$$E(x) = \frac{dU}{dx} = \frac{I\rho_0}{S}\left(1 + \frac{2x}{d}\right) = \frac{C_0U_0}{2\mathscr{E}_0 S}\left(1 + \frac{2x}{d}\right). \tag{38}$$

At $x = 0$ the field $E_1 = \frac{C_0U_0}{2\varepsilon_0 S}$ and the charge equals

$$q_1 = S\sigma_1 = SE_1\mathscr{E}_0 = \frac{C_0U_0}{2}. \tag{39}$$

At $x = d$ the field $E_2 = \frac{3C_0U_0}{2\varepsilon_0 S}$ and the charge equals

$$q_2 = -S\sigma_2 = -SE_2\mathscr{E}_0 = -\frac{3C_0U_0}{2}. \tag{40}$$

3. The net charge, including the charges on the plates and the volume charge in the medium between the plates, stored by the capacitor vanishes:

$$q_1 + q_2 + q = 0.$$

Therefore,

$$q = C_0 U_0.$$

4. The total energy stored by the capacitor is determined by the electric energy density $w_e = \varepsilon_0 E^2/2$:

$$W_e = \int_0^d \frac{\varepsilon_0 E^2(x)}{2} S dx = \frac{C_0 U_0^2}{8d} \int_0^d \left(1 + \frac{2x}{d}\right)^2 dx = \frac{13}{24} C_0 U_0^2.$$

54. Piston Stability

S. Karmazin

1. Let S be a cross-sectional area of the lower cylinder. Then the volume occupied by helium equals $V = (L-h)S$. Since the helium temperature is constant, one can apply the Boyle-Mariotte law:

$$p(L-h)S = \text{const}, \qquad \text{or} \quad p = \frac{(p_0 + \rho_{\text{Hg}} g h_1)(L - h_1)}{L - h}. \tag{41}$$

The helium pressure in equilibrium is $p = p_0 + \rho_{\text{Hg}} g h$, which yields a quadratic equation for h:

$$(p_0 + \rho_{\text{Hg}} g h_1)(L - h_1) = (p_0 + \rho_{\text{Hg}} g h)(L - h). \tag{42}$$

Therefore, there are two positions of equilibrium. The first position $h = h_1$ is given in the problem statement. By solving the quadratic equation (42), one finds the second position $h_2 = 360$ mm.

2. Let us study the stability of the equilibrium positions. Let $p_\uparrow$ be a pressure of mercury just above the piston and $p_\downarrow$ be a pressure of helium just below the piston. An equilibrium will be stable if the restoring force resulting from a small displacement of the piston tends to return it in the equilibrium position:

$$\frac{d(p_\downarrow - p_\uparrow)}{dh} > 0. \tag{43}$$

The pressure of mercury equals

$$p_\uparrow = p_0 + \rho_{\mathrm{Hg}} g h,$$

the pressure of helium $p_\downarrow = p$ follows from (41). By differentiation, one obtains

$$\frac{(p_0 + \rho_{\mathrm{Hg}} g h_1)(L - h_1)}{(L-h)^2} - \rho_{\mathrm{Hg}} g > 0.$$

Equation (42) holds at equilibrium, therefore,

$$\frac{p_0 + \rho_{\mathrm{Hg}} g h}{L-h} - \rho_{\mathrm{Hg}} g > 0.$$

Finally, the condition of stability of an equilibrium position is

$$h > \frac{1}{2}\left(L - \frac{p_0}{\rho_{\mathrm{Hg}} g}\right) = 370 \text{ mm}. \tag{44}$$

Thus, the first position of equilibrium $h_1 = 380$ mm is stable, while the second one, $h_2 = 360$ mm, is not.

55. Planar Waveguide

S. Kozel

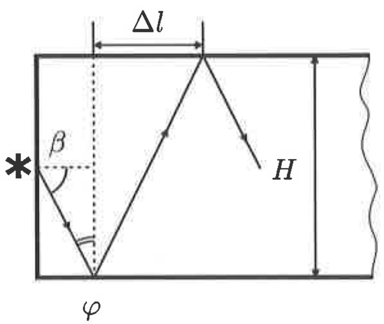

Figure 98

Consider refraction of the source rays on the left butt of the plate (fig. 98). The maximum refraction angle $\beta_{\max}$ corresponds to the angle of incidence $\alpha = 90°$:

$$\sin\beta_{\max} = \frac{1}{n}.$$

The minimum angle of incidence on a lateral side is

$$\varphi_{\min} = 90° - \beta_{\max}.$$

A path in the plate depends on the relation between the angle $\varphi_{\min}$ and the limiting angle of the total internal reflection $\varphi_{\lim}$, where $\sin\varphi_{\lim} = \frac{1}{n}$.

Case 1. If $\varphi_{\min} \geq \varphi_{\lim}$, all rays incident on the lateral sides will undergo the total internal reflection, so none of them escapes the plate. This is true providing $\sin\varphi_{\min} \geq \sin\varphi_{\lim} = \frac{1}{n}$, i.e.

$$\sin\varphi_{\min} = \cos\beta_{\max} = \frac{\sqrt{n^2-1}}{n} \geq \frac{1}{n} \Rightarrow \sqrt{n^2-1} \geq 1 \Rightarrow n \geq \sqrt{2}.$$

The minimum separation (Δl) between two succesive ray reflections on the opposite sides is

$$(\Delta l)_{\min} = H\tan\varphi_{\min} = H\cot\beta_{\max} = H\frac{\sqrt{n^2-1}/n}{1/n} = H\sqrt{n^2-1}\,.$$

The maximum number of reflections is

$$N_1 = \left[\frac{L}{(\Delta l)_{\min}} + \frac{1}{2}\right] = \left[\frac{L}{H}\frac{1}{\sqrt{n^2-1}} + \frac{1}{2}\right].$$

The term 1/2 is added since a ray travels a distance $\Delta l/2$ along the plate before the first reflection. The first case is realised for $n = n_1 = 1.73 > \sqrt{2}$, then $N_1 = 71$.

Case 2. If $\varphi_{\min} < \varphi_{\lim}$, the rays incident on a lateral side at an angle between $\varphi_{\min}$ and $\varphi_{\lim}$ leave the plate and do not reach the right butt. This happens if $n < \sqrt{2}$. In this case,

$$(\Delta l)_{\min} = H\tan\varphi_{\lim} = \frac{H}{\sqrt{n^2-1}}.$$

The maximum number of reflections is:

$$N_2 = \left[\frac{L}{(\Delta l)_{\min}} + \frac{1}{2}\right] = \left[\frac{L}{H}\sqrt{n^2-1} + \frac{1}{2}\right].$$

The second case is realised for $n = n_2 = 1.3 < \sqrt{2}$, so the number is $N_2 = 100 \cdot 0.83 = 83$.

56. Bag of Flour

V. Ataulin

After the oscillations decayed, the scale readings showed the bag mass, so it equals $m = m_0$. The bag speed right before the impact is $v = \sqrt{2gh}$. The impact is completely inelastic, so the bag and the pan velocities right after the impact are the same and equal to

$$V = \frac{m_0}{m_0 + M} v. \tag{45}$$

The only load on the scale spring right before the impact was due to the pan, so the spring extension was equal to

$$x_1 = \frac{Mg}{k}.$$

The maximum spring extension equals

$$x_2 = \frac{Mg}{k} + \frac{m_1 g}{k} = \frac{(M + m_1)g}{k}. \tag{46}$$

Let us use the conservation of energy to equate the net energy at the moment right after the impact and at the maximum spring extension:

$$\frac{(M + m_0)V^2}{2} + \frac{kx_1^2}{2} = \frac{kx_2^2}{2} - (M + m_0)g(x_2 - x_1).$$

Now, let us apply (45) and (46):

$$\frac{m_0^2 v^2}{2(M + m_0)} + \frac{M^2 g^2}{2k} = \frac{(M + m_1)^2 g^2}{2k} - (M + m_0) \cdot g \cdot \frac{m_1 g}{k}.$$

Simplifying this equation, one obtains:

$$\frac{m_0^2 v^2}{2(M + m_0)} = \frac{m_1 g^2}{k}(m_1 - 2m_0).$$

Substitution of $v = \sqrt{2gh}$ yields the answer:

$$M = \frac{2km_0^2 h}{m_1 g(m_1 - 2m_0)} - m_0 = 2 \text{ kg}.$$

57. Magnetism

A. Apolonskii

An increment in the area of $ABCD$ contour formed by the rods and the rails induces an emf which pulls a current in the system (fig. 99). The ensuing Ampere's forces decelerate AB and accelerate CD. When v_{AB} and v_{CD} velocities are equated, the magnetic flux through $ABCD$ contour does not change anymore, the induction emf vanishes, and so do the forces. After that the rods travel at the same speed in the same direction.

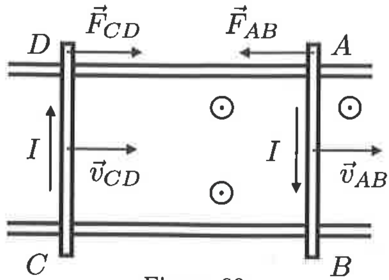

Figure 99

An increment in the contour area per a small time interval Δt is

$$\Delta S = (v_{AB} - v_{CD})l\Delta t,$$

and the resulting increment in the magnetic flux is

$$\Delta\Phi = B\Delta S = B(v_{AB} - v_{CD})l\Delta t.$$

Then the induction emf is

$$|\mathscr{E}_i| = \frac{\Delta\Phi}{\Delta t} = B(v_{AB} - v_{CD})l,$$

the current in the contour is

$$I_{AB} = I_{CD} = \frac{|\mathscr{E}_i|}{2R} = \frac{B(v_{AB} - v_{CD})l}{2R},$$

and Ampere's forces are equal to

$$F_{AB} = F_{CD} = \frac{B^2l^2(v_{AB} - v_{CD})}{2R}.$$

Accelerations of the rods are opposite and equal to

$$a = \frac{B^2l^2(v_{AB} - v_{CD})}{2mR}.$$

Taking into account the signs of the accelerations, one writes

$$\frac{\Delta v_{AB}}{\Delta t} = -\frac{B^2 l^2 (v_{AB} - v_{CD})}{2mR}, \qquad \frac{\Delta v_{CD}}{\Delta t} = \frac{B^2 l^2 (v_{AB} - v_{CD})}{2mR}.$$

Multiplying both sides by Δt and writing the increment of a distance between the rods as $\Delta d = (v_{AB} - v_{CD})\Delta t$, one obtains:

$$\Delta v_{AB} = -\frac{B^2 l^2}{2mR}\Delta d, \qquad \Delta v_{CD} = \frac{B^2 l^2}{2mR}\Delta d.$$

Then

$$v_{AB} = v_0 - \frac{B^2 l^2}{2mR}\Delta d, \qquad v_{CD} = \frac{B^2 l^2}{2mR}\Delta d.$$

When the velocities equate,

$$v_0 - \frac{B^2 l^2}{2mR}\Delta d = \frac{B^2 l^2}{2mR}\Delta d,$$

hence

$$\Delta d = \frac{m v_0 R}{B^2 l^2}.$$

Finally, the separation between the rods tends to

$$d' = d + \Delta d = d + \frac{m v_0 R}{B^2 l^2}.$$

The velocities are

$$v_{AB} = v_{CD} = v_0/2.$$

The heat produced by the system equals

$$Q = \frac{m v_0^2}{2} - 2\frac{m (v_0/2)^2}{2} = \frac{m v_0^2}{4}.$$

58. Linear Process

V. Slobodyanin

Let us write an equation of the linear process BD as

$$p(V) = p_1 + k(V - V_1), \qquad (47)$$

where p_1 and V_1 are the gas pressure and volume at D. Hence,

$$k = -\frac{p_1}{V_0 - V_1}, \qquad (48)$$

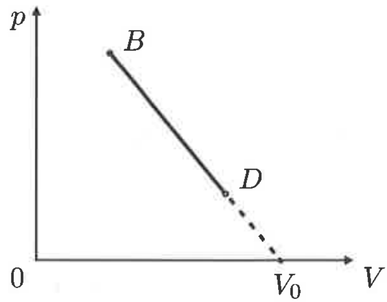

Figure 100

where V_0 is a point of intersection of the straight line BD with the volume axis.

