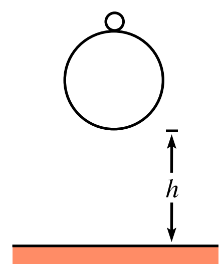
Week 1 (9/16/02)

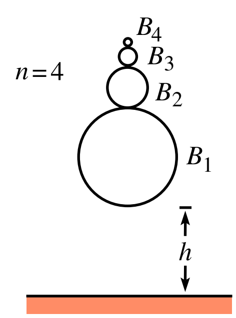
Basketball and tennis ball

(a) A tennis ball with (small) mass m_2 sits on top of a basketball with (large) mass m_1. The bottom of the basketball is a height h above the ground, and the bottom of the tennis ball is a height $h + d$ above the ground. The balls are dropped. To what height does the tennis ball bounce?



Note: Work in the approximation where m_1 is much larger than m_2, and assume that the balls bounce elastically.

(b) Now consider n balls, $B_1, \ldots, B_n$, having masses $m_1, m_2, \ldots, m_n$ (with $m_1 \gg m_2 \gg \cdots \gg m_n$), sitting in a vertical stack. The bottom of B_1 is a height h above the ground, and the bottom of B_n is a height $h + \ell$ above the ground. The balls are dropped. In terms of n, to what height does the top ball bounce?



Note: Work in the approximation where m_1 is much larger than m_2, which is much larger than m_3, etc., and assume that the balls bounce elastically.

If $h = 1$ meter, what is the minimum number of balls needed for the top one to bounce to a height of at least 1 kilometer? To reach escape velocity? Assume that the balls still bounce elastically (which is a bit absurd here). Ignore wind resistance, etc., and assume that ℓ is negligible.

Week 2 (9/23/02)

Green-eyed dragons

You visit a remote desert island inhabited by one hundred very friendly dragons, all of whom have green eyes. They haven't seen a human for many centuries and are very excited about your visit. They show you around their island and tell you all about their dragon way of life (dragons can talk, of course).

They seem to be quite normal, as far as dragons go, but then you find out something rather odd. They have a rule on the island which states that if a dragon ever finds out that he/she has green eyes, then at precisely midnight on the day of this discovery, he/she must relinquish all dragon powers and transform into a long-tailed sparrow. However, there are no mirrors on the island, and they never talk about eye color, so the dragons have been living in blissful ignorance throughout the ages.

Upon your departure, all the dragons get together to see you off, and in a tearful farewell you thank them for being such hospitable dragons. Then you decide to tell them something that they all already know (for each can see the colors of the eyes of the other dragons). You tell them all that at least one of them has green eyes. Then you leave, not thinking of the consequences (if any). Assuming that the dragons are (of course) infallibly logical, what happens?

If something interesting does happen, what exactly is the new information that you gave the dragons?

Week 3 (9/30/02)

Balancing a pencil

Consider a pencil that stands upright on its tip and then falls over. Let's idealize the pencil as a mass m sitting at the end of a massless rod of length ℓ.[1]

(a) Assume that the pencil makes an initial (small) angle θ_0 with the vertical, and that its initial angular speed is ω_0. The angle will eventually become large, but while it is small (so that $\sin\theta \approx \theta$), what is θ as a function of time?

(b) You might think that it would be possible (theoretically, at least) to make the pencil balance for an arbitrarily long time, by making the initial θ_0 and ω_0 sufficiently small.

It turns out that due to Heisenberg's uncertainty principle (which puts a constraint on how well we can know the position and momentum of a particle), it is impossible to balance the pencil for more than a certain amount of time. The point is that you can't be sure that the pencil is initially both at the top *and* at rest. The goal of this problem is to be quantitative about this. The time limit is sure to surprise you.

Without getting into quantum mechanics, let's just say that the uncertainty principle says (up to factors of order 1) that $\Delta x \Delta p \geq \hbar$ (where $\hbar = 1.06 \cdot 10^{-34}\,\text{Js}$ is Planck's constant). The precise implications of this are somewhat vague, but we'll just take it to mean that the initial conditions satisfy $(\ell\theta_0)(m\ell\omega_0) \geq \hbar$.

With this condition, find the maximum time it can take your solution in part (a) to become of order 1. In other words, determine (roughly) the maximum time the pencil can balance. (Assume $m = 0.01\,\text{kg}$, and $\ell = 0.1\,\text{m}$.)

[1] It actually involves only a trivial modification to do the problem correctly using the moment of inertia and the torque. But the point-mass version will be quite sufficient for the present purposes.

Week 4 (10/7/02)

Passing the spaghetti

At a dinner party, there are N people seated around a table. A plate of spaghetti starts at the head of the table. The person sitting there takes some spaghetti and then passes the (very large) plate at random to his/her right or left. Henceforth each person receiving the plate takes some spaghetti and then passes the plate at random to his/her right or left. (Diners who have already received the plate can simply pass it on, without taking any more.) When all the diners have finally received their spaghetti, the plate stops being passed, and the eating begins.

(a) What are the chances of being the last to be served, as a function of position (relative to the head) at the table of N people?

(b) If this procedure is repeated over the course of many dinners, what is the average number of times the plate is passed?

Week 5 (10/14/02)

The raindrop

Assume that a cloud consists of tiny water droplets suspended (uniformly distributed, and at rest) in air, and consider a raindrop falling through them. What is the acceleration of the raindrop? (Assume that when the raindrop hits a water droplet, the droplet's water gets added to the raindrop. Also, assume that the raindrop is spherical at all times.)

Week 6 (10/21/02)

Flipping a coin

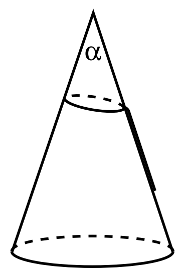
(a) Consider the following game. You flip a coin until you get a tails. The number of dollars you win equals the number of coins you end up flipping. (So if you immediately get a tails, you win one dollar; if you get one heads before a tails, you win two dollars, etc.) What is the expectation value of your winnings?

(b) Play the same game, except now let the number of dollars you win be equal to 2^{n-1}, where n is the number of coins you end up flipping. How much do you expect to win now? Does your answer make sense?

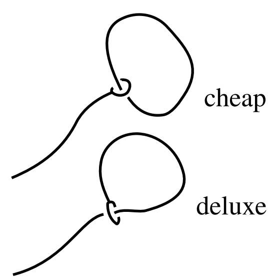
Week 7 (10/28/02)

Mountain climber

A mountain climber wishes to climb up a frictionless conical mountain. He wants to do this by throwing a lasso (a rope with a loop) over the top and climbing up along the rope. Assume that the mountain climber is of negligible height, so that the rope lies along the mountain, as shown.



At the bottom of the mountain are two stores. One sells "cheap" lassos (made of a segment of rope tied to loop of rope of *fixed* length). The other sells "deluxe" lassos (made of one piece of rope with a loop of *variable* length; the loop's length may change without any friction of the rope with itself).



When viewed from the side, this conical mountain has an angle α at its peak. For what angles α can the climber climb up along the mountain if he uses:

(a) a "cheap" lasso?

(b) a "deluxe" lasso?

Week 8 (11/4/02)

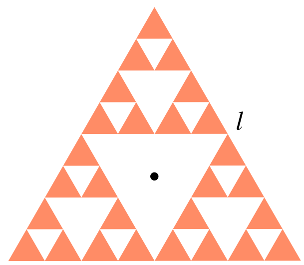
Sub-rectangles

A rectangle is divided into many smaller rectangles, each of which has the property that at least one pair of sides has integer length. Prove that the original rectangle also has this property.

Week 9 (11/11/02)

Fractal moment

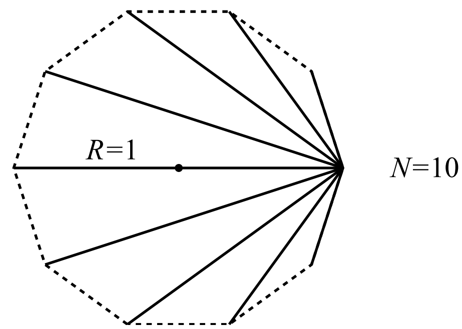
Take an equilateral triangle of side ℓ, and remove the "middle" triangle (1/4 of the area). Then remove the "middle" triangle from each of the remaining three triangles (as shown), and so on, forever. Let the final object have mass m. Find the moment of inertia of this object, around an axis through its center and perpendicular to its plane.



Week 10 (11/18/02)

Product of lengths

Inscribe a regular N-gon in a circle of radius 1. Draw the $N-1$ segments connecting a given vertex to the $N-1$ other vertices. Show that the product of the lengths of these $N-1$ segments equals N. The figure below shows the case where $N = 10$; the product of the lengths of the 9 segments is 10.



Week 11 (11/25/02)

Break or not break?

Two spaceships float in space and are at rest relative to each other. They are connected by a string. The string is strong, but it cannot withstand an arbitrary amount of stretching. At a given instant, the spaceships simultaneously (with respect to their initial inertial frame) start accelerating (along the direction of the line between them) with the same acceleration. (Assume they bought identical engines from the same store, and they put them on the same setting.)

Will the string eventually break?

Week 12 (12/2/02)

Decreasing numbers

Pick a random number (evenly distributed) between 0 and 1. Continue picking random numbers as long as they keep decreasing; stop picking when you obtain a number that is greater than the previous one you picked. What is the expected number of numbers you pick?

Week 13 (12/9/02)

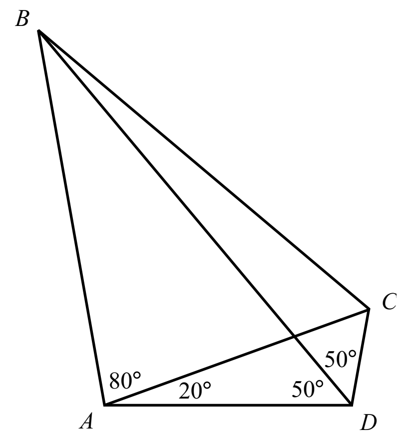
Unchanged velocity

A ball rolls without slipping on a table. It rolls onto a piece of paper. You slide the paper around in an arbitrary (horizontal) manner. (It's fine if there are abrupt, jerky motions, so that the ball slips with respect to the paper.) After you allow the ball to come off the paper, it will eventually resume rolling without slipping on the table. Show that the final velocity equals the initial velocity.

Week 14 (12/16/02)

Find the angles

Quadrilateral $ABCD$ has angles $\angle BDA = \angle CDB = 50°$, $\angle DAC = 20°$, and $\angle CAB = 80°$. Find angles $\angle BCA$ and $\angle DBC$.



Week 15 (12/23/02)

Maximal gravity

Given a point P in space, and given a piece of malleable material of constant density, how should you shape and place the material in order to create the largest possible gravitational field at P?

Week 16 (12/30/02)

Letters in envelopes

You are given N addressed letters and N addressed envelopes. If you randomly put one letter in each envelope, what is the probability that no letter ends up in the correct envelope?

Week 17 (1/6/03)

Icosahedron of resistors

Each edge of an icosahedron is a 1Ω resistor. Find the effective resistance between two adjacent vertices. (An icosahedron consists of 20 equilateral triangles. It has 12 vertices and 30 edges, with 5 edges meeting at each vertex.)

Week 18 (1/13/03)

Distribution of primes

Let $P(N)$ be the probability that a randomly chosen integer, N, is prime. Show that

$$P(N) = \frac{1}{\ln N}\,.$$

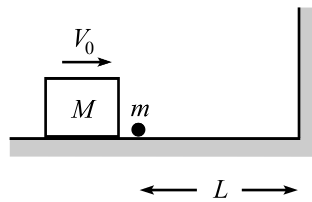
Note: Assume that N is very large, and ignore terms in your answer that are of subleading order in N. Also, make the assumption that the probability that N is divisible by a prime p is exactly $1/p$ (which is essentially true, for a large enough sample size of numbers).

Correction: (Thanks to Bob Silverman for pointing this out.) The assumption, "the probability that N is divisible by a prime p is exactly $1/p$," it is actually *not* valid. More precisely, when dealing with a sufficiently large number of primes, the probability that a prime p divides N is *not* independent of the probability that a prime q divides N. (This is related to Mertens' theorem, which you can look up.) The solution I posted for this problem is therefore incorrect, even though it does end up giving the correct result of $P(N) = 1/\ln N$. (A correction factor ends up canceling out.) However, you might still find it interesting to solve the problem using the given (invalid) assumption.

Week 19 (1/20/03)

Block and bouncing ball

A block with large mass M slides with speed V_0 on a frictionless table towards a wall. It collides elastically with a ball with small mass m, which is initially at rest at a distance L from the wall. The ball slides towards the wall, bounces elastically, and then proceeds to bounce back and forth between the block and the wall.



(a) How close does the block come to the wall?

(b) How many times does the ball bounce off the block, by the time the block makes its closest approach to the wall?

Assume that $M \gg m$, and give your answers to leading order in m/M.

Week 20 (1/27/03)

Collinear points

You are given a finite number of points in space with the property that any line that contains two of these points contains three of them. Prove that all the points must lie on a straight line.

Week 21 (2/3/03)

Ball on turntable

A ball with uniform mass density rolls without slipping on a turntable. Show that the ball moves in a circle (as viewed from the inertial lab frame), with a frequency equal to 2/7 times the frequency of the turntable.

Week 22 (2/10/03)

Trading envelopes

(a) I give you an envelope containing a certain amount of money, and you open it. I then put into a second envelope either twice this amount or half this amount, with a fifty-fifty chance of each. You are given the opportunity to trade envelopes. Should you?

(b) I put two sealed envelopes on a table. One contains twice as much money as the other. You pick an envelope and open it. You are then given the opportunity to trade envelopes. Should you?

(c) If your answers to (a) and (b) are the same, explain why. If they are different, explain why.

Week 23 (2/17/03)

$V(x)$ versus a hill

A bead, under the influence of gravity, slides along a frictionless wire whose height is given by the function $V(x)$. Find an expression for the bead's horizontal acceleration. (It can depend on whatever quantities you need it to depend on.)

You should find that the result is *not* the same as the $\ddot{x}$ for a particle moving in one dimension in the potential $mgV(x)$, in which case $\ddot{x} = -gV'$. But if you grab hold of the wire, is there any way you can move it so that the bead's $\ddot{x}$ is equal to the $\ddot{x} = -gV'$ result due to the one-dimensional potential, $mgV(x)$?

Week 24 (2/24/03)

Verifying weights

(a) You have a balance scale and wish to verify the weights of items that come in weights from 1 to 121 (all integral). What is the minimum number of fixed weights you need? What are the weights?

(b) Using n wisely-chosen fixed weights, what is the largest integer W, for which you can verify all the integral weights less than or equal to W? What fixed weights should you choose?

Week 25 (3/3/03)

Maximum deflection angle

A mass M collides elastically with a stationary mass m. If $M < m$, then it is possible for M to bounce directly backwards. However, if $M > m$, then there is a maximum angle of deflection of M. Find this angle.

Week 26 (3/10/03)

Drunken walk

A drunk performs a random walk along a street. At one end of the street is a river, and at the other end is a police station. If he gets to either of these ends, he remains there. He starts n steps from the river, and there are N total steps between the river and the police station.

(a) What is the probability that he ends up at the river? At the police station?

(b) What is the expected total number of steps he takes?

Week 27 (3/17/03)

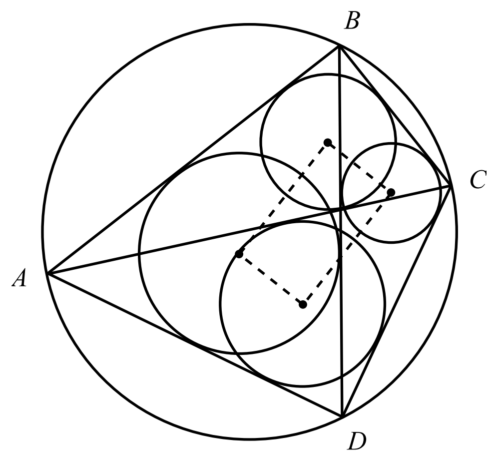
Relativistic cookies

Cookie dough (chocolate chip, of course) lies on a conveyor belt which moves along at speed v. A circular stamp stamps out cookies as the dough rushed by beneath it. When you buy these cookies in a store, what shape are they? That is, are they squashed in the direction of the belt, stretched in that direction, or circular?

Week 28 (3/24/03)

Rectangle in a circle

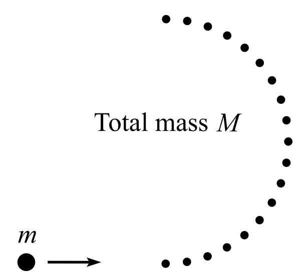
Given a cyclic quadrilateral $ABCD$, draw the diagonals AC and BD. Prove that the centers of the inscribed circles of triangles ABC, BCD, CDA, and DAB are the vertices of a rectangle.



Week 29 (3/31/03)

Balls in a semicircle

N identical balls lie equally spaced in a semicircle on a frictionless horizontal table, as shown. The total mass of these balls is M. Another ball of mass m approaches the semicircle from the left, with the proper initial conditions so that it bounces (elastically) off all N balls and finally leaves the semicircle, heading directly to the left.



(a) In the limit $N \to \infty$ (so the mass of each ball in the semicircle, M/N, goes to zero), find the minimum value of M/m that allows the incoming ball to come out heading directly to the left.

(b) In the minimum M/m case found in part (a), show that the ratio of m's final speed to initial speed equals $e^{-\pi}$.

Week 30 (4/7/03)

Difference of Powers

Find the smallest positive value of $33^m - 7^n$, where m and n are positive integers.

Week 31 (4/14/03)

Simultaneous claps

With respect to the ground, A moves to the right at speed $c/\sqrt{3}$, and B moves to the left, also at speed $c/\sqrt{3}$. At the instant they are a distance L apart (as measured in the ground frame), A claps his hands. B then claps his hands simultaneously (as measured by B) with A's clap. A then claps his hands simultaneously (as measured by A) with B's clap. B then claps his hands simultaneously (as measured by B) with A's second clap, and so on. As measured in the ground frame, how far apart are A and B when A makes his nth clap? What is the answer if $c/\sqrt{3}$ is replaced by a general speed v?

Week 32 (4/21/03)

The game show

A game show host offers you the choice of three doors. Behind one of these is the grand prize, and behind the other two are goats. The host announces that after you select a door (without opening it), he will open one of the other two doors and reveal a goat. You select a door. The host then opens one of the other doors and reveals a goat, and offers you the chance to switch your choice to the remaining door. Should you switch?

Week 33 (4/28/03)

Ball rolling in a cone

A ball (with moment of inertia $I = (2/5)MR^2$) rolls without slipping on the inside surface of a fixed cone, whose tip points downward. The half-angle at the vertex of the cone is θ. Initial conditions have been set up so that the ball travels around the cone in a horizontal circle of radius ℓ, with the contact points (the points on the ball that touch the cone) tracing out a circle (not necessarily a great circle) on the ball.

What should the radius of the circle of these contact points be, if you want the sphere to travel around the cone as fast as possible? (You may work in the approximation where R is much less than ℓ. Also, assume that the coefficient of friction between the ball and the cone is arbitrarily large.)

Week 34 (5/5/03)

Counterfeit coin

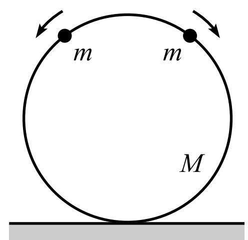
(a) You are given twelve coins, eleven of which have the same weight, and one of which has a weight different from the others (either heavier or lighter, you do not know). You have a balance scale. What is the minimum number of weighings it takes to determine which coin has the different weight, and also whether it is heavier or lighter than the rest?

(b) You are given N coins, $N-1$ of which have the same weight, and one of which has a weight different from the others (either heavier or lighter, you do not know). You are allowed W weighings on a balance scale. What is the maximum value for N, as a function of W, for which you can determine which coin has the different weight, and also whether it is heavy or light?

Week 35 (5/12/03)

Rising hoop

Two beads of mass m are positioned at the top of a frictionless hoop of mass M and radius R, which stands vertically on the ground. The beads are given tiny kicks, and they slide down the hoop, one to the right and one to the left, as shown. What is the smallest value of m/M for which the hoop will rise up off the ground at some time during the motion?



Week 36 (5/19/03)

Monochromatic Triangle

(a) Seventeen points, no three of which are collinear, are connected by all the possible lines between them (136, in fact). Each line is colored either red, green, or blue. Prove that within the resulting network of lines, there is at least one triangle all of whose sides are the same color.

(b) Let $\lceil a \rceil$ denote the smallest integer greater than or equal to a. Let $\lceil n!e \rceil$ points, no three of which are collinear, be connected by all the possible lines between them. Each line is colored one of n colors. Prove that within the resulting network of lines, there is at least one triangle all of whose sides are the same color.

Week 37 (5/26/03)

Bouncing down a plane

Consider a ball (with moment of inertia $I = (2/5)MR^2$) which bounces elastically off a surface. Assume that the ball's speed in the direction perpendicular to the surface is the same before and after a bounce. Also, assume that the ball is made of a type of rubber which allows it to not slip on the surface (which has friction) during the bounce. (This implies that the angular and linear motions may affect each other.)

The ball is projected from the surface of a plane which is inclined at angle θ. The initial velocity of the ball is perpendicular to the plane and has magnitude V. The initial angular velocity is zero. Find the component of the ball's velocity along the plane, immediately after the nth bounce.

Week 38 (6/2/03)

Sum over 1

(a) You are given a random number (evenly distributed) between 0 and 1. To this, you add a second such random number. Keep adding numbers until the sum exceeds 1, and then stop. How many numbers, on average, will you need?

(b) When the sum finally exceeds 1 and the game stops, what is the average result for the sum?

Week 39 (6/9/03)

Viewing the spokes

A wheel with spokes rolls on the ground. A stationary camera takes a picture of the wheel. Due to the nonzero exposure time of the camera, the spokes will generally appear blurred. At what location(s) in the picture does (do) the spoke(s) *not* appear blurred?

Week 40 (6/16/03)

Staying ahead

In a two-way election, candidate A receives a votes and candidate B receives b votes, with $a > b$. If the ballots are removed one at a time from the ballot box, and a running total of the score is kept, what is the probability that at all times A's sub-total is greater than or equal to B's sub-total?

Week 41 (6/23/03)

Speedy travel

A straight tube is drilled between two points (not necessarily diametrically opposite) on the earth. An object is dropped into the tube. How much time does it take to reach the other end? Ignore friction, and assume (erroneously) that the density of the earth is constant.

Week 42 (6/30/03)

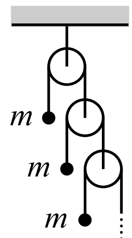
How much change?

You are out shopping one day with \$$N$, and you find an item whose price has a random value between \$0 and \$$N$. You buy as many of these items as you can with your \$$N$. What is the expected value of the money you have left over? (You may assume that \$$N$ is large compared to a penny, so that the distribution of prices is essentially continuous.)

Week 43 (7/7/03)

Infinite Atwood's machine

Consider the infinite Atwood's machine shown below. A string passes over each pulley, with one end attached to a mass and the other end attached to another pulley. All the masses are equal to m, and all the pulleys and strings are massless. The masses are held fixed and then simultaneously released. What is the acceleration of the top mass?



(We'll define this infinite system as follows. Consider it to be made of N pulleys, with a non-zero mass replacing what would have been the $(N+1)$st pulley. Then take the limit as $N \to \infty$.)

Week 44 (7/14/03)

Relatively prime numbers

What is the probability that two randomly chosen positive integers are relatively prime?

Week 45 (7/21/03)

Sliding along a plane

A block is placed on a plane inclined at angle θ. The coefficient of friction between the block and the plane is $\mu = \tan\theta$. The block is given a kick so that it initially moves with speed V horizontally along the plane (that is, in the direction perpendicular to the direction pointing straight down the plane). What is the speed of the block after a very long time?

Week 46 (7/28/03)

The birthday problem

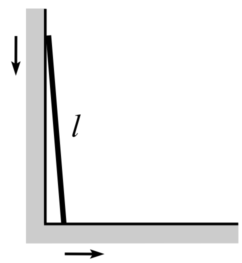
(a) How many people must be in a room in order for the probability to be greater than 1/2 that at least two of them have the same birthday? (By "same birthday", we mean the same day of the year; the year may differ.) Ignore leap years.

(b) Assume there is some large number, N, of days in a year. How many people are now necessary for the odds to favor a common birthday? Equivalently, assuming a normal 365-day year, how many people are required for the probability to be greater than 1/2 that at least two of them were born in the same hour on the same date? Or in the same minute of the same hour on the same date? Neglect terms in your answer that are of subleading order in N.

Week 47 (8/4/03)

Sliding ladder

A ladder of length ℓ and uniform mass density stands on a frictionless floor and leans against a frictionless wall. It is initially held motionless, with its bottom end an infinitesimal distance from the wall. It is then released, whereupon the bottom end slides away from the wall, and the top end slides down the wall. When it loses contact with the wall, what is the horizontal component of the velocity of the center of mass?



Week 48 (8/11/03)

The hotel problem

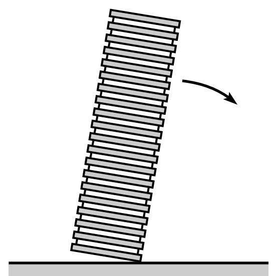
You are driving down a one-way road and pass a strip of a large number, N, of hotels. These all have different rates, arranged randomly. You want to maximize your chance of choosing the cheapest hotel, but you can't return to one you've passed up. Assume that your only goal is to obtain the cheapest one (the second cheapest is of no more value to you than the most expensive). If your strategy is to proceed past a certain fraction, x, of them and then pick the next one that is cheaper than all the ones you've seen so far, what should x be? What, then, is the probability of success? Assume that N is very large, and ignore terms in your answer that are of subleading order in N.

Week 49 (8/18/03)

Falling chimney

A chimney initially stands upright. It is given a tiny kick, and it topples over. At what point along its length is it most likely to break?

In doing this problem, work with the following two-dimensional simplified model of a chimney. Assume that the chimney consists of boards stacked on top of each other, and that each board is attached to the two adjacent ones with tiny rods at each end, as shown below. The goal is to determine which rod in the chimney has the maximum tension. (Work in the approximation where the width of the chimney is very small compared to the height.)



Week 50 (8/25/03)

Equal segments

Given a line segment, a line parallel to it, and a straightedge, divide the segment into N equal segments. (With a straightedge, you are allowed only to draw straight lines. You are not allowed to mark off distances on the straightedge.)

Week 51 (9/1/03)

Accelerating spaceship

A spaceship is initially at rest with respect to frame S. At a given instant, it starts to accelerate with constant proper acceleration, a. (The *proper acceleration* is the acceleration with respect to the instantaneous inertial frame the spaceship was just in. Equivalently, if an astronaut has mass m and is standing on a scale, then the scale reads a force of $F = ma$.) What is the relative speed of the spaceship and frame S when the spaceship's clock reads time t?

Week 52 (9/8/03)

Construct the center

Construct the center of a given circle, using only a compass. With a compass, you are allowed to mark points with the needle, and to draw arcs of circles (which may intersect at new points).

Week 53 (9/15/03)

Circles on the ice

A puck slides with speed v on frictionless ice. The surface is "level", in the sense that it is perpendicular to the direction of a hanging plumb bob at all points. Show that the puck moves in a circle, as seen in the earth's rotating frame. What is the radius of the circle? What is the frequency of the motion? Assume that the radius of the circle is small compared to the radius of the earth.

Week 54 (9/22/03)

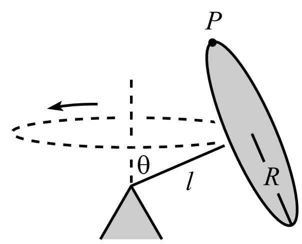
Rolling the die

Two players alternately roll an N-sided die. The player who fails to improve upon the previous roll loses. What is the probability that the first player wins?

Week 55 (9/29/03)

Fixed highest point

Consider a top made of a uniform disk of radius R, connected to the origin by a massless stick (which is perpendicular to the disk) of length ℓ, as shown below. Paint a dot on the top at its highest point, and label this as point P. You wish to set up uniform circular precession, with the stick making a constant angle θ with the vertical, and with P always being the highest point on the top. What relation between R and ℓ must be satisfied for this motion to be possible? What is the frequency of precession, Ω?



Week 56 (10/6/03)

Stirling's formula

Using $N! = \int_0^\infty x^N e^{-x} dx$ (which you can prove by induction), derive Stirling's formula,

$$N! \approx N^N e^{-N} \sqrt{2\pi N}.$$

Also, find the order-$1/N$ correction to this (and the order-$1/N^2$ correction, if you really want to).

Week 57 (10/13/03)

Throwing a beach ball

A beach ball is thrown upward with initial speed v_0. Assume that the drag force from the air is $F = -m\alpha v$. What is the speed of the ball, v_f, when it hits the ground? (An implicit equation is sufficient.) Does the ball spend more time or less time in the air than it would if it were thrown in vacuum?

Week 58 (10/20/03)

Coins and Gaussians

Flip a coin $2N$ times, where N is large. Let $P(x)$ be the probability of obtaining exactly $N + x$ heads. Show that

$$P(x) \approx \frac{e^{-x^2/N}}{\sqrt{\pi N}}\,.$$

Week 59 (10/27/03)

Getting way ahead

A rocket with proper length L accelerates from rest, with proper acceleration g (where $gL \ll c^2$). Clocks are located at the front and back of the rocket. If we look at this setup in the frame of the rocket, then the general-relativistic time-dilation effect tells us that the times on the two clocks are related by $t_f = (1 + gL/c^2)t_b$. Therefore, if we look at things in the ground frame, then the times on the two clocks are related by

$$t_f = t_b\left(1 + \frac{gL}{c^2}\right) - \frac{Lv}{c^2}\,,$$

where the last term is the standard special-relativistic lack-of-simultaneity result. Derive the above relation by working entirely in the ground frame.

Note: You may find this relation surprising, because it implies that the front clock will eventually be an arbitrarily large time ahead of the back clock, in the ground frame. (The subtractive Lv/c^2 term is bounded by L/c and will therefore eventually become negligible compared to the additive, and unbounded, $(gL/c^2)t_b$ term.) But both clocks are doing basically the same thing relative to the ground frame, so how can they eventually differ by so much? Your job is to find out.

Week 60 (11/3/03)

Cereal box prizes

In each box of a certain cereal there is a prize, which is one of N colors. Assuming that the prize in each box has equal odds of being any color (even after you've bought many boxes), how many boxes do you need to buy, on average, before you collect all the different colors?

Let $P(n)$ be the probability that you collect the final color in the nth box. For what value of n, in terms of N, is $P(n)$ maximum? Assume that N is large, and ignore terms in your answer that are of subleading order in N.

Week 61 (11/10/03)

Falling rope

(a) A rope of length L lies in a straight line on a frictionless table, except for a very small piece at one end which hangs down through a hole in the table. This piece is released, and the rope slides down through the hole. What is the speed of the rope at the instant it loses contact with the table?

(b) A rope of length L lies in a heap on a table, except for a very small piece at one end which hangs down through a hole in the table. This piece is released, and the rope unravels and slides down through the hole. What is the speed of the rope at the instant it loses contact with the table? (Assume that the rope is greased, so that it has no friction with itself.)

Week 62 (11/17/03)

Leftover dental floss

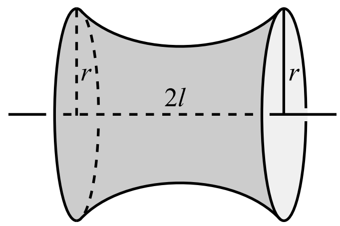
Two rolls of dental floss initially have equal lengths, L. Each day, a person chooses one of the rolls at random and cuts off a fixed small length, d. This continues until one of the rolls runs out of floss. How much floss, on average, is left on the other roll at this time? Assume that $N \equiv L/d$ is very large, and ignore terms in your answer that are of subleading order in N.

Week 63 (11/24/03)

Minimal surface

Consider a soap bubble that stretches between two identical circular rings of radius r, as shown below. The planes of the rings are parallel, and the distance between them is 2ℓ. Find the shape of the soap bubble.

What is the largest value of ℓ/r for which a stable soap bubble exists? You will have to solve something numerically here.



Week 64 (12/1/03)

Attracting bugs

N bugs are initially located at the vertices of a regular N-gon, whose sides have length ℓ. At a given moment, they all begin crawling with equal speeds in the clockwise direction, directly toward the adjacent bug. They continue to walk directly toward the adjacent bug, until they finally all meet at the center of the original N-gon. What is the total distance each bug travels? How many times does each bug spiral around the center?

Week 65 (12/8/03)

Relativistic cart

A long cart moves at relativistic speed v. Sand is dropped into the cart at a rate $dm/dt = \sigma$ in the ground frame. Assume that you stand on the ground next to where the sand falls in, and you push on the cart to keep it moving at constant speed v. What is the force between your feet and the ground? Calculate this force in both the ground frame (your frame) and the cart frame, and show that the results are equal (as should be the case for longitudinal forces).

Week 66 (12/15/03)

Bowl of spaghetti

A bowl contains N spaghetti noodles. You reach into the bowl and grab two free ends at random and attach them. You do this N times until there are no free ends left. On average, how many loops are formed by this process?

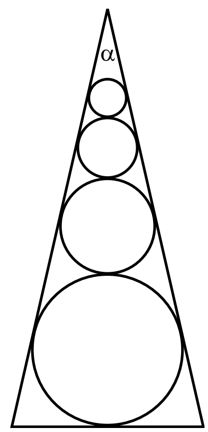
Inverted pendulum

A pendulum consists of a mass m at the end of a massless stick of length ℓ. The other end of the stick is made to oscillate vertically with a position given by $y(t) = A\cos(\omega t)$, where $A \ll \ell$. It turns out that if ω is large enough, and if the pendulum is initially nearly upside-down, then it will, surprisingly, *not* fall over as time goes by. Instead, it will (sort of) oscillate back and forth around the vertical position. Explain why the pendulum doesn't fall over, and find the frequency of the back and forth motion.

Week 68 (12/29/03)

Tower of circles

Consider N circles stacked on top of each other inside an isosceles triangle, as shown below for the case $N = 4$. Let A_C be the sum of the areas of the N circles, and let A_T be the area of the triangle. In terms of N, what should the vertex angle, α, be so that the ratio A_C/A_T is maximized? Assume that N is large, and ignore terms in your answer that are of subleading order in N.



Week 69 (1/5/04)

Compton scattering

A photon collides with a stationary electron. If the photon scatters at an angle θ, show that the resulting wavelength, λ', is given in terms of the original wavelength, λ, by

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta),$$

where m is the mass of the electron. Note: The energy of a photon is $E = h\nu = hc/\lambda$.

Week 70 (1/12/04)

Painting a funnel

Consider the curve $y = 1/x$, from $x = 1$ to $x = \infty$. Rotate this curve around the x-axis to create a funnel-like surface of revolution. The volume of this funnel is

$$V = \int_1^\infty \frac{\pi}{x^2}\,dx = \pi,$$

which is finite. The surface area, however, is

$$A = \int_1^\infty \frac{2\pi\sqrt{1+y'^2}}{x}\,dx \quad > \quad \int_1^\infty \frac{2\pi}{x}\,dx,$$

which is infinite. So it seems like you can fill up the funnel with paint, but you can't paint it. What is the solution to this apparent paradox?

Week 71 (1/19/04)

Maximum trajectory length

A ball is thrown at speed v from zero height on level ground. At what angle should it be thrown so that the distance traveled *through the air* is maximum. (You will have to solve something numerically.)

Week 72 (1/26/04)

Find the foci

Using a straightedge and compass, construct (1) the foci of a given ellipse, (2) the focus of a given parabola, and (3) the foci of a given hyperbola.

Week 73 (2/2/04)

Chain on a scale

A chain of length L and mass density σ is held such that it hangs vertically just above a scale. It is then released. What is the reading on the scale, as a function of the height of the top of the chain?

Week 74 (2/9/04)

Comparing the numbers

The numbers 1 through N are put in a hat. You and $N-1$ other people each pick a number. You then compare your number with the others, one at a time, until you find one that is smaller than yours. This procedure is repeated many times. How many numbers, on average, will you have to check in order to find one that is smaller than yours? (Ignore the situations where you have the number "1".) Consider two cases:

(a) You ask the other people randomly. That is, at all times you have equal probabilities of asking each person. This could be arranged, for example, by demanding that you have a very bad memory, so that you may ask a given person more than once.

(b) You have a good memory. In other words, you don't ask a given person more than once.

Week 75 (2/16/04)

Hanging chain

A chain of uniform mass density per unit length hangs between two given points on two walls. Find the shape of the chain. (Aside from an arbitrary additive constant, the function describing the shape should contain one unknown constant.)

Week 76 (2/23/04)

Crawling ant

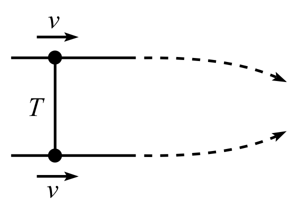
A rubber band with initial length L has one end tied to a wall. At $t = 0$, the other end is pulled away from the wall at speed V (assume that the rubber band stretches uniformly). At the same time, an ant located at the end not attached to the wall begins to crawl toward the wall, with speed u relative to the band. Will the ant reach the wall? If so, how much time will it take?

Week 77 (3/1/04)

Relativistic momentum paradox

Two equal masses are connected by a massless string with tension T. (By "massless", we mean that it has no mass in its unstretched, zero-length state.) The masses are constrained to move with speed v along parallel lines, as shown below. The constraints are then removed, and the masses are drawn together. They collide and make one blob which continues to move to the right. Is the following reasoning correct? If your answer is "no", then state what is invalid about whichever of the four sentences is/are invalid.

"The forces on the masses point in the y-direction. Therefore, there is no change in momentum in the x-direction. But the mass of the resulting blob is greater than the sum of the initial masses (because they collided with some relative speed). Therefore, the speed of the resulting blob must be less than v (to keep p_x constant), so the whole apparatus slows down in the x-direction."



Week 78 (3/8/04)

Infinite square roots

Find the value of

$$\sqrt{1-\sqrt{\frac{17}{16}-\sqrt{1-\sqrt{\frac{17}{16}-\sqrt{1-\cdots}}}}}\,.$$

Assume, when taking the limit, that deep within the nested square roots we choose to begin with a "1" (starting with a "17/16" would yield a complex number).

Week 79 (3/15/04)

Propelling a car

For some odd reason, you decide to throw baseballs at a car of mass M, which is free to move frictionlessly on the ground. You throw the balls at the back of the car at speed u, and at a mass rate of σ kg/s (assume the rate is continuous, for simplicity). If the car starts at rest, find its speed and position as a function of time, assuming that the back window is open, so that the balls collect inside the car.

Week 80 (3/22/04)

Nine divisible by 9

Given any seventeen integers, show that there is at least one subset of nine integers whose sum is divisible by 9.

Week 81 (3/29/04)

Rainbows

Assuming that the index of refraction of water is 4/3 and that raindrops are spherical, show that the location of a rainbow is approximately 42° above the line from the sun to you. If you see a double rainbow, what is the angle of the second one? Even triple rainbows are possible, although they are difficult to see; where is the third one?

Week 82 (4/5/04)

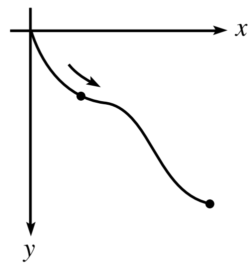
Standing in a line

N people are standing in a line, facing forward down the line. How many of them, on average, are able to say, “I am taller than everyone in front of me.”?

Week 83 (4/12/04)

The brachistochrone

A bead is released from rest at the origin and slides down a frictionless wire that connects the origin to a given point, as shown. What shape should the wire take so that the bead reaches the endpoint in the shortest possible time?



Week 84 (4/19/04)

Poisson and Gaussian

Throw N balls at random into B boxes. Let a be the average number of balls, N/B, in a box. Let $P(x)$ be the probability that a given box has exactly x balls in it.

(a) Show that

$$P(x) \approx \frac{a^x e^{-a}}{x!}\,.$$

Certain assumptions are needed for this expression to be valid. What are they?

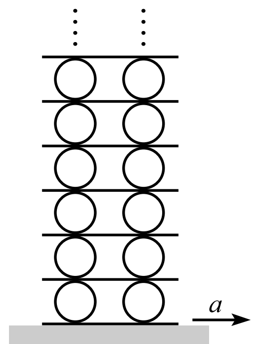
(b) Show that if a is large, the above Poisson distribution essentially becomes a Gaussian distribution,

$$P(x) = \frac{a^x e^{-a}}{x!} \approx \frac{e^{-(x-a)^2/2a}}{\sqrt{2\pi a}}\,.$$

Week 85 (4/26/04)

Tower of cylinders

Consider the infinitely tall system of identical massive cylinders and massless planks shown below. The moment of inertia of the cylinders is $I = MR^2/2$. There are two cylinders at each level, and the number of levels is infinite. The cylinders do not slip with respect to the planks, but the bottom plank is free to slide on a table. If you pull on the bottom plank so that it accelerates horizontally with acceleration a, what is the horizontal acceleration of the bottom row of cylinders?



Week 86 (5/3/04)

Shifted intervals

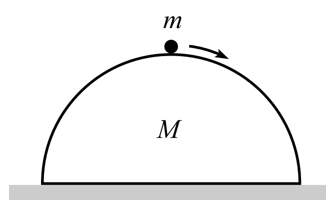
Let $\epsilon \equiv 1/N$. Choose a number at random between 0 and 1. Choose a second number between ϵ and $1+\epsilon$. Choose a third number between 2ϵ and $1+2\epsilon$. Continue this process, until you choose an Nth number between $1-\epsilon$ and $2-\epsilon$. What is the probability that the first number you choose is the smallest of all the numbers? Assume that N is very large, and make suitable approximations.

Week 87   (5/10/04)

Leaving the hemisphere

A point particle of mass m sits at rest on top of a frictionless hemisphere of mass M, which rests on a frictionless table, as shown. The particle is given a tiny kick and slides down the hemisphere. At what angle θ (measured from the top of the hemisphere) does the particle lose contact with the hemisphere?

In answering this question for $m \neq M$, it is sufficient for you to produce an equation that θ must satisfy (it will be a cubic). However, for the special case of $m = M$, this equation can be solved without too much difficulty; find the angle in this case.



Week 88 (5/17/04)

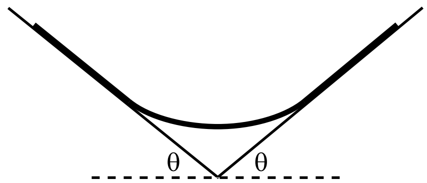
Ladder envelope

A ladder initially stands vertically against a wall. Its bottom end is given a sideways kick, causing the ladder to slide down. Assume that the bottom end is constrained to keep contact with the ground, and the top end is constrained to keep contact with the wall. Describe the envelope of the ladder's positions.

Week 89 (5/24/04)

Rope between inclines

A rope rests on two platforms which are both inclined at an angle θ (which you are free to pick), as shown. The rope has uniform mass density, and its coefficient of friction with the platforms is 1. The system has left-right symmetry. What is the largest possible fraction of the rope that does not touch the platforms? What angle θ allows this maximum value?



Week 90 (5/31/04)

The game of NIM

Determine the best strategy for each player in the following two-player game. There are three piles, each of which contains some number of coins. Players alternate turns, each turn consisting of removing any (non-zero) number of coins from a single pile. The goal is to be the person to remove the last coin(s).

Solution

Week 1 (9/16/02)

Basketball and tennis ball

(a) For simplicity, assume that the balls are separated by a very small distance, so that the relevant bounces happen a short time apart. This assumption isn't necessary, but it makes for a slightly cleaner solution.

Just before the basketball hits the ground, both balls are moving downward with speed (using $mv^2/2 = mgh$)

$$v = \sqrt{2gh}. \tag{1}$$

Just after the basketball bounces off the ground, it moves upward with speed v, while the tennis ball still moves downward with speed v. The relative speed is therefore $2v$. After the balls bounce off each other, the relative speed is still $2v$. (This is clear if you look at things in the frame of the basketball, which is essentially a brick wall.[1]) Since the upward speed of the basketball essentially stays equal to v, the upward speed of the tennis ball is $2v + v = 3v$. By conservation of energy, it will therefore rise to a height of $H = d + (3v)^2/(2g)$. But $v^2 = 2gh$, so we have

$$H = d + 9h. \tag{2}$$

(b) Just before B_1 hits the ground, all of the balls are moving downward with speed $v = \sqrt{2gh}$.

We will inductively determine the speed of each ball after it bounces off the one below it. If B_i achieves a speed of v_i after bouncing off B_{i-1}, then what is the speed of B_{i+1} after it bounces off B_i? The relative speed of B_{i+1} and B_i (right before they bounce) is $v + v_i$. This is also the relative speed after they bounce. Since B_i is still moving upwards at essentially speed v_i, the final upward speed of B_{i+1} is therefore $(v + v_i) + v_i$. Thus,

$$v_{i+1} = 2v_i + v. \tag{3}$$

Since $v_1 = v$, we obtain $v_2 = 3v$ (in agreement with part (a)), $v_3 = 7v$, $v_4 = 15v$, etc. In general,

$$v_n = (2^n - 1)v, \tag{4}$$

which is easily seen to satisfy eq. (3), with the initial value $v_1 = v$.

From conservation of energy, B_n will bounce to a height of

$$H = \ell + \frac{\left((2^n - 1)v\right)^2}{2g} = \ell + (2^n - 1)^2 h. \tag{5}$$

[1] It turns out that the relative speed is the same before and after any elastic collision, independent of what the masses are. This is easily seen by working in the center-of-mass frame, where the masses simply reverse their velocities.

If h is 1 meter, and we want this height to equal 1000 meters, then (assuming ℓ is not very large) we need $2^n - 1 > \sqrt{1000}$. Five balls won't quite do the trick, but six will, and in this case the height is almost four kilometers.

Escape velocity from the earth (which is $v_{\text{esc}} = \sqrt{2gR} \approx 11,200\,\text{m/s}$) is reached when

$$v_n \ge v_{\text{esc}} \quad \Longrightarrow \quad (2^n - 1)\sqrt{2gh} \ge \sqrt{2gR} \quad \Longrightarrow \quad n \ge \ln_2 \left(\sqrt{\frac{R}{h}} + 1 \right). \tag{6}$$

With $R = 6.4 \cdot 10^6\,\text{m}$ and $h = 1\,\text{m}$, we find $n \ge 12$. Of course, the elasticity assumption is absurd in this case, as is the notion that one can find 12 balls with the property that $m_1 \gg m_2 \gg \cdots \gg m_{12}$.

Solution

Week 2 (9/23/02)

Green-eyed dragons

Let's start with a smaller number of dragons, N, instead of one hundred, to get a feel for the problem.

If $N = 1$, and you tell this dragon that at least one of the dragons has green eyes, then you are simply telling him that he has green eyes, so he must turn into a sparrow at midnight.

If $N = 2$, let the dragons be called A and B. After your announcement that at least one of them has green eyes, A will think to himself, "If I do *not* have green eyes, then B can see that I don't, so B will conclude that she must have green eyes. She will therefore turn into a sparrow on the first midnight." Therefore, if B does not turn into a sparrow on the first midnight, then on the following day A will conclude that he himself must have green eyes, and so he will turn into a sparrow on the second midnight. The same thought process will occur for B, so they will both turn into sparrows on the second midnight.

If $N = 3$, let the dragons be called A, B, and C. After your announcement, C will think to himself, "If I do *not* have green eyes, then A and B can see that I don't, so as far as they are concerned, they can use the reasoning for the $N = 2$ situation, in which case they will both turn into sparrows on the second midnight." Therefore, if A and B do not turn into sparrows on the second midnight, then on the third day C will conclude that he himself must have green eyes, and so he will turn into a sparrow on the third midnight. The same thought process will occur for A and B, so they will all turn into sparrows on the third midnight. The pattern now seems clear.

Claim: *Consider N dragons, all of whom have green eyes. If you announce to all of them that at least one of them has green eyes, they will all turn into sparrows on the Nth midnight.*

Proof: We will prove this by induction. We will assume the result is true for N dragons, and then we will show that it is true for $N + 1$ dragons. We saw above that it holds for $N = 1, 2, 3$.

Consider $N + 1$ dragons, and pick one of them, called A. After your announcement, she will think to herself, "If I do *not* have green eyes, then the other N dragons can see that I don't, so as far as they are concerned, they can use the reasoning for the situation with N dragons, in which case they will all turn into sparrows on the Nth midnight." Therefore, if they do not all turn into sparrows on the Nth midnight, then on the $(N + 1)$st day A will conclude that she herself must have green eyes, and so she will turn into a sparrow on the $(N + 1)$st midnight. The same thought process will occur for the other N dragons, so they will all turn into sparrows on the $(N + 1)$st midnight. ■

Hence, in our problem all one hundred dragons will turn into sparrows on the 100th midnight.

Although we've solved the problem, you may be troubled by the fact that your seemingly useless information did indeed have major consequences. How could this be, when surely all the dragons already knew what you told them? Did you really give them new information? The answer is "yes". Let's see what this new information is.

Consider the case $N = 1$. Here it is clear that you provided new information, since you essentially told the one dragon that he has green eyes. But for the cases $N \geq 2$, the new information is slightly more subtle.

Consider the case $N = 2$. Prior to your announcement, A knows that B has green eyes, and B knows that A has green eyes. That is the extent of the knowledge, and they can't conclude anything else from it. But after you tell them that at least one of them has green eyes, then A knows *two* things: He knows that B has green eyes, *and* he knows that B knows that there is at least one dragon with green eyes (because A knows that B heard your information). B gains a similar second piece of information. This second piece of information is critical, as we saw above in the reasoning for the $N = 2$ case.

Consider the case $N = 3$. A knows that B green eyes, and he also knows that B knows that there is at least one dragon with greens eyes (because A can see that B can see C). So the two bits of information in the $N = 2$ case above are already known before you speak. What new information is gained after you speak? Only after you speak is it true that A knows that B knows that C knows that there is at least one dragon with green eyes.

The analogous result holds for a general number N. There is no paradox here. Information *is* gained by your speaking. More information is added to the world than the information you gave.[1] And it turns out, as seen in the proof of Claim 1, that the new information is indeed enough to allow all the dragons to eventually figure out their eye color.

To sum up: Before you make your announcement, the following statement is true for N dragons: A_1 knows that A_2 knows that A_3 knows that ... that A_{N-2} knows that A_{N-1} knows that there is at least one dragon with green eyes. This is true because A_{N-1} can see A_N; and A_{N-2} can see that A_{N-1} can see A_N; and so on, until lastly A_1 can see that A_2 can see that ... that A_{N-1} can see A_N. The same result holds, of course, for any group of $N - 1$ dragons. The point is that it is only after you make your announcement that the chain is extended the final step to the Nth dragon. The fact that the Nth dragon heard your statement is critical to the truth of this complete chain.

So, in the end, it turns out to be of great importance how far the chain, "A knows that B knows that C knows that ..." goes.

Note that if one of the dragons missed your farewell announcement (which was "At least one the 100 dragons on this island has green eyes"), then they will all happily remain dragons throughout the ages.

[1] For example, A knows that you made your statement *while* stepping onto your boat *and* wearing a blue shirt. Or, more relevantly, A knows that you made your statement in front of all the other dragons. In short, it's not just what you say; it's how you say it.

Solution

Week 3 (9/30/02)

Balancing a pencil

(a) The component of gravity in the tangential direction is $mg\sin\theta \approx mg\theta$. Therefore, the tangential $F = ma$ equation is $mg\theta = m\ell\ddot{\theta}$, which may be written as $\ddot{\theta} = (g/\ell)\theta$. The general solution to this equation is

$$\theta(t) = Ae^{t/\tau} + Be^{-t/\tau}, \qquad \text{where } \tau \equiv \sqrt{\ell/g}. \tag{1}$$

The constants A and B are found from the initial conditions,

$$\begin{aligned} \theta(0) = \theta_0 \qquad &\Longrightarrow \qquad A + B = \theta_0, \\ \dot{\theta}(0) = \omega_0 \qquad &\Longrightarrow \qquad (A - B)/\tau = \omega_0. \end{aligned} \tag{2}$$

Solving for A and B, and then plugging into eq. (1) gives

$$\theta(t) = \frac{1}{2}\left(\theta_0 + \omega_0\tau\right)e^{t/\tau} + \frac{1}{2}\left(\theta_0 - \omega_0\tau\right)e^{-t/\tau}. \tag{3}$$

(b) The constants A and B will turn out to be small (they will each be of order $\sqrt{\hbar}$). Therefore, by the time the positive exponential has increased enough to make θ of order 1, the negative exponential will have become negligible. We will therefore ignore the latter term from here on. In other words,

$$\theta(t) \approx \frac{1}{2}\left(\theta_0 + \omega_0\tau\right)e^{t/\tau}. \tag{4}$$

The goal is to keep θ small for as long as possible. Hence, we want to minimize the coefficient of the exponential, subject to the uncertainty-principle constraint, $(\ell\theta_0)(m\ell\omega_0) \geq \hbar$. This constraint gives $\omega_0 \geq \hbar/(m\ell^2\theta_0)$. Hence,

$$\theta(t) \geq \frac{1}{2}\left(\theta_0 + \frac{\hbar\tau}{m\ell^2\theta_0}\right)e^{t/\tau}. \tag{5}$$

Taking the derivative with respect to θ_0 to minimize the coefficient, we find that the minimum value occurs at

$$\theta_0 = \sqrt{\frac{\hbar\tau}{m\ell^2}}\,. \tag{6}$$

Substituting this back into eq. (5) gives

$$\theta(t) \geq \sqrt{\frac{\hbar\tau}{m\ell^2}}\, e^{t/\tau}. \tag{7}$$

Setting $\theta \approx 1$, and then solving for t gives (using $\tau \equiv \sqrt{\ell/g}$)

$$t \leq \frac{1}{4}\sqrt{\frac{\ell}{g}}\ln\left(\frac{m^2\ell^3 g}{\hbar^2}\right). \tag{8}$$

With the given values, $m = 0.01\,\mathrm{kg}$ and $\ell = 0.1\,\mathrm{m}$, along with $g = 10\,\mathrm{m/s}^2$ and $\hbar = 1.06 \cdot 10^{-34}\,\mathrm{Js}$, we obtain

$$t \le \frac{1}{4}(0.1\,\mathrm{s})\ln(9 \cdot 10^{61}) \approx 3.5\,\mathrm{s}. \tag{9}$$

No matter how clever you are, and no matter how much money you spend on the newest, cutting-edge pencil-balancing equipment, you can never get a pencil to balance for more than about four seconds.

REMARKS: This smallness of this answer is quite amazing. It is remarkable that a quantum effect on a macroscopic object can produce an everyday value for a time scale. Basically, the point here is that the fast exponential growth of θ (which gives rise to the log in the final result for t) wins out over the smallness of $\hbar$, and produces a result for t of order 1. When push comes to shove, exponential effects always win.

The above value for t depends strongly on ℓ and g, through the $\sqrt{\ell/g}$ term. But the dependence on m, ℓ, and g in the log term is very weak. If m were increased by a factor of 1000, for example, the result for t would increase by only about 10%. Note that this implies that any factors of order 1 that we neglected throughout this problem are completely irrelevant. They will appear in the argument of the log term, and will thus have negligible effect.

Note that dimensional analysis, which is generally a very powerful tool, won't get you too far in this problem. The quantity $\sqrt{\ell/g}$ has dimensions of time, and the quantity $\eta \equiv m^2\ell^3 g/\hbar^2$ is dimensionless (it is the only such quantity), so the balancing time must take the form

$$t \approx \sqrt{\frac{\ell}{g}}\,f(\eta), \tag{10}$$

where f is some function. If the leading term in f were a power (even, for example, a square root), then t would essentially be infinite ($t \approx 10^{30}\,\mathrm{s}$ for the square root). But f in fact turns out to be a log (which you can't determine without solving the problem), which completely cancels out the smallness of $\hbar$, reducing an essentially infinite time down to a few seconds.

Solution

Week 4 (10/7/02)

Passing the spaghetti

(a) For the case of $n = 3$, it is obvious that the two people not at the head of the table have equal 1/2 chances of being the last served (BTLS).

For the case of $n = 4$, label the diners as A,B,C,D (with A being the head), and consider D's probability of BTLS. The various paths of spaghetti that allow D to be the last served are:

$$\text{ABC...},\ \text{ABABC...},\ \text{ABABABC...},\ \text{etc.} \tag{1}$$

The sum of the probabilities of these is

$$\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots = \frac{1/4}{1 - 1/4} = \frac{1}{3}\,. \tag{2}$$

By symmetry, B also has a 1/3 chance of BLTS, and then that leaves a 1/3 chance for C. Hence, B, C, and D all have equal 1/3 chances of BLTS.

The probabilities for $n = 5$ are a bit tedious to calculate in this same manner, so at this point we will (for lack of a better option) make the following guess:

Claim: *For arbitrary n, all diners not at the head of the table have equal $1/(n-1)$ chances of being the last served (BTLS).*

This seems a bit counterintuitive (because you might think that the diners further from the head have a greater chance of BTLS), but it is in fact correct.

Proof: Two things must happen to a given diner for BTLS:

(1) The plate must approach the given diner from the right or left and reach the person next to that diner.

(2) The plate must then reverse its direction and make its way (in whatever manner) all the way around the table until it reaches the person on the other side of the given diner.

For any of the (non-head) diners, the probability that the first of these conditions will be satisfied is 1. This condition will therefore not differentiate the probabilities of BTLS.

Given that (1) has happened, there is some definite probability of (2) happening, independent of where the diner is located. This is true because the probability of traveling all the way around the table does not depend on where this traveling starts. Hence, (2) also does not differentiate between the $n - 1$ (non-head) probabilities of BTLS.

Thus, all the $n - 1$ (non-head) probabilities of BTLS are equal, and are therefore equal to $1/(n - 1)$. ∎

(b) This problem is equivalent to asking how many steps it takes, on average, for a random walk in one dimension to hit n sites.

Let f_n be this expected number of steps. And let g_n be defined as follows. Assume that n sites have been visited, and that the present position is at one of the ends of this string of n sites. Then g_n is the expected number of steps it takes to reach a new site.

We then have

$$f_n = f_{n-1} + g_{n-1}. \tag{3}$$

This is true because in order to reach n sites, you must first reach $n-1$ sites (which takes f_{n-1} steps, on average). And then you must reach one more site, starting at the end of the string of $n-1$ sites (which takes g_{n-1} steps, on average).

Claim: $g_n = n$.

Proof: Let the sites which have been visited be labeled $1, 2, \ldots, n$. Let the present position be site 1.

There is a 1/2 chance that the next step will be to site number 0, in which case it only takes one step to reach a new site.

There is a 1/2 chance that the next step will be to site number 2. By considering this site to be an end-site of the string $2, 3, \ldots, n-1$ (which has size $n-2$), we see that it takes g_{n-2} steps (on average) to reach sites 1 or n. And then from each of these, it takes of g_n steps (on average) to reach a new site. Putting this together gives $g_n = \frac{1}{2}(1) + \frac{1}{2}(1 + g_{n-2} + g_n)$, or

$$g_n = g_{n-2} + 2. \tag{4}$$

Since we obviously have $g_1 = 1$, and since it is easy to see from the above reasoning that $g_2 = 2$ (or equivalently, that $g_0 = 0$), we inductively obtain $g_n = n$. ∎

Therefore, $f_n = f_{n-1} + (n-1)$. Using $f_1 = 0$, we see by induction that

$$f_n = \frac{n(n-1)}{2}. \tag{5}$$

Solution

Week 5 (10/14/02)

The raindrop

Let ρ be the mass density of the raindrop, and let λ be the average mass density in space of the water droplets. Let $r(t)$, $M(t)$, and $v(t)$ be the radius, mass, and speed of the raindrop, respectively.

We need three equations to solve for the above three unknowns. The equations we will use are two different expressions for dM/dt, and the $F = dp/dt$ expression for the raindrop.

The first expression for $\dot{M}$ is obtained by simply taking the derivative of $M = (4/3)\pi r^3\rho$, which gives

$$\begin{aligned} \dot{M} &= 4\pi r^2 \dot{r}\rho \qquad (1) \\ &= 3M\frac{\dot{r}}{r}\,. \qquad (2) \end{aligned}$$

The second expression for $\dot{M}$ is obtained by noting that the change in M is due to the acquisition of water droplets. The raindrop sweeps out volume at a rate given by its cross-sectional area times its velocity. Therefore,

$$\dot{M} = \pi r^2 v\lambda. \qquad (3)$$

The force on the droplet is Mg, and the momentum is Mv. Therefore, $F = dp/dt$ gives

$$Mg = \dot{M}v + M\dot{v}. \qquad (4)$$

We now have three equations involving the three unknowns, r, M, and v.[1]

Our goal is to find $\dot{v}$. We will do this by first finding $\ddot{r}$. Eqs. (1) and (3) give

$$\begin{aligned} v &= \frac{4\rho}{\lambda}\dot{r} \qquad (5) \\ \Longrightarrow \quad \dot{v} &= \frac{4\rho}{\lambda}\ddot{r}. \qquad (6) \end{aligned}$$

Plugging eqs. (2, 5, 6) into eq. (4) gives

$$Mg = \left(3M\frac{\dot{r}}{r}\right)\left(\frac{4\rho}{\lambda}\dot{r}\right) + M\left(\frac{4\rho}{\lambda}\ddot{r}\right). \qquad (7)$$

Therefore,

$$\tilde{g}r = 12\dot{r}^2 + 4r\ddot{r}, \qquad (8)$$

where we have defined $\tilde{g} \equiv g\lambda/\rho$, for convenience. The only parameter in eq. (8) is $\tilde{g}$. Therefore, $r(t)$ can depend only on $\tilde{g}$ and t. Hence, by dimensional analysis, r must take the form

$$r = A\tilde{g}t^2, \qquad (9)$$

[1] Note that we *cannot* write down the naive conservation-of-energy equation (which would say that the decrease in the water's potential energy equals the increase in its kinetic energy), because mechanical energy is *not* conserved. The collisions between the raindrop and the droplets are completely inelastic. The raindrop will, in fact, heat up. See the remark at the end of the solution.

where A is a numerical constant, to be determined. Plugging this expression for r into eq. (8) gives

$$\begin{aligned} \tilde{g}(A\tilde{g}t^2) &= 12(2A\tilde{g}t)^2 + 4(A\tilde{g}t^2)(2A\tilde{g}) \\ \Longrightarrow \quad A &= 48A^2 + 8A^2. \end{aligned} \tag{10}$$

Therefore, $A = 1/56$, and so $\ddot{r} = 2A\tilde{g} = \tilde{g}/28 = g\lambda/28\rho$. Eq. (6) then gives the acceleration of the raindrop as

$$\dot{v} = \frac{g}{7}\,, \tag{11}$$

independent of ρ and λ.

Remark: A common invalid solution to this problem is the following, which (incorrectly) uses conservation of energy.

The fact that v is proportional to $\dot{r}$ (shown in eq. (5)) means that the volume swept out by the raindrop is a cone. The center-of-mass of a cone is 1/4 of the way from the base to the apex (as you can show by integrating over horizontal circular slices). Therefore, if M is the mass of the raindrop after it has fallen a height h, then an (incorrect) application of conservation of energy gives

$$\frac{1}{2}Mv^2 = Mg\frac{h}{4} \qquad \Longrightarrow \qquad v^2 = \frac{gh}{2}\,. \tag{12}$$

Taking the derivative of this (or equivalently, using $v^2 = 2ad$), we obtain

$$\dot{v} = \frac{g}{4}\,. \qquad \text{(Incorrect)} \tag{13}$$

The reason why this solution is invalid is that the collisions between the raindrop and the droplets are completely inelastic. Heat is generated, and the overall kinetic energy of the raindrop is smaller than you would otherwise expect.

Let's calculate how much mechanical energy is lost (and therefore how much the raindrop heats up) as a function of the height fallen. The loss in mechanical energy is

$$E_{\text{lost}} = Mg\frac{h}{4} - \frac{1}{2}Mv^2. \tag{14}$$

Using $v^2 = 2(g/7)h$, this becomes

$$\Delta E_{\text{int}} = E_{\text{lost}} = \frac{3}{28}Mgh, \tag{15}$$

where ΔE_{int} is the gain in internal thermal energy.

The energy required to heat 1g of water by 1 C° is 1 calorie (= 4.18 Joules). Therefore, the energy required to heat 1 kg of water by 1 C° is ≈ 4200 J. In other words,

$$\Delta E_{\text{int}} = 4200\, M\, \Delta T, \tag{16}$$

where M is measured in kilograms, and T is measured in Celsius. Eqs. (15) and (16) give the increase in temperature as a function of h,

$$4200\, \Delta T = \frac{3}{28}gh. \tag{17}$$

How far must the raindrop fall before it starts to boil? If we assume that the water droplets' temperature is near freezing, then the height through which the raindrop must fall to have $\Delta T = 100$ C° is found to be

$$h \approx 400\,\text{km}. \tag{18}$$

We have, of course, idealized the problem. But needless to say, there is no need to worry about getting burned by the rain.

A typical value for h more like a few kilometers, which would raise the temperature only by about one degree. This effect, of course, is washed out by many other factors.

Solution

Week 6 (10/21/02)

Flipping a coin

(a) There is a 1/2 chance that you win one dollar, a 1/4 chance that you win two dollars, a 1/8 chance that you win three dollars, etc. Therefore, the average value of your winnings is

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots. \tag{1}$$

This may be written as

$$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)+\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)+\left(\frac{1}{8}+\frac{1}{16}+\cdots\right)+\cdots, \tag{2}$$

which equals

$$(1)+\left(\frac{1}{2}\right)+\left(\frac{1}{4}\right)+\cdots=2. \tag{3}$$

So you expect to win an average of two dollars each time you play this game.

