THEORETICAL COMPETITION

January 13, 2010

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*.** Write only inside the bordered area.
6. Begin each question on a separate sheet.
7. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Question Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
    - Used ***Writing sheets*** in order;
    - The sheets you do not wish to be evaluated
    - Unused sheets and the printed question.

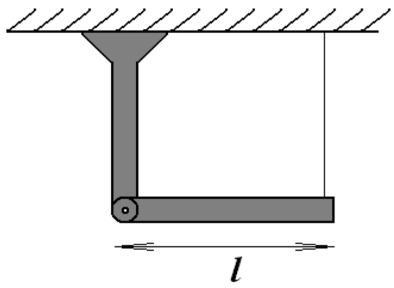
   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

Problem 1 (10 points)

This problem consists of three unrelated parts.

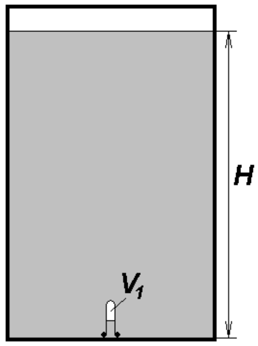
1A (3 points)

One end of a homogeneous rigid rod with mass m and length l is suspended on the vertical support by an ideal joint, and the other end hangs on a thread such that the rod is in horizontal position. At a certain time point the thread is cut. Find the dependence of reaction force of the joint on the angle α of the rod deviation from the horizontal position.



1B (4 points)

Liquid is poured in a closed cylindrical vessel with thick walls. The height of the liquid level is H. The mass density of the liquid decreases linearly with the height from ρ_{max} to small value which can be assumed to be equal to zero. Thin layer of vapor is saturated over the liquid surface but its pressure can be neglected with respect to the hydrostatic pressure of the liquid. Inverted test-tube of the volume V_0 is placed at the bottom of the vessel. The mass M of the test-tube is concentrated at its neck, and its length is small compared to H. The gas of volume V_1 with negligible mass is placed inside the test-tube. The temperature of the system is kept constant. Can the test-tube stay motionless in the liquid at a certain height from the bottom? If it is possible, what conditions are to be satisfied? What is the height of the test-tube position over the bottom in this case? Is this state stable? The volume of walls of the test-tube is small compared to V_1.



1C (3 points)

Rectangular light source of size $c \times b$ is fixed on the ceiling of the room of height $L = 3.0\ m$.

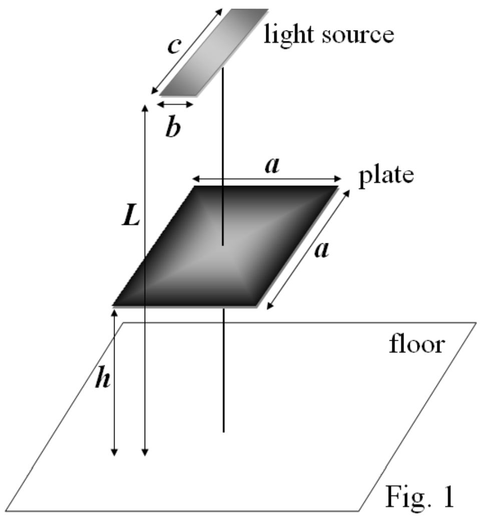


Fig. 1

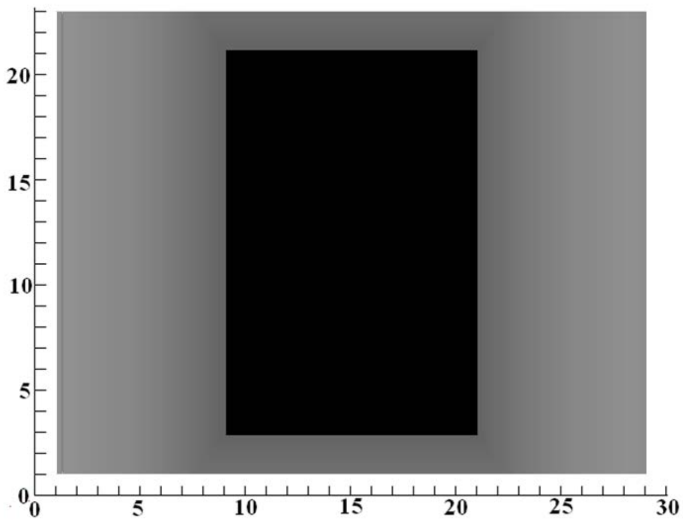
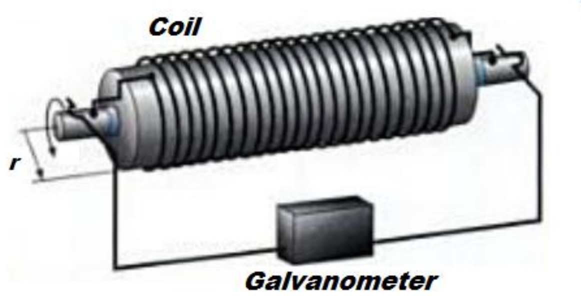


Fig. 2

Opaque square plate with side $a = 10\ cm$ is placed horizontally under the lamp at a height h above the floor (Fig. 1). Fig. 2 schematically shows the shape of the shadow casted by the plate on the room floor. The shadow consists of a dark rectangle surrounded by a lighter rectangle, and when approaching the edges of the outer rectangle the shade turns lighter. The value of the scale division in Fig. 2 is $1.0\ cm$. Find the sizes of the light source *b* and *c*, and the height *h* of the plate position. Specify the exact orientation of the light source with respect to the plate shadow.

Problem 2 (10 points)
The Tolmen-Stewart experiment



In 1916 Tolmen and Stewart carried out their famous experiment proving that the electric current in metals was caused by freely moving electrons. The sketch of the experimental setup is shown on the left hand side.

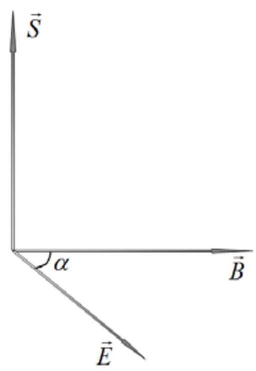
A long coil of the radius r and the length h has an inertia moment J_0. The coil is reeled up by a single layer of the metal wire of the length ℓ and mass m such that the number of loops on the unit length equals n. The both ends of the wire are abridged to the galvanometer by the sliding contacts. The coil together with the wire is set rotating with the angular velocity ω_0 and, then, the friction force with the torque M is applied to stop the coil rotation. The total resistance of the circuit is R and its capacity is assumed negligible.

Part 1. In this part of the problem you can neglect the inductance of the coil and assume that the Ohm's law holds at any time.

1. **[1 point]** Determine the total inertia moment J of the coil together with the wire. Express your answer in terms of J_0, m, r.
2. **[1 point]** Determine the dependence of the angular velocity $\omega(t)$ of the coil on time t. Express your answer in terms of J, M, ω_0.
3. **[1 point]** Determine the dependence of the electric current strength $I(t)$ on time t. Express your answer in terms of J, M, r, ℓ, R with m_e, e being the mass and the electric charge of electron, respectively.
4. **[2 points]** During the experiment the galvanometer has registered the electric charge Q passed in the circuit. Express the charge-to-mass ratio of the electron e/m_e in terms of ω_0, r, ℓ, R, Q.

Part 2. In this part of the problem you have to take into account the small inductance of the coil which is supposed to be quite long to neglect the side effects.

5. **[1 point]** Find the maximal electric current strength $I_{\max}$ in the coil. Express your answer in terms of J, M, r, ℓ, R, m_e, e. Plot a qualitative dependence $I(t)$.
6. **[1 point]** Find the maximal energy W_0 stored in the coil during the experiment. Express your answer in terms of $J, M, r, \ell, R, m_e, e, n, h$ and the magnetic constant μ_0.
7. **[3 points]** The flux S of the electromagnetic energy through the unit area is determined by the Pointing vector which is perpendicular to both electric and magnetic fields with the module $S = \dfrac{1}{\mu_0} EB \sin\alpha$, where E is the electric field strength vector, B is the magnetic induction vector and α is the angle between them (see the picture on the right). Find the electromagnetic energy W passing through the lateral surface for the time period while the electric current increases and the electromagnetic



energy W' passing through the lateral surface for the time period while the electric current decreases. Express your answer in terms of $\ell, r, M, J, n, R, m_e, e, \mu_0$.

Problem 3 (10 points)
The relic radiation and cosmic rays

According to modern cosmology, our universe is filled with electromagnetic radiation, remained after the Big Bang and called the cosmic **relict** radiation, or cosmic microwave background (shortly, CMB). With good accuracy, the CMB is homogeneous and isotropic in the reference system associated with our Galaxy which we take as the laboratory reference system. The frequency distribution of the CMB coincides with the spectrum of a blackbody at temperature $T = 2.7\ K$. Discovery and study of properties of the CMB were awarded Nobel Prizes in Physics in 1978 and 2006.

Relict photons are extremely numerous and, therefore, may affect Galaxy radiation of another nature called **cosmic rays** (shortly, CR). It is believed that CR are formed in stellar explosions. CR consist mainly of protons whose energy can exceed by many orders the energy of terrestrial accelerators. The mechanism of generation of ultrahigh energy CR is not entirely clear, but the experimentally observed energy distribution of CR is limited above by the energy $E_p^{max} = 10^{21}\ eV$. In this problem it is assumed that this limitation is caused by energy loss in the process of interaction between protons and the CMB. Protons can participate in the Compton scattering

$$p + \gamma \rightarrow p + \gamma \qquad (1)$$

and cause the reaction

$$p + \gamma \rightarrow \Delta \rightarrow \pi^0 + p \qquad (2)$$

where p is a proton, γ is a relict photon, Δ is the lightest, as compared to the nucleon, baryon with the rest mass $m_\Delta = 1.232 \times 10^6\ eV/c^2$, which quickly decays into pi-meson π^0 and proton p. (Δ-particle decays also to π^+-meson and neutron. Neutron quickly turns into proton by β-decay, so the consideration of this reaction channel is not essential in this Problem). Due to the birth of Δ-particles, the probability of interaction of protons with γ-quanta increases dramatically.

The purpose of this Problem is to determine the upper limit of the observed energy spectrum of CR, supposing that reaction (2) is the main mechanism of energy loss of CR, and compare the loss of energy protons in the reactions (1) and (2).

In the following questions, you have to measure the energy in electron-volts eV, and momenta in eV/c, where c is the speed of light in vacuum. The rest mass of the proton is $m_p = 938 \times 10^6 eV/c^2$, the rest mass of pi-meson is $m_\pi = 140 \times 10^6 eV/c^2$. Boltzmann constant is $k = 1.38 \times 10^{-23}$ Дж $\cdot K^{-1}$.

1. Estimate most probable energy E_γ and the corresponding momentum p_γ of relict photons, given that they correspond to the blackbody radiation at temperature $T = 2.7\ K$.

It is assumed in the following questions that the initial photon in reactions (1) and (2) has the energy and the momentum found Question 1.

2. The rest mass m of a particle is related to the total relativistic energy E and momentum $\vec{p}$ of this particle in an arbitrary inertial reference system as $E^2/c^2 - \vec{p}^2 = m^2c^2$. Here, the quantity mc^2 does not depend on the reference system and is a complete internal energy of the particle. Write a corresponding expression for the total internal energy of a system of two non-interacting particles (i.e. the total energy in the reference frame in which the total

momentum of physical system is equal to zero) having the total energies E_1, E_2 and momenta $\vec{p}_1$, $\vec{p}_2$.

For further analysis you may need the relativistic law of transformation of momentum and energy of the particle. In the transition from the inertial reference system S to an inertial system S', moving along the positive direction of the axis Z $(OZ \uparrow\uparrow OZ)$ with the speed V_0 in the reference system, energy and momentum transform like the coordinates of space-time point $(x, y, z, t) \rightarrow (x', y', z', t')$:

$$p_z c = G(p_z' c + \frac{V_0}{c} E'),$$
$$p_x = p_x',$$
$$p_y = p_y',$$
$$E = G(E' c + \frac{V_0}{c} p_z' c). \qquad (3)$$

Here,

$$E'/c = \sqrt[2]{m^2 c^2 + p_x'^2 + p_y'^2 + p_z'^2}, \quad G = 1/\sqrt[2]{1 - (V/c)^2}.$$

3. Determine the lowest possible energy of the proton at which reaction (2) turns possible:
   a) in the center of mass of proton and photon $p + \gamma$ (i.e., the reference system in which the total momentum of the proton and photon is zero);
   b) in the Galaxy system of reference.

Starting from the obtained results, determine the maximum energy of protons in cosmic rays.

4. Assuming that in the Galaxy system a proton has the maximum energy E_p^{max} and commits a head-to-head collision with a relict photon, determine the momentum of pi-meson π^0 in reaction (2) in this frame of reference for the case when the emitted particles move along or against the direction of the initial momentum of the proton. What is the change of the momentum of the proton in this case?

5. What is the value of momentum of the outgoing photon in the Compton scattering (1) under the same conditions as in the Question 4?

6. Is reaction (2) with the cosmic relict radiation possible, if the initial momenta of the proton and the photon are parallel in the Galaxy system? If the reaction is possible, what is the minimum value of the photon momentum in the Galaxy system?