According to the problem statement, the temperature decreases during the process and the gas receives heat all the way. Let us find out, what these two conditions mean.

The ideal gas law for 1 mole of gas is:

$$pV = RT. \qquad (49)$$

The temperature decreases, therefore $\Delta(pV) = p\Delta V + V\Delta p < 0$. Using equation (47) for BD, one can write this condition as

$$[p_1 + k(V - V_1)]\Delta V + Vk\Delta V < 0.$$

where $\Delta V > 0$. Substituting k from (48) and simplifying, one obtains:

$$V > \frac{V_0}{2}.$$

The gas receives heat $\delta Q > 0$ during the whole process BD. Since

$$\delta Q = \frac{1}{R}(C_P p\Delta V + C_V V\Delta p),$$

then using (47) again, one has

$$\gamma(p_1 + k(V - V_1)) + Vk > 0,$$

where $\gamma = C_P/C_V = 4/3$ (the gas is polyatomic). Similarly, substituting k from (48) and simplifying, one gets

$$V < \frac{\gamma}{\gamma+1}V_0 = \frac{4}{7}V_0.$$

The work is maximal if $V_B = V_0/2$ and $V_D = 4V_0/7$. In this case, $k = -\frac{7}{3}\frac{p_1}{V_0}$ and from (47) one obtains $p_B = 7p_1/6$. The maximum work equals

$$A_{\max} = \frac{1}{2}(p_B+p_D)(V_D-V_B) = \frac{1}{2}\left(1+\frac{6}{7}\right)p_B\left(\frac{4}{7}-\frac{1}{2}\right)V_0 = \frac{13}{98}p_BV_B.$$

It follows from (49) that

$$A_{\max} = \frac{13}{98}RT_B = 540.15 \text{ J},$$

where $T_B = 490$ K.

59. Circle of Lenses

I. Erofeev, M. Osin

1. According to the problem statement, the lens planes form a regular polygon. Draw two rays from the centre C (CD and CE), so that they

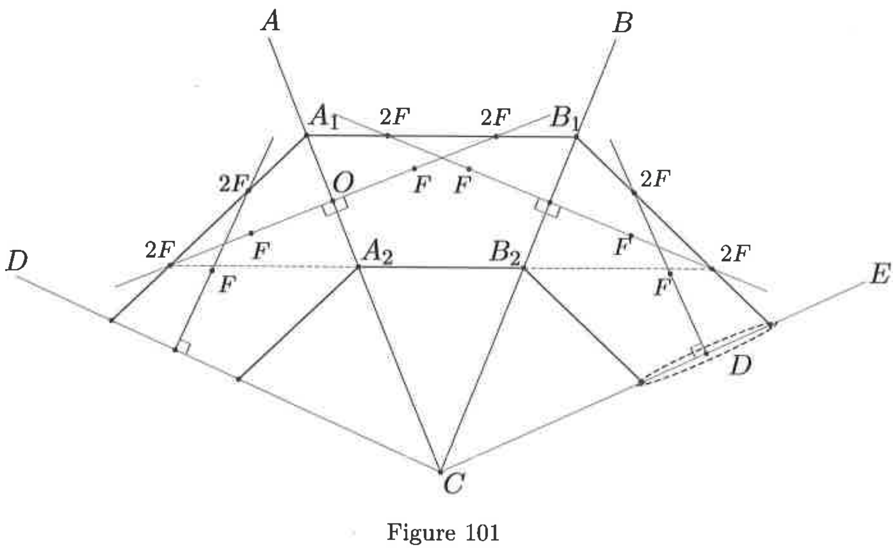

Figure 101

make an angle $\alpha = 2\pi/N$ with the adjacent straight lines (CA and CB). These are projections of the lens planes on the diagram plane.

2. Suppose the focal point F belongs to the left lens. Draw a perpendicular to this lens (fig. 101). The point of intersection of the perpendicular and the lens plane is the lens optical centre O. Construct optical centres of the lenses on the remaining planes.

3. The polygon is regular, therefore the ray path is symmetric with respect to a lens plane. In the case of a converging lens, the ray intersects its main optical axis at a distance equal to the double focal length (DFL) according to the thin lens formula. If the lens is diverging, then the ray imaginary extension intersects this point. Plot the DFLs on the main optical axes of lenses. A ray passing through a DFL inside ACB will extend beyond the lens optical centres with respect to C. This corresponds to a converging lens. A ray passing through DFLs outside of ACB, will travel nearer than the lens optical centres with respect to C. This corresponds to a diverging lens. Let these rays intersect the lens planes (CA and CB) at A_1, B_1 and A_2, B_2. Plotting a distance CB_1 from the centre C, one restores the points through which the ray passes if the lenses are diverging. One can see that in both cases the ray passes through the lenses (of a radius $D/2$).

4. Now suppose that the focal point F belongs to the right lens. Going through the same steps as before, one can see that the ray misses the lenses (passes at a distance exceeding its radius $D/2$).

5. Thus, two answers are possible. A lens can be either converging or diverging. The focal point F belongs to the left lens.

60. Dipole in Electric Field *M. Proskurin*

A force $F(x)$ exerted on the dipole by the electric field is a sum of the forces acting on the charges $+q$ and $-q$. Let x be a coordinate of the dipole centre, then

$$F(x) = qE(x + l/2) - qE(x - l/2) = q\frac{dE}{dx}l = -2qlE_0\frac{x}{L^2}.$$

The second Newton's law for the dipole is

$$m\ddot{x} = F(x) = -2\frac{qlE_0}{L^2}x. \tag{50}$$

This is the equation of a simple harmonic motion with an angular frequency ω:

$$\omega = \sqrt{2\frac{qlE_0}{mL^2}}.$$

Let the dipole enter the region of electric field at $t = 0$. Then the solution to (50) is

$$x(t) = A\sin(\omega t + \varphi), \qquad v(t) = A\omega\cos(\omega t + \varphi).$$

The initial conditions are

$$x(0) = -L = A\sin\varphi,$$
$$v(0) = v_0 = A\omega\cos\varphi,$$

therefore,

$$\varphi = -\arctan\frac{\omega L}{v_0}, \qquad A = L\sqrt{1 + \left(\frac{v_0}{\omega L}\right)^2}$$

The maximum dipole speed follows from the equation of motion:

$$v_{max} = A\omega = \sqrt{v_0^2 + 2\frac{qlE_0}{m}}.$$

Due to symmetry, a time of flight t through the region of electric field equals a doubled time of flight from $x = -L$ to the origin, i.e.

$$t = 2\frac{|\varphi|}{\omega} = 2\arctan\frac{\omega L}{v_0} = 2\arctan\sqrt{\frac{2qlE_0}{mv_0^2}}.$$

61. Three in A Field, not Counting Capacitor *V. Chivilev*

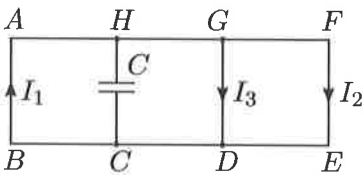

Figure 102

Eventually, the capacitor will be charged, so there will be no current through it and the currents in the whole circuit will be set. Suppose the currents are those shown (fig. 102). Kirchhoff's laws for the contours are

$$AFEB: \quad \mathscr{E}_{AFEB} = 5I_1r + 3I_2r, \tag{51}$$
$$AGDB: \quad \mathscr{E}_{AGDB} = 5I_1r + I_3r, \tag{52}$$

and the first Kirchhoff's law for the point D is

$$I_1 = I_2 + I_3.$$

According to Faraday's law of induction,

$$\mathscr{E}_{AFEB} = \frac{d(B \cdot 3a^2)}{dt} = 3ka^2, \quad \mathscr{E}_{AGDB} = 2ka^2. \tag{53}$$

Let $ka^2/r = I_0$. Then, solving the set of equations (51), (52), and (53), one obtains:

$$I_1 = \frac{9}{23}I_0, \quad I_2 = \frac{8}{23}I_0, \quad I_3 = \frac{1}{23}I_0.$$

These equations together with a condition $k > 0$ indicate that a current I_1 through AB flows from B to A.
Consider a contour $AHCB$:

$$I_0 r = ka^2 = \mathscr{E}_{AHCB} = I_1 \cdot 3r + U_C,$$

where U_C is a voltage across the capacitor plates. Therefore,

$$U_C = -\frac{4}{23}ka^2 \quad \text{and} \quad Q = \frac{4}{23}Cka^2;$$

the charges of the upper and lower plate are negative and positive, respectively.
A Joule heat generated by the circuit equals the sum of the heats generated by all wires:

$$N = 5I_1^2 r + 3I_2^2 r + I_3^2 r = \frac{26}{23}\frac{k^2a^4}{r}, \qquad \text{therefore} \quad W = N\tau = \frac{26}{23}\frac{k^2a^4}{r}\tau.$$

62. Spoke Oscillation *V. Slobodyanin*

Consider spoke rotation around the stand left edge. The kinetic energy of the system equals the sum of kinetic energies of the spheres:

$$K = \frac{m}{2}\left(\frac{L-l}{2}\right)^2 \dot{\varphi}^2 + \frac{m}{2}\left(\frac{L+l}{2}\right)^2 \dot{\varphi}^2 = \frac{m\left(L^2+l^2\right)}{4}\dot{\varphi}^2 \approx \frac{mL^2}{4}\dot{\varphi}^2.$$

The potential energy of the spoke is:

$$\Pi = -mg\left(\frac{L-l}{2}\right)\varphi + mg\left(\frac{L+l}{2}\right)\varphi = mgl\varphi.$$

The mechanical energy is conserved, so

$$\dot{K} + \dot{\Pi} = \frac{mL^2}{2}\ddot{\varphi}\dot{\varphi} + mgl\dot{\varphi} = 0.$$

Dividing the latter by $\dot{\varphi}$ (the solution $\dot{\varphi} = 0$ is trivial), one obtains an equation of uniformly accelerated motion:

$$\ddot{\varphi} = -\frac{2gl}{L^2}.$$

A time τ required to return the spoke in the horizontal position can be found from the condition:

$$\frac{\ddot{\varphi}\tau^2}{2} = -\varphi_0, \qquad \text{hence,} \quad \tau = \sqrt{\frac{\varphi_0}{gl}}L.$$

After the spoke returns to the horizontal position, it will rotate around the right edge of the stand, then around the left one again, etc. The period of this motion equals

$$T = 4\tau = 4L\sqrt{\frac{\varphi_0}{gl}}.$$

63. Soda Pressure *V. Babintsev*

When temperature increases, the gas pressure in the empty volume V between water and the cap increases. This happens due to reduced solubility of carbon dioxide and its escape from water. The pressure inside the bottle must not exceed the maximum p_2 at the maximum temperature t_2.