(b) There is a 1/2 chance that you win one dollar, a 1/4 chance that you win two dollars, a 1/8 chance that you win four dollars, etc. So the average value of your winnings is now

$$\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\frac{8}{16}+\cdots=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots=\infty. \tag{4}$$

You will quickly discover, however, that you will not win an infinite amount of money playing this game. We seem to have a paradox. The expectation value is infinite, but certainly no one is going to put up an infinite amount of money, or even a million dollars, for the opportunity to play this game once. What is the solution to this paradox?

The answer is that an expectation value is defined to be an average over an *infinite* number of trials (or the limit towards an infinite number), but you are simply not going to play an infinite number of games. In other words, the calculated expectation value doesn't agree with your experiment, because your experiment has nothing whatsoever to do with the precise definition of an expectation value. To be sure, if you did somehow play an infinite number of games, then you would indeed have an infinite average for your winnings. The whole paradox arises from trying to make "expectation value" mean something it doesn't.

This might not be a very satisfying explanation, so let us get a better feeling for the problem by looking at a situation where someone plays $N = 2^n$ games. How much money would a "reasonable" person be willing to put up front for the opportunity to play these N games?

Well, in about 2^{n-1} games he will win one dollar; in about 2^{n-2} he will win two dollars; in about 2^{n-3} games he will win four dollars; etc., until in about one game he will win 2^{n-1} dollars. In addition, there are the "fractional" numbers of games where he wins much larger quantities of money (for example, in half a game he will win 2^n dollars, etc.), and this is indeed where the infinite expectation value comes from, in the calculation above. But let us forget about these for the moment, in order to just get a lower bound on what a reasonable person should put on the table. Adding up the above cases gives the total winnings as $2^{n-1}(1)+2^{n-2}(2)+2^{n-3}(4)+\cdots+1(2^{n-1}) = 2^{n-1}n$. The average value of these winnings in the $N = 2^n$ games is therefore $2^{n-1}n/2^n = n/2 = (\log_2 N)/2$.

A reasonable person should therefore expect to win at least $(\log_2 N)/2$ dollars per game. (By "expect", we mean that if the player plays a very large number of sets of N games, and then takes an average over these sets, he will win at least $2^{n-1}n$ dollars per set.) This clearly increases with N, and goes to infinity as N goes to infinity. It is nice to see that we can obtain this infinite limit without having to worry about what happens in the infinite number of "fractional" games. Remember, though, that this quantity, $(\log_2 N)/2$, has nothing to do with a true expectation value, which is only defined for $N \to \infty$.

Someone may still not be satisfied and want to ask, "But what if I play only N games? I will never ever play another game. How much money do I expect to win?" The proper answer is that the question has no meaning. It is not possible to define how much one expects to win, if one is not willing to take an average over a arbitrarily large number of trials.

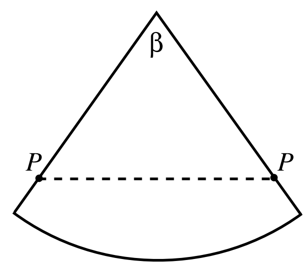
Solution

Week 7 (10/28/02)

Mountain climber

(a) We will take advantage of the fact that a cone is "flat", in the sense that we can make one out of a piece of paper, without crumpling the paper.

Cut the cone along a straight line emanating from the peak and passing through the knot of the lasso, and roll the cone flat onto a plane. Call the resulting figure, which is a sector of a circle, S.



If the cone is very sharp, then S will look like a thin "pie piece". If the cone is very wide, with a shallow slope, then S will look like a pie with a piece taken out of it. Points on the straight-line boundaries of the sector S are identified with each other. Let P be the location of the lasso's knot. Then P appears on each straight-line boundary, at equal distances from the tip of S. Let β be the angle of the sector S.

The key to this problem is to realize that the path of the lasso's loop must be a straight line on S, as shown in the above figure. (The rope will take the shortest distance between two points since there is no friction. And rolling the cone onto a plane does not change distances.) A straight line between the two identified points P is possible if and only if the sector S is smaller than a semicircle. The restriction on the mountain is therefore $\beta < 180°$.

What is this restriction, in terms of the angle of the peak, α? Let C denote a cross-sectional circle, a distance d (measured along the cone) from the top of the mountain. A semicircular S implies that the circumference of C equals πd. This then implies that the radius of C equals $d/2$. Therefore,

$$\sin(\alpha/2) < \frac{d/2}{d} = \frac{1}{2} \qquad \Longrightarrow \qquad \alpha < 60° \tag{1}$$

is the condition under which the mountain is climbable. In short, having $\alpha < 60°$ guarantees that there is a loop around the cone of shorter length than the distance straight to the peak and back.

Remark: When viewed from the side, the rope will appear perpendicular to the side of the mountain at the point opposite the lasso's knot. A common mistake is to assume that this implies $\alpha < 90°$. This is not the case, because the loop does not lie in a plane. Lying in a plane, after all, would imply an elliptical loop; but the loop must certainly have a kink in it where the knot is, since there must exist a vertical component to the tension. (If we had posed the problem with a planar, triangular mountain, then the answer would be $\alpha < 90°$.)

(b) Use the same strategy. Roll the cone onto a plane. If the mountain is very steep, then the climber's position can fall by means of the loop growing larger. If the mountain has a shallow slope, the climber's position can fall by means of the loop growing smaller. The only situation in which the climber will not fall is the one where the change in position of the knot along the mountain is exactly compensated by the change in length of the loop.

In terms of the sector S in a plane, this condition requires that if we move P a distance ℓ up (or down) along the mountain, the distance between the identified points P decreases (or increases) by ℓ. In the above figure, we must therefore have an equilateral triangle, and so $\beta = 60°$.

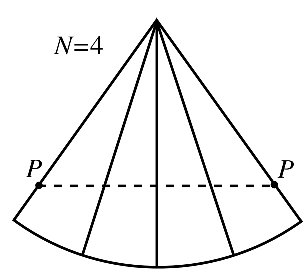
What peak-angle α does this correspond to? As above, let C be a cross-sectional circle, a distance d (measured along the cone) from the top of the mountain. Then $\beta = 60°$ implies that the circumference of C equals $(\pi/3)d$. This then implies that the radius of C equals $d/6$. Therefore,

$$\sin(\alpha/2) = \frac{d/6}{d} = \frac{1}{6} \qquad \Longrightarrow \qquad \alpha \approx 19° \tag{2}$$

is the condition under which the mountain is climbable. We see that there is exactly one angle for which the climber can climb up along the mountain.

REMARK: Another way to see the $\beta = 60°$ result is to note that the three directions of rope emanating from the knot must all have the same tension, since the deluxe lasso is one continuous piece of rope. They must therefore have $120°$ angles between themselves (to provide zero net force on the massless knot). This implies that $\beta = 60°$.

FURTHER REMARKS: For each type of lasso, for what angles can the mountain be climbed if the lasso is looped N times around the top of the mountain? The solution here is similar to that above. For the "cheap" lasso of part (a), roll the cone N times onto a plane, as shown for $N = 4$.



The resulting figure S_N is a sector of a circle divided into N equal sectors, each representing a copy of the cone. As above, S_N must be smaller than a semicircle. The circumference of the circle C (defined above) must therefore be less than $\pi d/N$. Hence, the radius of C must be less than $d/2N$. Thus,

$$\sin(\alpha/2) < \frac{d/2N}{d} = \frac{1}{2N} \qquad \Longrightarrow \qquad \alpha < 2\sin^{-1}\Big(\frac{1}{2N}\Big). \tag{3}$$

For the “deluxe” lasso of part (b), again roll the cone N times onto a plane. From the reasoning in part (b), we must have $N\beta = 60°$. The circumference of C must therefore be $\pi d/3N$, and so its radius must be $d/6N$. Therefore,

$$\sin(\alpha/2) = \frac{d/6N}{d} = \frac{1}{6N} \qquad \Longrightarrow \qquad \alpha = 2\sin^{-1}\left(\frac{1}{6N}\right). \tag{4}$$

Solution

Week 8 (11/4/02)

Sub-rectangles

Put the rectangle in the x-y plane, with its sides parallel to the x and y axes, and consider the function,

$$f(x,y) = e^{2\pi i x} e^{2\pi i y}. \tag{1}$$

Claim: *The integral of $f(x,y)$ over a rectangle, whose sides are parallel to the axes, is zero if and only if at least one pair of sides has integer length.*

Proof:

$$\begin{aligned} \int_a^b \int_c^d e^{2\pi i x} e^{2\pi i y}\, dx\, dy &= \int_a^b e^{2\pi i x}\, dx \int_c^d e^{2\pi i y}\, dy \\ &= -\frac{1}{4\pi^2}\Big(e^{2\pi i b} - e^{2\pi i a}\Big)\Big(e^{2\pi i d} - e^{2\pi i c}\Big) \\ &= -\frac{e^{2\pi i a} e^{2\pi i c}}{4\pi^2}\Big(e^{2\pi i (b-a)} - 1\Big)\Big(e^{2\pi i (d-c)} - 1\Big). \end{aligned} \tag{2}$$

This equals zero if and only if at least one of the factors in parentheses is zero. In other words, it equals zero if and only if at least one of $(b-a)$ and $(d-c)$ is an integer. ■

Since we are told that each of the smaller rectangles has at least one integer pair of sides, the integral of f over each of these smaller rectangles is zero. Therefore, the integral of f over the whole rectangle is also zero. Hence, from the Claim, we see that the whole rectangle has at least one integer pair of sides.

REMARKS:

1. This same procedure works for the analogous problem in higher dimensions. Given an N-dimensional rectangular parallelepiped which is divided into smaller ones, each of which has the property that at least one edge has integer length, then the original parallelepiped must also have this property. In three dimensions, for example, this follows from considering integrals of the function $f(x,y,z) = e^{2\pi i x} e^{2\pi i y} e^{2\pi i z}$, in the same manner as above.

2. We can also be a bit more general and make the following statement. Given an N-dimensional rectangular parallelepiped which is divided into smaller ones, each of which has the property that at least n "non-equivalent" edges (that is, ones that aren't parallel) have integer lengths, then the original parallelepiped must also have this property.

   This follows from considering the $\binom{N}{m}$ functions (where $m \equiv N + 1 - n$) of the form,

   $$f(x_{j_1}, x_{j_2}, \ldots, x_{j_m}) = e^{2\pi i x_{j_1}} e^{2\pi i x_{j_2}} \cdots e^{2\pi i x_{j_m}}, \tag{3}$$

   where the indices $j_1, j_2, \ldots j_m$ are a subset of the indices $1, 2, \ldots, N$. (For example, the previous remark deals with $n = 1$, $m = N$, and only $\binom{N}{N} = 1$ function.) We'll leave it to you to show that the integrals of these $\binom{N}{m}$ functions, over an N-dimensional rectangular parallelepiped, are *all* equal to zero if and only if at least n "non-equivalent" edges of the parallelepiped have integer length.

Solution

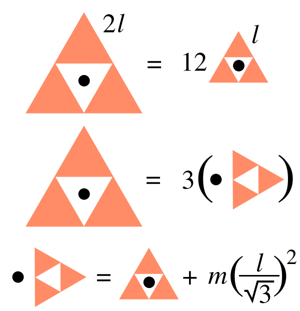
Week 9 (11/11/02)

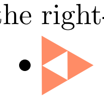
Fractal moment

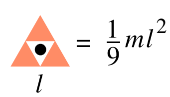
The strategy here will be to compute the moment of inertia by using scaling arguments, along with the parallel-axis theorem. To do this, we will need to compare the I for our triangle of side ℓ to that of a triangle of side 2ℓ. So let us scale up our triangle by a factor of 2 and examine what happens to the integral $I = \int r^2\,dm$. We get a simple factor of 2^2 from the r^2, but happens to the dm?

A solid triangle would yield a factor of $2^2 = 4$ in the dm (since area is proportional to length squared), but our fractal object is a bit different. The mass scales in a strange way. Doubling the size of our triangle increases its mass by a factor of only 3. This is true because the doubled triangle is simply made up of *three* of the smaller ones, plus an empty triangle in the middle. Thus, the dm picks up a factor of 3, and so the I for a fractal triangle of side 2ℓ is $4 \cdot 3 = 12$ times that of a fractal triangle of side ℓ (where the axes pass through any two corresponding points).

In what follows, we'll use pictures to denote the I's of the fractal objects around the dots shown. In terms of these pictures, we have,



The first line comes from the scaling argument, the second is obvious (moments of inertia simply add), and the third comes from the parallel-axis theorem (you can show that the distance between the dots is $\ell/\sqrt{3}$). Equating the right-hand sides of the first two equations, and then using the third to eliminate  , gives



Remarks: This result is larger than the I for a uniform triangle (which happens to be $m\ell^2/12$), because the mass is generally further away from the center in the fractal case.

When we increase the side length of our fractal triangle by a factor of 2, the factor of 3 in the dm is between the factor of $2^1 = 2$ relevant to a one-dimensional object, and the factor of $2^2 = 4$ relevant to a two-dimensional object. So in some sense our object has a dimension between 1 and 2. It is reasonable to define the dimension, d, of an object as the number for which f^d is the increase in "volume" when the dimensions are increased by a factor f. For our fractal triangle, we have $2^d = 3$, and so $d = \log_2 3 \approx 1.58$.

Solution

Week 10 (11/18/02)

Product of lengths

(This clever solution and generalization comes from Mike Robinson.)

Put the circle in the complex plane, with its center at the origin. Let the given vertex of the N-gon be located at the point $(1,0)$. Let $a \equiv e^{2\pi i/N}$, so that $a^N = 1$. Then the other vertices are located at the points a^n, where $n = 1, \ldots, N-1$.

Let the distance between the vertex at $(1,0)$ and the vertex at a^n be ℓ_n. Then the desired product (call it P_N) of the $N-1$ segments from the given vertex to the other vertices is

$$\begin{aligned} P_N &= \ell_1 \ell_2 \ldots \ell_{N-1} \\ &= |1-a||1-a^2| \cdots |1-a^{N-1}| \\ &= (1-a)(1-a^2)\cdots(1-a^{N-1}), \end{aligned} \tag{1}$$

where the third line comes from the fact that the product is real, because $(1-a^k)$ is the complex conjugate of $(1-a^{N-k})$, so the phases in the product cancel in pairs.

Consider the function,

$$F(z) \equiv z^N - 1. \tag{2}$$

One factorization of $F(z)$ is

$$F(z) = (z-1)(z^{N-1} + z^{N-2} + \cdots + 1). \tag{3}$$

Another factorization is

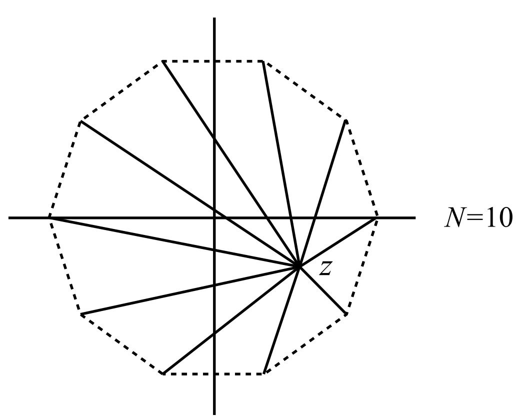
$$F(z) = (z-1)(z-a)(z-a^2)\cdots(z-a^{N-1}), \tag{4}$$

because the righthand side is simply the factorization that yields the zeros of $z^N - 1$ (namely, all the numbers of the form a^n). These two factorizations give

$$(z-a)(z-a^2)\cdots(z-a^{N-1}) = z^{N-1} + z^{N-2} + \cdots + 1. \tag{5}$$

This equality holds for any value of z. In particular, if we set $z = 1$ we obtain $P_N = N$, as desired.

Remark: Consider the product of the N lengths from an arbitrary point z in the complex plane, to *all* N vertices of the N-gon.



This product equals the absolute value of the righthand side of eq. (4). Hence, it equals $|F(z)| = |z^N - 1|$. Note what this gives in the $N \to \infty$ limit. If z equals any of the Nth roots of 1, we obtain zero, of course. But if z is any point inside the unit circle, we obtain $|0 - 1| = 1$, independent of both z and N.

The product of all the lengths *except* the length to the point (1,0), as in the original statement of the problem, equals

$$\left| \frac{z^N - 1}{z - 1} \right| = |z^{N-1} + z^{N-2} + \cdots + 1|. \tag{6}$$

Note that as $N \to \infty$, this goes to infinity for $z = 1$ (it's just our original result of N, by looking at the righthand side). But it goes to the finite number $1/|z - 1|$ for any z inside the unit circle (by looking at the lefthand side).

Solution

Week 11 (11/25/02)

Break or not break?

There are two possible reasonings, which seem to create a paradox:

- To an observer in the original rest frame, the spaceships stay the same distance, d, apart. Therefore, in the frame of the spaceships, the distance between them, d', must equal γd. This is true because d' is the distance that gets length-contracted down to d. After a long enough time, γ will differ appreciably from 1, so the string will be stretched by a large factor. Therefore, it *will* break.

- Let A be the rear spaceship, and let B be the front spaceship. From A's point of view, it looks like B is doing exactly what he is doing (and vice versa). A says that B has the same acceleration that he has. So B should stay the same distance ahead of him. Therefore, the string should *not* break.

The first reasoning is correct. The string *will* break. So that's the answer to our problem. But as with any good relativity paradox, we shouldn't feel at ease until we've explained what's wrong with the wrong reasoning.

The problem with the second reasoning is that A does *not* see B doing exactly what he is doing. Rather, A sees B's clock running fast. Perhaps the easiest way to show this is via the gravitational time-dilation effect. Since A and B are accelerating, they may be considered (by the Equivalence Principle) to be in a gravitational field, with B "higher" in the field. But high clocks run fast in a gravitational field. Hence, A sees B's clock running fast (and B sees A's clock running slow). A therefore sees B's engine running faster, and so B pulls away from A. Therefore, the string eventually breaks.

Remarks:

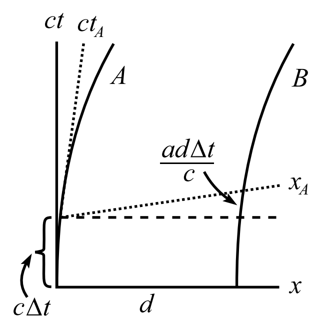
1. There is one slight (inconsequential) flaw in the first reasoning above. There is not one "frame of the spaceships". Their frames differ, since they measure a relative speed between themselves. It is therefore not clear exactly what is meant by the "length" of the string, because it is not clear what frame the measurement should take place in. This ambiguity, however, does not change the fact that A and B observe their separation to be (essentially) γd.

   If we want there to eventually be a well-defined "frame of the spaceships", we can simply modify the problem by stating that after a while, the spaceships stop accelerating simultaneously, as measured by someone in the original inertial frame. (Equivalently, A and B turn off their engines after equal proper times.) What A will see is the following. B pulls away from A. B then turns off his engine. The gap continues to widen. But A continues to fire his engine until be reaches B's speed. They then sail onward, in a common frame, keeping a constant separation (which is greater than the original separation, by a factor γ.)

2. The main issue in this problem is that it depends exactly on how we choose to accelerate an extended object. If we accelerate a stick by pushing on the back end (or by pulling on the front end), its length will remain essentially the same in its own frame, and it will become shorter in the original frame. But if we arrange for each end (or perhaps a number of points on the stick) to speed up in such a way that they always

move at the same speed with respect to the original frame, then the stick will be torn apart.

3. There is no need to invoke the Equivalence Principle to show that A sees B's clock run fast. We can demonstrate this effect completely within the realm of special relativity. Consider the following Minkowski diagram (about which we'll invoke a few standard facts).



After a small time Δt, the x_A-axis (which consists of simultaneous events, as observed by A) tilts upward with slope $v/c = a\,\Delta t/c$. It therefore intersects B's worldline at a ct value of

$$c\,\Delta t + \frac{ad\,\Delta t}{c} = c\,\Delta t\left(1 + \frac{ad}{c^2}\right). \tag{1}$$

A therefore sees B's clock run fast by a factor $(1 + ad/c^2)$, which is the standard gravitational time-dilation result.

Another derivation of the ad/c^2 result is the following. Consider the situation a short time after the start. An outside observer sees A's and B's clocks showing the same time. Therefore, by the usual vd/c^2 loss-of-simultaneity result in special relativity, B's clock must read vd/c^2 more that A's, in the moving frame. The increase per unit time, as viewed by A, must therefore be $(vd/c^2)/t = ad/c^2$, as above.

At any later time, we can repeat (roughly) either of the above two derivations in the instantaneous rest frame of A. Note that any special-relativistic time-dilation or length-contraction effects will be second order in (v/c), and hence negligible for small v.

Solution

Week 12 (12/2/02)

Decreasing numbers

First Solution: Let $E(x)$ be the expected number of numbers you have *yet to pick*, given that you have just picked the number x. Then, for example, $E(0) = 1$, because the next number you pick is guaranteed to be greater than $x = 0$, whereupon the game stops. Let's calculate $E(x)$.

Imagine picking the next number, having just picked x. There is a $(1-x)$ chance that this next number is greater than x, in which case the game stops. So in this case it takes you just one pick after the number x. If, on the other hand, you pick a number, y, which is less than x, then you can expect to pick $E(y)$ numbers after that. So in this case it takes you an average of $E(y)+1$ total picks after the number x. These two scenarios my be combined to give the equation,

$$\begin{aligned} E(x) &= 1 \cdot (1-x) + \int_0^x (E(y)+1)\,dy \\ &= 1 + \int_0^x E(y)\,dy \end{aligned} \tag{1}$$

Differentiating this with respect to x gives $E'(x) = E(x)$. Therefore, $E(x) = Ae^x$, where A is some constant. The condition $E(0) = 1$ gives $A = 1$. Hence

$$E(x) = e^x. \tag{2}$$

The total number of picks, T, is simply $T = E(1)$, because the first pick is automatically less than 1, so the number of picks *after* starting a game with the number 1 is equal to the total number of picks in a game starting with a random number. Since $E(1) = e$, we have

$$T = e. \tag{3}$$

Second Solution: Let the first number you pick be x_1, the second x_2, the third x_3, and so on. There is a $p_2 = 1/2$ chance that $x_2 < x_1$. There is a $p_3 = 1/3!$ chance that $x_3 < x_2 < x_1$. There is a $p_4 = 1/4!$ chance that $x_4 < x_3 < x_2 < x_1$, and so on.

You must make at least two picks in this game. The probability that you make exactly two picks is equal to the probability that $x_2 > x_1$, which is $1 - p_2 = 1/2$.

The probability that you make exactly three picks is equal to the probability that $x_2 < x_1$ and $x_3 > x_2$. This equals the probability that $x_2 < x_1$ minus the probability that $x_3 < x_2 < x_2$, that is, $p_2 - p_3$.

The probability that you make exactly four picks is equal to the probability that $x_3 < x_2 < x_1$ and $x_4 > x_3$. This equals the probability that $x_3 < x_2 < x_1$ minus the probability that $x_4 < x_3 < x_2 < x_2$, that is, $p_3 - p_4$.

Continuing in this manner, we find that the expected total number of picks, T, is

$$T = 2(1-p_2) + 3(p_2 - p_3) + 4(p_3 - p_4) + \cdots$$

$$
\begin{aligned}
&= 2\left(1-\frac{1}{2!}\right)+3\left(\frac{1}{2!}-\frac{1}{3!}\right)+4\left(\frac{1}{3!}-\frac{1}{4!}\right)+\cdots \\
&= 2+\frac{(3-2)}{2!}+\frac{(4-3)}{3!}+\frac{(5-4)}{4!}+\cdots \\
&= 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots \\
&= e. \qquad (4)
\end{aligned}
$$

Third Solution: Let $p(x)\,dx$ be the probability that a number between x and $x+dx$ is picked as part of the decreasing sequence. Then we may find $p(x)$ by adding up the probabilities, $p_j(x)\,dx$, that a number between x and $x+dx$ is picked on the jth pick.

The probability that such a number is picked first is dx. The probability that it is picked second is $(1-x)dx$, because $1-x$ is the probability that the first number is greater than x. The probability that it is picked third is $(1/2)(1-x)^2dx$, because $(1-x)^2$ is the probability that the first two numbers are greater than x, and $1/2$ is the probability that these numbers are picked in decreasing order. Likewise, the probability that it is picked fourth is $(1/3!)(1-x)^3dx$. Continuing in this manner, we see that the probability that it is picked sooner or later in the decreasing sequence is

$$
\begin{aligned}
p(x)\,dx &= \left(1+(1-x)+\frac{(1-x)^2}{2!}+\frac{(1-x)^3}{3!}+\cdots\right)dx \\
&= e^{1-x}\,dx. \qquad (5)
\end{aligned}
$$

The expected number of numbers picked in the decreasing sequence is therefore $\int_0^1 e^{1-x}dx = e-1$. Adding on the last number picked (which is not in the decreasing sequence) gives a total of e numbers picked, as above.

REMARKS:

1. What is the average value of the smallest number you pick? The probability that the smallest number is between x and $x+dx$ equals $e^{1-x}(1-x)\,dx$. This is true because $p(x)\,dx = e^{1-x}\,dx$ is the probability that you pick a number between x and $x+dx$ as part of the decreasing sequence (from the third solution above), and then $(1-x)$ is the probability that the next number you pick is larger. The average value, s, of the smallest number you pick is therefore $s=\int_0^1 e^{1-x}(1-x)x\,dx$. Letting $y\equiv 1-x$ for convenience, and integrating (say, by parts), we have

$$
\begin{aligned}
s &= \int_0^1 e^y y(1-y)\,dy \\
&= \left(-y^2e^y+3ye^y-3e^y\right)\Big|_0^1 \\
&= 3-e \\
&\approx 0.282. \qquad (6)
\end{aligned}
$$

Likewise, the average value of the final number you pick is $\int_0^1 e^{1-x}(1-x)(1+x)/2\,dx$, which you can show equals $2-e/2\approx 0.64$. The $(1+x)/2$ in this integral arises from the fact that if you do pick a number greater than x, its average value will be $(1+x)/2$.

2. We can also ask questions such as: Continue the game as long as $x_2 < x_1$, and $x_3 > x_2$, and $x_4 < x_3$, and $x_5 > x_4$, and so on, with the numbers alternating in size. What is the expected number of numbers you pick?

   We can apply the method of the first solution here. Let $A(x)$ be the expected number of numbers you have yet to pick, for $x = x_1, x_3, x_5, \ldots$. And let $B(x)$ be the expected number of numbers you have yet to pick, for $x = x_2, x_4, x_6, \ldots$. From the reasoning in the first solution, we have

$$\begin{aligned} A(x) &= 1\cdot(1-x) + \int_0^x (B(y)+1)\,dy &&= 1 + \int_0^x B(y)\,dy, \\ B(x) &= 1\cdot x + \int_x^1 (A(y)+1)\,dy &&= 1 + \int_x^1 A(y)\,dy. \end{aligned} \tag{7}$$

   Differentiating these two equations yields $A'(x) = B(x)$ and $B'(x) = -A(x)$. If we then differentiate the first of these and substitute the result into the second, we obtain $A''(x) = -A(x)$. Likewise, $B''(x) = -B(x)$. The solutions to these equations may be written as

$$A(x) = \alpha \sin x + \beta \cos x \qquad \text{and} \qquad B(x) = \alpha \cos x - \beta \sin x. \tag{8}$$

   The condition $A(0) = 1$ yields $\beta = 1$. The condition $B(1) = 1$ then gives $\alpha = (1 + \sin 1)/\cos 1$. The desired answer to the problem equals $B(0)$, since we could imagine starting the game with someone picking a number greater than 0, which is guaranteed. (Similarly, the desired answer also equals $A(1)$.) So the expected total number of picks is $B(0) = (1 + \sin 1)/\cos 1$. This has a value of about 3.41, which is greater than the $e \approx 2.72$ answer to our original problem.

Solution

Week 13 (12/9/02)

Unchanged velocity

Our strategy will be to produce (and equate) two different expressions for the total change in the angular momentum of the ball (relative to its center). The first comes from the effects of the friction force on the ball. The second comes from looking at the initial and final motion.

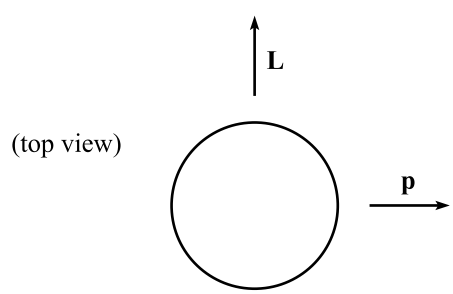
To produce our first expression for $\Delta\mathbf{L}$, note that the normal force provides no torque, so we may ignore it. The friction force, $\mathbf{F}$, from the paper changes both $\mathbf{p}$ and $\mathbf{L}$, according to,

$$\begin{aligned}\Delta\mathbf{p} &= \int \mathbf{F}\,dt,\\ \Delta\mathbf{L} &= \int \boldsymbol{\tau}\,dt \;=\; \int (-R\hat{\mathbf{z}})\times\mathbf{F}\,dt \;=\; (-R\hat{\mathbf{z}})\times\int\mathbf{F}\,dt. \qquad (1)\end{aligned}$$

Both of these integrals run over the entire slipping time, which may include time on the table after the ball leaves the paper. In the second line above, we have used the fact that the friction force always acts at the same location, namely $(-R\hat{\mathbf{z}})$, relative to the center of the ball. The two above equations yield

$$\Delta\mathbf{L} = (-R\hat{\mathbf{z}})\times\Delta\mathbf{p}. \qquad (2)$$

To produce our second equation for $\Delta\mathbf{L}$, let's examine how $\mathbf{L}$ is related to $\mathbf{p}$ when the ball is rolling without slipping, which is the case at both the start and the finish. When the ball is not slipping, we have the following situation (assume the ball is rolling to the right):



The magnitudes of p and L are given by

$$\begin{aligned}p &= mv,\\ L &= I\omega \;=\; \frac{2}{5}mR^2\omega \;=\; \frac{2}{5}Rm(R\omega) \;=\; \frac{2}{5}Rmv \;=\; \frac{2}{5}Rp, \qquad (3)\end{aligned}$$

where we have used the non-slipping condition, $v = R\omega$. (The actual $I = (2/5)mR^2$ value for a solid sphere will not be important for the final result.) It is easy to see that the directions of $\mathbf{L}$ and $\mathbf{p}$ can be combined with the above $L = 2Rp/5$ scalar relation to give

$$\mathbf{L} = \frac{2}{5}R\hat{\mathbf{z}}\times\mathbf{p}, \qquad (4)$$

where $\hat{\mathbf{z}}$ points out of the page. Since this relation is true at both the start and the finish, it must also be true for the differences in $\mathbf{L}$ and $\mathbf{p}$. That is,

$$\Delta\mathbf{L} = \frac{2}{5}R\hat{\mathbf{z}} \times \Delta\mathbf{p}. \tag{5}$$

Eqs. (2) and (5) give

$$\begin{aligned} (-R\hat{\mathbf{z}}) \times \Delta\mathbf{p} &= \frac{2}{5}R\hat{\mathbf{z}} \times \Delta\mathbf{p} \\ \Longrightarrow \qquad 0 &= \hat{\mathbf{z}} \times \Delta\mathbf{p}. \end{aligned} \tag{6}$$

There are three ways this cross product can be zero:

- $\Delta\mathbf{p}$ is parallel to $\hat{\mathbf{z}}$. But it isn't, since $\Delta\mathbf{p}$ lies in the horizontal plane.
- $\hat{\mathbf{z}} = 0$. Not true.
- $\Delta\mathbf{p} = 0$. So this must be true. Therefore, $\Delta\mathbf{v} = 0$, as we wanted to show.

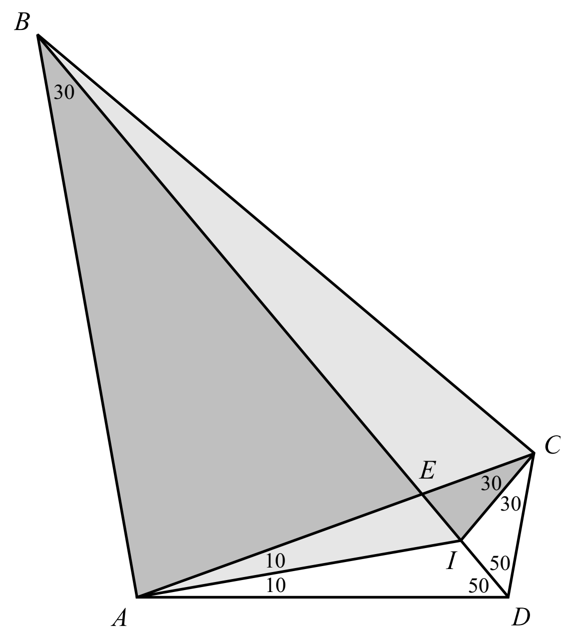
Solution

Week 14 (12/16/02)

Find the angles

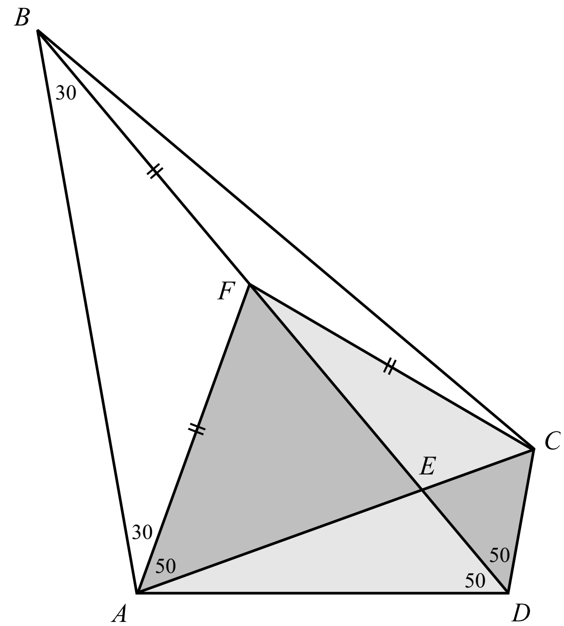
Although this problem seems simple at first glance, angle chasing won't provide the answer. Something a bit more sneaky is required. At the risk of going overboard, we'll give four solutions. You can check that all of the solutions rely on the equality of the two given $50°$ angles, and on the fact that $2(80°) + 20° = 180°$.

First Solution: In the figure below, note that $\angle ACD = 60°$ and $\angle ABD = 30°$. Let AC and BD intersect at E. Draw the angle bisectors of triangle ACD. They meet at the incenter, I, located along segment ED. Since $\angle ECI = 30° = \angle EBA$, triangles ECI and EBA are similar. Therefore, triangles EBC and EAI are also similar. Thus, $\angle EBC = \angle EAI = 10°$. We then easily find $\angle ECB = 60°$.



Second solution: In the figure below, note that $\angle ABD = 30°$. Let AC and BD intersect at E. Draw segment AF, with F on BE, such that $\angle EAF = 50°$. We then have $\angle FAB = 30°$. So triangle FAB is isosceles, with $FA = FB$.

Since $\angle EDC = \angle EAF$, triangles EDC and EAF are similar. Therefore, triangles EAD and EFC are also similar. Hence, $\angle ECF = 50°$, and triangle FCA is isosceles with $FC = FA$. Thus, $FC = FA = FB$, and so triangle FBC is also isosceles, with $\angle FBC = \angle FCB$. Since it is easy to show that these two angles must sum to $20°$, they must each be $10°$. Therefore, $\angle FBC = 10°$ and $\angle ECB = 60°$.



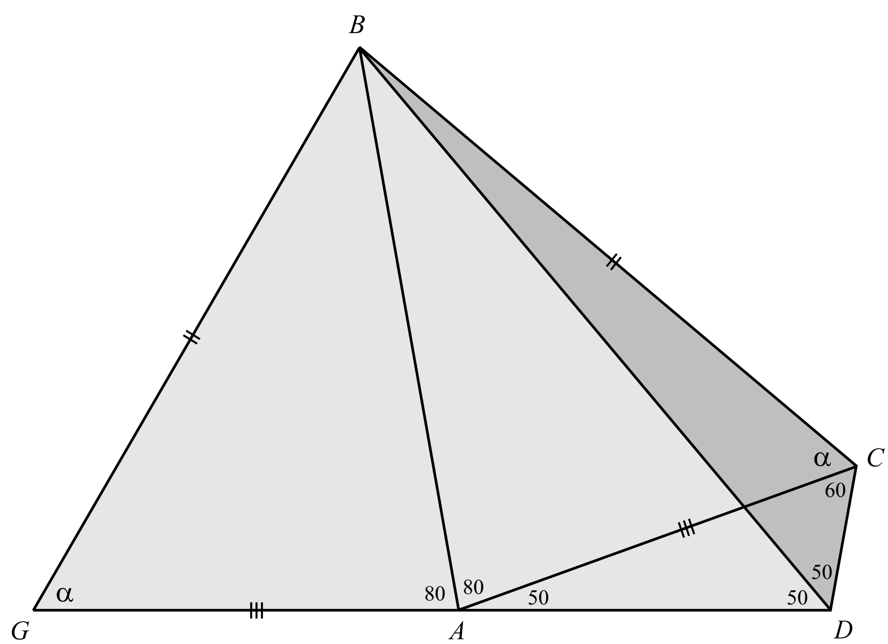
Third Solution: In the figure below, note that $\angle ACD = 60°$. Reflect triangle ABC across AB to yield triangle ABG. Note that D, A, and G are collinear. From the law of sines in triangle DBC, we have

$$\frac{\sin 50°}{BC} = \frac{\sin(60° + \alpha)}{BD}. \tag{1}$$

From the law of sines in triangle DBG, we have

$$\frac{\sin 50°}{BG} = \frac{\sin \alpha}{BD}. \tag{2}$$

But $BC = BG$, so we have $\sin(60° + \alpha) = \sin \alpha$. Therefore, $60° + \alpha$ and α must be supplementary angles, which gives $\alpha = 60°$. We then easily obtain $\angle DBC = 10°$.



Fourth Solution: We now present the brute-force method using the law of sines, just to show that it can be done.

In the figure below, let AC and BD intersect at E. Let the length of AB be 1 unit. Then the law of sines in triangle AED gives

$$a = \frac{\sin 50^\circ}{\sin 110^\circ}, \qquad \text{and} \qquad d = \frac{\sin 20^\circ}{\sin 110^\circ}. \tag{3}$$

The law of sines in triangles AEB and DEC then gives

$$b = \left(\frac{\sin 80^\circ}{\sin 30^\circ}\right)\left(\frac{\sin 50^\circ}{\sin 110^\circ}\right), \qquad \text{and} \qquad c = \left(\frac{\sin 50^\circ}{\sin 60^\circ}\right)\left(\frac{\sin 20^\circ}{\sin 110^\circ}\right). \tag{4}$$

The law of sines in triangle BEC finally gives

$$\left(\frac{\sin 80^\circ \sin 50^\circ}{\sin 30^\circ \sin 110^\circ}\right) \Big/ \sin\alpha = \left(\frac{\sin 50^\circ \sin 20^\circ}{\sin 60^\circ \sin 110^\circ}\right) \Big/ \sin\beta. \tag{5}$$

Substituting $70^\circ - \alpha$ for β yields (after some algebra)

$$\tan\alpha = \frac{\sin 60^\circ \sin 80^\circ \sin 70^\circ}{\sin 60^\circ \sin 80^\circ \cos 70^\circ + \sin 30^\circ \sin 20^\circ}. \tag{6}$$

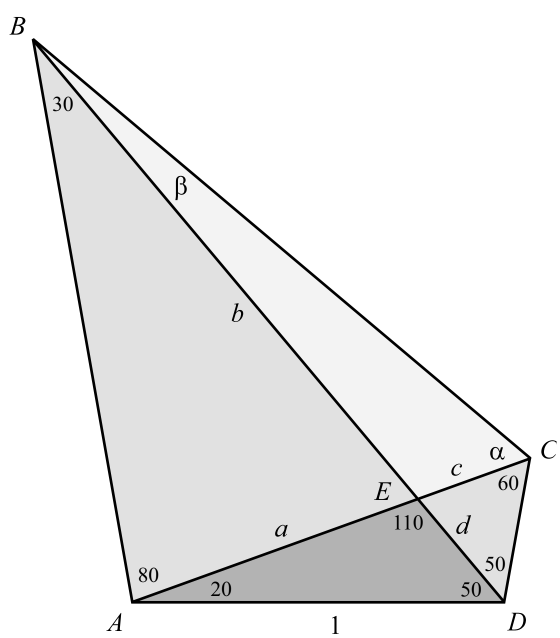
Using $\sin 20^\circ = 2\sin 10^\circ \cos 10^\circ = 2\sin 10^\circ \sin 80^\circ$, along with $\sin 30^\circ = 1/2$, gives

$$\tan\alpha = \frac{\sin 60^\circ \sin 70^\circ}{\sin 60^\circ \cos 70^\circ + \sin 10^\circ}. \tag{7}$$

Finally, expanding $\sin 10^\circ = \sin(70^\circ - 60^\circ)$ gives the result

$$\tan\alpha = \tan 60^\circ. \tag{8}$$

Hence $\alpha = 60^\circ$, and so $\beta = 10^\circ$.



Solution

Week 15 (12/23/02)

Maximal gravity

Assume that the material has been shaped and positioned so that the field at P is maximum. Let this field point in the x-direction. The key to this problem is to realize that all the small elements of mass dm on the surface of the material must give equal contributions to the x-component of the field at P. If this were not the case, then we could simply move a tiny piece of the material from one point on the surface to another, thereby increasing the field at P, in contradiction to our assumption that the field at P is maximum.

Label the points on the surface by their distance r from P, and by the angle θ that the line of this distance subtends with the x-axis. Then a small mass dm on the surface provides an x-component of the gravitational field equal to

$$F_x = \cos\theta \frac{G\,dm}{r^2}\,. \tag{1}$$

Since this contribution cannot depend on the location of the mass dm on the surface, we must have $r^2 \propto \cos\theta$. The surface may therefore be described by the equation,

$$r^2 = a^2\cos\theta, \tag{2}$$

where the constant a^2 depends on the volume of the material. The desired shape clearly exhibits cylindrical symmetry around the x-axis, so let us consider a cross section in the x-y plane. In terms of x and y (with $x^2+y^2=r^2$ and $\cos\theta = x/r$), eq. (2) becomes

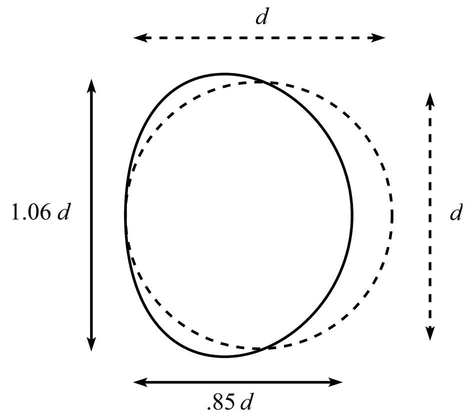
$$r^3 = a^2 x \qquad \Longrightarrow \qquad r^2 = a^{4/3}x^{2/3} \qquad \Longrightarrow \qquad y^2 = a^{4/3}x^{2/3} - x^2. \tag{3}$$

To get a sense of what this surface looks like, note that $dy/dx = \infty$ at both $x=0$ and $x=a$ (the point on the surface furthest from P). So the surface is smooth and has no cusps. We can easily calculate the volume in terms of a, and we find

$$V = \int_0^a \pi y^2\,dx = \int_0^a \pi(a^{4/3}x^{2/3} - x^2)\,dx = \frac{4\pi}{15}a^3. \tag{4}$$

Since the diameter of a sphere of volume V is $(6V/\pi)^{1/3}$, we see that a sphere with the same volume would have a diameter of $(8/5)^{1/3}a \approx 1.17a$. Hence, our shape is squashed by a factor of $(5/8)^{1/3} \approx 0.85$ along the x-direction, compared to a sphere of the same volume.

We may also calculate the maximum height in the y-direction. You can show that is occurs at $x = 3^{-3/4}a \approx 0.44a$ and has a value of $2(4/27)^{1/3}a \approx 1.24a$. Hence, our shape is stretched by a factor of $2(4/27)^{1/4}(5/8)^{1/3} \approx 1.24/1.17 \approx 1.06$ in the y-direction, compared to a sphere of the same volume. Cross sections of our shape and a sphere with the same volume are shown below.


d
1.06 d
d
.85 d

Solution

Week 16 (12/30/02)

Letters in envelopes

First Solution: (This solution is due to Aravi Samuel.) We will use induction on N. Let B_N denote the number of "bad" configurations where none of the N letters end up in the correct envelope. We claim that $B_{N+1} = N(B_N + B_{N-1})$. This can be seen as follows. In proceeding inductively from N to $N+1$ letters, there are two possible ways we can generate bad configurations:

- Given a bad configuration with N letters, we can create a bad configuration with $N+1$ letters by simply placing down the $(N+1)$st letter in its envelope, and then trading that letter with any of the other N letters. This provides us with NB_N bad configurations.
- We can also create a bad configuration with $N+1$ letters by taking a configuration of N letters where exactly one letter is in the correct envelope (there are NB_{N-1} such configurations) and then trading that letter with the $(N+1)$st letter. This provides us with NB_{N-1} bad configurations.

We therefore see that $B_{N+1} = N(B_N + B_{N-1})$. The probability of obtaining a bad configuration with N letters is $P_N = B_N/N!$. Hence, $B_N = N!\,P_N$, and so

$$\begin{aligned} (N+1)!\,P_{N+1} &= N\Big(N!\,P_N + (N-1)!\,P_{N-1}\Big) \\ \Longrightarrow \qquad (N+1)P_{N+1} &= NP_N + P_{N-1}. \end{aligned} \tag{1}$$

To solve this recursion relation, we can write it in the more suggestive form,

$$P_{N+1} - P_N = -\frac{1}{N+1}\Big(P_N - P_{N-1}\Big). \tag{2}$$

Since $P_1 = 0$ and $P_2 = 1/2$, we have $P_2 - P_1 = 1/2$. We then find inductively that $P_k - P_{k-1} = (-1)^k/k!\,$. Therefore,

$$\begin{aligned} P_N &= P_1 + \sum_{k=2}^{N} (P_k - P_{k-1}) \\ &= 1 - 1 + \sum_{k=2}^{N} \frac{(-1)^k}{k!} \\ &= \sum_{k=0}^{N} \frac{(-1)^k}{k!}\,. \end{aligned} \tag{3}$$

This is simply the partial series expansion for e^{-1}. So for large N, it approaches $1/e \approx 37\%$. This series expansion for $1/e$ converges very rapidly, so N does not have to be very large for the approximation $P_N \approx 1/e$ to be valid. For example, if $N = 5$ we have $P_5 - 1/e \approx 0.001$.

Remark: This $1/e$ result in the large-N limit can also be seen in the following way. The probability that a given letter does not end up in its corresponding envelope is $1 - 1/N$.

Therefore, if we ignore the fact that the placements of the letters are related (they are related because two letters cannot end up in the same envelope), then the probability that no letter ends up in the correct envelope is

$$\left(1 - \frac{1}{N}\right)^N \approx \frac{1}{e}\,. \tag{4}$$

It is not obvious that the correlations between the letters can be neglected here, but in view of the above result, this is apparently the case.

Second Solution: Let P_N be the probability that none of the N letters end up in the correct envelope. Let L_i and E_i denote the ith letter and corresponding envelope, respectively.

Consider a given letter, L_{a_1}, and assume that no letter ends up in the correct envelope. Then L_{a_1} must end up in some E_{a_2}, with $a_2 \neq a_1$. L_{a_2} will then end up in some E_{a_3}. L_{a_3} will then end up in some E_{a_4}, and so on. Eventually, one of the envelopes in this chain must be E_{a_1}. Let it be $E_{a_{n+1}}$. We may describe this situation by saying that L_{a_1} belongs to a "loop" of length n. If no letter ends up in the correct envelope, then n can be any number from 2 to N.

Claim: *The probability that the loop containing L_{a_1} has length n is equal to $1/N$, independent of n.*

Proof: L_{a_1} has an $(N-1)/N$ probability of ending up in some E_{a_2}, with $a_2 \neq a_1$. L_{a_2} then has an $(N-2)/(N-1)$ probability of ending up in some E_{a_3}, with $a_3 \neq a_2, a_1$. This continues until $L_{a_{n-1}}$ then has an $(N-(n-1))/(N-(n-2))$ probability of ending up in some E_{a_n}, with $a_n \neq a_{n-1}, \cdots, a_2, a_1$. Finally, L_{a_n} has a $1/(N-(n-1))$ probability of ending up in $E_{a_{n+1}} = E_{a_1}$. The probability that L_{a_1} belongs to a loop of length n is therefore equal to

$$\left(\frac{N-1}{N}\right)\left(\frac{N-2}{N-1}\right)\cdots\left(\frac{N-(n-1)}{N-(n-2)}\right)\left(\frac{1}{N-(n-1)}\right) = \frac{1}{N}\,. \quad \blacksquare \tag{5}$$

Given that a loop of length n is formed, which happens with probability $1/N$, the probability that all the $N-n$ other letters end up in the wrong envelopes is simply P_{N-n}. We therefore arrive at the relation,

$$P_N = \frac{1}{N}\Big(P_{N-2} + P_{N-3} + \cdots + P_1 + P_0\Big). \tag{6}$$

There is no P_{N-1} term in this equation, because a loop of length 1 would mean that L_{a_1} went into E_{a_1}. Note that $P_1 = 0$, and that $P_0 = 1$ here.

Multiplying eq. (6) through by N, and then subtracting the analogous equation for P_{N-1} (after multiplying through by $N-1$), gives

$$\begin{aligned} NP_N - (N-1)P_{N-1} &= P_{N-2} \\ \Longrightarrow \quad P_N - P_{N-1} &= -\frac{1}{N}\left(P_{N-1} - P_{N-2}\right). \end{aligned} \tag{7}$$

This is the same as eq. (2), with $N+1$ replaced by N. The solution proceeds as above.

Third Solution: We will find P_N by counting the number of cases that have no letter in the correct envelope, and then dividing this by the total number of possible arrangements, $N!$.

We may count these cases in the following manner. There are $N!$ total combinations. To count the number that have no letter in the correct envelope, we must subtract from $N!$ the number of combinations with, for example, (at least) L_1 in the correct envelope; there are $(N-1)!$ of these combinations. Likewise for the situations where another letter is in the correct envelope. So there seem to be $N! - N(N-1)!$ combinations with no letter in the correct envelope. However, we have double-counted some of the cases. For example, a combination which has (at least) L_1 and L_2 in the correct envelopes has been subtracted twice; there are $(N-2)!$ of these. Likewise for all the other pairs of letters. So we must add on $\binom{N}{2}(N-2)!$ combinations. But now a combination which has (at least) L_1, L_2, and L_3 in the correct envelopes has not been counted at all (because we have subtracted it off three times, and then added it on three times); there are $(N-3)!$ of these. Likewise for the other triplets. So we must subtract off $\binom{N}{3}(N-3)!$ combinations. Now, however, the combinations with (at least) L_1, L_2, L_3, and L_4 in the correct envelopes have been counted $-\binom{4}{1}+\binom{4}{2}-\binom{4}{3} = -2$ times. Likewise for the other quadruplets. So we must add on $\binom{N}{4}(N-4)!$ combinations.

In general, if we have done this procedure up to $(k-1)$-tuplets, then a combination having (at least) k letters in the correct envelopes has been counted T times, where

$$T = -\binom{k}{1} + \binom{k}{2} - \cdots + (-1)^{k-1}\binom{k}{k-1}. \tag{8}$$

However, the binomial expansion gives

$$\begin{aligned} 0 &= (1-1)^k \\ &= 1 - \binom{k}{1} + \binom{k}{2} + \cdots + (-1)^{k-1}\binom{k}{k-1} + (-1)^k \\ &= 1 + T + (-1)^k. \end{aligned} \tag{9}$$

Therefore, $T = -2$ for even k, and $T = 0$ for odd k. So we have either undercounted by one, or overcounted by one. Hence, the total number of combinations having no letter in the correct envelope is

$$N! - \binom{N}{1}(N-1)! + \binom{N}{2}(N-2)! + \cdots = \sum_{k=0}^{N} \frac{(-1)^k N!}{k!}. \tag{10}$$

To obtain the probability, P_N, that no letter is in the correct envelope, we must divide this result by $N!$. Therefore,

$$P_N = \sum_{k=0}^{N} \frac{(-1)^k}{k!}. \tag{11}$$

Remarks:

1. What is the probability (call it P_N^l) that exactly l out of the N letters end up in the correct envelopes? (With this notation, P_N^0 equals the P_N from above.) We can find P_N^l as follows.

   The probability that a given set of l letters goes into the correct envelopes is $1/\big(N(N-1)(N-2)\cdots(N-l+1)\big)$. The probability that the remaining $N-l$ letters all go into the wrong envelopes is P_{N-l}^0. This situation can happen in $\binom{N}{l}$ ways. Therefore,

$$\begin{aligned} P_N^l &= \frac{\binom{N}{l}}{N(N-1)\cdots(N-l+1)}\, P_{N-l}^0 \\ &= \frac{1}{l!} P_{N-l}. \end{aligned} \tag{12}$$

   Hence, using eq. (3),

$$P_N^l = \frac{1}{l!}\sum_{k=0}^{N-l}\frac{(-1)^k}{k!}\,. \tag{13}$$

   For large N, we have $P_N^l \approx 1/(l!e)$. The fact that this falls off so rapidly with l means that we are essentially guaranteed of having just a few letters in the correct envelopes. For example, we find (for large N) that the probability of having four or fewer letters in the correct envelopes is about 99.7%.

2. It is interesting to note that the equality, $P_N^l = \frac{1}{l!}P_{N-l}$, may directly yield the large-N result, $P_N \approx 1/e$, without having to go through all the work of the original problem. To see this, note that

$$1 = \sum_{l=0}^{N} P_N^l = \sum_{l=0}^{N}\frac{1}{l!}P_{N-l}. \tag{14}$$

   Since the terms with small l values dominate this sum, we may (for large N) replace the P_{N-l} values with $\lim_{M\to\infty} P_M$. Hence,

$$1 \approx \sum_{l=0}^{N}\frac{1}{l!}\Big(\lim_{M\to\infty} P_M\Big). \tag{15}$$

   Therefore,

$$\begin{aligned} \lim_{M\to\infty} P_M &\approx \left(\sum_{l=0}^{N}\frac{1}{l!}\right)^{-1} \\ &\approx \frac{1}{e}\,. \end{aligned} \tag{16}$$

3. Let's check that the P_N^l given by eq. (13) do indeed satisfy $\sum_{l=0}^{N} P_N^l = 1$. This may be done as follows:

$$\begin{aligned} \sum_{l=0}^{N} P_N^l &= \sum_{l=0}^{N}\sum_{k=0}^{N-l}\frac{1}{l!}\frac{(-1)^k}{k!} \\ &= \sum_{l=0}^{N}\sum_{s=l}^{N}\frac{1}{l!}\frac{(-1)^{s-l}}{(s-l)!} \qquad \text{(with } s = l+k\text{)} \\ &= \sum_{s=0}^{N}\sum_{l=0}^{s}\frac{1}{l!}\frac{(-1)^{s-l}}{(s-l)!} \qquad \text{(rewriting the limits in the } s,l \text{ plane)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{s=0}^{N} \frac{1}{s!} \sum_{l=0}^{s} \frac{s!}{l!(s-l)!}(-1)^{s-l} \\
&= \sum_{s=0}^{N} \frac{1}{s!}(1-1)^{s} \\
&= 1 \qquad \text{(only } s=0 \text{ contributes)}. \qquad (17)
\end{aligned}
$$

4. What is the average number, A, of letters in the correct envelopes? If the setup of the problem is repeated many times, then the average number of times a given letter ends up in the correct envelope is $1/N$. Since there are N letters, the average total number of correct envelopes is $N(1/N) = 1$.

   You can check that the expression for P_N^l in eq. (13) leads to $A = 1$. For finite N, the sum gets a little messy, but the result in eq. (17) will help simplify things a bit if you want to work it out.

   For large N, where we have $P_N^l \approx 1/(l!e)$, the sum is easy, and we obtain

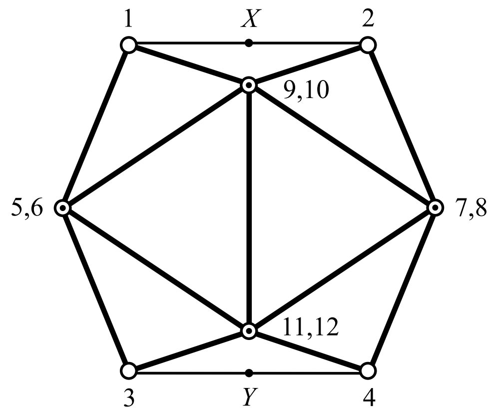
$$
A = \sum_{l=0}^{N} l P_N^l \approx \frac{1}{e} \sum_{l=1}^{N} \frac{1}{(l-1)!} \approx 1. \qquad (18)
$$

Solution

Week 17 (1/6/03)

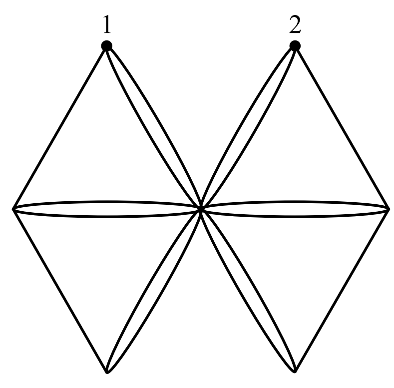
Icosahedron of resistors

First Solution: We will calculate the effective resistance between vertices 1 and 2 in the figure below. When the icosahedron is viewed from the angle shown, four vertices lie directly behind four other vertices; each of these pairs is represented by a dot inside a circle. And thirteen edges lie directly behind thirteen other edges; each of these pairs is represented by a bold line. The remaining two (of the 30 total) edges not represented in the figure are the ones connecting vertices 5 and 6, and 7 and 8. For future reference, points X and Y are defined to be the midpoints of the edges shown.

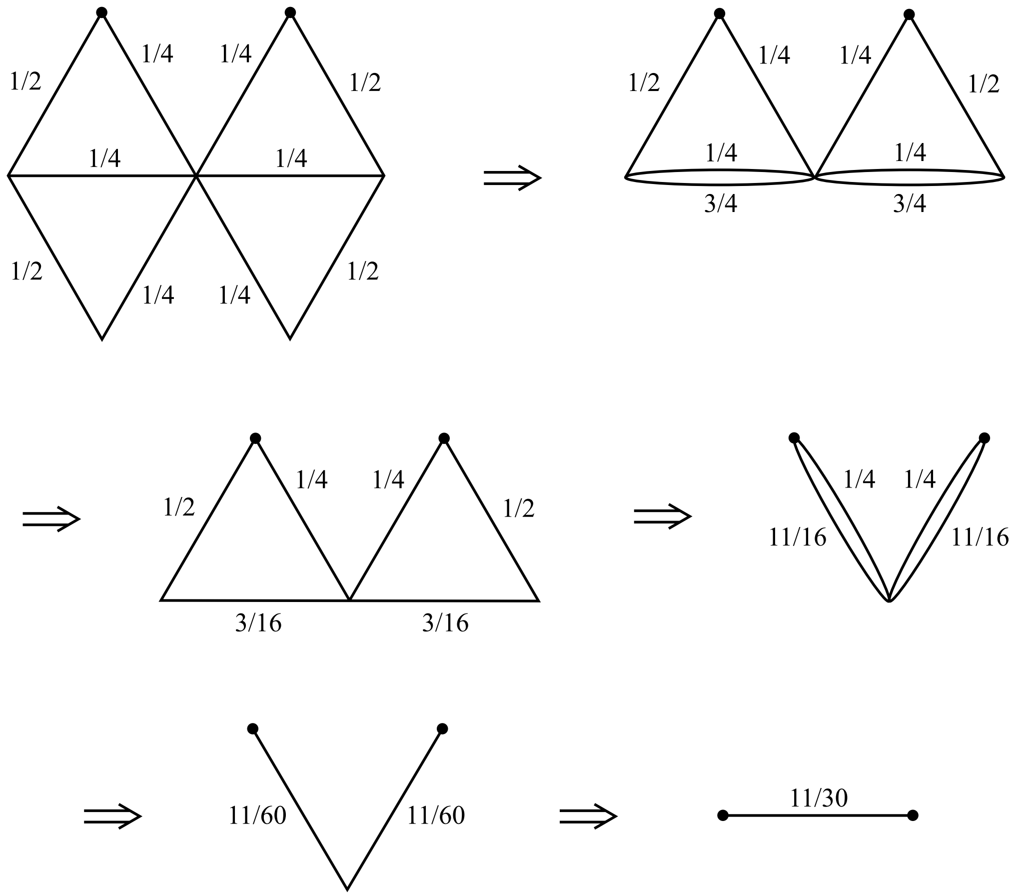


If a potential difference is created between vertices 1 and 2, then the potentials at vertices 5 and 6 are equal, and likewise for the other three pairs of vertices. We may therefore bring each of these four pairs of points together and identify each pair as one point. The resulting circuit is simply the figure above, now planar, where each bold line represents a $(1/2)\Omega$ resistor (because it arises from two 1Ω resistors in parallel).

We now note that all points on the vertical bisector of the circuit (that is, X, Y, the 9-10 pair, and the 11-12 pair) are at equal potentials, so we may bring them all together and identify them as one point. Since X and Y split the top and bottom resistors into two $(1/2)\Omega$ resistors, we arrive at the following circuit, where every line in the figure represents a $(1/2)\Omega$ resistor.



This circuit may be reduced as follows:



The effective resistance between two adjacent vertices is therefore $(11/30)\Omega$.

Second Solution: Let vertices 1 and 2 be adjacent. Consider the setup where a current $1A$ enters through vertex 1, and a current $(1/11)A$ leaves through the other 11 vertices. Note that, due to symmetry, a current $(1/5)A$ flows through each of the 5 edges leaving vertex 1. Hence, the voltage difference between vertices 1 and 2 is

$$V_1 - V_2 = \left(\frac{1}{5}A\right)(1\Omega) = \frac{1}{5}V. \tag{1}$$

Consider a second setup, where a current $1A$ *leaves* through vertex 2, and a current $(1/11)A$ *enters* through the other 11 vertices. Again, note that a current $(1/5)A$ flows through each of the 5 edges entering vertex 2. Hence, the voltage difference between vertices 1 and 2 is

$$V_1 - V_2 = \left(\frac{1}{5}A\right)(1\Omega) = \frac{1}{5}V. \tag{2}$$

If we superimpose these two setups on each other, then we arrive at the setup where:

- A current $(12/11)A$ enters through vertex 1,

- A current $(12/11)A$ leaves through vertex 2,
- No current enters or leaves through the other 10 vertices, and
- The potential difference between vertices 1 and 2 is

$$V_1 - V_2 = \frac{1}{5}V + \frac{1}{5}V = \frac{2}{5}V. \tag{3}$$

We have therefore constructed precisely the experimental setup that serves to determine the effective resistance between vertices 1 and 2. That is, we have put a current in at vertex 1, taken it out at vertex 2, and measured the voltage difference between the two points. The effective resistance between vertices 1 and 2 is therefore given by

$$V = IR \qquad \Longrightarrow \qquad \frac{2}{5}V = \left(\frac{12}{11}A\right) R_{\text{eff}} \qquad \Longrightarrow \qquad R_{\text{eff}} = \frac{11}{30}\Omega. \tag{4}$$

Remark: Note that the sum of the effective resistances across all of the 30 edges in the icosahedron is $11\Omega = (N-1)\Omega$, where N is the number of vertices. You can quickly use the method of the second solution above to calculate the sum of the effective resistances across all of the edges in any of the other regular polyhedra. You will find that the result is always $(N-1)\Omega$. Does this result hold for more general figures? Stay tuned for a future problem of the week...

Solution

Week 18 (2/13/03)

Distribution of primes

A necessary and sufficient condition for N to be prime is that N have no prime factors less than or equal to $\sqrt{N}$. Therefore, under the assumption that a prime p divides N with probability $1/p$, the probability that N is prime is

$$P(N) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\cdots\left(1-\frac{1}{p_{(\sqrt{N})}}\right), \tag{1}$$

where $p_{(\sqrt{N})}$ denotes the largest prime less than or equal to $\sqrt{N}$. Our strategy for solving for $P(N)$ will be to produce a differential equation for it.

Consider $P(N+n)$, where n is an integer that satisfies $\sqrt{N} \ll n \ll N$. We have

$$P(N+n) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\cdots\left(1-\frac{1}{p_{(\sqrt{N+n})}}\right), \tag{2}$$

where $p_{(\sqrt{N+n})}$ denotes the largest prime less than or equal to $\sqrt{N+n}$. Eq. (2) may be written as

$$P(N+n) = P(N)\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p^{(\sqrt{N+n})}}\right), \tag{3}$$

where the p_i are all the primes between $\sqrt{N}$ and $\sqrt{N+n}$. Let there be k of these primes. Since $n \ll N$, we have $\sqrt{N+n}/\sqrt{N} \approx 1$. Therefore, the p_i are multiplicatively all roughly the same. To a good approximation, we may therefore set them all equal to $\sqrt{N}$ in eq. (3). This gives

$$P(N+n) \approx P(N)\left(1-\frac{1}{\sqrt{N}}\right)^k. \tag{4}$$

We must now determine k. The number of numbers between $\sqrt{N}$ and $\sqrt{N+n}$ is

$$\begin{aligned}
\sqrt{N+n}-\sqrt{N} &= \sqrt{N}\sqrt{1+\frac{n}{N}}-\sqrt{N} \\
&\approx \sqrt{N}\left(1+\frac{n}{2N}\right)-\sqrt{N} \\
&= \frac{n}{2\sqrt{N}}\,.
\end{aligned} \tag{5}$$

Each of these numbers has roughly a $P(\sqrt{N})$ chance of being prime. Therefore, there are approximately

$$k \approx \frac{P(\sqrt{N})n}{2\sqrt{N}} \tag{6}$$

prime numbers between $\sqrt{N}$ and $\sqrt{N+n}$.