SOLUTIONS FOR THEORETICAL COMPETITION

Theoretical Question 1

1A

Potential energy of the rigid rod $U=mgl/2\sin\alpha$ transforms to the kinetic energy of its rotation $E=J\omega^2/2$, where $J=ml^2/3$ is its inertia moment with respect to the vertical support, ω is the angular velocity. Making balance of these energies, one can find angular velocity of the center of mass around the axis of the support, and after that one can get instantaneous angular velocity of rotation of the center of mass around the axis of the support, and with the help of the obtained expression we find normal acceleration

$$a_n = \omega^2\left(\frac{l}{2}\right) = \frac{3}{2}g\sin\alpha$$

We find the tangent acceleration from the dynamics equation $M=\beta J$, where $M=mgl/2\cos\alpha$ is momentum of the gravity force with respect to the axis of rotation, β is angular acceleration of the center of mass related to tangent acceleration $a_t = \beta l/2$:

$$a_t = \frac{3}{4}g\cos\alpha$$

Center of mass acceleration **a** is found from the equation

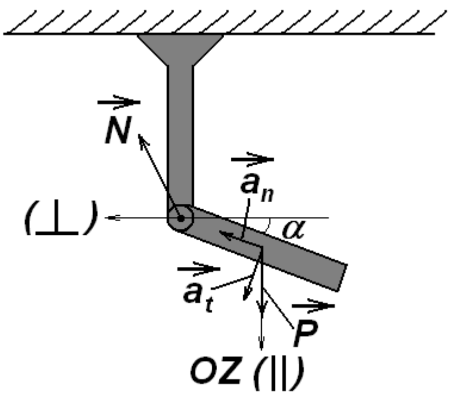
$$\mathbf{P}+\mathbf{N}=m\mathbf{a}$$

where **P** is gravity force, **N** is reaction force of the support. Decomposing this equation into vertical and horizontal components, and accounting for

$$a_{n||} = -a_n\cos\alpha\ ,\ a_{n\perp} = a_n\sin\alpha,$$
$$a_{t||} = a_t\sin\alpha\ ,\ a_{t\perp} = a_t\sin\alpha,$$

we find

$$N_{||} = mg\left(\frac{3}{4}\cos^2\alpha - \frac{3}{2}\sin^2\alpha - 1\right),$$
$$N_{\perp} = \frac{9}{4}mg\cos\alpha\ \sin\alpha.$$



Marking scheme

No.	Items	Points
1	Writing down balance equation for the kinetic and potential energies and determination of the angular velocity of rotation of the center of mass	**0.5**

2	Determination of the normal acceleration	**0.5**
3	Determination of the tangent acceleration from the equation of dynamics of rotation	**0.75**
4	Determination of the component of the reaction force of the support from the 2nd Newton law	**1.25**

1B

First of all let us find dependencies of the liquid density and pressure upon the height measured from the vessel bottom:

$$\rho_{жидкости}(h) = \rho_{\max}\left(1-\frac{h}{H}\right); \tag{1}$$

$$p(h) = \int_h^H \rho(x) g dx = \frac{1}{2}\rho_{\max} gH\left(1-\frac{h}{H}\right)^2. \tag{2}$$

One can see that pressure goes to zero at the top level of the vessel. It means that in this region volume of the gas will exceed the volume of the test-tube, so the gas bulbs will come out. Until the gas doesn't come out from the test-tube, its isothermal expansion takes place, so

$$p(h)V(h) = \frac{1}{2}\rho_{\max} gHV_1. \tag{3}$$

From (2) and (3) one can obtain:

$$V(h) = \frac{V_1}{\left(1-\frac{h}{H}\right)^2}. \tag{4}$$

Formula (4) is valid for $V(h) < V_0$, i.e. for

$$h < H\left(1-\sqrt{\frac{V_1}{V_0}}\right). \tag{5}$$

One can write down the average density of the gas in the test-tube taking into account the mass of its walls from (4)-(5):

$$\rho_{газа}(h) = \frac{M}{V(h)} = \begin{cases} \rho_1\left(1-\frac{h}{H}\right)^2, & h < H\left(1-\sqrt{\frac{V_1}{V_0}}\right); \\ \rho_0, & h \ge H\left(1-\sqrt{\frac{V_1}{V_0}}\right), \end{cases} \tag{6}$$

where $\rho_{0,1} = M/V_{0,1}$.

Various kinds of dependencies of $\rho/\rho_{\max}$ upon h/H are plotted on Fig.1-3. Curve 1 corresponds to the liquid, while curve 2 corresponds to the average density of the gas in the test-tube taking into account the mass of its walls.

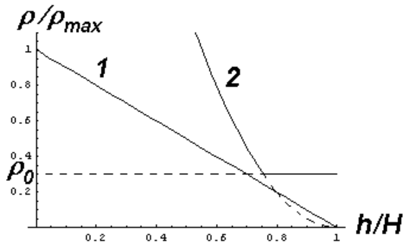


Fig.1

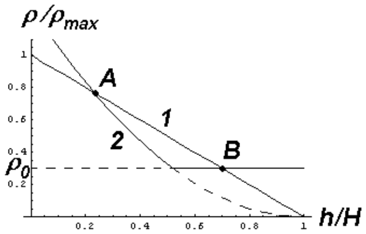


Fig.2

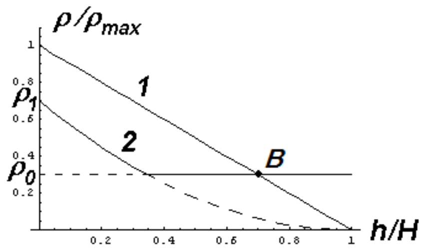


Fig.3

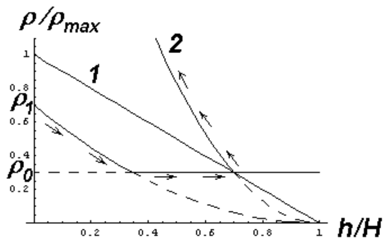


Fig.4

Curves 1 and 2 don't intersect (Fig.1), if the condition $\rho_{газа} > \rho_{жидкости}$ is fulfilled for $h = H\left(1-\sqrt{\dfrac{V_1}{V_0}}\right)$, i.e.

$$\frac{\rho_0}{\rho_{max}} > \sqrt{\frac{V_1}{V_0}}. \tag{7}$$

In this case the test-tube will always sink.

If the conditions

$$\frac{\rho_0}{\rho_{max}} < \sqrt{\frac{V_1}{V_0}}, \qquad \rho_1 > \rho_{max}, \tag{8}$$

are fulfilled in the same time, curves intersect in two points A and B (Fig.2):

$$h = H\left(1-\frac{\rho_{max}}{\rho_1}\right) \quad \text{(point A);} \tag{9}$$

$$h = H\left(1-\frac{\rho_0}{\rho_{max}}\right) \quad \text{(point B).} \tag{10}$$

Point A is unstable because the test-tube will sink due to the shift down, and it will rise due to the shift up. Analysis of stability of the point B will be presented below.

If conditions

$$\rho_1 < \rho_{max}, \qquad \frac{\rho_0}{\rho_{max}} < \sqrt{\frac{V_1}{V_0}}, \tag{11}$$

are fulfilled, only the intersection point B exists (Fig.3). But one should take into account that at the horizontal part of curve 2 motion to the right along this curve results to the gas flow out of the test-tube. Consequently, motion to the left along the curve will be quite different. It will take place along the parabola corresponding to (6) (upper line of the formula), but with the new (larger) value of ρ_1 corresponding to the amount of gas remained in the test-tube.

Grading scheme

1. Pressure dependency upon the height	**0.5**
2. Dependency of the average gas density in the test-tube upon the height:	
a) taking into account the gas flow from the test-tube (complete answer)	**1.0**
b) without taking into account the gas flow from the test-tube (incomplete answer)	0.5
3. Comparison of dependencies of liquid and gas densities upon the height:	
a) for three cases (complete answer)	**1.25**
b) for two cases (incomplete answer)	0.75
c) for one case (incomplete answer)	0.5
4. Determination of the height corresponding to the intersection points:	
a) two points (complete answer)	**0.5**
b) one point (incomplete answer)	0.25
5. Study of stability for point A	**0.25**

6. Study of stability for point B **0.5**

Totally (maximum) **4.0**

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1C

The form of shape described in the problem is explained by the appearance of the full shade (dark rectangle) and the semi-shadow (lighter outer rectangle). Fig. 1 illustrates the course of outer rays forming a shadow C_1C_2 and the semi-shadow (C_1D_1 and C_2D_2) in cross section perpendicular to one sides of the source. Denote the full width of the shadow C_1C_2 - x_1,, the width of the semi-shadow D_1D_2 - x_2. These values can be expressed through the geometric dimensions of the source and the plate.

From the similarity of the triangles $A_2B_1B_2$ and $A_2D_1C_2$ it follows

$$\frac{x_1+x_2}{2a} = \frac{L}{L-h} \qquad (1)$$

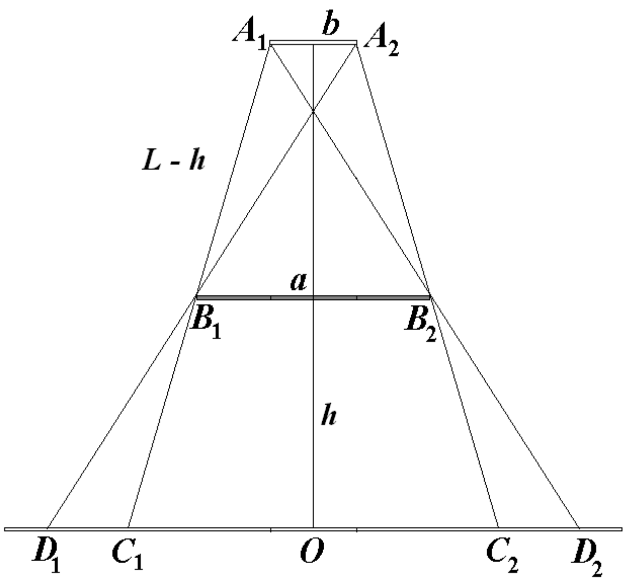

Рис. 1

From the similarity of the triangles $A_1A_2B_2$ and $B_2C_2D_2$ it follows

$$\frac{x_2-x_1}{2b} = \frac{h}{L-h} = \frac{L}{L-h} - 1 \qquad (2)$$

From the drawing of shadows, we define the required sizes

$x_1 = 12см,\qquad x_2 = 28см$

Using formula (1) we find

$\frac{L}{L-h} = 2$

Hence, we find $h = \frac{L}{2} = 1{,}5м$. From formula (2) we find one of the transverse source size $b = 8{,}0см$. Similar calculations for the perpendicular cross section gives the following results:

$x_1 = 18см,\qquad x_2 = 22см$

Then, it follows that $c = 2{,}0см$.

The results indicate that the long side of the source is placed horizontally (if you use fig.2 from the conditions of the problem).

Marking scheme.

No.	Solution item number	Points
1	Plotting of the ray tracing to explain the emergence of the shadow and semi-shadow	0,5
2	Geometric relationships between size of the source, location and size of the plate and size of the shadow and the semi-shadow (1)-(2)	2x0,25
3	Calculation of height of the plate above the floor (with a numerical value)	0,5
4	Calculation of the dimensions of the source (with numerical	2x0,5

	values)	
5	Position of the source with respect to the shadow	0,5

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Theoretical Question 2

Solution

1. **[1 point]** The total inertia moment with respect to the rotation axis is a sum of the inertia moment of the coil itself and the metallic wire

$$J = J_0 + mr^2 . \tag{1}$$

2. **[1 point]** The equation of the coil rotation as a rigid bode takes the form

$$J\varepsilon = J\frac{d\omega}{dt} = -M , \tag{2}$$

where ε is the angular acceleration.

It follows from equation (2) that the coil stops at the time moment

$$t_0 = \frac{\omega_0 J}{M} . \tag{3}$$

Finally, the dependence of the angular velocity on time t is found as

$$\omega(t) = \begin{cases} \omega_0 - \dfrac{M}{J}t, & t < t_0 = \dfrac{\omega_0 J}{M} \\ 0, & t \geq t_0 \end{cases} . \tag{4}$$

3. **[1 point]** At the stoppage of the coil, electrons keep on moving due to their inertia, as a result the galvanometer registers the electric current.. Let $a = \varepsilon r$ be the linear acceleration of the coil rim. If the coil is tightly reeled up and the wire is rather thin that linear acceleration is directed along the wire. At the stoppage process electrons are subjected to the inertial force $-m_e a$ opposite to the linear acceleration of the coil. This inertial force can be interpreted as an effective electric field

$$E_{eff} = -\frac{m_e a}{e} . \tag{5}$$

Thus, the effective electromotive force in the coil caused by the inertia of freely moving electrons is obtained as

$$\mathrm{Emf} = E_{eff}\ell = -\frac{m_e}{e}a\ell . \tag{6}$$

Therefore, the Ohm's law for the electric circuit is written as

$$IR = -\mathrm{Emf} = \frac{m_e a\ell}{e} . \tag{7}$$

Taking into account solution of 2, one gets

$$I(t) = \begin{cases} \dfrac{Mm_e r\ell}{eJR}, & t < t_0 = \dfrac{\omega_0 J}{M} \\ 0, & t \geq t_0 \end{cases} . \tag{8}$$

4. **[2 points]** The electric charge, registered by the galvanometer, is found from (8) as

$$Q = It_0 = \frac{m_e \omega_0 r\ell}{eR} . \tag{9}$$

The charge-to-mass ratio of electron is simply obtained as

$$\frac{e}{m_e} = \frac{\omega_0 r\ell}{RQ} . \tag{10}$$

5. **[1 point]** In this case equation (7) is rewritten as follows

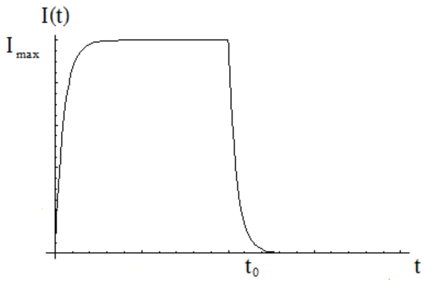
$$L\frac{dI}{dt}+IR=-\mathrm{Emf}=\frac{m_e a\ell}{e}. \tag{11}$$

where $L=\mu_0 n^2\pi r^2 h$ is the coil inductance.