Let us determine a mass of carbon dioxide which will be added to V if the temperature increases from t_1 to t_2. To this end, determine the solubility of carbon dioxide at these temperatures using the plot and Henry's law:

$$\sigma_1 = \sigma_1'\frac{p_1}{p_0} \approx 4.50 \text{ g/kg}, \qquad \sigma_2 = \sigma_2'\frac{p_2}{p_0} \approx 4.07 \text{ g/kg},$$

where $\sigma'_1 = 3$ g/kg and $\sigma'_2 = 1.1$ g/kg are solubilities at t_1 and t_2 at the atmospheric pressure determined from the plot. Thus, the temperature increment increases a mass of carbon dioxide in V by

$$\Delta m = (\sigma_1 - \sigma_2) m_w = 0.86 \text{ g}.$$

It is also necessary to take into account a mass m_1 of carbon dioxide dissolved in water when the bottle was just sealed.
Then the net mass of carbon dioxide is

$$m_2 = m_1 + \Delta m = \frac{p_1 V \mu}{R T_1} + \Delta m.$$

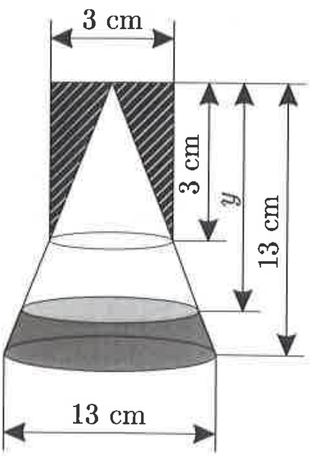

Figure 103

The maximum pressure of carbon dioxide equals p_2:

$$p_2 V = \frac{m_2}{\mu} R T_2 = p_1 V \frac{T_2}{T_1} + \frac{\Delta m}{\mu} R T_2.$$

Thus, the minimum empty volume of the bottle is

$$V = \frac{\Delta m}{\mu} \frac{R T_1 T_2}{p_2 T_1 - p_1 T_2} \approx 250 \text{ cm}^3.$$

The volume V is a sum of the upper cylinder volume

$$V_1 = \frac{1}{4} \pi d^2 (H - h) \approx 21.2 \text{ cm}^3,$$

and the lower cone (frustum) volume V_2 (fig. 103). For computational convenience, let us divide the volume shape differently, namely, by the cone volume V_c and the volume V_0 left after the cone has been removed from the cylinder (the shaded part in the diagram):

$$V_0 = 21.2 \text{ cm}^3 - \frac{1}{12} \pi d^2 (H - h) \approx 14.1 \text{ cm}^3.$$

Thus,

$$V = 14.1 \text{ cm}^3 + \frac{\pi y^3}{12} \approx 250 \text{ cm}^3, \qquad \text{hence} \quad y \approx 10 \text{ cm}.$$

Finally, the allowed level of water is $x = H - y \approx 20$ cm.

64. Lens Recovery

A. Apolonskii

Suppose that the rod with sources is a triangle side (e.g., AB) and the points A_1 and B_1 are their images (fig. 104). Then the beams AA_1 and BB_1 intersect at the middle of BB_1 at the point O which is the lens centre. A beam going from A through B is refracted in the lens plane and goes further through their images A_1 and B_1. Then the midpoint K of AB lies in the lens plane. In this case the lens plane OK intersects the rod AB, which is impossible. Therefore, the triangle centre corresponds to a rod endpoint.

Consider this possibility. The straight lines AB and A_1B_1 intersect at the midpoint K of A_1B_1 in the lens plane (fig. 105). Thus, OK is the lens plane and O is its centre. The straight line PQ passing through O perpendicular to OK is the main optical axis. Points A_1 and B_1 are image endpoints. The image itself is «broken» and is represented by two rays lying on the straight line A_1B_1 and going to infinity from the points A_1 and B_1.

The lens must be converging since only such a lens can form a «broken» image. It is not difficult to determine positions of the focal points and the focal length. Let a beam AM to travel from the point A parallel to the lens axis PQ until its intersection with the lens plane at M. After refraction the beam passes through the image A_1. The intersection MA_1 with PQ is the main focal point K of the lens. Since OK is the middle line of the triangle and O is the midpoint of AA_1, $OF = h/4$, where h is the triangle height, and $h = 3AB/2 = 3l/2$. Thus, the focal length is $OF = 3l/8$.

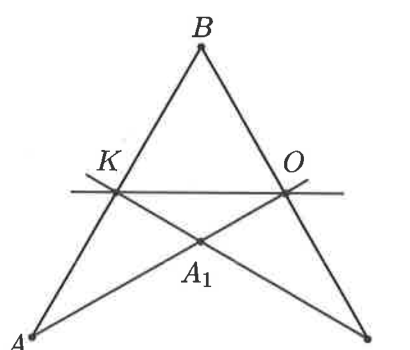

Figure 104

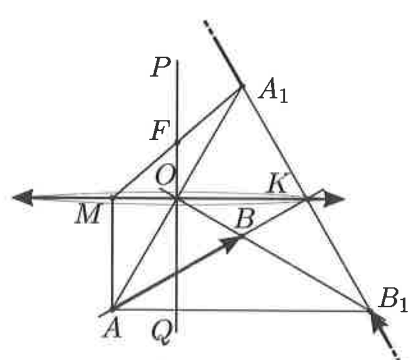

Figure 105

65. Faulty Rocket

I. Erofeev

If a source of sound is moving, the frequency registered by a stationary observer can differ from the emitted frequency. This is Doppler's effect. Let us derive the formula of Doppler's effect for a stationary observer and a moving source.

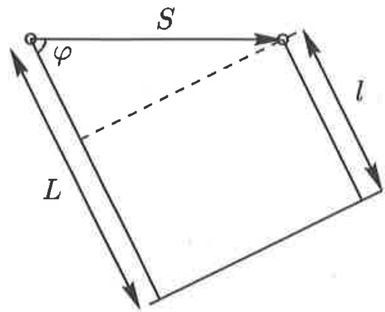

Figure 106

Let T be the period of a sound wave; the source travels a distance $S = vT$ while a wavefront approaches the observer by a distance $L = cT$ (fig. 106). Thus, the distance between fronts in the direction toward the observer equals $l = L - S\cos\varphi = (c - v\cos\varphi)T$ and the frequency is

$$f = \frac{c}{l} = \frac{c}{(c - v\cos\varphi)T} = \frac{f_0}{1 - (v/c)\cos\varphi}. \tag{54}$$

1. If the rocket is coming directly to a sensor, then $\varphi = 0$,

$$f_{\max} = \frac{f_0}{1 - v/c},$$

and the sound frequency increases.

At an arbitrary angle φ,

$$f = \frac{f_0}{1 - (v/c)\cos\varphi}.$$

The frequency $f > f_0$ for a positive velocity component in the direction to the sensor ($\cos\varphi > 0$), otherwise $f < f_0$. If $\varphi = \pi/2$, $f = f_0$.

2. There are two basic positions of a sensor: inside the circular rocket path and outside it.* Let us consider the case of a sensor outside the path. In this case, the points from which the maximum and minimum frequencies are coming to the sensor are the ends (M_1 and M_2, respectively) of the tangent segments drawn from the sensor location A to the rocket path (fig. 107).

*There is one more possibility: the sensor is located on the rocket path. However, in this case there would be a sharp peak in the frequency plot which is not observed.

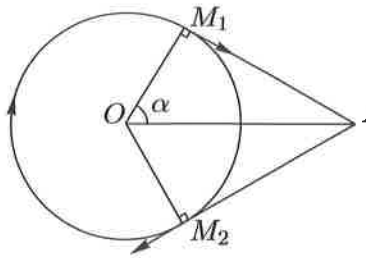

Figure 107

Due to symmetry, $AM_1 = AM_2$, the delays between emission and detection of the sound are the same for M_1 and M_2. This means that M_1 and M_2 divide the path circumference in the same ratio as the points of maximum and minimum frequency divide a signal period on the plot. The maximum frequency is $f^{\text{out}}_{\max} = f_{\max}$.

If a sensor is located inside the path (fig. 108), the angle φ between the rocket velocity and a direction to the sensor cannot be 0 and π anymore. To determine its maximum and minimum, consider a triangle OBN_1. Here $ON_1 = r$ is a circle radius, $OB = b$ is a distance from the path centre to the sensor, $\angle ON_1B = \theta = \pi/2 - \varphi$, and $\angle OBN_1 = \gamma$. Note that θ is acute for any position of N_1; according to the law of sines:

$$\sin\theta = \frac{b}{r}\sin\gamma.$$

Since $b < r$, $\sin\theta$ is maximal when $\sin\gamma = 1$, i.e. $\gamma = \pi/2$. And, since $|\cos\varphi| = |\sin\theta|$, the maximum and minimum of $\cos\varphi$ are at N_1 and N_2, the ends of an arc centred at B, (fig. 109).

Again, due to symmetry and an equality $BN_1 = BN_2$, the points N_1 and N_2 divide the circumference in the same ratio as the points of maximum and minimum frequency divide a period on the plot.

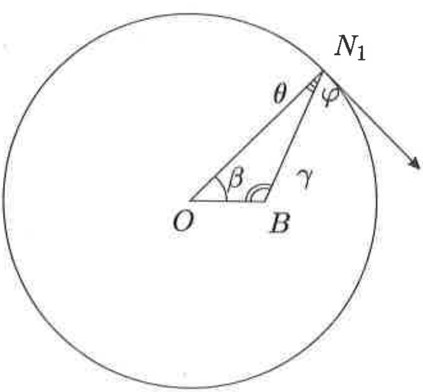

Figure 108

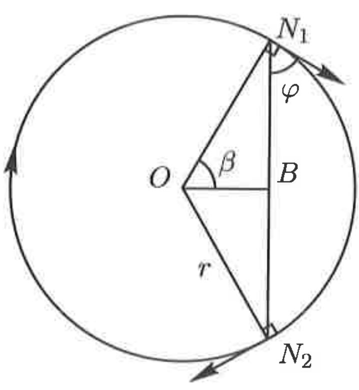

Figure 109

In this case, the maximum frequency is

$$f_{\max}^{\text{in}} = \frac{f_0}{1 - \dfrac{v}{c}\cos\varphi} = \frac{f_0}{1 - \dfrac{v\,b}{c\,r}} < f_{\max}.$$

Now let us take a look at the plot and notice that for both sensors the time interval between the maximum and minimum frequency equals $t_1 = t_2 = T/4 = 6\ s$, where $T = 24\ s$ is the signal period (the period of rocket rotation). Thus, in both cases $2\alpha = 2\beta = \pi/2$. Further, since the maximum frequency of sensor 1 exceeds the maximum frequency of sensor 2, this is possible only if sensor 1 is outside the path and sensor 2 is inside.