Since $n \ll N$, we see that $k \ll \sqrt{N}$. Therefore, we may approximate the $(1-1/\sqrt{N})^k$ term in eq. (4) by $1 - k/\sqrt{N}$. Using the value of k from eq. (6), and writing $P(N+n) \approx P(N) + P'(N)n$, we can rewrite eq. (4) as

$$P(N) + P'(N)n \approx P(N)\left(1 - \frac{P(\sqrt{N})n}{2N}\right). \tag{7}$$

We therefore arrive at the differential equation,

$$P'(N) \approx -\frac{P(N)P(\sqrt{N})}{2N}\,. \tag{8}$$

It is easy to check that the solution for P is

$$P(N) \approx \frac{1}{\ln N}\,, \tag{9}$$

as we wanted to show.

Remarks:

1. It turns out (under the assumption that a prime p divides N with probability $1/p$) that the probability that N has exactly n prime factors is

$$P_n(N) \approx \frac{(\ln\ln N)^{n-1}}{(n-1)!\ln N}\,. \tag{10}$$

Our original problem dealt with the case $n = 1$, and eq. (10) does indeed reduce to eq. (9) when $n = 1$. Eq. (10) can be proved by induction on n, but the proof I have is rather messy. If anyone has a clean proof, let me know.

2. We should check that $P_1(N) + P_2(N) + P_3(N) + \cdots = 1$. The sum must equal 1, of course, because every number N has *some* number of divisors. Indeed (letting the sum go to infinity, with negligible error),

$$\begin{aligned}
\sum_{n=1}^{\infty} P_n(N) &= \sum_{n=1}^{\infty} \frac{(\ln\ln N)^{n-1}}{(n-1)!\ln N} \\
&= \frac{1}{\ln N}\sum_{m=0}^{\infty} \frac{(\ln\ln N)^m}{m!} \\
&= \frac{e^{\ln\ln N}}{\ln N} \\
&= 1.
\end{aligned} \tag{11}$$

3. We can also calculate the expected number, $\overline{n}$, of divisors of N. To do this, let's calculate $\overline{n-1}$ (which is a little cleaner), and then add 1.

$$\begin{aligned}
\overline{n-1} &= \sum_{n=1}^{\infty} (n-1)P_n(N) \\
&\approx \sum_{n=2}^{\infty} \frac{(\ln\ln N)^{n-1}}{(n-2)!\ln N} \\
&= \frac{\ln\ln N}{\ln N}\sum_{k=0}^{\infty} \frac{(\ln\ln N)^k}{k!} \\
&= \ln\ln N.
\end{aligned} \tag{12}$$

We can now add 1 to this to obtain $\overline{n}$. However, all our previous results have been calculated to leading order in N, so we have no right to now include an additive term of 1. To leading order in N, we therefore have

$$\overline{n} \approx \ln \ln N. \tag{13}$$

4. There is another way to calculate $\overline{n}$, without using eq. (10). Consider a group of M numbers, all approximately equal to N. The number of prime factors among all of these M numbers (which equals $M\overline{n}$ by definition) is given by[1]

$$M\overline{n} = \frac{M}{2} + \frac{M}{3} + \frac{M}{5} + \frac{M}{7} + \cdots. \tag{14}$$

Since the primes in the denominators occur with frequency $1/\ln x$, this sum may be approximated by the integral,

$$M\overline{n} \approx M \int_1^N \frac{dx}{x \ln x} = M \ln \ln N. \tag{15}$$

Hence, $\overline{n} \approx \ln \ln N$, in agreement with eq. (13).

5. For which n is $P_n(N)$ maximum? Since $P_{n+1}(N) = (\ln \ln N/n) P_n(N)$, we see that increasing n increases $P_n(N)$ if $n < \ln \ln N$. But increasing n decreases $P_n(N)$ if $n > \ln \ln N$. So the maximum $P_n(N)$ is obtained when

$$n \approx \ln \ln N. \tag{16}$$

6. The probability distribution in eq. (10) is a Poisson distribution, for which the results in the previous remarks are well known. A Poisson distribution is what arises in a random process such as throwing a large number of balls into a group of boxes. For the problem at hand, if we take $M(\ln \ln N)$ primes and throw them down onto M numbers (all approximately equal to N), then the distribution of primes (actually, the distribution of primes minus 1) will be (roughly) correct.

[1] We've counted multiple factors of the same prime only once. For example, we've counted 16 as having only one prime factor. To leading order in N, this method of counting gives the same $\overline{n}$ as assigning four prime factors to 16 gives (due to the fact that $\sum(1/p^k)$ converges for $k \geq 2$).

Solution

Week 19 (1/20/03)

Block and bouncing ball

(a) Consider one of the collisions. Let it occur at a distance ℓ from the wall, and let v and V be the speeds of the ball and block, respectively, *after* the collision. We claim that the quantity $\ell(v-V)$ is invariant. That is, it is the same for each collision. This can be seen as follows.

The time to the next collision is given by $Vt + vt = 2\ell$ (because the sum of the distances traveled by the two objects is 2ℓ). Therefore, the next collision occurs at a distance ℓ' from the wall, where

$$\ell' = \ell - Vt = \ell - \frac{2\ell V}{V+v} = \frac{\ell(v-V)}{v+V}\,. \tag{1}$$

Therefore,

$$\ell'(v+V) = \ell(v-V). \tag{2}$$

We now make use of the fact that in an elastic collision, the relative speed before the collision equals the relative speed after the collision. (This is most easily seen by working in the center of mass frame, where this scenario clearly satisfies conservation of E and p.) Therefore, if v' and V' are the speeds after the next collision, then

$$v + V = v' - V'. \tag{3}$$

Using this in eq. (2) gives

$$\ell'(v'-V') = \ell(v-V), \tag{4}$$

as we wanted to show.

What is the value of this invariant? After the first collision, the block continues to move at speed V_0, up to corrections of order m/M. And the ball acquires a speed of $2V_0$, up to corrections of order m/M. (This can be seen by working in the frame of the heavy block, or equivalently by using eq. (3) with $V' \approx V = V_0$ and $v = 0$.) Therefore, the invariant $\ell(v-V)$ is essentially equal to $L(2V_0 - V_0) = LV_0$.

Let L_{min} be the closest distance to the wall. When the block reaches this closest point, its speed is (essentially) zero. Hence, all of the initial kinetic energy of the block now belongs to the ball. Therefore, $v = V_0\sqrt{M/m}$, and our invariant tells us that $LV_0 = L_{\text{min}}(V_0\sqrt{M/m} - 0)$. Thus,

$$L_{\text{min}} = L\sqrt{\frac{m}{M}}\,. \tag{5}$$

(b) (This solution is due to Slava Zhukov)

With the same notation as in part (a), conservation of momentum in a given collision gives

$$MV - mv = MV' + mv'. \tag{6}$$

This equation, along with eq. (3),[1] allows us to solve for V' and v' in terms of V and v. The result, in matrix form, is

$$\begin{pmatrix} V' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{M-m}{M+m} & \frac{-2m}{M+m} \\ \frac{2M}{M+m} & \frac{M-m}{M+m} \end{pmatrix} \begin{pmatrix} V \\ v \end{pmatrix}. \tag{7}$$

The eigenvectors and eigenvalues of this transformation are

$$\begin{aligned} A_1 = \begin{pmatrix} 1 \\ -i\sqrt{\frac{M}{m}} \end{pmatrix}, \qquad \lambda_1 &= \frac{(M-m) + 2i\sqrt{Mm}}{M+m} \equiv e^{i\theta}, \\ A_2 = \begin{pmatrix} 1 \\ i\sqrt{\frac{M}{m}} \end{pmatrix}, \qquad \lambda_2 &= \frac{(M-m) - 2i\sqrt{Mm}}{M+m} \equiv e^{-i\theta}, \end{aligned} \tag{8}$$

where

$$\theta \equiv \arctan\left(\frac{2\sqrt{Mm}}{M-m}\right) \approx 2\sqrt{\frac{m}{M}}\,. \tag{9}$$

The initial conditions are

$$\begin{pmatrix} V \\ v \end{pmatrix} = \begin{pmatrix} V_0 \\ 0 \end{pmatrix} = \frac{V_0}{2}(A_1 + A_2). \tag{10}$$

Therefore, the speeds after the nth bounce are given by

$$\begin{aligned} \begin{pmatrix} V_n \\ v_n \end{pmatrix} &= \frac{V_0}{2}(\lambda_1^n A_1 + \lambda_2^n A_2) \\ &= \frac{V_0}{2}\left(e^{in\theta}\begin{pmatrix} 1 \\ -i\sqrt{\frac{M}{m}} \end{pmatrix} + e^{-in\theta}\begin{pmatrix} 1 \\ i\sqrt{\frac{M}{m}} \end{pmatrix}\right) \\ &= V_0\begin{pmatrix} \cos n\theta \\ \sqrt{\frac{M}{m}}\sin n\theta \end{pmatrix}. \end{aligned} \tag{11}$$

Let the block reach its closest approach to the wall at the Nth bounce. Then $V_N = 0$, and so eq. (11) tells us that $N\theta = \pi/2$. Using the definition of θ from eq. (9), we have

$$\begin{aligned} N &= \frac{\pi/2}{\arctan\frac{2\sqrt{Mm}}{M-m}} \\ &\approx \frac{\pi}{4}\sqrt{\frac{M}{m}}\,. \end{aligned} \tag{12}$$

Remark: This solution is exact, up to the second line in eq. (12), where we finally use $M \gg m$. We can use the first line of eq. (12) to determine the relation between m and M for which the Nth bounce leaves the block exactly at rest at its closest

[1] Alternatively, you could use conservation of energy, but this is a quadratic statement in the velocities, which makes things messy. Conservation of energy is built into the linear eq. (3).

approach to the wall. For this to happen, we need the N in eq. (12) to be an integer. Letting $m/M \equiv r$, we can rewrite eq. (12) as

$$\frac{2\sqrt{r}}{1-r} = \tan\frac{\pi}{2N} \equiv \sqrt{\frac{1-\cos\beta}{1+\cos\beta}}\,, \tag{13}$$

where we have used the tan half-angle formula, with $\beta \equiv \pi/N$. Squaring both sides and solving the resulting quadratic equation for r gives

$$r = \frac{3+\cos\beta - 2\sqrt{2+2\cos\beta}}{1-\cos\beta}\,. \tag{14}$$

If we want the block to come to rest after $N = 1$ bounce, then $\beta = \pi$ gives $r = 1$, which is correct. If we want $N = 2$, then $\beta = \pi/2$ gives $r = 3 - 2\sqrt{2} \approx 0.172$. If we want $N = 3$, then $\beta = \pi/3$ gives $r = 7 - 4\sqrt{3} \approx 0.072$. For general N, eq. (14) must be computed numerically. For large N, the second line in eq. (12) shows that r goes like $1/N^2$. More precisely, $r \approx \pi^2/(16N^2)$.

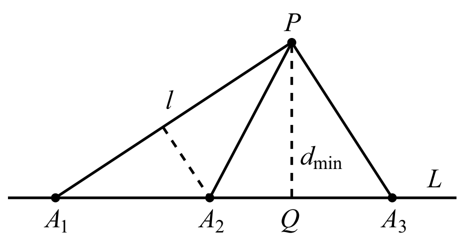
Solution

Week 20 (1/27/03)

Collinear points

Draw all the lines determined by the points. From the assumption of the problem, there are at least three points on each of these lines. Consider all of the distances between any of the points and any of the lines. (Many of these distances are zero, of course, for points lying on a given line.) Since there is a finite number of points and lines, there is a finite number of these distances.

Assume that all the points do not lie on one line, so that some of these distances are nonzero. Then there is a smallest nonzero distance, $d_{\min}$ (it may occur more than once). Consider a point P, and line L, associated with $d_{\min}$, as shown.



Let Q be the projection of point P onto L. Since L contains at least three points, at least two of them must lie on the same side of Q (or one may coincide with Q). Call these points A_1 and A_2. Let ℓ be the line through P and A_1. Then the distance from A_2 to ℓ is less than $d_{\min}$ (because this distance is less than or equal to the distance from Q to ℓ, which is strictly less than the distance from Q to P). But this contradicts our assumption that $d_{\min}$ was the smallest nonzero distance. Hence, there can be no smallest nonzero distance. Therefore, all the distances are zero, and all the points must lie on one line.

Solution

Week 21 (2/3/03)

Ball on turntable

Let the angular velocity of the turntable be $\Omega\hat{\mathbf{z}}$, and let the angular velocity of the ball be $\boldsymbol{\omega}$. If the ball is at position $\mathbf{r}$ (with respect to the lab frame), then its velocity (with respect to the lab frame) may be broken up into the velocity of the turntable (at position $\mathbf{r}$) plus the ball's velocity with respect to the turntable. The non-slipping condition says that this latter velocity is given by $\boldsymbol{\omega} \times (a\hat{\mathbf{z}})$, where a is the radius of the ball. The ball's velocity with respect to the lab frame is therefore

$$\mathbf{v} = (\Omega\hat{\mathbf{z}}) \times \mathbf{r} + \boldsymbol{\omega} \times (a\hat{\mathbf{z}}). \tag{1}$$

The important point to realize in this problem is that the friction force from the turntable is responsible for changing both the ball's linear momentum and its angular momentum. In particular, $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}\,. \tag{2}$$

And the angular momentum of the ball is $\mathbf{L} = I\boldsymbol{\omega}$, so $\boldsymbol{\tau} = d\mathbf{L}/dt$ (relative to the center of the ball) gives

$$(-a\hat{\mathbf{z}}) \times \mathbf{F} = I\frac{d\boldsymbol{\omega}}{dt}\,, \tag{3}$$

because the force is applied at position $-a\hat{\mathbf{z}}$ relative to the center.

We will now use the previous three equations to demonstrate that the ball undergoes circular motion. Our goal will be to produce an equation of the form,

$$\frac{d\mathbf{v}}{dt} = \Omega'\hat{\mathbf{z}} \times \mathbf{v}, \tag{4}$$

since this describes circular motion, with frequency Ω' (to be determined). Plugging the expression for $\mathbf{F}$ from eq. (2) into eq. (3) gives

$$\begin{aligned} (-a\hat{\mathbf{z}}) \times \left(m\frac{d\mathbf{v}}{dt}\right) &= I\frac{d\boldsymbol{\omega}}{dt} \\ \Longrightarrow \qquad \frac{d\boldsymbol{\omega}}{dt} &= -\left(\frac{am}{I}\right)\hat{\mathbf{z}} \times \frac{d\mathbf{v}}{dt}\,. \end{aligned} \tag{5}$$

Taking the derivative of eq. (1) gives

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}} \times \frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times (a\hat{\mathbf{z}}) \\ &= \Omega\hat{\mathbf{z}} \times \mathbf{v} - \left(\left(\frac{am}{I}\right)\hat{\mathbf{z}} \times \frac{d\mathbf{v}}{dt}\right) \times (a\hat{\mathbf{z}}). \end{aligned} \tag{6}$$

Since the vector $d\mathbf{v}/dt$ lies in the horizontal plane, it is easy to work out the cross-product in the right term (or just use the identity $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$) to obtain

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}} \times \mathbf{v} - \left(\frac{ma^2}{I}\right)\frac{d\mathbf{v}}{dt} \\ \Longrightarrow \qquad \frac{d\mathbf{v}}{dt} &= \left(\frac{\Omega}{1 + (ma^2/I)}\right)\hat{\mathbf{z}} \times \mathbf{v}. \end{aligned} \tag{7}$$

For a uniform sphere, $I = (2/5)ma^2$, so we obtain

$$\frac{d\mathbf{v}}{dt} = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}} \times \mathbf{v}. \tag{8}$$

Therefore, in view of eq. (4), we see that the ball undergoes circular motion, with a frequency equal to 2/7 times the frequency of the turntable. This result for the frequency does not depend on initial conditions.

REMARKS:

1. Integrating eq. (8) from the initial time to some later time gives

$$\mathbf{v} - \mathbf{v}_0 = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}_0). \tag{9}$$

This may be written (as you can show) in the more suggestive form,

$$\mathbf{v} = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}} \times \left(\mathbf{r} - \left(\mathbf{r}_0 + \frac{7}{2\Omega}(\hat{\mathbf{z}} \times \mathbf{v}_0)\right)\right). \tag{10}$$

This equation describes circular motion, with the center located at the point,

$$\mathbf{r}_c = \mathbf{r}_0 + \frac{7}{2\Omega}(\hat{\mathbf{z}} \times \mathbf{v}_0), \tag{11}$$

and with radius,

$$R = |\mathbf{r}_0 - \mathbf{r}_c| = \frac{7}{2\Omega}|\hat{\mathbf{z}} \times \mathbf{v}_0| = \frac{7v_0}{2\Omega}. \tag{12}$$

2. There are a few special cases to consider:

   - If $v_0 = 0$ (that is, if the spinning motion of the ball exactly cancels the rotational motion of the turntable), then $R = 0$ and the ball remains in the same place (of course).
   - If the ball is initially not spinning, and just moving along with the turntable, then $v_0 = \Omega r_0$. The radius of the circle is therefore $R = (7/2)r_0$, and its center is located at (from eq. (11))

   $$\mathbf{r}_c = \mathbf{r}_0 + \frac{7}{2\Omega}(-\Omega\mathbf{r}_0) = -\frac{5\mathbf{r}_0}{2}. \tag{13}$$

   The point on the circle diametrically opposite to the initial point is therefore at a radius $r_c + R = (5/2)r_0 + (7/2)r_0 = 6r_0$.
   - If we want the center of the circle be the center of the turntable, then eq. (11) says that we need $(7/2\Omega)\hat{\mathbf{z}} \times \mathbf{v}_0 = -\mathbf{r}_0$. This implies that $\mathbf{v}_0$ has magnitude $v_0 = (2/7)\Omega r_0$ and points tangentially in the same direction as the turntable moves. (That is, the ball moves at 2/7 times the velocity of the turntable beneath it.)

3. The fact that the frequency $(2/7)\Omega$ is a rational multiple of Ω means that the ball will eventually return to the same point on the turntable. In the lab frame, the ball will trace out two circles in the time it takes the turntable to undergo seven revolutions. From the point of view of someone on the turntable, the ball will "spiral" around five times before returning to the original position.

4. If we look at a ball with moment of inertia $I = \eta ma^2$ (so a uniform sphere has $\eta = 2/5$), then you can show that the "2/7" in eq. (8) gets replaced by "$\eta/(1+\eta)$". If a ball has most of its mass concentrated at its center (so that $\eta \to 0$), then the frequency of the circular motion goes to 0, and the radius goes to ∞ (as long as $v_0 \neq 0$).

Solution

Week 22 (2/10/03)

Trading envelopes

(a) Let your envelope contain N dollars. Then the other envelope contains either $2N$ or $N/2$ dollars. If you switch, the expected value of your assets is $\frac{1}{2}(2N)+\frac{1}{2}(N/2)=5N/4$. This is greater than N. Therefore, you should switch.

(b) There are (at least) two possible modes of reasoning, yielding different results:

- It seems that we should be able to use the same reasoning as in part (a). If you have N dollars in your envelope, then the other one has either $2N$ or $N/2$. Since you had a 50-50 chance of picking either envelope, the other envelope should have a 50-50 chance of containing $2N$ or $N/2$ dollars. If you switch, there is a $1/2$ chance you win N dollars, and a $1/2$ chance you lose $N/2$ dollars. Therefore, the expectation value for your gain is $N/4$ dollars. So you should switch.
- If the correct strategy is to switch (that is, if there is an average gain from trading), then if person A picks one envelope and person B picks the other, then they are both better off if they switch. This cannot be true. Likewise, it cannot be true that they are both better off if they do not switch. Therefore, it doesn't matter whether or not they switch.

The second reasoning is correct. The flaw in the first reasoning is that the other envelope does *not* have a 50-50 chance of containing $2N$ or $N/2$ dollars. Such a 50-50 distribution would yield a zero probability of the envelopes containing a finite and nonzero quantity (as we'll explain below). In a nutshell, it is incorrect to assume that because you have a 50-50 chance of picking each envelope, the envelope you don't pick has a 50-50 chance of having twice or half the amount in your envelope.

(c) As we have stated, the fundamental difference between the scenarios in parts (a) and (b) is that the second envelope in scenario (b) does not have a 50-50 chance of containing half or twice the amount in your envelope. Let's look at this further.

Consider the following slightly modified game, which has all the essentials of the original one. Consider a game where powers of 2 (positive, negative, or zero) are the only numbers of dollars allowed in the envelopes.

The fact that in scenario (b) there is not a 50-50 chance that the other envelope has $2N$ or $N/2$ dollars is most easily seen by looking at the simplest distribution of money in the envelopes, the case where only two quantities are used. Let's say that I always put \$4 in one envelope and \$8 in the other. (And assume that you have a bad memory and can't remember anything from one game to the next.) If your strategy is to switch, and if you initially have \$4, then you will definitely win \$4 on the switch. And if you initially have \$8, then you will definitely lose \$4 on the switch. Since you have a 50-50 chance of starting with the \$4 or \$8 envelope, you will on average neither win nor lose any money. In this example, it is clear that if you have, for example, \$4, there is *not* a 50-50 chance that the other envelope contains \$2 or \$8. Rather, there is a 100% chance that it contains \$8.

You can try make a situation in scenario (b) that comes "close" to always having a 50-50 chance that the other envelope has twice or half the amount in your envelope. For example, let there be a $1/n$ chance that the envelopes contain 2^k and 2^{k+1} dollars, for all k from 1 to n. Then indeed if there are 2^m dollars in your envelope, for $m = 2, \dots, n-1$, then there is a 50-50 chance that the other envelope has twice or half that amount. In all these $n-2$ cases, you will win money, on average, if you switch. And you will certainly win money if you switch in the case where you have the minimum amount, 2^1 dollars. You will, however, lose a great deal of money if you happen to start out with 2^{n+1} dollars. This only happens $1/(2n)$ of the time, but it in fact precisely cancels, on average, the winnings from all the other $n-1$ cases (as you can show). Therefore, it doesn't matter if you switch.

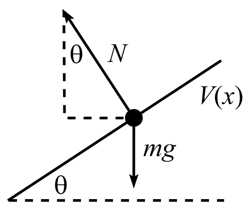
If you want to produce a 50-50 chance that the other envelope has twice or half the amount in your envelope for all m, then you have to assign equal probabilities to all of the $(2^k, 2^{k+1})$ pairs, for $-\infty < k < \infty$. But the assignment of equal probabilities to an infinite set requires that all of these probabilities are zero, which means that there is a zero chance of putting a finite amount of money in the envelopes. Since it's stated that there *is* some amount of money in the envelopes, we conclude that all the probabilities of the $(2^k, 2^{k+1})$ pairs are not equal. The setup in part (b) is therefore not the same as in part (a), and there is no paradox.

Solution

Week 23 (2/17/03)

$V(x)$ versus a hill

Quick solution: Consider the normal force, N, acting on the bead at a given point. Let θ be the angle that the tangent to $V(x)$ makes with the horizontal, as shown.



The horizontal $F = ma$ equation is

$$-N \sin\theta = m\ddot{x}. \tag{1}$$

The vertical $F = ma$ equation is

$$N\cos\theta - mg = m\ddot{y} \qquad \Longrightarrow \qquad N\cos\theta = mg + m\ddot{y}. \tag{2}$$

Dividing eq. (1) by eq. (2) gives

$$-\tan\theta = \frac{\ddot{x}}{g+\ddot{y}}\,. \tag{3}$$

But $\tan\theta = V'(x)$. Therefore,

$$\ddot{x} = -(g+\ddot{y})V'. \tag{4}$$

We see that this is not equal to $-gV'$. In fact, there is in general no way to construct a curve with height $y(x)$ which gives the same horizontal motion that a 1-D potential $V(x)$ gives, for all initial conditions. We would need $(g+\ddot{y})y' = V'$, for all x. But at a given x, the quantities V' and y' are fixed, whereas $\ddot{y}$ depends on the initial conditions. For example, if there is a bend in the wire, then $\ddot{y}$ will be large if $\dot{y}$ is large. And $\dot{y}$ depends (in general) on how far the bead has fallen.

Eq. (4) holds the key to constructing a situation that does give the $\ddot{x} = -gV'$ result for a 1-D potential $V(x)$. All we have to do is get rid of the $\ddot{y}$ term. So here's what we do. We grab our $y = V(x)$ wire and then move it up (and/or down) in precisely the manner that makes the bead stay at the same height with respect to the ground. (Actually, constant vertical speed would be good enough.) This will make the $\ddot{y}$ term vanish, as desired. (Note that the vertical movement of the curve doesn't change the slope, V', at a given value of x.)

Remark: There is one case where $\ddot{x}$ is (approximately) equal to $-gV'$, even when the wire remains stationary. In the case of small oscillations of the bead near a minimum of $V(x)$, $\ddot{y}$ is small compared to g. Hence, eq. (4) shows that $\ddot{x}$ is approximately equal to

$-gV'$. Therefore, for small oscillations, it is reasonable to model a particle in a 1-D potential $mgV(x)$ as a particle sliding in a valley whose height is given by $y = V(x)$.

Long solution: The component of gravity along the wire is what causes the change in speed of the bead. That is,

$$-g\sin\theta = \frac{dv}{dt}\,, \tag{5}$$

where θ is given by

$$\tan\theta = V'(x) \qquad \Longrightarrow \qquad \sin\theta = \frac{V'}{\sqrt{1+V'^2}}\,, \qquad \cos\theta = \frac{1}{\sqrt{1+V'^2}}\,. \tag{6}$$

We are, however, not concerned with the rate of change of v, but rather with the rate of change of $\dot{x}$. In view of this, let us write v in terms of $\dot{x}$. Since $\dot{x} = v\cos\theta$, we have $v = \dot{x}/\cos\theta = \dot{x}\sqrt{1+V'^2}$. (Dots denote d/dt. Primes denote d/dx.) Therefore, eq. (5) becomes

$$\begin{aligned}\frac{-gV'}{\sqrt{1+V'^2}} &= \frac{d}{dt}\Big(\dot{x}\sqrt{1+V'^2}\Big) \\ &= \ddot{x}\sqrt{1+V'^2} + \frac{\dot{x}V'(dV'/dt)}{\sqrt{1+V'^2}}\,.\end{aligned} \tag{7}$$

Hence, $\ddot{x}$ is given by

$$\ddot{x} = \frac{-gV'}{1+V'^2} - \frac{\dot{x}V'(dV'/dt)}{1+V'^2}\,. \tag{8}$$

We'll simplify this in a moment, but first a remark.

REMARK: A common incorrect solution to this problem is the following. The acceleration along the curve is $g\sin\theta = -g(V'/\sqrt{1+V'^2})$. Calculating the horizontal component of this acceleration brings in a factor of $\cos\theta = 1/\sqrt{1+V'^2}$. Therefore, we might think that

$$\ddot{x} = \frac{-gV'}{1+V'^2}\,. \tag{9}$$

But we have missed the second term in eq. (8). Where is the mistake? The error is that we forgot to take into account the possible change in the curve's slope. (Eq. (9) is true for straight lines.) We addressed only the acceleration due to a change in *speed*. We forgot about the acceleration due to a change in the *direction* of motion. (The term we missed is the one with dV'/dt.) Intuitively, if we have sharp enough bend in the wire, then $\dot{x}$ can change at an arbitrarily large rate. In view of this fact, eq. (9) is definitely incorrect, because it is bounded (by $g/2$, in fact).

To simplify eq. (8), note that $V' \equiv dV/dx = (dV/dt)/(dx/dt) \equiv \dot{V}/\dot{x}$. Therefore,

$$\begin{aligned}\dot{x}V'\frac{dV'}{dt} &= \dot{x}V'\frac{d}{dt}\left(\frac{\dot{V}}{\dot{x}}\right) \\ &= \dot{x}V'\left(\frac{\dot{x}\ddot{V} - \dot{V}\ddot{x}}{\dot{x}^2}\right) \\ &= V'\ddot{V} - V'\ddot{x}\left(\frac{\dot{V}}{\dot{x}}\right) \\ &= V'\ddot{V} - V'^2\ddot{x}.\end{aligned} \tag{10}$$

Substituting this into eq. (8), we obtain

$$\ddot{x} = -(g + \ddot{V})V', \tag{11}$$

in agreement with eq. (4), since $y(x) = V(x)$.

Eq. (11) is valid for a curve $V(x)$ that remains fixed. If we grab the wire and start moving it up and down, then the above solution is invalid, because the starting point, eq. (5), rests on the assumption that gravity is the only force that does work on the bead. But if we move the wire, then the normal force also does work.

It turns out that for a moving wire, we simply need to replace the $\ddot{V}$ in eq. (11) by $\ddot{y}$. This can be seen by looking at things in the (instantaneously inertial) vertically-moving frame in which the wire is at rest. In this new frame, the normal force does no work, so the above solution is valid. And in this new frame, $\ddot{y} = \ddot{V}$. Eq. (11) therefore becomes $\ddot{x} = -(g + \ddot{y})V'$. Shifting back to the lab frame (which moves at constant speed with respect to the instantaneous inertial frame of the wire) doesn't change $\ddot{y}$. We thus arrive at eq. (4), valid for a stationary or vertically moving wire.

Solution

Week 24 (2/24/03)

Verifying weights

(a) Each weight may be used in three different ways. It may be put on the left side, the right side, or not used at all. Therefore, if we have n weights, they may be combined in 3^n ways. This is true because in adding the weights, there are three choices for the coefficient of each weight in the sum: a plus sign if it's on the left, a minus sign if it's on the right, and a zero if it's not used at all.

There are, however, duplicates among these 3^n combinations. For every positive number, there is its negative (where the left and right scales are simply reversed), which represents the same weight on the balance scale. Since it is, in principle, possible for the number 0 to not be repeated, an upper bound on the number of positive integer weights that may be weighed with n fixed weights is $(3^n - 1)/2$. Therefore, to weigh all weights up to 121, we must have $n \geq 5$.

We claim that 5 weights are sufficient. For 5 weights to do the job, we must have no duplications. Therefore, the m smallest fixed weights must yield all values up to $(3^m - 1)/2$. Hence, the first weight must be 1 (to get values up to 1), the second weight must be 3 (to get values up to 4), the third weight must be 9 (to get values up to 13), the fourth weight must be 27 (to get values up to 40), and the fifth weight must be 81 (to get values up to 121). The fixed weights are therefore powers of 3.

(b) In general, the n weights of 1, 3, 9, ..., 3^{n-1} may be used to verify all integral weights from 1 to $W_n \equiv (3^n - 1)/2$. This may be prove by induction on n, as follows.

Assume that all weights up to $W = (3^n - 1)/2$ can be verified with n fixed weights. If the next fixed weight is chosen to be 3^n, then we can additionally verify all weights from $3^n - W_n$ to $3^n + W_n$. Using the above form of W_n, this range can be rewritten as $(3^n + 1)/2$ to $(3^{n+1} - 1)/2$. But this is exactly the range needed to be able to verify any weight up to $W_{n+1} = (3^{n+1} - 1)/2$.

Therefore, we have shown that if we can verify up to W_n by using n fixed weights, then we can verify up to W_{n+1} by using $n+1$ fixed weights. Since we can clearly verify up to $W_1 = 1$ by using one weight, the result holds for all n.

$W_n \equiv (3^n - 1)/2$ is the largest number for which this result holds, by the upper-bound argument in part (a). Note that the inductive process shows that all the fixed weights must be powers of 3.

Remarks:

(a) To rephrase the inductive argument in part (b), powers of 3 have the relevant property that $(1 + 3 + 3^2 + \cdots 3^{n-1}) + 1 = 3^n - (1 + 3 + 3^2 + \cdots 3^{n-1})$. The importance of this can be seen by looking at, for example, $n = 4$. We see that 41, which is not quite obtainable with four weights, can be obtained by taking the fifth weight, 81, and subtracting off the highest possible sum with four weights, namely 40.

(b) Consider the following question: You wish to pick n pairs of equal weights (for example, if $n = 3$, you might pick weights of $1, 1, 4, 4, 11, 11$) such that you can verify any positive integer weight up to W. How should you choose the weights in order to maximize W?

Along the lines of the reasoning in part (a), we observe that each pair of weights may be used in five ways: We may put (1) both on the left side, (2) one on the left and none on the right, (3) one on each side or use neither, (4) one on the right and none on the left, or (5) both on the right.

Therefore, the weights may be used in 5^n ways. As above, however, there are duplicates among these 5^n combinations. For every positive number, there is its negative, which represents the same weight on the balance scale. Since it is, in principle, possible for the number 0 to not be repeated, an upper bound on the number of positive integer weights that may be weighed with n pairs of fixed weights is $(5^n - 1)/2$.

It is indeed possible to achieve $W = (5^n - 1)/2$, by choosing the weights to be powers of 5. The weights should be $1, 1, 5, 5, 25, 25, \ldots$. Powers of 5 have the relevant property that $2(1+5+5^2+\cdots 5^{n-1})+1 = 5^n - 2(1+5+5^2+\cdots 5^{n-1})$. For example, if $n = 3$. We see that 63, which is not quite obtainable with three pairs of weights, can be obtained by taking a weight from the fourth pair, namely 125, and subtracting off the highest possible sum obtainable with three pairs, namely 62.

In general, if we use n k-tuplets of weights, it is possible to verify any positive integer weight up to $[(2k+1)^n - 1]/2$. The weights should be powers of $(2k+1)$.

Solution

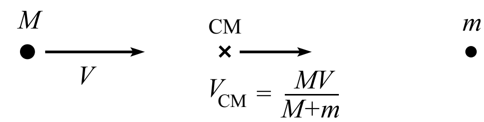
Week 25 (3/3/03)

Maximum deflection angle

First Solution: Although it is possible to solve this problem by working in the lab frame (see the second solution below), it is much easier to make use of the center-of-mass frame. Let M have initial speed V in the lab frame. Then the CM moves with speed

$$V_{\text{CM}} = \frac{MV}{M+m}\,, \tag{1}$$

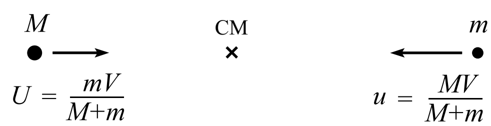
as shown.



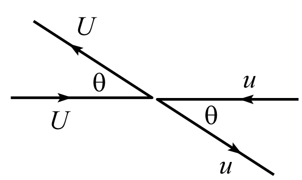
The speeds of the masses in the CM frame are therefore equal to

$$U = V - V_{\text{CM}} = \frac{mV}{M+m}\,, \qquad \text{and} \qquad u = |-V_{\text{CM}}| = \frac{MV}{M+m}\,, \tag{2}$$

as shown.

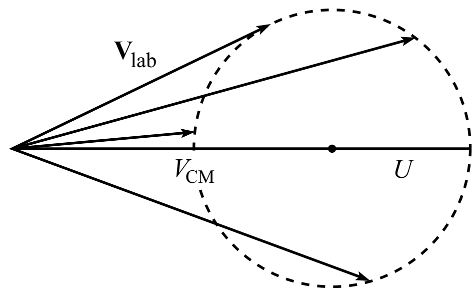


In the CM frame, the collision is simple. The particles keep the same speeds, but simply change their directions (while still moving in opposite directions), as shown.

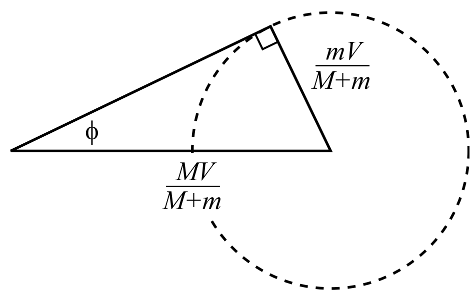


The angle θ is free to have any value. This scenario clearly satisfies conservation of energy and momentum; therefore, it is what happens.

The important point to note is that since θ can have any value, the tip of the $\mathbf{U}$ velocity vector can be located anywhere on a circle of radius U. If we then shift back to the lab frame, we see that the final velocity of M with respect to the lab frame, $\mathbf{V}_{\text{lab}}$, is obtained by adding $\mathbf{V}_{\text{CM}}$ to the vector $\mathbf{U}$ (which can point anywhere on the dotted circle below). A few possibilities for $\mathbf{V}_{\text{lab}}$ are shown.



The largest angle of deflection is obtained when $\mathbf{V}_{\text{lab}}$ is tangent to the dotted circle, in which case we have the following situation.



The maximum angle of deflection, ϕ, is therefore given by

$$\sin\phi = \frac{U}{V_{\text{CM}}} = \frac{\frac{mV}{M+m}}{\frac{MV}{M+m}} = \frac{m}{M}\,. \tag{3}$$

Second Solution: Let V' and v' be the final speeds, and let ϕ and γ be the scattering angles of M and m, respectively, in the lab frame. Then conservation of p_x, p_y, and E give

$$\begin{aligned} MV &= MV'\cos\phi + mv'\cos\gamma, && (4)\\ 0 &= MV'\sin\phi - mv'\sin\gamma, && (5)\\ \frac{1}{2}MV^2 &= \frac{1}{2}MV'^2 + \frac{1}{2}mv'^2. && (6) \end{aligned}$$

Putting the ϕ terms on the left-hand sides of eqs. (4) and (5), and then squaring and adding these equations, gives

$$M^2(V^2 + V'^2 - 2VV'\cos\phi) = m^2v'^2. \tag{7}$$

Equating this expression for $m^2v'^2$ with the one obtained by multiplying eq. (6) through by m gives

$$\begin{aligned} M(V^2 + V'^2 - 2VV'\cos\phi) &= m(V^2 - V'^2) \\ \Longrightarrow \quad (M+m)V'^2 - (2MV\cos\phi)V' + (M-m)V^2 &= 0. && (8) \end{aligned}$$

A solution to this quadratic equation in V' exists if and only if the discriminant is non-negative. Therefore, we must have

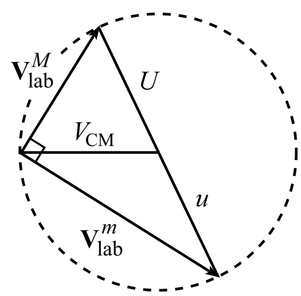
$$\begin{aligned}
(2MV\cos\phi)^2 - 4(M+m)(M-m)V^2 &\geq 0 \\
\Longrightarrow \qquad m^2 &\geq M^2(1-\cos^2\phi) \\
\Longrightarrow \qquad m^2 &\geq M^2\sin^2\phi \\
\Longrightarrow \qquad \frac{m}{M} &\geq \sin\phi. \qquad (9)
\end{aligned}$$

REMARKS: If $M < m$, then eq. (9) says that any value of ϕ is possible. In particular, it it possible for M to bounce directly backwards. In the language of the first solution above, if $M < m$ then $V_{\rm CM} < U$, so the dotted circle passes to the left of the left vertex of the triangle. This means that ϕ can take on any value.

The method of the first solution provides an easy way to demonstrate the result that if the two masses are equal, then they always scatter at a relative angle of 90° (a familiar result in billiards). If $M = m$, then

$$u = U = V_{\rm CM} = \frac{MV}{M+M} = \frac{V}{2}\,. \qquad (10)$$

Therefore, the $\mathbf{u}$ and $\mathbf{U}$ vectors in the figure below form a diameter of the dotted circle, which means that the final velocities of M and m in the lab frame are perpendicular.



Solution

Week 26 (3/10/03)

Drunken walk

(a) **First Solution:** Imagine a large number of copies of the given setup proceeding simultaneously. After each drunk takes his first step in all of the copies, the average position of all of them remains the same (namely, n steps from the river), because each one had a 50-50 chance of moving either way. Likewise, the average position remains unchanged after each successive step. This is true because the drunks who are still moving won't change the average position (because of their random motion), and the drunks who have stopped at an end of the street certainly won't change the average position (because they aren't moving). Therefore, the average position is always n steps from the river.

Let the drunks keep moving until all of them have stopped at either end. Let $P_r(n)$ and $P_p(n)$ be the probabilities of ending up at the river and police station, respectively, having started n steps from the river. Then after all the drunks have stopped, their average distance from the river is $0 \cdot P_r(n) + N \cdot P_p(n)$. But this must equal n. Hence, $P_p(n) = n/N$, and so $P_r(N) = 1 - (n/N)$.

Second Solution: Let the river and police station be located at positions 0 and N, respectively. Let $P_r(n)$ be the probability of ending up at the river, given a present position of n. Since after one step the drunk is equally likely to be at $n-1$ and $n+1$, we must have

$$P_r(n) = \frac{1}{2}P_r(n-1) + \frac{1}{2}P_r(n+1). \tag{1}$$

Therefore, $P_r(n)$ is a linear function of n. Invoking the requirements that $P_r(0) = 1$ and $P_r(N) = 0$, we find $P_r(n) = 1 - n/N$. The probability of ending up at the police station, $P_p(n)$, is then $P_p(n) = 1 - P_r(n) = n/N$.

(b) Let $g(k)$ be the expected number of steps it takes to reach an end of the street, having started k steps from the river. After one step from position k, there is a $1/2$ chance of being at position $k-1$, and a $1/2$ chance of being at position $k+1$. Therefore, the $g(k)$ are related by

$$g(k) = \frac{1}{2}g(k-1) + \frac{1}{2}g(k+1) + 1, \tag{2}$$

where $g(0) = g(N) = 0$. We must now solve this recursion relation.

Multiplying through by 2, and then summing all the eqs. (2) for values of k from 1 to m gives

$$\begin{aligned} g(1) + g(m) &= g(m+1) + 2m \\ \Longrightarrow \qquad g(m+1) &= g(1) + g(m) - 2m. \end{aligned} \tag{3}$$

Note that if we set $m = N - 1$, we obtain $0 = g(1) + g(N-1) - 2(N-1)$. Since $g(1) = g(N-1)$ by symmetry, we find $g(1) = g(N-1) = N-1$.

Summing all the eqs. (3) for values of m from 1 to $n-1$ gives

$$\begin{aligned} g(n) &= n \cdot g(1) - 2\sum_{1}^{n-1} m \\ &= n \cdot g(1) - n(n-1). \end{aligned} \tag{4}$$

Using $g(1) = N - 1$, we find

$$\begin{aligned} g(n) &= n(N-1) - n(n-1) \\ &= n(N-n). \end{aligned} \tag{5}$$

This can be written as

$$g(n) = \left(\frac{N}{2}\right)^2 - \left(\frac{N}{2} - n\right)^2, \tag{6}$$

which is just an inverted parabola.

Solution

Week 27 (3/17/03)

Relativistic cookies

Let the diameter of the cookie cutter be L, and consider the two following reasonings.

- In the lab frame, the dough is length-contracted, so the diameter L corresponds to a distance larger than L (namely γL) in the dough's frame. Therefore, when you buy a cookie, it is stretched out by a factor γ in the direction of the belt.[1]
- In the frame of the dough, the cookie cutter is length-contracted in the direction of motion. It has length L/γ. So in the frame of the dough, the cookies have a length of only L/γ. Therefore, when you buy a cookie, it is squashed by a factor γ in the direction of the belt.

Which reasoning is correct? The first one is. The cookies are stretched out. The fallacy in the second reasoning is that the various parts of the cookie cutter do *not* strike the dough simultaneously in the dough frame. What the dough sees is this: The cutter moves to, say, the left. The right side of the cutter stamps the dough, then nearby parts of the cutter stamp it, and so on, until finally the left side of the cutter stamps the dough. But by this time the front (that is, the left) of the cutter has moved farther to the left. So the cookie turns out to be longer than L. It takes a little work to demonstrate that the length is actually γL, but let's do that now (by working in the dough frame).

Consider the moment when the the rightmost point of the cutter strikes the dough. In the dough frame, a clock at the rear (the right side) of the cutter reads Lv/c^2 more than a clock at the front (the left side). The front clock must therefore advance by Lv/c^2 by the time it strikes the dough. (This is true because all points on the cutter strike the dough simultaneously in the cutter frame. Hence, all cutter clocks read the same when they strike.) But due to time dilation, this takes a time $\gamma(Lv/c^2)$ in the dough frame. During this time, the cutter travels a distance $v(\gamma Lv/c^2)$. Since the front of the cutter was initially a distance L/γ (due to length contraction) ahead of the back, the total length of the cookie in the dough frame equals

$$\begin{aligned}\ell &= \frac{L}{\gamma} + v\left(\frac{\gamma L v}{c^2}\right) \\ &= \gamma L\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) \\ &= \gamma L\left(\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right) \\ &= \gamma L,\end{aligned}$$

as we wanted to show.

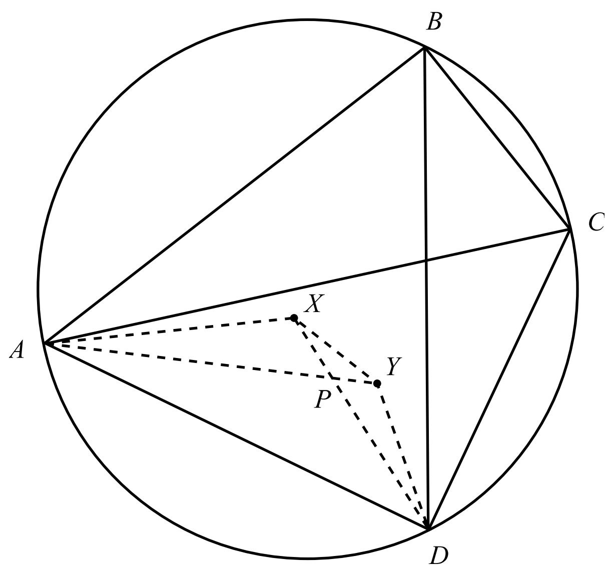
[1] The shape is an ellipse, since that's what a stretched-out circle is. The eccentricity of an ellipse is the focal distance divided by the semi-major axis length. As an exercise, you can show that this equals $\beta \equiv v/c$ here.

Solution

Week 28 (3/24/03)

Rectangle in a circle

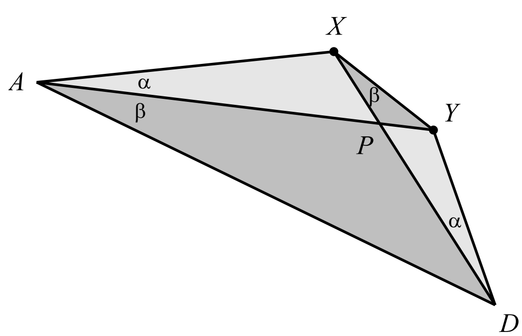
In the figure below, let the incenters of triangles ADB and ADC be X and Y, respectively.



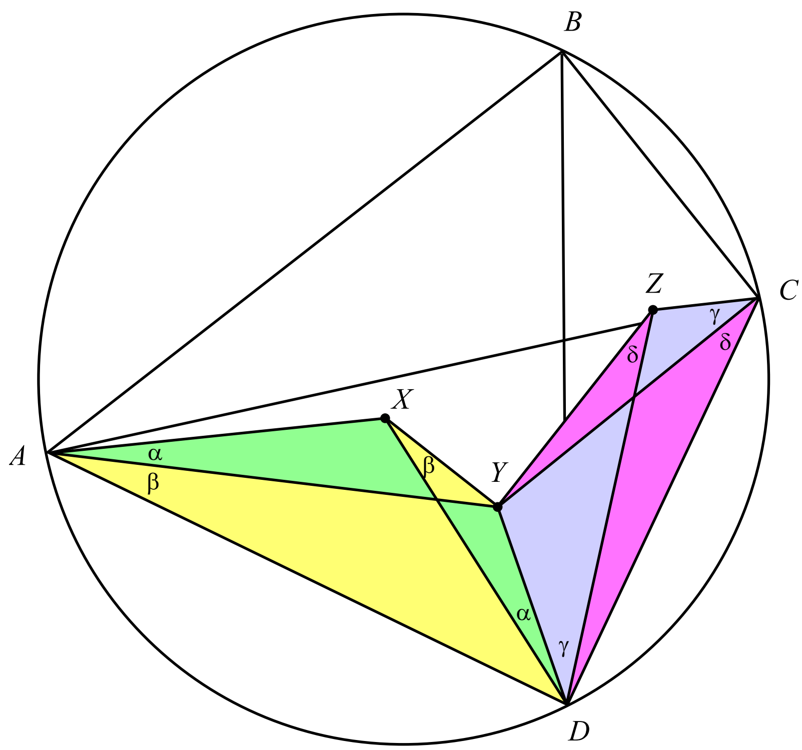
Angle $\angle XAY$ can be written as

$$\begin{aligned}\angle XAY &= \angle XAD - \angle YAD \\ &= \frac{1}{2}\angle BAD - \frac{1}{2}\angle CAD \\ &= \frac{1}{2}\angle BAC \\ &= \frac{1}{4}(\widehat{BC}).\end{aligned}$$

A similar argument (with A, B, X interchanged with D, C, Y) shows that angle $\angle YDX$ also equals $(1/4)(\widehat{BC})$. This equality of angles $\angle XAY$ and $\angle YDX$ implies that triangles XAP and YDP are similar. This in turn implies that triangles PXY and PAD are similar. Therefore, $\angle PXY = \angle PAD$. These results may be summarized in the following figure.



We may now repeat the above procedure with the incenters (Y and Z) of triangles DCA and DCB. The result is two more pairs of equal angles, as shown.



The four angles shown have the values,

$$\begin{aligned}
\alpha &= (1/4)(\widehat{BC}), \\
\beta &= (1/2)\angle CAD = (1/4)(\widehat{CD}), \\
\gamma &= (1/4)(\widehat{AB}), \\
\delta &= (1/2)\angle ACD = (1/4)(\widehat{AD}).
\end{aligned}$$

Therefore,

$$\alpha + \beta + \gamma + \delta = \frac{1}{4}(\widehat{BC} + \widehat{CD} + \widehat{AB} + \widehat{AD}) = \frac{1}{4}(360^\circ) = 90^\circ. \tag{1}$$

We now note that angle $\angle XYZ$ is given by

$$\begin{aligned}
\angle XYZ &= 360^\circ - \angle XYD - \angle ZYD \\
&= 360^\circ - (180^\circ - \alpha - \beta) - (180^\circ - \gamma - \delta) \\
&= \alpha + \beta + \gamma + \delta \\
&= 90^\circ.
\end{aligned}$$

The same reasoning holds for the other three vertices of the incenter quadrilateral. Therefore, this quadrilateral is a rectangle, as we wanted to show.

Solution

Week 29 (3/31/03)

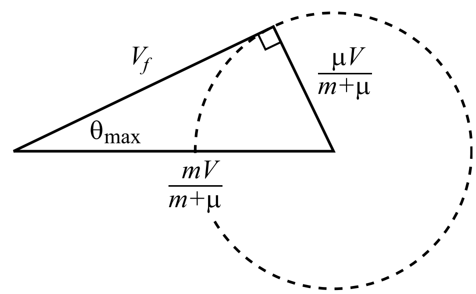
Balls in a semicircle

(a) Let $\mu \equiv M/N$ be the mass of each ball in the semicircle. We need the deflection angle in each collision to be $\theta = \pi/N$. However, if the ratio μ/m is too small, then this angle of deflection is not possible.

From Problem of the Week #25, the maximal angle of deflection in each collision is given by $\sin\theta = \mu/m$. (We'll just invoke this result here.) Since we want $\theta = \pi/N$ here, this $\sin\theta \le \mu/m$ condition becomes (using $\sin\theta \approx \theta$, for the small angle θ)

$$\theta \le \frac{\mu}{m} \qquad \Longrightarrow \qquad \frac{\pi}{N} \le \frac{M/N}{m} \qquad \Longrightarrow \qquad \pi \le \frac{M}{m}\,. \tag{1}$$

(b) Referring back to the solution to Problem #25, we see that m's speed after the first bounce is obtained from the following figure.



Looking at the right triangle, we see that the speed after the bounce is

$$V_f = V\,\frac{\sqrt{m^2-\mu^2}}{m+\mu}\,. \tag{2}$$

To first order in the small quantity μ/m, this equals

$$V_f \approx \frac{mV}{m+\mu} \approx V\left(1-\frac{\mu}{m}\right). \tag{3}$$

The same reasoning holds for each successive bounce, so the speed decreases by a factor of $(1-\mu/m)$ after each bounce. In the minimum M/m case found in part (a), we have

$$\frac{\mu}{m} = \frac{M/N}{m} = \frac{M/m}{N} = \frac{\pi}{N}\,. \tag{4}$$

Therefore, the ratio of m's final speed to initial speed equals

$$\frac{V_{\text{final}}}{V_{\text{initial}}} \approx \left(1-\frac{\pi}{N}\right)^N \approx e^{-\pi}. \tag{5}$$

That's a nice result, if there ever was one! Since $e^{-\pi}$ is roughly equal to $1/23$, only about 4% of the initial speed remains.

Solution

Week 30 (4/7/03)

Difference of Powers

A value of 26 is obtainable with $m = n = 1$. By considering the remainder when $33^m - 7^n$ is divided by certain numbers, we will show that no value smaller than 26 is possible. We will use the "mod" notation for convenience, where $a \equiv b \pmod{c}$ means that a leaves a remainder of b when divided by c.

- Consider divisions by 16. We have $33 \equiv 1 \pmod{16}$, and $7^n \equiv 7$ or $1 \pmod{16}$ because $7^2 \equiv 1 \pmod{16}$. Therefore, $33^m - 7^n \equiv 0$ or $10 \pmod{16}$. So the only possible answers to the problem are 0, 10, 16, and 26.

- Now consider divisions by 3. We have $33 \equiv 0 \pmod{3}$, and $7 \equiv 1 \pmod{3}$. Therefore, $33^m - 7^n \equiv 2 \pmod{3}$. This leaves 26 as the only possibility.

Solution

Week 31 (4/14/03)

Simultaneous claps

First Solution: The relative speed of A and B is obtained from the velocity-addition formula, which yields

$$\beta_{\text{rel}} = \frac{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}}{2} \,. \tag{1}$$

In B's frame, A's clock runs slow by a factor $\gamma = 1/\sqrt{1-\beta_{\text{rel}}^2} = 2$. Hence, if A claps her hands when her clock reads T (which happens to be $L/\sqrt{2}c$, but we won't need the actual value), then B claps her hands when her clock reads $2T$. But likewise, in A's frame, B's clock runs slow by a factor $\gamma = 2$. Hence, A will make her second clap at $4T$, and so on. A and B therefore increase their separation by a factor of 4 between successive claps of A. Their separation after A's nth clap therefore equals

$$d_n = 4^{n-1}L. \tag{2}$$

In the case where A and B move at a general speed v, their relative speed equals

$$\beta_{\text{rel}} = \frac{2\beta}{1+\beta^2} \,. \tag{3}$$

The associated γ factor is

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{2\beta}{1+\beta^2}\right)^2}} = \frac{1+\beta^2}{1-\beta^2} \,. \tag{4}$$

From the reasoning above, the separation after A's nth clap equals

$$d_n = L\left(\frac{1+\beta^2}{1-\beta^2}\right)^{2(n-1)} \,. \tag{5}$$

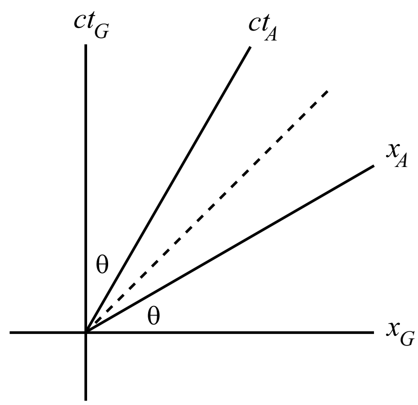
This agrees with eq. (2) when $\beta = 1/\sqrt{3}$.

Second Solution: We can also solve this problem by using Minkowski diagrams. Such diagrams show what the x and ct axes of one frame look like with respect to the x and ct axes of another frame. As we will see below, these diagrams make things very easy to visualize, and provide an easy geometrical way of determining various quantities.

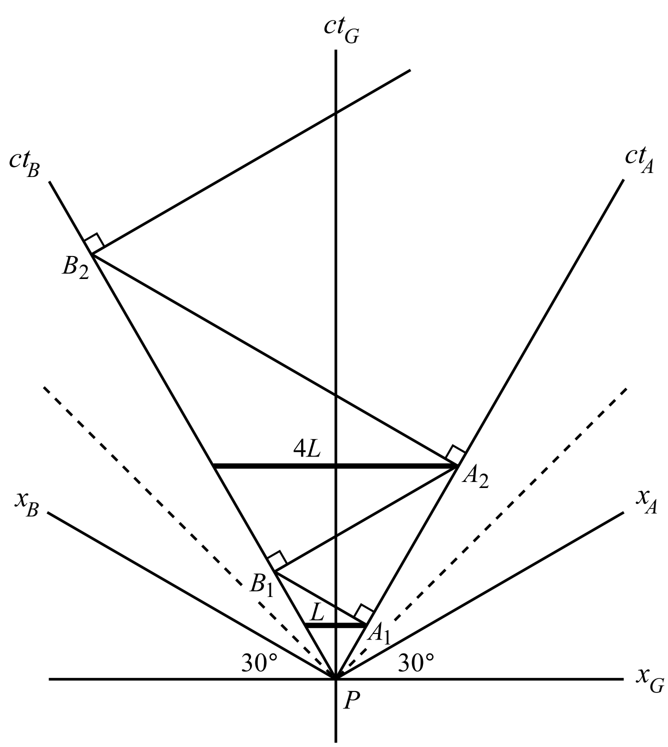
A basic Minkowski diagram is shown below. This diagram is what arises when A travels to the right at speed v. Her x_A and ct_A axes are tilted inward at an angle θ with respect to the x_G and ct_G axes of the ground frame. The angle θ satisfies $\tan\theta = v/c$. These facts (and many others) of a Minkowski diagram can be derived from the Lorentz transformations,

$$\begin{aligned} x_A &= \gamma(x_G - \beta c t_G) \\ ct_A &= \gamma(ct_G - \beta x_G), \end{aligned} \tag{6}$$

where $\beta \equiv v/c$.



An important point to note in this diagram is that all events on the x_A-axis have $t_A = 0$. Therefore, all events on the x_A-axis are simultaneous in A's frame. Likewise, all events on any line parallel to the x_A-axis are simultaneous with each other in A's frame. Armed with this fact, we can easily solve the given problem. The following diagram shows what the axes of A's and B's frames look like with respect to the ground frame.



A makes her first clap at event A_1. At this event, she is a distance L from B, as measured in the ground frame. B's first clap (which occurs simultaneously with A_1, as measured by B) is obtained by drawing a line through A_1 parallel to the x_B-axis. Since the x_B-axis is perpendicular to the ct_A-axis (due to the $30° = \tan^{-1}(1/\sqrt{3})$ angles in the diagram), we obtain the right angle shown.

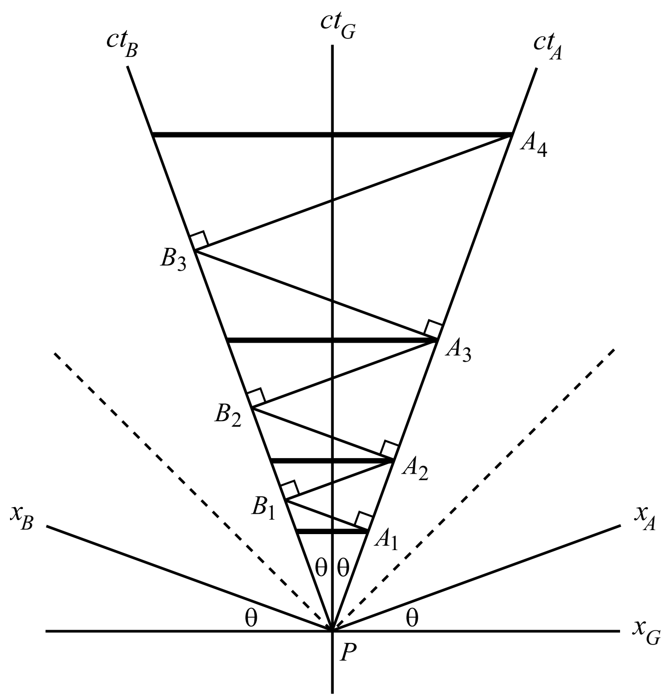
Similarly, A's second clap (which occurs simultaneously with B_1, as measured by A) is obtained by drawing a line through B_1 parallel to the x_A-axis. Since the x_A-axis is perpendicular to the ct_B-axis, we obtain the right angle shown.

Continuing in this manner, we can locate all subsequent claps. Making use of the plethora of 30°-60°-90° triangles in the figure, we see that $PB_1 = 2PA_1$, and $PA_2 = 2PB_1$, and so on. Therefore, $PA_2 = 4PA_1$, and $PA_3 = 4PA_2$, and so on. Looking now at the equilateral triangles whose top sides (the bold lines in the figure) give the distances between A and B (as measured in the ground frame), we see that these distances increase by a factor of 4 during each interval between A's claps. The separation after A's nth clap therefore equals

$$d_n = 4^{n-1}L, \tag{7}$$

in agreement with eq. (2).

In the case where A and B move at a general speed v, we obtain the following figure.



Using the same reasoning above, but now with a plethora of right triangles with an angle of 2θ, we see that $PA_2 = PA_1/\cos^2 2\theta$, and so on. Looking now at the isosceles triangles with vertex angle 2θ, whose top sides (the bold lines in the figure) give the distances between A and B (as measured in the ground frame), we see that these distances increase by a factor of $1/\cos^2 2\theta$ during each interval between A's claps. The separation after A's nth clap therefore equals

$$d_n = \frac{L}{(\cos 2\theta)^{2(n-1)}}\,. \tag{8}$$

Using $\tan\theta = \beta$, you can show that $\cos 2\theta = (1-\beta^2)/(1+\beta^2)$. Therefore,

$$d_n = L\left(\frac{1+\beta^2}{1-\beta^2}\right)^{2(n-1)}, \tag{9}$$

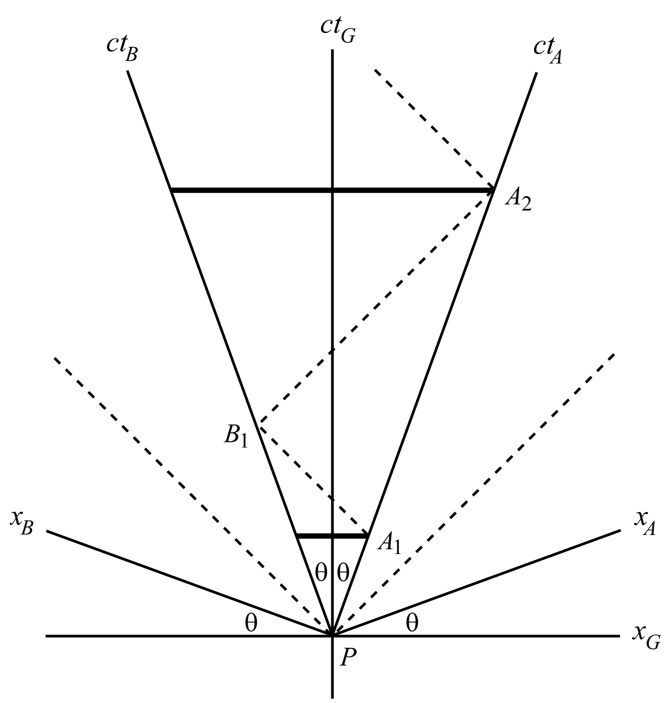
in agreement with eq. (5).

REMARK: In the above problem, each person clapped her hands simultaneously (as measured by her) with the other person's clap. We may also consider the (entirely different) setup where each person claps her hands when she *sees* the other person's clap. In this new setup, we are concerned with the actual travel of photons from one person to the other, whereas in the original problem, the travel of any photons was irrelevant.

The ratio of the B's clock reading when she makes her first clap, to A's clock reading when she makes her first clap, is simply the (inverse of the) longitudinal doppler factor $\sqrt{(1-\beta_{\rm rel})/(1+\beta_{\rm rel})}$. (This is clear if you imagine A continually sending out a series of light flashes to B.) Using the $\beta_{\rm rel}$ from eq. (3), we find this ratio to be $(1+\beta)/(1-\beta)$. We can use this same reasoning on successive claps, and so the separation between A and B after A's nth clap equals

$$d_n = L\left(\frac{1+\beta}{1-\beta}\right)^{2(n-1)} . \tag{10}$$

We can also solve this new setup by using a Minkowski diagram. The solution proceeds just as it did with the Minkowski diagrams above, except that now we must draw the 45° lines which describe a photon's travel, instead of the lines of simultaneity (which were parallel to the x_A and x_B axes) that we drew above. These 45° lines appear as follows.



We must find the ratio of PB_1 to PA_1. You can show that the law of signs in triangle PB_1A_1 gives

$$\frac{PB_1}{PA_1} = \frac{\sin(135^\circ - \theta)}{\sin(45^\circ - \theta)} . \tag{11}$$

Using the trig sum formula for sine, we find

$$\frac{PB_1}{PA_1} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{1+\tan\theta}{1-\tan\theta} = \frac{1+\beta}{1-\beta} . \tag{12}$$

Likewise, this is the ratio of PA_2 to PB_1. Therefore, $PA_2/PA_1 = (1+\beta)^2/(1-\beta)^2$, and so the separation between A and B after A's nth clap equals

$$d_n = L\left(\frac{1+\beta}{1-\beta}\right)^{2(n-1)}, \tag{13}$$

in agreement with eq. (10). As you can check, this is larger than the answer to the original problem, given in eq. (9). This must be the case, because the photons take some nonzero time to travel between the A and B.

Solution

Week 32 (4/21/03)

The game show

If you do not switch, your probability of winning equals 1/3. No actions taken by the host can change the fact that if you play a large number, N, of these games, then (roughly) $N/3$ of them will have the prize behind the door you pick.

If you switch, your probability of winning turns out to be greater. It increases to 2/3. This can be seen as follows. Without loss of generality, assume that you pick the first door. There are three equally likely possibilities for what is behind the three doors: PGG, GPG, and GGP (where P denotes prize, and G denotes goat). If you do not switch, then in only the first of these three cases do you win, so your odds of winning are 1/3. If you do switch, then in the first case you lose, but in the other two you win (because the door not opened by the host has the prize). Therefore, your odds of winning are 2/3, so you do in fact want to switch.