It follows from equation (11) that the maximal electric current strength is

$$I_{\max}=\frac{Mm_e r\ell}{eJR}. \tag{12}$$

The qualitative dependence of the electric current strength is plotted below



6. **[1 point]** The maximal electromagnetic energy stored in the coil equals

$$W_0=\frac{LI_{\max}^2}{2}=\frac{\mu_0\pi h}{2}\left(\frac{nMm_e r^2\ell}{eJR}\right)^2. \tag{13}$$

7. **[3 points]** In the stationary regime the magnetic field inductance

$$B=\mu_0 nI \tag{14}$$

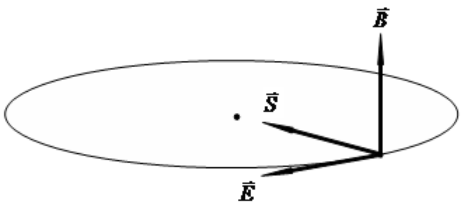
remains constant in the coil and the electric field is absent. This is not true for initial time moments while the electric current increases from 0 to its maximal value determined by formula (12). According to (14) the varying magnetic field generates the vortex electric field which causes the flux of the electromagnetic energy to appear. The strength of the vortex electric field at the lateral surface of the coil is found from the electromagnetic induction law of Faradey

$$Emf=E2\pi r=-\frac{d\Phi}{dt}=\frac{d}{dt}\left(B\pi r^2\right), \tag{15}$$

as

$$E=\frac{r}{2}\frac{dB}{dt}=\frac{\mu_0 nr}{2}\frac{dI}{dt}. \tag{16}$$

The mutual orientation of the vectors $\vec{E},\vec{B}$ и $\vec{S}$ is shown below.



Substituting expressions (14) and |(16) into the expression for the Pointing vector and taking into account that the vectors $\vec{E}$ and $\vec{B}$ are perpendicular, one obtains

$$S=\frac{\mu_0 n^2 r}{2}I\frac{dI}{dt}. \tag{17}$$

Thus, the electromagnetic energy, going inward the lateral surface of the coil while the electric current increases, is given by the summation (or integrating) of (17) as

$$W=\frac{\mu_0 n^2 r}{4}I_{\max}^2 2\pi rh=\frac{\mu_0\pi h}{2}\left(\frac{nMm_e r^2\ell}{eJR}\right)^2. \tag{18}$$

It is obvious that the same amount of the electromagnetic energy goes outward while the electric current strength decreases

$$W' = \frac{\mu_0 n^2 r}{4} I_{\max}^2 2\pi r h = \frac{\mu_0 \pi h}{2}\left(\frac{nMm_e r^2 \ell}{eJR}\right)^2 . \tag{19}$$

Marking scheme

№	Content	Points
1	Total inertia moment (1)	1
2	Equation of motion (2)	0.25
3	Stoppage time (3)	0.25
4	Dependence (4) of the angular velocity on time t	0.5
5	Expressions for the effective electric field (5) or (6)	0.25
6	The Ohm's law (7)	0.25
7	Dependence (8) of the electric current strength on time t	0.5
8	Charge (9) registered by the galvanometer	1
9	Charge-to-mass ratio (10) for electron	1
10	Equation (11) for the electric current strength	0.25
11	Maximal electric current strength (12)	0.25
12	Qualitative dependence of the electric current strength	0.5
13	Maximal energy (13)	1
14	Magnetic field induction (14)	0.5
15	Electromagnetic induction law (15)	0.5
16	Vortex electric field strength (16)	0.5
17	Pointing vector (17)	0.5
18	Electromagnetic energy (18)	0.5
19	Electromagnetic energy (19)	0.5

EXPERIMENTAL COMPETITION

14 January, 2010

Please read the instructions first:

1. The Experimental part consists of one problem. This part of the competition lasts 3 hours.
2. Please only use the pen that will be given to you.
3. You can use your own non-programmable calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet and additional papers***. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*****.** Write only inside the bordered area.
6. Begin each question on a separate sheet.
7. Write on the blank ***writing sheets*** whatever you consider is required for the solution of the question.
8. Fill the boxes at the top of each sheet of paper with your country (***Country*****),** your student code (***Student Code***), the question number (***Question Number*****)**, the progressive number of each sheet **(*****Page Number***), and the total number of ***Writing sheets*** (***Total Number of Pages***). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
9. At the end of the exam, arrange all sheets for each problem in the following order:
   - Used ***Writing sheets*** in order;
   - The sheets you do not wish to be evaluated
   - Unused sheets and the printed question.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper or equipment out of the room

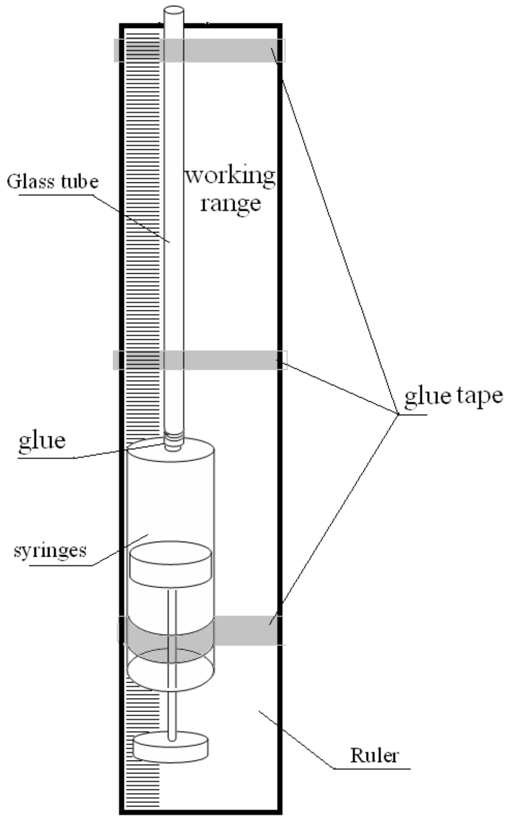
Slow motion

The aim of this work is to study the motion of bodies in a viscous medium.

Equipment List:

1. Tripod
2. Ruler of 30 cm
3. Stopwatch
4. Disposable syringes 10 ml
5. Liquid Gel (liquid soap) - 20 ml
6. Glass tube with the inner diameter – **D=4.6 mm**
7. Wooden sticks (diameter 3 mm)
8. A set of metal rods, length 4 cm, diameters **d= 2.5; 3.0; 3.5; 4.0 mm**
9. Disposable glass
10. Napkins to wipe the table and hands

Apparatus



In this experiment use the following installation. The glass tube is glued to the tip of the syringe. Tube with a syringe is attached to the ruler with an adhesive tape. Using the syringe you can fill up the tube with the gel. This setup can be fixed to the tripod vertically, either with the tube pointing upward or downward. Inside the tube you can place metal or wooden rods, the motion of which you will explore.

For successful completion of the experiment, strictly follow the instructions below:
1. Make sure that the tube is completely filled with the gel and there are no air bubbles inside it;
2. Change the gel in the syringe and the tube only when necessary;
3. Once you put a metal rod (or wooden stick) in the receiver, wait until it starts to move steadily, for this it must travel a distance of about 4-5 cm;
4. The results of measurements of the characteristics of motion have significant dispersion, so all measurements should be conducted several times;
5. **Be careful of the sharp edge of the tube!**

Part 1. Metal

Place the tube vertically with the open end pointing upward, so that you could put the metal rods into the tube. If necessary, conduct a second experiment, simply overturning the tube: when the rod sinks down, turn the tube upside down and you can repeat the measurement.

1. Prove experimentally that the motion of the rod inside the tube is uniform. Use only metal rod of diameter of 4.0 mm in this case.
Determine the velocity of the rod sinking in the gel as accurately as possible. Calculate the error of the velocity measured.

2. Measure the dependence of the velocity of the rods (with the same length) on their diameters. Plot a graph of the obtained dependence.

3. Theoretically, we can show that the velocity V of the rod depends on the thickness h of the gap between the tube walls and the lateral surface of the rod (if the gap is small compared to the radius of the tube) according to the law

$$V = Ch^{\gamma}, \tag{1}$$

where C is a constant.
Using your experimental data, check whether this dependence holds.
4. Determine the power index γ in formula (1) which most accurately fits the experimental data. Make error estimation of the obtained value of this power index γ.
5. Give a brief theoretical justification for the result obtained in Question 4.

Part 2. Wood

Overturn the syringe with the tube pointing downward. Insert a wooden rod into the tube (from below) then it will slowly emerge. You have to study the motion of the wooden rod. The wooden rod should be completely immersed in the fluid when it moves.

1. Prove experimentally that the motion of the rod inside the tube is uniform.
Measure the velocity of the rod in the gel as accurately as possible. Make error estimation of the obtained value of the velocity.
To conduct this experiment, use stick of maximal length.

2. Investigate a dependence of rod velocity on its length. Plot a graph of the obtained dependence. Make error estimation of the obtained values.

3. Give a brief theoretical explanation of the result obtained in Question 2.

Experimental competition

Solution

Part 1.

1. It is necessary to obtain the motion law of the rod. In this experiment it is preferable to measure time intervals at which the rod passes preliminary fixed paths, say 1 sm. Table 1 shows the corresponding measurements and Fig.1 graphically demonstrates the same dependence.

Table 1.

x, sm	***t, s***
0	0
1	6,9
2	14,3
3	21,8
4	29,2
5	36,8
6	44,1
7	51,0
8	58,4
9	65,7
10	72,2
11	79,0
12	85,9
13	92,9
14	100,3
15	107,7
16	114,6

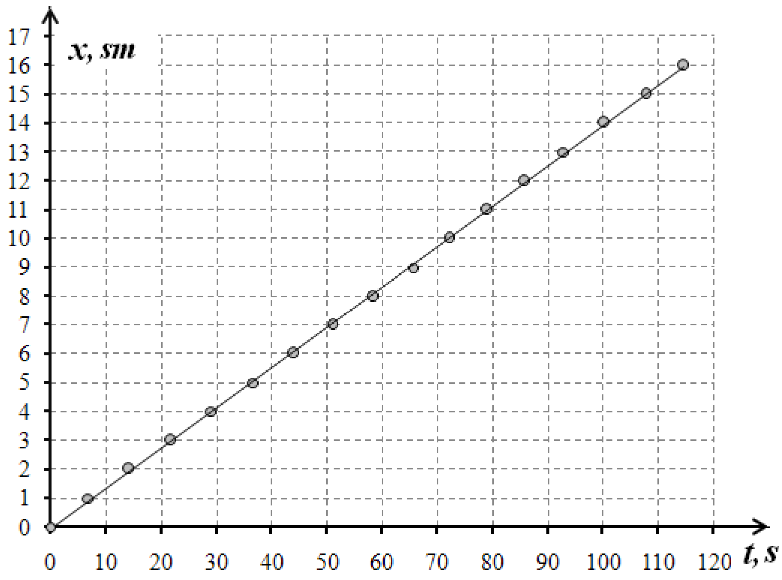


Fig.1 The motion law of the rod

The experimental line in Fig. 1 is linear with high accuracy which proves that the rod really moves uniformly. The most accurate technique of determining average velocity of the rod is least-squares method. The slope of the straight line in Fig.1 gives an average velocity which in this particular case is

$$V = (0{,}140 \pm 0{,}001) sm/s$$

2. In this part of the experiment it is necessary to measure the rod velocity several times and to guarantee maximal accuracy the fixed path for the rod to pass should be over 10 sm. Table 2 presents the measurement results for the time intervals ***t*** for the rods of different diameters and the calculated velocities ***V.***

Table 2.

***d*, mm**	***x*, sm**	***t₁*,s**	***t₂*,s**	***t₃*,s**	**<*t*>, *s***	**Δ*t*, *s***	***V*, *sm/s***	**Δ*V*, *sm/s***
4,0	10	77,5	76,6	81,7	78,60	5,4	**0,13**	0,01
3,5	10	11,7	11,6	12,0	11,76	0,48	**0,85**	0,03
3,0	10	3,4	3,3	3,3	3,35	0,12	**2,99**	0,11
2,5	20	3,3	3,5	3,2	3,33	0,30	**6,01**	0,54

To calculate the average velocity the following formula is used

$$V = \frac{x}{\langle t \rangle}, \quad (1)$$

where $\langle t \rangle$ is the average of the measured time intervals.

The experimental error can be evaluated as

$$\Delta t = 2\sqrt{\frac{\sum (t_i - \langle t \rangle)^2}{n}}, \qquad \Delta V = V\frac{\Delta t}{\langle t \rangle} \quad (2)$$

The corresponding dependence is plotted in Fig.2.