Consider the signal of sensor 1, (fig. 53):

$$f_{\max}^{\text{out}} = \frac{f_0}{1 - v/c} = 1.32\ \text{kHz}, \qquad f_{\min}^{\text{out}} = \frac{f_0}{1 + v/c} = 0.88\ \text{kHz},$$

hence, $f_0 = 1.056$ kHz and the rocket velocity is $v = c/5 = 66$ m/s. Using the period $T = 24$ s and the rocket speed v, one determines the radius of its path:

$$r = \frac{vT}{2\pi} = 252\ \text{m}.$$

Now a distance from the path centre to a sensor can be evaluated:

$$a = \frac{r}{\cos\alpha} = \sqrt{2}r = 357\ \text{m}, \qquad b = r\cos\beta = r/\sqrt{2} = 178\ \text{m}.$$

Finally, one has to determine the angle ψ between OA and OB which results in a phase shift $\tau = T/3 = 8$ s between the plot maxima. Since $a \neq b$, a different time of propagation from the rocket to sensors contributes to this shift. Thus,

$$\psi = \frac{\tau}{T} + \frac{AM_1}{c} - \frac{BN_1}{c} = \frac{2\pi}{3} + \frac{2 - \sqrt{2}}{10} = 2.15\ \text{rad}.$$

Note that the contribution due to a signal delay is about 3 %, which is of the same order of magnitude as the accuracy of time evaluation from the plot.

The distance between the sensors is determined by the law of cosines:

$$L = \sqrt{a^2 + b^2 - 2ab\cos\psi} = 478\ \text{m}.$$

66. Spring Stretching *A. Gudenko*

The period of longitudinal oscillations of a weight attached to a spring of length l and with a spring coefficient k is

$$T_{\|}^2 = 4\pi^2 \frac{m}{4k},$$

where m is the weight mass. The equation of weight motion in the transverse direction is

$$m\ddot{\varphi}\frac{l}{2} = 2F_{\text{elasticity}}\,\varphi = -2k\Delta l\varphi \quad \Longleftrightarrow \quad \ddot{\varphi} + \frac{4k\Delta l}{ml}\varphi = 0.$$

This is the equation of a simple harmonic motion with an angular frequency $\omega_{\perp}^2 = (4k\Delta l)/(ml)$, the period is

$$T_{\perp}^2 = 4\pi^2 \frac{ml}{4k\Delta l} = 4\pi^2 \frac{m}{4k}\left(1 + \frac{l_0}{\Delta l}\right).$$

The ratio of the periods is

$$\frac{T_{\perp}^2}{T_{\|}^2} = n^2 = 1 + \frac{l_0}{\Delta l}, \qquad \text{hence} \quad \frac{\Delta l}{l_0} = \frac{1}{n^2 - 1}.$$

Since $\Delta x = \Delta l_2 - \Delta l_1$,

$$\frac{\Delta x}{l_0} = \frac{1}{n_2^2 - 1} - \frac{1}{n_1^2 - 1} = \frac{n_1^2 - n_2^2}{(n_2^2 - 1)(n_1^2 - 1)},$$

and

$$l_0 = \frac{(n_2^2 - 1)(n_1^2 - 1)}{n_1^2 - n_2^2}\Delta x = 60 \text{ cm},$$

$$\Delta l_1 = \frac{l_0}{n_1^2 - 1} = 4 \text{ cm},$$

$$\Delta l_2 = \frac{l_0}{n_2^2 - 1} = 7.5 \text{ cm}.$$

67. Nanomelting *A. Gudenko*

A reduction of the melting point temperature for a nano-object is related to a relatively high fraction of atoms in the surface layer (as the volume decreases) which have an extra energy ΔU compared to atoms of the bulk. For samples of various shapes these fractions are different. For a sphere of a radius $R = d/2$ the fraction of atoms in the layer of a thickness δ is

$$\Delta N/N = \Delta V/V = \frac{4\pi R^2 \delta}{4\pi R^3/3} = 3\delta/R = 6\delta/d.$$

This equation is valid providing $\delta \ll d$. Let us estimate the surface layer thickness. A volume per a tin atom is

$$v = \frac{1}{\rho}\frac{\mu}{N_A},$$

therefore, an average interatomic distance equals $a = \sqrt[3]{v} \approx 0.30$ nm. In this case the surface layer is less than 1 nm, which is sufficiently less than 20 nm.

The heat of fusion decreases by an amount of extra energy brought by all surface layer atoms:

$$\Delta q_0 = \Delta U \Delta N = \Delta U N \cdot \frac{6\delta}{d}.$$

New heat of fusion is

$$q = q_0 - \Delta q_0 = q_0 - \Delta U N \cdot \frac{6\delta}{d}.$$

The heat of fusion per atom is

$$\lambda = \frac{q}{N} = \frac{q_0}{N} - \Delta U \frac{6\delta}{d}.$$

According to the problem statement, $\lambda \sim t$. Let us write $\lambda = q/N = \alpha t_d$ and $\lambda_0 = q_0/N = \alpha t_0$, where α is a proportionality factor. Therefore, the relative decrement of the melting point temperature compared to a bulk sample is

$$\frac{\Delta t_d}{t_0} = \frac{\Delta U}{\alpha t_0}\frac{6\delta}{d}.$$

The fraction of atoms in the surface layer of a tin foil of a thickness δ and an area S is

$$\frac{\Delta N}{N} = \frac{\Delta V}{V} = \frac{2S\delta}{Sh} = \frac{2\delta}{h} = \frac{2\delta}{d}.$$

Then the reduction in the melting point temperature of the foil is

$$\frac{\Delta t_h}{t_0} = \frac{\Delta U}{\alpha T_0}\frac{2\delta}{d}.$$

Thus,

$$\frac{\Delta t_d}{\Delta t_h} = 3.$$

This prediction is in a good agreement with experiment. Therefore, $\Delta t_h = \Delta t_d/3 \approx 8.30\,^\circ\text{C}$ and the answer follows:

$$t_h = t_0 - \Delta t_h \approx 223.7\,^\circ\text{C}.$$

68. Figure-Eight of Lord Kelvin

I. Erofeev

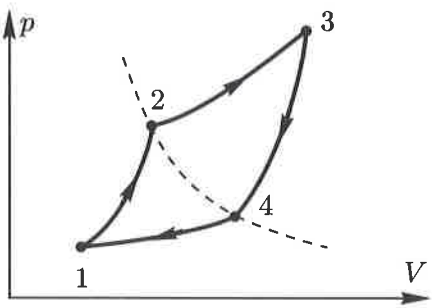

Figure 110

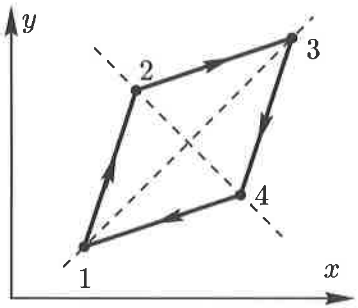

Figure 111

Let us sketch the process in familiar (p, V) coordinates (fig. 110). The plot consists of four polytropic processes: a process ab is represented by the line 12, ef is represented by 23, cb by 34, and ed by 41. It is important that points 2 and 4 belong to the same isotherm ($T_2 = T_4 = T_b$).

1. Since a heat transferred in a polytropic process equals $Q = C\Delta T$, the heat is equal to the area bounded by the curve in CT-coordinates.

According to the plot in the problem statement, a heat is received in processes 12 and 23 and disposed of in processes 34 and 41. Let us determine the received heat $Q+$ and the disposed heat Q_-:

$$Q_+ = Q_{12} + Q_{23} = C_a(T_2 - T_1) + C_d(T_3 - T_2) = 271.5 \text{ J};$$
$$Q_- = Q_{34} + Q_{41} = -C_a(T_3 - T_2) - C_d(T_2 - T_1) = -243 \text{ J}.$$

According to the first law of thermodynamics, the net heat received in a cycle equals to the work done:

$$A = Q_+ + Q_- = 28.5 \text{ J}, \qquad \text{then} \quad \eta = \frac{A}{Q_+} = 0.105.$$

2. The equation of polytropic process and the ideal gas law $pV/T = \text{const}$ allow one to derive the relation between T and V:

$$TV^{n-1} = \text{const}, \qquad \text{or} \quad \left(\frac{T}{T_0}\right)^{\frac{1}{1-n}} = \frac{V}{V_0}. \tag{55}$$

Let us write (55) for the four processes:

$$\left(\frac{T_1}{T_2}\right)^{\frac{1}{1-n_a}} = \frac{V_1}{V_2}, \qquad \left(\frac{T_2}{T_3}\right)^{\frac{1}{1-n_d}} = \frac{V_2}{V_3},$$
$$\left(\frac{T_3}{T_4}\right)^{\frac{1}{1-n_a}} = \frac{V_3}{V_4}, \qquad \left(\frac{T_4}{T_1}\right)^{\frac{1}{1-n_d}} = \frac{V_4}{V_1}.$$

Multiplying the above equalities, one obtains:

$$\left(\frac{T_1T_3}{T_2T_4}\right)^{\frac{n_a-n_d}{(1-n_a)(1-n_d)}} = 1, \qquad \text{so} \quad T_1T_3 = T_2T_4 = T_2^2. \tag{56}$$

Now, let us write (56) using $T_2 = T_1 + T_0$ and $T_3 = T_1 + 3T_0$, where $T_0 = 100$ K:

$$T_1(T_1 + 3T_0) = (T_1 + T_0)^2, \qquad \text{then} \quad T_1 = T_0.$$

Thus, $T_1 = 100$ K, $T_2 = 200$ K, and $T_3 = 400$ K.

3. Let us find a relation between the polytropic index n and a heat capacity C. Since $pV/T = \text{const}$ and $pV^n = \text{const}$,

$$\frac{T}{V}V^n = TV^{n-1} = \text{const}. \tag{57}$$

Differentiating (57), one obtains ΔV and ΔT:

$$V^{n-1}\Delta T + (n-1)TV^{n-2}\Delta V = 0, \qquad \text{hence} \quad \Delta V = \frac{\Delta T}{1-n}\frac{V}{T}.$$

According to the first law of thermodynamics,

$$\Delta Q = C\Delta T = \Delta U + A = C_V\Delta T + p\Delta V = C_V\Delta T + p\frac{\Delta T}{1-n}\frac{V}{T},$$

$$(C - C_V)\Delta T = \frac{\Delta T}{1-n}\frac{pV}{T} = \frac{\nu R\Delta T}{1-n}.$$

Therefore,

$$n = 1 - \frac{\nu R}{C - C_V} = \frac{C - C_p}{C - C_V}.$$

Taking the logarthim of the polytropic equation, one obtains:

$$\ln\frac{p}{p_0} + n\ln\frac{V}{V_0} = \text{const},$$

whence it follows that a polytropic process plotted in coordinates xy, where $x = \ln V/V_0$ and $y = \ln p/p_0$, is represented by a straight line with a slope $-n$. Therefore, the plot of the whole cycle is a parallelogram which points 2 and 4 lie on a straight line $x + y = \text{const}$ (fig. 111). The condition $p_1/p_3 = V_1/V_3$ implies that points 1 and 3 lie on a straight line $x - y = \text{const}$.

Therefore, the parallelogram diagonals are perpendicular, so the cycle is a rhombus. The diagonal 13 of the rhombus is a bisector, therefore, the sum of the slope angles of the polytropic lines equals 90° and the product of the slopes equals 1:

$$n_a n_d = \frac{C_p - C_a}{C_a - C_V}\cdot\frac{C_p - C_d}{C_d - C_V} = 1,$$

hence

$$C_a + C_d = C_p + C_V = \nu(c_p + c_V),$$

where c_p and c_V are the molar heat capacities. Finally,

$$\nu = \frac{C_a + C_d}{c_p + c_V} = 34.4 \text{ mmol.}$$

Note, that $C_a = \nu c_v$, and $C_d = \nu c_p$, therefore the cycle consists of two isobars and two isochors (fig. 112).