Remarks:

1. After the host reveals a goat, there is one prize and one goat behind the unopened doors. You might think that this implies that the probability of winning the prize is 1/2, independent of whether or not a switch is made. This is incorrect, because the above reasoning shows that there is only a 1/3 chance that the door you initially picked has the prize, and a 2/3 chance that the other unopened door has the prize. (The fact that there are two possibilities doesn't mean that their probabilities have to be equal, of course.)

2. It should be no surprise that the odds are different for the two strategies *after* the host has opened a door (the odds are obviously the same, equal to 1/3, whether or not a switch is made *before* the host opens a door), because the host gave you some of the information he had about the locations of things.

3. To make the above reasoning more believable, imagine a situation with 1000 doors (containing one prize and 999 goats). After you pick a door, the host opens 998 other doors to reveal 998 goats. In this setup, if you do not switch, your chances of winning are 1/1000. If you do switch, your chances of winning are 999/1000 (which can be seen by listing out the 1000 cases, as we did with the three cases above). In this case it is clear that the switch should be made, because the only case where you lose after you switch is the case where you had initially picked the prize (and that happens only 1/1000 of the time).

4. The clause in the statement of the problem, "The host announces that after you select a door (without opening it), he will open one of the other two doors and reveal a goat," is crucial. If it is omitted, and it is simply stated that, "The host then opens one of the other doors and reveals a goat," then it is impossible to state a preferred strategy. If the host doesn't announce his actions beforehand, then for all you know, he *always* reveals a goat (in which case you should switch). Or he *randomly* opens a door, and just happened to pick a goat (in which case it doesn't matter whether or not you switch, as you

can show). Or he opens a door and reveals a goat if and only if your initial door has the prize (in which case you definitely should not switch).

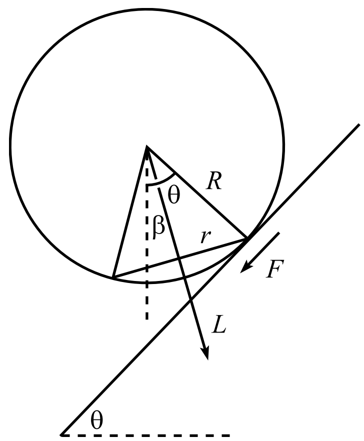
5. This problem is infamous for the intense arguments it so easily lends itself to. The common incorrect answer is that there are equal 1/2 chances of winning whether or not you switch. Now, there's nothing bad about getting the wrong answer, nor is there anything bad about not believing the correct answer for a while. But concerning the arguments that drag on and on, I think it should be illegal to argue about this problem for more than ten minutes, because at that point everyone should simply stop and *play the game*. Three coins with a dot on the bottom of one of them is all you need. Not only will the actual game give the correct answer (if you play enough times so that things average out), but the patterns that form when playing will undoubtedly convince the skeptic of the correct reasoning. Arguing endlessly about an experiment, when you can actually *do* the experiment, is as silly as arguing endlessly about what's behind a door, when you can simply *open* the door.

Solution

Week 33 (4/28/03)

Ball rolling in a cone

It turns out that the ball can move arbitrarily fast around the cone. As we will see, the plane of the contact circle (represented by the chord in the figure below) will need to be tilted downward from the contact point, so that the angular momentum has a rightward horizontal component, as shown.



Let's first look at $F = ma$ along the plane. Let Ω be the angular frequency of the ball's motion around the cone. Then the ball's horizontal acceleration is $m\ell\Omega^2$ to the left. So $F = ma$ along the plane gives (where F_f is the friction force)

$$mg\sin\theta + F_f = m\ell\Omega^2\cos\theta. \tag{1}$$

Now let's look at $\boldsymbol{\tau} = d\mathbf{L}/dt$. To get a handle on how fast the ball is spinning, consider what the setup looks like in the rotating frame in which the center of the ball is stationary (so the ball just spins in place as the cone spins around). Since there is no slipping, the contact points on the ball and the cone must have the same speed. That is,

$$\omega r = \Omega\ell \qquad \Longrightarrow \qquad \omega = \frac{\Omega\ell}{r}\,, \tag{2}$$

where ω is the angular speed of the ball in the rotating frame, and r is the radius of the contact circle on the ball.[1] The angular momentum of the ball in the lab frame equals $L = I\omega$ (at least for the purposes here[2]), and it points in the direction shown above.

[1]If the center of the ball travels in a circle of radius ℓ, then the ℓ here should actually be replaced with $\ell + R\sin\theta$, which is the radius of the contact circle on the cone. But since we're assuming that $R \ll \ell$, we can ignore the $R\sin\theta$ part.

[2]This $L = I\omega$ result isn't quite correct, because the angular velocity of the ball in the lab frame equals the angular velocity in the rotating frame (which tilts downwards with the ω magnitude we just found) plus the angular velocity of the rotating frame with respect to the lab frame (which points straight up with magnitude Ω). This second part of the angular velocity simply yields an additional vertical component of the angular momentum. But the vertical component of $\mathbf{L}$ doesn't change with time as the ball moves around the cone. It is therefore irrelevant, since we will be concerned only with $d\mathbf{L}/dt$ in what follows.

The $\mathbf{L}$ vector precesses around a cone in $\mathbf{L}$-space with the same frequency, Ω, as the ball moves around the cone. Only the horizontal component of $\mathbf{L}$ changes, and it traces out a circle of radius $L_{\text{hor}} = L \sin\beta$, at frequency Ω. Therefore,

$$\left|\frac{d\mathbf{L}}{dt}\right| = L_{\text{hor}}\Omega = (I\omega\sin\beta)\Omega = \frac{I\Omega^2\ell\sin\beta}{r}\,, \tag{3}$$

and the direction of $d\mathbf{L}/dt$ is into the page.

The torque on the ball (relative to its center) is due to the friction force, F_f. Hence, $|\boldsymbol{\tau}| = F_f R$, and its direction is into the page. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives (with $I = \eta m R^2$, where $\eta = 2/5$ in this problem)

$$\begin{aligned} F_f R &= \frac{I\Omega^2\ell\sin\beta}{r} \\ \Longrightarrow \qquad F_f &= \frac{\eta m R\Omega^2\ell\sin\beta}{r}\,. \end{aligned} \tag{4}$$

Using this F_f in eq. (1) gives

$$mg\sin\theta + \frac{\eta m R\Omega^2\ell\sin\beta}{r} = m\ell\Omega^2\cos\theta. \tag{5}$$

Solving for Ω gives

$$\Omega^2 = \frac{g\sin\theta}{\ell\left(\cos\theta - \frac{\eta R\sin\beta}{r}\right)}\,. \tag{6}$$

We see that it is possible for the ball to move around the cone infinitely fast if

$$\cos\theta = \frac{\eta\sin\beta}{x}\,, \tag{7}$$

where $x \equiv r/R$. But from the above figure, we see that β is given by

$$\beta = \theta - \sin^{-1}(r/R). \tag{8}$$

Therefore, eq. (7) gives

$$\begin{aligned} \cos\theta &= \frac{\eta}{x}\sin(\theta - \sin^{-1}x) \\ \Longrightarrow \qquad x\cos\theta &= \eta\sin\theta\cos(\sin^{-1}x) - \eta\cos\theta\sin(\sin^{-1}x) \\ \Longrightarrow \qquad x\cos\theta &= \eta\sin\theta\sqrt{1-x^2} - \eta\cos\theta x \\ \Longrightarrow \qquad x(1+\eta)\cos\theta &= \eta\sin\theta\sqrt{1-x^2}. \end{aligned} \tag{9}$$

Squaring and solving for x^2 gives

$$x^2 = \frac{\eta^2\sin^2\theta}{(1+\eta)^2\cos^2\theta + \eta^2\sin^2\theta}\,. \tag{10}$$

In the problem at hand, we have $\eta = 2/5$, so

$$\frac{r}{R} \equiv x = \sqrt{\frac{4\sin^2\theta}{49\cos^2\theta + 4\sin^2\theta}}\,. \tag{11}$$

Remarks:

1. What value of θ allows largest the tilt angle of the contact circle (that is, the largest β)? From eq. (7), we see that maximizing β is equivalent to maximizing $x\cos\theta$, or equivalently $x^2\cos^2\theta$. Using the value of x^2 in eq. (10), we see that we want to maximize

$$x^2\cos^2\theta = \frac{\eta^2\sin^2\theta\cos^2\theta}{(1+\eta)^2\cos^2\theta+\eta^2\sin^2\theta}\,. \tag{12}$$

Taking the derivative with respect to θ and going through a bit of algebra, we find that the maximum is achieved when

$$\sin\theta = \sqrt{\frac{1+\eta}{1+2\eta}} = \sqrt{\frac{7}{9}} \qquad \Longrightarrow \qquad \theta = 61.9^\circ. \tag{13}$$

You can then show that

$$\sin\beta_{\text{max}} = \frac{1}{1+2\eta} = \frac{5}{9} \qquad \Longrightarrow \qquad \beta_{\text{max}} = 33.7^\circ. \tag{14}$$

2. Let's consider three special cases for the contact circle, namely, when it is a horizontal circle, a great circle, or a vertical circle.

   (a) *Horizontal circle*: In this case, we have $\beta = 0$, so eq. (6) gives

   $$\Omega^2 = \frac{g\tan\theta}{\ell}\,. \tag{15}$$

   In this case, $\mathbf{L}$ points vertically, which means that $d\mathbf{L}/dt$ is zero, which means that the torque is zero, which means that the friction force is zero. Therefore, the ball moves around the cone with the same speed as a particle sliding without friction. (You can show that such a particle does indeed have $\Omega^2 = g\tan\theta/\ell$.) The horizontal contact-point circle ($\beta = 0$) is the cutoff case between the sphere moving faster or slower than a frictionless particle.

   (b) *Great circle*: In this case, we have $r = R$ and $\beta = -(90^\circ - \theta)$. Hence, $\sin\beta = -\cos\theta$, and eq. (6) gives

   $$\Omega^2 = \frac{g\tan\theta}{\ell(1+\eta)}\,. \tag{16}$$

   This reduces to the frictionless-particle case when $\eta = 0$, as it should.

   (c) *Vertical circle*: In this case, we have $r = R\cos\theta$ and $\beta = -90^\circ$, so eq. (6) gives

   $$\Omega^2 = \frac{g\tan\theta}{\ell\left(1+\frac{\eta}{\cos^2\theta}\right)}\,. \tag{17}$$

   Again, this reduces to the frictionless-particle case when $\eta = 0$, as it should. But for $\theta \to 90^\circ$, Ω goes to zero, whereas in the other two cases above, Ω goes to ∞.

Solution

Week 34 (5/5/03)

Counterfeit coin

(a) Note that there are three possible outcomes to each weighing: left side heavier, right side heavier, or both sides equal. In order to do the given task in as few weighings as possible, we will need as much information from each weighing as possible. Hence, all three possibilities should be realizable for each weighing (except for the final weighing in some scenarios, as we will see below). So, for example, an initial weighing of six coins against six coins is probably not a good idea, because it is not possible for the scale to balance. We should expect to have to switch coins from one side of the scale to the other, from one weighing to the next, in order to make the three possibilities realizable for a given weighing. Having said that, here is one scheme that does the task in three weighings (there are other variations that also work):

Weigh four coins (labelled A_1, A_2, A_3, A_4) against four others (B_1, B_2, B_3, B_4). The remaining four will be labelled C_1, C_2, C_3, C_4. There are three possible outcomes to this weighing:

(1) The A group is heavier than the B group. We know in this case that the C coins are "good", and the "bad" coin is either an A or a B. If the bad coin is an A, it is heavy. If the bad coin is a B, it is light.

Now weigh (A_1, A_2, B_1) against (A_3, A_4, B_2). There are three possible outcomes:

- If the (A_1, A_2, B_1) side is heavier, the bad coin must be A_1, A_2, or B_2. Weigh A_1 against A_2. If A_1 is heavier, it is the bad (heavy) coin; if A_2 is heavier, it is the bad (heavy) coin; if they are equal, B_1 is the bad (light) coin.
- If the (A_3, A_4, B_2) side is heavier, the bad coin must be A_3, A_4, or B_1. Use the same strategy as in the previous case.
- If they are equal, the bad coin must be B_3 or B_4. Simply weigh B_3 against a good coin.

(2) The B group is heavier than the A group. This case is the same as the previous one, but with "heavy" switched with "light".

(3) The A and B groups balance. So the bad coin is a C.

Weigh (C_1, C_2) against (C_3, good-coin). There are three possible outcomes:

- If the (C_1, C_2) side is heavier, weigh C_1 against C_2. If C_1 is heavier, it is the bad (heavy) coin; if C_2 is heavier, it is the bad (heavy) coin; if they are equal, C_3 is the bad (light) coin.
- If the (C_3, good-coin) side is heavier, this is equivalent to the previous case, with "heavy" switched with "light".
- If they are equal, the bad coin is C_4. Weigh C_4 against a good coin to determine if it is heavy or light.

(b) **Lemma:** *Let there be N coins, about which our information is the following: The N coins may be divided into two sets, $\{H\}$ and $\{L\}$, such that i) if a coin is in $\{H\}$ and it turns out to be the bad coin, it is heavy; and ii) if a coin is in $\{L\}$ and it turns out to be the bad coin, it is light. Then, given n weighings, the maximum value of N for which we can identify the bad coin, and also determine whether it is heavy or light, is $N = 3^n$.*

Proof: For the case $n = 0$, the lemma is obviously true, because by assumption we know which of the two sets, $\{H\}$ and $\{L\}$, the one coin is in. We will show by induction that the lemma is true for all n.

Assume the lemma true for n weighings. Let us show that it is then true for $n+1$ weighings. We will do this by first showing that $N = 3^{n+1}$ is solvable, and then showing that $N = 3^{n+1} + 1$ is not always solvable.

By assumption, the $N = 3^{n+1}$ coins are divided into $\{H\}$ and $\{L\}$ sets. On both sides of the scale, put h coins from $\{H\}$ and l coins from $\{L\}$, with $h + l = 3^n$. (Either h or l may be zero, if necessary.) There are then 3^n coins left over.

There are three possible outcomes to this weighing:

- If the left side is heavier, the bad coin must be one of the h H-coins from the left or one of the l L-coins from the right.
- If the right side is heavier, the bad coin must be one of the h H-coins from the right or one of the l L-coins from the left.
- If the scale balances, the bad coin must be one of the 3^n leftover coins.

In each of these cases, the problem is reduced to a setup with 3^n coins which are divided into $\{H\}$ and $\{L\}$ sets. But this is assumed to be solvable with n weighings, by induction. Therefore, since $N = 3^0 = 1$ is solvable for $n = 0$, we conclude that $N = 3^n$ is solvable for all n.

Let us now show that $N = 3^{n+1}+1$ is not always solvable with $n+1$ weighings. Assume inductively that $N = 3^n + 1$ is not always solvable with n weighings. ($N = 3^0 + 1 = 2$ is certainly not solvable for $n = 0$.) For the first weighing, the leftover pile can have at most 3^n coins in it, since the bad coin may end up being there. There must therefore be at least $2 \cdot 3^n + 1$ total coins on the scale (which then implies that there must be at least $2 \cdot 3^n + 2$ total coins on the scale, since the number must be even). Depending on how the $\{H\}$ and $\{L\}$ coins are distributed on the scale, the first weighing will (assuming the scale doesn't balance) tell us that the bad coin is either in a subset containing s coins, or in the complementary subset containing $(2 \cdot 3^n + 2) - s$ coins. One of these sets will necessarily have at least $3^n + 1$ coins in it, which by assumption is not necessarily solvable. ∎

Returning to the original problem, let us first consider a modified setup where we have an additional known good coin at our disposal.

Claim: *Given N coins and W weighings, and given an additional known good coin, the maximum value for N for which we can identify the bad coin,* *and also determine whether it is heavy or light, is* $N_W^{\rm g} = (3^W - 1)/2$*, where the superscript "g" signifies that we have a known good coin available.*

Proof: The claim is true for $W = 1$. Assume inductively that it is true for W weighings. We will show that it is then true for $W + 1$ weighings.

In the first of our $W+1$ weighings, we may have (by the inductive assumption) at most $(3^W - 1)/2$ leftover coins not involved in the weighing, since the bad coin may end up being there (in which case we have many good coins from the scale at out disposal).

From the above lemma, we may have at most 3^W suspect coins on the scale. We can indeed have this many, if we bring in a known good coin to make the number of weighed coins, $3^W + 1$, even (so that we can have an equal number on each side). If the scale doesn't balance, the 3^W suspect coins satisfy the hypotheses of the lemma (they can be divided into $\{H\}$ and $\{L\}$ sets), so if the bad coin is among these 3^W coins, it can be determined in W weighings.

Therefore,

$$N_{W+1}^{\rm g} = N_W^{\rm g} + 3^W = \frac{3^W - 1}{2} + 3^W = \frac{3^{W+1} - 1}{2}\,, \tag{1}$$

as we wanted to show. ■

Corollary: *Given N coins and W weighings (and not having an additional known good coin available), the maximum value of N for which we can identify the bad coin, and also determine whether it is heavy or light, is*

$$N_w^{\rm ng} = \frac{3^W - 1}{2} - 1, \tag{2}$$

where the superscript "ng" signifies that we do not have a known good coin available.

Proof: If we are not given a known good coin, the only modification to the reasoning in the above claim is that we can't put a total of 3^W suspect coins on the scale, because 3^W is odd. We are limited to a total of $3^W - 1$ coins on the scale, so we now obtain

$$N_{W+1}^{\rm ng} = N_W^{\rm g} + (3^W - 1) = \frac{3^W - 1}{2} + (3^W - 1) = \frac{3^{W+1} - 1}{2} - 1. \tag{3}$$

Note that if the scale balances, so that we know the bad coin is a leftover coin, then from that point on, we do indeed have a known good coin at our disposal (any coin on the scale), so $N_W^{\rm g} = (3^W - 1)/2$ is indeed what appears in the above equation. ■

Therefore, N_W is decreased by one if we don't have a known good coin at the start.

Remark: It is possible to write down an upper bound for $N_W^{\rm g}$ and $N_W^{\rm ng}$, without going through all of the above work. We may do this by considering the number of possible outcomes of the W weighings. There are 3 possibilities for each weighing (left side heavier, right side heavier, or both sides equal), so there are at most 3^W possible outcomes. Each of these outcomes may be labelled by a string of W letters, for example, $LLRER$ if $W = 5$ (with "L" for "left", etc.).

However, the $EEEEE$ string (where the scale always balances) does not give enough information to determine whether the bad coin is heavy or light. Also, the "mirror image" outcome (namely $RRLEL$, for the $LLRER$ case above) corresponds to equivalent information, with "left" and "right" simply reversed. Therefore, there are $(3^W - 1)/2$ effectively different strings. Hence, we can start with no more than $(3^W - 1)/2$ coins (because we can imagine initially labelling each coin with a string, so that if that particular string is the one obtained, then the corresponding coin is the bad coin).

As we saw above, this upper bound of $(3^W - 1)/2$ is obtainable if we have an additional known good coin at our disposal. But we fall short of the bound by one, if we initially do not have an additional good coin.

Note that although there may be different possible strategies for coin placement at various points in the weighing process (so that there are actually far more than 3^W possible outcomes, taking into consideration different placements of coins), only 3^W of the possible outcomes are realizable in a given scheme. Whatever weighing strategy you pick, you can write down an "if, then" tree before you start the process. Once you pick a scheme, there are 3^W possible outcomes (and thus $(3^W - 1)/2$ effectively different outcomes).

Solution

Week 35 (5/12/03)

Rising hoop

Let θ be the angle through which the bead has fallen, and let N be the normal force from the hoop on the bead, with inward taken to be positive. Then the radial $F = ma$ equation for the bead is

$$N + mg\cos\theta = \frac{mv^2}{R}\,. \tag{1}$$

The height the bead has fallen is $R - R\cos\theta$, so conservation of energy gives

$$\frac{mv^2}{2} = mgR(1-\cos\theta) \qquad \Longrightarrow \qquad v^2 = 2gR(1-\cos\theta). \tag{2}$$

Therefore, the radial $F = ma$ equation becomes

$$\begin{aligned} N &= \frac{mv^2}{R} - mg\cos\theta \\ &= 2mg(1-\cos\theta) - mg\cos\theta \\ &= mg(2 - 3\cos\theta). \end{aligned} \tag{3}$$

By Newton's third law, this is the force from the bead on the hoop, with outward taken to be positive. Note that this force is positive (that is, the bead pulls outward on the hoop) if $\theta > \cos^{-1}(2/3) \approx 48.2^\circ$.

Since there are two beads, the total upward force on the hoop from the beads is

$$2N\cos\theta = 2mg(2-3\cos\theta)\cos\theta. \tag{4}$$

The θ that yields the maximum value of this upward force is obtained by taking the derivative, which gives

$$\begin{aligned} 0 &= \frac{d}{d\theta}(2\cos\theta - 3\cos^2\theta) \\ &= -2\sin\theta + 6\sin\theta\cos\theta. \end{aligned} \tag{5}$$

Therefore, the maximum value is achieved when $\cos\theta = 1/3$, in which case the upward force equals

$$2mg\left(2 - 3\left(\frac{1}{3}\right)\right)\left(\frac{1}{3}\right) = \frac{2mg}{3}\,. \tag{6}$$

The hoop will rise up off the ground if this maximum upward force is larger than the weight of the hoop. That is, if

$$\frac{2mg}{3} > Mg \qquad \Longrightarrow \qquad \frac{m}{M} > \frac{3}{2}\,. \tag{7}$$

REMARK: Alternatively, we can solve for the minimum value of m/M by setting the upward force, $2mg(2-3\cos\theta)\cos\theta$, equal to the weight of the hoop, Mg, and then using the quadratic formula to solve for $\cos\theta$. A solution for $\cos\theta$ exists only if the discriminant is positive, which is the case only if $m/M > 3/2$.

Solution

Week 36 (5/19/03)

Monochromatic Triangle

(a) Let us try to avoid forming a monochromatic triangle, a task that we will show is impossible. Consider one point and the sixteen lines drawn from it to the other sixteen points. From the pigeonhole principle (if you have n pigeons and $n-1$ pigeonholes, then at least two pigeons must go in one pigeonhole), we see that at least six of these lines must be of the same color. Let this color be red.

Now consider the six points at the ends of these red lines. Look at the lines going from one of these points to the other five. In order to not form a red triangle, each of these five lines must be either green or blue. Hence (by the pigeonhole principle) at least three of them must be of the same color. Let this color be green.

Finally, consider the three points at the ends of the three green lines. If any one of the three lines connecting them is red, a red triangle is formed. And if any one of the three lines connecting them is green, a green triangle is formed. Therefore, they must all be blue, and a blue triangle is formed.

(b) Consider the problem for the case of $n = 4$, in order to get an idea of how the solution generalizes. We claim that 66 points will necessitate a monochromatic triangle. As in the case of $n = 3$, isolate one point and paint all the lines from it to the other points. Since we have 65 other points and 4 colors, the pigeonhole principle requires that at least 17 of these lines be of the same color. In order to not have a monochromatic triangle, the lines joining the endpoints of these 17 lines must use only the remaining three colors, and the problem is reduced to the case of $n = 3$.

Generalizing this reasoning yields the following result:

Claim: *If n colors and P_n points necessitate a monochromatic triangle, then $n+1$ colors and*

$$P_{n+1} = (n+1)(P_n - 1) + 2 \tag{1}$$

points also necessitate a monochromatic triangle.

Proof: Isolate one point, and paint each of the $(n+1)(P_n-1)+1$ lines to the other points one of $n+1$ colors. From the pigeonhole principle, at least P_n of these lines must be the same color. In order to not have a monochromatic triangle, the points at the ends of these P_n lines must be joined by the n other colors. But by hypothesis, there must then be a monochromatic triangle. ■

If we use the above recursion relation in itself (that is, if we write the P_n in eq. (1) in terms of P_{n-1}, and then write P_{n-1} in terms of P_{n-2}, and so on), the pattern becomes clear. Using the initial condition $P_1 = 3$, we arrive at the following expression for P_n:

$$P_n = n!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) + 1, \tag{2}$$

as you can easily verify. The sum in the parentheses is smaller than e by a margin that is less than $1/n!$. Therefore, P_n does indeed equal the smallest integer greater than $n!e$.

REMARK: For $n = 1, 2, 3$, the numbers $\lceil n!e \rceil$ (which equal 3,6,17, respectively) are the smallest numbers which necessitate a monochromatic triangle. (For $n = 1$, two points don't even form a triangle. And for $n = 2$, you can easily construct a diagram that doesn't contain a monochromatic triangle. For $n = 3$, things are much more difficult, but in 1968 Kalbfleisch and Stanton showed that 16 points do *not* necessitate a monochromatic triangle.) For $n \geq 4$, the problem of finding the smallest number of points that necessitate a monochromatic triangle is unsolved, I believe.

Solution

Week 37 (5/26/03)

Bouncing down a plane

Let us ignore the tilt of the plane for a moment and determine how the ω_f and v_f after a bounce are related to the ω_i and v_i before the bounce (where v denotes the velocity component parallel to the plane). Let the positive directions of velocity and force be to the right along the plane, and let the positive direction of angular velocity be counterclockwise. If we integrate the force and torque over the small time of a bounce, we obtain

$$\begin{aligned} F &= \frac{dP}{dt} \qquad \Longrightarrow \qquad \int F\,dt = \Delta P, \\ \tau &= \frac{dL}{dt} \qquad \Longrightarrow \qquad \int \tau\,dt = \Delta L. \end{aligned} \tag{1}$$

But $\tau = RF$. And since R is constant, we have

$$\Delta L = \int RF\,dt = R\int F\,dt = R\Delta P. \tag{2}$$

Therefore,

$$I(\omega_f - \omega_i) = Rm(v_f - v_i). \tag{3}$$

But conservation of energy gives

$$\begin{aligned} \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2 &= \frac{1}{2}mv_i^2 + \frac{1}{2}I\omega_i^2 \\ \Longrightarrow \qquad I(\omega_f^2 - \omega_i^2) &= m(v_i^2 - v_f^2). \end{aligned} \tag{4}$$

Dividing this equation by eq. (3) gives[1]

$$R(\omega_f + \omega_i) = -(v_f + v_i). \tag{5}$$

We can now combine this equation with eq. (3), which can be rewritten (using $I = (2/5)mR^2$) as

$$\frac{2}{5}R(\omega_f - \omega_i) = v_f - v_i. \tag{6}$$

Given v_i and ω_i, the previous two equations are two linear equations in the two unknowns, v_f and ω_f. Solving for v_f and ω_f, and then writing the result in matrix notation, gives

$$\left(\begin{array}{c} v_f \\ R\omega_f \end{array}\right) = \frac{1}{7}\left(\begin{array}{cc} 3 & -4 \\ -10 & -3 \end{array}\right)\left(\begin{array}{c} v_i \\ R\omega_i \end{array}\right) \equiv \mathcal{A}\left(\begin{array}{c} v_i \\ R\omega_i \end{array}\right). \tag{7}$$

[1]We have divided out the trivial $\omega_f = \omega_i$ and $v_f = v_i$ solution, which corresponds to slipping motion on a frictionless plane. The nontrivial solution we will find shortly is the non-slipping one. Basically, to conserve energy, there must be no work done by friction. But since work is force times distance, this means that either the plane is frictionless, or that there is no relative motion between ball's contact point and the plane. Since we are given that the plane has friction, the latter (non-slipping) case must be the one we are concerned with.

Note that

$$\mathcal{A}^2 = \frac{1}{49}\begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix} = \mathcal{I}. \tag{8}$$

Now let us consider the effects of the tilted plane. Since the ball's speed perpendicular to the plane is unchanged by each bounce, the ball spends the same amount of time in the air between any two successive bounces. This time equals $T = 2V/g\cos\theta$, because the component of gravity perpendicular to the plane is $g\cos\theta$. During this time, the speed along the plane increases by $(g\sin\theta)T = 2V\tan\theta \equiv V_0$.

Let $\mathbf{Q}$ denote the $(v, R\omega)$ vector at a given time (where v denotes the velocity component parallel to the plane). The ball is initially projected with $\mathbf{Q} = \mathbf{0}$. Therefore, right before the first bounce, we have $\mathbf{Q}_1^{\textbf{before}} = (V_0, 0) \equiv \mathbf{V}_0$. (We have used the fact that ω doesn't change while the ball is in the air.) Right after the first bounce, we have $\mathbf{Q}_1^{\textbf{after}} = \mathcal{A}\mathbf{V}_0$. We then have $\mathbf{Q}_2^{\textbf{before}} = \mathcal{A}\mathbf{V}_0 + \mathbf{V}_0$, and so $\mathbf{Q}_2^{\textbf{after}} = \mathcal{A}(\mathcal{A}\mathbf{V}_0 + \mathbf{V}_0)$. Continuing in this manner, we see that

$$\begin{aligned} \mathbf{Q}_n^{\textbf{before}} &= (\mathcal{A}^{n-1} + \cdots + \mathcal{A} + \mathcal{I})\mathbf{V}_0, \quad \text{and} \\ \mathbf{Q}_n^{\textbf{after}} &= (\mathcal{A}^{n} + \cdots + \mathcal{A}^2 + \mathcal{A})\mathbf{V}_0. \end{aligned} \tag{9}$$

However, $\mathcal{A}^2 = \mathcal{I}$, so all the even powers of $\mathcal{A}$ equal $\mathcal{I}$. The value of $\mathbf{Q}$ after the nth bounce is therefore given by

$$\begin{aligned} n \text{ even} \quad \Longrightarrow \quad \mathbf{Q}_n^{\textbf{after}} &= \frac{n}{2}(\mathcal{A} + \mathcal{I})\mathbf{V}_0. \\ n \text{ odd} \quad \Longrightarrow \quad \mathbf{Q}_n^{\textbf{after}} &= \frac{1}{2}\Big((n+1)\mathcal{A} + (n-1)\mathcal{I}\Big)\mathbf{V}_0. \end{aligned} \tag{10}$$

Using the value of $\mathcal{A}$ defined in eq. (7), we find

$$\begin{aligned} n \text{ even} \quad \Longrightarrow \quad \begin{pmatrix} v_n \\ R\omega_n \end{pmatrix} &= \frac{n}{7}\begin{pmatrix} 5 & -2 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} V_0 \\ 0 \end{pmatrix}. \\ n \text{ odd} \quad \Longrightarrow \quad \begin{pmatrix} v_n \\ R\omega_n \end{pmatrix} &= \frac{1}{7}\begin{pmatrix} 5n-2 & -2n-2 \\ -5n-5 & 2n-5 \end{pmatrix}\begin{pmatrix} V_0 \\ 0 \end{pmatrix}. \end{aligned} \tag{11}$$

Therefore, the speed along the plane after the nth bounce equals (using $V_0 \equiv 2V\tan\theta$)

$$v_n = \frac{10nV\tan\theta}{7} \quad (n \text{ even}), \qquad \text{and} \qquad v_n = \frac{(10n-4)V\tan\theta}{7} \quad (n \text{ odd}). \tag{12}$$

Remark: Note that after an even number of bounces, eq. (11) gives $v = -R\omega$. This is the "rolling" condition. That is, the angular speed exactly matches up with the translation speed, so v and ω are unaffected by the bounce. (The vector $(1, -1)$ is an eigenvector of $\mathcal{A}$.) At the instant that an even-n bounce occurs, the v and ω are the same as they would be for a ball that simply rolls down the plane. At the instant after an odd-n bounce, the v is smaller than it would be for the rolling ball, but the ω is larger. (Right before an odd-n bounce, the v is larger but the ω is smaller.)

Solution

Week 38 (6/2/03)

Sum over 1

(a) **First Solution:** We will use the following fact: Given n random numbers between 0 and 1, the probability $P_n(1)$ that their sum does not exceed 1 equals $1/n!$.[1] To prove this, let us prove a slightly stronger result:

Theorem: *Given n random numbers between 0 and 1, the probability $P_n(s)$ that their sum does not exceed s equals $s^n/n!$ (for all $s \le 1$).*

Proof: Assume inductively that the result holds for a given n. (It clearly holds for all $s \le 1$ when $n = 1$.) What is the probability that $n+1$ numbers sum to no more than t (with $t \le 1$)? Let the $(n+1)$st number have the value x. Then the probability $P_{n+1}(t)$ that all $n+1$ numbers sum to no more than t equals the probability $P_n(t-x)$ that the first n numbers sum to no more than $t - x$. But $P_n(t-x) = (t-x)^n/n!$. Integrating this probability over all x from 0 to t gives

$$P_{n+1}(t) = \int_0^t \frac{(t-x)^n}{n!}\,dx = -\frac{(t-x)^{n+1}}{(n+1)!}\Big|_0^t = \frac{t^{n+1}}{(n+1)!}\,. \tag{1}$$

We see that if the theorem holds for n, then it also holds for $n+1$. Therefore, since the theorem holds for all $s \le 1$ when $n = 1$, it holds for all $s \le 1$ for any n. ■

We are concerned with the special case $s = 1$, for which $P_n(1) = 1/n!$.

The probability that it takes exactly n numbers for the sum to exceed 1 equals $1/(n-1)! - 1/n!$. This is true because the first $n-1$ numbers must sum to less than 1, and the nth number must push the sum over 1, so we must subtract off the probability that is does not.

The expected number of numbers, N, to achieve a sum greater than 1, is therefore

$$\begin{aligned} N &= \sum_2^\infty n\left(\frac{1}{(n-1)!} - \frac{1}{n!}\right) \\ &= \sum_2^\infty \frac{1}{(n-2)!} \\ &= e. \end{aligned} \tag{2}$$

Could it really have been anything else?

Second Solution: We will use the result, $P_n(s) = s^n/n!$, from the first solution. Let $F_n(s)\,ds$ be the probability that the sum of n numbers is between

[1] The number $1/n!$ is the volume of the region in n dimensions bounded by the coordinate planes and the hyperplane $x_1 + x_2 + \cdots + x_n = 1$. For example, in two dimensions we have a triangle with area 1/2; in three dimensions we have a pyramid with volume 1/6; etc.

s and $s + ds$. Then $F_n(s)$ is simply the derivative of $P_n(s)$, with respect to s. Therefore, $F_n(s) = s^{n-1}/(n-1)!$.

In order for it to take exactly m numbers for the sum to exceed 1, two things must happen: (1) the sum of the first $m-1$ numbers must equal a number, s, less than 1; this occurs with probability density $s^{m-2}/(m-2)!$. And (2) the mth number must push the sum over 1, that is, the mth number must be between $1-s$ and 1; this occurs with probability s.

The probability that it takes exactly m numbers for the sum to exceed 1 is therefore $\int_0^1 s(s^{m-2}/(m-2)!)\, ds$. The expected number of numbers, N, to achieve a sum greater than 1, therefore equals

$$\begin{aligned} N &= \sum_2^\infty m \int_0^1 \frac{s^{m-1}}{(m-2)!}\, ds \\ &= \sum_2^\infty m \frac{1}{m(m-2)!} \\ &= e. \end{aligned} \tag{3}$$

(b) **First Solution:** After n numbers have been added, the probability that their sum is between s and $s + ds$ is, from above, $F_n(s) = s^{n-1}/(n-1)!$ (for $s \le 1$). There is a probability of s that the $(n+1)$st number pushes the sum over 1. If this happens, then (since this last number must be between $1-s$ and 1, and is evenly distributed) the average result will be equal to $s + (1 - s/2) = 1 + s/2$. The expected sum therefore equals

$$\begin{aligned} S &= \sum_1^\infty \left(\int_0^1 \left(1 + \frac{s}{2}\right) s \frac{s^{n-1}}{(n-1)!}\, ds \right) \\ &= \sum_1^\infty \frac{3n^2 + 5n}{2(n+2)!} \\ &= \sum_1^\infty \frac{3(n+2)(n+1) - 4(n+2) + 2}{2(n+2)!} \\ &= \sum_1^\infty \left(\frac{3}{2n!} - \frac{2}{(n+1)!} + \frac{1}{(n+2)!} \right) \\ &= \frac{3}{2}(e-1) - 2(e-2) + (e - 5/2) \\ &= \frac{e}{2}. \end{aligned} \tag{4}$$

Second Solution: Each of the random numbers has an average value of 1/2. Therefore, since it takes (on average) e numbers for the sum to exceed 1, the average value of the sum will be $e/2$.

This reasoning probably strikes you as being either completely obvious or completely mysterious. In the case of the latter, imagine playing a large number of games in succession, writing down each of the random numbers in one long sequence. (You can note the end of each game by, say, putting a mark after the final number, but this is not necessary.) If you play N games (with N very large), then the result from part (a) shows that there will be approximately Ne numbers listed in the sequence. Each number is a random number between 0 and 1, so the average value is $1/2$. The sum of all of the numbers in the sequence is therefore approximately $Ne/2$. Hence, the average total per game is $e/2$.

Solution

Week 39 (6/9/03)

Viewing the spokes

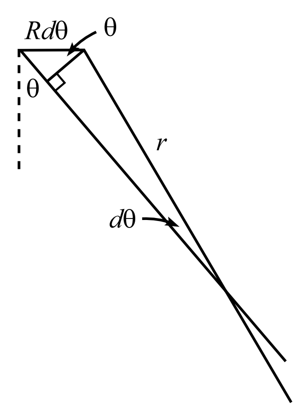
The contact point on the ground does not look blurred, because it is instantaneously at rest. But although this is the only point on the wheel that is at rest, there will be other locations in the picture where the spokes do not appear blurred.

The relevant property of a point in the picture where a spoke does not appear blurred is that the point lies on the spoke throughout the duration of the camera's exposure. (The point in the picture need not, however, actually correspond to the same atom on the spoke.)

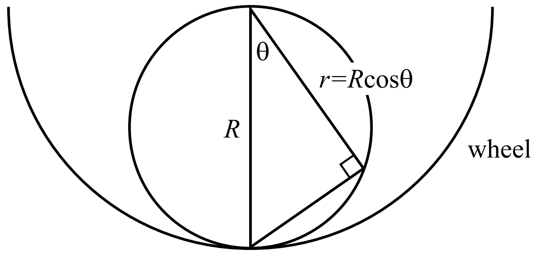
At a given time, consider a spoke in the lower half of the wheel. A short time later, the spoke will have moved (via both translation and rotation), but it will intersect its original position. The spoke will not appear blurred at this intersection point. We must therefore find the locus of these intersections. We can do this in two ways.

First method: Let R be the radius of the wheel. Consider a spoke that makes an angle of θ with the vertical at a given time. If the wheel then rolls through a small angle $d\theta$, the center moves a distance $R\,d\theta$. The spoke's motion is a combination of a translation through this distance $R\,d\theta$, plus a rotation through the angle $d\theta$ (around the top end).

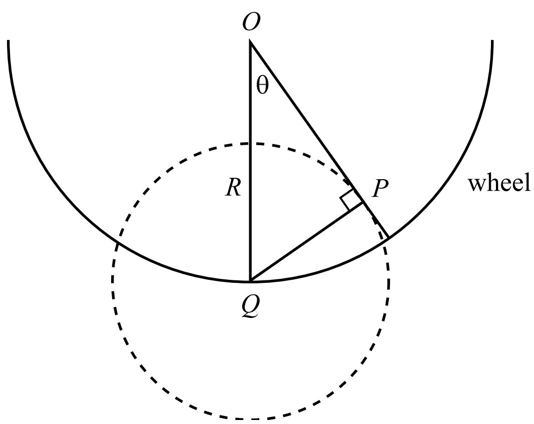
Let r be the radial position of the intersection of the initial and final positions of the spoke. In the following figure, we can write down two expressions for the short segment drawn perpendicular to the initial position of the spoke.



The two expression are $(R\,d\theta)\cos\theta$, and $r\,d\theta$ (to first order in $d\theta$). Equating these gives $r = R\cos\theta$. This describes a circle whose diameter is the lower vertical radius of the wheel, as shown below.



Second method: Since the wheel's contact point with the ground is instantaneously at rest, the wheel may be considered to be instantaneously rotating around this point. This means that every atom in the wheel (both in the spokes and the rim) instantaneously traces out the arc of a circle centered at the contact point. For the point P in the figure below, this circle is the dotted one drawn (with Q being the contact point). A spoke will not appear blurred at the point where this circular motion is along the direction of the spoke. That is, a spoke will not appear blurred at the point where the dotted circle is tangent to the spoke, as shown.



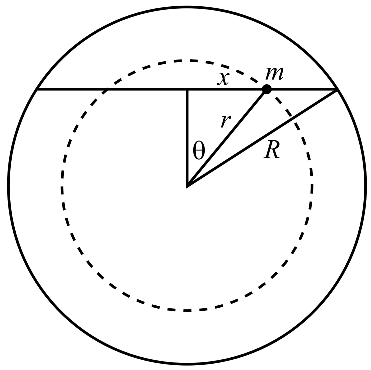
We are therefore concerned with the locus of all points P such that the segments PQ and PO are perpendicular. As seen in the previous diagram above, this locus is the circle whose diameter is the lower vertical radius of the wheel.

Solution

Week 41 (6/23/03)

Speedy travel

Let the earth's mass be M, and let its radius be R. Consider the object when it is a distance x from the center of the tube, at radius r, as shown.



The gravitational force on the object is due to the mass of the earth that is inside the radius r. Since mass is proportional to volume, the mass inside the radius r is $M(r/R)^3$. The force on the object is therefore

$$F = \frac{G\Big(M(r/R)^3\Big)m}{r^2} = \frac{GMmr}{R^3}\,. \tag{1}$$

We are interested in the component of the force along the tube, which is

$$F_x = -F\sin\theta = -\frac{GMmr}{R^3}\left(\frac{x}{r}\right) = -\left(\frac{GMm}{R^3}\right)x. \tag{2}$$

Therefore, $F = ma$ along the tube gives

$$-\left(\frac{GMm}{R^3}\right)x = m\ddot{x} \qquad \Longrightarrow \qquad \ddot{x} = -\left(\frac{GM}{R^3}\right)x. \tag{3}$$

This equation describes simple harmonic motion, with frequency

$$\omega = \sqrt{\frac{GM}{R^3}}\,. \tag{4}$$

Therefore, the roundtrip time is

$$\begin{aligned} T = \frac{2\pi}{\omega} &= 2\pi\sqrt{\frac{R^3}{GM}} \\ &= 2\pi\sqrt{\frac{(6.38\cdot 10^6\,\mathrm{m})^3}{(6.67\cdot 10^{-11}\,\mathrm{m}^3/\mathrm{kg}\cdot\mathrm{s}^2)(5.98\cdot 10^{24}\,\mathrm{kg})}} \\ &\approx 5060\,\mathrm{s} \\ &\approx 84 \text{ minutes.} \end{aligned} \tag{5}$$

The one-way time to the other end of the tube is therefore 42 minutes. That's quick! Note that this result is independent of where chord is. The chord can be a diameter of the earth, or it can be a straight tube spanning the 20-foot width of a room. Neglecting friction, an object will take 42 minutes to go from one side of the room to the other. The very slight component of gravity along the tube is enough to do the trick.

Solution

Week 42 (6/30/03)

How much change?

If the item costs between $N/2$ and N dollars, then you can buy only one item. These extremes will produce remainders of $N/2$ and 0, respectively. The average amount of money left over in this region, which has length $N(1-1/2)=N/2$, is therefore $N/4$.

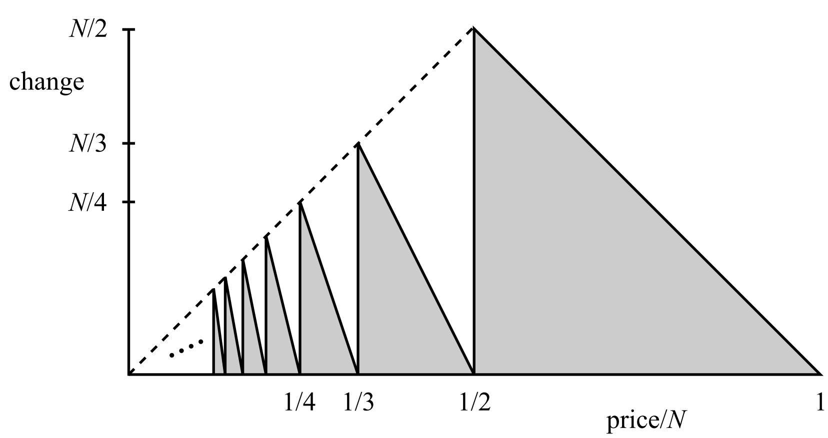
Likewise, if the item costs between $N/3$ and $N/2$ dollars, then you can buy only two items. These extremes will produce remainders of $N/3$ and 0, respectively. The average amount of money left over in this region, which has length $N(1/2-1/3)=N/6$, is therefore $N/6$.

Continuing in this manner, we see that if the item costs between $N/(n+1)$ and N/n, then you can buy only n items. These extremes will produce remainders of $N/(n+1)$ and 0, respectively. The average amount of money left over in this region, which has length $N(1/n-1/(n+1))=N/n(n+1)$, is therefore $N/2(n+1)$.

The expected amount, M, of money left over is therefore

$$\begin{aligned}
M &= \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\frac{N}{2(n+1)} \\
&= \frac{N}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}-\frac{1}{(n+1)^2}\right) \\
&= \frac{N}{2}\sum_{n=1}^{\infty}\left(\left[\frac{1}{n}-\frac{1}{n+1}\right]-\frac{1}{(n+1)^2}\right) \\
&= \frac{N}{2}\left(1-\left(\frac{\pi^2}{6}-1\right)\right) \\
&= N\left(1-\frac{\pi^2}{12}\right).
\end{aligned}$$

In the third line, we have used the fact that the sum in brackets telescopes to 1, and also that $\sum_{n=1}^{\infty}1/n^2=\pi^2/6$. Since $\pi^2/12\approx 0.82$, the amount of money left over is roughly $(0.18)N$ dollars. Note that what we have essentially done in this problem is find the area under the graph in the following figure.



Solution

Week 43 (7/7/03)

Infinite Atwood's machine

First Solution: If the strength of gravity on the earth were multiplied by a factor η, then the tension in all of the strings in the Atwood's machine would likewise be multiplied by η. This is true because the only way to produce a quantity with the units of tension (that is, force) is to multiply a mass by g. Conversely, if we put the Atwood's machine on another planet and discover that all of the tensions are multiplied by η, then we know that the gravity there must be ηg.

Let the tension in the string above the first pulley be T. Then the tension in the string above the second pulley is $T/2$ (because the pulley is massless). Let the downward acceleration of the second pulley be a_2. Then the second pulley effectively lives in a world where gravity has strength $g - a_2$.

Consider the subsystem of all the pulleys except the top one. This infinite subsystem is identical to the original infinite system of all the pulleys. Therefore, by the arguments in the first paragraph above, we must have

$$\frac{T}{g} = \frac{T/2}{g - a_2}\,, \tag{1}$$

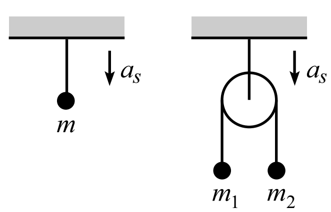
which gives $a_2 = g/2$. But a_2 is also the acceleration of the top mass, so our answer is $g/2$.

REMARKS: You can show that the relative acceleration of the second and third pulleys is $g/4$, and that of the third and fourth is $g/8$, etc. The acceleration of a mass far down in the system therefore equals $g(1/2 + 1/4 + 1/8 + \cdots) = g$, which makes intuitive sense.

Note that $T = 0$ also makes eq. (1) true. But this corresponds to putting a mass of zero at the end of a finite pulley system (see the following solution).

Second Solution: Consider the following auxiliary problem.

Problem: Two setups are shown below. The first contains a hanging mass m. The second contains a pulley, over which two masses, m_1 and m_2, hang. Let both supports have acceleration a_s downward. What should m be, in terms of m_1 and m_2, so that the tension in the top string is the same in both cases?



Answer: In the first case, we have

$$mg - T = ma_s. \tag{2}$$

In the second case, let a be the acceleration of m_2 relative to the support (with downward taken to be positive). Then we have

$$\begin{aligned} m_1 g - \frac{T}{2} &= m_1(a_s - a), \\ m_2 g - \frac{T}{2} &= m_2(a_s + a). \end{aligned} \tag{3}$$

Note that if we define $g' \equiv g - a_s$, then we may write the above three equations as

$$\begin{aligned} mg' &= T, \\ m_1 g' &= \frac{T}{2} - m_1 a, \\ m_2 g' &= \frac{T}{2} + m_2 a. \end{aligned} \tag{4}$$

Eliminating a from the last two of these equations gives $4m_1m_2g' = (m_1 + m_2)T$. Using this value of T in the first equation then gives

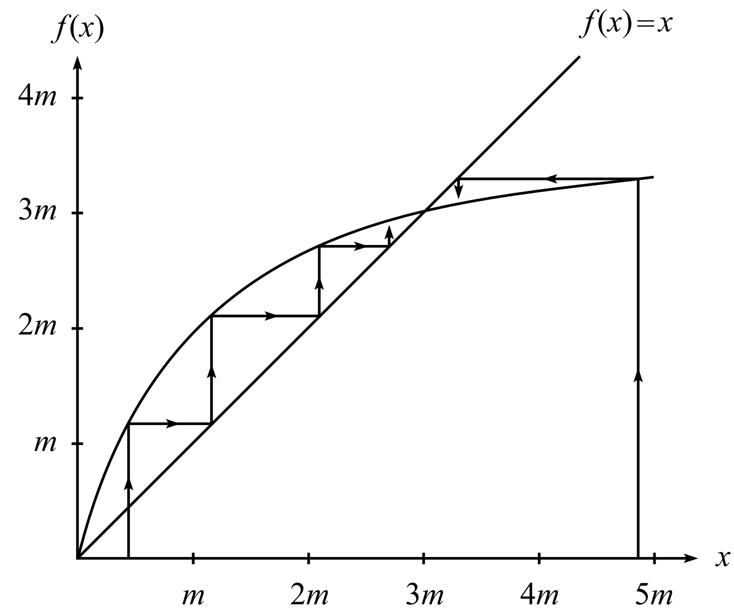
$$m = \frac{4m_1m_2}{m_1 + m_2}\,. \tag{5}$$

Note that the value of a_s is irrelevant. (We effectively have a fixed support in a world where the acceleration from gravity is g'.) This auxiliary problem shows that the two-mass system in the second case may be equivalently treated as a mass m, given by eq. (5), as far as the upper string is concerned. ∎

Now let's look at our infinite Atwood's machine. Start at the bottom. (Assume that the system has N pulleys, where $N \to \infty$.) Let the bottom mass be x. Then the auxiliary problem shows that the bottom two masses, m and x, may be treated as an effective mass $f(x)$, where

$$\begin{aligned} f(x) &= \frac{4mx}{m + x} \\ &= \frac{4x}{1 + (x/m)}\,. \end{aligned} \tag{6}$$

We may then treat the combination of the mass $f(x)$ and the next m as an effective mass $f(f(x))$. These iterations may be repeated, until we finally have a mass m and a mass $f^{(N-1)}(x)$ hanging over the top pulley. So we must determine the behavior of $f^N(x)$, as $N \to \infty$. This behavior is clear if we look at the following plot of $f(x)$.



Note that $x = 3m$ is a fixed point of f. That is, $f(3m) = 3m$. This plot shows that no matter what x we start with, the iterations approach $3m$ (unless we start at $x = 0$, in which case we remain there). These iterations are shown graphically by the directed lines in the plot. After reaching the value $f(x)$ on the curve, the line moves horizontally to the x value of $f(x)$, and then vertically to the value $f(f(x))$ on the curve, and so on.

Therefore, since $f^N(x) \to 3m$ as $N \to \infty$, our infinite Atwood's machine is equivalent to (as far as the top mass is concerned) just two masses, m and $3m$. You can then quickly show that that the acceleration of the top mass is $g/2$.

Note that as far as the support is concerned, the whole apparatus is equivalent to a mass $3m$. So $3mg$ is the upward force exerted by the support.

Solution

Week 44 (7/14/03)

Relatively prime numbers

The probability that two random numbers both have a given prime p as a factor is $1/p^2$. The probability that they do not have p as a common factor is thus $1 - 1/p^2$. Therefore, the probability that two numbers have no common prime factors is

$$P = (1 - 1/2^2)(1 - 1/3^2)(1 - 1/5^2)(1 - 1/7^2)(1 - 1/11^2)\cdots. \tag{1}$$

Using

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \tag{2}$$

this can be rewritten as

$$P = \Big((1 + 1/2^2 + 1/2^4 + \cdots)(1 + 1/3^2 + 1/3^4 + \cdots)\cdots\Big)^{-1}. \tag{3}$$

By the Unique Factorization Theorem (every positive integer is expressible as the product of primes in exactly one way), we see that the previous expression is equivalent to

$$P = (1 + 1/2^2 + 1/3^2 + 1/4^2 + 1/5^2 + 1/6^2 + \cdots)^{-1}. \tag{4}$$

And since the sum of the squares of the reciprocals of all of the positive integers is known to be $\pi^2/6$, the desired probability is $P = 6/\pi^2 \approx 61\%$.

Remarks:

1. The probability that n random numbers all have a given prime p as a factor is $1/p^n$. So the probability that they do not all have p as a common factor is $1 - 1/p^n$. In exactly the same manner as above, we find that the probability, P_n, that n numbers have no common factor among all of them is

$$P_n = (1 + 1/2^n + 1/3^n + 1/4^n + 1/5^n + 1/6^n + \cdots)^{-1}. \tag{5}$$

This is, by definition, the Riemann zeta function, $\zeta(n)$. It can be calculated exactly for even values of n, but only numerically for odd values. For the case of $n = 4$, we can use the known value $\zeta(4) = \pi^4/90$ to see that the probability that four random numbers do not all have a common factor is $P_4 = 90/\pi^4 \approx 92\%$.

2. We can also perform the somewhat silly exercise of applying this result to the case of $n = 1$. The question then becomes: What is the probability, P_1, that a randomly chosen positive integer does not have a factor? Well, 1 is the only positive integer without any factors, so the probability is $1/\infty = 0$. And indeed,

$$\begin{aligned} P_1 &= (1 - 1/2)(1 - 1/3)(1 - 1/5)(1 - 1/7)\ldots \\ &= (1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6\cdots)^{-1} \\ &= 1/\infty, \end{aligned} \tag{6}$$

because the sum of the reciprocals of all of the positive integers diverges.

3. Let $\phi(n)$ equal the number of integers less than n that are relatively prime to n. Then $\phi(n)/n$ equals the probability that a randomly chosen integer is relatively prime to n. (This is true because any integer is relatively prime to n if and only if its remainder, when divided by n, is relatively prime to n.) The result of our original problem therefore tells us that the average value of $\phi(n)/n$ is $6/\pi^2$.

4. To be precise about what we mean by probabilities in this problem, we really should word the question as: Let N be a very large integer. Pick two random integers less than or equal to N. What is the probability that these numbers are relatively prime, in the limit where N goes to infinity?

   The solution would then be slightly modified, in that the relevant primes p would be cut off at N, and "edge effects" due to the finite size of N would have to be considered. It is fairly easy to see that the answer obtained in this limit is the same as the answer obtained above.

Solution

Week 45 (7/21/03)

Sliding along a plane

The normal force from the plane is $N = mg\cos\theta$. Therefore, the friction force on the block is $\mu N = (\tan\theta)N = mg\sin\theta$. This force acts in the direction opposite to the motion. The block also feels the gravitational force of $mg\sin\theta$ pointing down the plane.

Because the magnitudes of the friction force and the gravitational force along the plane are equal, the acceleration along the direction of motion equals the negative of the acceleration in the direction down the plane. Therefore, in a small increment of time, the speed that the block loses along its direction of motion exactly equals the speed that it gains in the direction down the plane. Letting v be the speed of the block, and letting v_y be the component of the velocity in the direction down the plane, we therefore have

$$v + v_y = C, \tag{1}$$

where C is a constant. C is given by its initial value, which is $V + 0 = V$. The final value of C is $V_f + V_f = 2V_f$ (where V_f is the final speed of the block), because the block is essentially moving straight down the plane after a very long time. Therefore,

$$2V_f = V \qquad \Longrightarrow \qquad V_f = \frac{V}{2}\,. \tag{2}$$

Solution

Week 46 (7/28/03)

The birthday problem

(a) Given n people, the probability, P_n, that there is *not* a common birthday among them is

$$P_n = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\cdots\left(1 - \frac{n-1}{365}\right). \tag{1}$$

The first factor is the probability that two given people do not have the same birthday. The second factor is the probability that a third person does not have a birthday in common with either of the first two. This continues until the last factor is the probability that the nth person does not have a birthday in common with any of the other $n-1$ people.

We want $P_n < 1/2$. If we simply multiply out the above product with successive values of n, we find that $P_{22} = .524$, and $P_{23} = .493$. Therefore, there must be at least 23 people in a room in order for the odds to favor at least two of them having the same birthday.

REMARK: This answer of $n = 23$ is much smaller than most people expect, so it provides a nice betting opportunity. For $n = 30$, the odds of a common birthday increase to 70.6%, and most people still find it hard to believe that among 30 people there are probably two who have the same birthday. The table below lists various values of n and the probabilities, $1 - P_n$, that at least two people have a common birthday.

n	10	20	23	30	50	60	70	100
$1 - P_n$	11.7%	41.1%	50.7%	70.6%	97.0%	99.4%	99.92%	99.9994%

Even for $n = 50$, most people would be happy to bet, at even odds, that no two people have the same birthday. If they seem a bit hesitant, however, you can simply offer them the irrefusable odds of 10 to 1. (That is, you put down, for example, \$10 on the table, and they put down \$1. If there is a common birthday, you take all the money. If there is no common birthday, they take all the money.) Since there is a 97.0% chance of a common birthday among 50 people, a quick calculation shows that you will gain, on average, 67 cents for every \$10 you put down.

One reason why many people do not believe the $n = 23$ answer is correct is that they are asking themselves a different question, namely, "How many people need to be present for there to be a 1/2 chance that someone else has *my* birthday?" The answer to this question is indeed much larger than 23. The probability that no one out of n people has a birthday on a given day is $(1 - 1/365)^n$. For $n = 252$, this is just over 1/2. And for $n = 253$, it is just under 1/2. Therefore, you need to come across 253 other people in order to expect that at least one of them has *your* birthday.

(b) **First Solution:** Given n people, and given N days in a year, the reasoning in part (a) shows that the probability that no two people have the same birthday is

$$P_n = \left(1 - \frac{1}{N}\right)\left(1 - \frac{2}{N}\right)\cdots\left(1 - \frac{n-1}{N}\right). \tag{2}$$

If we take the natural log of this equation and use the expansion,

$$\ln(1-x) = -(x + x^2/2 + \cdots), \tag{3}$$

then the requirement $P_n \le 1/2$ becomes

$$-\left(\frac{1}{N} + \frac{2}{N} + \cdots \frac{n-1}{N}\right) - \frac{1}{2}\left(\frac{1}{N^2} + \frac{4}{N^2} + \cdots \frac{(n-1)^2}{N^2}\right) - \cdots \le -\ln 2. \tag{4}$$

Using the sums,

$$\sum_1^n k = \frac{n(n+1)}{2}, \qquad \text{and} \qquad \sum_1^n k^2 = \frac{n(n+1)(2n+1)}{6}, \tag{5}$$

we can rewrite eq. (4) as

$$\frac{n(n-1)}{2N} + \frac{n(n-1)(2n-1)}{12N^2} + \cdots \ge \ln 2. \tag{6}$$

For large N, the first term will be of order 1 when $n \approx \sqrt{N}$, in which case the second- and higher-order terms are negligible. Therefore, keeping only the first term (which behaves like $n^2/2N$), we find that P_n is equal to 1/2 when

$$n \approx \sqrt{2\ln 2}\sqrt{N}. \tag{7}$$

Let's look at a few cases:

- For $N = 365$, eq. (7) gives $n = 22.5$. Since we must have an integral number of people, this agrees with the exact result, $n = 23$.
- For $N = 24 \cdot 365 = 8760$ (that is, for births in the same hour), we find $n = 110.2$. This agrees with the exact result, $n = 111$, obtained by multiplying out eq. (2).
- For $N = 60 \cdot 24 \cdot 365 = 525{,}600$ (that is, for births in the same minute), we find $n = 853.6$. This agrees with the exact result, $n = 854$, obtained by multiplying out eq. (2) (not by hand!). This is a very small number compared to the more than half a million minutes in a year.

REMARKS:

(1) If we wish to ask how many people need to be in a room in order for the probability to be at least p that some two have the same birthday, then the above derivation is easily modified to yield

$$n \approx \sqrt{2\ln\left(\tfrac{1}{1-p}\right)}\sqrt{N}. \tag{8}$$

(2) The alternative question introduced in part (a), "How many people need to be present for there to be a 1/2 chance that someone else has *my* birthday?", may also be answered in the large-N limit. The probability that no one out of n people has a birthday on a given day is

$$\left(1 - \frac{1}{N}\right)^n = \left(1 - \frac{1}{N}\right)^{N(n/N)} \approx e^{-n/N}. \tag{9}$$

Therefore, if $n > N\ln 2$, you can expect that at least one of the n people has your birthday. For $N = 365$, we find that $N\ln 2$ is slightly less than 253, so this agrees with the result obtained in part (a). Note that this result is linear in N, whereas the result of the original problem in eq. (7) behaves like $\sqrt{N}$. The reason for this square-root behavior can be seen in the next solution.

Second Solution: Given a large number, n, of people, there are $\binom{n}{2} = n(n-1)/2 \approx n^2/2$ pairs of people. The probability that a given pair has the same birthday is $1/N$, so the probability that they do not have the same birthday is $1 - 1/N$.[1] Therefore, the probability that no pair has a common birthday is

$$P_n = \left(1 - \frac{1}{N}\right)^{n^2/2} \approx \left(1 - \frac{1}{N}\right)^{N(n^2/2N)} \approx e^{-n^2/(2N)}. \tag{10}$$

This equals 1/2 when

$$n \approx \sqrt{2\ln 2}\sqrt{N}. \tag{11}$$

in agreement with eq. (7).

Remark: We assumed above that all the pairs are independent, as far as writing down the $1 - 1/N$ probability goes. Let us now show that this approximately true. We will show that for large N and n, the assumptions on the coincidence of birthdays in some pairs do not significantly affect the probability of coincidence in other pairs. More precisely, the relation $P_n \approx e^{-n^2/(2N)}$ is true if $n \ll N^{2/3}$. The reasoning is as follows.

Consider n people and N days in a year. Assumptions on the coincidence of birthdays in some pairs will slightly affect the probability of coincidence in other pairs, because the given assumptions restrict the possible birthdays of these other pairs. For example, if it is given that A and B do not have the same birthday, and also that B and C do not, then the probability that A and C do not have the same birthday is $1 - 1/364$ (instead of $1 - 1/365$), because they are both restricted from having a birthday on B's birthday, whatever it may be.

The maximal restriction that can be placed on the possible birthdays of a given pair occurs when it is known that neither of them has a birthday in common with any of the other $n-2$ people. In this case, the possible birthdays for the last two people are restricted to $N - (n-2)$ days of the year. Therefore, the probability that the last pair does not have a common birthday is $1 - 1/(N-n+2) > 1 - 1/(N-n)$. This is the most that any such probability can differ from the naive $1 - 1/N$. Therefore, we may say that

$$\begin{aligned} \left(1 - \frac{1}{N-n}\right)^{n^2/2} &\leq P_n \leq \left(1 - \frac{1}{N}\right)^{n^2/2} \\ \Longrightarrow \quad e^{-n^2/2(N-n)} &\leq P_n \leq e^{-n^2/(2N)}. \end{aligned} \tag{12}$$

The ratio of these upper and lower bounds on P_n is

$$\exp\left(-\frac{n^2}{2N} + \frac{n^2}{2(N-n)}\right) = \exp\left(\frac{n^3}{2N(N-n)}\right). \tag{13}$$

[1] This is not quite correct for all the pairs, because two pairs are not independent if, for example, they share a common person. But it is accurate enough for our purposes in the large-N limit. See the remark below.

For large N, this ratio is essentially equal to 1, provided that $n \ll N^{2/3}$. Therefore, $P_n \approx e^{-n^2/(2N)}$ if $n \ll N^{2/3}$. And since the result in eq. (11) is in this realm, it is therefore valid.

Extension: We can also ask the following question: How many people must be in a room in order for the probability to be greater than 1/2 that at least b of them have the same birthday? (Assume that there is a very large number, N, of days in a year, and ignore effects that are of subleading order in N.)

We can solve this problem in the manner of the second solution above. Given a large number, n, of people, there are $\binom{n}{b}$ groups of b people. This is approximately equal to $n^b/b!$ (assuming that $b \ll n$).

The probability that a given group of b people all have the same birthday is $1/N^{b-1}$, so the probability that they do not all have the same birthday is $1-(1/N^{b-1})$.[2] Therefore, the probability, $P_n^{(b)}$, that no group of b people all have the same birthday is

$$P_n^{(b)} \approx \left(1-\frac{1}{N^{b-1}}\right)^{n^b/b!} \approx e^{-n^b/b!N^{b-1}}. \tag{14}$$

This equals 1/2 when

$$n \approx (b! \ln 2)^{1/b} N^{1-1/b}. \tag{15}$$

Remarks:

(1) Eq. (15) holds in the large-N limit. If we wish to make another approximation, that of large b, we see that the quantity $(b! \ln 2)^{1/b}$ goes like b/e, for large b. (This follows from Sterling's formula, $m! \approx m^m e^{-m}\sqrt{2\pi m}$.) Therefore, for large N, n, and b (with $b \ll n \ll N$), we have $P_n^{(b)} = 1/2$ when

$$n \approx (b/e)N^{1-1/b}. \tag{16}$$

(2) The right-hand side of equation eq. (15) scales with N according to $N^{1-1/b}$. This means that if we look at the numbers of people needed to have a greater than 1/2 chance that pairs, triplets, etc., have common birthdays, we see that these numbers scale like

$$N^{1/2},\ N^{2/3},\ N^{3/4},\ \cdots. \tag{17}$$

For large N, these numbers are multiplicatively far apart. Therefore, there are values of n for which we can say, for example, that we are virtually certain that there are pairs and triplets with common birthdays, but also that we are virtually certain that there are no quadruplets with a common birthday. For example, if $n = N^{17/24}$ (which satisfies $N^{2/3} < n < N^{3/4}$), then eq. (14) shows that the probability that there is a common birthday triplet is $1-e^{-\frac{1}{6}N^{1/8}} \approx 1$, whereas the probability that there is a common birthday quadruplet is $1-e^{-\frac{1}{24}N^{-1/6}} \approx \frac{1}{24}N^{-1/6} \approx 0$.