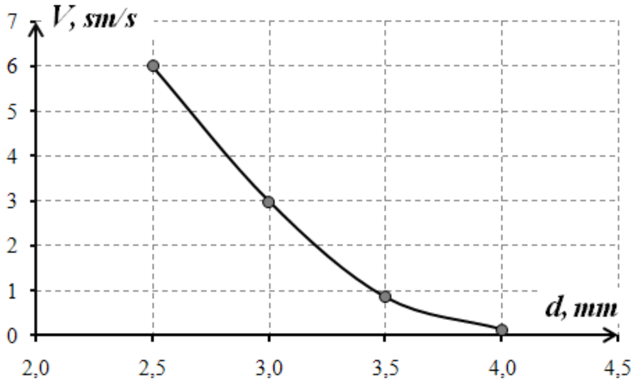


Fig.2 The dependence rod velocity on the rod diameter

3. To check the validity of the formula written in the experimental task it is necessary to plot in logarithmic scale the average velocity dependence on the difference between the diameters of the test tube and the rod. It is done in Fig.3.

The linearity of the obtained line clearly demonstrates the power dependence between the chosen values, i.e. proves the validity of formula (1) written in the experimental task// It is possible that the last point in Fig.3 deviates slightly from the straight line due to the big gap between the rod and the test tube walls.

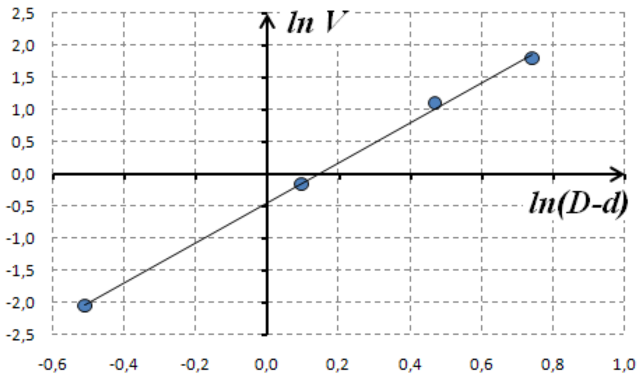


Fig. 3 The rod velocity versus the size of the gap (log-log scale)

4. The power index in formula (1) is numerically equal to the slope of the straight line in Fig.3. Calculation with the least-squares method gives rise to the following value

$\gamma = 3{,}11 \pm 0{,}16$

5. When the rod sinks the liquid has to flow between the lateral surface of the rod and the test tube walls (Fig.4). To overcome the viscous friction of the liquid, the pressure difference $\Delta P = P_0 - P_1$ should appear between the lower and upper ends of the rod. It is this difference in the pressure that produces an additional force acting on the sinking rod. Besides, the rod is slowed down by the above mentioned viscous friction acting on its lateral surface.

It is reasonable to assume that the viscous friction force acting on the liquid in the gap from the tube walls is proportional to the average velocity of the flowing liquid and inversely proportional to the gap size.

$$F_v = \beta \frac{v}{h} \quad (3)$$

At uniform flowing of the liquid this force is compensated by the pressure difference (the weight of the liquid in the gap is negligible due to the small size of the gap)

$$\beta \frac{v}{h} \approx S_1 \Delta P, \quad (4)$$

where $S_1 = 2\pi R h$ is the area of the transversal section of the gap. Since the rod moves slowly, one can neglect the viscous friction and write

$$mg = S_0 \Delta P, \quad (5)$$

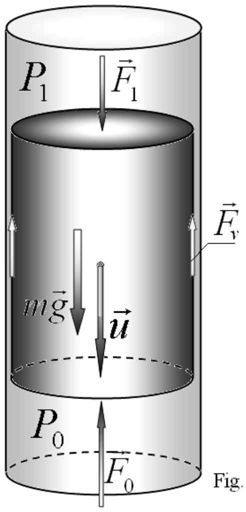


Fig. 4

where $S_0 = \pi R^2$ is the area of the rod transversal section. It follows from formulas (4)-(5) that the average velocity of the liquid in the gap is found as

$$v = \frac{mg}{R\beta} h^2 \quad (6)$$

The rod velocity u is related to the average velocity of the liquid in the gap as $S_0 u = S_1 v$, thusпоэтому

$$u = \frac{2h}{R} v \sim h^3. \quad (7)$$

The results obtained above verify formula (7) within the experimental error.

Part 2.

1. The experimental technique used in this task is quite analogous to the same task in Part 1. In Table 3 and Fig.5 the results are presented for the wood stick.

Table 3.

***x*, см**	***t*, с**
0	0,0
1	13,1
2	25,6
3	37,0
4	48,8
5	61,3
6	73,9
7	86,1
8	98,2
9	109,6
10	121,7
11	134,0
12	146,0

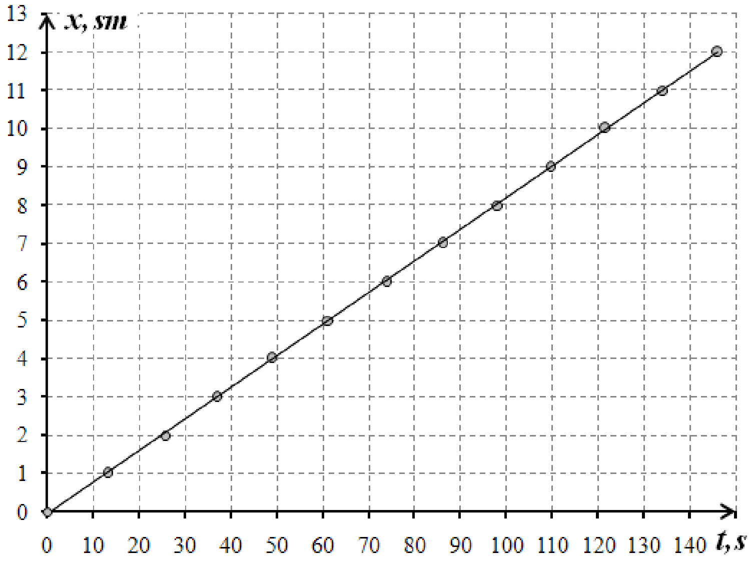


Fig.5 The motion law of the wood stick

It is seen that the wood stick also emerges uniformly and its average velocity, calculated by the least-squares method from Table 3, is

$$V = (0{,}0825 \pm 0{,}0004) cm/c$$

2. To obtain velocities it is enough to measure time intervals for a wood stick to pass a fixed path several times. Results of the measurements shows that within the experimental error the emerging velocity does not depend on the stick length and equals to

$$V = (0{,}08 \pm 0{,}01) sm/s$$

Thus, the plot is just a horizontal straight line.

3. In this case the diameters of sticks are all equal. It is obvious that the viscous friction is proportional to the stick length. $\beta \sim l$. The mass and the pressure difference is also proportional to the length l. Consequently, it follows from formula (6) that the velocity of the flowing liquid and, thus, the velocity of the stick does not depend on the length.

THEORETICAL COMPETITION

January 16, 2011

Please read this first:

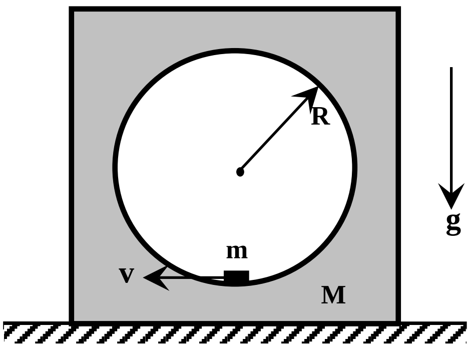
1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets.*** Write only inside the boxed area.
6. Begin each question on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Question Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
    - Used ***Writing sheets*** in order;
    - The sheets you do not wish to be evaluated
    - Unused sheets and the printed question.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

Problem 1 (10 points)

This problem consists of three independent parts.

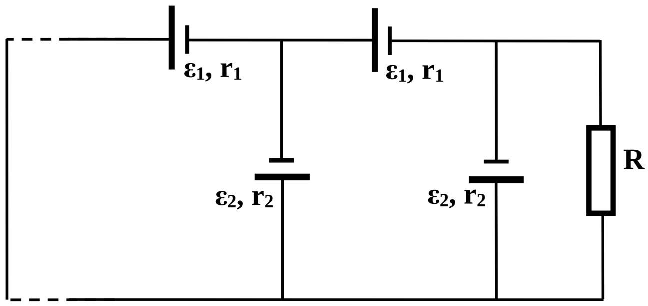
1A (3.5 points)



The body is a cube in the center of which a spherical cavity of radius R is carved out. Inside the spherical cavity at its bottom there is a motionless puck whose geometric sizes are negligible. Find the minimal horizontal velocity (at all possible cube-to-puck mass ratios) which has to be imposed on the puck so as the cubic body should jump up from the table surface in the motion process followed. Friction in the system is completely absent. At which value of the cube-to-puck mass ratio M/m that minimal velocity is achieved?

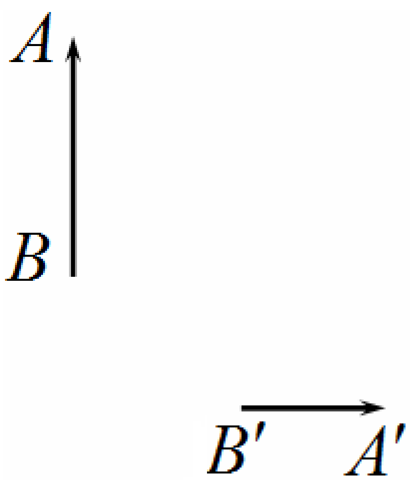
1B (4 points)

The resistance $R = 2.0\,\Omega$ is connected to an infinite number of current sources obtained by repetition as shown in the figure on the right. Find the electric current flowing through the resistance R. The emfs' and internal resistances of the sources are known to be $\varepsilon_1 = 2.0\,\text{V}$, $r_1 = 1.0\,\Omega$, and $\varepsilon_2 = 1.0\,\text{V}$, $r_2 = 2.0\,\Omega$.



1C (2.5 points)

In Figure on the right b, you can see the object AB and its image A'B' in a thin lens. By drawing method, please find:
a) the optical center of the lens (0.5 points);
б) the lens plane (1 point);
в) the main focuses of the lens (0.5 points).
Is the lens concave (diverging) or convex (collecting)? Please write down the answer. (0.5 points).



Problem 2 (10 points)
Electrical conductivity of metals (10 points)

Ohm's law

Conductors are materials, usually metals, in which an ordered motion of free charges called an electric current is possible in the presence of an external electric field. The law relating the electric current strength I flowing through the conductor with a voltage U applied to its ends was experimentally discovered by Georg Ohm (1787-1854) and has the following form:

$$I = \frac{U}{R}, \tag{1}$$

where R is called the resistance of the conductor.

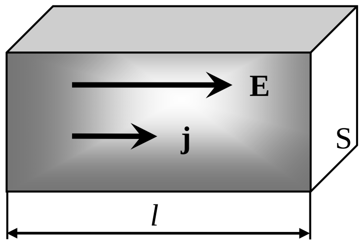


Fig.1

Consider a small element of the metallic material with the length l and the cross section S, whose ends are subject to the voltage U. Let σ be the specific electrical conductivity of the substance which is the quantity inverse to the specific electrical resistivity ρ. The resistance of the conductor and the electric current strength flowing through it are written as

$$R = \rho\frac{l}{S} = \frac{1}{\sigma}\frac{l}{S}, \qquad I = jS, \tag{2}$$

where the current density j is introduced, representing the amount of the electric charge that passes through the unit of the cross section in the unit of time. The current density depends on the electron number density and electron **average ordered velocity**.

Taking into account that $E = U/l$ stands for the electric field strength, the local (differential) form of Ohm's law is obtained from Eqs. (1) and (2) as

$$j = \sigma E. \tag{3}$$

Accounting for the same direction of the electric field strength and the current density vectors, relation (3) can be rewritten in vector form

$$\mathbf{j} = \sigma\mathbf{E}. \tag{4}$$

1. **[1 point]** Starting from the Joule-Lenz law, discovered first by James Joule and later by Heinrich Lenz, determine the volume density of the thermal power P_V released in the conductor, i.e. the amount of heat generated by the electric current in the unit volume 1 m^3 in the unit of time 1 s. Express your answer in terms of E and σ.

The Drude model

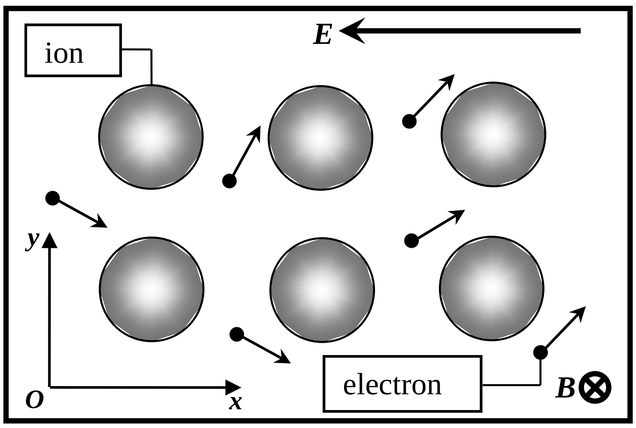


Fig.2

After the discovery of the electron in 1900 by Joseph John Thompson, German physicist Paul Drude proposed the so-called classical theory of electrical conductivity of metals. According to this theory, the electrons with the number density n, the mass m and the electric charge $-e$ can move freely in the ionic crystal lattice of metal, occasionally colliding with ions located at the sites and thereby transferring their kinetic energy to ions.