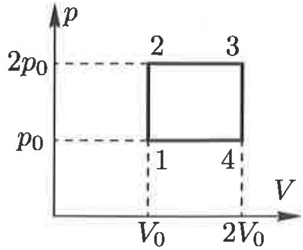

Figure 112

69. From Snell's Archive *V. Slobodyanin*

First, let us analyse where the lens can be located and of which type it can be.

Diverging lens. An image formed by a diverging lens is always virtual and is located on the same side of the lens as the source. The distance between the lens and the image is less than the distance between the lens and the source. Thus, the problem statement is satisfied by a diverging lens located to the right of S_1 (fig. 113).

Converging lens. An image can be both virtual and real. A virtual (and magnified) image is always located further from the lens than a source, so a lens which forms a virtual image is located to the left of S_0 (fig. 114).

If a lens is located between the source S_0 and the image S_1, the image is real. Since a lens has two focal points, the point F can be both to the right (fig. 115) and to the left of the lens (fig. 116).

Thus, four solutions are possible!

Now let us determine the precise coordinates of optical centres of the lenses and do the drawings. Let a distance from the light source to a focal point be d, to the image be L, and to the lens be x. The thin lens formula is

$$\frac{1}{|x|} \pm \frac{1}{|L-x|} = \pm \frac{1}{|d-x|}, \tag{58}$$

where the plus sign on the left-hand side corresponds to real image and the minus sign to virtual one. The plus/minus sign at the lens optical power corresponds to converging/diverging lens, respectively.

Points S_0, F, and S_1 divide the optical axis into four intervals, the moduli in equation (58) have different signs in these intervals.

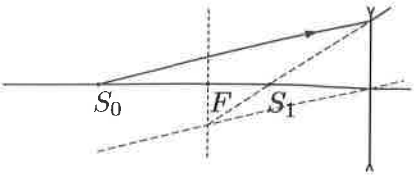

Figure 113

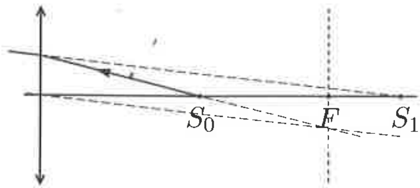

Figure 114

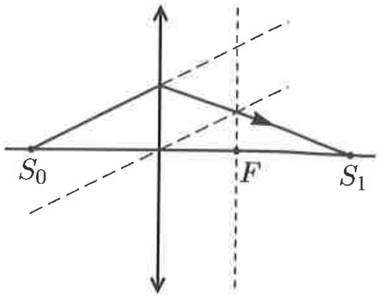

Figure 115

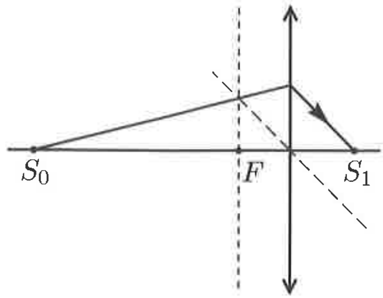

Figure 116

Let us find all solutions of (58). Fix the signs of moduli on the interval S_0F. We have already noticed that only a converging lens can be located here, so

$$\frac{1}{x}+\frac{1}{L-x}=\frac{1}{d-x},$$
$$x^2-2Lx+Ld=0,$$

therefore

$$L-x=\mp\sqrt{L(L-d)},$$

and the solution with minus sign belongs to the considered interval. Similarly, one solves (58) on the remaining intervals. One finds that two more converging lenses are at the points $x=\pm\sqrt{Ld}$ and the converging lens is at $L-x=\sqrt{L(L-d)}$.

All constructions in this problem are reduced to construction of square roots of products of interval lengths. According to a geometry theorem, if projections of catheti of a right triangle on the hypotenuse are equal to a and b, then the triangle height drawn from the right angle equals $\sqrt{ab}$. Using this theorem, one constructs intervals

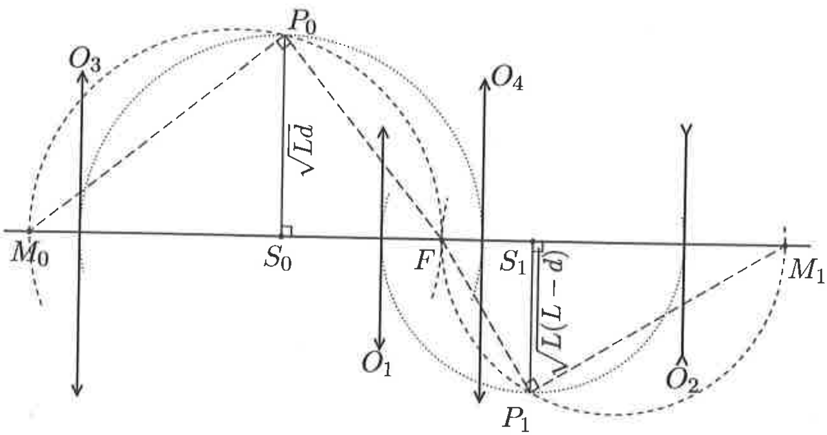

Figure 117

of a length $\sqrt{S_0S_1 \cdot FS_1}$ and $\sqrt{S_0S_1 \cdot S_0F}$ and plot them from the point S_0 to the right and from the point S_1 to both sides, respectively.

Step by step instruction of the construction (fig. 117):

1. Plot the interval S_1M_1 of a length $S_1M_1 = S_0S_1 = L$ on the optical axis, so that the point M_1 is to the right of S_1.

2. Determine the geometric mean of lengths $FS_1 = L - d$ and L. To do this, draw a circle on the diameter FM_1. Find the point of intersection P_1 of the circle and a perpendicular to the optical axis passing through S_1. The length of S_1P_1 equals $\sqrt{L(L-d)}$.

3. Plot intervals of a length S_1P_1 to the left and right of S_1. The left lens O_1 will be converging and the right lens O_2 will be diverging.

4. Plot an interval S_0M_0 of a length $S_0M_0 = S_0S_1 = L$, so that M_0 is to the right of S_0.

5. Similarly to **2**, draw a circle on the diameter M_0F and a perpendicular S_0P_0, which length is equal to $\sqrt{Ld}$.

6. Plot intervals of a length S_0P_0 to the left and right of S_0. Lenses O_3 and O_4 are converging.

70. Electroshock *S. Varlamov*

1. The period of free oscillations of a sphere equals $T = 2\pi\sqrt{l/g} = 2.0$ s.

2. After the voltage has been applied, a sphere accumulates a charge $\pm q$. Since $r \ll d$, a sphere can be considered as a point-like charge. Suppose the spheres approached, so each wire had deflected from the vertical by a small angle α. For a small angle, $\sin\alpha \approx \tan\alpha \approx \alpha$. The Coulomb force of attraction

$$F_q = \frac{kq^2}{(d-2\alpha l)^2},$$

is balanced by a net force of gravity and wire tension $F_p = mg\alpha$:

$$k\frac{q^2}{(d-2\alpha l)^2} = mg\alpha \quad \Rightarrow \quad k\frac{q^2}{mg} = \alpha(d-2\alpha l)^2.$$

The function $f(\alpha) = \alpha(d-2\alpha l)^2$ has a maximum at $\alpha_0 = d/(6l)$. This corresponds to the maximum charge of a sphere:

$$kq_{\max}^2 = mg\alpha_0(d-2\alpha_0 l)^2 = \frac{2mgd^3}{27l} \quad \Rightarrow \quad q_{\max} = \sqrt{\frac{2mgd^3}{27kl}}.$$

A smaller charge is necessary to deflect a wire by an angle $\alpha > \alpha_0$, until the spheres collide. Therefore, if $q_{\max}$ is exceeded by a tiny amount, the spheres will collide quickly enough due to Coulomb attraction, discharge, and move apart again until they accumulate the critical charge $q_{\max}$ required for the collision. The potential difference between the spheres at $q = q_{\max}$ is equal to the minimum voltage of the source required for the spheres to collide:

$$U_{\min} = \varphi_+ - \varphi_- = k\frac{q_{\max}}{r} - \left(-k\frac{q_{\max}}{r}\right) = 2\sqrt{\frac{2kmgd^3}{27lr^2}} = 64.6 \text{ kV}.$$

3. Since $U_{\min} \ll U_0 = 1.0\cdot 10^6$ V, the current can be considered constant and equal to $I = U_0/R$. Then the time required to charge the spheres to $q_{\max}$ equals

$$t_0 = \frac{q_{\max}}{I} = \frac{U_{\min} r}{2k}\frac{R}{U_0} = 18 \text{ s}.$$

71. Oscillations

S. Varlamov, A. Gudenko

Consider a uniformly charged sphere. It is known that electric field within such a sphere vanishes. Consider points A and B which are symmetric with respect to the sphere centre (fig. 118). A field at B is a sum of the fields due to the lower ($\vec{E}_1^B$) and upper ($\vec{E}_2^B$) semi-spheres:

$$\vec{E} = \vec{E}_1^B + \vec{E}_2^B = 0. \tag{59}$$

From symmetry, the field $\vec{E}_1^A$ at the point A in the lower semi-sphere is equal in magnitude and opposite to $\vec{E}_2^B$ produced at the point B in the upper semi-sphere: $\vec{E}_1^A = -\vec{E}_2^B$. Using (59), one obtains $\vec{E}_1^A = \vec{E}_1^B$, i.e. the fields produced by the lower semi-sphere at the points symmetric with respect to its centre are equal.

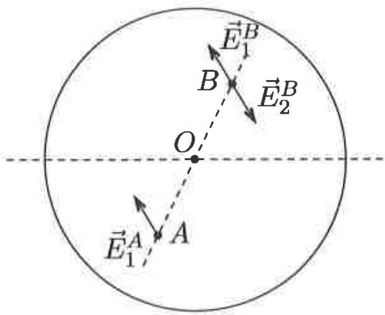

Figure 118

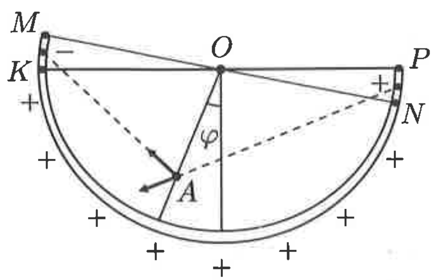

Figure 119

Let us determine a direction of the field component perpendicular to the pendulum thread at A (fig. 119). Consider a plane MN drawn through the sphere centre perpendicular to the thread. If all charges from NP are moved to MK, the field at A will be parallel to the thread. Therefore, the field produced at A by the original semi-sphere $KDPO$ can be represented as a sum of the fields due to a semi-sphere $MDNO$, a positively charged part PON, and a negatively charged part MOK. The net field produced by PON and MOK is almost perpendicular to the thread and directed to a pendulum displacement. In the first approximation the field is proportional to the charges of these parts which are proportional to a deviation angle φ of the pendulum. One can see that under a small deviation from the vertical the restoring force is proportional to the deviation angle.