[2] Again, this is not quite correct, because the groups are not all independent. But I believe it is accurate enough for our purposes in the large-N limit. I have a proof which I think is correct, but which is very ugly. If anyone has a nice clean proof, let me know.

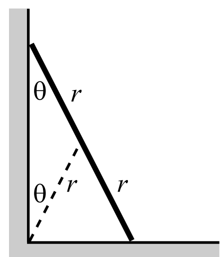
Solution

Week 47 (8/4/03)

Sliding ladder

The important point to realize in this problem is that the ladder loses contact with the wall before it hits the ground. Let's find where this loss of contact occurs.

Let $r = \ell/2$, for convenience. While the ladder is in contact with the wall, its CM moves in a circle of radius r. (This follows from the fact that the median to the hypotenuse of a right triangle has half the length of the hypotenuse). Let θ be the angle between the wall and the radius from the corner to the CM. (This is also the angle between the ladder and the wall.)



We'll solve this problem by assuming that the CM always moves in a circle, and then determining the position at which the horizontal CM speed starts to decrease, that is, the point at which the normal force from the wall would have to become negative. Since the normal force of course can't be negative, this is the point where the ladder loses contact with the wall.

By conservation of energy, the kinetic energy of the ladder equals the loss in potential energy, which is $mgr(1-\cos\theta)$. This kinetic energy may be broken up into the CM translational energy plus the rotation energy. The CM translational energy is simply $mr^2\dot{\theta}^2/2$ (because the CM travels in a circle of radius r). The rotational energy is $I\dot{\theta}^2/2$. (The same $\dot{\theta}$ applies here as in the CM translational motion, because θ is the angle between the ladder and the vertical, and thus is the angle of rotation of the ladder.) Letting $I \equiv \eta mr^2$ to be general ($\eta = 1/3$ for our ladder), the conservation of energy statement is $(1+\eta)mr^2\dot{\theta}^2/2 = mgr(1-\cos\theta)$. Therefore, the speed of the CM, which is $v = r\dot{\theta}$, equals

$$v = \sqrt{\frac{2gr}{1+\eta}}\sqrt{(1-\cos\theta)}\,. \tag{1}$$

The horizontal component of this is

$$v_x = \sqrt{\frac{2gr}{1+\eta}}\sqrt{(1-\cos\theta)}\cos\theta. \tag{2}$$

Taking the derivative of $\sqrt{(1-\cos\theta)}\cos\theta$, we see that the horizontal speed is maximum when $\cos\theta = 2/3$. Therefore the ladder loses contact with the wall when

$$\cos\theta = \frac{2}{3} \qquad \Longrightarrow \qquad \theta \approx 48.2^\circ. \tag{3}$$

Note that this is independent of η. This means that, for example, a dumbbell (two masses at the ends of a massless rod, with $\eta = 1$) will lose contact with the wall at the same angle.

Plugging this value of θ into eq. (2), and using $\eta = 1/3$, we obtain a final horizontal speed of

$$v_x = \frac{\sqrt{2gr}}{3} \equiv \frac{\sqrt{g\ell}}{3}\,. \tag{4}$$

Note that this is 1/3 of the $\sqrt{2gr}$ horizontal speed that the ladder would have if it were arranged (perhaps by having the top end slide down a curve) to eventually slide horizontally along the ground.

Remark: The normal force from the wall is zero at the start and finish, so it must reach a maximum at some intermediate value of θ. Let's find this θ. Taking the derivative of v_x in eq. (2) to find a_x, and then using $\dot{\theta} \propto \sqrt{1-\cos\theta}$ from eq. (1), we see that the force from the wall is proportional to

$$a_x \;\propto\; \frac{\sin\theta(3\cos\theta - 2)}{\sqrt{1-\cos\theta}}\,\dot{\theta} \;\propto\; \sin\theta(3\cos\theta - 2). \tag{5}$$

Taking the derivative of this, we find that the force from the wall is maximum when

$$\cos\theta = \frac{1+\sqrt{19}}{6} \qquad \Longrightarrow \qquad \theta \approx 26.7^\circ. \tag{6}$$

Solution

Week 48 (8/11/03)

The hotel problem

In figuring out the probability for success (choosing the cheapest hotel), it is convenient to organize the different cases according to what the highest ranking hotel (in order of cheapness) in the first fraction x is. Let H_1 denote the cheapest hotel, H_2 the second cheapest, etc.

Assume that H_1 is among the first x, which happens with probability x. In this case, there is failure.

Assume that H_2 is the cheapest among the first x, which happens with probability $x(1-x)$. This is the probability that H_2 is in the first x, times the probability that H_1 is not.[1] In this case, we have success.

Assume that H_3 is the cheapest among the first x, which happens with probability $x(1-x)^2$ (again, see the remark below). This is the probability that H_3 is in the first x, times the probability that H_2 is not, times the probability that H_1 also is not. In this case, we have success 1/2 of the time, because only 1/2 of the time H_1 will come before H_2.

Continuing in this fashion, we see that the probability for success, P, is

$$\begin{aligned} P(x) &= x(1-x) + \frac{1}{2}x(1-x)^2 + \frac{1}{3}x(1-x)^3 + \cdots \\ &= \sum_{k=1} \frac{1}{k}\, x(1-x)^k. \end{aligned} \tag{1}$$

The $1/k$ factor comes from the probability that H_1 is first among the top k hotels which lie in the final $(1-x)$ fraction. Using the expansion $\ln(1-y) = -(y + y^2/2 + y^3/3 + \cdots)$, with $y = 1-x$, we obtain

$$P(x) = -x \ln x. \tag{2}$$

Taking the derivative, we see that $P(x)$ is maximized when $x = 1/e$, in which case the value is $1/e$. Therefore, you want to pass up $1/e \approx 37\%$ of the hotels, and then pick the next one that is better than all the ones you've seen. Your chance of getting the best one is $1/e \approx 37\%$.

Remark: For sufficiently large N, the probabilities in eq. (1) are arbitrarily close to $x(1-x)^k/k$, for small values of k. But each successive term in eq. (1) is suppressed by a factor of at least $(1-1/e)$. The terms therefore become negligibly small at a k value that is independent of N. The only significant contribution to the sum therefore comes from small values of k, for which the given probabilities are essentially correct.

[1] The $(1-x)$ factor is technically not correct, because there are only $N-1$ spots available for H_1, given that H_2 has been placed. But the error is negligible for large N. See the remark below.

Solution

Week 49 (8/18/03)

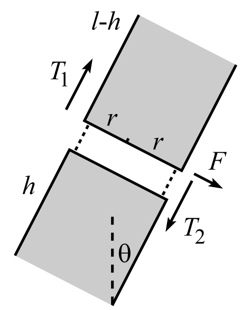
Falling chimney

Before we start dealing with the forces in the rods, let's first determine $\ddot{\theta}$ as a function of θ (the angle through which the chimney has fallen). Let ℓ be the height of the chimney. Then the moment of inertia around the pivot point on the ground is $m\ell^2/3$ (if we ignore the width), and the torque (around the pivot point) due to gravity is $\tau = mg(\ell/2)\sin\theta$. Therefore, $\tau = dL/dt$ gives $mg(\ell/2)\sin\theta = (1/3)m\ell^2\ddot{\theta}$, or

$$\ddot{\theta} = \frac{3g\sin\theta}{2\ell}\,. \tag{1}$$

Let's now determine the forces in the rods. Our strategy will be to imagine that the chimney consists of a chimney of height h, with another chimney of height $\ell - h$ placed on top of it. We'll find the forces in the rods connecting these two "sub-chimneys", and then we'll maximize one of these forces (T_2, defined below) as a function of h.

The forces on the top piece are gravity and the forces from the two rods at each end of the bottom board. Let's break these latter forces up into transverse and longitudinal forces along the chimney. Let T_1 and T_2 be the two longitudinal components, and let F be the sum of the transverse components, as shown.



We have picked the positive directions for T_1 and T_2 such that positive T_1 corresponds to a compression in the left rod, and positive T_2 corresponds to a tension in the right rod (which is what the forces will turn out to be, as we'll see). It turns out that if the width (which we'll call $2r$) is much less than the height, then $T_2 \gg F$ (as we will see below), so the tension in the right rod is essentially equal to T_2. We will therefore be concerned with maximizing T_2.

In writing down the force and torque equations for the top piece, we have three equations (the radial and tangential $F = ma$ equations, and $\tau = dL/dt$ around the CM), and three unknowns (F, T_1, and T_2). If we define the fraction $f \equiv h/\ell$, then the top piece has length $(1-f)\ell$ and mass $(1-f)m$, and its CM travels in a circle of radius $(1+f)\ell/2$, Therefore, our three force and torque equations are, respectively,

$$\begin{aligned} T_2 - T_1 + (1-f)mg\cos\theta &= (1-f)m\left(\frac{(1+f)\ell}{2}\right)\dot{\theta}^2, \\ F + (1-f)mg\sin\theta &= (1-f)m\left(\frac{(1+f)\ell}{2}\right)\ddot{\theta}, \end{aligned}$$

$$(T_1+T_2)r - F\frac{(1-f)\ell}{2} = (1-f)m\left(\frac{(1-f)^2\ell^2}{12}\right)\ddot{\theta}. \tag{2}$$

At this point, we could plow forward and solve this system of three equations in three unknowns. But things simplify greatly in the limit where $r \ll \ell$. The third equation says that $T_1 + T_2$ is of order $1/r$, and the first equation says that $T_2 - T_1$ is of order 1. These imply that $T_1 \approx T_2$, to leading order in $1/r$. Therefore, we may set $T_1 + T_2 \approx 2T_2$ in the third equation. Using this approximation, along with the value of $\ddot{\theta}$ from eq. (1), the second and third equations become

$$\begin{aligned} F + (1-f)mg\sin\theta &= \frac{3}{4}(1-f^2)mg\sin\theta, \\ 2rT_2 - F\frac{(1-f)\ell}{2} &= \frac{1}{8}(1-f)^3 mg\ell\sin\theta. \end{aligned} \tag{3}$$

This first of these equations gives

$$F = \frac{mg\sin\theta}{4}(-1+4f-3f^2), \tag{4}$$

and then the second gives

$$T_2 \approx \frac{mg\ell\sin\theta}{8r} f(1-f)^2. \tag{5}$$

As stated above, this is much greater than F (because $\ell/r \gg 1$), so the tension in the right rod is essentially equal to T_2. Taking the derivative of T_2 with respect to f, we see that it is maximum at

$$f \equiv \frac{h}{\ell} = \frac{1}{3}\,. \tag{6}$$

Therefore, the chimney is most likely to break at a point one-third of the way up (assuming that the width is much less than the height). Interestingly, $f = 1/3$ makes the force F in eq. (4) exactly equal to zero.

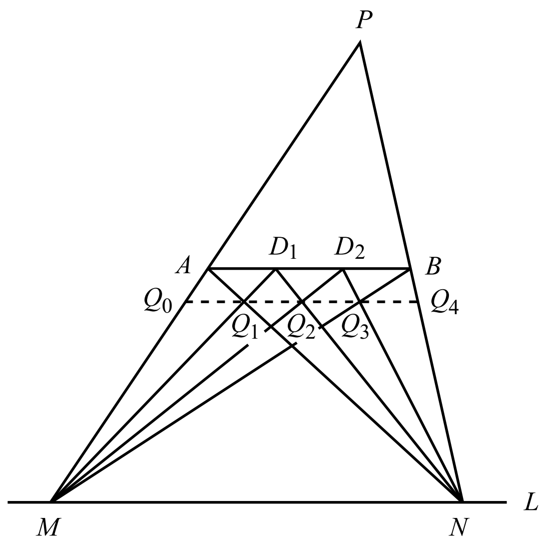
Solution

Week 50 (8/25/03)

Equal segments

The construction will proceed inductively. Given a segment divided into N equal segments, we will show how to divide it into $N+1$ equal segments. For purposes of concreteness and having manageable figures, we will just do the case $N = 3$. Generalization to arbitrary N will be clear.

In the figure below, let the segment AB be divided into three equal segments by D_1 and D_2. From an arbitrary point P (assume that P is on the side of AB opposite to the infinite line L, although it need not be), draw lines through A and B, which meet L at points M and N, respectively. Draw segments MD_1, MD_2, MB, NA, ND_1, and ND_2. Let the resulting intersections (the ones closest to segment AB) be Q_1, Q_2, and Q_3, as shown.



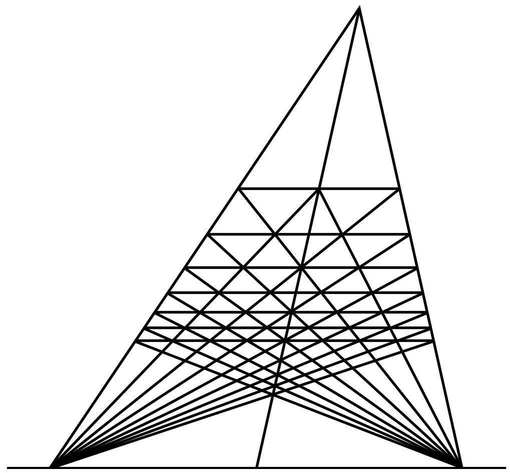
Claim: *The lines PQ_1, PQ_2, and PQ_3 divide AB into four equal segments.*

Proof: First, note that Q_1, Q_2, and Q_3 are collinear on a line parallel to AB and L. This is true because the ratio of similar triangles AQ_1D_1 and NQ_1M is the same as the ratio of similar triangles $D_1Q_2D_2$ and NQ_2M (because $AD_1 = D_1D_2$). Therefore, the altitude from Q_1 to AD_1 equals the altitude from Q_2 to D_1D_2. The same reasoning applies to Q_3, so all the Q_i are equal distances from AB. Let the line determined by the Q_i intersect PM and PN at Q_0 and Q_4, respectively.

We now claim that the distances Q_iQ_{i+1}, $i = 0, \ldots, 3$ are equal. They are equal because the ratio of similar triangles Q_0AQ_1 and MAN is the same as the ratio of similar triangles $Q_1D_1Q_2$ and MD_1N (because the ratio of the altitudes from A in the first pair is the same as the ratio of the altitudes from D_1 in the second pair). Hence, $Q_0Q_1 = Q_1Q_2$. Likewise for the other Q_iQ_{i+1}.

Therefore, since Q_0Q_4 is parallel to AB, the intersections of the lines PQ_i with AB divide AB into four equal segments. ■

Remark: To divide AB into five equal segments, we can use the same figure, with most of the work having already been done. The only new lines we need to draw are NQ_0 and MQ_4, to give a total of four intersections on a horizontal line one "level" below the Q_i. If we continue with this process, we obtain figures looking like the one below. The horizontal lines in this figure are divided into equal parts by the intersections of the diagonal lines. The initial undivided segment is the top one.[1]



We leave for you the following exercise: Given a segment of length ℓ, and a line parallel to it, construct a segment with a length equal to an arbitrary multiple of ℓ, using only a ruler.

[1]This segment must be used again for the $N = 2$ segment, because the above procedure yields only one point Q_1, and this single point doesn't determine a line parallel to L.

Solution

Week 51 (9/1/03)

Accelerating spaceship

We will solve this problem by considering two nearby times and using the velocity-addition formula,

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}\,. \tag{1}$$

Using the definition of the proper acceleration, a, we have (with $v_1 \equiv v(t)$ and $v_2 \equiv a\,dt$)

$$v(t + dt) = \frac{v(t) + a\,dt}{1 + v(t)a\,dt/c^2}\,. \tag{2}$$

Expanding both sides to first order in dt yields[1]

$$\frac{dv}{dt} = a\left(1 - \frac{v^2}{c^2}\right). \tag{3}$$

Separating variables and integrating gives, using $1/(1-z^2) = 1/2(1-z) + 1/2(1+z)$,

$$\int_0^v \left(\frac{1}{1 - v/c} + \frac{1}{1 + v/c}\right) dv = 2a\int_0^t dt. \tag{4}$$

This yields $\ln\left((1+v/c)/(1-v/c)\right) = 2at/c$. Exponentiating, and solving for v, gives

$$v(t) = c\left(\frac{e^{2at/c} - 1}{e^{2at/c} + 1}\right) = c\tanh(at/c). \tag{5}$$

Note that for small a or small t (more precisely, for $at/c \ll 1$), we obtain $v(t) \approx at$, as we should. And for $at/c \gg 1$, we obtain $v(t) \approx c$, as we should.

Remarks: If a happens to be a function of time, $a(t)$, then we can't move the a outside the integral in eq. (4), so we instead end up with the general formula,

$$v(t) = c\tanh\left(\frac{1}{c}\int_0^t a(t)\,dt\right). \tag{6}$$

If we define the *rapidity*, ϕ, by

$$\phi(t) \equiv \frac{1}{c}\int_0^t a(t)\,dt, \tag{7}$$

then we have

$$v = c\tanh\phi \qquad \Longleftrightarrow \qquad \tanh\phi = \frac{v}{c}\,. \tag{8}$$

Note that whereas v has c as a limiting value, ϕ can become arbitrarily large. The ϕ associated with a given v is simply $1/mc$ times the time integral of the force (felt by the astronaut) needed to bring the astronaut up to speed v. By applying a force for an arbitrarily long time, we can make ϕ arbitrarily large.

[1] Equivalently, just take the derivative of $(v + w)/(1 + vw/c^2)$ with respect to w, and then set $w = 0$.

The quantity ϕ is very useful because many expressions in relativity (which we'll just invoke here) take on a particularly nice form when written in terms of ϕ. Consider, for example, the velocity-addition formula. Let $\beta_1 = \tanh\phi_1$ and $\beta_2 = \tanh\phi_2$. Then if we add β_1 and β_2 using the velocity-addition formula, eq. (1), we obtain

$$\frac{\beta_1+\beta_2}{1+\beta_1\beta_2} = \frac{\tanh\phi_1+\tanh\phi_2}{1+\tanh\phi_1\tanh\phi_2} = \tanh(\phi_1+\phi_2), \tag{9}$$

where we have used the addition formula for $\tanh\phi$ (which can be proved by writing things in terms of the exponentials $e^{\pm\phi}$). Therefore, while the velocities add in the strange manner of eq. (1), the rapidities add by standard addition.

The Lorentz transformation,

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}, \tag{10}$$

also takes a nice form when written in terms of the rapidity. Note that γ can be written as

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\tanh^2\phi}} = \cosh\phi, \tag{11}$$

and so

$$\gamma\beta \equiv \frac{\beta}{\sqrt{1-\beta^2}} = \frac{\tanh\phi}{\sqrt{1-\tanh^2\phi}} = \sinh\phi. \tag{12}$$

Therefore, the Lorentz transformation becomes

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}. \tag{13}$$

This looks similar to a rotation in a plane, which is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}, \tag{14}$$

except that we now have hyperbolic trig functions instead of the usual trig functions. The fact that the *invariant interval*, $s^2 \equiv c^2t^2 - x^2$, does not depend on the frame is clear from eq. (13), because the cross terms in the squares cancel, and $\cosh^2\phi - \sinh^2\phi = 1$. (Compare with the invariance of $r^2 \equiv x^2 + y^2$ for rotations in a plane.)

Quantities associated with a Minkowski diagram also take a nice form when written in terms of the rapidity. In particular, the angle between the axes of the two relevant frames happens to be $\tan\theta = \beta$, where βc is the relative speed between the frames. But $\beta = \tanh\phi$, so the angle between the axes is given by

$$\tan\theta = \tanh\phi. \tag{15}$$

The integral $\int a(t)\,dt$ (which is c times the rapidity) may be described as the naive, incorrect speed. That is, it is the speed the astronaut might *think* he has, if he has his eyes closed and knows nothing about the theory of relativity. (And indeed, his thinking would be essentially correct for small speeds.) The quantity $\int a(t)\,dt$ seems like a reasonably physical thing, so if there is any justice in the world, $\int a(t)\,dt = \int F(t)\,dt/m$ should have *some* meaning. And indeed, although it doesn't equal v, all you have to do to get v is take a tanh and throw in some factors of c.

The fact that rapidities add via simple addition when using the velocity-addition formula, as we saw in eq. (9), is evident from eq. (6). There is really nothing more going on here than the fact that

$$\int_{t_0}^{t_2} a(t)\,dt = \int_{t_0}^{t_1} a(t)\,dt + \int_{t_1}^{t_2} a(t)\,dt. \tag{16}$$

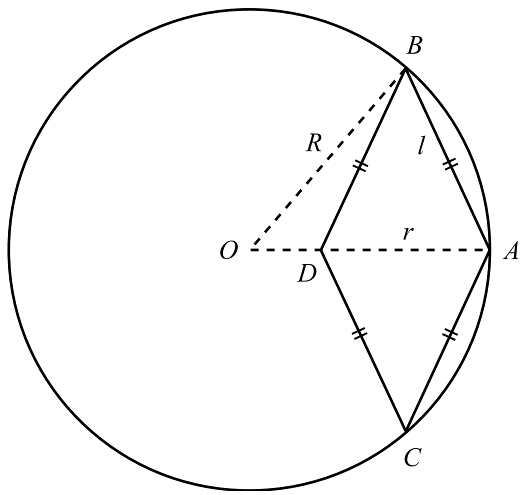
To be explicit, let a force be applied from t_0 to t_1 that brings a mass up to speed $\beta_1 = \tanh\phi_1 = \tanh(\int_{t_0}^{t_1} a\,dt)$, and then let an additional force be applied from t_1 to t_2 that adds on an additional speed of $\beta_2 = \tanh\phi_2 = \tanh(\int_{t_1}^{t_2} a\,dt)$ (relative to the speed at t_1). Then the resulting speed may be looked at in two ways: (1) it is the result of relativistically adding the speeds $\beta_1 = \tanh\phi_1$ and $\beta_2 = \tanh\phi_2$, and (2) it is the result of applying the force from t_0 to t_2 (you get the same final speed, of course, whether or not you bother to record the speed along the way at t_1), which is $\beta = \tanh(\int_{t_0}^{t_2} adt) = \tanh(\phi_1 + \phi_2)$, where the last equality comes from the obvious statement, eq. (16). Therefore, the relativistic addition of $\tanh\phi_1$ and $\tanh\phi_2$ gives $\tanh(\phi_1 + \phi_2)$, as we wanted to show.

Solution

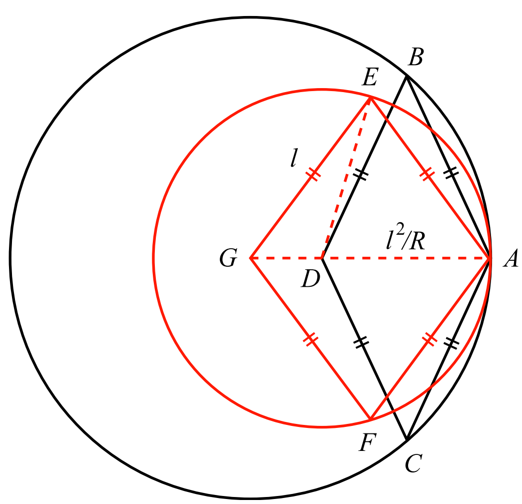
Week 52 (9/8/03)

Construct the center

Pick an arbitrary point A on the circle, as shown below. Construct points B and C on the circle, with $AB = AC = \ell$, where ℓ is arbitrary.[1] Construct point D with $DB = DC = \ell$. Let the distance DA be r. If O is the location (which we don't know yet) of the center of the circle, then triangles AOB and ABD are similar isosceles triangles (because they have $\angle BAD$ in common). Therefore, $OA/BA = BA/DA$, which gives $r = \ell^2/R$.



The above construction shows that if we are given a length ℓ and a circle of radius R, then we can construct the length ℓ^2/R. Therefore, we can produce the length R by simply repeating the above construction with the same length ℓ, but now with a circle of radius ℓ^2/R (which we just produced). In the following figure, we obtain $GA = \ell^2/(\ell^2/R) = R$. Hence, G is the center of the circle.



[1]However, this construction will not work if ℓ is too large or too small. We will determine these bounds below.

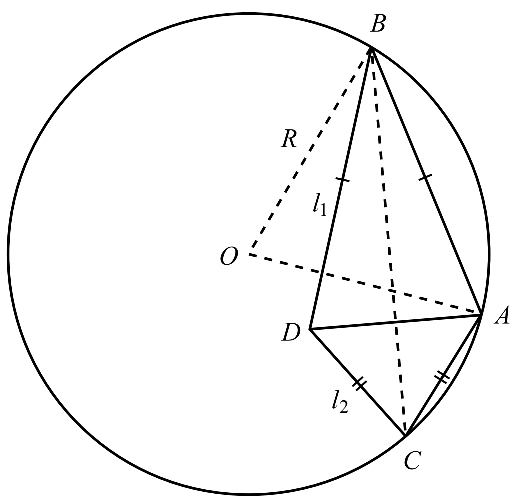
If you want to go through the similar-triangles argument, note that triangles ADE and AEG are similar isosceles triangles (because they have $\angle EAD$ in common). Therefore, $DA/EA = EA/GA$, which gives $GA = \ell^2/(\ell^2/R) = R$.

RESTRICTIONS: In order for this construction to work, it is necessary (and sufficient) for $R/2 < \ell < 2R$. The upper limit on ℓ comes from the requirement that a circle of radius ℓ (centered at A) intersects the given circle of radius R.[2] This gives $\ell < 2R$. The lower limit on ℓ comes from the requirement that a circle of radius ℓ (centered at A) intersects the circle of radius ℓ^2/R (centered at D). This gives $\ell < 2\ell^2/R$, which implies $R/2 < \ell$.

REMARK: The above solution can be extended to solve the following problem: Given three points, construct the circle passing through them.

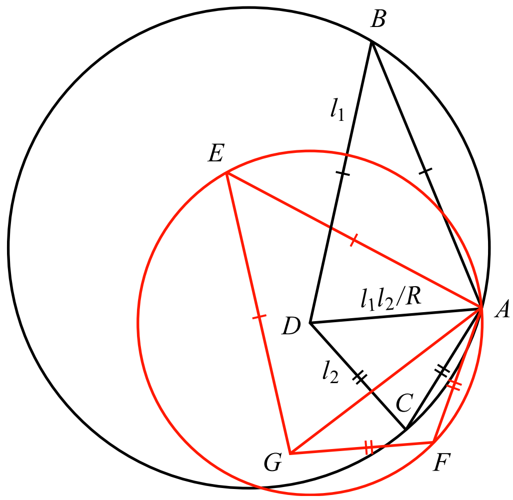
The solution proceeds along the lines of the above solution. In the figure below, construct point D with $DB = AB = \ell_1$ and $DC = AC = \ell_2$. Let O be the location (which we don't know yet) of the center of the desired circle. Then $\angle BOA = \widehat{BA}$. Also, $\angle DCA = 2(\angle BCA) = 2(\widehat{BA}/2) = \widehat{BA}$. Therefore, $\angle BOA = \angle DCA$, and so triangles BOA and DCA are similar isosceles triangles. Hence,

$$\frac{BO}{BA} = \frac{DC}{DA} \qquad \Longrightarrow \qquad \frac{R}{\ell_1} = \frac{\ell_2}{DA} \qquad \Longrightarrow \qquad DA = \frac{\ell_1 \ell_2}{R}\,. \tag{1}$$



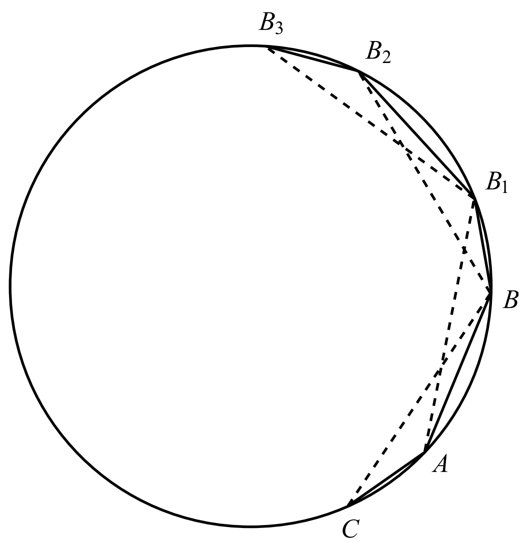
As in the above solution, we can apply this construction again, with the same lengths ℓ_1 and ℓ_2, but now with a circle of radius $\ell_1\ell_2/R$ (which we just produced). In the following figure, we obtain $GA = \ell_1\ell_2/(\ell_1\ell_2/R) = R$. And having found R, we can easily construct the center of the circle.

[2]The construction still works even if the intersection points are nearly diametrically opposite to A.



Restrictions: In order for this construction to work, we must be able to construct points E and F on the circle of radius $\ell_1\ell_2/R$ (centered at D). In order for these points to exist, the diameter of this circle must be larger than both ℓ_1 and ℓ_2. That is, $2\ell_1\ell_2/R > \max(\ell_1, \ell_2)$. This can be rewritten as $\min(\ell_1, \ell_2) > R/2$. This is the condition that must be satisfied. (There are no upper bounds on ℓ_1 and ℓ_2.)

What do we do if this lower bound is not satisfied? Simply construct more points on the circle until some three of them satisfy the constraint. For example, as shown below, construct point B_1 with $B_1B = CA$ and $B_1A = CB$. Then triangle B_1BA is congruent to triangle CAB, so point B_1 also lies on the circle. In a similar manner we can construct B_2 as shown, and then B_3, etc., to obtain an arbitrary number of points on the circle. After constructing many points, we will eventually be able to pick three of them that satisfy the constraint, $\min(\ell_1, \ell_2) > R/2$.



Of course, after constructing these new points on the circle, it is easy to pick three of them that have $\ell_1 = \ell_2$ (for example B, B_{2n}, and B_{4n}, in the notation in the figure). We can then use the easier, symmetrical solution in part (a) to find the center of the circle.

Solution

Week 53 (9/15/03)

Circles on the ice

By construction, the normal force from the ice exactly cancels all effects of the gravitational and centrifugal forces in the rotating frame of the earth (because the plumb bob hangs in the direction of the "effective gravity" force, which is the sum of the gravitational and centrifugal forces). We therefore need only concern ourselves with the Coriolis force. This force equals $\mathbf{F}_{\text{cor}} = -2m\boldsymbol{\omega} \times \mathbf{v}$.

Let the angle down from the north pole be θ (we assume the circle is small enough so that θ is essentially constant throughout the motion). Then the component of the Coriolis force that points horizontally along the surface has magnitude $f = 2m\omega v \cos\theta$ and is perpendicular to the direction of motion. (The vertical component of the Coriolis force will simply modify the required normal force.) Because this force is perpendicular to the direction of motion, v does not change. Therefore, f is constant. But a constant force perpendicular to the motion of a particle produces a circular path. The radius of the circle is given by

$$2m\omega v \cos\theta = \frac{mv^2}{r} \qquad \Longrightarrow \qquad r = \frac{v}{2\omega\cos\theta}\,. \tag{1}$$

The frequency of the circular motion is

$$\omega' = \frac{v}{r} = 2\omega\cos\theta. \tag{2}$$

REMARKS: To get a rough idea of the size of the circle, you can show (using $\omega \approx 7.3 \cdot 10^{-5}\,\text{s}^{-1}$) that $r \approx 10\,\text{km}$ when $v = 1\,\text{m/s}$ and $\theta = 45^\circ$. Even the tiniest bit of friction will clearly make this effect essentially impossible to see.

For the $\theta \approx \pi/2$ (that is, near the equator), the component of the Coriolis force along the surface is negligible, so r becomes large, and ω' goes to 0.

For the $\theta \approx 0$ (that is, near the north pole), the Coriolis force essentially points along the surface. The above equations give $r \approx v/(2\omega)$, and $\omega' \approx 2\omega$. For the special case where the center of the circle is the north pole, this $\omega' \approx 2\omega$ result might seem incorrect, because you might want to say that the circular motion should be achieved by having the puck remain motionless in the inertial fame, while the earth rotates beneath it (thus making $\omega' = \omega$). The error in this reasoning is that the "level" earth is not spherical, due to the non-radial direction of the effective gravity (the combination of the gravitational and centrifugal forces). If the puck starts out motionless in the inertial frame, it will be drawn toward the north pole, due to the component of gravity in that direction. In order to not fall toward the pole, the puck needs to travel with frequency ω (relative to the inertial frame) in the direction opposite to the earth's rotation.[1] The puck therefore moves at frequency 2ω relative to the frame of the earth.

[1] In the rotating frame of the puck, the puck then feels the same centrifugal force that it would feel if it were at rest on the earth, spinning with it. It therefore happily stays at the same θ value on the "level" surface, just as a puck at rest on the earth does.

Solution

Week 54 (9/22/03)

Rolling the die

To get a feel for the problem, you can work out the answer for small values of N. For $N = 1$, the probability that the first player wins is 1. For $N = 2$, it is 3/4. And for $N = 3$, it is 19/27. A pattern in these numbers is more evident if we instead list the probabilities that the first player loses. These are 0, 1/4, and 8/27. (And if you work things out for $N = 4$, you'll obtain 81/256.) We therefore guess that the probability, P_L, that the first player *loses* is

$$P_L = \left(1 - \frac{1}{N}\right)^N. \tag{1}$$

We'll prove this by proving the following more general claim. Eq. (1) is the special case of the claim with $r = 0$.

Claim: *Let L_r be the probability that a player loses, given that a roll of r has just occurred. Then*

$$L_r = \left(1 - \frac{1}{N}\right)^{N-r}. \tag{2}$$

Proof: Assume that a roll of r has just occurred. To determine the probability, L_r, that the player who goes next loses, let's consider the probability, $1 - L_r$, that she wins. In order to win, she must roll a number, a, greater than r (each of which occurs with probability $1/N$); and her opponent must then lose, given that he has to beat a roll of a (this occurs with probability L_a). So the probability of winning, given that one must beat a roll of r, is

$$1 - L_r = \frac{1}{N}(L_{r+1} + L_{r+2} + \cdots + L_N). \tag{3}$$

If we write down the analogous equation using $r - 1$ instead of r,

$$1 - L_{r-1} = \frac{1}{N}(L_r + L_{r+1} + \cdots + L_N), \tag{4}$$

and then subtract eq. (4) from eq. (3), we obtain

$$L_{r-1} = \left(1 - \frac{1}{N}\right)L_r, \tag{5}$$

for all r from 1 to N. Using $L_N = 1$, we find that

$$L_r = \left(1 - \frac{1}{N}\right)^{N-r} \qquad (0 \le r \le N). \ \blacksquare \tag{6}$$

We may consider the first player to start out with a roll of $r = 0$ having just occurred. Therefore, the probability that the first player wins is $1 - (1 - 1/N)^N$. For large N, this probability approaches $1 - 1/e \approx 63.2\%$. For a standard die with $N = 6$, it takes the value $1 - (5/6)^6 \approx 66.5\%$.

Note that the probability that the first player wins can be written as

$$1-\left(1-\frac{1}{N}\right)^{N}=\frac{1}{N}\left(\left(1-\frac{1}{N}\right)^{N-1}+\left(1-\frac{1}{N}\right)^{N-2}+\cdots+\left(1-\frac{1}{N}\right)^{1}+1\right). \quad (7)$$

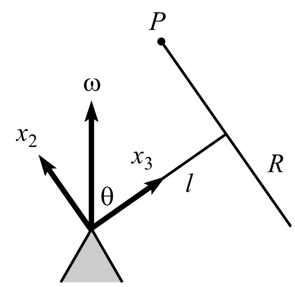
The right-hand side shows explicitly the probabilities of winning, depending on what the first roll is. For example, the first term on the right-hand side is the probability, $1/N$, that the first player rolls a 1, times the probability, $(1-1/N)^{N-1}$, that the second player loses given that he must beat a 1.

Solution

Week 55 (9/29/03)

Fixed highest point

For the desired motion, the important thing to note is that every point in the top moves in a fixed circle around the $\hat{\mathbf{z}}$-axis. Therefore, $\boldsymbol{\omega}$ points vertically. Hence, if Ω is the frequency of precession, we have $\boldsymbol{\omega} = \Omega\hat{\mathbf{z}}$, as shown.



We'll need to use $\boldsymbol{\tau} = d\mathbf{L}/dt$ to solve this problem, so let's first calculate $\mathbf{L}$. With the pivot as the origin, the principle axes are shown above (and $\hat{\mathbf{x}}_1$ points into the page, but it won't come into play here). The principal moments are

$$I_3 = \frac{MR^2}{2}, \qquad \text{and} \qquad I \equiv I_1 = I_2 = M\ell^2 + \frac{MR^2}{4}, \tag{1}$$

where we have used the parallel-axis theorem to obtain the latter. The components of $\boldsymbol{\omega}$ along the principal axes are $\omega_3 = \Omega\cos\theta$, and $\omega_2 = \Omega\sin\theta$. Therefore, we have

$$\begin{aligned} \mathbf{L} &= I_3\omega_3\hat{\mathbf{x}}_3 + I_2\omega_2\hat{\mathbf{x}}_2 \\ &= I_3\Omega\cos\theta\hat{\mathbf{x}}_3 + I\Omega\sin\theta\hat{\mathbf{x}}_2, \end{aligned} \tag{2}$$

where we have kept things in terms of the moments, I_3 and I, to be general for now. The horizontal component of $\mathbf{L}$ is

$$\begin{aligned} L_\perp &= L_3\sin\theta - L_2\cos\theta \\ &= (I_3\Omega\cos\theta)\sin\theta - (I\Omega\sin\theta)\cos\theta \\ &= (I_3 - I)\Omega\cos\theta\sin\theta, \end{aligned} \tag{3}$$

with rightward taken to be positive. This horizontal component spins around in a circle with frequency Ω. Therefore, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = L_\perp\Omega = \Omega^2\sin\theta\cos\theta(I_3 - I), \tag{4}$$

and it is directed into the page (or out of the page, if this quantity is negative). $d\mathbf{L}/dt$ must equal the torque, which has magnitude $|\boldsymbol{\tau}| = Mg\ell\sin\theta$ and is directed into the page. Therefore,

$$\Omega = \sqrt{\frac{Mg\ell}{(I_3 - I)\cos\theta}}\,. \tag{5}$$

We see that for a general symmetric top, the desired precessional motion (where the same "side" always points up) is possible only if

$$I_3 > I. \tag{6}$$

Note that this condition is independent of θ. For the problem at hand, I_3 and I are given in eq. (1), so we find

$$\Omega = \sqrt{\frac{4g\ell}{(R^2 - 4\ell^2)\cos\theta}}\,. \tag{7}$$

The necessary condition for the desired motion to exist is therefore

$$R > 2\ell. \tag{8}$$

REMARKS:

1. Given that the desired motion does indeed exist, it is intuitively clear that Ω should become very large as $\theta \to \pi/2$. But it is by no means intuitively clear (at least to me) that such motion should exist at all for angles near $\pi/2$.
2. Ω approaches a non-zero constant as $\theta \to 0$, which isn't entirely obvious.
3. If both R and ℓ are scaled up by the same factor, we see that Ω decreases. This also follows from dimensional analysis.
4. The condition $I_3 > I$ can be understood in the following way. If $I_3 = I$, then $\mathbf{L} \propto \boldsymbol{\omega}$, and so $\mathbf{L}$ points vertically along $\boldsymbol{\omega}$. If $I_3 > I$, then $\mathbf{L}$ points somewhere to the right of the $\hat{\mathbf{z}}$-axis (at the instant shown in the figure above). This means that the tip of $\mathbf{L}$ is moving into the page, along with the top. This is what we need, because $\boldsymbol{\tau}$ points into the page. If, however, $I_3 < I$, then $\mathbf{L}$ points somewhere to the left of the $\hat{\mathbf{z}}$-axis, so $d\mathbf{L}/dt$ points out of the page, and hence cannot be equal to $\boldsymbol{\tau}$.

Solution

Week 56 (10/6/03)

Stirling's formula

Let's first prove the result,

$$N! = \int_0^\infty x^N e^{-x} dx \equiv I_N. \tag{1}$$

The proof by induction proceeds as follows. Integrating by parts gives

$$\int_0^\infty x^N e^{-x} dx = -x^N e^{-x} dx \Big|_0^\infty + N \int_0^\infty x^{N-1} e^{-x} dx. \tag{2}$$

The first term on the right-hand side is zero, so we have $I_N = N I_{N-1}$. Therefore, if $I_{N-1} = (N-1)!$, then $I_N = N!$. Since it is indeed true that $I_0 = 0! = 1$, we see that $I_N = N!$ for all N.

Let us now write $x^N e^{-x}$ as $\exp(N \ln x - x) \equiv \exp(f(x))$, and then expand $f(x)$ in a Taylor series about its maximum, which occurs at $x = N$. Computing the first two derivatives of $f(x)$, evaluated at N, gives the following Taylor series in the exponent.

$$\begin{aligned} N! &= \int_0^\infty \exp\left(-N + N \ln N - \frac{(x-N)^2}{2N} + \cdots\right) dx \\ &\approx N^N e^{-N} \int_0^\infty \exp\left(-\frac{(x-N)^2}{2N}\right) dx. \end{aligned} \tag{3}$$

If N is very large, we can let the integral run from $-\infty$ to ∞, with negligible error. Letting $y \equiv x - N$, we have

$$\begin{aligned} N! &\approx N^N e^{-N} \int_{-\infty}^\infty e^{-y^2/2N} dy \\ &= N^N e^{-N} \sqrt{2\pi N}, \end{aligned} \tag{4}$$

as desired.

Remark: We have used the fact that $\int_{-\infty}^\infty e^{-y^2} dy = \sqrt{\pi}$. This can be proved in the following way, where we make use of a change of variables from cartesian to polar coordinates. Let I be the desired integral. Then

$$\begin{aligned} I^2 &= \int_{-\infty}^\infty e^{-x^2} dx \int_{-\infty}^\infty e^{-y^2} dy \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)}\, dx\, dy \\ &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r\, dr\, d\theta \\ &= 2\pi \left(-\frac{e^{-r^2}}{2}\right) \Big|_0^\infty \\ &= \pi. \end{aligned} \tag{5}$$

A change of variables then gives $\int_{-\infty}^{\infty} e^{-y^2/n}\,dy = \sqrt{n\pi}$.

The calculation of the higher-order corrections is a bit messier, because we have to keep track of more terms in the Taylor expansion of $f(x)$. Let's find the order-$1/N$ correction first. Our strategy will be to write the integrand in eq. (1) as a gaussian plus small corrections. Computing the first four derivatives of $f(x)$, evaluated at N, gives us the following Taylor series in the exponent (letting $y \equiv x - N$, and letting the limits of integration run from $-\infty$ to ∞).

$$\begin{aligned} N! &= \int_{\infty}^{\infty} \exp\left(-N + N\ln N - \frac{y^2}{2N} + \frac{y^3}{3N^2} - \frac{y^4}{4N^3} + \cdots\right) dy \\ &\approx N^N e^{-N} \int_{\infty}^{\infty} \exp\left(-\frac{y^2}{2N}\right) \exp\left(\frac{y^3}{3N^2} - \frac{y^4}{4N^3}\right) dy \\ &\approx N^N e^{-N} \int_{\infty}^{\infty} \exp\left(-\frac{y^2}{2N}\right) \left(1 + \left[\frac{y^3}{3N^2} - \frac{y^4}{4N^3}\right]\right. \\ &\qquad\qquad \left. + \frac{1}{2}\left[\frac{y^3}{3N^2} - \frac{y^4}{4N^3}\right]^2 + \cdots\right) dy. \end{aligned} \tag{6}$$

Since terms with odd powers of y integrate to zero, we obtain (to leading orders in $1/N$),

$$N! \approx N^N e^{-N} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2N}\right) \left(1 - \frac{y^4}{4N^3} + \frac{1}{2}\left[\frac{y^3}{3N^2}\right]^2 + \cdots\right) dy. \tag{7}$$

At this point, we need to know how to calculate integrals of the form $\int_{-\infty}^{\infty} x^{2n} e^{-ax^2}\,dx$. Using $\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\pi}a^{-1/2}$, and successively differentiating with respect to a, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-ax^2}\,dx &= \sqrt{\pi}a^{-1/2}, \\ \int_{-\infty}^{\infty} x^2 e^{-ax^2}\,dx &= \frac{1}{2}\sqrt{\pi}a^{-3/2}, \\ \int_{-\infty}^{\infty} x^4 e^{-ax^2}\,dx &= \frac{3}{4}\sqrt{\pi}a^{-5/2}, \\ \int_{-\infty}^{\infty} x^6 e^{-ax^2}\,dx &= \frac{15}{8}\sqrt{\pi}a^{-7/2}. \end{aligned} \tag{8}$$

Letting $a \equiv 1/2N$ here, eq. (7) gives

$$\begin{aligned} N! &\approx N^N e^{-N} \sqrt{\pi}\left((2N)^{1/2} - \frac{1}{4N^3}\frac{3}{4}(2N)^{5/2} + \frac{1}{18N^4}\frac{15}{8}(2N)^{7/2}\right) \\ &= N^N e^{-N} \sqrt{2\pi N}\left(1 + \frac{1}{12N}\right). \end{aligned} \tag{9}$$

Note that to obtain all the terms of order $1/N$, it is necessary to include the $(y^3/3N^2)^2$ term in eq. (7). This is an easy term to forget.

If you like these sorts of calculations, you can go a step further and find the order-$1/N^2$ correction. It turns out that you have to keep terms out to the $-y^6/6N^5$ term in the expansion in the first line of eq. (6). Furthermore, you must keep terms out to the $[\cdots]^4/4!$ term in the expansion in the last line of eq. (6). The relevant extra terms that take the place of the "$\cdots$" in eq. (7) then turn out to be

$$\begin{aligned}&\left[-\frac{y^6}{6N^5}\right]+\frac{1}{2}\left[\left(-\frac{y^4}{4N^3}\right)^2+2\left(\frac{y^3}{3N^2}\right)\left(\frac{y^5}{5N^4}\right)\right]\\&+\frac{1}{3!}\left[3\left(\frac{y^3}{3N^2}\right)^2\left(-\frac{y^4}{4N^3}\right)\right]+\frac{1}{4!}\left[\left(\frac{y^3}{3N^2}\right)^4\right],\end{aligned} \tag{10}$$

where we have grouped these terms via square brackets according to which term in the e^z series expansion in the last line of eq. (6) they come from. To do all of the necessary integrals in the modified eq. (7), we'll need the next three integrals in the list in eq. (8). They are

$$\begin{aligned}\int_{-\infty}^{\infty}x^8e^{-ax^2}\,dx &= \frac{3\cdot5\cdot7}{2^4}\sqrt{\pi}a^{-9/2},\\ \int_{-\infty}^{\infty}x^{10}e^{-ax^2}\,dx &= \frac{3\cdot5\cdot7\cdot9}{2^5}\sqrt{\pi}a^{-11/2},\\ \int_{-\infty}^{\infty}x^{12}e^{-ax^2}\,dx &= \frac{3\cdot5\cdot7\cdot9\cdot11}{2^6}\sqrt{\pi}a^{-13/2}.\end{aligned} \tag{11}$$

Putting the terms of eq. (10) in place of the "$\cdots$" in eq. (7), we find that the coefficient of $N^Ne^{-N}\sqrt{2\pi N}$ equals $1/N^2$ times

$$\begin{aligned}&-\frac{1}{6}\frac{3\cdot5}{2^3}2^3+\frac{1}{2}\left(\frac{1}{16}+\frac{2}{15}\right)\frac{3\cdot5\cdot7}{2^4}2^4-\frac{1}{3!}\frac{3}{36}\frac{3\cdot5\cdot7\cdot9}{2^5}2^5+\frac{1}{4!}\frac{1}{81}\frac{3\cdot5\cdot7\cdot9\cdot11}{2^6}2^6\\&\quad=\frac{1}{288}\,.\end{aligned} \tag{12}$$

Therefore, we may write Stirling's formula as

$$N!\approx N^Ne^{-N}\sqrt{2\pi N}\left(1+\frac{1}{12N}+\frac{1}{288N^2}\right). \tag{13}$$

This result of 1/288 is rather fortuitous, because it is the third term in the Taylor series for $e^{1/12}$. This means that we can write $N!$ as

$$\begin{aligned}N! &= N^Ne^{-N}\sqrt{2\pi N}\Big(e^{1/12N}+\mathcal{O}(1/N^3)\Big)\\ &\approx N^Ne^{-N+1/12N}\sqrt{2\pi N}.\end{aligned} \tag{14}$$

It turns out that the order-$1/N^3$ correction is *not* equal to $1/(3!\cdot12^3)$, which is the next term in the expansion for $e^{1/12}$.

Solution

Week 57 (10/13/03)

Throwing a beach ball

On both the way up and the way down, the total force on the ball is

$$F = -mg - m\alpha v. \tag{1}$$

On the way up, v is positive, so the drag force points downward, as it should. And on the way down, v is negative, so the drag force points upward.

Our strategy for finding v_f will be to produce two different expressions for the maximum height, h, and then equate them. We'll find these two expressions by considering the upward and then the downward motion of the ball. In doing so, we will need to write the acceleration of the ball as

$$a = \frac{dv}{dt} = \frac{dy}{dt}\frac{dv}{dy} = v\frac{dv}{dy}\,. \tag{2}$$

For the upward motion, $F = ma$ gives

$$\begin{aligned} -mg - m\alpha v &= mv\frac{dv}{dy} \\ \Longrightarrow \qquad \int_0^h dy &= -\int_{v_0}^0 \frac{v\,dv}{g+\alpha v}\,. \end{aligned} \tag{3}$$

where we have taken advantage of the fact that we know that the speed of the ball at the top is zero. Writing $v/(g+\alpha v)$ as $[1 - g/(g+\alpha v)]/\alpha$, we may evaluate the integral to obtain

$$h = \frac{v_0}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 + \frac{\alpha v_0}{g}\right). \tag{4}$$

Now let us consider the downward motion. Let v_f be the final speed, which is a positive quantity. The final velocity is then the negative quantity, $-v_f$. Using $F = ma$, we similarly obtain

$$\int_h^0 dy = -\int_0^{-v_f} \frac{v\,dv}{g+\alpha v}\,. \tag{5}$$

Performing the integration (or just replacing the v_0 in eq. (4) with $-v_f$) gives

$$h = -\frac{v_f}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 - \frac{\alpha v_f}{g}\right). \tag{6}$$

Equating the expressions for h in eqs. (4) and (6) gives an implicit equation for v_f in terms of v_0,

$$v_0 + v_f = \frac{g}{\alpha}\ln\left(\frac{g+\alpha v_0}{g - \alpha v_f}\right). \tag{7}$$

Remarks: In the limit of small α (more precisely, in the limit $\alpha v_0/g \ll 1$), we can use $\ln(1+x) = x - x^2/2 + \cdots$ to obtain approximate values for h in eqs. (4) and (6). The results are, as expected,

$$h \approx \frac{v_0^2}{2g}\,, \qquad \text{and} \qquad h \approx \frac{v_f^2}{2g}\,. \tag{8}$$

We can also make approximations for large α (or large $\alpha v_0/g$). In this limit, the log term in eq. (4) is negligible, so we obtain $h \approx v_0/\alpha$. And eq. (6) gives $v_f \approx g/\alpha$, because the argument of the log must be very small in order to give a very large negative number, which is needed to produce a positive h on the left-hand side. There is no way to relate v_f and h is this limit, because the ball quickly reaches the terminal velocity of $-g/\alpha$ (which is the velocity that makes the net force equal to zero), independent of h.

Let's now find the times it takes for the ball to go up and to go down. We'll present two methods for doing this.

First method: Let T_1 be the time for the upward path. If we write the acceleration of the ball as $a = dv/dt$, then $F = ma$ gives

$$\begin{aligned} -mg - m\alpha v &= m\frac{dv}{dt} \\ \Longrightarrow \qquad \int_0^{T_1} dt &= -\int_{v_0}^0 \frac{dv}{g + \alpha v}\,. \end{aligned} \tag{9}$$

$$T_1 = \frac{1}{\alpha}\ln\left(1 + \frac{\alpha v_0}{g}\right). \tag{10}$$

In a similar manner, we find that the time T_2 for the downward path is

$$T_2 = -\frac{1}{\alpha}\ln\left(1 - \frac{\alpha v_f}{g}\right). \tag{11}$$

Therefore,

$$T_1 + T_2 = \frac{1}{\alpha}\ln\left(\frac{g + \alpha v_0}{g - \alpha v_f}\right). \tag{12}$$

Using eq. (7), we have

$$T_1 + T_2 = \frac{v_0 + v_f}{g}\,. \tag{13}$$

This is shorter than the time in vacuum (namely $2v_0/g$) because $v_f < v_0$.

Second method: The very simple form of eq. (13) suggests that there is a cleaner way to calculate the total time of flight. And indeed, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way up, we obtain $-v_0 = -gT_1 - \alpha h$ (because $\int v\,dt = h$). Likewise, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way down, we obtain $-v_f = -gT_2 + \alpha h$ (because $\int v\,dt = -h$). Adding these two results gives eq. (13). This procedure only works, of course, because the drag force is proportional to v.

Remarks: The fact that the time here is shorter than the time in vacuum isn't obvious. On one hand, the ball doesn't travel as high in air as it would in vacuum (so you might think that $T_1 + T_2 < 2v_0/g$). But on the other hand, the ball moves slower in air (so you might think that $T_1 + T_2 > 2v_0/g$). It isn't obvious which effect wins, without doing a calculation.

For any α, you can use eq. (10) to show that $T_1 < v_0/g$. But T_2 is harder to get a handle on, because it is given in terms of v_f. But in the limit of large α, the ball quickly reaches terminal velocity, so we have $T_2 \approx h/v_f \approx (v_0/\alpha)/(g/\alpha) = v_0/g$. Interestingly, this is the same as the downward (and upward) time for a ball thrown in vacuum.

Solution

Week 58 (10/20/03)

Coins and Gaussians

There are $\binom{2N}{N+x}$ ways to obtain $N+x$ heads in $2N$ flips. Therefore, the probability of obtaining $N+x$ heads is

$$P(x) = \frac{1}{2^{2N}}\binom{2N}{N+x}. \tag{1}$$

Our goal is to find an approximate expression for $\binom{2N}{N+x}$ when N is large. Using Stirling's formula,[1] $N! \approx N^N e^{-N}\sqrt{2\pi N}$, we have

$$\begin{aligned}
\binom{2N}{N+x} &= \frac{(2N)!}{(N+x)!(N-x)!} \\
&\approx \frac{(2N)^{2N}\sqrt{2N}}{(N+x)^{N+x}(N-x)^{N-x}\sqrt{2\pi}\sqrt{N^2-x^2}} \\
&= \frac{2^{2N}\sqrt{N}}{\left(1+\frac{x}{N}\right)^{N+x}\left(1-\frac{x}{N}\right)^{N+x}\sqrt{\pi}\sqrt{N^2-x^2}}\,.
\end{aligned} \tag{2}$$

Now,

$$\begin{aligned}
\left(1+\frac{x}{N}\right)^{N+x} &= \exp\left((N+x)\ln\left(1+\frac{x}{N}\right)\right) \\
&= \exp\left((N+x)\left(\frac{x}{N}-\frac{x^2}{2N^2}+\cdots\right)\right) \\
&\approx \exp\left(x+\frac{x^2}{2N}\right).
\end{aligned} \tag{3}$$

Likewise,

$$\left(1-\frac{x}{N}\right)^{N+x} \approx \exp\left(-x+\frac{x^2}{2N}\right). \tag{4}$$

Therefore, eq. (2) becomes

$$\binom{2N}{N+x} \approx \frac{2^{2N}e^{-x^2/N}}{\sqrt{\pi N}}\,, \tag{5}$$

where we have set $\sqrt{N^2-x^2} \approx N$, because the exponential factor shows that only x up to order $\sqrt{N}$ contribute significantly. Note that it is necessary to expand the log in eq. (3) to second order to obtain the correct result. Using eq. (5) in eq. (1) gives the desired result,

$$P(x) \approx \frac{e^{-x^2/N}}{\sqrt{\pi N}}\,. \tag{6}$$

Note that if we integrate this probability over x, we do indeed obtain 1, because $\int_{-\infty}^{\infty} e^{-x^2/N}\,dx = \sqrt{\pi N}$ (see the solution to Problem of the Week 56).

[1] From Problem of the Week 56.

Remark: To find where eq. (6) is valid, we can expand the log factor in eq. (3) to fourth order in x. We then obtain

$$P(x) \approx \frac{e^{-x^2/N}}{\sqrt{\pi N}} e^{-x^4/6N^3}. \tag{7}$$

Therefore, when $x \sim N^{3/4}$, eq. (6) is not valid. However, when $x \sim N^{3/4}$, the $e^{-x^2/N}$ factor in $P(x)$ makes it negligibly small, so $P(x)$ is essentially zero in any case.

Solution

Week 59 (10/27/03)

Getting way ahead

The explanation of why the two clocks show different times in the ground frame is the following. The rocket becomes increasingly length contracted in the ground frame, which means that the front end isn't traveling as fast as the back end. Therefore, the time-dilation factor for the front clock isn't as large as that for the back clock. So the front clock loses less time relative to the ground, and hence ends up ahead of the back clock. Of course, it's not at all obvious that everything works out quantitatively, and that the front clock eventually ends up an arbitrarily large time ahead of the back clock. In fact, it's quite surprising that this is the case, since the above difference in speeds is rather small. But let's now show that the above explanation does indeed account for the difference in the clock readings.

Let the back of the rocket be located at position x. Then the front is located at position $x+L\sqrt{1-v^2}$, due to the length contraction. (We will set $c=1$ throughout this solution.) Taking the time derivatives of the two positions, we see that the speeds of the back and front are (with $v \equiv dx/dt$)[1]

$$v_b = v, \qquad v_f = v(1 - L\gamma\dot{v}), \qquad \text{where} \;\; \gamma \equiv 1/\sqrt{1-v^2}. \tag{1}$$

We must now find v in terms of the time, t, in the ground frame. The quickest way to do this is to use the fact that longitudinal forces are independent of the frame.[2] The force on, say, the astronaut, is $f = mg$ in the spaceship frame, so it must also be mg in the ground frame. Therefore, $F = dp/dt$ in the ground frame gives (using the fact that g is constant)

$$mg = \frac{d(m\gamma v)}{dt} \qquad \Longrightarrow \qquad gt = \gamma v \qquad \Longrightarrow \qquad v = \frac{gt}{\sqrt{1+(gt)^2}}\,. \tag{2}$$

Remark: We can also find $v(t)$ by using the result of Problem of the Week 51, which says that in terms of the proper time, τ, of a uniformly-accelerated particle, its speed is given by $v(\tau) = \tanh(g\tau)$. This yields $\gamma = 1/\sqrt{1-v^2} = \cosh(g\tau)$. If we then integrate $dt = \gamma\, d\tau$, we obtain $gt = \sinh(g\tau)$. This gives

$$v(t) = \tanh(g\tau) = \frac{\sinh(g\tau)}{\cosh(g\tau)} = \frac{\sinh(g\tau)}{\sqrt{1+\sinh^2(g\tau)}} = \frac{gt}{\sqrt{1+(gt)^2}}\,. \quad \blacksquare \tag{3}$$

Having found v, we must now find the γ-factors associated with the speeds of the front and back of the rocket. The γ-factor associated with the speed of the back (namely v) is

$$\gamma_b = \frac{1}{\sqrt{1-v^2}} = \sqrt{1+(gt)^2}\,. \tag{4}$$

[1] Since these speeds are not equal, there is of course an ambiguity concerning which speed we should use in the length-contraction factor, $\sqrt{1-v^2}$. Equivalently, the rocket actually doesn't have one inertial frame that describes all of it. But you can show that any differences arising from this ambiguity are of higher order in gL/c^2 than we need to be concerned with.

[2] This takes a little effort to prove, but we'll just accept it here. If you want a self-contained method for finding v, we'll give another one in the remark below.

The γ-factor associated with the speed of the front, $v_f = v(1 - L\gamma\dot{v})$, is a little harder to obtain. We must first calculate $\dot{v}$. From eq. (2), we find $\dot{v} = g/(1+g^2t^2)^{3/2}$, which gives

$$v_f = v(1 - L\gamma\dot{v}) = \frac{gt}{\sqrt{1+(gt)^2}}\left(1 - \frac{gL}{1+g^2t^2}\right). \tag{5}$$

The γ-factor (or rather $1/\gamma$, which is what we'll be concerned with) associated with this speed can now be found as follows. In the first line below, we ignore the higher-order $(gL)^2$ term, because it is really $(gL/c^2)^2$, and we are assuming that gL/c^2 is small. And in obtaining the third line, we use the Taylor-series approximation $\sqrt{1-\epsilon} \approx 1 - \epsilon/2$.

$$\begin{aligned}
\frac{1}{\gamma_f} = \sqrt{1 - v_f^2} &\approx \sqrt{1 - \frac{g^2t^2}{1+g^2t^2}\left(1 - \frac{2gL}{1+g^2t^2}\right)} \\
&= \frac{1}{\sqrt{1+g^2t^2}}\sqrt{1 + \frac{2g^3t^2L}{1+g^2t^2}} \\
&\approx \frac{1}{\sqrt{1+g^2t^2}}\left(1 + \frac{g^3t^2L}{1+g^2t^2}\right). \qquad (6)
\end{aligned}$$

We can now calculate the time that each clock shows, at time t in the ground frame. The time on the back clock changes according to $dt_b = dt/\gamma_b$, so eq. (4) gives

$$t_b = \int_0^t \frac{dt}{\sqrt{1+g^2t^2}}. \tag{7}$$

The integral[3] of $1/\sqrt{1+x^2}$ is $\sinh^{-1} x$. Letting $x \equiv gt$, this gives

$$gt_b = \sinh^{-1}(gt), \tag{8}$$

in agreement with the results in the above remark. The time on the front clock changes according to $dt_f = dt/\gamma_f$, so eq. (6) gives

$$t_f = \int_0^t \frac{dt}{\sqrt{1+g^2t^2}} + \int_0^t \frac{g^3t^2L\,dt}{(1+g^2t^2)^{3/2}}. \tag{9}$$

The integral[4] of $x^2/(1+x^2)^{3/2}$ is $\sinh^{-1} x - x/\sqrt{1+x^2}$. Letting $x \equiv gt$, this gives

$$gt_f = \sinh^{-1}(gt) + (gL)\left(\sinh^{-1}(gt) - \frac{gt}{\sqrt{1+g^2t^2}}\right). \tag{10}$$

Using eqs. (8) and (2), we may rewrite this as

$$gt_f = gt_b(1 + gL) - gLv. \tag{11}$$

Dividing by g, and putting the c's back in to make the units correct, gives

$$t_f = t_b\left(1 + \frac{gL}{c^2}\right) - \frac{Lv}{c^2}, \tag{12}$$

as we wanted to show.

[3] To derive this, make the substitution $x \equiv \sinh\theta$.

[4] Again, to derive this, make the substitution $x \equiv \sinh\theta$.

Solution

Week 60 (11/3/03)

Cereal box prizes

First Solution: Assume that you have collected c of the colors, and let B_c be the number of boxes it takes to get the next color. The average value of B_c, which we will denote by $\overline{B}_c$, may be found as follows. The probability that a box yields a new color is $1-c/N$, and the probability that it does not is c/N. The expected number of boxes to get the next prize is therefore

$$\overline{B}_c = 1\left(1-\frac{c}{N}\right) + 2\left(\frac{c}{N}\right)\left(1-\frac{c}{N}\right) + 3\left(\frac{c}{N}\right)^2\left(1-\frac{c}{N}\right) + \cdots. \tag{1}$$

Letting $x \equiv c/N$, we have

$$\begin{aligned}
\overline{B}_c &= (1-x)(1+2x+3x^2+4x^3+\cdots) \\
&= (1-x)\Big((1+x+x^2+\cdots)+(x+x^2+\cdots)+(x^2+\cdots)+\cdots\Big) \\
&= (1-x)\left(\frac{1}{1-x}+\frac{x}{1-x}+\frac{x^2}{1-x}\right)+\cdots \\
&= \frac{1}{1-x}\,.
\end{aligned} \tag{2}$$

Therefore,

$$\overline{B}_c = \frac{N}{N-c}\,. \tag{3}$$

Note that the above $1+2x+3x^2+\cdots = 1/(1-x)^2$ result, for $0 \le x < 1$, may also be derived by taking the derivative of the equality $1+x+x^2+\cdots = 1/(1-x)$.

We see that (with $c=0$) it takes one box, of course, to get the first color. Then (with $c=1$) it takes an average of $N/(N-1)$ boxes to get the second color. Then (with $c=2$) it takes an average of $N/(N-2)$ boxes to get the third color. This continues until (with $c=N-1$) it takes an average of N boxes to get the last color.

We wish to find the average, $\overline{B}$, of the total number of boxes needed to get all the colors. $\overline{B}$ can be viewed as the average of the sum of the waiting times for each new color. But this equals the sum of the average waiting times, which were calculated above. In other words, letting B be the total number of boxes in a particular trial, and letting B_0, B_1, …, B_{N-1} be the waiting times during that particular trial, we have (with a bar denoting average)

$$\begin{aligned}
\overline{B} &= \overline{(B_0+B_1+\cdots+B_{N-1})} \\
&= \overline{B}_0+\overline{B}_1+\cdots+\overline{B}_{N-1} \\
&= N\left(\frac{1}{N}+\frac{1}{N-1}+\cdots+\frac{1}{2}+1\right).
\end{aligned} \tag{4}$$

For large N, this goes like

$$\overline{B} \approx N(\ln N + \gamma), \tag{5}$$

where $\gamma \approx 0.577$ is Euler's constant.