Real motion of an electron is very complicated because of the chaotic thermal motion. Under the influence of the external field, all electrons acquire the same acceleration and thus gain some extra velocity. This results in an ordered motion of electrons causing an electrical current to appear. We are only interested in that ordered motion of electrons which is superimposed on their random thermal walks.

Since the real picture of the electrical conductivity is very complicated, we adopt the following simplified model. Assume that an electron with an initial zero velocity accelerates over time interval τ and then collides with an ion at the site, thereby transferring all the acquired kinetic energy. Then it again starts to accelerate and over time interval τ collides with another ion and so on. Keeping this in mind find answers to the following: *In those processes the electrons do not interact with each other.*

2. **[1 point]** Determine the vector of average ordered velocity of electrons $\mathbf{u}$. Express your answer in terms of e, $\mathbf{E}$, m, and τ.

3. **[1 point]** The current density in the sample is determined by the average velocity component parallel to the external electric field strength E. Show that Ohm's law holds in this simplified model and determine the specific conductivity σ of the metal. Express your answer in terms of e, n, m, and τ.

4. **[1 point]** Determine the amount of the kinetic energy Q_V transferred by electrons to the crystal lattice in the unit volume 1 m^3 and in the unit of time 1 s. Express your answer in terms of e, $\mathbf{E}$, n, m, and τ.

Magnetoresistance

One of the important galvanomagnetic phenomenon is the change in the conductivity of a conductor that is subject to an external transverse magnetic field. This phenomenon is called a magnetoresistance effect. Due to experiments, the relative deviation of the specific conductivity $\Delta\sigma/\sigma$ at not very strong magnetic field with the induction B is given by the formula

$$\frac{\Delta\sigma}{\sigma}=\frac{\sigma(B)-\sigma(B=0)}{\sigma(B=0)}=\mu B^{\nu}, \qquad (5)$$

where μ and ν are some constants.

Making use of the Drude model described above, solve the following problems. Carefully examine figure 2 presented above, since it represents the system of coordinates in use and displays directions of all vectors.

5. **[1 point]** Find the dependences on time t of the projections $u_x(t)$ and $u_y(t)$ of the electron velocity between two consecutive collisions. Express your answer in terms of e, E, B, m, and t.

6. **[2 points]** The current density in the sample is determined by the average velocity component parallel to the external electric field strength E. Assuming that the magnetic field induction B is small enough, find the constants μ and ν in formula (5). Express your answer in terms of e, m, and τ.

The Hall effect

In 1879, Edwin Hall discovered the phenomenon of appearance of the transverse potential difference, later called the Hall voltage, by placing the current-carrying conductor in a constant transverse magnetic field.

In the simplest consideration, the Hall effect is described as follows. Suppose that an electric current flows through a metal bar due to the applied external electric field of the strength E and is placed in a weak transverse magnetic field of induction B. The magnetic field deflects electrons from their straight-line motion to one of the bar faces. Thus, the Lorentz force, in contrast to the magnetoresistance effect, causes the accumulation of the negative charge near one of the bar faces and the positive charge near the opposite face. The accumulation of charge persists until the transverse electric field E_H, generated by the accumulated charges themselves (directed along the axis Oy as shown in the figure above), **does totally compensate** for the transverse displacement of electrons over the time interval τ.

Making use of the Drude model, described above, solve the following problems. Carefully examine figure 2 presented above, since it represents the system of coordinates in use and displays directions of all vectors.

7. **[0.5 points]** Look carefully at the second figure above. Near which of the faces, the top or the bottom, is the negative charge accumulated?

8. **[1.5 points]** Find the dependences on time t of the projections $u_x(t)$ and $u_y(t)$ of the electron velocity between two consecutive collisions. Express your answer in terms of e, E, E_H, B, m, and t.

9. **[1 point]** Find the Hall electric field strength E_H. Express your answer in terms of e, E, B, m, and τ, and then in terms of e, j, B, and n.

In solving these problems you can use the following approximate formulae valid for small values of x:

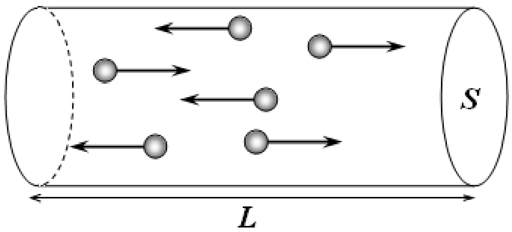
$$\sin x \approx x - \frac{x^3}{6}$$
$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Problem 3 (10 points)

Thermodynamics of simple quantum ideal gas

In classical physics, the energy of the system varies continuously. In the physics of the microworld, most physical parameters are quantized, i.e., they can take a discrete set of values. Quantization of energy can result in actually observed macroscopic effects. In this problem, you are to consider the simplest model of a quantum ideal gas.

Model



Let a gas consist of N identical atoms of mass m, which are placed in a long cylindrical vessel of length L and cross-section S. Atoms can only move along the axis of the vessel. The kinetic energy of atoms is quantized, i.e. it can take a discrete set of values determined by

$$E_n = n\varepsilon, \quad (1)$$

where $n = 1, 2, 3...$ and ε is a known constant value. Assume that the kinetic energy of an atom is expressed by the formula of classical physics.

The vessel is brought into contact with the thermostat so that the gas temperature in the vessel is T. The value of kinetic energy of a single atom is changed due to contact with the thermostat. Assume that the atom number density is such low that collisions between atoms are rare and can be neglected.

At thermodynamic equilibrium the number of atoms, which occupy the level with the energy E_n, is determined by the Boltzmann distribution function of the form

$$N_n = C\exp\left(-n\frac{\varepsilon}{k_B T}\right), \qquad (2)$$

where k_B denotes the Boltzmann constant, C is a normalization factor that you have to determine by yourself.

Subproblems:

1 **[1 point]** Find the number of atoms N_n that occupy the energy level E_n. Express your answer in terms of N, ε, T, and k_B.

2 **[3 points]** Find the expression for the internal energy U of the gas. Express your answer in terms of N, ε, T, and k_B. Obtain approximate formulae for the internal energy of the gas in two limiting cases: $k_B T >> \varepsilon$ (**high temperature limit or classical limit**) and $k_B T << \varepsilon$ (**low temperature limit**).

3 **[3 points]** Calculate the molar heat capacity of gas at constant volume. Express your answer in terms of N, ε, T, and k_B. Obtain approximate formulae for the molar heat capacity at constant volume both in the classical limit and the limit of low temperatures. Draw a schematic plot of the molar heat capacity dependence against the gas temperature.

4 **[3 points]** Find the pressure P exerted by the gas on the vessel walls. Express your answer in terms of N, ε, T, and k_B. Obtain approximate formulae for the pressure both in the classical limit and the limit of low temperatures. Draw a schematic plot of the pressure dependence against the gas temperature.

In solving these problems you can use the following formulae:

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x},$$

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2},$$

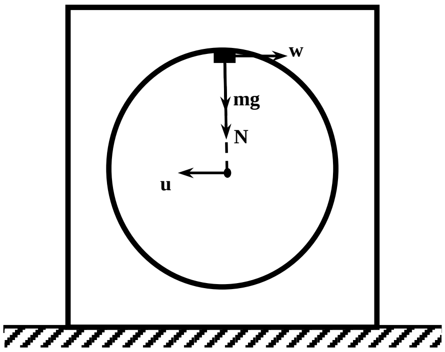
$$\exp(x) \approx 1 + x, \quad x<<1,$$

$$\frac{1}{1-x} \approx 1 + x, \quad |x| << 1.$$

SOLUTIONS FOR THEORETICAL COMPETITION

Theoretical Question 1 (10 points)

1A (3.5 points)



It is elementary to show that the jump of the cube appears at the puck position depicted in the picture on the left hand side. Let u be the cube velocity of mass M at this time moment and let w be the horizontal relative velocity of the puck of mass m with respect to the cube. Since the friction in the system is totally absent, the horizontal projection of the total momentum of the system is conserved,

$$m\mathrm{v} = Mu + m(u-w)\,, \tag{1}$$

as well as with the total mechanical energy,

$$\frac{m\mathrm{v}^2}{2} = \frac{Mu^2}{2} + \frac{m}{2}\left[w^2 + (u-w)^2\right] + 2mgR\,. \tag{2}$$

In the instant frame of reference associated with the cube, the puck moves with the velocity w along the circle of radius R and its equation of motion projected on the radial direction is given by

$$N + mg = \frac{mw^2}{R}\,. \tag{3}$$

It is rather obvious that the condition of the cube's jump from the plane of the table is found, according to Newton's third law, as

$$N = Mg\,. \tag{4}$$

Solving the set of equations (1)-(4), the puck velocity is obtained as

$$\mathrm{v} = \sqrt{gR}\sqrt{5 + \frac{M}{m} + 4\frac{m}{M}}\,. \tag{5}$$

The minimal velocity of the puck is derived from relation (5) by differentiating over M/m,

$$\mathrm{v}_{\min} = 3\sqrt{gR} \tag{6}$$

and it is achieved at the mass ratio

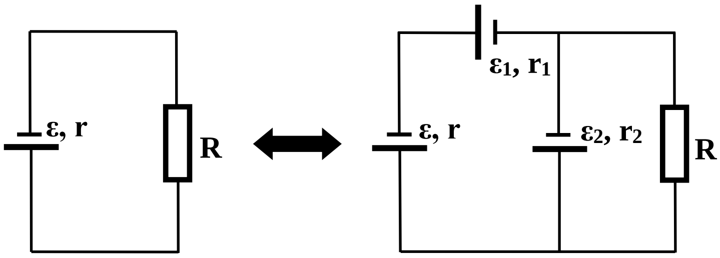
$$M/m = 2\,. \tag{7}$$

Marking scheme

№	Content	points
1	Formula (1)	0.5
2	Formula (2)	0.5
3	Formula (3)	0.5
4	Formula (4)	0.5
5	Formula (5)	0.5
6	Formula (6)	0.5
7	Formula (7)	0.5

1B (4 points)

We can replace the infinite circuit of current sources by an effective current source with an emf ε and an internal resistance r. Thus, we obtain the circuit shown in the figure on the left hand side. Then, we disconnect the resistance R, add another two current sources and connect back the resistance R. Hence, we obtain the circuit shown in the figure on the right hand side. Since the number of cells with the sources is infinite, then both circuits should be equivalent at any value of R.



It can be shown from the direct current laws that the following two statements are valid:

1. Let us take two current sources with ε_1, r_1 and ε_2, r_2, connected in series. Then, they can be replaced by a single source with $\varepsilon = \varepsilon_1 + \varepsilon_2$ and $r = r_1 + r_2$.
2. Let us take two current sources with ε_1, r_1 and ε_2, r_2, connected in parallel. Then, they can be replaced by a single source with $\varepsilon = (\varepsilon_1 r_2 + \varepsilon_2 r_1)/(r_1 + r_2)$ and $r = r_1 r_2/(r_1 + r_2)$.

Now, applying 1 and 2 to the circuit shown on the right hand side, we should obtain the circuit shown on the left hand side, thus the following relations must be satisfied:

$$\varepsilon = \frac{(\varepsilon + \varepsilon_1) r_2 + \varepsilon_2 (r + r_1)}{r + r_1 + r_2}, \tag{1}$$

$$r = \frac{r_2 (r + r_1)}{r + r_1 + r_2}. \tag{2}$$

Solution is given by

$$\varepsilon = \varepsilon_2 + \frac{\varepsilon_1}{2}\left(\sqrt{1 + \frac{4r_2}{r_1}} - 1\right) = 3.0\,\mathrm{V}, \tag{3}$$

$$r = \frac{r_1}{2}\left(\sqrt{1 + \frac{4r_2}{r_1}} - 1\right) = 1.0\ \Omega. \tag{4}$$