After the sphere charge has been changed, a position of sphere equilibrium must be higher than the pivot, otherwise the equality of the old and new periods is impossible. When the sphere is higher than the pivot, a restoring force arising under a sphere displacement from equilibrium can be only due to electrostatic forces, so the sphere charge must be positive $q_2 > 0$ (fig. 120).

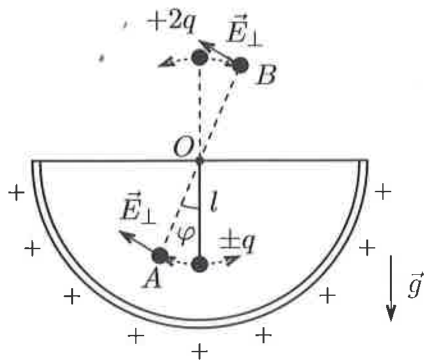

Figure 120

When the pendulum deviates from the lower point of equilibrium by a small angle φ, a restoring force f_1 arises. A contribution to this force due to electrostatic interaction between the sphere and the semi-sphere is proportional to the charge and the deviation angle, therefore:

$$f_1 = (\alpha q_1 - mg)\varphi.$$

The restoring force at the upper point (the sphere charge is q_2) equals

$$f_2 = (-\alpha q_2 + mg)\varphi.$$

The corresponding equations are those of simple harmonic oscillations (here l is the pendulum length):

$$ml\ddot{\varphi} + (mg - \alpha q_1)\,\varphi = 0 \quad \text{for the lower point,}$$
$$ml\ddot{\varphi} + (\alpha q_2 - mg)\,\varphi = 0 \quad \text{for the upper point.}$$

Then the frequencies of oscillation at the lower and upper positions of equilibrium will be equal to

$$\omega_1^2 = \omega_0^2 - \beta q_1,$$
$$\omega_2^2 = -\omega_0^2 + \beta q_2,$$

where $\omega_0^2 = g/l$ is the frequency of oscillation inside a neutral semi-sphere, and $\beta = \alpha/(ml)$ is a constant.

The problem statement can be realised in two cases: $q_1 > 0$ and $q_1 < 0$. In both cases, the pendulum oscillates first near the lower and then near the upper positions of equilibrium.

Case 1. Consider $q_1 > 0$. Then $q_2 = 2q_1$. According to the problem statement, $\omega_1 = \omega_2 = \omega$, hence

$$\omega_0^2 - \beta q_1 = -\omega_0^2 + 2\beta q_1, \qquad \text{and} \quad \beta q_1 = \frac{2}{3}\omega_0^2.$$

Therefore, $\omega^2 = \frac{1}{3}\omega_0^2$ and the period of oscillations is

$$T = \frac{2\pi}{\omega} = T_0\sqrt{3} = 1.73 \text{ s}.$$

Here $T_0 = 2\pi/\omega_0$ is the period of oscillations in the neutral semi-sphere.

Case 2. Consider $q_1 < 0$. Then $q_2 = -2q_1$. Similarly to case 1, one equates frequencies ω_1 and ω_2, which gives $\beta q_1 = -2\omega_0^2$. Therefore, $\omega^2 = 3\omega_0^2$ and the period of oscillations is

$$T = \frac{2\pi}{\omega} = \frac{T_0}{\sqrt{3}} = 0.58 \text{ s}.$$

72. Little Conducting Cube

E. Savinov

Assume a potential at the vertex B to be zero. According to the problem statement, when the current I flows through A and B, the potential of A equals $\varphi_A = Ir$.

From symmetry, the potential at M equals $\varphi_M = \frac{1}{2}\varphi_A = \frac{1}{2}Ir$, and the potential at C equals

$$\varphi_C = \varphi_M - U = \frac{1}{2}Ir - U.$$

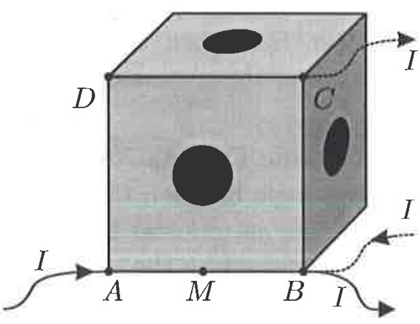

Figure 121

By symmetry, $\varphi_D - \varphi_M = -(\varphi_C - \varphi_M)$, hence $\varphi_D = \frac{1}{2}Ir + U$.
Now disconnect a wire from A and connect it to C. As before, assume the potential at B to be zero. If a potential at C is $\varphi'_C = -Ir$ then,

by symmetry, the same current I will flow through the cube, flowing in at B and flowing out at C. In so doing we have reproduced the initial circuit rotated by $90°$, but potentials at all points of the circuit are now less by Ir. Point A of the new circuit corresponds to the point D of the old circuit, so $\varphi'_A = \varphi_D - Ir = U - \frac{1}{2}Ir$.
Let us employ the principle of superposition and «superimpose» both circuits. The current in the net circuit flows in at A and flows out at C, this is the circuit for a measurement of R_{AC}. Let us determine potentials at A and C in the new circuit.

$$\varphi''_A = \varphi_A + \varphi'_A = U + \frac{1}{2}Ir;$$

$$\varphi''_C = \varphi_C + \varphi'_C = -U - \frac{1}{2}Ir.$$

Here the voltage across AC equals $U''_{AC} = \varphi''_A - \varphi''_C = 2U + Ir$. According to Ohm's law,

$$R_{AC} = \frac{U''_{AC}}{I} = \frac{2U}{I} + r = 36\ k\Omega.$$

Let the cube length be a and the plate thickness be d. Note that a cube resistance is $R \sim L/S \sim a/ad = 1/d$, where $L \sim a$ is a typical size and $S \sim ad$ is a typical transverse cross-section of a cube plate. It follows that $R \sim a/ad = 1/d$, i.e. the cube resistance is independent of the edge length a. Therefore, the resistances R_{AB} and R_{AC} do not vary under a change of the plate side if its thickness d stays constant.

73. Cosmic Object

I. Vorob'ev

Let the angle between the direction of the object motion and the direction to the astronomer be φ. Then, pulses transmitted in a period T_0 will be received by the observer separated by a period

$$T = T_0(1 + \frac{v}{c}\cos\varphi),$$

where v is the object speed.
Indeed, if the first pulse is transmitted from a point A, the next one will be transmitted from a point B in a time T_0 at a distance vT_0.

The paths to the observer are almost parallel and the path difference is $BC = AB\cos\varphi = vT_0\cos\varphi$ (fig. 122) traveled in a time $(vT_0\cos\varphi)/c$, where c is the speed of light. Together with T_0 this gives T. As long as the angle φ can be considered almost constant, the same relation will hold for any intervals Δt_0 and Δt of signal transmission and reception:

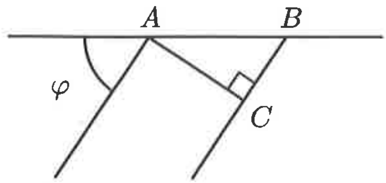

Figure 122

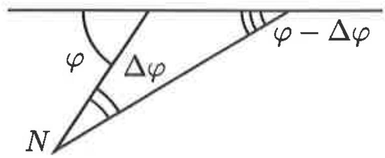

Figure 123

$$\Delta t = \Delta t_0(1 + \frac{v}{c}\cos\varphi).$$

If a direction from N (the observer) to the cosmic object has turned by an angle $\Delta\varphi$, the angle between the direction of motion and the direction to observer became $\varphi - \Delta\varphi$ (fig. 123). Let us calculate an increment of the period of received pulses under this change of the angle:

$$\Delta T = \frac{dT}{d\varphi}(-\Delta\varphi) = T_0\frac{v}{c}\sin\varphi\Delta\varphi$$

Then one has for a time of displacement Δt_0 corresponding to the time of observation Δt: $\Delta\varphi = v\Delta t_0\sin\varphi/r$, where r is the required distance. Eliminating the speed from the equation for ΔT, one obtains:

$$\Delta T = \frac{T_0}{\Delta t_0}\frac{(\Delta\varphi)^2 r}{c} = \frac{T}{\Delta t}\frac{(\Delta\varphi)^2 r}{c},$$

because $T_0/T = \Delta t_0/\Delta t$. Finally,

$$r = \frac{c\Delta T\Delta t}{T(\Delta\varphi)^2}.$$

The solution was obtained in the reference frame of the observer, therefore, the results are valid even if the object speed does not satisfy the condition $v \ll c$.

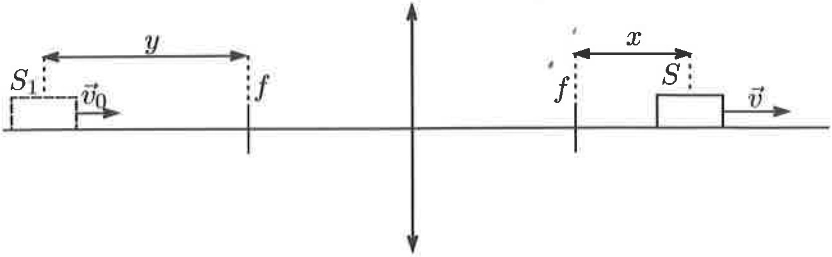

Figure 124

74. «Millicar» *S. Varlamov*

Let the distance from the source S to the front focal point of the lens be x and the distance from its image to the back focal point be y. According to Newton, the thin lens formula can be written as

$$xy = f^2.$$

This equation easily follows from the ordinary thin lens formula $1/a + 1/b = 1/f$ in which substitutions $a = x + f$ and $b = y + f$ are made. For small displacements Δx and Δy one obtains

$$x\Delta y + y\Delta x = 0 \Rightarrow \frac{\Delta x}{\Delta y} = \frac{v}{v_0} = -\frac{x}{y} = -\frac{xy}{y^2} = -\frac{f^2}{y^2}.$$

Therefore, the car speed is $v = -v_0 f^2/y^2$. The minus sign means that if the car is receding from the focal point (x increases, i.e. $v > 0$), its image approaches the focal point (y decreases, $v_0 < 0$). The car acceleration is

$$a = \frac{dv}{dt} = 2\frac{v_0 f^2}{y^3}\frac{dy}{dt} = 2\frac{v_0^2 f^2}{y^3} = 2\frac{v_0^2 x^3}{f^4}.$$

The acceleration cannot exceed (in magnitude) the value $a_{\max} = \mu g$. This gives an inequality

$$\frac{2v_0^2|x|^3}{f^4} \le \mu g, \qquad \text{whence} \quad |x| \le f\sqrt[3]{\frac{\mu g f}{2v_0^2}},$$

i.e. the car cannot move from the focal point at the distance greater than $f\sqrt[3]{\mu g f/(2v_0^2)}$.