Second Solution: We will calculate the probability, $P(n)$, that the final color is obtained in the nth box. The desired expectation value is then $\sum_0^\infty nP(n)$.

Claim: $P(n) = \sum_{k=1}^{N-1} (-1)^{k-1} \binom{N-1}{k-1} \left(1 - \frac{k}{N}\right)^{n-1} \qquad (n = 2, 3, \ldots).$

Proof: Let us first calculate the probability, $p(n)$, that you have obtained all the colors by the time you have looked in the nth box. $P(n)$ is then given by $P(n) = p(n) - p(n-1)$.

Assume that n boxes have been bought. Then there is a total of N^n equally likely possibilities for the way the colors can turn up in these n boxes. In finding $p(n)$, we will need to subtract from N^n the number of possibilities that do *not* have at least one prize of each color. We can count this number is the following way.

If (at least) color *1* is missing, then there are $(N-1)^n$ different combinations. Likewise for the situations where another color is missing. So there seem to be $N(N-1)^n$ combinations missing a color. However, we have double-counted some of the cases. For example, a combination which has (at least) *1* and *2* missing has been counted twice; there are $(N-2)^n$ of these. Likewise for all the other pairs of colors. So we must subtract off $\binom{N}{2}(N-2)^n$ combinations. But now a combination which has (at least) *1*, *2*, and *3* missing has not been counted at all (because we have included it three times, and then subtracted it off three times); there are $(N-3)^n$ of these. Likewise for the other triplets. So we must add on $\binom{N}{3}(N-3)^n$ combinations. Now, however, the combinations with (at least) *1*, *2*, *3*, and *4* missing have been counted $\binom{4}{1} - \binom{4}{2} + \binom{4}{3} = 2$ times. Likewise for the other quadruplets. So we must subtract off $\binom{N}{4}(N-4)^n$ combinations.

In general, if we have done this procedure up to $(k-1)$-tuplets, then the combinations missing (at least) k of the colors have been counted T times, where

$$T = \binom{k}{1} - \binom{k}{2} + \cdots + (-1)^k \binom{k}{k-1}. \tag{6}$$

However, the binomial expansion gives

$$\begin{aligned} 0 &= (1-1)^k \\ &= 1 - \binom{k}{1} + \binom{k}{2} + \cdots + (-1)^{k-1}\binom{k}{k-1} + (-1)^k \\ &= 1 - T + (-1)^k. \end{aligned} \tag{7}$$

Therefore, $T = 2$ for even k, and $T = 0$ for odd k. So we have either overcounted by one, or undercounted by one. Hence, the total number of combinations missing at least one color is

$$\binom{N}{1}(N-1)^n - \binom{N}{2}(N-2)^n + \cdots + (-1)^N \binom{N}{N-1}(1)^n. \tag{8}$$

To obtain the probability, $p(n)$, that you have collected all the colors by the nth box, we must subtract this number from N^n, and then divide the result by N^n. We obtain

$$p(n) = \sum_{k=0}^{N-1} (-1)^k \binom{N}{k} \left(1 - \frac{k}{N}\right)^n, \qquad (n = 1, 2, 3, \ldots). \tag{9}$$

Remark: $p(n)$ must of course equal zero, for $n = 1, 2, \ldots, N-1$, because you need to buy at least N boxes to get all the colors. To show this explicitly, first note that $p(n)$ may be written as

$$p(n) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \left(1 - \frac{k}{N}\right)^n, \tag{10}$$

where we have let the sum run up to $k = N$, because the $k = N$ term is zero anyway. It is therefore sufficient to demonstrate that $\sum_{k=0}^{N} (-1)^k \binom{N}{k} k^m = 0$, for $m = 0, 1, \ldots N-1$, because this identity will make all the separate terms arising from the binomial expansion of $(1 - k/N)^n$ in eq. (10) equal to zero on their own. You can prove this identity by taking successive derivatives of the relation $(1-x)^N = \sum (-1)^k \binom{N}{k} x^k$, and setting $x = 1$.

The probability, $P(n)$, that the final color is obtained in the nth box is

$$\begin{aligned}
P(n) &= p(n) - p(n-1) \\
&= \sum_{k=0}^{N-1} (-1)^k \binom{N}{k} \left(1 - \frac{k}{N}\right)^{n-1} \left(\left(1 - \frac{k}{N}\right) - 1\right) \\
&= \sum_{k=0}^{N-1} (-1)^k \frac{N!}{k!(N-k)!} \left(1 - \frac{k}{N}\right)^{n-1} \left(-\frac{k}{N}\right) \\
&= \sum_{k=1}^{N-1} (-1)^{k-1} \binom{N-1}{k-1} \left(1 - \frac{k}{N}\right)^{n-1}, \qquad (n = 2, 3, \ldots).
\end{aligned} \tag{11}$$

$P(n)$ is zero for $n = 2, 3, \ldots, N-1$. And $P(1)$ is of course also zero, but it cannot be expressed as in eq. (11), because eq. (9) is not valid for $n = 0$. ■

Having found $P(n)$, we can write the average number, $\overline{B}$, of required boxes as

$$\overline{B} = \sum_{n=2}^{\infty} n \sum_{k=1}^{N-1} (-1)^{k-1} \binom{N-1}{k-1} \left(1 - \frac{k}{N}\right)^{n-1}. \tag{12}$$

We have let the sum over n range from 2 to ∞, instead of N to ∞, because it will simplify our calculations a bit (and all the terms from $n = 2$ to $n = N-1$ are zero). Switching the order of the sums, and performing the sum over n by using a technique similar to the one used in eq. (2), we obtain

$$\begin{aligned}
\overline{B} &= \sum_{k=1}^{N-1} (-1)^{k-1} \binom{N-1}{k-1} \left(\frac{N^2}{k^2} - 1\right) \\
&= \sum_{k=1}^{N-1} (-1)^{k-1} \frac{(N-1)!}{(k-1)!(N-k)!} \left(\frac{(N+k)(N-k)}{k^2}\right) \\
&= -\sum_{k=1}^{N-1} (-1)^k \binom{N-1}{k} \left(1 + \frac{N}{k}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{k=1}^{N-1}(-1)^k\binom{N-1}{k} - \sum_{k=1}^{N-1}(-1)^k\binom{N-1}{k}\frac{N}{k} \\
&= -\Big((1-1)^{N-1}-1\Big) - N\sum_{k=1}^{N-1}(-1)^k\binom{N-1}{k}\frac{1}{k}\,. \\
&= 1 - N\sum_{k=1}^{N-1}(-1)^k\binom{N-1}{k}\frac{1}{k}\,. \qquad (13)
\end{aligned}
$$

At this point, we need the following claim.

Claim: $\displaystyle\sum_{k=1}^{M}(-1)^k\binom{M}{k}\frac{1}{k} = -\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{M}\right).$

Proof: We will prove this by using a technique similar to the one mentioned in the above Remark, except that now we will find it necessary to take an integral instead of a derivative, because we want to produce a k in the denominator. Starting with the binomial expansion, we have

$$
\begin{aligned}
\sum_{k=0}^{M}(-1)^k\binom{M}{k}x^k &= (1-x)^M \qquad (14) \\
\Longrightarrow \int\sum_{k=1}^{M}(-1)^k\binom{M}{k}x^{k-1}\,dx &= \int\frac{1}{x}\Big((1-x)^M-1\Big)\,dx \\
\Longrightarrow \sum_{k=1}^{M}(-1)^k\binom{M}{k}\frac{x^k}{k} &= -\int\frac{1-(1-x)^M}{1-(1-x)}\,dx \\
\Longrightarrow \sum_{k=1}^{M}(-1)^k\binom{M}{k}\frac{x^k}{k} &= -\int\Big(1+(1-x)+\cdots+(1-x)^{M-1}\Big)\,dx \\
\Longrightarrow \sum_{k=1}^{M}(-1)^k\binom{M}{k}\frac{x^k}{k} &= (1-x)+\frac{1}{2}(1-x)^2+\cdots+\frac{1}{M}(1-x)^M + C,
\end{aligned}
$$

where C is a constant of integration. Setting $x=0$ in the last line yields

$$
C = -\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{M}\right). \qquad (15)
$$

Setting $x=1$ in the last line then proves the claim. ■

Using the claim in eq. (13) gives

$$
\begin{aligned}
\overline{B} &= 1+N\left(\frac{1}{N-1}+\cdots+\frac{1}{2}+1\right) \\
&= N\left(\frac{1}{N}+\frac{1}{N-1}+\cdots+\frac{1}{2}+1\right), \qquad (16)
\end{aligned}
$$

in agreement with eq. (4).

Now let us maximize $P(n)$, as given in eq. (11). For convenience, we will instead maximize $P(n+1)$, so that there will be a simpler "n" in the exponent in eq. (11), instead of "$n-1$". Of course, such a distinction is irrelevant in the limit of large N. Let us assume for the moment (we will justify this below) that the values of k that contribute significantly to the sum in eq. (11) are small enough so that we may use the approximation,

$$\left(1-\frac{k}{N}\right)^n \approx \left(e^{-n/N}\right)^k. \tag{17}$$

The expression for $P(n+1)$ in eq. (11) then becomes (letting the sum run up to $k=N$, since the $k=N$ term is zero)

$$\begin{aligned}
P(n+1) &\approx e^{-n/N}\sum_{k=1}^{N}(-1)^{k-1}\binom{N-1}{k-1}\left(e^{-n/N}\right)^{k-1} \\
&= e^{-n/N}\sum_{j=0}^{N-1}(-1)^{j}\binom{N-1}{j}\left(e^{-n/N}\right)^{j} \\
&= e^{-n/N}\left(1-e^{-n/N}\right)^{N-1} \\
&\approx e^{-n/N}e^{-(N-1)e^{-n/N}}.
\end{aligned} \tag{18}$$

To maximize $P(n)$, we therefore want to minimize $n/N+(N-1)e^{-n/N}$. Taking the derivative of this with respect to n, we find that the minimum is achieved when

$$n_{\text{max}} = N\ln(N-1) \approx N\ln N, \tag{19}$$

where we have dropped terms of order 1. Using this result in eq. (18), we see that the maximum value of $P(n)$, which is obtained at $n \approx N\ln N$, equals $1/(Ne)$.

Remarks:

1. Let us now justify eq. (17). Using

$$\ln\left(1-\frac{k}{N}\right)^n = n\ln\left(1-\frac{k}{N}\right) \approx n\left(-\frac{k}{N}-\frac{k^2}{2N^2}-\cdots\right), \tag{20}$$

we have

$$\left(1-\frac{k}{N}\right)^n \approx e^{-nk/N}e^{-nk^2/2N^2}. \tag{21}$$

Therefore, eq. (17) is valid if $nk^2/N^2 \ll 1$. For n near $N\ln N$, which is the general size of n we are concerned with, this requires that $k \ll \sqrt{N/\ln N}$. But for large N and for $n \approx N\ln N$, the first few terms in the second (and hence first) line of eq. (18) dominate the sum. This is true because the binomial coefficient goes like $N^j/j!$, and the $(e^{-n/N})^j$ term goes like $1/N^j$ (using $n \approx N\ln N$). Hence, the the jth term goes like $1/j!$. Therefore, only the first few terms contribute, so the relevant k values easily satisfy the bound $k \ll \sqrt{N/\ln N}$.

Note also that the step in going from the third to fourth line in eq. (18) is valid because $e^{-n/N} \approx 1/N$ is sufficiently small.

2. For large N, the average number of boxes, and the point where $P(n)$ is maximum (given in eqs. (5) and (19), respectively) are

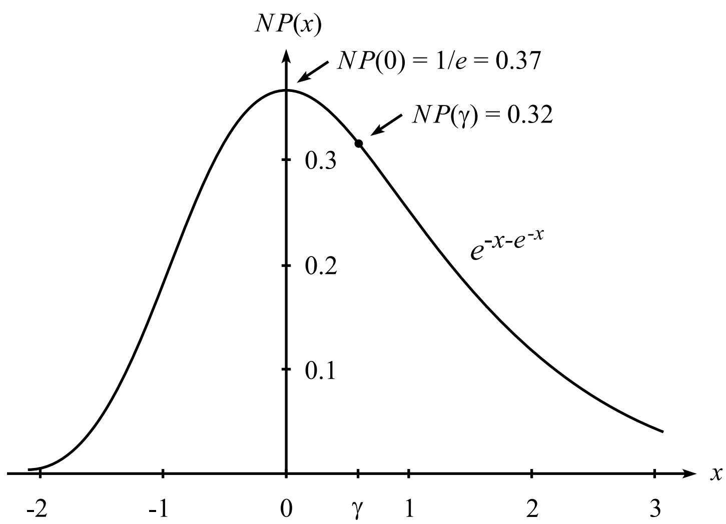
$$\overline{B} = N(\ln N + \gamma) \qquad \text{and} \qquad n_{\max} = N \ln N. \tag{22}$$

The difference between these is $\gamma N \approx (0.577)N$. For example, when $N = 100$, the maximum occurs at $N \ln N \approx 460$, while the average number is $N \ln N + \gamma N \approx 518$.

3. If we let $n \equiv N \ln N + xN$, then we can rewrite eq. (18), to leading order in N, as

$$NP(x) \approx e^{-x-e^{-x}}. \tag{23}$$

A plot of $NP(x)$ vs. x is shown below. This is simply a plot of $NP(n)$ vs. n, centered at $N \ln N$, and with the horizontal axis on a scale of N.



For all large N, we obtain the same shape. Virtually all of the graph is contained within a few units of N from the maximum. Compared to a $P(n)$ vs. n graph, this graph has its vertical axis multiplied by N and its horizontal axis multiplied by $1/N$, so the total integral should still be 1. And indeed,

$$\int_{-\infty}^{\infty} e^{-x-e^{-x}} dx = e^{-e^{-x}} \Big|_{-\infty}^{\infty} = 1. \tag{24}$$

How much area under the curve lies to the left of, say, $x = -2$? Letting $x = -a$ to be general, the integral in eq. (24) gives an area (in other words, a probability) of $e^{-e^{-x}}|_{-\infty}^{-a} = e^{-e^{a}}$. This decreases very rapidly as a grows. Even for $a = 2$, we find that there is a probability of only 0.06% of obtaining all the colors before you hit the $n = N \ln N - 2N$ box.

How much area under the curve lies to the right of, say, $x = 3$? Letting $x = b$ to be general, we find an area of $e^{-e^{-x}}|_{b}^{\infty} = 1 - e^{-e^{-b}} \approx 1 - (1 - e^{-b}) = e^{-b}$. This decreases as b grows, but not as rapidly as in the above case. For $b = 3$, we find that there is a probability of 5% that you haven't obtained all the colors by the time you hit the $n = N \ln N + 3N$ box.

4. You might be tempted to fiddle around with a saddle-point approximation in this problem. That is, you might want to approximate $P(n)$ as a Gaussian around its maximum at $n_{\text{max}} \approx N \ln N$. However, this will *not* work in this problem, because for any (large) N, $P(n)$ will always keep its same lopsided shape. The average will always be a significant distance (namely γN, which is comparable to the spread of the graph) from the maximum, and the ratio of the height at the average to the height at the maximum will always be

$$\frac{P(\gamma)}{P(0)} = \frac{e^{-\gamma-e^{-\gamma}}}{e^{-0-e^{-0}}} \approx 0.87. \tag{25}$$

Solution

Week 61 (11/10/03)

Falling rope

(a) **First Solution:** Let σ be the mass density of the rope. From conservation of energy, we know that the rope's final kinetic energy, which is $(\sigma L)v^2/2$, equals the loss in potential energy. This loss equals $(\sigma L)(L/2)g$, because the center of mass falls a distance $L/2$. Therefore,

$$v = \sqrt{gL}\,. \tag{1}$$

This is the same as the speed obtained by an object that falls a distance $L/2$. Note that if the initial piece hanging down through the hole is arbitrarily short, then the rope will take an arbitrarily long time to fall down. But the final speed will be always be (arbitrarily close to) $\sqrt{gL}$.

Second Solution: Let x be the length that hangs down through the hole. The gravitational force on this length, which is $(\sigma x)g$, is responsible for changing the momentum of the entire rope, which is $(\sigma L)\dot{x}$. Therefore, $F = dp/dt$ gives $(\sigma x)g = (\sigma L)\ddot{x}$, which is simply the $F = ma$ equation. Hence, $\ddot{x} = (g/L)x$, and the general solution to this equation is

$$x(t) = Ae^{t\sqrt{g/L}} + Be^{-t\sqrt{g/L}}. \tag{2}$$

Note that if ϵ is the initial value for x, then $A = B = \epsilon/2$ satisfies the initial conditions $x(0) = \epsilon$ and $\dot{x}(0) = 0$, in which case we may write $x(t) = \epsilon\cosh(t\sqrt{g/L})$. But we won't need this information in what follows.

Let T be the time for which $x(T) = L$. If ϵ is very small, then T will be very large. But for large t,[1] we may neglect the negative-exponent term in eq. (2). We then have

$$x \approx Ae^{t\sqrt{g/L}} \qquad \Longrightarrow \qquad \dot{x} \approx Ae^{t\sqrt{g/L}}\sqrt{g/L} \approx x\sqrt{g/L} \qquad \text{(for large } t\text{)}. \tag{3}$$

When $x = L$, we obtain

$$\dot{x}(T) = L\sqrt{g/L} = \sqrt{gL}\,, \tag{4}$$

in agreement with the first solution.

(b) Let σ be the mass density of the rope, and let x be the length that hangs down through the hole. The gravitational force on this length, which is $(\sigma x)g$, is responsible for changing the momentum of the rope. This momentum is $(\sigma x)\dot{x}$, because only the hanging part is moving. Therefore, $F = dp/dt$ gives

$$\sigma x g = \frac{d(\sigma x\dot{x})}{dt} \quad \Longrightarrow \quad xg = x\ddot{x} + \dot{x}^2. \tag{5}$$

[1] More precisely, for $t \gg \sqrt{L/g}$.

Note that $F = ma$ gives the wrong equation, because it neglects the fact that the moving mass, σx, is changing. It therefore misses the second term on the right-hand side of eq. (5). In short, the momentum of the rope increases because it is speeding up (which gives the $x\ddot{x}$ term) *and* because additional mass is continually being added to the moving part (which gives the $\dot{x}^2$ term, as you can show).

To solve eq. (5) for $x(t)$, note that g is the only parameter in the equation. Therefore, the solution for $x(t)$ can involve only g's and t's.[2] By dimensional analysis, $x(t)$ must then be of the form $x(t) = bgt^2$, where b is a numerical constant to be determined. Plugging this expression for $x(t)$ into eq. (5) and dividing by g^2t^2 gives $b = 2b^2 + 4b^2$. Therefore, $b = 1/6$, and our solution may be written as

$$x(t) = \frac{1}{2}\left(\frac{g}{3}\right)t^2. \tag{6}$$

This is the equation for something that accelerates downward with acceleration $g' = g/3$. The time the rope takes to fall a distance L is then given by $L = g't^2/2$, which yields $t = \sqrt{2L/g'}$. The final speed in thus

$$v = g't = \sqrt{2Lg'} = \sqrt{\frac{2gL}{3}}\,. \tag{7}$$

This is smaller than the $\sqrt{gL}$ result from part (a). We therefore see that although the total time for the scenario in part (a) is very large, the final speed in that case is in fact larger than that in the present scenario.

Remarks: Using eq. (7), you can show that 1/3 of the available potential energy is lost to heat. This inevitable loss occurs during the abrupt motions that suddenly bring the atoms from zero to non-zero speed when they join the moving part of the rope. The use of conservation of energy is therefore *not* a valid way to solve this problem.

You can show that the speed in part (a)'s scenario is smaller than the speed in part (b)'s scenario for x less than $2L/3$, but larger for x greater than $2L/3$.

[2] The other dimensionful quantities in the problem, L and σ, do not appear in eq. (5), so they cannot appear in the solution. Also, the initial position and speed (which will in general appear in the solution for $x(t)$, because eq. (5) is a second-order differential equation) do not appear in this case, because they are equal to zero.

Solution

Week 62 (11/17/03)

Leftover dental floss

Let (x, y) denote the occurrence where x segments have been cut off the left roll, and y segments have been cut off the right roll. In solving this problem, we'll need to first calculate the probability that the process ends at (N, n), which leaves a length $(N-n)d$ on the right roll. For this to happen, the process must first get to $(N-1, n)$, and then the left roll must be chosen for the last segment. The probability of reaching the point $(N-1, n)$ is

$$P_{N-1,n} = \frac{1}{2^{N-1+n}}\binom{N-1+n}{n}, \tag{1}$$

because the binomial coefficient gives the number of different ways the $N-1+n$ choices of roll (each of which occurs with probability 1/2) can to end up at the point $(N-1, n)$. The probability of then choosing the left roll for the next piece is 1/2. Therefore, the probability of ending the process at (N, n) is

$$P^{\text{end}}_{N,n} = \frac{1}{2^{N+n}}\binom{N-1+n}{n}. \tag{2}$$

The leftover length in this case is $(N-n)d$, so we see that the average leftover length at the end of the process is

$$\begin{aligned}
\ell &= 2d\sum_{n=0}^{N-1}(N-n)P^{\text{end}}_{N,n} \\
&= 2d\sum_{n=0}^{N-1}\frac{N-n}{2^{N+n}}\binom{N-1+n}{n},
\end{aligned} \tag{3}$$

where the factor of 2 out front comes from the fact that either roll may be the one that runs out.

In order to simplify this result, we will use the standard result that for large N, a binomial coefficient can be approximated by a Gaussian function. From Problem of the Week 58, we have, for large N and $x \ll N$,

$$\binom{2M}{M-x} \approx \frac{2^{2M}}{\sqrt{\pi M}}e^{-x^2/M}. \tag{4}$$

To make use of this, we'll first need to rewrite eq. (3) as

$$\begin{aligned}
\ell &= 2d\sum_{n=0}^{N-1}\frac{N-n}{2^{N+n}}\frac{N}{N+n}\binom{N+n}{n} \\
&= 2d\sum_{z=1}^{N}\frac{z}{2^{2N-z}}\frac{N}{2N-z}\binom{2N-z}{N-z} \qquad (\text{with } z \equiv N-n) \\
&= 2d\sum_{z=1}^{N}\frac{z}{2^{2N-z}}\frac{N}{2N-z}\binom{2(N-z/2)}{(N-z/2)-z/2}.
\end{aligned} \tag{5}$$

Using eq. (4) to rewrite the binomial coefficient gives (with $M \equiv N - z/2$, and $x \equiv z/2$)

$$\begin{aligned}
\ell &\approx 2d \sum_{z=1}^{N} \frac{Nz}{2N - z} \frac{e^{-z^2/4(N-z/2)}}{\sqrt{\pi(N - z/2)}} \\
&\approx \frac{d}{\sqrt{\pi N}} \sum_{z=0}^{N} z e^{-z^2/(4N)} \\
&\approx \frac{d}{\sqrt{\pi N}} \int_0^{\infty} z e^{-z^2/(4N)}\, dz \\
&= 2d\sqrt{\frac{N}{\pi}}\,. \qquad (6)
\end{aligned}$$

In obtaining the second line above, we have kept only the terms of leading order in N. The exponential factor guarantees that only z values up to order $\sqrt{N}$ will contribute. Hence, z is negligible when added to N.

Without doing all the calculations, it's a good bet that the answer should go like $\sqrt{N}\, d$, but it takes some effort to show that the exact coefficient is $2/\sqrt{\pi}$.

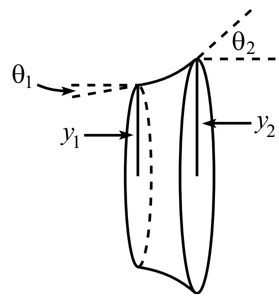
In terms of the initial length of floss, $L \equiv Nd$, the average leftover amount can be written as $\ell \approx (2/\sqrt{\pi})\sqrt{Ld}$. So we see that ℓ is proportional to the geometric mean of L and d.

Solution

Week 63 (11/24/03)

Minimal surface

First Solution: By symmetry, the surface is obtained by rotating a certain function $y(x)$ around the x-axis. Our goal is to find $y(x)$. Consider a thin vertical cross-sectional ring of the surface, as shown below. The ratio of the circumferences of the circular boundaries of the ring is y_2/y_1.



The condition that the bubble be in equilibrium is that the tension (force per unit length) throughout the surface is constant, because otherwise there would be a net force on some little patch. Therefore, the requirement that the horizontal forces on the ring cancel is $y_1 \cos\theta_1 = y_2 \cos\theta_2$, where the θ's are the angles of the surface, as shown. In other words, $y\cos\theta$ is constant throughout the surface. But $\cos\theta = 1/\sqrt{1+y'^2}$, so we have

$$\frac{y}{\sqrt{1+y'^2}} = \text{Constant} \qquad \Longrightarrow \qquad 1 + y'^2 = By^2, \tag{1}$$

where B is some constant. At this point, motivated by the facts that $1+\sinh^2 z = \cosh^2 z$ and $d(\cosh z)/dz = \sinh z$, we can guess that the solution to this differential equation is

$$y(x) = \frac{1}{b}\cosh b(x+d), \tag{2}$$

where $b = \sqrt{B}$, and d is a constant of integration. Or, we can do things from scratch by solving for $y' \equiv dy/dx$ and then separating variables to obtain (again with $b = \sqrt{B}$)

$$dx = \frac{dy}{\sqrt{(by)^2 - 1}}\,. \tag{3}$$

We can then use the fact that the integral of $1/\sqrt{z^2-1}$ is $\cosh^{-1} z$, to obtain the same result as in eq. (2).

Eq. (2) gives the general solution in the case where the rings may have unequal radii. The constants b and d are determined by the boundary conditions (the facts that the y values equal the radii of the rings at the x-values of the rings). In the special case at hand where the radii are equal, the two boundary conditions give $r = (1/b)\cosh b(\pm\ell + d)$, where $x = 0$ has been chosen to be midway between the rings. Therefore, $d = 0$, and so the constant b is determined from

$$r = \frac{1}{b}\cosh b\ell. \tag{4}$$

Our solution for $y(x)$ then

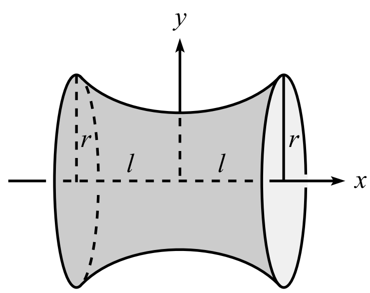
$$y(x) = \frac{1}{b} \cosh bx. \tag{5}$$

There is, however, an ambiguity in this solution, in that there may be two solutions for b in eq. (4). We'll comment on this in the first "Remark" at the end of the problem.

Second Solution: We can also solve this problem by using a "principle of least action" type of argument, which takes advantage of the fact that the surface is the one with the minimum area. There are two ways of going about this. One is quick, and the other is lengthy. A sketch of the lengthy way is the following. The area of the surface in the following figure is

$$\int_{-\ell}^{\ell} 2\pi y \sqrt{1 + y'^2}\, dx, \tag{6}$$

where $y' \equiv dy/dx$.



In analogy with the principle of least action, our "Lagrangian" (from a physicist's point of view) is $L = 2\pi y \sqrt{1 + y'^2}$, and in order for the area to be minimized, L must satisfy the Euler-Lagrange equation,

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y}\,. \tag{7}$$

It is, alas, rather tedious to work through all of the necessary differentiations here. If you so desire, you can show that this equation does in fact lead to eq. (1). But let's instead just do things the quick way. If we consider x to be a function of y (there's no need to worry about any double-value issues, because the Euler-Lagrange formalism deals with local variations), we may write the area as

$$\int_{-\ell}^{\ell} 2\pi y \sqrt{1 + x'^2}\, dy, \tag{8}$$

where $x' \equiv dx/dy$. Our "Lagrangian" is now $L = 2\pi y \sqrt{1 + x'^2}$, and the Euler-Lagrange equation gives

$$\frac{d}{dy}\left(\frac{\partial L}{\partial x'}\right) = \frac{\partial L}{\partial x} \qquad \Longrightarrow \qquad \frac{d}{dy}\left(\frac{yx'}{\sqrt{1 + x'^2}}\right) = 0. \tag{9}$$

The zero on the right-hand side makes everything nice and easy, because it tells us that $yx'/\sqrt{1 + x'^2}$ is constant. Defining this constant to be $1/b$, and then solving for x' and separating variables, gives

$$dx = \frac{dy}{\sqrt{(by)^2 - 1}}\,, \tag{10}$$

which is identical to eq. (3). The solution then concludes as above.

Let us now determine the maximum value of ℓ/r for which the minimal surface exists. If ℓ/r is too large, then we will see that there is no solution for b in eq. (4). In short, the minimal "surface" turns out to be the two given circles, attached by a line, which isn't a nice two-dimensional surface. If you perform the experiment with soap bubbles (which want to minimize their area), and if you pull the rings too far apart, then the surface will break and disappear, as it tries to form the two circles.

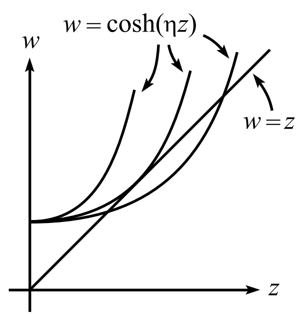
Define the dimensionless quantities,

$$\eta \equiv \frac{\ell}{r}\,, \qquad \text{and} \qquad z \equiv br. \tag{11}$$

Then eq. (4) becomes

$$z = \cosh \eta z. \tag{12}$$

If we make a rough plot of the graphs of $w = z$ and $w = \cosh \eta z$ for a few values of η, as shown below, we see that there is no solution for z if η is too big. The limiting value of η for which there exists a solution occurs when the curves $w = z$ and $w = \cosh \eta z$ are tangent; that is, when the slopes are equal in addition to the functions being equal.



Let η_0 be the limiting value of η, and let z_0 be the place where the tangency occurs. Then equality of the values and the slopes gives

$$z_0 = \cosh(\eta_0 z_0), \qquad \text{and} \qquad 1 = \eta_0 \sinh(\eta_0 z_0). \tag{13}$$

Dividing the second of these equations by the first gives

$$1 = (\eta_0 z_0) \tanh(\eta_0 z_0). \tag{14}$$

This must be solved numerically. The solution is

$$\eta_0 z_0 \approx 1.200. \tag{15}$$

Plugging this into the second of eqs. (13) gives

$$\left(\frac{\ell}{r}\right)_{\max} \equiv \eta_0 \approx 0.663. \tag{16}$$

Note also that $z_0 = 1.200/\eta_0 = 1.810$. We see that if ℓ/r is larger than 0.663, then there is no solution for $y(x)$ that is consistent with the boundary conditions. Above this value of ℓ/r, the soap bubble minimizes its area by heading toward the shape of just two disks, but it will pop well before it reaches that configuration.

REMARKS:

1. As mentioned at the end of the first solution above, there may be more than one solution for the constant b in eq. (5). In fact, the preceding graph shows that for any $\eta < 0.663$, there are two solutions for z in eq. (12), and hence two solutions for b in eq. (4). This means that there are two possible surfaces that might solve our problem. Which one do we want? It turns out that the surface corresponding to the smaller value of b is the one that minimizes the area, while the surface corresponding to the larger value of b is the one that (in some sense) maximizes the area.

   We say "in some sense" because the surface is actually a saddle point for the area. It can't be a maximum, after all, because we can always make the area larger by adding little wiggles to it. It's a saddle point because there does exist a class of variations for which is has the maximum area, namely ones where the "dip" in the curve is continuously made larger (just imagine lowering the midpoint in a smooth manner). The reason why this curve arises in the first solution above is that we simply demanded that the surface be in equilibrium; it just happens to be an *unstable* equilibrium in this case. The reason why it arises in the second solution above is that the Euler-Lagrange technique simple sets the "derivative" equal to zero and thus does not differentiate between maxima, minima, and saddle points.

2. How does the area of the limiting surface (with $\eta_0 = 0.663$) compare with the area of the two circles? The area of the two circles is

$$A_{\rm c} = 2\pi r^2. \tag{17}$$

   The area of the limiting surface is

$$A_{\rm s} = \int_{-\ell}^{\ell} 2\pi y\sqrt{1+y'^2}\,dx. \tag{18}$$

   Using eq. (5), this becomes

$$\begin{aligned} A_{\rm s} &= \int_{-\ell}^{\ell} \frac{2\pi}{b}\cosh^2 bx\,dx \\ &= \int_{-\ell}^{\ell} \frac{\pi}{b}(1+\cosh 2bx)\,dx \\ &= \frac{2\pi\ell}{b} + \frac{\pi\sinh 2b\ell}{b^2}\,. \end{aligned} \tag{19}$$

   But from the definitions of η and z, we have $\ell = \eta_0 r$ and $b = z_0/r$ for the limiting surface. Therefore, $A_{\rm s}$ can be written as

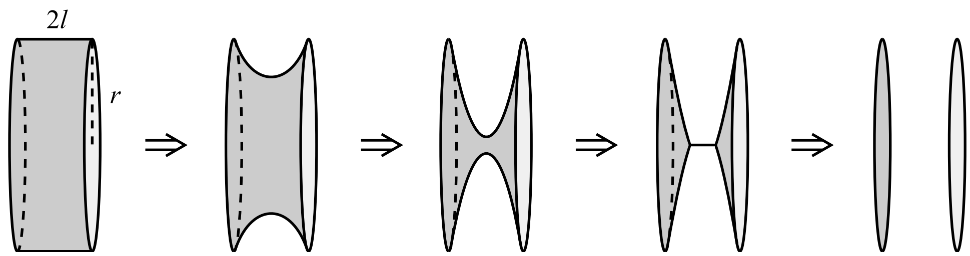
$$A_{\rm s} = \pi r^2\left(\frac{2\eta_0}{z_0} + \frac{\sinh 2\eta_0 z_0}{z_0^2}\right). \tag{20}$$

   Plugging in the numerical values ($\eta_0 \approx 0.663$ and $z_0 \approx 1.810$) gives

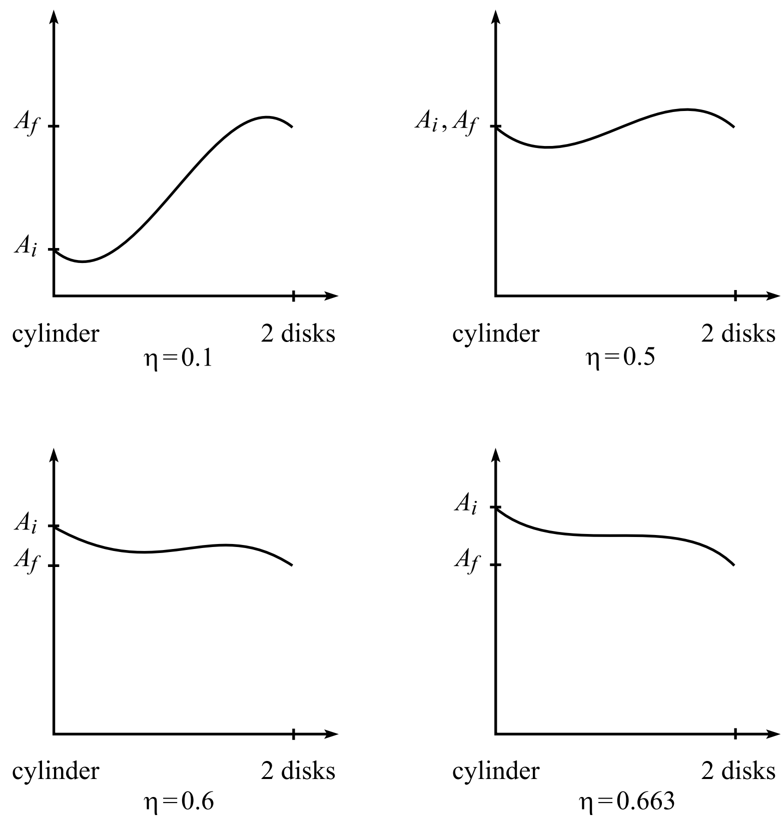
$$A_{\rm c} \approx (6.28)r^2, \qquad \text{and} \qquad A_{\rm s} \approx (7.54)r^2. \tag{21}$$

   The ratio of $A_{\rm s}$ to $A_{\rm c}$ is approximately 1.2 (it's actually $\eta_0 z_0$, as you can show). The limiting surface therefore has a larger area. This is expected, of course, because for $\ell/r > \eta_0$ the surface tries to run off to one with a smaller area, and there are no other stable configurations besides the cosh solution we found.

3. How does the area of the surface change if we gradually transform it from a cylinder to the two disks? There are many ways to go about doing this transformation, but let's just be vague and say that we pick a nice smooth method that passes through the two cosh solutions (if $\eta < 0.663$) that we found above. The transformation might look something like:



The area of the starting cylinder is $A_i = (2\ell)(2\pi r) = 4\pi r\ell$, and the area of the ending two disks is $A_f = 2\pi r^2$. Note that the ratio of these is $A_i/A_f = 2\ell/r \equiv 2\eta$. For a given $\eta \equiv \ell/r$, what does the plot of the changing area look like? Below are four qualitative plots, for four values of η. We've imagined changing η by keeping r (and hence A_f) fixed and changing ℓ. Ignore the actual measure along the axes; just look at the general shape.



We see that $\eta = 0.663$ is the value for which the maximum and minimum merge into one point with zero slope. For higher values of η, there are no points on the curve that have zero slope.

The actual values of the area along these curves is nebulous, because we haven't been quantitative about exactly how we're varying the surface. But the area at the maximum and minimum (at which points we have one of our cosh surfaces) can be found from eq. (20), which says that

$$A_{\rm s} = \pi r^2 \left(\frac{2\eta}{z} + \frac{\sinh 2\eta z}{z^2} \right), \tag{22}$$

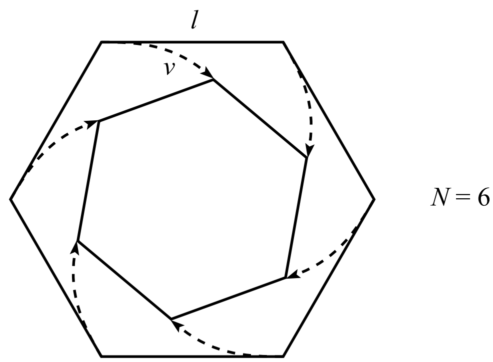
for general $\eta \equiv \ell/r$ and $z \equiv br$. For a given η, the two solutions for z are found from eq. (12). Plugging each of these into eq. (22) gives the areas at the maximum and minimum.

Solution

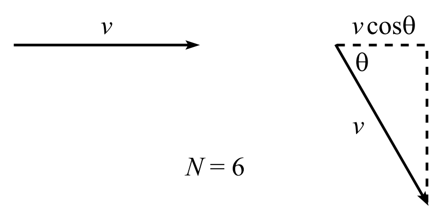
Week 64 (12/1/03)

Attracting bugs

First Solution: In all of these solutions, the key point to realize is that at any time, the bugs form the vertices of a regular N-gon, as shown below for $N = 6$. This is true because this is the only configuration that respects the symmetry of the N bugs. The N-gon will rotate and shrink until it becomes a point at the center.



The important quantity in this first solution is the relative speed of two adjacent bugs. This relative speed is constant, because the relative angle of the bugs' motions is always the same. If the bugs' speed is v, then we see from the figure below that the relative speed is $v_r = v(1 - \cos\theta)$, where $\theta = 2\pi/N$. This is the rate at which the separation between two adjacent bugs decreases.[1]



For example, if $N = 3$ we have $v_r = 3v/2$; if $N = 4$ we have $v_r = v$; and if $N = 6$ we have $v_r = v/2$. Note also that for $N = 2$ (which does not give not much of a polygon, being just a straight line) we have $v_r = 2v$, which is correct for two bugs walking directly toward each other. And if $N \to \infty$ we have $v_r \to 0$, which is correct for bugs walking around in a circle.

If two bugs start a distance ℓ apart, and if they always walk at a relative speed of $v(1 - \cos\theta)$, then the time it takes for them to meet is $t = \ell/(v(1 - \cos\theta))$. Therefore, since the bugs walk at speed v, they will each travel a total distance of

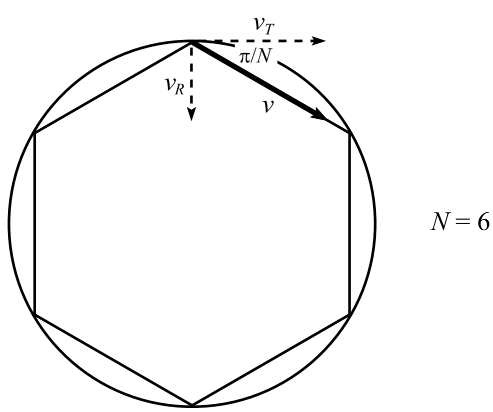
$$vt = \frac{\ell}{1 - \cos(2\pi/N)}\,. \tag{1}$$

[1] The transverse $v\sin\theta$ component of the front bug's velocity is irrelevant here, because it provides no first-order change in the distance between the bugs, for small increments in time.

Note that for a square, this distance equals the length of a side, ℓ. For large N, the approximation $\cos \approx 1 - \theta^2/2$ gives $vt \approx N^2\ell/(2\pi^2)$.

The bugs will spiral around an infinite number of times. This is true because the future path of the bugs at any time must simply be a scaled-down version of the future path at the start (because any point in time may be considered to be the start time, with a scaled-down version of the initial separation). This would not be possible if the bugs hit the center after spiraling around only a finite number of times. We will see in the third solution below that the bugs' distance from the center deceases by a factor $e^{-2\pi \tan(\pi/N)}$ after each revolution.[2]

Second Solution: In this solution, we will determine how quickly the bugs approach the center of the N-gon. A bug's velocity may be separated into radial and tangential components, v_R and v_T, as shown below. Because at any instant the bugs all lie on the vertices of a regular N-gon, they always walk at the same angle relative to a circular motion. Therefore, the magnitudes of v_R and v_T remain constant.



What is the radial component, v_R, in terms of v? The angle between a bug's motion and a circular motion is π/N, so we have

$$v_R = v \sin(\pi/N). \tag{2}$$

What is the radius, R_0, of the initial N-gon? A little geometry shows that

$$R_0 = \frac{\ell}{2\sin(\pi/N)}\,. \tag{3}$$

The time it takes a bug to reach the center is then $t = R_0/v_R = (\ell/v)/(2\sin^2(\pi/N))\,.$ Therefore, each bug travels a total distance of

$$vt = \frac{\ell}{2\sin^2(\pi/N)}\,. \tag{4}$$

[2] Of course, bugs of non-zero size would hit each other before they reach the center. If the bugs happen to be very, very small, then they would eventually require arbitrarily large friction with the floor, in order to provide the centripetal acceleration needed to keep them in a spiral with a very small radius.

This agrees with the result of the first solution, due to the half-angle formula, $\sin^2(\theta/2) = (1-\cos\theta)/2$. The same reasoning used in the first solution shows that the bugs spiral around an infinite number of times.

Third Solution: In this solution, we will parametrize a bug's path, and then integrate the differential arclength. Let us find a bug's distance, $R(\phi)$, from the center, as a function of the angle ϕ through which it has travelled. The angle between a bug's motion and a circular motion is π/N. Therefore, the change in radius, dR, divided by the change in arclength along the circle, $R\,d\phi$, is $dR/(R\,d\phi) = -\tan(\pi/N)$. Separating variables and integrating gives

$$\begin{aligned} \int_{R_0}^{R} \frac{dR}{R} &= -\int_0^{\phi} \tan(\pi/N)\, d\phi \\ \Longrightarrow \quad \ln(R/R_0) &= -\phi\tan(\pi/N) \\ \Longrightarrow \quad R(\phi) &= R_0\, e^{-\phi\tan(\pi/N)}, \end{aligned} \tag{5}$$

where R_0 is the initial distance from the center, equal to $\ell/(2\sin(\pi/N))$. We now see, as stated in the first solution, that one revolution decreases R by the factor $e^{-2\pi\tan(\pi/N)}$, and that an infinite number of revolutions is required for R to become zero. Having found $R(\phi)$, we may find the total distance travelled by integrating the arclength:

$$\begin{aligned} \int \sqrt{(R\,d\phi)^2 + (dR)^2} &= \int_0^{\infty} \sqrt{R^2 + (dR/d\phi)^2}\, d\phi \\ &= \int_0^{\infty} \frac{R_0\, e^{-\phi\tan(\pi/N)}\, d\phi}{\cos(\pi/N)} \\ &= \frac{\ell}{2\sin^2(\pi/N)}\,. \end{aligned} \tag{6}$$

Remark: In the first solution, we found that for large N the total distance travelled goes like $\ell N^2/(2\pi^2)$. This result can also be found in the following manner. For large N, a bug's motion can be approximated by a sequence of circles, C_n, with radii $R_n = R_0 e^{-n(2\pi)\tan(\pi/N)} \approx R_0 e^{-n(2\pi^2/N)}$. To leading order in N, the total distance travelled is therefore the sum of the geometric series,

$$\begin{aligned} \sum_{n=0}^{\infty} 2\pi R_n &\approx \sum_{n=0}^{\infty} 2\pi R_0 e^{-n(2\pi^2/N)} \\ &= \frac{2\pi R_0}{1 - e^{-2\pi^2/N}} \\ &\approx \frac{2\pi R_0}{2\pi^2/N} \\ &\approx \frac{N^2\ell}{2\pi^2}\,, \end{aligned} \tag{7}$$

where we have used $R_0 = \ell/(2\sin(\pi/N)) \approx N\ell/(2\pi)$.

Solution

Week 65 (12/8/03)

Relativistic cart

Ground frame (your frame): In your frame, the force at your feet is responsible for changing the momentum of the cart-plus-sand-inside system. Let's label this system as "C".

To find the dp/dt of C, let's determine how fast the mass of C increases. We claim that the rate of mass increase is $\gamma\sigma$. This can be seen as follows. Assume that C has a mass M at a given time. A mass $\sigma\, dt$ falls into the cart during a time dt. The energy of the resulting C′ is $\gamma M + \sigma\, dt$ (we'll drop the c's here), while the momentum is still $\gamma M v$. Using $E^2 = p^2 + m^2$, we see that the resulting mass equals

$$M' = \sqrt{(\gamma M + \sigma\, dt)^2 - (\gamma M v)^2} \approx \sqrt{M^2 + 2\gamma M\sigma\, dt}, \tag{1}$$

where we have dropped the second-order dt^2 terms. Using the Taylor series $\sqrt{1+\epsilon} \approx 1 + \epsilon/2$, we may approximate M' as

$$M' \approx M\sqrt{1 + \frac{2\gamma\sigma\, dt}{M}} \approx M\left(1 + \frac{\gamma\sigma\, dt}{M}\right) = M + \gamma\sigma\, dt. \tag{2}$$

Therefore, C's mass increases at a rate $\gamma\sigma$.[1] Intuitively, this rate of increase must certainly be greater than the nonrelativistic result of "σ", because heat is generated during the collision, and this heat shows up as mass in the final object.

Having found the rate at which the mass increases, we see that the rate at which the momentum increases is (using the fact that v is constant)

$$\frac{dp}{dt} = \gamma\left(\frac{dM}{dt}\right)v = \gamma(\gamma\sigma)v = \gamma^2\sigma v. \tag{3}$$

Since $F = dp/dt$, this is the force that you exert on the cart. Therefore, it is also the force that the ground exerts on your feet (because the net force on you is zero).

Cart frame: The sand-entering-cart events happen at the same location in the ground frame, so time dilation says that the sand enters the cart at a slower rate in the cart frame; that is, at a rate σ/γ. The sand flies in at speed v, and then eventually comes at rest on the cart, so its momentum decreases at a rate $\gamma(\sigma/\gamma)v = \sigma v$. This is the force that your hand applies to the cart.

If this were the only change in momentum in the problem, then we would have a problem, because the force on your feet would be σv in the cart frame, whereas we found above that it is $\gamma^2\sigma v$ in the ground frame. This would contradict the fact that longitudinal forces are the same in different frames. What is the resolution of this apparent paradox?

[1] This result is easier to see if we work in the frame where C is at rest. In this frame, a mass $\sigma\, dt$ comes flying in with energy $\gamma\sigma\, dt$, and essentially all of this energy shows up as mass (heat) in the final object. That is, essentially none of it shows up as overall kinetic energy of the object, which is a general result for when a small object hits a stationary large object.

The resolution is that while you are pushing on the cart, *your mass is decreasing*. You are moving with speed v in the cart frame, and mass is continually being transferred from you (who are moving) to the cart (which is at rest). This is the missing change in momentum we need. Let's be quantitative about this.

Go back to the ground frame for a moment. We found above that the mass of C increases at rate $\gamma\sigma$ in the ground frame. Therefore, the energy of C increases at a rate $\gamma(\gamma\sigma)$ in the ground frame. The sand provides σ of this energy, so you must provide the remaining $(\gamma^2 - 1)\sigma$ part. Therefore, since you are losing energy at this rate, you must also be losing mass at this rate in the ground frame (because you are at rest there).

Now go back to the cart frame. Due to time dilation, you lose mass at a rate of only $(\gamma^2 - 1)\sigma/\gamma$. This mass goes from moving at speed v (that is, along with you), to speed zero (that is, at rest on the cart). Therefore, the rate of decrease in momentum of this mass is $\gamma((\gamma^2 - 1)\sigma/\gamma)v = (\gamma^2 - 1)\sigma v$.

Adding this result to the σv result due to the sand, we see that the total rate of decrease in momentum is $\gamma^2\sigma v$. This is therefore the force that the ground applies to your feet, in agreement with the calculation in the ground frame.

Solution

Week 66 (12/15/03)

Bowl of spaghetti

Assume that we have reached into the bowl and pulled out one end. Then there are $2N-1$ free ends left in the bowl. Therefore, there is a $1/(2N-1)$ chance that a loop is formed by choosing the other end of the noodle that we are holding. And there is a $(2N-2)/(2N-1)$ chance that a loop is not formed. In the former case, we end up with one loop and $N-1$ strands. In the latter case, we just end up with $N-1$ strands, because we have simply created a strand of twice the original length, and the length of a strand is irrelevant in this problem.

Therefore, after the first step, we see that no matter what happens, we end up with $N-1$ strands and, on average, $1/(2N-1)$ loops. We can now repeat this reasoning with $N-1$ strands. After the second step, we are guaranteed to be left with $N-2$ strands and, on average, another $1/(2N-3)$ loops. This process continues until we are left with one strand, whereupon the final Nth step leaves us with zero strands, and we (definitely) gain one more loop.

Adding up the average number of loops gained at each stage, we obtain an average total number of loops equal to

$$n = \frac{1}{2N-1} + \frac{1}{2N-3} + \cdots + \frac{1}{3} + 1. \tag{1}$$

This grows very slowly with N. It turns out that we need $N = 8$ noodles in order to expect at least two loops. If we use the ordered pair (n, N) to signify that N noodles are needed in order to expect n loops, we can numerically show that the first few integer-n ordered pairs are: $(1, 1)$, $(2, 8)$, $(3, 57)$, $(4, 419)$, and $(5, 3092)$.

For large N, we can say that the average number of loops given in eq. (1) is roughly equal to $1/2$ times the sum of the reciprocals up to $1/N$. So it approximately equals $(\ln N)/2$. To get a better approximation, let S_N denote the sum of the integer reciprocals up to $1/N$. Then we have (using $S_N \approx \ln N + \gamma$, where $\gamma \approx 0.577$ is Euler's constant)

$$\begin{aligned}
n + \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2N-2} + \frac{1}{2N}\right) &= S_{2N} \\
\Longrightarrow \qquad n + \frac{1}{2}S_N &= S_{2N} \\
\Longrightarrow \qquad n + \frac{1}{2}(\ln N + \gamma) &\approx \ln(2N) + \gamma \\
\Longrightarrow \qquad n &\approx \frac{1}{2}(\ln N + \ln 4 + \gamma) \\
\Longrightarrow \qquad N &\approx \frac{e^{2n-\gamma}}{4}\,.
\end{aligned} \tag{2}$$

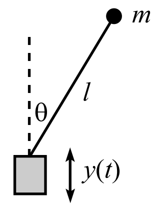
You can show that this relation between n and N agrees with the above numerical results, except for the $n = 1$ case. You will need to use the more precise value of $\gamma \approx 0.5772$ for the $n = 5$ case.

Solution

Week 67 (12/22/03)

Inverted pendulum

Let θ be defined as shown below. We'll use the Lagrangian method to determine the equation of motion for θ.



With $y(t) = A\cos(\omega t)$, the position of the mass m is given by

$$(X, Y) = (\ell \sin\theta, y + \ell\cos\theta). \tag{1}$$

Taking the derivatives of these coordinates, we see that the square of the speed is

$$V^2 = \dot{X}^2 + \dot{Y}^2 = \ell^2\dot{\theta}^2 + \dot{y}^2 - 2\ell\dot{y}\dot{\theta}\sin\theta. \tag{2}$$

The Lagrangian is therefore

$$L = K - U = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{y}^2 - 2\ell\dot{y}\dot{\theta}\sin\theta) - mg(y + \ell\cos\theta). \tag{3}$$

The equation of motion for θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta} \qquad \Longrightarrow \qquad \ell\ddot{\theta} - \ddot{y}\sin\theta = g\sin\theta. \tag{4}$$

Plugging in the explicit form of $y(t)$, we have

$$\ell\ddot{\theta} + \sin\theta\Big(A\omega^2\cos(\omega t) - g\Big) = 0. \tag{5}$$

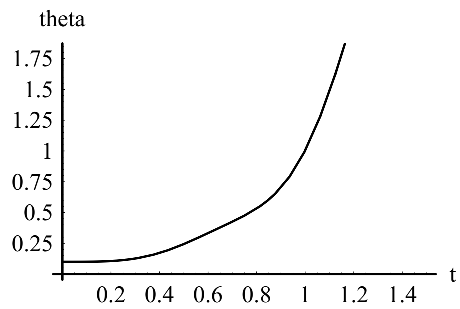
In retrospect, this makes sense. Someone in the reference frame of the support, which has acceleration $\ddot{y} = -A\omega^2\cos(\omega t)$, may as well be living in a world where the acceleration from gravity is $g - A\omega^2\cos(\omega t)$ downward. Eq. (5) is simply the $F = ma$ equation in the tangential direction in this accelerated frame.

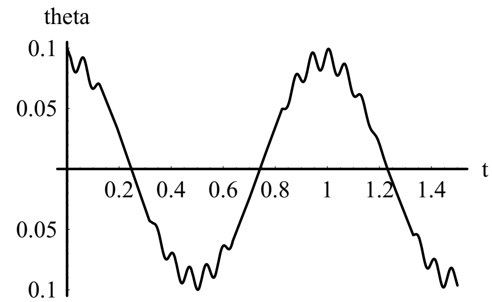
Assuming θ is small, we may set $\sin\theta \approx \theta$, which gives

$$\ddot{\theta} + \theta\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big) = 0, \tag{6}$$

where $\omega_0 \equiv \sqrt{g/\ell}$, and $a \equiv A/\ell$. Eq. (6) cannot be solved exactly, but we can still get a good idea of how θ depends on time. We can do this both numerically and (approximately) analytically.

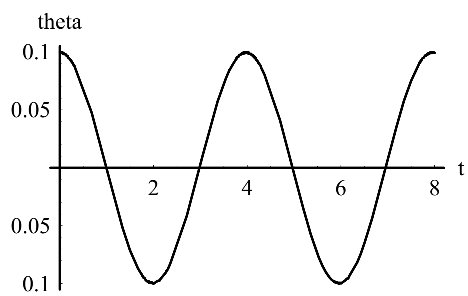
The figures below show how θ depends on time for parameters with values $\ell = 1\,\text{m}$, $A = 0.1\,\text{m}$, and $g = 10\,\text{m/s}^2$ (so $a = 0.1$, and $\omega_0^2 = 10\,\text{s}^{-2}$). In the first plot, $\omega = 10\,\text{s}^{-1}$. And in the second plot, $\omega = 100\,\text{s}^{-1}$. The stick falls over in first case, but undergoes oscillatory motion in the second case. Apparently, if ω is large enough the stick will not fall over.

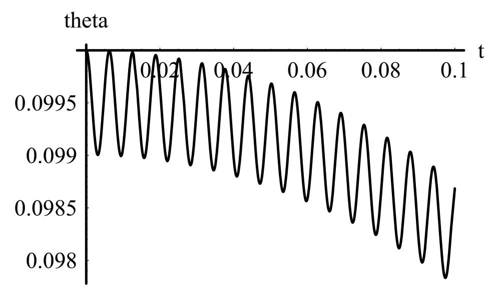




Let's now explain this phenomenon analytically. At first glance, it's rather surprising that the stick stays up. It seems like the average (over a few periods of the ω oscillations) of the tangential acceleration in eq. (6), namely $-\theta(a\omega^2\cos(\omega t) - \omega_0^2)$, equals the positive quantity $\theta\omega_0^2$, because the $\cos(\omega t)$ term averages to zero (or so it appears). So you might think that there is a net force making θ increase, causing the stick fall over.

The fallacy in this reasoning is that the average of the $-a\omega^2\theta\cos(\omega t)$ term is *not* zero, because θ undergoes tiny oscillations with frequency ω, as seen below. Both of these plots have $a = 0.005$, $\omega_0^2 = 10\,\mathrm{s}^{-2}$, and $\omega = 1000\,\mathrm{s}^{-1}$ (we'll work with small a and large ω from now on; more on this below). The second plot is a zoomed-in version of the first one near $t = 0$.





The important point here is that the tiny oscillations in θ shown in the second plot are correlated with $\cos(\omega t)$. It turns out that the θ value at the t where $\cos(\omega t) = 1$ is larger than the θ value at the t where $\cos(\omega t) = -1$. So there is a net negative contribution to the $-a\omega^2\theta\cos(\omega t)$ part of the acceleration. And it may indeed be large enough to keep the pendulum up, as we will now show.

To get a handle on the $-a\omega^2\theta\cos(\omega t)$ term, let's work in the approximation where ω is large and $a \equiv A/\ell$ is small. More precisely, we will assume $a \ll 1$ and $a\omega^2 \gg \omega_0^2$, for reasons we will explain below. Look at one of the little oscillations in the second of the above plots. These oscillations have frequency ω, because they are due simply to the support moving up and down. When the support moves up, θ increases; and when the support moves down, θ decreases. Since the average position of the pendulum doesn't change much over one of these small periods, we can look for an approximate solution to eq. (6) of the form

$$\theta(t) \approx C + b\cos(\omega t), \tag{7}$$

where $b \ll C$. C will change over time, but on the scale of $1/\omega$ it is essentially constant, if $a \equiv A/\ell$ is small enough.

Plugging this guess for θ into eq. (6), and using $a \ll 1$ and $a\omega^2 \gg \omega_0^2$, we find that $-b\omega^2 \cos(\omega t) + Ca\omega^2 \cos(\omega t) = 0$, to leading order.[1] So we must have $b = aC$. Our approximate solution for θ is therefore

$$\theta \approx C\Big(1 + a\cos(\omega t)\Big). \tag{8}$$

Let's now determine how C gradually changes with time. From eq. (6), the average acceleration of θ, over a period $T = 2\pi/\omega$, is

$$\begin{aligned}
\overline{\ddot{\theta}} &= \overline{-\theta\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big)} \\
&\approx \overline{-C\Big(1 + a\cos(\omega t)\Big)\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big)} \\
&= -C\Big(a^2\omega^2\overline{\cos^2(\omega t)} - \omega_0^2\Big) \\
&= -C\left(\frac{a^2\omega^2}{2} - \omega_0^2\right) \\
&\equiv -C\Omega^2,
\end{aligned} \tag{9}$$

where

$$\Omega = \sqrt{\frac{a^2\omega^2}{2} - \frac{g}{\ell}}\,. \tag{10}$$

But if we take two derivatives of eq. (7), we see that $\overline{\ddot{\theta}}$ simply equals $\ddot{C}$. Equating this value of $\overline{\ddot{\theta}}$ with the one in eq. (9) gives

$$\ddot{C}(t) + \Omega^2 C(t) \approx 0. \tag{11}$$

This equation describes nice simple-harmonic motion. Therefore, C oscillates sinusoidally with the frequency Ω given in eq. (10). This is the overall back and forth motion seen in the first of the above plots. Note that we must have $a\omega > \sqrt{2}\omega_0$ if this frequency is to be real so that the pendulum stays up. Since we have assumed $a \ll 1$, we see that $a^2\omega^2 > 2\omega_0^2$ implies $a\omega^2 \gg \omega_0^2$, which is consistent with our initial assumption above.

If $a\omega \gg \omega_0$, then eq. (10) gives $\Omega \approx a\omega/\sqrt{2}$. Such is the case if we change the setup and simply have the pendulum lie flat on a horizontal table where the acceleration from gravity is zero. In this limit where g is irrelevant, dimensional analysis implies that the frequency of the C oscillations must be a multiple of ω, because ω is the only quantity in the problem with units of frequency. It just so happens that the multiple is $a/\sqrt{2}$.

---

[1] The reasons for the $a \ll 1$ and $a\omega^2 \gg \omega_0^2$ qualifications are the following. If $a\omega^2 \gg \omega_0^2$, then the $a\omega^2\cos(\omega t)$ term dominates the ω_0^2 term in eq. (6). The one exception to this is when $\cos(\omega t) \approx 0$, but this occurs for a negligibly small amount of time if $a\omega^2 \gg \omega_0^2$. If $a \ll 1$, then we can legally ignore the $\ddot{C}$ term when eq. (7) is substituted into eq. (6). We will find below, in eq. (9), that our assumptions lead to $\ddot{C}$ being roughly proportional to $a^2\omega^2$. Since the other terms in eq. (6) are proportional to $a\omega^2$, we need $a \ll 1$ in order for the $\ddot{C}$ term to be negligible. In short, $a \ll 1$ is the condition under which C varies slowly on the time scale of $1/\omega$.

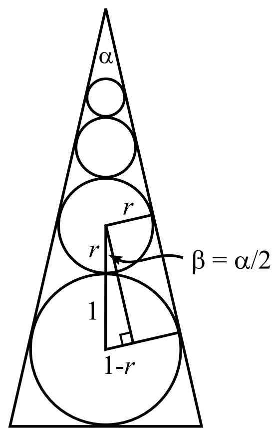
Solution

Week 68 (12/29/03)

Tower of circles

Let the bottom circle have radius 1, and let the second circle have radius r. From the following figure, we have

$$\sin\beta = \frac{1-r}{1+r}\,, \qquad \text{where } \beta \equiv \alpha/2. \tag{1}$$



In solving this problem, it is easier to work with r, instead of the angle α. So we will find the value of r for which A_C/A_T is maximum, and then use eq. (1) to obtain α. Note that r is the ratio of the radii of any two adjacent circles. This follows from the fact that we could have drawn the above thin little right triangle by using any two adjacent circles. Alternatively, it follows from the fact that if we scale up the top $N-1$ circles by the appropriate factor, then we obtain the bottom $N-1$ circles.

The area, A_T, of the triangle may be calculated in terms of r and N as follows. Since we could imagine stacking an infinite number of circles up to the vertex of the triangle, we see that the the height of the triangle is

$$h = 2 + 2r + 2r^2 + 2r^3 + \cdots = \frac{2}{1-r}\,. \tag{2}$$

The length of half the base, b, of the triangle is give by $b/2 = h\tan\beta$. But from eq. (1) we have $\tan\beta = (1-r)/(2\sqrt{r})$. Therefore,

$$b = \frac{2h}{\tan\beta} = \frac{2}{\sqrt{r}}\,. \tag{3}$$

The area of the triangle is then

$$A_T = \frac{bh}{2} = \frac{2}{(1-r)\sqrt{r}}\,. \tag{4}$$

The total area of the circles is

$$
\begin{aligned}
A_C &= \pi\Big(1 + r^2 + r^4 + \cdots r^{2(N-1)}\Big) && (5)\\
&= \pi\frac{1 - r^{2N}}{1 - r^2}\,. && (6)
\end{aligned}
$$

Therefore, the ratio of the areas is

$$
\frac{A_C}{A_T} = \frac{\pi}{2}\frac{\sqrt{r}(1 - r^{2N})}{1 + r}\,. \tag{7}
$$

Setting the derivative of this equal to zero to obtain the maximum, we find

$$
(1 - r) - (4N + 1)r^{2N} - (4N - 1)r^{2N+1} = 0. \tag{8}
$$

In general, this can only be solved numerically for r. But if N is very large, we can obtain an approximate solution. To leading order in N, we may set $4N \pm 1 \approx 4N$. We may also set $r^{2N+1} \approx r^{2N}$, because r must be very close to 1 (otherwise there would be nothing to cancel the "1" term in eq. (8)). For convenience, let us write $r \equiv 1 - \epsilon$, where ϵ is very small. Eq. (8) then yields

$$
8N(1 - \epsilon)^{2N} \approx \epsilon. \tag{9}
$$

But $(1 - \epsilon)^{2N} \approx e^{-2N\epsilon}$.[1] Hence,

$$
e^{-2N\epsilon} \approx \frac{\epsilon}{8N}\,. \tag{10}
$$

Taking the log of both sides gives

$$
\begin{aligned}
\epsilon &\approx \frac{1}{2N}\ln\left(\frac{8N}{\epsilon}\right)\\
&\approx \frac{1}{2N}\ln\left(\frac{8N}{\frac{1}{2N}\ln\left(\frac{8N}{\epsilon}\right)}\right), \qquad \text{etc.} && (11)
\end{aligned}
$$

Therefore, to leading order in N, we have

$$
\epsilon \approx \frac{1}{2N}\ln\left(\frac{16N^2}{\mathcal{O}(\ln N)}\right) = \frac{\ln N - \mathcal{O}(\ln\ln N) + \cdots}{N} \approx \frac{\ln N}{N}\,. \tag{12}
$$

Note that for large N, this ϵ is much less than $1/\sqrt{N}$, so eq. (10) is indeed valid. Hence, $r = 1 - \epsilon \approx 1 - (\ln N)/N$. Eq. (1) then gives

$$
\alpha = 2\beta \approx 2\sin\beta = 2\frac{1 - r}{1 + r} \approx \frac{2\epsilon}{2} = \epsilon, \tag{13}
$$

and so

$$
\alpha \approx \frac{\ln N}{N}\,. \tag{14}
$$

This is the desired answer to leading order in N, in the sense that as N becomes very large, this answer becomes multiplicatively arbitrarily close to the true answer.