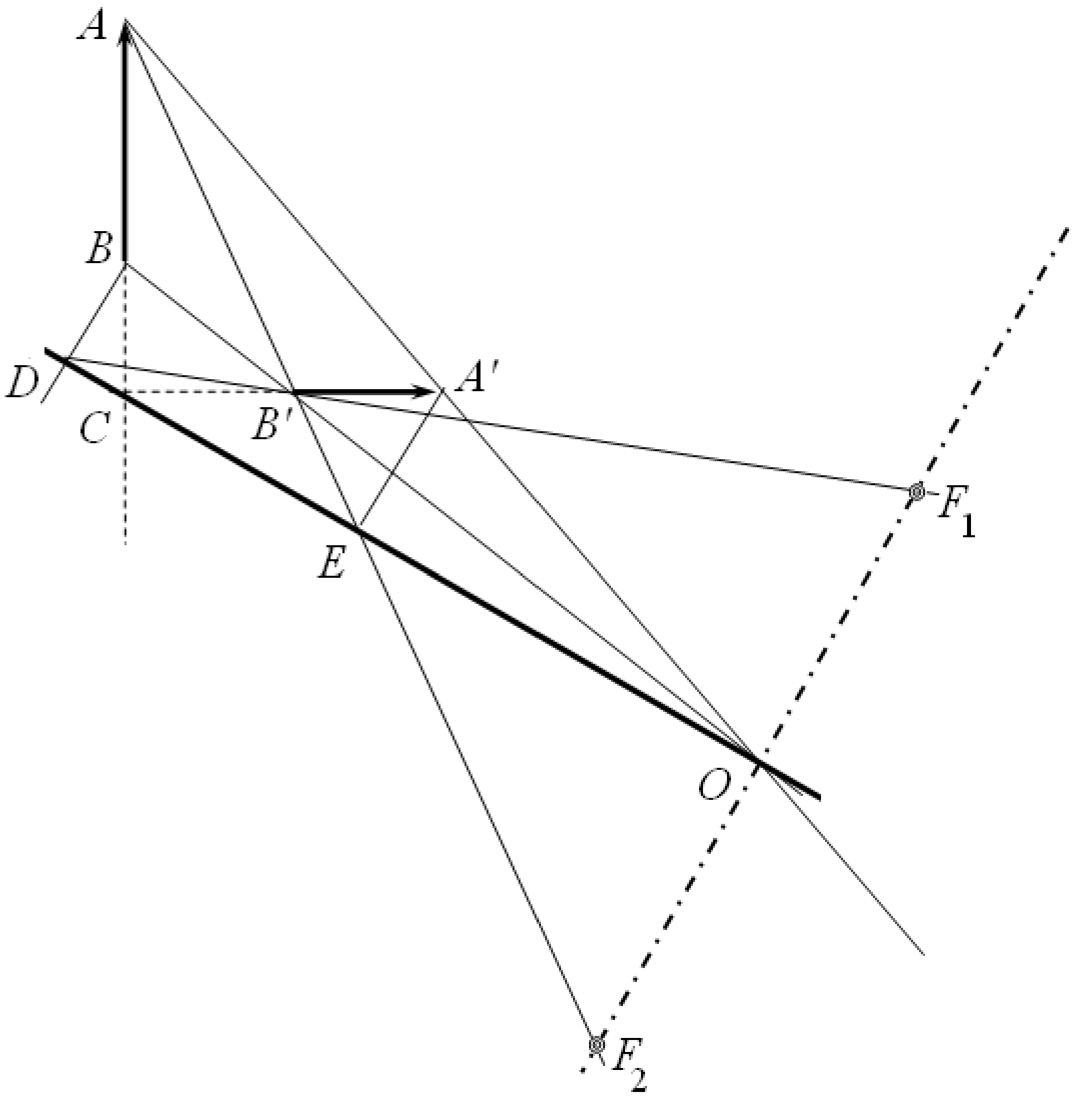
Therefore, the current flowing through the resistance R is found as

$$I = \frac{\varepsilon}{R + r} = 1.0\,\mathrm{A}. \tag{5}$$

Marking scheme

№	Content	Points
1	Equivalent circuit	1,0
2	Rule 1	0.5
3	Rule 2	0.5
4	Formula (1)	0.5
5	Formula (2)	0.5
6	Formula (3)	0.25
7	Formula (4)	0.25
8	Formula (5)	0.5

1C (2.5 points)



All the rays eradiated from the point A have to pass through the point A' after refraction in the lens; all the rays eradiated from the point B have to pass through the point B' after refraction in the lens. Rays passing through the optical center of the lens do not change direction. Therefore, the point of intersection of lines AA' and BB' is the optical center O of the lens. If a ray passes through both the points A and B, then it should necessarily pass throug*h* the points A' and B'. Consequently, the point of the intersection of lines AB and A'B' lies in the plane of the lens. Thus, the plane of the lens passes through the points O and C. The main optical axis of the lens passes through its optical center and is perpendicular to the plane of the lens, Further constructions are traditional: we draw ray BD through the point B which is parallel to the main optical axis, and after refraction in the lens the ray (or its extension) should pass through B'. From its continuation to the intersection with the main optical axis, we find one of the main focuses F_1. Similarly, we find the second main focus F_2. The drawing above shows that the lens is concave (diverging).

Theoretical Question 2 (10 points)
Electrical conductivity of metals

Ohm's law

1. [1 point]

In accordance with the Joule-Lentz law, the heat power released in the conductor is found as

$$P = \frac{U^2}{R}, \tag{1}$$

which means that the specific heat power P_V is written as

$$P_V = \frac{U^2}{RV} = \frac{U^2}{RSl}. \tag{2}$$

With the aid of

$$R = \rho\frac{l}{S} = \frac{1}{\sigma}\frac{l}{S} \text{ and } E = \frac{U}{l}, \tag{3}$$

one gets

$$P_V = \sigma E^2. \tag{4}$$

The Drude model

2. [1 point]

The second law of Newton for the electron motion in a constant electric field is read as

$$m\mathbf{a} = \mathbf{F} = -e\mathbf{E}. \tag{5}$$

It follows from Eq.(5) that for the time interval τ the electron passes the distance

$$s = \frac{a\tau^2}{2}, \tag{6}$$

which means that the module of the average velocity of the electron is

$$u = \frac{s}{\tau} = \frac{a\tau}{2} = \frac{eE\tau}{2m}, \tag{7}$$

or, in the vector form,

$$\mathbf{u} = -\frac{e\tau}{2m}\mathbf{E}. \tag{8}$$

3. [1 point]

The current density depends on the electron number density, its electric charge, and its average velocity as follows:

$$\mathbf{j} = -ne\mathbf{u} = \frac{e^2 n\tau}{2m}\mathbf{E}, \tag{9}$$

which is Ohm’s law with the specific conductivity found as

$$\sigma = \frac{e^2 n\tau}{2m}. \tag{10}$$

4. [1 point]

Each electron transfers its kinetic energy at the end of the acceleration, i.e. at the moment of collision with an ion,

$$E_k = \frac{mu_{\max}^2}{2} = \frac{m}{2}\left(\frac{eE\tau}{m}\right)^2. \qquad (11)$$

By definition there are n electrons in the cubic meter of the conductor, and each of them transfers its kinetic energy (11) for the time interval τ. Thus, the total specific energy Q_V transferred by electrons to the crystal lattice in the unit of volume and in the unit of time,

$$Q_V = \frac{nE_k}{\tau} = \frac{nmu^2}{2\tau} = \frac{e^2 n\tau}{2m}E^2 = \sigma E^2. \qquad (12)$$

This expression coincides with Eq.(4), thus proving the validity of the Joule-Lenz law in the Drude model.

Magnetoresistance

5. [1 point]

In the presence of magnetic field the equation of motion for the electron is written as

$$m\frac{d\mathbf{u}}{dt} = -e\mathbf{E} - e\mathbf{u}\times\mathbf{B}. \qquad (13)$$

The projections on the coordinate axes are found as

$$m\frac{du_x}{dt} = eE + eBu_y, \qquad (14)$$

$$m\frac{du_y}{dt} = -eBu_x, \qquad (15)$$

$$m\frac{du_z}{dt} = 0. \qquad (16)$$

Eq.(16) shows that the electron trajectory lies in XY plane. Substituting $u_x^{'} = u_x$, $u_y^{'} = u_y + E/B$ into Eqs. (14)-(15), we obtain

$$m\frac{du_x^{'}}{dt} = eBu_y, \qquad (17)$$

$$m\frac{du_y^{'}}{dt} = -eBu_x^{'}. \qquad (18)$$

Solutions to Eqs. (17) and (18) are derived as harmonic oscillations of the form

$$u_x^{'} = A\cos(\omega t + \alpha), \qquad (19)$$

$$u_y^{'} = A\sin(\omega t + \alpha), \qquad (20)$$

or, in terms of the previous variables,

$$u_x = A\cos(\omega t + \alpha), \qquad (21)$$

$$u_y = A\sin(\omega t + \alpha) - \frac{E}{B}, \qquad (22)$$

where $\omega = eB/m$.

From initial conditions $u_x = 0$ and $u_y = 0$, we determine the constants $A = E/B$ and $\alpha = \pi/2$. Substitution into Eqs. (21) and (22) yields

$$u_x(t) = \frac{E}{B}\sin\left(\frac{eB}{m}t\right), \qquad (23)$$

$$u_y(t) = -\frac{E}{B}\left[1 - \cos\left(\frac{eB}{m}t\right)\right]. \qquad (24)$$

6. [2 points]

At small magnitude of the magnetic field induction, Eq. (23) takes the form

$$u_x = \frac{eE}{m}t - \frac{e^3EB^2}{6m^3}t^3. \tag{25}$$

The displacement of the electron along the *OX* axis over the time interval τ equals

$$s = \frac{eE}{2m}\tau^2 - \frac{e^3EB^2}{24m^3}\tau^4, \tag{26}$$

and the average speed is found as

$$u_{av} = \frac{s}{\tau} = \frac{eE}{2m}\tau - \frac{e^3EB^2}{24m^3}\tau^3. \tag{27}$$

Thus, we are able to determine the relative deviation of the specific conductivity as

$$\frac{\Delta\sigma}{\sigma} = \frac{neu_{av}(B) - neu_{av}(B=0)}{neu_{av}(B=0)} = -\frac{1}{12}\left(\frac{e\tau B}{m}\right)^2, \tag{28}$$

and, therefore,

$$\mu = -\frac{1}{12}\left(\frac{e\tau}{m}\right)^2, \qquad \nu = 2. \tag{29}$$

The Hall effect

7. [0.5 points]

The Lorentz force acting on the electrons is directed downward, therefore the negative charge is accumulated near the bottom face.

8. [1.5 points]

Since the electrons are accumulated near the bottom face of the bar, the Hall electric field is oppositely directed with respect to the *OY* axis. Hence, the electron equation of motion (13) is rewritten as

$$m\frac{du_x}{dt} = eE + eBu_y, \tag{30}$$

$$m\frac{du_y}{dt} = eE_H - eBu_x, \tag{31}$$

$$m\frac{du_z}{dt} = 0. \tag{32}$$

Again, the electron trajectory lies in the *XY* plane. Making substitution $u_x' = u_x - E_H/B$, $u_y' = u_y + E/B$ in Eqs. (30) and (31), one gets

$$m\frac{du_x'}{dt} = eBu_y', \tag{33}$$

$$m\frac{du_y'}{dt} = -eBu_x'. \tag{34}$$

Solutions to Eqs. (33) and (34) are again derived as harmonic oscillations of the form

$$u_x' = A\cos(\omega t + \alpha), \tag{35}$$

$$u_y' = A\sin(\omega t + \alpha), \tag{36}$$

or, in terms of the previous variables,

$$u_x = A\cos(\omega t + \alpha) + \frac{E_H}{B}, \tag{37}$$

$$u_y = A\sin(\omega t + \alpha) - \frac{E}{B}. \tag{38}$$

From initial conditions $u_x = 0$ and $u_y = 0$, we obtain the following final solution

$$u_x(t) = \frac{E}{B}\sin\left(\frac{eB}{m}t\right) + \frac{E_H}{B}\left[1 - \cos\left(\frac{eB}{m}t\right)\right], \tag{39}$$

$$u_y(t) = \frac{E_H}{B}\sin\left(\frac{eB}{m}t\right) - \frac{E}{B}\left[1 - \cos\left(\frac{eB}{m}t\right)\right]. \tag{40}$$

9. [1 point]

At small magnitudes of the magnetic field induction, the condition for zero final displacement $y(\tau) = 0$ along the OY axis at the time moment τ

$$\int_0^\tau u_y(t)dt = 0 \quad \Rightarrow \quad E_H = \frac{eE\tau}{3m}B, \tag{41}$$

or

$$E_H = \frac{2j}{3ne}B. \tag{42}$$

Marking scheme

	Content	points
1	The Joule-Lenz law (1)	0.25
2	The specific heat power (2)	0.25
3	Formulae (3)	0.25
4	Final result (4)	0.25
5	Equation of motion (5)	0.25
6	Path (6)	0.25
7	Average speed (7)	0.25
8	Vector of the average velocity (8)	0.25
9	Current density (9)	0.5
10	Specific conductivity (10)	0.5
11	Kinetic energy of electrons (11)	0.5
12	Total heat transferred (12)	0.5
13	Equation of motion (13)	0,25
14	Equations of motion (14)-(16)	0,25
15	Velocity (23)	0,25
16	Velocity (24)	0,25
17	Expansion of the velocity (25)	0,25
18	Displacement (26)	0,25
19	Average speed (27)	0.5
20	Final result (29)	2*0.5
21	The correct face stated	0.5
22	Equations of motion (30)-(32)	0.5
23	Velocity (39)	0.5
24	Velocity (40)	0.5
25	The Hall electric field strength (41)	0.5
26	The Hall electric field strength (42)	0.5

Theoretical Question 3

1 [1 point] The constant C is found from the condition that the total number of particles is equal to N:

$$\sum_{n=1}^{\infty} N_n = N . \tag{1}$$

Substituting the expression for the Boltzmann distribution function and obtaining summation, we get

$$\begin{gathered} N=\sum_{n=1}^{\infty} N_n=\sum_{n=1}^{\infty} C\exp\left(-n\frac{\varepsilon}{k_B T}\right)=C\frac{\exp\left(-\frac{\varepsilon}{k_B T}\right)}{1-\exp\left(-\frac{\varepsilon}{k_B T}\right)} \quad \Rightarrow \\ N_n=N\frac{1-\exp\left(-\frac{\varepsilon}{k_B T}\right)}{\exp\left(-\frac{\varepsilon}{k_B T}\right)}\exp\left(-n\frac{\varepsilon}{k_B T}\right) \end{gathered} \tag{2}$$