If $v_0 > \sqrt{\mu g f/2}$, the distance l from the car to the lens can vary within the limits:

$$f\left(1 - \sqrt[3]{\frac{\mu g f}{2v_0^2}}\right) \le l < f, \qquad \text{or} \quad f < l \le f\left(1 + \sqrt[3]{\frac{\mu g f}{2v_0^2}}\right).$$

The image can be either virtual ($l < f$) or real ($l > f$).
If $v_0 < \sqrt{\mu g f/2}$, one obtains

$$0 < l < f \quad \text{or} \quad f < l \le f\left(1 + \sqrt[3]{\frac{\mu g f}{2v_0^2}}\right).$$

75. Friction Drive *E. Meilikhov, A. Gudenko*

The friction force changes its sign at a distance $r_0 = \omega_0 R_0/\omega$ from the disk centre. The net torque exerted on the disk in a stationary regime vanishes.
For a dry friction, $F_{\mathrm{fr}} = F = \mathrm{const}$:

$$\int_0^{r_0} \frac{rF}{R} dr = \int_{r_0}^{R_0} \frac{rF}{R} dr, \qquad \frac{r_0^2}{2} = \frac{R^2 - r_0^2}{2},$$

$$\text{whence} \qquad r_0 = \frac{R}{\sqrt{2}}, \qquad \omega_\mu = \sqrt{2}\omega_0 \frac{R_0}{R}.$$

For a lubricated friction, $F_{\mathrm{fr}} = \beta v_{\mathrm{rel}}$, and the corresponding torque of the friction force is

$$\int_0^R r\beta(\omega r - \omega_0 R_0) dr = 0, \qquad \text{since} \ \ \omega r_0 = \omega_0 R_0, \ \ \text{then}$$

$$\int_0^R r\beta(\omega r - \omega r_0) dr = \beta\omega \int_0^R r(r - r_0) dr = 0,$$

$$\text{so} \qquad \frac{R^3}{3} = \frac{R^2 r_0}{2}, \qquad \omega_\eta = \frac{\omega_0 R_0}{r_0} = \frac{3}{2}\omega_0 \frac{R_0}{R}.$$

The velocity ratio is

$$k = \frac{\omega_\eta}{\omega_\mu} = \frac{3/2}{\sqrt{2}} \approx 1.06.$$

Thus, the stationary rotation velocity ω_η of lubricated friction exceeds the stationary velocity ω_μ of dry friction by 6 %.

76. Solar Sail *S. Varlamov*

Consider reflection of a photon from the sail mirror. Let the mirror velocity be v and the photon momentum before and after the collision be p_1 and p_2, respectively. Let the sail velocity change by $\Delta v \ll v$ after the collision. According to the conservation of energy and momentum in the mirror-and-photon system,

$$p_1 + mv = -p_2 + m(v + \Delta v), \tag{60}$$

$$p_1 c + \frac{mv^2}{2} = p_2 c + \frac{m(v + \Delta v)^2}{2}. \tag{61}$$

Since $\Delta v \ll v$, these equations can be reduced to

$$p_1 + p_2 = m\Delta v, \tag{62}$$

$$(p_1 - p_2)c = mv\Delta v, \tag{63}$$

which allows one to determine the change of the sail momentum after a single collision:

$$\Delta p = m\Delta v = p_1 + p_2 = 2p_1 \frac{c}{c+v}.$$

Suppose n photons collide with a mirror at rest per a unit of time and the energy of a single photon flying toward the mirror is E_1. Then $WS = nE_1$. A solar energy emitted in a given solid angle is constant. The solid angle subtended by an area, which dimensions are much less than the distance to the Sun, is inversely proportional to the distance squared, therefore, $W \propto 1/R^2$, i.e. $W = W_0 R_0^2 / R^2$.
Since the sail travels at some speed v, the number of collisions per a unit of time reduces to $n_1 = n\Delta t(c - v)/c$. A momentum transferred

per a time Δt equals $n_1 \Delta t \Delta p$. A force of light pressure on the sail equals

$$F_W = \frac{\Delta p n_1 \Delta t}{\Delta t} = \frac{W_0 S}{E_1} \frac{R_0^2}{R^2} \frac{2cp_1}{c+v} \frac{c-v}{c} = \frac{2W_0 S}{c} \frac{c-v}{c+v} \frac{R_0^2}{R^2}.$$

It is directed opposite to the Sun.
A force of gravitational attraction is

$$F_G = G\frac{Mm}{R^2} = \frac{4\pi^2 R_0^3}{T^2} \frac{m}{R^2},$$

where T is the period of Earth rotation around the Sun. For a body velocity to remain constant, the net force exerted on the body must vanish. Note that F_G and F_W are proportional to $1/R^2$, therefore, the motion at a constant speed is possible at an arbitrary distance from the Sun. Let us determine v by equating F_G and F_W:

$$\frac{4\pi^2 R_0^3}{T^2} \frac{m}{R^2} = \frac{2W_0 S}{c} \frac{c-v}{c+v} \frac{R_0^2}{R^2},$$

so

$$v = \frac{W_0 S T^2 - 2\pi^2 R_0 mc}{W_0 S T^2 + 2\pi^2 R_0 mc} c = -1.19 \cdot 10^7 \text{ m/s},$$

i.e. the sail is speeding toward the Sun. In one hour of flight a distance to the Sun will be

$$R_1 = R_0 - |v|t = 1.07 \cdot 10^{11} \text{ m} = 0.72 \text{ au}.$$

77. Circular Process

V. Slobodyanin

Consider one mole of an ideal gas. By definition, its heat capacity is

$$C = \frac{\delta Q}{dT} = \frac{dU + p\,dV}{dT}.$$

For an ideal gas,

$$dU = C_V dT, \qquad R\,dT = p\,dV + V\,dp.$$

Therefore, the heat capacity is

$$C = C_V + R\frac{p\,dV}{p\,dV + V\,dp} = C_V + \frac{R}{1 + \dfrac{V}{p}\dfrac{dp}{dV}}.$$

Tangents to a circle at any diametrically opposite points A and B have the same slope:

$$\left(\frac{dp}{dV}\right)_A = \left(\frac{dp}{dV}\right)_B.$$

Therefore, heat capacities can be equal when dp/dV either vanishes, or diverges. This corresponds to $C = C_p$ or $C = C_V$, respectively. The equality also takes place if

$$\frac{V_A}{p_A} = \frac{V_B}{p_B},$$

i.e. when points A, B, and the circle centre lie on the same straight line passing through the origin. Therefore,

$$\frac{V_A}{p_A} = \frac{V_0}{p_0}.$$

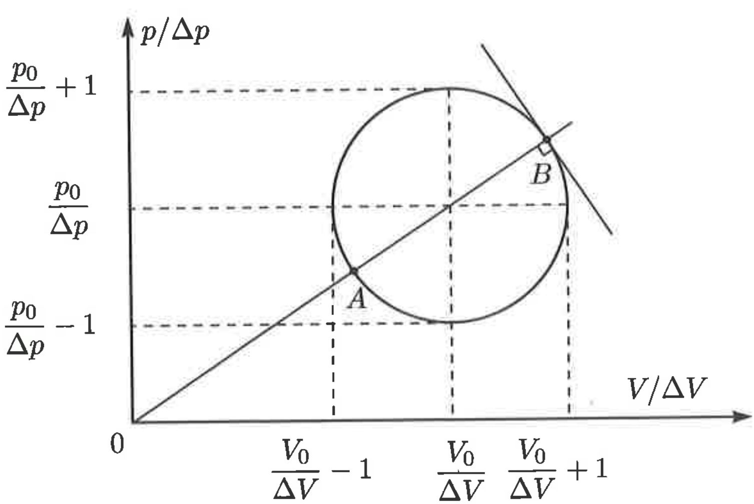

Figure 125

Let us plot the process in dimensionless coordinates (fig. 125). Draw a straight line through A and B. Draw a tangent at B. It is perpendicular to AB. Thus, if the slope of AB equals k, the slope of the tangent is $k' = -1/k$, therefore

$$\frac{dp/\Delta p}{dV/\Delta V} = -\frac{V_0/\Delta V}{p_0/\Delta p}, \qquad \text{and} \quad \frac{dp}{dV} = -\frac{V_0}{p_0}\left(\frac{\Delta p}{\Delta V}\right)^2.$$

The heat capacity C equals

$$C = C_V + \frac{R}{1 - \left(\frac{V_0}{p_0}\right)^2 \left(\frac{\Delta p}{\Delta V}\right)^2}.$$

If $p_0/V_0 = \Delta p/\Delta V$, then $c = \pm\infty$, i.e. the tangent points belong to isotherms.

Let us compare heat capacities at points C and D lying in quadrants 2 and 4, respectively. Since

$$\left(\frac{dp}{dV}\right)_C = \left(\frac{dp}{dV}\right)_D > 0,$$

a heat capacity is greater where the ratio V/p is less:

$$\frac{V_C}{p_C} < \frac{V_D}{p_D}.$$

For the case at hand, $C_C > C_D$.

78. AC Y-circuit *A. Apolonskii*

When two circuit elements connected in-series are connected to an AC power source, the element voltages add up like vectors with the net vector magnitude being equal to a voltage measured by a voltmeter. Let an angle between an abscissa and this vector be 0° for a resistor. Then the angle equals −90° for a capacitor and +90° for an inductor. Note that the net voltage never exceeds the sum of voltages. Suppose that one of the elements 1 or 2 is an inductor and the other one is a capacitor. Then the net voltage across them equals

$$U_{12} = 80\text{ V} - 45\text{ V} = 35\text{ V},$$

which is equal to a source output voltage. Note that $21^2 + 28^2 = 35^2$, hence, element 3 is a resistor.
For this circuit it is impossible to tell which element 1 or 2 is a capacitor and which is an inductor. Indeed, swapping the elements and keeping their impedances intact, one does not change the terminal voltages. Consider all remaining cases. The net voltage across elements 1 and 2 is determined either as a sum of their voltages (if the elements are of the same type), or as a square root of a sum of their voltages squared (if one of them is a resistor and the other one is either capacitor, or an inductor). Then the net voltage, which is equal to the source output voltage, will exceed, at least, 80 V in any case. However, this is greater than the sum of voltages if the source is connected to terminals 1 and 2. This contradiction shows that one of these elements is a capacitor and another one is an inductor.
Addition of impedances is similar to addition of voltages. The ratio of absolute values of impedances of two elements connected in-series equals the ratio of voltages across these elements, therefore, $Z_1 : Z_3 = 4 : 3$ (the power source connected to terminals 1 and 3), $Z_3 : Z_2 = 4 : 3$ (the power source connected to terminals 2 and 3). Let $Z_2 = 9$, $Z_3 = 12$, and $Z_1 = 16$ in arbitrary units. Using the rule of impedance addition, one obtains $Z_{12} = 7$, $Z_{13} = 20$, and $Z_{23} = 15$. A ratio of the currents is inversely proportional to the impedance ratio, hence,

$$I_{12} : I_{13} : I_{23} = \frac{1}{7} : \frac{1}{20} : \frac{1}{15} = 60 : 21 : 28.$$

79. I'm Little Rain Cloud... *A. Gudenko, A. Sheronov*

Consider a mole of gas moving in the atmosphere. According to the conservation of energy, in the adiabatic approximation an external work done over the mole of gas is divided between a change in its internal energy U and potential energy $\mu g z$. Thus,

$$P_1 V_1 - P_2 V_2 = U_2 - U_1 + \mu g (z_2 - z_1).$$

Rearranging the terms, one obtains

$$c_p \Delta T = -\mu g \Delta z.$$

This gives an air temperature as a function of elevation:

$$T = T_0 - \frac{\mu g}{c_p} z = T_0 - \frac{2\mu g}{7R} z.$$

The height of Earth atmosphere can be estimated by equating the temperature to the absolute zero:

$$H \approx \frac{7RT_0}{2\mu g} \approx 30 \text{ km}.$$

The lower edge of the clouds forms at a dew point, i.e at the elevation h_0 at which the partial pressure of water vapour becomes equal to the pressure of saturated vapour $P(z)$, assuming that at the ground level the vapour pressure is $P_0 = \varphi P_H(T_0)$.
According to hydrostatics, the partial pressure of water vapour depends on elevation as

$$\frac{\partial P}{\partial z} = -\rho g.$$

Since $h_0 \ll H$, the temperature and pressure of water vapour can be considered to be small, so its density is almost constant and equal to $\rho \approx P_0 \mu_{H_2O}/RT_0$. Then the pressure varies linearly:

$$\frac{P(z)}{P_H(T_0)} \approx \frac{P_0}{P_H(T_0)} - \frac{\rho g z}{P_H(T_0)} = \varphi\left(1 - \frac{\mu_{H_2O} g z}{RT_0}\right).$$

Using the tabulated dependence of saturated water vapour on temperature and the dependence of temperature on elevation, plot the pressure of saturated vapour versus elevation $P_H(z)/P_H(T_0)$ (fig. 126). Then plot the curve of the partial water vapour pressure $P(z)/P_H(T_0)$. The abscissa of the intersection point of these curves corresponds to the required elevation. From the plots one obtains $h_0 \approx 0.43$ km. Note that the partial pressure of water vapour at this elevation has lessened by 6% compared to the pressure at the ground level and the temperature has lessened by 2%, which justifies our assumptions.