[1] This follows from taking the log of $(1 - \epsilon)^{2N}$, to obtain $\ln((1 - \epsilon)^{2N}) = 2N\ln(1 - \epsilon) \approx -2N(\epsilon + \epsilon^2/2 + \cdots)$. This is approximately equal to $-2N\epsilon$, provided that the second term in the expansion is small, which is the case when $\epsilon \ll 1/\sqrt{N}$, which we will find to be true.

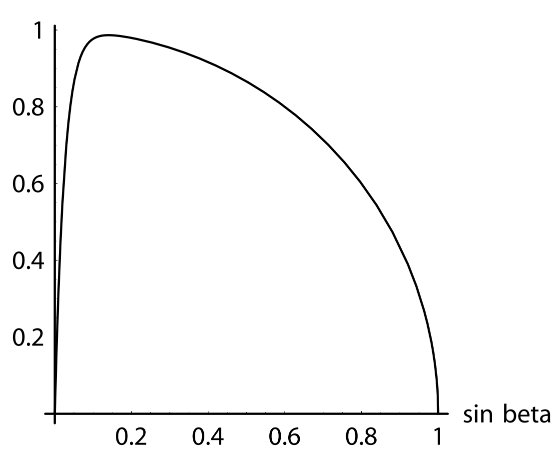
Remarks:

1. The radius of the top circle in the stack is
$$R_N = r^{N-1} \approx r^N = (1-\epsilon)^N \approx e^{-\ln N \epsilon}. \tag{15}$$
Using eq. (10) and then eq. (12), we have
$$R_N \approx \sqrt{\frac{\epsilon}{8N}} \approx \frac{\sqrt{\ln N}}{2\sqrt{2}N}\,. \tag{16}$$

2. The distance from the center of the top circle to the vertex equals
$$\frac{R_N}{\sin\beta} \approx \frac{R_N}{\beta} \approx \frac{\frac{\sqrt{\ln N}}{2\sqrt{2}N}}{\frac{1}{2N}\ln N} = \frac{1}{\sqrt{2\ln N}}\,, \tag{17}$$
which goes to zero (but very slowly) for large N.

3. For $r \approx 1 - (\ln N)/N$, eq. (7) yields $A_C/A_T \approx \pi/4$. This is the expected answer, because if we look at a small number of adjacent circles, they appear to be circles inside a rectangle (because the long sides of the isosceles triangle are nearly parallel for small α), and it is easy to see that $\pi/4$ is the answer for the rectangular case.

4. Using eq. (7), along with $r = (1-\sin\beta)/(1+\sin\beta)$ from eq. (1), we can make a plot of $(4/\pi)(A_C/A_T)$ as a function of $\sin\beta$. The figure below shows the plot for $N = 20$. In the limit of very large N, the left part of the graph approaches a vertical segment. The rest of the curve approaches a quarter circle, as N goes to infinity. That is, $(4/\pi)(A_C/A_T) \approx \cos\beta$, for $N \to \infty$. This is true because if N is large, and if β is larger than order $(1/N)\ln N$, then we effectively have an infinite number of circles in the triangle. In this infinite case, the ratio A_C/A_T is given by the ratio of the area of a circle to the area of a circumscribing trapezoid whose sides are tilted at an angle β. You can show that this ratio is $(\pi/4)\cos\beta$.



5. We can also consider the more general case of higher dimensions. For example, instead of stacking N circles inside a triangle, we can stack N spheres inside a cone. Let α be the angle at the peak of the cone. Then the α for which the ratio of the total volume of the spheres to the volume of the cone is maximum is $\alpha \approx (2\ln N)/(3N)$. And the answer in the general case of d dimensions (with $d \geq 2$) is $\alpha \approx (2\ln N)/(dN)$. This agrees with eq. (14) for the $d = 2$ case. We can show this general result as follows.

   As in the original problem, the height and base radius of the generalized "cone" are still

$$h = \frac{2}{1-r}, \qquad \text{and} \qquad b = \frac{2}{\sqrt{r}}. \tag{18}$$

   Therefore, the "volume" of the cone is proportional to

$$V_{\text{cone}} \propto b^{d-1}h \propto \frac{1}{r^{(d-1)/2}(1-r)}. \tag{19}$$

   The total volume of the "spheres" is proportional to

$$\begin{aligned} V_{\text{spheres}} &\propto 1 + r^d + r^{2d} + \cdots r^{(N-1)d} \qquad (20) \\ &= \frac{1-r^{Nd}}{1-r^d}. \qquad (21) \end{aligned}$$

   Therefore,

$$\frac{V_{\text{spheres}}}{V_{\text{cone}}} \propto \frac{(1-r^{Nd})r^{(d-1)/2}(1-r)}{1-r^d}. \tag{22}$$

   To maximize this, it is easier to work with the small quantity $\epsilon \equiv 1 - r$. In terms of ϵ, we have (using the binomial expansion)

$$\begin{aligned} \frac{V_{\text{spheres}}}{V_{\text{cone}}} &\propto \frac{(1-(1-\epsilon)^{Nd})(1-\epsilon)^{(d-1)/2}\epsilon}{1-(1-\epsilon)^d}. \\ &\approx \frac{(1-e^{-Nd\epsilon})\left[1 - \left(\frac{d-1}{2}\right)\epsilon + \left(\frac{(d-1)(d-3)}{8}\right)\epsilon^2 - \cdots\right]}{d\left[1 - \left(\frac{d-1}{2}\right)\epsilon + \left(\frac{(d-1)(d-2)}{6}\right)\epsilon^2 - \cdots\right]}. \qquad (23) \end{aligned}$$

   The terms in the square brackets in the numerator and the denominator differ at order ϵ^2, so we have

$$\frac{V_{\text{spheres}}}{V_{\text{cone}}} \propto (1-e^{-Nd\epsilon})(1 - A\epsilon^2 + \cdots), \tag{24}$$

   where A happens to equal $(d^2-1)/24$, but we won't need this. Taking the derivative of eq. (24) with respect to ϵ to obtain the maximum, we find

$$(1-e^{-Nd\epsilon})(-2A\epsilon) + Nde^{-Nd\epsilon}(1-A\epsilon^2) = 0. \tag{25}$$

   The N in the second term tells us that $e^{-Nd\epsilon}$ must be at most order $1/N$. Therefore, we can set $1 - e^{-Nd\epsilon} \approx 1$ in the first term. Also, we can set $1 - A\epsilon^2 \approx 1$ in the second term. This gives

$$\begin{aligned} e^{-Nd\epsilon} &\approx \frac{2A\epsilon}{Nd} \\ \Longrightarrow \qquad \epsilon &\approx \frac{1}{Nd}\ln\left(\frac{Nd}{2A\epsilon}\right). \qquad (26) \end{aligned}$$

In the same manner as in eq. (12), we find, to leading order in N,

$$\epsilon \approx \frac{2\ln N}{dN}\,. \tag{27}$$

And since $\alpha = \epsilon$ from eq. (13), we obtain $\alpha \approx (2\ln N)/(dN)$, as desired.

6. Eq. (8) can be solved numerically for r, for any value of N. A few results are:

N	r	α (deg)	α (rad)	$(\ln N)/N$	$(4/\pi)(A_C/A_T)$
1	.333	60	1.05	0	.770
2	.459	43.6	.760	.347	.887
3	.539	34.9	.609	.366	.931
10	.754	16.1	.282	.230	.987
100	.953	2.78	.0485	.0461	.999645
1000	.9930	.400	$6.98 \cdot 10^{-3}$	$6.91 \cdot 10^{-3}$	$1 - 6.96 \cdot 10^{-6}$
10^6	.9999864	$7.76 \cdot 10^{-4}$	$1.36 \cdot 10^{-5}$	$1.38 \cdot 10^{-5}$	$1 - 2.47 \cdot 10^{-11}$

For large N, we see that

$$\alpha \text{ (rad)} \approx \frac{\ln N}{N}\,, \qquad \text{and} \qquad r \approx 1 - \frac{\ln N}{N}\,. \tag{28}$$

Also, using eqs. (7) and (12), you can show that to leading order in N,

$$\frac{4}{\pi}\frac{A_C}{A_T} \approx 1 - \frac{(\ln N)^2 + \ln N}{8N^2}\,, \tag{29}$$

which agrees well with the numerical results, for large N.

Solution

Week 69 (1/5/04)

Compton scattering

We will solve this problem by making use of 4-momenta. The *4-momentum* of a particle is given by

$$P \equiv (P_0, P_1, P_2, P_3) \equiv (E, p_x c, p_y c, p_z c) \equiv (E, \mathbf{p}c). \tag{1}$$

In general, the inner-product of two *4-vectors* is given by

$$A \cdot B \equiv A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3. \tag{2}$$

The square of a 4-momentum (that is, the inner product of a 4-momentum with itself) is therefore

$$P^2 \equiv P \cdot P = E^2 - |\mathbf{p}|^2 c^2 = m^2 c^4. \tag{3}$$

Let's now apply these idea to the problem at hand. We will actually be doing nothing here other than applying conservation of energy and momentum. It's just that the language of 4-vectors makes the whole procedure surprisingly simple. Note that conservation of E and $\mathbf{p}$ during the collision can be succinctly written as

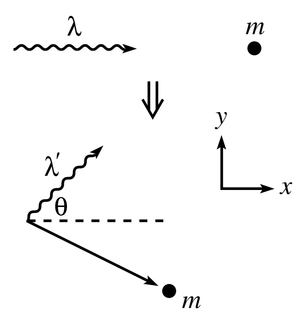
$$P_{\text{before}} = P_{\text{after}}. \tag{4}$$

Referring to the figure below, the 4-momenta before the collision are

$$P_\gamma = \left(\frac{hc}{\lambda}, \frac{hc}{\lambda}, 0, 0\right), \qquad P_m = (mc^2, 0, 0, 0). \tag{5}$$

And the 4-momenta after the collision are

$$P'_\gamma = \left(\frac{hc}{\lambda'}, \frac{hc}{\lambda'}\cos\theta, \frac{hc}{\lambda'}\sin\theta, 0\right), \qquad P'_m = (\text{we won't need this}). \tag{6}$$



If we wanted to, we could write P'_m in terms of its momentum and scattering angle. But the nice thing about this 4-momentum method is that we don't need to introduce any quantities that we're not interested in.

Conservation of energy and momentum give $P_\gamma + P_m = P'_\gamma + P'_m$. Therefore,

$$\begin{aligned}
(P_\gamma + P_m - P'_\gamma)^2 &= P'^2_m \\
\Longrightarrow P^2_\gamma + P^2_m + P'^2_\gamma + 2P_m(P_\gamma - P'_\gamma) - 2P_\gamma P'_\gamma &= P'^2_m \\
\Longrightarrow 0 + m^2c^4 + 0 + 2mc^2\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) - 2\frac{hc}{\lambda}\frac{hc}{\lambda'}(1-\cos\theta) &= m^2c^4. \qquad (7)
\end{aligned}$$

Multiplying through by $\lambda\lambda'/(2hmc^3)$ gives the desired result,

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta). \qquad (8)$$

The ease of this solution arose from the fact that all the unknown garbage in P'_m disappeared when we squared it.

REMARKS:

1. If $\theta \approx 0$ (that is, not much scattering), then $\lambda' \approx \lambda$, as expected.
2. If $\theta = \pi$ (that is, backward scattering) and additionally $\lambda \ll h/mc$ (that is, $mc^2 \ll hc/\lambda = E_\gamma$), then $\lambda' \approx 2h/mc$, so

$$E'_\gamma = \frac{hc}{\lambda'} \approx \frac{hc}{\frac{2h}{mc}} = \frac{1}{2}mc^2. \qquad (9)$$

Therefore, the photon bounces back with an essentially fixed E'_γ, independent of the initial E_γ (as long as E_γ is large enough). This isn't all that obvious.

Solution

Week 70 (1/12/04)

Painting a funnel

It is true that the volume is finite, and that you can fill it up with paint. It is also true that the surface area is infinite, but you actually *can* paint it.

The apparent paradox arises from essentially comparing apples and oranges. In our case we are comparing areas (things of dimension two) with volumes (things of dimension three). When someone says that the funnel can't be painted, she is saying that it would take an infinite *volume* of paint to cover it. But the fact that the surface area is infinite does *not* imply that it takes an infinite volume of paint to cover it. To be sure, if we try to paint the funnel with a given fixed thickness of paint, then we would indeed need an infinite volume of paint. But is this case we would essentially have a tube of paint of fixed radius, for large values of x, with the funnel taking up a negligible volume at the center of the tube. This tube certainly has an infinite volume.

But what if we paint the funnel with a decreasing thickness of paint, as x gets larger? For example, if we make the thickness go like $1/x$, then the volume of paint goes like $\int_1^\infty (1/x^2)\,dx$, which is finite. In this manner, we can indeed paint the funnel. To sum up, we buy paint by the gallon, not by the square meter. And a gallon of paint can cover an infinite area, as long as we make the thickness go to zero fast enough.

Solution

Week 71 (1/19/04)

Maximum trajectory length

Let θ be the angle at which the ball is thrown. Then the coordinates are given by $x = (v\cos\theta)t$ and $y = (v\sin\theta)t - gt^2/2$. The ball reaches its maximum height at $t = v\sin\theta/g$, so the length of the trajectory is

$$\begin{aligned} L &= 2\int_0^{v\sin\theta/g}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}\,dt \\ &= 2\int_0^{v\sin\theta/g}\sqrt{(v\cos\theta)^2+(v\sin\theta-gt)^2}\,dt \\ &= 2v\cos\theta\int_0^{v\sin\theta/g}\sqrt{1+\left(\tan\theta-\frac{gt}{v\cos\theta}\right)^2}\,dt. \end{aligned} \tag{1}$$

Letting $z \equiv \tan\theta - gt/v\cos\theta$, we obtain

$$L = -\frac{2v^2\cos^2\theta}{g}\int_{\tan\theta}^{0}\sqrt{1+z^2}\,dz. \tag{2}$$

Letting $z \equiv \tan\alpha$, and switching the order of integration, gives

$$L = \frac{2v^2\cos^2\theta}{g}\int_0^{\theta}\frac{d\alpha}{\cos^3\alpha}\,. \tag{3}$$

You can either look up this integral, or you can derive it (see the remark at the end of the solution). The result is

$$\begin{aligned} L &= \frac{2v^2\cos^2\theta}{g}\cdot\frac{1}{2}\left(\frac{\sin\theta}{\cos^2\theta}+\ln\left(\frac{1+\sin\theta}{\cos\theta}\right)\right) \\ &= \frac{v^2}{g}\left(\sin\theta+\cos^2\theta\ln\left(\frac{\sin\theta+1}{\cos\theta}\right)\right). \end{aligned} \tag{4}$$

As a double-check, you can verify that $L = 0$ when $\theta = 0$, and $L = v^2/g$ when $\theta = 90^\circ$. Taking the derivative of eq. (4) to find the maximum, we obtain

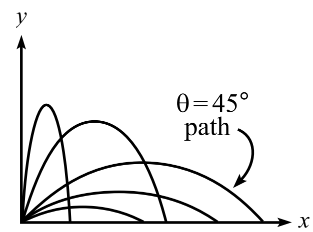
$$0 = \cos\theta - 2\cos\theta\sin\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right) + \cos^2\theta\left(\frac{\cos\theta}{1+\sin\theta}\right)\frac{\cos^2\theta+(1+\sin\theta)\sin\theta}{\cos^2\theta}\,. \tag{5}$$

This reduces to

$$1 = \sin\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right). \tag{6}$$

Finally, you can show numerically that the solution for θ is $\theta_0 \approx 56.5^\circ$.

A few possible trajectories are shown below. Since it is well known that $\theta = 45^\circ$ provides the maximum *horizontal* distance, it follows from the figure that the θ_0 yielding the arc of maximum *length* must satisfy $\theta_0 \geq 45^\circ$. The exact angle, however, requires the above detailed calculation.



Remark: Let's now show that the integral in eq. (3) is given by

$$\int \frac{d\alpha}{\cos^3\alpha} = \frac{1}{2}\left(\frac{\sin\alpha}{\cos^2\alpha} + \ln\left(\frac{1+\sin\alpha}{\cos\alpha}\right)\right). \tag{7}$$

Letting $c \equiv \cos\alpha$ and $s \equiv \sin\alpha$ for convenience, and dropping the $d\alpha$ in the integrals, we have

$$\begin{aligned}
\int \frac{1}{c^3} &= \int \frac{c}{c^4} \\
&= \int \frac{c}{(1-s^2)^2} \\
&= \frac{1}{4}\int c\left(\frac{1}{1+s} + \frac{1}{1-s}\right)^2 \\
&= \frac{1}{4}\int \left(\frac{c}{(1+s)^2} + \frac{c}{(1-s)^2}\right) + \frac{1}{2}\int \frac{c}{(1-s^2)} \\
&= \frac{1}{4}\left(\frac{-1}{1+s} + \frac{1}{1-s}\right) + \frac{1}{4}\int\left(\frac{c}{1+s} + \frac{c}{1-s}\right) \\
&= \frac{s}{2(1-s^2)} + \frac{1}{4}\Big(\ln(1+s) - \ln(1-s)\Big) \\
&= \frac{s}{2c^2} + \frac{1}{4}\ln\left(\frac{1+s}{1-s}\right) \\
&= \frac{1}{2}\left(\frac{s}{c^2} + \ln\left(\frac{1+s}{c}\right)\right),
\end{aligned} \tag{8}$$

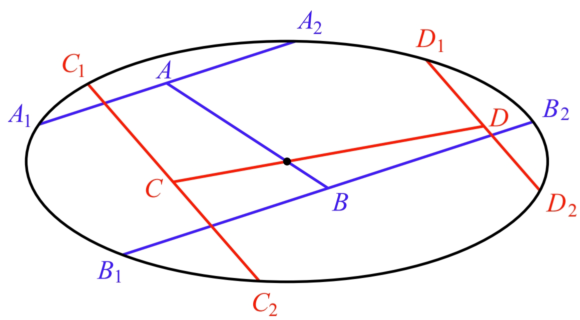
as we wanted to show.

Solution

Week 72 (1/26/04)

Find the foci

Ellipse: Let us first find the center of the ellipse. In the figure below, draw two arbitrary parallel lines that each meet the ellipse at two points. Call these points A_1, A_2 on one line, and B_1, B_2 on the other. Bisect segments A_1A_2 and B_1B_2 to yield points A and B. Now repeat this construction with two other parallel lines to give two new bisection points C and D.



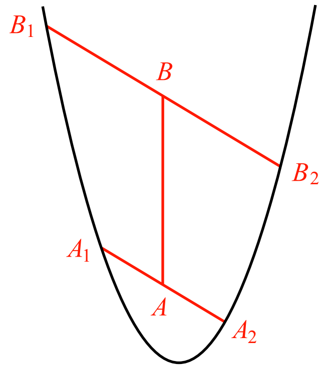
Claim 1 *The intersection of segments AB and CD is the center of the ellipse.*

Proof: Given a circle, the segment joining the midpoints of two parallel chords passes through the center of the circle. An ellipse is simply a stretched circle, and in this stretching process, all midpoints of segments remain midpoints. (If this doesn't satisfy you, we'll give an analytical proof when we get to the hyperbola case.) ∎

Having found the center, we can now find the major and minor axes by drawing a circle, with its center at the center of the ellipse, which meets the ellipse at four points (the vertices of a rectangle). The axes of the ellipse are the lines parallel to the sides of this rectangle, through the center of the ellipse.

Having found the axes, the foci are the points on the major axis that are one-half of a major-axis distance from the endpoints of the minor axis.

Parabola: Let us first find the axis of the parabola. Draw two arbitrary parallel lines that each meet the parabola at two points. Call these points A_1, A_2 on one line, and B_1, B_2 on the other. Bisect segments A_1A_2 and B_1B_2 to yield points A and B.



Claim 2 *Segment AB is parallel to the axis of the parabola.*

Proof: This follows from the reasoning in the ellipse case, along with the fact that a parabola is simply an ellipse with its center at infinity. (Again, if this doesn't satisfy you, we'll give an analytical proof when we get to the hyperbola case.) ■

To obtain the axis, draw a line perpendicular to AB, which meets the parabola at points C and D. The perpendicular bisector of CD is the axis of the parabola.

Having found the axis, the focus may be found as follows. Call the axis the y-axis of a coordinate system, with the parabola opening up in the positive y-direction. Let the vertex of the parabola be at $(0, 0)$, and let the focus be at $(0, a)$. Then a horizontal line through the focus must meet the parabola at points $(\pm 2a, a)$, because the absolute value of the x-coordinate (that is, the distance from these points to the focus) must equal the distance from these points to the directrix (which is the horizontal line located a distance a below the vertex), which equals $2a$. This also follows from writing the parabola in the form $x^2 = 4ay$, where a is the focal distance.

The focus of the parabola may therefore be found by drawing lines through the vertex, with slopes $1/2$ and $-1/2$. These two lines meet the parabola at points E and F. The intersection of segment EF with the axis is the focus.

Hyperbola: Let is first find the center of the hyperbola. The same construction works here as did for the ellipse, but we will now present an analytical proof.

Claim 3 *The center of a conic section is the intersection of two lines, each containing the midpoints of two parallel chords of the conic section.*

Proof: Let the conic section be written as

$$rx^2 + sy^2 = 1. \tag{1}$$

This describes an ellipse if r and s are positive, and a hyperbola if r and s have opposite sign. A parabola is obtained in the limit $r/s \to 0, \pm\infty$. Consider a line of the form

$$y = ax + b. \tag{2}$$

If this line meets the conic section in two points, you can show that the midpoint of the resulting chord has coordinates

$$\left(-\frac{sab}{r + sa^2}, \frac{rb}{r + sa^2}\right). \tag{3}$$

Note that when solving the quadratic equation for the intersection of the line and the conic section, we can ignore the discriminant in the quadratic formula, because we are concerned only with the midpoint between the intersections. This simplifies things greatly.

The slope of the line joining this point to the center of the conic section (which is the origin) equals $-r/(sa)$. This is independent of the constant b, so another parallel chord (that is, another chord with the same a but a different b) will also have its midpoint lying on this same line through the origin. ■

Having found the center, we can now find the axes by drawing a circle, with its center at the center of the hyperbola, which meets the hyperbola at four points (the vertices of a rectangle). The axes of the hyperbola are the lines parallel to the sides of this rectangle, through the center of the hyperbola.

Let us now, for convenience, assume that the hyperbola is written in the form

$$\frac{x^2}{m^2} - \frac{y^2}{n^2} = 1. \tag{4}$$

Then the focal length is well known to be $c = \sqrt{m^2 + n^2}$. We have already found m, which is the distance from the center to an intersection of the major axis with the hyperbola. So we simply need to find n, which may be found by noting that the point $(\sqrt{2}m, n)$ is on the hyperbola. We may therefore construct the foci as follows. Knowing the length m, we can construct the length $\sqrt{2}m$, and then the point $(\sqrt{2}m, 0)$. We can then draw a vertical line to obtain the point $(\sqrt{2}m, n)$. Then we can draw a horizontal line to obtain the point (m, n). This yields the diagonal distance $\sqrt{m^2 + n^2}$. The foci are the points $(\pm\sqrt{m^2 + n^2}, 0)$.

Solution

Week 73 (2/2/04)

Chain on a scale

First solution: Let y be the height of the top of the chain, and let F be the desired force applied by the scale. The net force on the whole chain is $F - (\sigma L)g$ (with upward taken to be positive). The momentum of the chain is $(\sigma y)\dot{y}$. Note that this is negative, since $\dot{y}$ is negative. Equating the net force with the change in momentum gives

$$\begin{aligned} F - \sigma L g &= \frac{d(\sigma y \dot{y})}{dt} \\ &= \sigma y \ddot{y} + \sigma \dot{y}^2. \end{aligned} \tag{1}$$

The part of the chain that is still above the scale is in free-fall. Therefore, $\ddot{y} = -g$. And $\dot{y} = \sqrt{2g(L-y)}$, which is the usual result for a falling object. Putting these into eq. (1) gives

$$\begin{aligned} F &= \sigma L g - \sigma y g + 2\sigma(L-y)g \\ &= 3\sigma(L-y)g. \end{aligned} \tag{2}$$

This answer has the expected property of equaling zero when $y = L$, and also the interesting property of equaling $3(\sigma L)g$ right before the last bit touches the scale. Once the chain is completely on the scale, the reading will suddenly drop down to the weight of the chain, namely $(\sigma L)g$.

Second solution: The normal force from the scale is responsible for doing two things. It holds up the part of the chain that already lies on the scale, and it also changes the momentum of the atoms that are suddenly brought to rest when they hit the scale. The first of these two parts of the force is simply the weight of the chain already on the scale, which is $F_{\text{weight}} = \sigma(L-y)g$.

To find the second part of the force, we need to find the change in momentum, dp, of the part of the chain that hits the scale during a given time, dt. The amount of mass that hits the scale in a time dt is $dm = \sigma|dy| = \sigma|\dot{y}|\,dt = -\sigma\dot{y}\,dt$. This mass initially has velocity $\dot{y}$, and then it is abruptly brought to rest. Therefore, the change in momentum is $dp = 0 - (dm)\dot{y} = \sigma\dot{y}^2\,dt$. The force required to cause this change in momentum is thus

$$F_{dp/dt} = \frac{dp}{dt} = \sigma\dot{y}^2. \tag{3}$$

But as in the first solution, we have $\dot{y} = \sqrt{2g(L-y)}$. Therefore, the total force from the scale is

$$\begin{aligned} F &= F_{\text{weight}} + F_{dp/dt} \\ &= \sigma(L-y)g + 2\sigma(L-y)g \\ &= 3\sigma(L-y)g. \end{aligned} \tag{4}$$

Solution

Week 74 (2/9/04)

Comparing the numbers

(a) Let your number be n. We will average over the equally likely values of n (excluding $n = 1$) at the end of the calculation.

For convenience, let $p \equiv (n-1)/(N-1)$ be the probability that a person you ask has a number smaller than yours. And let $1-p \equiv (N-n)/(N-1)$ be the probability that a person you ask has a number larger than yours.

Let A_n be the average number of people you have to ask in order to find a number smaller than yours, given that you have the number n. A_n may be calculated as follows.

There is a probability p that it takes only one check to find a smaller number.

There is a probability $1-p$ that the first person you ask has a larger number. From this point on, you have to ask (by definition) an average of A_n people in order to find a smaller number. In this scenario, you end up asking a total of $A_n + 1$ people.

Therefore, it must be true that

$$A_n = p \cdot 1 + (1-p)(A_n + 1). \tag{1}$$

This gives

$$A_n = \frac{1}{p} = \frac{N-1}{n-1}\,. \tag{2}$$

All values of n, from 2 to N, are equally likely, so we simply need to find the average of the numbers A_n, for n ranging from 2 to N. This average is

$$\begin{aligned} A &= \frac{1}{N-1}\sum_{n=2}^{N}\frac{N-1}{n-1} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N-1}\,. \end{aligned} \tag{3}$$

This expression for A is the exact answer to the problem. To obtain an approximate answer, note that the sum of the reciprocals of the numbers from 1 to M equals $\ln M + \gamma$, where $\gamma \equiv 0.577...$ is Euler's constant. So if N is large, you have to check about $\ln N + \gamma$ other numbers before you find one that is smaller than yours.

(b) Let your number be n. As in part (a), we will average over the equally likely values of n (excluding $n = 1$) at the end of the calculation.

Let B_n^N be the average number of people you have to ask in order to find a smaller number, given that you have the number n among the N numbers. B_n^N may be calculated as follows.

There is a probability $(n-1)/(N-1)$ that it takes only one check to find a smaller number.

There is a probability $(N-n)/(N-1)$ that the first person you ask has a larger number. From this point on, you have to ask (by definition) an average of B_n^{N-1} people in order to find a smaller number. In this scenario, you end up asking a total of $B_n^{N-1}+1$ people.

Therefore, it must be true that

$$\begin{aligned} B_n^N &= \frac{n-1}{N-1}\cdot 1 + \frac{N-n}{N-1}\Big(B_n^{N-1}+1\Big) \\ &= 1+\left(\frac{N-n}{N-1}\right)B_n^{N-1}. \end{aligned} \tag{4}$$

Using the fact that $B_n^N = 1$ when $N = n$, we can use eq. (4) to inductively increase N (while holding n constant) to obtain B_n^N for $N > n$. If you work out a few cases, you will quickly see that $B_n^N = N/n$. We can then easily check this by induction on N; it is true for $N = n$, so we simply need to verify in eq. (4)) that

$$\frac{N}{n} = 1+\left(\frac{N-n}{N-1}\right)\frac{N-1}{n}\,, \tag{5}$$

which is indeed true. Therefore,

$$B_n^N = \frac{N}{n}\,. \tag{6}$$

As in part (a), all values of n, from 2 to N, are equally likely, so we simply need to find the average of the numbers $B_n^N = N/n$, for n ranging from 2 to N. This average is

$$\begin{aligned} B &= \frac{1}{N-1}\sum_{n=2}^{N}\frac{N}{n} \\ &= \frac{N}{N-1}\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}\right). \end{aligned} \tag{7}$$

This expression for B is the exact answer to the problem. If N is large, then the result is approximately equal to $\ln N+\gamma-1$, due to the first term of “1” missing in the parentheses. This result is one person fewer than the result in part (a). So a good memory saves you, on average, one query.

REMARK: The continuum version of this problem (in which case the quality of your memory in irrelevant, to leading order) is the following.

Someone gives you a random number between 0 and 1, with a flat distribution. Pick successive random numbers until you finally obtain one that is smaller. How many numbers, on average, will you have to pick?

This is simply the original problem, in the limit $N\to\infty$. So the answer should be $\ln(\infty)$, which is infinite. And indeed, from the reasoning in part (a), if you start with the number x, then the average number of picks you need to make to find a smaller number is $1/x$, from eq. (2). Averaging these waiting times of $1/x$, over the equally likely values of x, gives an average waiting time of[1]

$$\int_0^1 \frac{dx}{x} = \infty. \tag{8}$$

We can get around this infinite answer by changing the probability distribution on the unit interval. For example, let the probability distribution be proportional to $1/\sqrt{x}$. Then the probability of someone having a number smaller than x is proportional to $\int_0^x dx/\sqrt{x} \propto \sqrt{x}$. Therefore, if you start with the number x, then the average number of picks you need to make to find a smaller number is proportional to $1/\sqrt{x}$, from eq. (2). Averaging these waiting times of $1/\sqrt{x}$, over the equally likely values of x, gives an average waiting time proportional to

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx \neq \infty. \tag{9}$$

In general, if the probability distribution is proportional to x^r, then $r = 0$ (that is, a flat distribution) is the cutoff case between having a finite or infinite expectation value for the number of necessary picks.

[1]However, if you play this game a few times, you will quickly discover that the average number of necessary picks is not infinite. If you find this unsettling, you are encouraged to look at Problem of the Week 6.

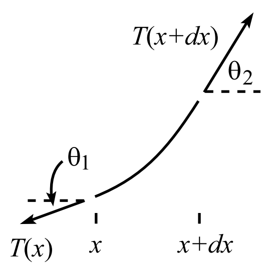
Solution

Week 75 (2/16/04)

Hanging chain

We'll present four solutions. The first one involves balancing forces. The other three involve various variations on a variational argument.

First solution: Let the chain be described by the function $y(x)$, and let the tension be described by the function $T(x)$. Consider a small piece of the chain, with endpoints at x and $x + dx$, as shown.



Let the tension at x pull downward at an angle θ_1 with respect to the horizontal, and let the tension at $x + dx$ pull upward at an angle θ_2 with respect to the horizontal. Balancing the horizontal and vertical forces on the small piece of chain gives

$$\begin{aligned} T(x + dx)\cos\theta_2 &= T(x)\cos\theta_1, \\ T(x + dx)\sin\theta_2 &= T(x)\sin\theta_1 + \frac{g\rho\, dx}{\cos\theta_1}, \end{aligned} \tag{1}$$

where ρ is the mass per unit length. The second term on the right is the weight of the small piece, because $dx/\cos\theta_1$ (or $dx/\cos\theta_2$, which is essentially the same) is its length. We must now somehow solve these two differential equations for the two unknown functions, $y(x)$ and $T(x)$. There are various ways to do this. Here is one method, broken down into three steps.

First step: Squaring and adding eqs. (1) gives

$$(T(x + dx))^2 = (T(x))^2 + 2T(x)g\rho\tan\theta_1\, dx + \mathcal{O}(dx^2). \tag{2}$$

Writing $T(x+dx) \approx T(x)+T'(x)\, dx$, and using $\tan\theta_1 = dy/dx \equiv y'$, we can simplify eq. (2) to (neglecting second-order terms in dx)

$$T' = g\rho y'. \tag{3}$$

Therefore,

$$T = g\rho y + c_1, \tag{4}$$

where c_1 is a constant of integration.

SECOND STEP: Let's see what we can extract from the first equation in eqs. (1). Using

$$\cos\theta_1 = \frac{1}{\sqrt{1+(y'(x))^2}}\,, \qquad \text{and} \qquad \cos\theta_2 = \frac{1}{\sqrt{1+(y'(x+dx))^2}}\,, \tag{5}$$

and expanding things to first order in dx, the first of eqs. (1) becomes

$$\frac{T+T'dx}{\sqrt{1+(y'+y''dx)^2}} = \frac{T}{\sqrt{1+y'^2}}\,. \tag{6}$$

All of the functions here are evaluated at x, which we won't bother writing. Expanding the first square root gives (to first order in dx)

$$\frac{T+T'dx}{\sqrt{1+y'^2}}\left(1-\frac{y'y''dx}{1+y'^2}\right) = \frac{T}{\sqrt{1+y'^2}}\,. \tag{7}$$

To first order in dx this yields

$$\frac{T'}{T} = \frac{y'y''}{1+y'^2}\,. \tag{8}$$

Integrating both sides gives

$$\ln T + c_2 = \frac{1}{2}\ln(1+y'^2), \tag{9}$$

where c_2 is a constant of integration. Exponentiating then gives

$$c_3^2T^2 = 1+y'^2, \tag{10}$$

where $c_3 \equiv e^{c_2}$.

THIRD STEP: We will now combine eq. (10) with eq. (4) to solve for $y(x)$. Eliminating T gives $c_3^2(g\rho y+c_1)^2 = 1+y'^2$. We can rewrite this is the somewhat nicer form,

$$1+y'^2 = \alpha^2(y+h)^2, \tag{11}$$

where $\alpha \equiv c_3 g\rho$, and $h = c_1/g\rho$. At this point we can cleverly guess (motivated by the fact that $1+\sinh^2 z = \cosh^2 z$) that the solution for y is given by

$$y(x)+h = \frac{1}{\alpha}\cosh\alpha(x+a). \tag{12}$$

Or, we can separate variables to obtain

$$dx = \frac{dy}{\sqrt{\alpha^2(y+h)^2-1}}\,, \tag{13}$$

and then use the fact that the integral of $1/\sqrt{z^2-1}$ is $\cosh^{-1} z$, to obtain the same result.

The shape of the chain is therefore a hyperbolic cosine function. The constant h isn't too important, because it simply depends on where we pick the $y=0$ height. Furthermore, we can eliminate the need for the constant a if we pick $x=0$ to be where the lowest point of the chain is (or where it would be, in the case where the slope is always nonzero). In this case, using eq. (12), we see that $y'(0) = 0$ implies $a = 0$, as desired. We then have (ignoring the constant h) the nice simple result,

$$y(x) = \frac{1}{\alpha}\cosh(\alpha x). \tag{14}$$

We'll show how to determine α at the end of the solutions.

Second solution: We can also solve this problem by using a variational argument. The chain will want to minimize its potential energy, so we want to find the function $y(x)$ that minimizes the integral,

$$U = \int (dm)gy = \int \Big(\rho\sqrt{1+y'^2}\,dx\Big)gy = \rho g\int y\sqrt{1+y'^2}\,dx, \tag{15}$$

subject to the constraint that the length of the chain is some given length ℓ. That is,

$$\ell = \int \sqrt{1+y'^2}\,dx. \tag{16}$$

Without this constraint, we could find $y(x)$ by simply using the Euler-Lagrange equation on the "Lagrangian" $y\sqrt{1+y'^2}$ given in eq. (15). But with the constraint, we must use the method of Lagrange multipliers. This works for functionals in the same way it works for functions. Basically, for any small variation in $y(x)$ near the minimum, we want the change in U to be proportional to the change in ℓ.[1] This means that there exists a linear combination of U and ℓ that doesn't change, to first order in any small variation in $y(x)$. In other words, the Lagrangian[2]

$$L = y\sqrt{1+y'^2} + h\sqrt{1+y'^2} = (y+h)\sqrt{1+y'^2} \tag{17}$$

satisfis the Euler-Lagrange equation, for some value of h. Therefore,

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(\frac{(y+h)y'}{\sqrt{1+y'^2}}\right) = \sqrt{1+y'^2}. \tag{18}$$

We must now perform some straightforward (although tedious) differentiations. Using the product rule on the left-hand side, and making copious use of the chain rule, we obtain

$$\frac{y'^2}{\sqrt{1+y'^2}} + \frac{(y+h)y''}{\sqrt{1+y'^2}} - \frac{(y+h)y'^2y''}{(1+y'^2)^{3/2}} = \sqrt{1+y'^2}. \tag{19}$$

Multiplying through by $(1+y'^2)^{3/2}$ and simplifying gives

$$(y+h)y'' = (1+y'^2). \tag{20}$$

[1] The reason for this is the following. Assume that we have found the desired function $y(x)$ that minimizes U, and consider two different variations in $y(x)$ that give the same change in ℓ, but different changes in U. Then the difference in these variations will produce no change in ℓ, while yielding a nonzero first-order change in U. This contradicts the fact that our $y(x)$ yielded an extremum of U.

[2] We'll use "h" for the Lagrange multiplier, to make the notation consistent with that in the first solution.

Having produced the Euler-Lagrange differential equation, we must now integrate it. If we multiply through by y' and rearrange, we obtain

$$\frac{y'y''}{1+y'^2} = \frac{y'}{y+h}\,. \tag{21}$$

Taking the dx integral of both sides gives $(1/2)\ln(1+y'^2) = \ln(y+h) + c_4$, where c_4 is a constant of integration. Exponentiation then gives (with $\alpha \equiv e^{c_4}$)

$$1+y'^2 = \alpha^2(y+h)^2. \tag{22}$$

in agreement with eq. (11).

Third solution: Let's use a variational argument again, but now with y as the independent variable. That is, let the chain be described by the function $x(y)$. Then the potential energy is

$$U = \int (dm)gy = \int \Big(\rho\sqrt{1+x'^2}\,dy\Big)gy = \rho g \int y\sqrt{1+x'^2}\,dy. \tag{23}$$

The constraint is

$$\ell = \int \sqrt{1+x'^2}\,dy. \tag{24}$$

Using the method of Lagrange multipliers as in the second solution above, the Lagrangian we want to consider is

$$L = y\sqrt{1+x'^2} + h\sqrt{1+x'^2} = (y+h)\sqrt{1+x'^2}. \tag{25}$$

Our Euler-Lagrange equation is then

$$\frac{d}{dy}\left(\frac{\partial L}{\partial x'}\right) = \frac{\partial L}{\partial x} \qquad \Longrightarrow \qquad \frac{d}{dy}\left(\frac{(y+h)x'}{\sqrt{1+x'^2}}\right) = 0. \tag{26}$$

The zero on the right-hand side makes things nice and easy, because it means that the quantity in parentheses is a constant. Calling this constant $1/\alpha$ (to end up with the notation in the second solution), we have $\alpha(y+h)x' = \sqrt{1+x'^2}$. Therefore,

$$x' = \frac{1}{\sqrt{\alpha^2(y+h)^2-1}}\,, \tag{27}$$

which is equivalent to eq. (13).

Fourth solution: Note that our "Lagrangian" in the second solution above, which is given in eq. (17) as

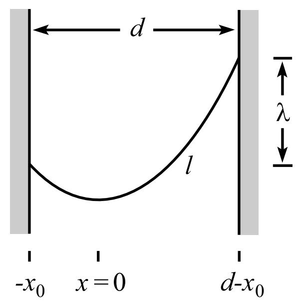
$$L = (y+h)\sqrt{1+y'^2}, \tag{28}$$

is independent of x. Therefore, in analogy with conservation of energy (which arises from a Lagrangian that is independent of t), the quantity

$$E \equiv y'\frac{\partial L}{\partial y'} - L = -\frac{y+h}{\sqrt{1+y'^2}} \tag{29}$$

is independent of x. Call it $1/\alpha$. Then we have reproduced eq. (11).

Remark: The constant α can be determined from the locations of the endpoints and the length of the chain. The position of the chain may be described by giving (1) the horizontal distance, d, between the two endpoints, (2) the vertical distance, λ, between the two endpoints, and (3) the length, ℓ, of the chain, as shown.



Note that it is not obvious what the horizontal distances between the ends and the minimum point (which we have chosen as the $x = 0$ point) are. If $\lambda = 0$, then these distances are simply $d/2$. But otherwise, they are not so clear.

If we let the left endpoint be located at $x = -x_0$, then the right endpoint is located at $x = d - x_0$. We now have two unknowns, x_0 and α. Our two conditions are[3]

$$y(d - x_0) - y(-x_0) = \lambda, \tag{30}$$

along with the condition that the length equals ℓ, which takes the form (using eq. (14))

$$\begin{aligned} \ell &= \int_{-x_0}^{d-x_0} \sqrt{1 + y'^2}\, dx \\ &= \frac{1}{\alpha} \sinh(\alpha x)\Big|_{-x_0}^{d-x_0}. \end{aligned} \tag{31}$$

Writing out eqs. (30) and (31) explicitly, using eq. (14), we have

$$\begin{aligned} \cosh\big(\alpha(d - x_0)\big) - \cosh(-\alpha x_0) &= \alpha\lambda, \qquad \text{and} \\ \sinh\big(\alpha(d - x_0)\big) - \sinh(-\alpha x_0) &= \alpha\ell. \end{aligned} \tag{32}$$

If we take the difference of the squares of these two equations, and use the hyperbolic identities $\cosh^2 x - \sinh^2 x = 1$ and $\cosh x \cosh y - \sinh x \sinh y = \cosh(x - y)$, we obtain

$$2 - 2\cosh(\alpha d) = \alpha^2(\lambda^2 - \ell^2). \tag{33}$$

We can now numerically solve this equation for α. Using a "half-angle" formula, you can show that eq. (33) may also be written as

$$2\sinh(\alpha d/2) = \alpha\sqrt{\ell^2 - \lambda^2}. \tag{34}$$

We can check some limits here. If $\lambda = 0$ and $\ell = d$ (that is, the chain forms a horizontal straight line), then eq. (34) becomes $2\sinh(\alpha d/2) = \alpha d$. The solution to this is $\alpha = 0$, which does indeed correspond to a horizontal straight line, because for small α, eq. (14) behaves like $\alpha x^2/2$ (up to an additive constant), which varies slowly with x for small α. Another limit is where ℓ is much larger than both d and λ. In this case, eq. (34) becomes $2\sinh(\alpha d/2) \approx \alpha\ell$. The solution to this is a very large α, which corresponds to a "droopy" chain, because eq. (14) varies rapidly with x for large α.

[3]We'll take the right end to be higher than the left end, without loss of generality.

Solution

Week 76 (2/23/04)

Crawling ant

At time t, the movable end of the band is a distance $\ell(t) = L + Vt$ from the wall. Let the ant's distance from the wall be $x(t)$, and consider the fraction of the length along the band, $F(t) = x(t)/\ell(t)$. The given question is equivalent to: For what value of t does the fraction $F(t)$ become zero (if at all)? To answer this, let us see how $F(t)$ changes with time.

After an infinitesimal time, dt, the ant's position, x, increases by $(x/\ell)V\,dt$ due to the stretching, and decreases by $u\,dt$ due to the crawling. Therefore,

$$\begin{aligned} F(t+dt) &= \frac{x + (x/\ell)V\,dt - u\,dt}{\ell + V\,dt} \\ &= \frac{x}{\ell} - \frac{u\,dt}{\ell + V\,dt}\,. \end{aligned} \tag{1}$$

To first order in dt, this yields

$$F(t+dt) = F(t) - \frac{u}{\ell}\,dt. \tag{2}$$

In other words, $F(t)$ decreases due to the fact that in a time dt the ant crawls a distance $u\,dt$ relative to the band, which has a length $\ell(t)$. Eq. (2) gives

$$\frac{dF(t)}{dt} = -\frac{u}{\ell}\,. \tag{3}$$

Using $\ell(t) = L + Vt$ and integrating eq. (3), we obtain

$$F(t) = 1 - \frac{u}{V}\ln\left(1 + \frac{V}{L}t\right), \tag{4}$$

where the constant of integration has been chosen to satisfy $F(0) = 1$. We now note that for *any* positive values of u and V, we can make $F(t) = 0$ by choosing

$$t = \frac{L}{V}\left(e^{V/u} - 1\right). \tag{5}$$

For large V/u, the time it takes the ant to reach the wall becomes exponentially large, but it does indeed reach it in a finite time. For small V/u, you can use $e^{\epsilon} \approx 1 + \epsilon$ to show that eq. (5) reduces to $t \approx L/u$, as it should.

Remark: If $u < V$, then the ant will initially get carried away from the wall before it eventually comes back and reaches it. What is the maximum distance it gets from the wall? The ant's distance from the wall is

$$x(t) = F(t)\ell(t) = \left(1 - \frac{u}{V}\ln\left(1 + \frac{V}{L}t\right)\right)\left(L + Vt\right). \tag{6}$$

Setting the derivative of this equal to zero gives

$$\left(1 - \frac{u}{V}\ln\left(1 + \frac{V}{L}t\right)\right)V - u = 0. \tag{7}$$

Note that we could have arrived at this by simply demanding that the speed of the ant be zero, which means that the $F(t)V$ speed due to the stretching cancels the u speed in the other direction due to the crawling. The fraction $F(t)$ is therefore simply u/V.

Solving eq. (7) for t gives

$$t_{\max} = \frac{L}{V}\left(e^{V/u-1} - 1\right). \tag{8}$$

This holds only if $V \geq u$. If $V < u$, then $t_{\max} = 0$ and $x_{\max} = L$. Plugging the $t_{\max}$ from eq. (8) into eq. (6) gives

$$x_{\max} = \frac{u}{V}\frac{L}{e}e^{V/u} \qquad (V \geq u). \tag{9}$$

If, for example, $V = 2u$, then $x_{\max} \approx (1.36)L$. And if $V = 10u$, then $x_{\max} \approx (800)L$.

Note that for large V/u, the $t_{\max}$ in eq. (8) is approximately $1/e$ times the time it takes the ant to reach the wall, given in eq. (5).

Solution

Week 77 (3/1/04)

Relativistic momentum paradox

The reasoning is not correct. The horizontal speed of the masses remains the same. The system does *not* slow down in the x-direction. We can see that this must be the case, by looking at the setup in the inertial frame that moves in the x-direction with the v_x that the masses have when they are still on the constraints (which happens to be the same v_x that they always have). In this moving frame, the masses feel a force only in the y-direction when they come off the constraints, so they won't move horizontally with respect to this frame. Therefore, they will always move with constant v_x in the lab frame.

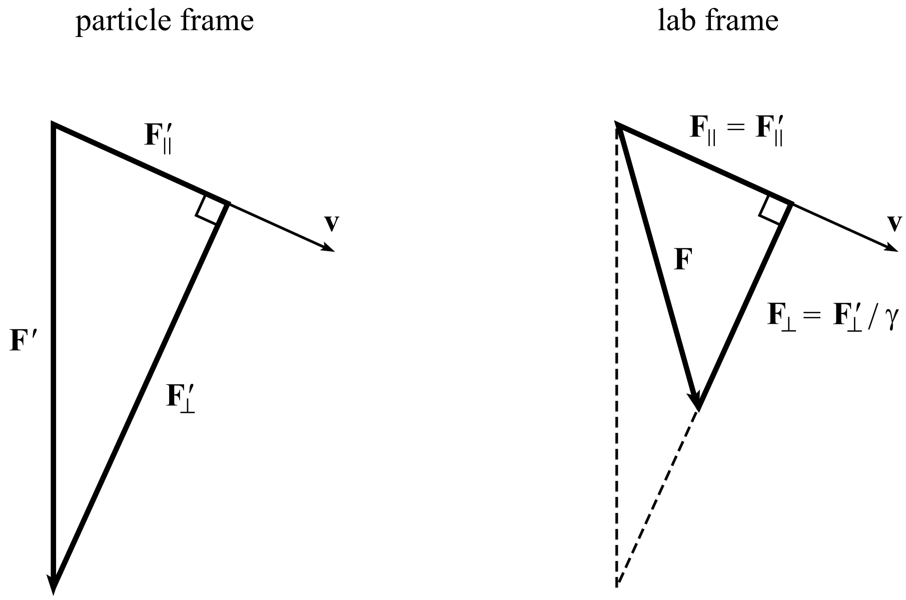
Since v_x is constant, the final p_x of the resulting blob is larger than the sum of the initial p_x's of the two masses, because the mass of the resulting blob is indeed greater than the sum of the initial masses. Where does the extra momentum come from? It comes from the string, which initially had some p_x because it had stored energy (because work was done to extend the string) and hence mass.

But where is the error in the reasoning stated in the problem? The first sentence is the one that is incorrect. In the lab frame, the force on each mass actually does *not* point in the y-direction. The reason for this is the following.

It turns out that in relativity, a transverse force is decreased by a factor γ when going from the particle's frame to another frame. (We'll just accept this fact here.) More precisely, consider an object flying past you in some direction. In the particle's frame (S'), let the force components be $F'_{\|}$ in that direction, and $F'_{\perp}$ in the orthogonal direction. Then the force components that you measure on the particle in the lab frame (S) are

$$F_{\|} = F'_{\|} \qquad \text{and} \qquad F_{\perp} = \frac{F'_{\perp}}{\gamma}\,. \tag{1}$$

Let's see what this implies. After the masses have been drawn together by the string a bit, they will be heading diagonally upward or downward, instead of directly in the x-direction. Let the upper mass be traveling in the direction $\mathbf{v}$ shown below. In the frame of the mass, the force $\mathbf{F}'$ points directly downward. But if we break this force into the components along the motion and perpendicular to the motion, we see that when transforming to the lab frame, the transverse component is decreased by a factor γ, as shown. The force $\mathbf{F}$ in the lab frame therefore points slightly forward, as shown. This forward component of the force is what increases the x-momentum of the masses. The x-momentum of each mass does indeed increase, because p_x takes the form $p_x = \gamma m v_x$, and γ increases due to the fact that the speed increases because of the increasing v_y.


particle frame
lab frame
$\mathbf{F}'_{\parallel}$
$\mathbf{v}$
$\mathbf{F}'$
$\mathbf{F}'_{\perp}$
$\mathbf{F}_{\parallel} = \mathbf{F}'_{\parallel}$
$\mathbf{v}$
$\mathbf{F}$
$\mathbf{F}_{\perp} = \mathbf{F}'_{\perp} / \gamma$

Solution

Week 78 (3/8/04)

Infinite square roots

Let

$$x=\sqrt{1-\sqrt{\frac{17}{16}-\sqrt{1-\sqrt{\frac{17}{16}-\sqrt{1-\cdots}}}}}. \tag{1}$$

Then $x=\sqrt{1-\sqrt{17/16-x}}$. Squaring a few times yields $x^4-2x^2+x-1/16=0$, or equivalently,

$$(2x-1)(8x^3+4x^2-14x+1)=0. \tag{2}$$

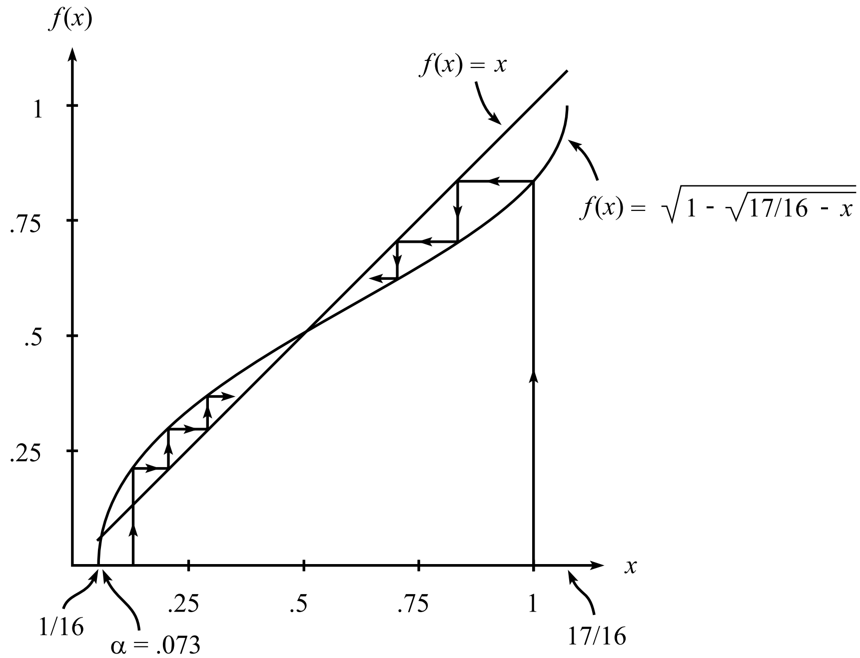
Therefore, either $x=1/2$, or x is a root of $8x^3+4x^2-14x+1=0$. Solving this cubic equation numerically, or fiddling around with the values at a few points, shows that it has three real roots. One is negative, between -1 and -2 (≈ -1.62). One is slightly greater than 1 (≈ 1.05). And one is slightly greater than $1/14$ ($\approx .073$).

The first two of these cannot be the answer to the problem, because x must be positive and less than one. So the only possibilities are $x=1/2$ and $x\approx .073$. A double-check shows that neither was introduced in the squaring steps above; both are correct solutions to $x=\sqrt{1-\sqrt{17/16-x}}$. But x clearly has one definite value, so which is it? A few iterations on a calculator, starting with a "1" under the innermost radical, shows that $x=1/2$ is definitely the answer. Why?

The answer lies in the behavior of $f(x)=\sqrt{1-\sqrt{17/16-x}}$ near the points $x=1/2$ and $x=\alpha\equiv .07318\ldots$. The point $x=1/2$ is stable, in the sense that if we start with an x value slightly different from $1/2$, then $f(x)$ will be closer to $1/2$ than x was. The point $x=\alpha$, on the other hand, is unstable, in the sense that if we start with an x value slightly different from α, then $f(x)$ will be farther from α than x was.

Said in another way, the slope of $f(x)$ at $x=1/2$ is less than 1 in absolute value (it is in fact $2/3$), while the slope at $x=\alpha$ is greater than 1 in absolute value (it is approximately 3.4). What this means is that points tend to head toward $1/2$, but away from α, under iteration by $f(x)$. In particular, if we start with the value 1, as we are supposed to do in this problem, then we will eventually get arbitrarily close to $1/2$ after many applications of f. Therefore, $1/2$ is the correct answer.

The above reasoning is perhaps most easily understood via the figure below. This figure shows graphically what happens to two initial values of x, under iteration by f. To find what happens to a given point x_0, draw a vertical line from x_0 to the curve $y=f(x)$; this gives $f(x_0)$. Then draw a horizontal line to the point $(f(x_0),f(x_0))$ on the line $y=x$. Then draw a vertical line to the curve $y=f(x)$; this gives $f(f(x_0))$. Continue drawing these horizontal and vertical lines to obtain successive iterations by f.



Using this graphical method, it is easy to see that if:

- $x_0 > 17/16$, then we immediately get imaginary values.
- $\alpha < x_0 \leq 17/16$, then iteration by f will lead to $1/2$.
- $x_0 = \alpha$ exactly, then we stay at α under iteration by f.
- $x_0 < \alpha$, then we will eventually get imaginary values. If $x_0 < 1/16$, then imaginary values will occur immediately, after one iteration; otherwise it will take more than one iteration.

Solution

Week 79 (3/15/04)

Propelling a car

Let the speed of the car be $v(t)$. Consider the time interval when a mass dm enters the car. Conservation of momentum gives

$$\begin{aligned} (dm)u + mv &= (m + dm)(v + dv) \\ \Longrightarrow \quad dm(u - v) &= m\,dv, \end{aligned} \tag{1}$$

where we have dropped the second-order $dm\,dv$ term. Separating variables and integrating gives

$$\begin{aligned} \int_M^m \frac{dm}{m} = \int_0^v \frac{dv}{u-v} \qquad &\Longrightarrow \qquad \ln\left(\frac{m}{M}\right) = -\ln\left(\frac{u-v}{u}\right) \\ &\Longrightarrow \qquad m = \frac{Mu}{u-v}\,. \end{aligned} \tag{2}$$

Note that $m \to \infty$ as $v \to u$, as it should.

How does m depend on time? Mass enters the car at a rate $\sigma(u-v)/u$, because although you throw the balls at speed u, the relative speed of the balls and the car is only $(u - v)$. Therefore,

$$\frac{dm}{dt} = \frac{(u-v)\sigma}{u}\,. \tag{3}$$

Substituting the m from eq. (2) into this equation gives

$$\begin{aligned} \int_0^v \frac{dv}{(u-v)^3} &= \int_0^t \frac{\sigma\,dt}{Mu^2} \\ \Longrightarrow \qquad \frac{1}{2(u-v)^2} - \frac{1}{2u^2} &= \frac{\sigma t}{Mu^2} \\ \Longrightarrow \qquad v(t) &= u\left(1 - \frac{1}{\sqrt{1 + \frac{2\sigma t}{M}}}\right). \end{aligned} \tag{4}$$

Note that $v \to u$ as $t \to \infty$, as it should. Integrating this speed to obtain the position gives

$$x(t) = ut - \frac{Mu}{\sigma}\sqrt{1 + \frac{2\sigma t}{M}} + \frac{Mu}{\sigma}\,. \tag{5}$$

We see that even though the speed approaches u, the car will eventually be an arbitrarily large distance behind a ball with constant speed u. For example, pretend that the first ball missed the car and continued to travel forward at speed u.

Solution

Week 80 (3/22/04)

Nine divisible by 9

First, consider the following simpler problem:

Problem: Given any five integers, show that there is at least one subset of three integers whose sum is divisible by 3.

Solution: Let us try to find a set of five integers that contains no subset of three integers whose sum is divisible by 3, a task that we will show is impossible. Each of the five integers is, for our purposes, equal to 0, 1, or 2, because we are concerned only with divisions by 3. We cannot have one of each of these, because $0 + 1 + 2$ is divisible by 3. We must therefore have at most two of the types. But the pigeonhole principle then implies that we have at least three of one of the types. The sum of these three integers is divisible by 3. ■

Returning to the original problem, pick five integers to obtain a triplet whose sum is divisible by 3. Then pick another five integers to obtain another such triplet. We can continue to do this for a total of five times, given the seventeen integers.

We now have five triplets, each of whose sum is divisible by 3. As far as divisions by 9 are concerned, these sums are equal to 0, 3, or 6. We can now use the same reasoning as in our auxiliary problem above (but with everything scaled up by a factor of 3) to show that we can find a set of three triplets that has a sum divisible by 9. In other words, we have found a set of nine integers whose sum is divisible by 9.

This result is a special case of the following theorem.

Theorem: Given any $2n - 1$ integers, there is at least one subset of n integers whose sum is divisible by n.

We will prove this theorem by demonstrating two lemmas.

Lemma 1: If the theorem is true for integers n_1 and n_2, then it is also true for the product n_1n_2.

Proof: Consider a set of $2n_1n_2 - 1$ integers. Under the assumption that the theorem is true for n_1, we can certainly pick a subset of n_1 integers whose sum is divisible by n_1. From the remaining $2n_1n_2 - 1 - n_1$ integers we can pick another such subset of n_1 integers, and so on. We can continue to do this until we have obtained $2n_2 - 1$ such subsets. This is true because after forming $2n_2 - 2$ such subsets, there are $[2n_1n_2 - 1] - [(2n_2 - 2)n_1] = 2n_1 - 1$ integers left over, from which we can pick one last such subset of n_1 integers.

Now consider these $2n_2 - 1$ sums divided by n_1. Assuming that the theorem holds for n_2, we can find n_2 of these sums (divided by n_1) that have a sum divisible by n_2. Bringing back in the factor of n_1, we see that we have found a set of n_1n_2 integers whose sum is divisible by n_1n_2. ■

In proving the general theorem, this first lemma shows that it is sufficient to prove the theorem for primes, p:

Lemma 2: If p is prime, then given $2p-1$ integers, there is at least one subset of p integers whose sum is divisible by p.

Proof: Consider all the possible $N \equiv \binom{2p-1}{p}$ subsets of p integers. Label the sums of these subsets as S_j, where $1 \leq j \leq N$ (these may be indexed in an arbitrary manner), and consider the sum

$$S = \sum_{j=1}^{N} S_j^{(p-1)}. \tag{1}$$

REMARK: The following proof isn't mine (I'm not sure where it came from originally). At first glance, it might seem that adding up these $(p-1)$st powers is a little out of the blue, but it's actually a fairly reasonable thing to do. There are two types of sums: "good" ones that are divisible by p, and "bad" ones that aren't. It would be nice to label them all in a sort of binary way, say, with a "0" for good, and a "1" for bad. Fermat's Little Theorem (which states that if $a \not\equiv 0 \pmod{p}$ then $a^{p-1} \equiv 1 \pmod{p}$) provides the perfect way for doing this.

We will prove this second lemma by demonstrating two claims:

Claim 1: S is be divisible by p.

Proof: Let the $2p-1$ integers be a_i, where $1 \leq i \leq 2p-1$. Expand all of the $S_j^{(p-1)}$ powers and collect all the like terms in S. The terms will have the form of some coefficient times $a_{i_1}^{b_1} a_{i_2}^{b_2} \cdots a_{i_k}^{b_k}$. The number, k, of different a_i's involved may be any number from 1 to $p-1$, and the b_j's must of course add up to $(p-1)$.

We will now show that the coefficient of an arbitrary $a_{i_1}^{b_1} a_{i_2}^{b_2} \cdots a_{i_k}^{b_k}$ term is divisible by p. The coefficient will depend on the b_i, but it will happen to always be divisible by p. The coefficient may be viewed as the product of two factors.

- Firstly, there is a multinomial coefficient from each $S_j^{(p-1)}$ in which the given $a_{i_1}^{b_1} a_{i_2}^{b_2} \cdots a_{i_k}^{b_k}$ occurs. This multinomial coefficient is $\binom{p-1}{b_1, b_2, \ldots, b_k}$, but it will turn out not to be important.

- Secondly, we must count the number of different $S_j^{(p-1)}$ in which the given $a_{i_1}^{b_1} a_{i_2}^{b_2} \cdots a_{i_k}^{b_k}$ occurs. This number may be found as follows. We know that k of the p integers in S_j must be $a_1, a_2, \ldots, a_k$. The remaining $p-k$ integers can be any subset of the other $2p-1-k$ integers. There are $\binom{2p-1-k}{p-k}$ such subsets.

The coefficient of the $a_{i_1}^{b_1} a_{i_2}^{b_2} \cdots a_{i_k}^{b_k}$ term in S is therefore $\binom{p-1}{b_1, b_2, \ldots, b_k}\binom{2p-1-k}{p-k}$. Writing the second factor in this as

$$\binom{2p-1-k}{p-k} = \frac{(2p-1-k)(2p-2-k)\cdots p}{(p-k)!} \tag{2}$$

demonstrates that every coefficient is divisible by p, independent of the values of the b_i. Therefore, S is divisible by p. Q.E.D.

Claim 2: If none of the S_j are divisible by p, then S is not divisible by p.

Proof: Assume that none of the S_j are divisible by p. Then by Fermat's Little Theorem (which states that if $a \not\equiv 0 \,(\text{mod } p)$ then $a^{p-1} \equiv 1 \,(\text{mod } p)$), we have

$$S \equiv \left(\sum_{i=1}^{N} 1 \right) \pmod{p} \equiv N \pmod{p}. \tag{3}$$

We now note that

$$N \equiv \binom{2p-1}{p} = \frac{(2p-1)(2p-2)\cdots(p+1)}{(p-1)!}\,, \tag{4}$$

which is not divisible by p. Therefore, S is not divisible by p. Q.E.D.

These two claims show that at least one of the S_j must be divisible by p. ■

REMARK: For the case where n is a prime number, p, it is possible to say a bit more about exactly how many of the S_j are divisible by p. We claim that either 1, $p+1$, $2p+1$, ..., of the S_j are divisible by p. The reasoning is as follows.

Fermat's Little Theorem implies that each S_j that is not divisible by p contributes 1 to S, while each S_j that is divisible by p contributes 0 to S. Under the (incorrect) assumption that none of the S_j are divisible by p, eq. (3) states that $S \equiv N \pmod{p}$. Using eq. (4), and noting that

$$(2p-1)(2p-2)\cdots(p+1) \equiv (p-1)! \pmod{p}, \tag{5}$$

we see that this assumption leads to the incorrect conclusion that $S \equiv 1 \pmod{p}$. But since we know from eq. (2) that S must actually be divisible by p, then either 1, $p+1$, $2p+1$, ..., of the S_j must be divisible by p, because each S_j that is divisible by p will contribute 0, instead of 1, to S. Thus, for example, given five integers, there are either one, four, seven, or ten subsets of three integers whose sum is divisible by 3.