2 [3 points] The internal energy of the gas is a sum of the kinetic energies of all atoms:

$$\begin{gathered} U=\sum_{n=1}^{\infty} E_n N_n=\sum_{n=1}^{\infty} Cn\varepsilon\exp\left(-n\frac{\varepsilon}{k_B T}\right)=C\frac{\exp\left(-\frac{\varepsilon}{k_B T}\right)}{\left(1-\exp\left(-\frac{\varepsilon}{k_B T}\right)\right)^2}= \\ =N\frac{\varepsilon}{1-\exp\left(-\frac{\varepsilon}{k_B T}\right)} \end{gathered} \tag{3}$$

In the classical limit $k_B T >> \varepsilon$, the argument of the exponent is small, it is thus justifiable to use the approximate formula $\exp\left(-\frac{\varepsilon}{k_B T}\right)\approx 1-\frac{\varepsilon}{k_B T}$. In this case, we obtain

$$U = N\,k_B T . \tag{4}$$

At low temperatures, the exponent itself is small, $\exp\left(-\frac{\varepsilon}{k_B T}\right)<<1$, hence

$$U=N\frac{\varepsilon}{1-\exp\left(-\frac{\varepsilon}{k_B T}\right)}\approx N\varepsilon\left(1+\exp\left(-\frac{\varepsilon}{k_B T}\right)\right). \tag{5}$$

3 [3 points] The molar heat capacity at fixed volume is found as

$$C_V=\frac{\partial U}{\partial T}. \tag{6}$$

In the most general case we derive

$$C_V = \frac{\partial U}{\partial T} = \frac{N_A\varepsilon}{\left(1-\exp\left(-\frac{\varepsilon}{k_BT}\right)\right)^2}\exp\left(-\frac{\varepsilon}{k_BT}\right)\frac{\varepsilon}{k_BT^2} = R\left(\frac{\varepsilon}{k_BT}\right)^2\frac{\exp\left(-\frac{\varepsilon}{k_BT}\right)}{\left(1-\exp\left(-\frac{\varepsilon}{k_BT}\right)\right)^2}. \qquad (7)$$

In order to approximate expressions in two limiting cases it is easier to use the expansions deduced in Subproblem 2. In the high temperature limit, we get

$$\begin{aligned} &k_BT >> \varepsilon \\ &U = N_A k_BT \quad \Rightarrow \quad C_V = R \end{aligned}. \qquad (8)$$

i.e. the molar heat capacity is a constant. Here N_A is the Avogadro constant, $N_A k_B = R$ stands for the universal gas constant.

At law temperatures,

$$\begin{aligned} &U = N_A\varepsilon\left(1+\exp\left(-\frac{\varepsilon}{k_BT}\right)\right) \quad \Rightarrow \\ &C_V = N_A\varepsilon\frac{\varepsilon}{k_BT^2}\exp\left(-\frac{\varepsilon}{k_BT}\right) = R\left(\frac{\varepsilon}{k_BT}\right)^2\exp\left(-\frac{\varepsilon}{k_BT}\right) \end{aligned}. \qquad (9)$$

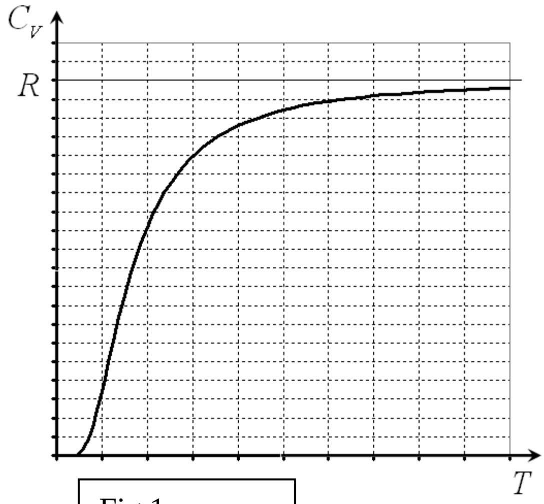


Fig.1

It is seen that the molar heat capacity goes to zero as the temperature vanishes. The schematic plot is drawn in figure 1.

4 [3 points] Calculation of the gas pressure can be conducted in different ways. For example, the average force exerted on the wall by a single atom is equal to the ratio of the moment transferred to the time interval between two consecutive collisions,

$$\langle f_n \rangle = \frac{\Delta p}{\Delta \tau} = \frac{2mv_n}{2\, L/v_n} = \frac{mv_n^2}{L} = 2\frac{E_n}{L}. \qquad (10)$$

To determine the pressure it is necessary to summarize those forces

$$P = \frac{\sum_n N_n \langle f_n \rangle}{S} = \frac{2}{SL}\sum_{n=1}^{\infty} N_n E_n = 2\frac{U}{V}. \qquad (11)$$

Substituting the formula for the internal gas energy (3), we obtain

$$P = 2\frac{N}{V}\frac{\varepsilon}{1-\exp\left(-\frac{\varepsilon}{k_BT}\right)}. \qquad (12)$$

In the two limiting cases the above obtained expressions for the internal energy should be used.

At $k_BT >> \varepsilon$

$$P = 2\frac{N}{V}k_BT, \qquad (13)$$

i.e. the pressure is proportional to the absolute temperature.

At low temperatures, we have

$$P = 2\frac{N\varepsilon}{V}\left(1+\exp\left(-\frac{\varepsilon}{k_B T}\right)\right). \qquad (14)$$

At temperatures going to zero, the pressure tends to a constant value

$$P_0 = 2\frac{N\varepsilon}{V}. \qquad (15)$$

The schematic plot of the pressure against the temperature is shown in figure 2.

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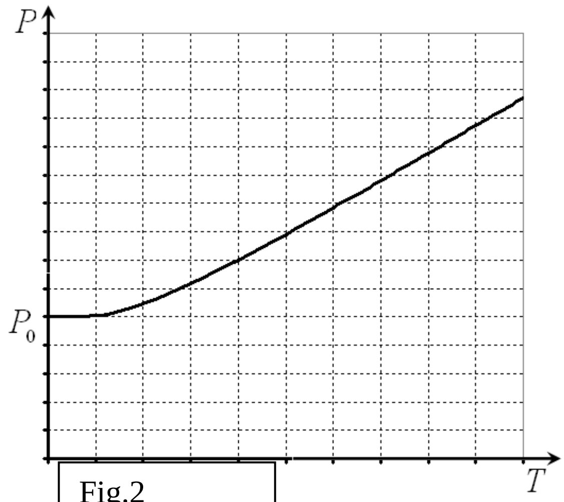


Fig.2

Marking scheme

№	Contents	points	
1	Normalizing condition (1)	0,5	**1**
2	Calculation of the number of particles (2)	0,5	
3	General expression for the internal energy U	0,5	
4	Calculation of the internal energy U (3)	1,0	**3**
5	Calculation of the classical limit of U (4)	0,5	
6	Calculation of the low temperature limit of U (5)	1,0	
7	General expression for the molar heat capacity C_V (6)	0,5	
8	Calculation of the molar heat capacity C_V (7)	1,0	
9	Calculation of the classical limit of C_V (8)	0,5	**3**
10	Calculation of the low temperature limit of C_V (9)	0,5	
11	Schematic plot for C_V	0,5	
12	General expression for average force (10)	0,5	
12	General expression for P (11)	0,5	
13	Calculation of the pressure P (12)	0,5	
14	Calculation of the classical limit of P (13)	0,5	**3**
15	Calculation of the low temperature limit of P (14)	0,5	
16	Schematic plot P	0,5	

EXPERIMENTAL COMPETITION

17 January, 2011

Please read the instructions first:

1. The Experimental competition consists of one problem. This part of the competition lasts 3 hours.
2. Please only use the pen that is provided to you.
3. You can use your own non-programmable calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet and additional papers***. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*.** Write only inside the bordered area.
6. Fill the boxes at the top of each sheet of paper with your country (***Country*),** your student code (***Student Code***), the question number (***Question Number***), the progressive number of each sheet **(*Page Number***), and the total number of ***Writing sheets*** (***Total Number of Pages***). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
7. At the end of the exam, arrange all sheets for each problem in the following order:
   - Used ***Writing sheets*** in order;
   - The sheets you do not wish to be evaluated
   - Unused sheets and the printed question.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper or equipment out of the room

Deformation, Hysteresis, and Bistability

Instruments and equipment: rubber cord, wooden stand with the column, wooden ruler, 6 weights by 100 g each, measuring tape, thread, and pins.

Part 1. Stretching (4.5 points)

1.1 Fix the rubber cord to one side of the wooden column with the pins provided (Fig. 1). Measure the dependence of the cord length L on the gravity force of hanging weights. Carry out your measurements in two different ways:
loading process, i.e. increasing step-by-step the number of weights from 0 to 6;
unloading process, i.e. decreasing step-by-step the number of weights from 6 to 0.

1.2 Plot in the same graph the relative elongations of the rubber cord as a function of the gravity force of hanging weights at loading and unloading processes.

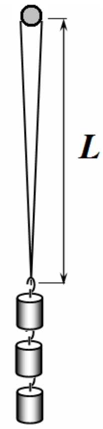


Fig. 1

Part 2. Equilibrium (7.5 points)

Place the ruler near the vertical wall of the column as shown in Fig. 2. Attach the rubber cord to the top of the ruler with the pins provided. Attach the other end of the cord to the column with the pins. The cord length l_0 in the unstrained state should be about 8 cm. The distance from the lower edge of the ruler to the top of the vertical column of the stand (see Fig. 2) is equal to the ruler length. Tie a thread to the top of the ruler so that the weights could be hanged on.

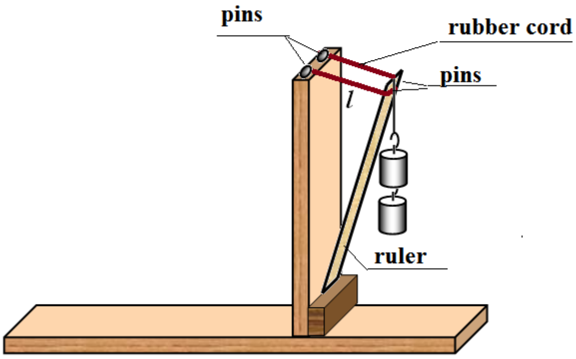


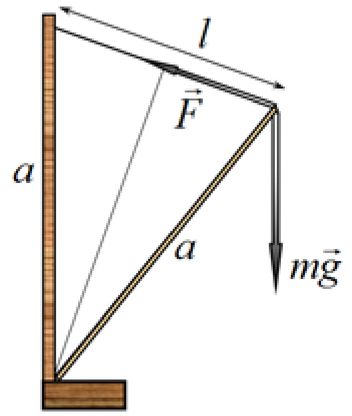
Fig. 2

Attention! *You will have to measure the length of the cord l with different number of hanging weights.*

Attaching the rubber cord, keep both sides to have the same length. For the sake of security wrap the rubber cord around the pins several times. Tie a knot at the end of the thread, attach it to the ruler upper side with the pins, and then put it over the upper edge of the ruler.

All measurements should be carried out very carefully: if you change the number of weights, hold the ruler by hand, then slowly move it downward or move upward until it comes into equilibrium, try to avoid oscillation of the weights. You can gently move the lower edge of the ruler horizontally along the column to find a more stable position of the ruler with the weights.

Theoretical description



2.1 Show that the condition of equilibrium of the ruler fixed as described above has the form:

$$F(l) = mg\frac{l}{a}, \qquad (1)$$

where l is the length of the cord, $F(l)$ denotes the elastic force of the cord at its length l, a stands for the length of the ruler, mg is the gravity force of hanging weights.

2.2 Using the data obtained in the first part of this experimental problem, plot in the same graph the dependences of the elastic force as a function of the length of the rubber cord at loading and unloading processes. **Please, do not forget that the initial length of the cord in this part of the experiment is different from that of the previous Part 1!** In the same graph, plot 6 graphs $f(l) = mg\frac{l}{a}$ for each of the six possible number of weights.

2.3 Using these plots find values of the cord length l corresponding to equilibrium positions at various number of weights in two cases: for the loading and for the unloading processes. Plot in the same graph the cord lengths in the equilibrium position against the gravity force of hanging weights for the loading (consecutive increase in the number of weights) and unloading processes (consecutive decrease in the number of weights).

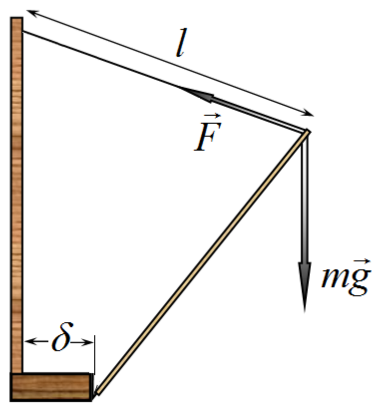
Experiment

2.4 Conduct measurements of the dependence of the cord length l in the equilibrium as a function of the gravity force of hanging weights. Measurements are to be made in two ways, i.e. for loading and unloading processes.

2.5 In the same plot you drawn in Section 2.3, plot the obtained experimental curves.

Part 3. Bistability (3 points)

If the lower edge of the ruler is shifted away from the vertical column (see Fig. 3), then at a certain length of the rubber cord and some number of weights it is possible to find two stable equilibrium positions of the ruler (bistability): the first one closer to the vertical, the second one closer to the horizon. In this Part you have to determine the conditions under which the bistability occurs.