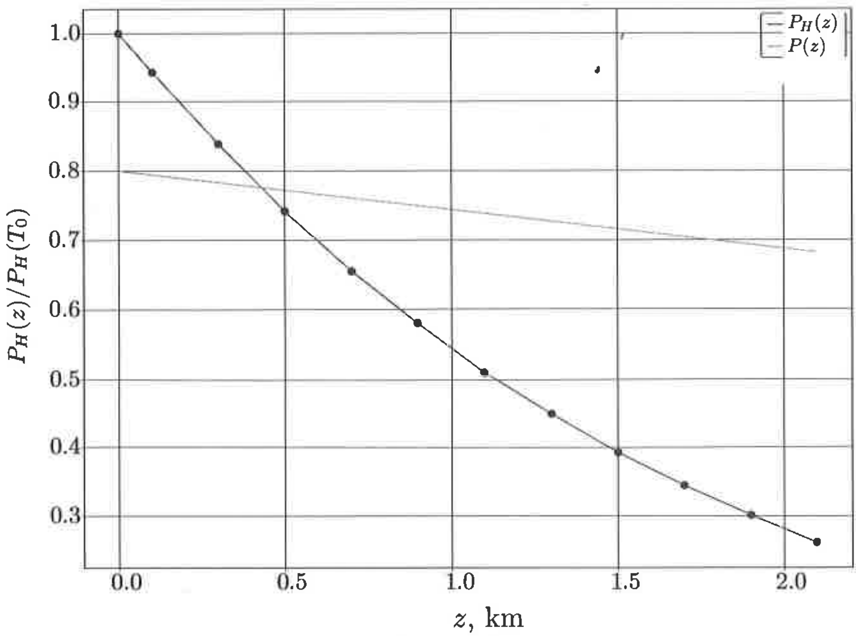

Figure 126

80. Slinky *A. Gudenko*

According to the law of centre-of-mass motion, the time of coil fall equals the time of fall of a point-like mass from a height $H = h + h_c$, where h_c is the coil centre-of-mass height relative to the lowest turn of the suspended coil (fig. 127). Let us determine this height.

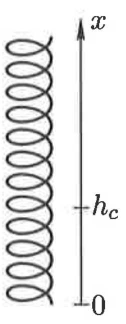

Figure 127

Assume the spring coefficient of a single turn to be k, its mass to be m, and the total number of turns to be N. Since an i-th turn is in equilibrium,

$$k\Delta x_i = (i-1)mg.$$

The coordinate of the i-it turn equals

$$x_i = \sum_1^i \Delta x_i = \frac{mgi(i-1)}{2k}.$$

Since the total number of turns $N \gg 1$, one obtains

$$l = x_N = \frac{mgN(N-1)}{2k} \approx \frac{mgN^2}{2k}.$$

Let us evaluate the center-of mass coordinate:

$$h_c = \frac{1}{mN}\sum_1^N mx_i = \frac{1}{N}\sum_1^N \frac{mgi(i-1)}{2k} =$$

$$= \frac{1}{N}\left(\frac{mgN^3}{6k} + O(N^2)\right) \approx \frac{mgN^2}{6k} = \frac{l}{3}.$$

The time of the coil fall is

$$\tau = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2\left(h+\frac{l}{3}\right)}{g}} \approx 0.55 \text{ s}.$$

Taking into account the length of collapsed coil yields a correction to the center-of-mass coordinate which does not exceed l_0, the corresponding correction for the time of fall equals

$$\frac{\Delta\tau}{\tau} \approx \frac{1}{2}\frac{\Delta H}{H} \approx \frac{l_0}{2\left(h+\frac{l}{3}\right)} \approx 2\% \quad \Rightarrow \quad \Delta\tau = 0.01\,\text{s}.$$

81. Small Bead *V. Plis*

Let us apply Gauss's theorem to determine a dependence $E_r(r)$ near the point of equilibrium: a flux of vector $\vec{E}$ through the surface of a small cylinder (a radius is r and a height is $2x$, where $x \ll r \ll R, H$), which is coaxial with the charged cylinder, vanishes. First, determine $E_x(x)$. This is a field of a uniformly charged ring of a width $2x$ which is located at the far end (relative to the point being considered) of the cylinder. A ring charge is $2xQ/H$. A distance to the point considered is $L \approx \sqrt{R^2 + H^2/4}$. Then the field of the ring on the axis is

$$E_x(x) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{H}2x\cdot\frac{1}{L^2}\cdot\frac{H}{2L} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{L^3}x.$$

Let us calculate the flux Φ_b through a base and the flux Φ_l through the lateral surface of the Gauss cylinder:

$$\Phi_\text{b} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{L^3}x\cdot\pi r^2, \qquad \Phi_\text{l} = 2\pi r\cdot 2x\cdot E_r(r).$$

According to Gauss's theorem, $2\Phi_b + \Phi_l = 0$, hence,

$$2\pi r \cdot 2x \cdot E_r(r) = -2 \cdot \frac{1}{4\pi\varepsilon_0} \frac{Q}{L^3} x \cdot \pi r^2.$$

It follows that the electric field in the ring plane at a distance r from the centre is proportional to r:

$$E_r(r) = -\frac{1}{4\pi\varepsilon_0} \frac{Q}{2L^3} r.$$

Equation of motion of the bead is

$$m\ddot{r} = qE_r(r) = -q\frac{1}{4\pi\varepsilon_0} \frac{Q}{2L^3} r.$$

The frequency of harmonic oscillations equals

$$\omega = \sqrt{\frac{1}{4\pi\varepsilon_0} \frac{q}{m} \frac{Q}{2L^3}} = \frac{1}{L}\sqrt{\frac{1}{4\pi\varepsilon_0} \frac{\gamma Q}{2L}},$$

The period equals

$$T = \frac{2\pi}{\omega} = 2\sqrt{2}\pi L\sqrt{\frac{4\pi\varepsilon_0 L}{\gamma Q}}.$$

82. Lunar Eclipse

V. Slobodyanin

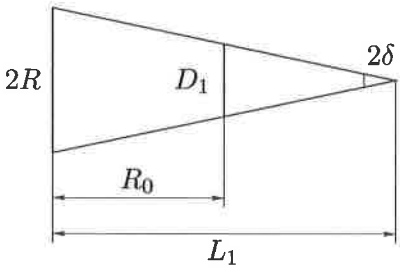

Figure 128

1. If there is no refraction, the rays converge at the same angle at which the Sun is subtended from Earth, i.e. at 2δ, therefore,

$$L_1 = R/\delta \approx 1.4 \cdot 10^6 \text{ km}.$$

Applying the second Newton's law to the Moon orbit of the radius R_0 and an angular frequency ω_0, one finds:

$$\omega_0 = \frac{2\pi}{T_0} = \sqrt{\frac{GM_E}{R_0^3}} = \sqrt{\frac{gR^2}{R_0^3}},$$

hence,

$$R_0 = \sqrt[3]{\frac{gT_0^2R^2}{4\pi^2}} \approx 384 \cdot 10^3 \text{ km}.$$

This gives the Moon diameter

$$D = 2\delta R_0 = 3.45 \cdot 10^3 \text{ km}$$

and the diameter of the dark spot at the Moon distance (fig. 128):

$$D_1 = 2R\left(1 - \frac{R_0}{L_1}\right) \approx 9.3 \cdot 10^3 \text{ km}.$$

Hence, the duration of total lunar eclipse is

$$T = \frac{D_1 - D_2}{\omega_0 R_0} = T_0 \frac{D_1 - D_2}{2\pi R_0} \approx 1.6 \text{ hr}.$$

2. Let us write Snell's law for a ray refracted at the atmosphere boundary:

$$\sin\left(\frac{\pi}{2} - \varphi\right) = n\left(\frac{\pi}{2} - \gamma\right).$$

Taking into account that $n = 1 + \Delta n$, where

$$\Delta n = 2.8 \cdot 10^{-4} \ll 1,$$

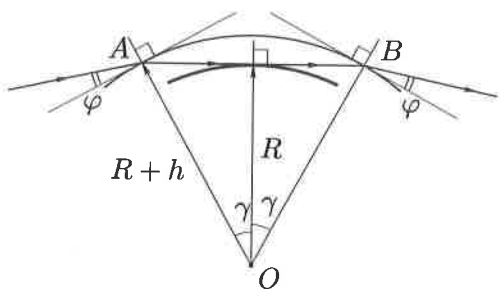

Figure 129

and using the small-angle approximation, one can rewrite the above formula as

$$1 - \frac{\varphi^2}{2} = (1 + \Delta n)\left(1 - \frac{\gamma^2}{2}\right).$$

Expanding the right-hand side and neglecting the third-order term, one obtains $(\gamma^2 - \varphi^2)/2 = \Delta n$, which can be approximated by $\gamma(\gamma - \varphi) = \Delta n$. Therefore, the ray refraction angle is

$$\Delta\varphi = \gamma - \varphi = \frac{\Delta n}{\gamma}.$$

According to the diagram (fig. 129), $\cos\gamma = R/(R+h)$, hence,

$$\gamma \approx \sin\gamma = \sqrt{1 - \left(\frac{R}{R+h}\right)^2} \approx \frac{\sqrt{2Rh}}{R+h} \approx \sqrt{\frac{2h}{R}}.$$

The ray undergoes two identical refractions: when entering the atmosphere and when leaving it, so the net angle is $2\Delta\varphi$. Thus, in the case of refraction, the angle ψ at which solar rays converge turns out to be $\psi = 2\delta + 4\varphi$. Finally,

$$L_2 = \frac{2R}{2\delta + 4\Delta\varphi} = \frac{R}{\delta + \Delta n\sqrt{2R/h}} \approx 408 \cdot 10^3 \text{ km}.$$

This allows one to determine the diameter of the dark spot on the Moon,

$$D_2 = 2R\left(1 - \frac{R_0}{L_2}\right) \approx 753 \text{ km},$$

and the required area ratio, $\varepsilon = (D_2/D)^2 \approx 4.8\,\%$.

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