It is easy to construct a case where only one of the S_j is divisible by p. We may pick $p-1$ of the $2p-1$ integers to be congruent to each other modulo p, and then pick the remaining p integers to also be congruent to each other (but not to the other $p-1$ integers) modulo p. Then the subset consisting of these latter p integers is the only subset of p integers whose sum is divisible by p.

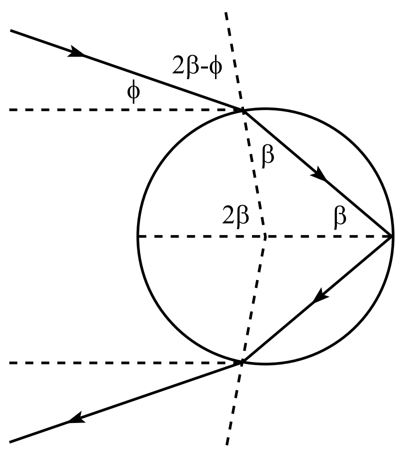
Solution

Week 81 (3/29/04)

Rainbows

Rainbows exist due to the fact that raindrops scatter light preferentially in certain directions. The effect of this "focusing" is to make the sky appear brighter in a certain region, as we will see in detail below. This brightness is the rainbow that you see. The colors of the rainbow are caused by the different indexes of refraction of the different colors; more on this below.

Primary rainbow: The preferred direction of the scattered light depends on how many internal reflections the light ray undergoes in the raindrop. Let's first consider the case of one internal reflection, as shown below. The light ray refracts into the raindrop, then reflects inside, and then refracts back out into the air. Let β and ϕ be defined as in the figure. Then the angles 2β and $2\beta - \phi$ are shown.



What is ϕ as a function of β? Snell's law at the refraction points gives

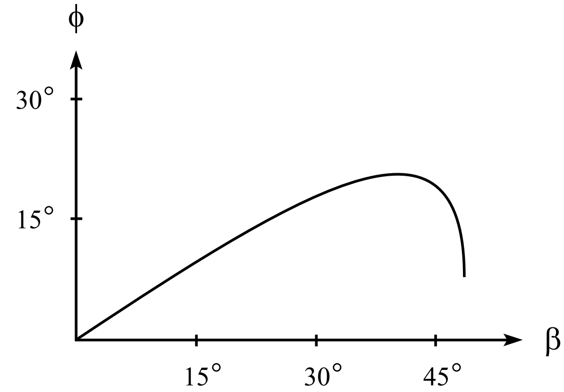
$$\sin(2\beta - \phi) = \frac{4}{3}\sin\beta. \tag{1}$$

Solving for ϕ yields

$$\phi = 2\beta - \arcsin\left(\frac{4}{3}\sin\beta\right). \tag{2}$$

The plot of ϕ vs. β looks like the following.[1]

[1]Note that β cannot be greater than $\arcsin(3/4) \approx 48.6^\circ$, because this would yield the acrsin of a number larger than 1. 48.6° is the *critical angle* for the air/water interface.



We see that ϕ has a maximum of roughly $\phi_{\text{max}} \approx 20°$ at $\beta_{\text{max}} \approx 40°$. To be more precise, we can set $d\phi/d\beta = 0$. The result is

$$0 = 2 - \frac{\frac{4}{3}\cos\beta}{\sqrt{1 - \frac{16}{9}\sin^2\beta}}. \tag{3}$$

Squaring and using $\cos^2\beta = 1 - \sin^2\beta$ yields

$$\sin\beta_{\text{max}} = \sqrt{\frac{5}{12}} \qquad \Longrightarrow \qquad \beta_{\text{max}} \approx 40.2°. \tag{4}$$

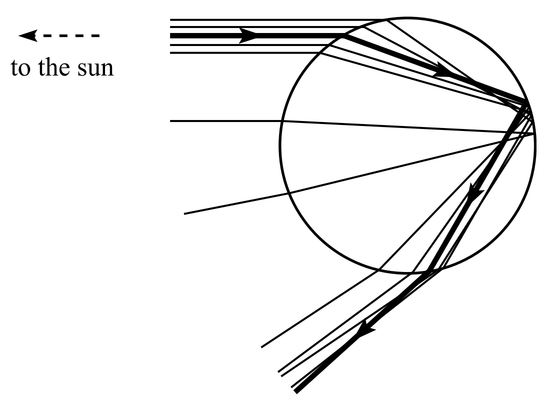
Substituting this back into eq. (2) gives

$$\phi_{\text{max}} \approx 21.0°. \tag{5}$$

The significance of this maximum is not that it is the largest value of ϕ, but rather that the slope of the $\phi(\beta)$ curve is zero, which means that there are many different values of β that yield essentially the same value ($\approx 21°$) of ϕ. The light therefore[2] gets "focused" into the total angle of

$$2\phi_{\text{max}} \approx 42°, \tag{6}$$

so the sky appears brighter at this angle (relative to the line from the sun to you). This brightness is the rainbow that you see. This reasoning is perhaps more clear if we draw a diagram with the incoming angle of the rays constant (which is in fact the case, because all of the sun's rays are parallel). A rough diagram of this is shown below. The bold line is the path with the maximum 2ϕ angle of 42°. You can see that the outgoing rays pile up at this angle.



[2]See the fourth remark below for elaboration on this reasoning.

Remarks:

1. The above reasoning explains why the rainbow is where it is. But why do we see the different colors? The colors arise from the fact that different wavelengths of light have different indexes of refraction at the air/water boundary. At one end of the visible spectrum, violet light has an index of 1.344. And at the other end, red light has an index of 1.332.[3] If you go through the above calculation, but now with these indexes in place of the "4/3" we used above, you will find that violet light appears at an angle of $2\phi_{\text{max}}^{\text{violet}} \approx 40.5°$, and red light appears at an angle of $2\phi_{\text{max}}^{\text{red}} \approx 42.2°$. Since red occurs at the larger angle, it is therefore the color at the top of the rainbow. Violet is at the bottom, and intermediate wavelengths are in between. The fact that red is at the top can be traced to the fact that $21°$ is the *maximum* value of ϕ.

2. A rainbow is actually a little wider than the approximately $2°$ spread we just found, because the sun isn't a point source. It subtends an angle of about half a degree, which adds half a degree to the rainbow's spread. Also, the rainbow's colors are somewhat washed out on the scale of half a degree, so a rainbow isn't as crisp as it would be if the sun were a point source.

3. In addition to the ordering of the colors, there is another consequence of the fact that $21°$ is the *maximum* value of ϕ. It is possible for a raindrop to scatter light at 2ϕ values smaller than $42°$ (for one internal reflection), but impossible for larger angles. Therefore, the region in the sky below the rainbow appears brighter than the region above it. Even though the focusing effect occurs only right at the rainbow, the simple scattering of light through 2ϕ values smaller than $42°$ makes the sky appear brighter below the rainbow.

4. There is one slight subtlety in the above reasoning we should address, even though turns out not to be important. We said above that if many different β values correspond to a certain value of ϕ, then there will be more light scattered into that value of ϕ. However, the important thing is not how many β values correspond to a certain ϕ, but rather the "cross section" of light that corresponds to that ϕ. Since the light hits the raindrop at an angle $2\beta - \phi$ with respect to the normal, the amount of light that corresponds to a given $d\beta$ interval is decreased by a factor of $\cos(2\beta - \phi)$. What we did in the above solution was basically note that the interval $d\beta$ that corresponds to a given interval $d\phi$ is

$$d\beta = \frac{d\phi}{d\phi/d\beta}\,, \tag{7}$$

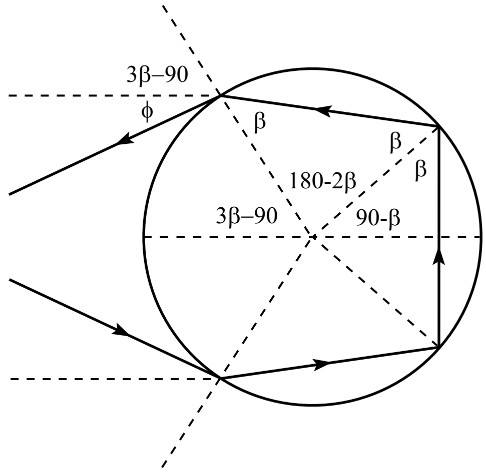
which diverges at ϕ_{max}, because $d\phi/d\beta = 0$ there. But since we are in fact concerned with the number of light rays that correspond to the interval $d\phi$, we now note that this number is proportional to

$$d\beta \cos(2\beta - \phi) = \frac{d\phi}{d\phi/d\beta} \cos(2\beta - \phi). \tag{8}$$

But $2\beta - \phi \neq 90°$ at ϕ_{max}, so this still diverges at ϕ_{max}. Our answer therefore remains the same.

Secondary rainbow: Now consider the secondary rainbow. This rainbow arises from the fact that the light may undergo two reflections inside the raindrop before it refracts back out. This scenario is shown below. With β defined as in the figure, the $180° - 2\beta$ and $90° - \beta$ angles follow, and then the $3\beta - 90°$ angle follows.

[3]The ends of the visible spectrum are somewhat nebulous, but I think these values are roughly correct.



Snell's law at the refraction points gives

$$\sin(3\beta - 90^\circ + \phi) = \frac{4}{3}\sin\beta. \tag{9}$$

Solving for ϕ yields

$$\phi = 90^\circ - 3\beta + \arcsin\left(\frac{4}{3}\sin\beta\right). \tag{10}$$

Taking the derivative to find the extremum (which is a minimum this time) gives

$$\sin\beta_{\text{min}} = \sqrt{\frac{65}{128}} \qquad \Longrightarrow \qquad \beta_{\text{min}} \approx 45.4^\circ. \tag{11}$$

Substituting this back into eq. (10) gives

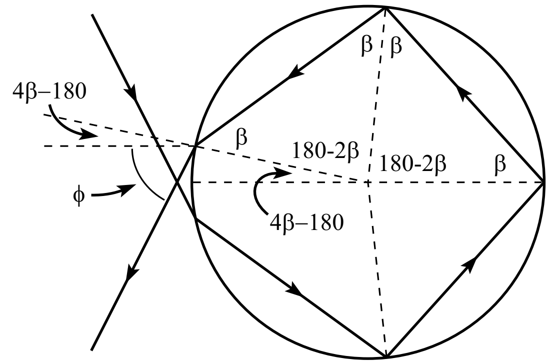
$$\phi_{\text{min}} \approx 25.4^\circ. \tag{12}$$

Using the same reasoning as above, we see that the light gets focused into the total angle of $2\phi_{\text{max}} \approx 51^\circ$, so the the sky appears brighter at this angle.

The fact that $\phi(\beta)$ has a minimum instead of a maximum means that the rainbow is inverted, with violet now on top. More precisely, if you go through the above calculation, but now with the red and violet indexes of refraction in place of the "4/3" used above, you will find that violet light appears at an angle of 53.8°, and red light appears at an angle of 50.6°. The spread here is a little larger than that for the primary rainbow above.

The secondary rainbow is fainter than the primary one for three reasons. First, the larger spread means that the light is distributed over a larger area. Second, additional light is lost at the second internal reflection. And third, the angle with respect to the normal at which the light ray hits the raindrop is larger for the secondary rainbow. For the primary rainbow it was $2\beta - \phi \approx 59^\circ$, while for the secondary rainbow it is $3\beta - 90^\circ + \phi \approx 72^\circ$.

Tertiary rainbow: Now consider the tertiary rainbow. This rainbow arises from three reflections inside the raindrop, as shown below. With β defined as in the figure, the two $180^\circ - 2\beta$ angles follow, and then the $4\beta - 180^\circ$ angle follows.



Snell's law at the refraction points gives

$$\sin(4\beta - 180^\circ + \phi) = \frac{4}{3}\sin\beta. \tag{13}$$

Solving for ϕ yields

$$\phi = 180^\circ - 4\beta + \arcsin\left(\frac{4}{3}\sin\beta\right). \tag{14}$$

Taking the derivative to find the extremum (which is a minimum) gives

$$\sin\beta_{\text{min}} = \sqrt{\frac{8}{15}} \qquad \Longrightarrow \qquad \beta_{\text{min}} \approx 46.9^\circ. \tag{15}$$

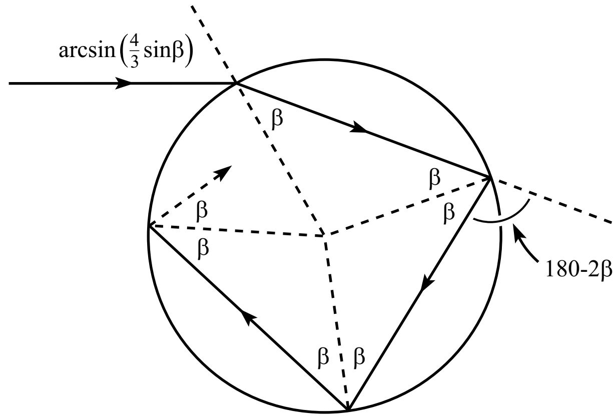
Substituting this back into eq. (10) gives

$$\phi_{\text{min}} \approx 69.2^\circ. \tag{16}$$

Using the same reasoning as above, we see that the light gets focused into the total angle of $2\phi_{\text{max}} \approx 138^\circ$, so the the sky appears brighter at this angle. The fact that this angle is larger than 90° means that the rainbow is actually behind you (if you are facing the primary and secondary rainbows), back towards the sun. It is a circle around the sun, located at an angle of $180^\circ - 138^\circ = 42^\circ$ relative to the line between you and the sun.

This rainbow is much more difficult to see, in part because of the increased effects of the three factors stated above in the secondary case, but also because you are looking back toward the sun, which essentially drowns out the light from the rainbow. Although I have never been able to seen this tertiary rainbow, it seems quite possible under (highly improbable) ideal conditions, namely, having rain fall generally all around you, except in a path between you and the sun, while at the same time having the sun eclipsed by a properly sized cloud.

Nth-order rainbow: We can now see what happens in general, when the light ray undergoes N reflections inside the raindrop.



With β defined as in the figure, the light gets deflected clockwise by an angle of $180 - 2\beta$ at each reflection point. In addition, it gets deflected clockwise by an angle of $\arcsin(\frac{4}{3}\sin\beta) - \beta$ at the two refraction points. The total angle of deflection is therefore

$$\Phi = 2\left(\arcsin\left(\frac{4}{3}\sin\beta\right) - \beta\right) + N(180^\circ - 2\beta). \tag{17}$$

Taking the derivative to find the extremum gives[4]

$$\sin\beta_0 = \sqrt{\frac{9(N+1)^2 - 16}{16[(N+1)^2 - 1]}}. \tag{18}$$

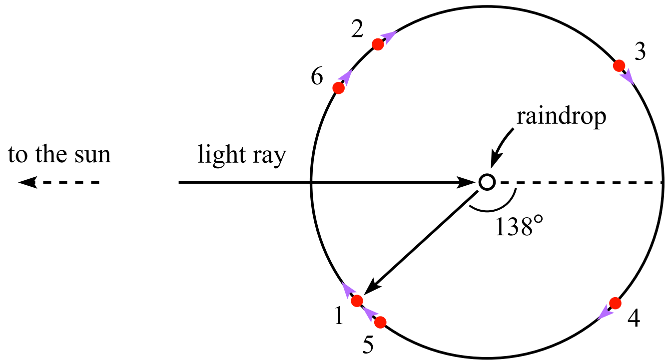
You can check that this agrees with the results in eqs. (4), (11), and (15). Substituting this back into eq. (17) gives the total angle of deflection, Φ, relative to the direction from the sun to you. See the discussion below for an explanation on how Φ relates to the angle ϕ we used above. A few values of Φ are given in the following table.

N	Φ	Φ (mod 360°)
1	138°	138°
2	231°	231°
3	318°	318°
4	404°	44°
5	488°	128°
6	572°	212°

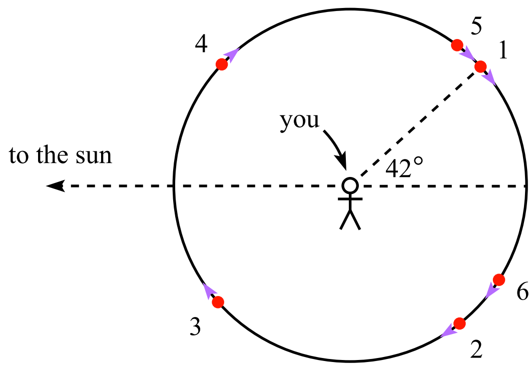
If you take the second derivative of Φ, you will find that it is always positive. Therefore, the extremum is always a minimum. It takes a little thought, though, to deduce the ordering of the colors from this fact. Since the red end of the spectrum has the smallest index of refraction, it is bent the least at the refraction points. Therefore, it has a smaller total angle of deflection than the other colors. The red light will therefore be located (approximately) at the angles given in the above table, while the other colors will occur at larger angles. The locations of the focusing angles

[4]Note that as $N \to \infty$, we have $\sin\beta_0 \to 3/4$. Therefore, β approaches the critical angle, 48.6°, which means that the light ray hits the raindrop at nearly 90° with respect to the normal. The cross section of the relevant light rays is therefore very small, which contributes to the faintness of the higher-order rainbows.

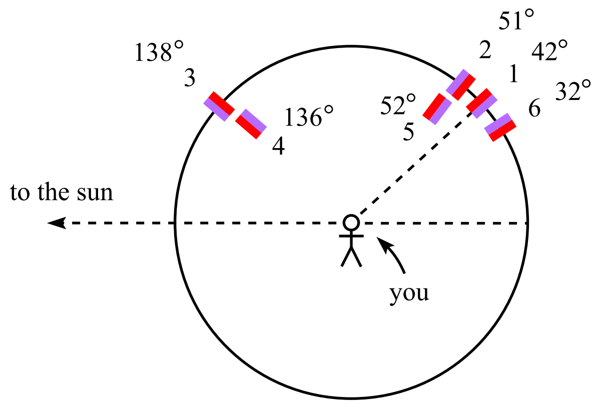
are shown below, where the arrows indicate the direction of the other colors relative to the red. All the arrows point clockwise, because all the extrema are minima.



Since the direction you see the light coming from is the opposite of the direction in which the light travels, you see the light at the following angles. This diagram is obtained by simply rotating the previous one by 180°.



We now note that in the figure preceding eq. (17), we could have had the light ray hitting the bottom part of the raindrop instead of the top, in which case the word "clockwise" would have been replaced with "counterclockwise". This would have the effect of turning Φ into $-\Phi$. The light is, of course, reflected in a whole circle (as long as the ground doesn't get in the way), and this circle is the rainbow that you see; the angles Φ and $-\Phi$ simply represent the intersection of the circle with a vertical plane. We can therefore label each rainbow with an angle between 0 and 180°. This simply means taking the dots and arrows below the horizontal line and reflecting them in the horizontal line. The result is the following figure, which gives the locations and orientations (red on one end, violet on the other) of the first six rainbows.



Note that in the $N \to \infty$ limit, the deflection angle in eq. (17) takes a simple form. In this limit, eq. (18) gives $\beta \approx 48.6°$. Therefore, $\arcsin(\frac{4}{3}\sin\beta) \approx 90°$, and eq. (17) gives

$$\begin{aligned} \Phi &\approx 2\,(90° - 48.6°) + N(180° - 2 \cdot 48.6°) \\ &= (N+1)(180° - 2 \cdot 48.6°) \\ &= (N+1)82.8°. \end{aligned} \tag{19}$$

Solution

Week 82 (4/5/04)

Standing in a line

First solution: Let T_N be the expected number of people who are able to make the given statement. If we consider everyone except the last person in line, then this group of $N-1$ people has by definition T_{N-1} people who are able to make the statement. Let us now add on the last person. There is a $1/N$ chance that she is the tallest, in which case she is able to make the statement. Otherwise she cannot. Therefore, we have

$$T_N = T_{N-1} + \frac{1}{N}\,. \tag{1}$$

Starting with $T_1 = 1$, we therefore inductively find

$$T_N = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\,. \tag{2}$$

For large N, this goes like $\ln N$, which grows very slowly with N.

Second solution: Let T_N be the desired average. Consider the location of the tallest person. If he is the last person in the line (which occurs with probability $1/N$), then the problem reduces to that for the $N-1$ people in front of him. So in this case, we can expect $1+T_{N-1}$ people who are able to make the given statement.

If the tallest person is the second to last person in the line (which occurs with probability $1/N$), then the problem reduces to that for the $N-2$ people in front of him (because the person behind him is not able to make the statement). So in this case, we can expect $1+T_{N-2}$ people who are able to make the given statement.

Continuing along these lines, and adding up all N possibilities for the location of the tallest person, we find

$$\begin{aligned} T_N &= \frac{1}{N}\Big((1+T_{N-1}) + (1+T_{N-2}) + \cdots + (1+T_1) + (1+T_0)\Big) \\ \Longrightarrow \quad NT_N &= N + T_{N-1} + T_{N-2} + \cdots + T_1. \end{aligned} \tag{3}$$

Writing down the analogous equation for $N-1$,

$$(N-1)T_{N-1} = (N-1) + T_{N-2} + T_{N-2} + \cdots + T_1, \tag{4}$$

and then subtracting this from eq. (3), yields

$$T_N = T_{N-1} + \frac{1}{N}\,, \tag{5}$$

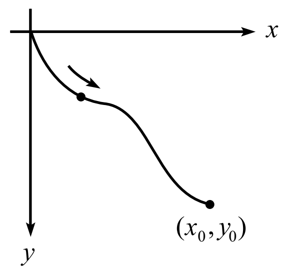
which agrees with the first solution.

Solution

Week 83 (4/12/04)

The brachistochrone

First solution: In the figure below, the boundary conditions are $y(0) = 0$ and $y(x_0) = y_0$, with downward taken to be the positive y direction.



From conservation of energy, the speed as a function of y is $v = \sqrt{2gy}$. The total time is therefore

$$T = \int_0^{x_0} \frac{ds}{v} = \int_0^{x_0} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\, dx. \tag{1}$$

Our goal is to find the function $y(x)$ that minimizes this integral, subject to the boundary conditions above. We can therefore apply the results of the variational technique, with a "Lagrangian" equal to

$$L \propto \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,. \tag{2}$$

The Euler-Lagrange equation is

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(\frac{1}{\sqrt{y}} \cdot y' \cdot \frac{1}{\sqrt{1+y'^2}}\right) = -\frac{\sqrt{1+y'^2}}{2y\sqrt{y}}\,. \tag{3}$$

Using the product rule on the three factors on the left-hand side, and making copious use of the chain rule, we obtain

$$-\frac{y'^2}{2y\sqrt{y}\sqrt{1+y'^2}} + \frac{y''}{\sqrt{y}\sqrt{1+y'^2}} - \frac{y'^2 y''}{\sqrt{y}(1+y'^2)^{3/2}} = -\frac{\sqrt{1+y'^2}}{2y\sqrt{y}}\,. \tag{4}$$

Multiplying through by $2y\sqrt{y}(1+y'^2)^{3/2}$ and simplifying gives

$$-2yy'' = 1 + y'^2. \tag{5}$$

We can integrate this equation if we multiply through by y' and rearrange to obtain

$$\begin{aligned} \int \frac{2y'y''}{1+y'^2} = -\int \frac{y'}{y}\,. \qquad &\Longrightarrow \qquad \ln(1+y'^2) = -\ln y + A \\ &\Longrightarrow \qquad 1 + y'^2 = \frac{B}{y}\,, \end{aligned} \tag{6}$$

where $B \equiv e^A$. We must now integrate one more time. Solving for y' and separating variables gives

$$\frac{\sqrt{y}\,dy}{\sqrt{B-y}} = \pm\, dx. \tag{7}$$

A helpful change of variables to get rid of the square root in the denominator is $y \equiv B\sin^2\phi$. Then $dy = 2B\sin\phi\cos\phi\; d\phi$, and eq. (7) simplifies to

$$2B\sin^2\phi\; d\phi = \pm\, dx. \tag{8}$$

We can now make use of the relation $\sin^2\phi = (1-\cos 2\phi)/2$ to integrate this. The result is $B(2\phi - \sin 2\phi) = \pm\, 2x - C$, where C is an integration constant.

Now note that we may rewrite our definition of ϕ (which was $y \equiv B\sin^2\phi$) as $2y = B(1-\cos 2\phi)$. If we then define $\theta \equiv 2\phi$, we have

$$x = \pm\, a(\theta - \sin\theta) \pm d, \qquad y = a(1-\cos\theta). \tag{9}$$

where $a \equiv B/2$, and $d \equiv C/2$.

The particle starts at $(x, y) = (0, 0)$. Therefore, θ starts at $\theta = 0$, since this corresponds to $y = 0$. The starting condition $x = 0$ then implies that $d = 0$. Also, we are assuming that the wire heads down to the right, so we choose the positive sing in the expression for x. Therefore, we finally have

$$x = a(\theta - \sin\theta), \qquad y = a(1-\cos\theta). \tag{10}$$

This is the parametrization of a *cycloid*, which is the path taken by a point on the rim of a rolling wheel. The initial slope of the $y(x)$ curve is infinite, as you can check.

Second solution: Let's use a variational argument again, but now with y as the independent variable. That is, let the chain be described by the function $x(y)$. The arclength is now given by $ds = \sqrt{1+x'^2}\,dy$. Therefore, instead of the Lagrangian in eq. (2), we now have

$$L \propto \frac{\sqrt{1+x'^2}}{\sqrt{y}}\,. \tag{11}$$

The Euler-Lagrange equation is

$$\frac{d}{dy}\left(\frac{\partial L}{\partial x'}\right) = \frac{\partial L}{\partial x} \qquad \Longrightarrow \qquad \frac{d}{dy}\left(\frac{1}{\sqrt{y}}\frac{x'}{\sqrt{1+x'^2}}\right) = 0. \tag{12}$$

The zero on the right-hand side makes things nice and easy, because it means that the quantity in parentheses is a constant. Call it D. We then have

$$\begin{aligned} \frac{1}{\sqrt{y}}\frac{x'}{\sqrt{1+x'^2}} = D \qquad &\Longrightarrow \qquad \frac{1}{\sqrt{y}}\frac{dx/dy}{\sqrt{1+(dx/dy)^2}} = D \\ &\Longrightarrow \qquad \frac{1}{\sqrt{y}}\frac{1}{\sqrt{(dy/dx)^2+1}} = D. \end{aligned} \tag{13}$$

This is equivalent to eq. (6), and the solution proceeds as above.

Third solution: Note that the "Lagrangian" in the first solution above, which is given in eq. (2) as

$$L = \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,, \tag{14}$$

is independent of x. Therefore, in analogy with conservation of energy (which arises from a Lagrangian that is independent of t), the quantity

$$E \equiv y'\frac{\partial L}{\partial y'} - L = \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}}{\sqrt{y}} = \frac{-1}{\sqrt{y}\sqrt{1+y'^2}} \tag{15}$$

is a constant. We have therefore again reproduced eq. (6).

Solution

Week 84 (4/19/04)

Poisson and Gaussian

(a) Consider a given box. The probability that exactly x balls end up in it is

$$P(x) = \binom{N}{x}\left(\frac{1}{B}\right)^x\left(1-\frac{1}{B}\right)^{N-x}. \tag{1}$$

This is true because the probability that a certain set of x balls ends up in the given box is $(1/B)^x$, and the probability that the other $N-x$ balls do not end up in the box is $(1-1/B)^{N-x}$, and there are $\binom{N}{x}$ ways to pick this certain set of x balls.

Let us now make approximations to $P(x)$. If N and B are much larger than x, then $N!/(N-x)! \approx N^x$, and $(1-1/B)^x \approx 1$ (we'll be more precise about these approximations below). Therefore,

$$\begin{aligned}
P(x) &= \frac{N!}{(N-x)!x!}\left(\frac{1}{B}\right)^x\left(1-\frac{1}{B}\right)^{N-x} \\
&\approx \frac{N^x}{x!}\left(\frac{1}{B}\right)^x\left(1-\frac{1}{B}\right)^{N} \\
&\approx \frac{1}{x!}\left(\frac{N}{B}\right)^x e^{-N/B} \\
&\equiv \frac{a^x e^{-a}}{x!}.
\end{aligned} \tag{2}$$

This result is called the Poisson distribution. For what x is $P(x)$ maximum? If we set $P(x) = P(x+1)$, we find $x = a-1$. Therefore, we are most likely to obtain $a-1$ or a balls in a box.

We can also consider eq. (2) to be a function of non-integer values of x, by using Stirling's formula, $x! \approx x^x e^{-x}\sqrt{2\pi x}$. This is valid for large x, which is generally the case we will be concerned with. (But note that eq. (2) is valid for small x, too.) Allowing non-integer values of x, the maximum $P(x)$ occurs halfway between $a-1$ and a, that is, at $x = a-1/2$. You can also show this by taking the derivative of eq. (2), with Stirling's expression in place of the $x!$. Furthermore, you can show that $x = a-1/2$ leads to a maximum $P(x)$ value of $P_{\text{max}} \approx 1/\sqrt{2\pi a}$.

In the real world, x can take on only integer values, of course. So it should be the case that the sum of the $P(x)$ probabilities, from $x=0$ to $x=\infty$, equals 1. And indeed,

$$\begin{aligned}
\sum_{x=0}^{\infty} P(x) &= \sum_{x=0}^{\infty}\frac{a^x e^{-a}}{x!} \\
&= e^{-a}\sum_{x=0}^{\infty}\frac{a^x}{x!} \\
&= e^{-a}e^{a} \\
&= 1.
\end{aligned} \tag{3}$$

Let's now be precise about the approximations we made above.

- $N!/(N-x)! \approx N^x$, because

$$\begin{aligned}\frac{N!}{(N-x)!} &= (1)\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{x-1}{N}\right) \\ &\approx N^x\left(1-\frac{x^2}{2N}\right).\end{aligned} \tag{4}$$

  So we need $x \ll \sqrt{N}$ for this approximation to be valid.

- $(1-1/B)^x \approx 1$, because $(1-1/B)^x \approx 1-x/B$. So we need $x \ll B$ for this approximation to be valid.

- In going from the second to the third line in eq. (2), we used the approximation $(1-1/B)^N \approx e^{-N/B}$. A more accurate statement is $(1-1/B)^N \approx e^{-N/B}e^{-N/2B^2}$, which you can verify by taking the log of both sides. So we need $N \ll B^2$ for this approximation to be valid.

In practice, the basic requirement is $N \ll B^2$. Given this, the $x_{\text{max}} \approx a \equiv N/B$ value that produces the maximum value of $P(x)$ does satisfy both $x_{\text{max}} \ll B$ and $x_{\text{max}} \ll \sqrt{N}$. So the approximations are valid in the region near x_{max}. These are generally the x values we are concerned with, because if x differs much from x_{max}, then $P(x)$ is essentially zero anyway. This will be clear after looking at the Gaussian expression that we'll derive in part (b).

(b) In showing that a Poisson distribution can be approximated by a Gaussian distribution, it will be easier to work with the log of $P(x)$. Let $y \equiv x-a$. The relevant y in this problem will turn out to be small compared to a, so we will eventually expand things in terms of the small quantity y/a. Using Stirling's formula, $x! \approx x^x e^{-x}\sqrt{2\pi x}$ (for large x, as we are assuming here), and also using the expansion $\ln(1+\epsilon) \approx \epsilon - \epsilon^2/2 + \epsilon^3/3 - \cdots$, we have

$$\begin{aligned}\ln P(x) &= \ln\left(\frac{a^x e^{-a}}{x!}\right) \\ &\approx \ln\left(\frac{a^x e^{-a}}{x^x e^{-x}\sqrt{2\pi x}}\right) \\ &= x\ln a - a - x\ln x + x - \ln\sqrt{2\pi x} \\ &= x\ln(a/x) + (x-a) - \ln\sqrt{2\pi x} \\ &= -(a+y)\ln\left(1+\frac{y}{a}\right) + y - \ln\sqrt{2\pi(a+y)} \\ &= -(a+y)\left(\frac{y}{a} - \frac{y^2}{2a^2} + \frac{y^3}{3a^3} + \cdots\right) + y - \ln\sqrt{2\pi(a+y)} \\ &\approx -\frac{y^2}{2a} + \frac{y^3}{6a^2} - \ln\sqrt{2\pi(a+y)}.\end{aligned} \tag{5}$$

The $y^3/6a^2$ term is much smaller than the $y^2/2a$ term (assuming $y \ll a$), so we may ignore it. Also, we may set $2\pi(a+y) \approx 2\pi a$, with negligible multiplicative error. Therefore, exponentiating bother sides of eq. (5) gives

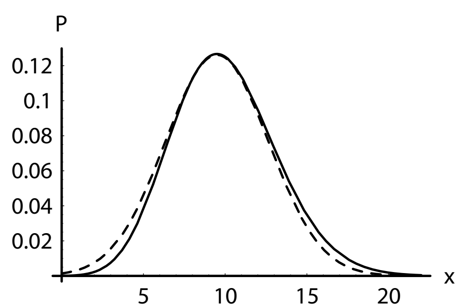
$$P(x) \approx \frac{e^{-(x-a)^2/2a}}{\sqrt{2\pi a}}\,, \tag{6}$$

which is the desired Gaussian distribution. Note that the maximum occurs at $x = a$. If you want to be a little more accuarate, you can include the correction from the $\ln\sqrt{2\pi(a+y)} = \ln\sqrt{2\pi a} + \ln\sqrt{1+(y/a)}$ term in eq. (5), to show that the Gaussian is actually centered at $x = a - 1/2$, that is,

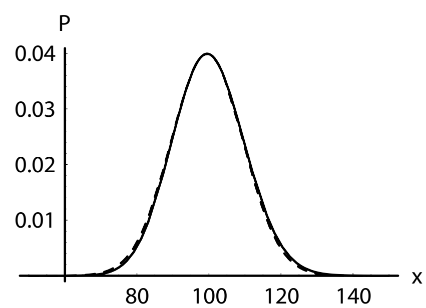
$$P(x) \approx \frac{e^{-\left(x-(a-1/2)\right)^2/2a}}{\sqrt{2\pi a}}\,. \tag{7}$$

The location of the maximum now agrees with the $x_{\max} = a - 1/2$ result that we obtained in part (a). However, for large a, the distinction between a and $a - 1/2$ is fairly irrelevant.

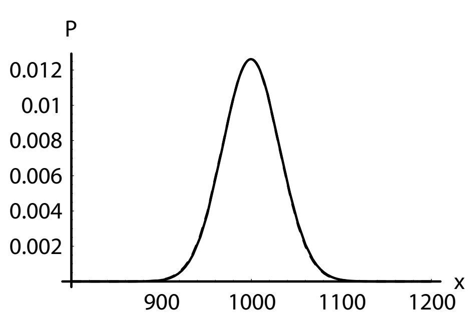
Note that the spread of the Gaussian is of the order $\sqrt{a}$. In our particular case, if we let $x - (a - 1/2) = \eta\sqrt{2a}$, then $P(x)$ is decreased by a factor of $e^{-\eta^2}$ relative to the maximum. $\eta = 1$ gives about 37% of the maximum, $\eta = 2$ gives about 2%, and $\eta = 3$ gives about .01%, which is quite negligible. If, for example, $a = 1000$, then virtually all of the non-negligible part of the graph is contained in the region $900 < x < 1100$, as shown below. The solid curve in these plots is the Poisson distribution from eq. (2), and the dotted curve is the Gaussian distribution from eq. (7). The two curves are essentially indistinguishable in the $a = 1000$ case.



$a = 10$



$a = 100$



$a = 1000$

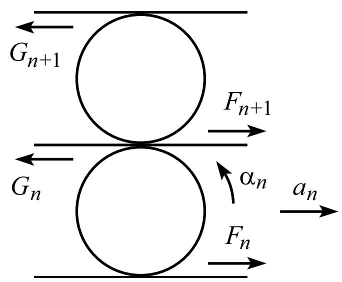
We should check that the Gaussian probability distribution in eq. (7) has an integral equal to 1. In doing this, we can let the integral run from $-\infty$ to ∞ with negligible error. Using the general result, $\int_{-\infty}^{\infty} e^{-y^2/b}\, dy = \sqrt{\pi b}$, and letting $b \equiv 2a$, we see that the integral is indeed equal to 1.

Solution

Week 85 (4/26/04)

Tower of cylinders

Both cylinders in a given row move in the same manner, so we may simply treat them as one cylinder with mass $m = 2M$. Let the forces that the boards exert on the cylinders be labelled as shown. "F" is the force from the plank below a given cylinder, and "G" is the force from the plank above it.



Note that by Newton's third law, we have $F_{n+1} = G_n$, because the planks are massless.

Our strategy will be to solve for the linear and angular accelerations of each cylinder in terms of the accelerations of the cylinder below it. Since we want to solve for two quantities, we will need to produce two equations relating the accelerations of two successive cylinders. One equation will come from a combination of $F = ma$, $\tau = I\alpha$, and Newton's third law. The other will come from the nonslipping condition.

With the positive directions for a and α defined as in the figure, $F = ma$ on the nth cylinder gives

$$F_n - G_n = ma_n, \tag{1}$$

and $\tau = I\alpha$ on the nth cylinder gives

$$(F_n + G_n)R = \frac{1}{2}mR^2\alpha_n \qquad \Longrightarrow \qquad F_n + G_n = \frac{1}{2}mR\alpha_n \tag{2}$$

Solving the previous two equations for F_n and G_n gives

$$\begin{aligned} F_n &= \frac{1}{2}\left(ma_n + \frac{1}{2}mR\alpha_n\right), \\ G_n &= \frac{1}{2}\left(-ma_n + \frac{1}{2}mR\alpha_n\right). \end{aligned} \tag{3}$$

But we know that $F_{n+1} = G_n$. Therefore,

$$a_{n+1} + \frac{1}{2}R\alpha_{n+1} = -a_n + \frac{1}{2}R\alpha_n. \tag{4}$$

We will now use the fact that the cylinders don't slip with respect to the boards. The acceleration of the board above the nth cylinder is $a_n - R\alpha_n$. But the acceleration of this same board, viewed as the board below the $(n+1)$st cylinder, is $a_{n+1} + R\alpha_{n+1}$. Therefore,

$$a_{n+1} + R\alpha_{n+1} = a_n - R\alpha_n. \tag{5}$$

Eqs. (4) and (5) are a system of two equations in the two unknowns, a_{n+1} and α_{n+1}, in terms of a_n and α_n. Solving for a_{n+1} and α_{n+1} gives

$$\begin{aligned} a_{n+1} &= -3a_n + 2R\alpha_n, \\ R\alpha_{n+1} &= 4a_n - 3R\alpha_n. \end{aligned} \tag{6}$$

We can write this in matrix form as

$$\begin{pmatrix} a_{n+1} \\ R\alpha_{n+1} \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix}. \tag{7}$$

We therefore have

$$\begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 4 & -3 \end{pmatrix}^{n-1} \begin{pmatrix} a_1 \\ R\alpha_1 \end{pmatrix}. \tag{8}$$

Consider now the eigenvectors and eigenvalues of the above matrix. The eigenvectors are found via

$$\begin{vmatrix} -3-\lambda & 2 \\ 4 & -3-\lambda \end{vmatrix} = 0 \qquad \Longrightarrow \qquad \lambda_\pm = -3 \pm 2\sqrt{2}. \tag{9}$$

The eigenvectors are then

$$\begin{aligned} V_+ &= \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \qquad \text{for } \lambda_+ = -3 + 2\sqrt{2}, \\ V_- &= \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}, \qquad \text{for } \lambda_- = -3 - 2\sqrt{2}. \end{aligned} \tag{10}$$

Note that $|\lambda_-| > 1$, so $\lambda_-^n \to \infty$ as $n \to \infty$. This means that if the initial $(a_1, R\alpha_1)$ vector has any component in the V_- direction, then the $(a_n, R\alpha_n)$ vectors will head to infinity. This violates conservation of energy. Therefore, the $(a_1, R\alpha_1)$ vector must be proportional to V_+.[1] That is, $R\alpha_1 = \sqrt{2}a_1$. Combining this with the fact that the given acceleration, a, of the bottom board equals $a_1 + R\alpha_1$, we obtain

$$a = a_1 + \sqrt{2}a_1 \qquad \Longrightarrow \qquad a_1 = \frac{a}{\sqrt{2}+1} = (\sqrt{2}-1)a. \tag{11}$$

Remark: Let us consider the general case where the cylinders have a moment of inertia of the form $I = \beta MR^2$. Using the above arguments, you can show that eq. (7) becomes

$$\begin{pmatrix} a_{n+1} \\ R\alpha_{n+1} \end{pmatrix} = \frac{1}{1-\beta} \begin{pmatrix} -(1+\beta) & 2\beta \\ 2 & -(1+\beta) \end{pmatrix} \begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix}. \tag{12}$$

And you can show that the eigenvectors and eigenvalues are

$$\begin{aligned} V_+ &= \begin{pmatrix} \sqrt{\beta} \\ 1 \end{pmatrix}, \qquad \text{for } \lambda_+ = \frac{\sqrt{\beta}-1}{\sqrt{\beta}+1}, \\ V_- &= \begin{pmatrix} \sqrt{\beta} \\ -1 \end{pmatrix}, \qquad \text{for } \lambda_- = \frac{\sqrt{\beta}+1}{\sqrt{\beta}-1}. \end{aligned} \tag{13}$$

[1]This then means that the $(a_n, R\alpha_n)$ vectors head to zero as $n \to \infty$, because $|\lambda_+| < 1$. Also, note that the accelerations change sign from one level to the next, because λ_+ is negative.

As above, we cannot have the exponentially growing solution, so we must have only the V_+ solution. We therefore have $R\alpha_1 = a_1/\sqrt{\beta}$. Combining this with the fact that the given acceleration, a, of the bottom board equals $a_1 + R\alpha_1$, we obtain

$$a = a_1 + \frac{a_1}{\sqrt{\beta}} \qquad \Longrightarrow \qquad a_1 = \left(\frac{\sqrt{\beta}}{1+\sqrt{\beta}}\right) a. \tag{14}$$

You can verify that all of these results agree with the $\beta = 1/2$ results obtained above.

Let's now consider a few special cases of the

$$\lambda_+ = \frac{\sqrt{\beta}-1}{\sqrt{\beta}+1} \tag{15}$$

eigenvalue, which gives the ratio of the accelerations in any level to the ones in the next level down.

- If $\beta = 0$ (all the mass of a cylinder is located at the center), then we have $\lambda_+ = -1$. In other words, the accelerations have the same magnitudes but different signs from one level to the next. The cylinders simply spin in place while their centers remain fixed. The centers are indeed fixed, because $a_1 = 0$, from eq. (14).
- If $\beta = 1$ (all the mass of a cylinder is located on the rim), then we have $\lambda_+ = 0$. In other words, there is no motion above the first level. The lowest cylinder basically rolls on the bottom side of the (stationary) plank right above it. Its acceleration is $a_1 = a/2$, from eq. (14).
- If $\beta \to \infty$ (the cylinders have long massive extensions that extend far out beyond the rim), then we have $\lambda_+ = 1$. In other words, all the levels have equal accelerations. This fact, combined with the $R\alpha_1 = a_1/\sqrt{\beta} \approx 0$ result, shows that there is no rotational motion at any level, and the whole system simply moves to the right as a rigid object with acceleration $a_1 = a$, from eq. (14).

Solution

Week 86 (5/3/04)

Shifted intervals

Because ϵ is very small, let us discretize each of the intervals into tiny units of length ϵ. If the first number is in the smallest of its possible ϵ-units (that is, between 0 and ϵ), then it is guaranteed to be the smallest of all the numbers. If it is in the second smallest ϵ-unit (between ϵ and 2ϵ), there is a $1-\epsilon$ chance that it is the smallest of all the numbers, because this is the probability that the second number is larger than it.[1] If it is in the third ϵ-unit, there is a $(1-\epsilon)(1-2\epsilon)$ chance that it is the smallest, because this is the probability that both the second and third numbers are larger than it. In general, if the first number is in the kth ϵ-unit, there is a

$$P_k = (1-\epsilon)(1-2\epsilon)(1-3\epsilon)\cdots(1-(k-1)\epsilon) \tag{1}$$

chance that it is the smallest. Since the first number has an equal probability of ϵ of being in any of the ϵ-units, the total probability that it is the smallest therefore equals

$$P = \epsilon + \epsilon P_1 + \epsilon P_2 + \epsilon P_3 + \cdots + \epsilon P_{1/\epsilon}. \tag{2}$$

For small ϵ, we can make an approximation to the P_k's, as follows. Take the log of P_k in eq. (1) to obtain

$$\begin{aligned}
\ln P_k &= \ln(1-\epsilon) + \ln(1-2\epsilon) + \ln(1-3\epsilon) + \cdots + \ln(1-(k-1)\epsilon) \\
&\approx \left(-\epsilon - \frac{\epsilon^2}{2}\right) + \left(-2\epsilon - \frac{2^2\epsilon^2}{2}\right) + \cdots \left(-(k-1)\epsilon - \frac{(k-1)^2\epsilon^2}{2}\right) \\
&= -\epsilon\Big(1+2+\cdots+(k-1)\Big) - \frac{\epsilon^2}{2}\Big(1+2^2+\cdots+(k-1)^2\Big) \\
&= -\epsilon\left(\frac{k(k-1)}{2}\right) - \frac{\epsilon^2}{2}\left(\frac{k(k-1)(2k-1)}{6}\right) \\
&\approx -\frac{\epsilon k^2}{2} - \frac{\epsilon^2 k^3}{6}\,.
\end{aligned} \tag{3}$$

In going from the first to the second line, we have used the first two terms in the Taylor series, $\ln(1-x) = -x - x^2/2 - \cdots$. And in going from the fourth to the fifth line, we have used the fact that the k values we will be concerned with will generally be large, so we have kept only the leading power of k. Exponentiating eq. (3) gives

$$P_k \approx e^{-\epsilon k^2/2} e^{-\epsilon^2 k^3/6}. \tag{4}$$

The second factor here is essentially equal to 1 if $\epsilon^2 k^3 \ll 1$, that is, if $k \ll 1/\epsilon^{2/3}$. But we are only concerned with k values up to the order of $1/\epsilon^{1/2}$, because if k is much larger than this, the first exponential factor in eq. (4) makes P_k essentially

[1] Technically, the probability is on average equal to $1-\epsilon/2$, because the average value of the first number in this case is $3\epsilon/2$. But the $\epsilon/2$ correction in this probability (and other analogous ones) is inconsequential.

equal to zero. Since $1/\epsilon^{1/2} \ll 1/\epsilon^{2/3}$, we see that whenever P_k is not essentially zero, we can set the second exponential factor equal to 1. So we have

$$P_k \approx e^{-\epsilon k^2/2}. \tag{5}$$

Eq. (2) then becomes

$$P \approx \epsilon \left(1 + e^{-\epsilon/2} + e^{-2^2\epsilon/2} + e^{-3^2\epsilon/2} + \cdots + e^{-(1/\epsilon)^2\epsilon/2}\right). \tag{6}$$

Since ϵ is small, we may approximate this sum by an integral. And since the terms eventually become negligibly small, we can let the integral run to infinity. We then have

$$P \approx \epsilon \int_0^\infty e^{-(\epsilon/2)x^2}\, dx. \tag{7}$$

Using the general result, $\int_{-\infty}^{\infty} e^{-y^2/b}\, dy = \sqrt{\pi b}$, we have

$$P \approx \frac{\epsilon}{2}\sqrt{\frac{2\pi}{\epsilon}} = \sqrt{\frac{\pi\epsilon}{2}} \equiv \sqrt{\frac{\pi}{2N}}\,. \tag{8}$$

Note that the P_k in eq. (5) is negligibly small if $k \gg 1\sqrt{\epsilon} \equiv \sqrt{N}$. Therefore, most of the terms in the sum in eq. (6) are negligible. The fraction of the terms that contribute goes like $\sqrt{N}/N = 1/\sqrt{N}$.

Remark: Eq. (8) shows that P scales like $1/\sqrt{N}$. If we consider the different setup where all the N intervals range from 0 to 1, instead of being successively shifted by ϵ, then the probability that the first number is the smallest is simply $1/N$, because the smallest number is equally likely to be in any of the N identical intervals. It makes sense that the above $1/\sqrt{N}$ result for the shifted intervals is larger than the $1/N$ result for the non-shifted intervals.

If you want to derive the "non-shifted" $1/N$ result by doing an integral, observe that if the first number equals x, then there is a $(1-x)^{N-1}$ chance that all the other $N-1$ numbers are larger than x. Therefore,

$$P = \int_0^1 (1-x)^{N-1}\, dx = \frac{1}{N}\,. \tag{9}$$

Solution

Week 87 (5/10/04)

Leaving the hemisphere

Assume that the particle slides off to the right. Let v_x and v_y be its horizontal and vertical velocities, with rightward and downward taken to be positive, respectively. Let V_x be the velocity of the hemisphere, with leftward taken to be positive. Conservation of momentum gives

$$mv_x = MV_x \qquad \Longrightarrow \qquad V_x = \left(\frac{m}{M}\right) v_x. \tag{1}$$

Consider the moment when the particle is located at an angle θ down from the top of the hemisphere. Locally, it is essentially on a plane inclined at angle θ, so the three velocity components are related by

$$\frac{v_y}{v_x + V_x} = \tan\theta \qquad \Longrightarrow \qquad v_y = \tan\theta \left(1 + \frac{m}{M}\right) v_x. \tag{2}$$

To see why this is true, look at things in the frame of the hemisphere. In this frame, the particle moves to the right with speed $v_x + V_x$, and downward with speed v_y. Eq. (2) represents the constraint that the particle remains on the hemisphere, which is inclined at an angle θ at the given location.

Let us now apply conservation of energy. In terms of θ, the particle has fallen a distance $R(1-\cos\theta)$, so conservation of energy gives

$$\frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}MV_x^2 = mgR(1-\cos\theta). \tag{3}$$

Using eqs. (1) and (2), we can solve for v_x^2 to obtain

$$v_x^2 = \frac{2gR(1-\cos\theta)}{(1+r)\Big(1+(1+r)\tan^2\theta\Big)}\,, \qquad \text{where} \;\; r \equiv \frac{m}{M}\,. \tag{4}$$

This function of θ starts at zero for $\theta = 0$ and increases as θ increases. It then achieves a maximum value before heading back down to zero at $\theta = \pi/2$. However, v_x *cannot* actually decrease, because there is no force available to pull the particle to the left. So what happens is that v_x initially increases due to the non-zero normal force that exists while contact remains. But then v_x reaches its maximum, which corresponds to the normal force going to zero and the particle losing contact with the hemisphere. The particle then sails through the air with constant v_x. Our goal, then, is to find the angle θ for which the v_x^2 in eq. (4) is maximum. Setting the derivative equal to zero gives

$$\begin{aligned}
& 0 &=& \Big(1+(1+r)\tan^2\theta\Big)\sin\theta - (1-\cos\theta)(1+r)\frac{2\tan\theta}{\cos^2\theta} \\
\Longrightarrow\quad & 0 &=& \Big(1+(1+r)\tan^2\theta\Big)\cos^3\theta - 2(1+r)(1-\cos\theta) \\
\Longrightarrow\quad & 0 &=& \cos^3\theta + (1+r)(\cos\theta - \cos^3\theta) - 2(1+r)(1-\cos\theta) \\
\Longrightarrow\quad & 0 &=& r\cos^3\theta - 3(1+r)\cos\theta + 2(1+r). \qquad (5)
\end{aligned}$$

This is the desired equation that determines θ. It is a cubic equation, so in general it can't be solved so easily for θ. But in the special case of $r = 1$, we have

$$0 = \cos^3\theta - 6\cos\theta + 4. \tag{6}$$

By inspection, $\cos\theta = 2$ is an (unphysical) solution, so we find

$$(\cos\theta - 2)(\cos^2\theta + 2\cos\theta - 2) = 0. \tag{7}$$

The physical root of the quadratic equation is

$$\cos\theta = \sqrt{3} - 1 \approx 0.732 \qquad \Longrightarrow \qquad \theta \approx 42.9^\circ. \tag{8}$$

Alternate solution: In the reference frame of the hemisphere, the horizontal speed of the particle $v_x + V_y = (1+r)v_x$. The total speed in this frame equals this horizontal speed divided by $\cos\theta$, so

$$v = \frac{(1+r)v_x}{\cos\theta}\,. \tag{9}$$

The particle leaves the hemisphere when the normal force goes to zero. The radial $F = ma$ equation therefore gives

$$mg\cos\theta = \frac{mv^2}{R}\,. \tag{10}$$

You might be concerned that we have neglected the sideways fictitious force in the accelerating frame of the hemisphere. However, the hemisphere is *not* accelerating beginning at the moment when the particle loses contact, because the normal force has gone to zero. Therefore, eq. (10) looks exactly like it does for the familiar problem involving a fixed hemisphere; the difference in the two problems is in the calculation of v.

Using eqs. (4) and (9) in eq. (10) gives

$$mg\cos\theta = \frac{m(1+r)^2}{R\cos^2\theta}\cdot\frac{2gR(1-\cos\theta)}{(1+r)\big(1+(1+r)\tan^2\theta\big)}\,. \tag{11}$$

Simplifying this yields

$$\big(1+(1+r)\tan^2\theta\big)\cos^3\theta = 2(1+r)(1-\cos\theta), \tag{12}$$

which is the same as the second line in eq. (5). The solution proceeds as above.

Remark: Let's look at a few special cases of the $r \equiv m/M$ value. In the limit $r \to 0$ (in other words, the hemisphere is essentially bolted down), eq. (5) gives

$$\cos\theta = 2/3 \qquad \Longrightarrow \qquad \theta \approx 48.2^\circ, \tag{13}$$

a result which may look familiar to you. In the limit $r \to \infty$, eq. (5) reduces to

$$0 = \cos^3\theta - 3\cos\theta + 2 \qquad \Longrightarrow \qquad 0 = (\cos\theta - 1)^2(\cos\theta + 2). \tag{14}$$

Therefore, $\theta = 0$. In other words, the hemisphere immediately gets squeezed out very fast to the left.

For other values of r, we can solve eq. (5) either by using the formula for the roots of a cubic equation (very messy), or by simply doing things numerically. A few numerical results are:

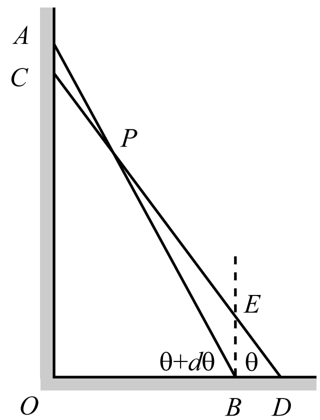
r	$\cos\theta$	θ
0	.667	48.2°
1/2	.706	45.1°
1	.732	42.9°
2	.767	39.9°
10	.858	30.9°
100	.947	18.8°
1000	.982	10.8°
∞	1	0°

Solution

Week 88 (5/17/04)

Ladder envelope

Let the ladder have length 1, for simplicity. In the figure below, let the ladder slide from segment AB to segment CD. Let CD make an angle θ with the floor, and let AB make an angle $\theta + d\theta$, with $d\theta$ very small. The given problem is then equivalent to finding the locus of intersections, P, of adjacent ladder positions AB and CD.



Put the ladder on a coordinate system with the floor as the x-axis and the wall as the y-axis. Let a vertical line through B intersect CD at point E. We will find the x and y coordinates of point P by determining the ratio of similar triangles ACP and BEP. We will find this ratio by determining the ratio of AC to BE. AC is given by

$$AC = \sin(\theta + d\theta) - \sin\theta \approx \cos\theta\, d\theta, \tag{1}$$

which is simply the derivative of $\sin\theta$ times $d\theta$. Similarly,

$$BD = \cos\theta - \cos(\theta + d\theta) \approx \sin\theta\, d\theta, \tag{2}$$

which is simply the negative of the derivative of $\cos\theta$ times $d\theta$. BE is then given by

$$BE = BD\tan\theta \approx \tan\theta\sin\theta\, d\theta. \tag{3}$$

The ratio of triangle ACP to triangle BEP is therefore

$$\frac{\triangle ACP}{\triangle BEP} = \frac{AC}{BE} \approx \frac{\cos\theta}{\tan\theta\sin\theta} = \frac{\cos^2\theta}{\sin^2\theta} \equiv r. \tag{4}$$

The x coordinate of P is then, using $OB = \cos(\theta + d\theta) \approx \cos\theta$,

$$P_x = \frac{r}{1+r}(OB) \approx \frac{\cos^2\theta}{\sin^2\theta + \cos^2\theta}(OB) \approx \cos^3\theta. \tag{5}$$

Likewise, the y coordinate of P is $P_y = \sin^3\theta$. The envelope of the ladder may therefore be described parametrically by

$$(x, y) = (\cos^3\theta, \sin^3\theta), \qquad \pi/2 \geq \theta \geq 0. \tag{6}$$

Equivalently, using $\cos^2\theta + \sin^2\theta = 1$, the envelope may be described by the equation,

$$x^{2/3} + y^{2/3} = 1. \tag{7}$$

Solution

Week 89 (5/24/04)

Rope between inclines

Let the total mass of the rope be m, and let a fraction f of it hang in the air. Consider the right half of this section. Its weight, $(f/2)mg$, must be balanced by the vertical component, $T\sin\theta$, of the tension at the point where it joins the part of the rope touching the right platform. The tension at this point therefore equals $T = (f/2)mg/\sin\theta$.

Now consider the part of the rope touching the right platform. This part has mass $(1-f)m/2$. The normal force from the platform is $N = (1-f)(mg/2)\cos\theta$, so the maximal friction force equals $(1-f)(mg/2)\cos\theta$, because $\mu = 1$. This fiction force must balance the sum of the gravitational force component along the plane, which is $(1-f)(mg/2)\sin\theta$, plus the tension at the lower end, which is the $(f/2)mg/\sin\theta$ we found above. Therefore,

$$\frac{1}{2}(1-f)mg\cos\theta = \frac{1}{2}(1-f)mg\sin\theta + \frac{fmg}{2\sin\theta}\,, \tag{1}$$

which gives

$$f = \frac{F(\theta)}{1+F(\theta)}\,, \qquad \text{where} \quad F(\theta) \equiv \cos\theta\sin\theta - \sin^2\theta. \tag{2}$$

This expression for f is a monotonically increasing function of $F(\theta)$, as you can check. The maximal f is therefore obtained when $F(\theta)$ is as large as possible. Using the double-angle formulas, we can rewrite $F(\theta)$ as

$$F(\theta) = \frac{1}{2}(\sin 2\theta + \cos 2\theta - 1). \tag{3}$$

The derivative of this is $\cos 2\theta - \sin 2\theta$, which equals zero when $\tan 2\theta = 1$. Therefore,

$$\theta_{\text{max}} = 22.5^\circ. \tag{4}$$

Eq. (3) then yields $F(\theta_{\text{max}}) = (\sqrt{2}-1)/2$, and so eq. (2) gives

$$f_{\text{max}} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = (\sqrt{2}-1)^2 = 3 - 2\sqrt{2} \approx 0.172. \tag{5}$$

Solution

Week 90 (5/31/04)

The game of NIM

As with many problems, this one can be solved in two possible ways. We can (1) write down the correct answer, through some stroke of genius, and then verify that it works, or (2) work out some simple cases, get a feel for the problem, and eventually wind our way around to the correct answer. For the problem at hand, let's proceed via the second method and try to arrive at the answer with some motivation.

We'll start by working out what happens in particular cases of small numbers of coins in the piles, and then we'll look for a pattern. A reasonable way to organize the results is to determine which combinations of numbers are guaranteed losing positions (assuming that both players are aware of the optimal strategy).

The most obvious losing position (LP) is piles with coins of numbers (1,1,0). If you encounter this setup, you must pick one coin, and then your opponent will pick the last one and thereby win. More generally, the combination $(N, N, 0)$ is an LP, because if you take off n coins from one pile, your opponent will take off n coins from the other. He will keep matching you on each turn, until finally the situation is (0,0,0), with his having removed the last coin(s). Note that situations of $(N, M, 0)$ and (N, N, M) are therefore winning positions (WP), because it is possible to turn them into an LP with one move.

This reasoning utilizes the following two obvious properties that an LP must have: (1) removal of any number of coins from one pile of an LP creates a non-LP, and (2) on the next turn it is always possible to bring the situation back to an LP again.

Consider now the cases where no two piles have the same numbers of coins. The smallest case is (1,2,x). $x = 3$ is the first possibility for an LP, and we quickly see that it is indeed an LP, because the removal of any number of coins from any of the piles yields a setup of the from $(N, M, 0)$ or (N, N, M), which are WP's, as we saw above.

Note that once we have found an LP, we know that any triplet that has two numbers in common with the LP, with its remaining number larger than that in the LP, must be a WP. This is true because it is possible to turn it into an LP by removing coins from this last pile.

If we look at other triplets in which the first number is 1, we find that (1,4,5), (1,6,7), (1,8,9), etc., are LP's, as you can check by showing that any move turns them into a WP.

Now consider cases where 2 is the smallest number of coins in a pile. We find, after a little fiddling, that (2,4,6), (2,5,7), (2,8,10), and (2,9,11), etc., are LP's, as you can check.

Similar fiddling, starting with a 3, gives (3,4,7), (3,5,6), (3,8,11), and (3,9,10), etc., as LP's.

Let's now make a table of this hodgepodge of results, for up to seven coins in a pile. The two axes will be the first two numbers in an LP triplet, and the entry in the table will be the third. The table is of course symmetric.

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

As a first guess at the key to this table, we might say that two numbers in an LP triplet must add up to the third. This, however, does not work for the (3,5,6) triplet. It also does not work for the (3,9,10) triplet that we found above. Continuing on to higher numbers, the guess seems to work for triplets starting with a 4. But then if we start with a 5, we eventually find the LP triplets (5,9,12) and (5,11,14), for which the sum of two numbers doesn't equal the third.

In an effort to find the key, let us exploit the patterns in the table, perhaps brought out best by the following grouping:

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

The entries in the upper right 4×4 box are 4 more than the corresponding entries in the upper left box. Likewise, within each 4×4 box, the entries in the upper right 2×2 box are 2 more than the entries in the upper left box. Similar results would be evident if we doubled the size of the box (out to 15), where we would see 8×8 boxes having entries differing by 8.

All this suggests that powers of 2 are important in this problem. We therefore should consider writing the numbers in a way where factors of 2 are evident, that is, in base 2. There is no guarantee that this will help, but let's try it and see what happens. Let's write down the troublesome triplets we've found (the ones for which two of the numbers don't add up to the third) in base 2:

3:	11	3:	11	5:	101	5:	101
5:	101	9:	1001	9:	1001	11:	1011
6:	110	10:	1010	12:	1100	14:	1110

What property do these triplets have? When written in the above form, we see that each column in base 2 contains an even number of 1's. After checking some other triplets, this appears to be true in general for an LP. We will prove this with the following theorem.

Theorem: *Call a triplet an E-triplet (the "E" stands for "even") if it has the following property: When the three numbers are written in base 2, there is an even number (that is, either zero or two) of 1's in each digit's place. Then a triplet is a losing position (LP) if and only if it is an E-triplet.*

Proof: Let us establish the following three facts concerning E-triplets:

1. Removal of any number of coins from one pile of an E-triplet will turn the triplet into a non-E-triplet.

2. Given a non-E-triplet, it is always possible to remove coins from one pile to turn the triplet into an E-triplet.

3. (0,0,0) is an E-triplet.

These facts may be demonstrated as follows:

1. This is true because any two numbers in an E-triplet uniquely determine the third.

2. We will describe how to turn any non-E-triplet into an E-triplet. Write the three numbers of coins in base 2, and put them in a column, with the unit's digits aligned, as we did above. Starting from the *left*, look at each digit's column until you find a column with an odd number (that is, either one or three) of 1's. Let this be the nth column (counting from the *right*).

   If there is one 1 in the nth column, label the number containing this 1 as A. If there are three 1's, then arbitrarily pick any of the numbers to be A. Remove coins from A by switching the 1 in the nth column to a 0, and also by switching any 1's to 0's, or 0's to 1's, in other columns to the right of the nth column, in order to produce an even number or 1's is all columns. We have now created an E-triplet.

   Note that this switching of 1's and 0's does indeed correspond to removing (as opposed to adding) coins from A, because even if all the columns to the right of the nth column involve switching 0's to 1's, this addition of $2^{n-1} - 1$ coins is still less than the subtraction of the 2^{n-1} coins arising from the 1-to-0 switch in the nth column.

3. This is true, by definition of an E-triplet.

The first two of these facts show that if player X receives an E-triplet on a given turn, then player Y can ensure that X receives an E-triplet on every subsequent turn. Therefore, X must always *create* a *non*-E-triplet, by the first of the three facts. X therefore cannot take the last coin, because he cannot create the E-triplet $(0, 0, 0)$. Therefore, an E-triplet is a losing position. ■

The best strategy in this game is therefore to give your opponent an E-triplet whenever you can. If both players are aware of this strategy, then the outcome of the game is determined by the initial piles of coins. If they form an E-triplet, then the player who goes first loses. If they do not form an E-triplet, then the player who goes first wins, because he can always create an E-triplet to give to his opponent.

Remarks: If the starting numbers of coins are random, then the player who goes first will most likely win, because most triplets are not E-triplets. We can demonstrate this fact by using the somewhat crude scenario where the three numbers are random numbers from 0 to $2^n - 1$, that is, they each have n digits in base 2 (many of which may be zero). Then there are $(2^n)^3$ possible triplets. But there are only 4^n possible E-triplets, because each of the n columns of three digits (when we write the three numbers on top of each other) has four E-triplet possibilities: 0,0,0; 1,1,0; 1,0,1; and 0,1,1. The fraction of E-triplets is therefore $4^n/2^{3n} = 2^{-n}$, which goes to zero for large n.

Note that there is nothing special about having three piles. We can have any number of piles (but still two players), and all the above reasoning still holds. Losing positions are the ones that have an even number of 1's in each column, when written in base 2. The three facts in the above theorem still hold.