We can show (but you do not need to do that!) that for the small displacement δ of the lower edge of the ruler, the equilibrium condition (1) is approximately described by the following formula:

$$F(l) = mg\frac{l-\delta}{a}. \qquad (2)$$

3.1 Plot schematically in the figure taken from section 2.3 such a function $f(l) = mg\frac{l-\delta}{a}$ that it represents the existence of two stable equilibrium positions and mark them on the figure. In the function $f(l)$ you can vary both parameters m and δ at your will.

3.2. **The length of the free part of the rubber cord should be the same as in Part 2.** Find experimentally two stable equilibrium positions of the ruler at the same number of weights. Find and write down at which number of weights those two stable positions really exist. Measure and write down the two lengths of the rubber cord at which these two equilibrium positions are achieved.

To search for the two equilibrium positions: a) put the ruler almost vertically and allow it to go down slightly holding by hand, b) stretch the cord until the ruler is almost in horizontal position and allow it to rise slowly holding gently by hand. Repeat these procedures several times!

When you try to find the bistability you are allowed to change slightly the free length of the rubber cord. If this length is changed, please, measure it and write down its new value (in cm).

SOLUTIONS FOR EXPERIMENTAL COMPETITION

Part 1

The measurement results are shown in Table 1. The last column gives the calculated values of the strain $\varepsilon = (l - l_0)/l_0$. The plot of the rubber cord length relative extension against the weights is shown in Fig. 1; 1 weight = 100 g.

Number of weights, N	L, cm	ε
0	20,8	0,00
1	23,1	0,11
2	28,1	0,35
3	37,2	0,79
4	49,7	1,39
5	61,3	1,95
6	71,2	2,42
5	65,2	2,13
4	55,3	1,66
3	42,1	1,02
2	31,5	0,51
1	24,6	0,18
0	20,8	0,00

Table 1

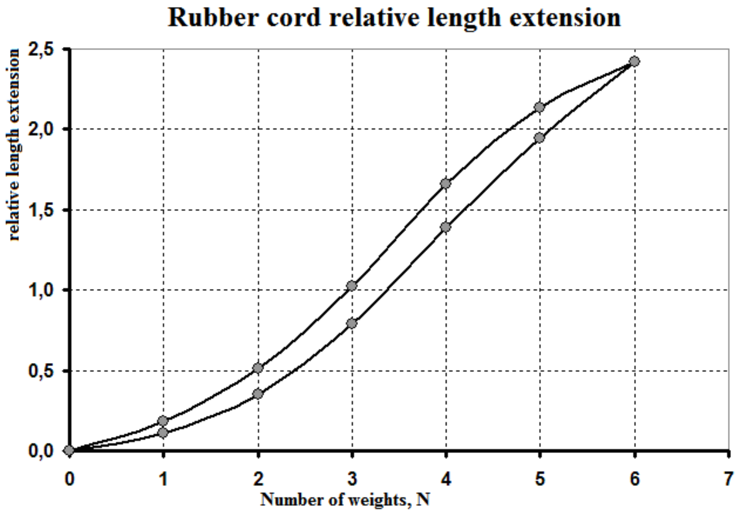


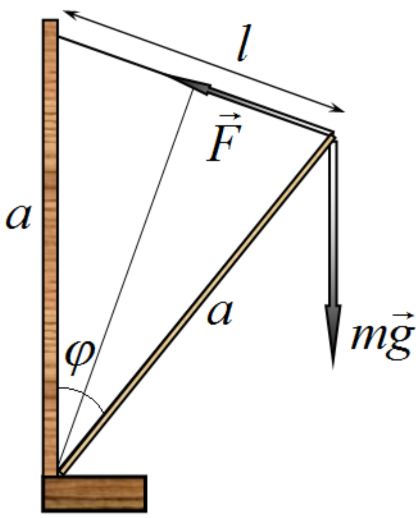
Figure 1

Part 2

2.1 The equilibrium condition follows from equating the torques of the elastic and the gravity forces written as

$$F(l)\,a\cos\frac{\varphi}{2} = mga\sin\varphi,$$

where φ is the angle of deviation from the vertical. Taking into account that $\sin\varphi = 2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}$ and $2a\sin\frac{\varphi}{2} = l$, we obtain the condition (1).



2.2 When plotting these graphs it is necessary to renormalize the length of the rubber cord for each value of the stretching force: $l = \frac{l_0}{L_0}L$. The curves of the functions $f(l) = mg\frac{l}{a}$ are straight lines passing through the origin. Required drawing is carried out in Fig. 2.

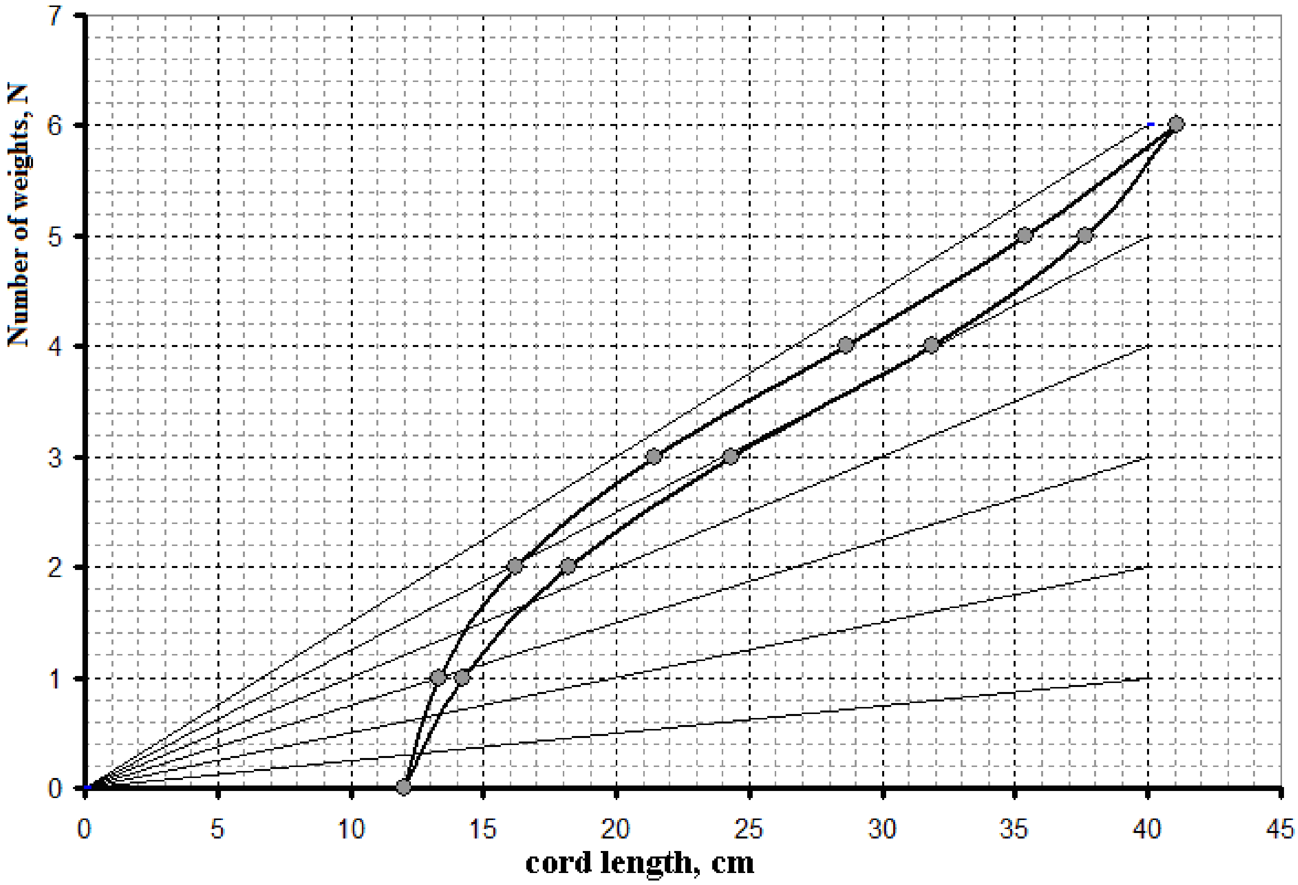


Figure 2

Equilibrium positions correspond to the intersection points of the graphs.

Table 2 represents the equilibrium positions found both with the help of Fig. 2 and obtained experimentally. Fig. 3 shows the corresponding curves, a fairly good agreement is achieved.

F, N	L, exp.	L, theory
0	11,6	12,0
1	11,7	12,5
2	12,0	12,7
3	13,1	13,2
4	14,6	14,0
5	18,4	16,5
6	51,0	47,0
5	48,7	40,0
4	18,2	17,0
3	14,6	14,0
2	13,0	13,5
1	11,8	12,5
0	11,6	12,0

Table 2

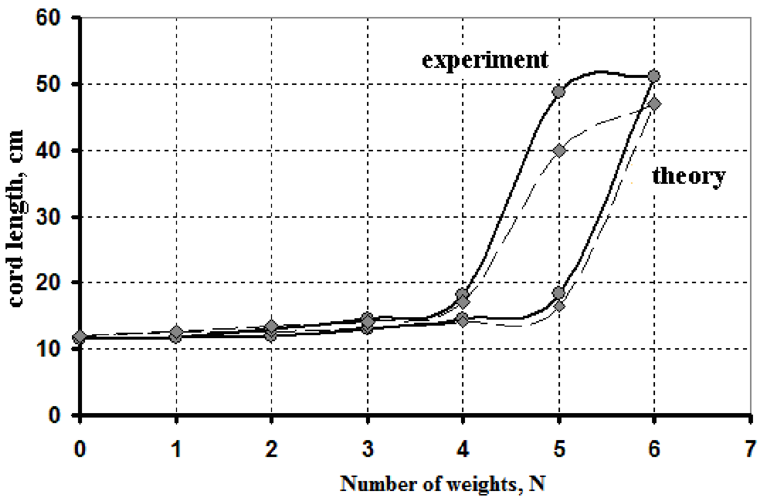


Figure 3

Part 3. Bistability.

The bistability is possible when the straight line $f(l) = mg\dfrac{l-\delta}{a}$ has three intersection points with the plot of the elastic force. A possible graph is shown in Fig. 4. Note that the intermediate position of equilibrium is unstable!

With the equipment provide the bistability is clearly observed at 5 or 6 weights (500 or 600 g).

The equilibrium positions correspond to the rubber cord lengths 13 - 14 cm and 36 - 44 cm.

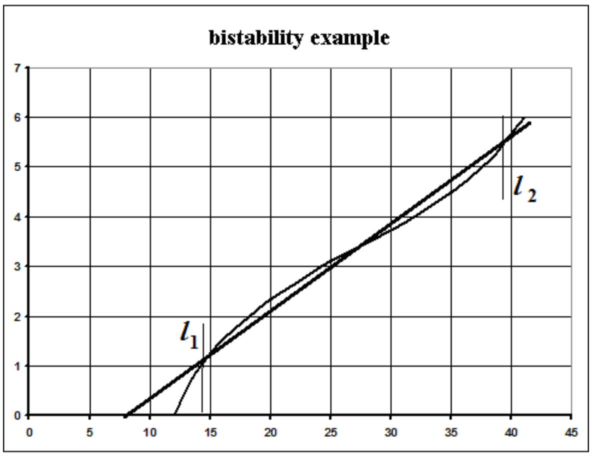


Figure 4

Marking scheme

	Content	Points
	Часть 1. Растяжение.	**4,5**
1.1	Measurements: - measurements made (12 points); - the hysteresis loop obtained; - measurement made with the accuracy less than 20% - measurement made with the accuracy less than 30%	 1 0,5 1,0 (0,5)
1.2	The plot of the unit elongation against the gravity of hanging weights: - calculation of the unit elongation for all experimental points; - the graph plotted (the axes are named and ticked; the points are placed according to the table; the approximate curve is drawn)	1 1
	Part 2. Equilibrium	**7,5**
2.1	The proof for the equilibrium position: - the torque of the gravity force; the torque of the elastic force; - geometrical relations and trigonometric transformations;	 0,5 0,5
2.2	The theoretical calculation of the cord length: - fornula for calculation of the cord length from the unit elongation; - calculation for all values of the elastic force; - the graph of the elastic force against the cord length (the axes are named and ticked; the points are placed according to the table; the approximate curve is drawn); - 6 straight lines drawn;	 0,5 0,5 0,5 0,5
2.3	-the values are taken from the graph (12 points); - the theoretical curve of the cord length against the gravity force (he axes are named and ticked; the points are placed according to the table; the approximate curve is drawn);	0,5 0,5

2.4	Measurements: - measurements conducted (10 points); - the hysteresis loop is obtained (differences at 4 and 5 weights); - measurement conducted with the accuracy less than 20%; - measurement conducted with the accuracy less than *50%;*	 1 0,5 1,0 *(0,5)*
2.5	The experimental graph drawn: - the points are placed according to the table; - the approximate curve is drawn.	 0,5 0,5
	Part 3. Bistability	**3**
3.1	The qualitative explanation of the bistability: - the approximate curve of the gravity force against the cord length; - the shifted straight line is drawn to have three intersection points; - the stable equilibrium positions are determined;	 0,5 0,5 0,5
3.2	the stable equilibrium positions are found experimentally: - at the weight 500 g (or 600 g is possible); - the cord lengths at equilibrium positions are obtained in the intervals stated above	 1 0,5

Problem 1 (10 points)

This task consists of three parts which are not related to each other.

Task 1.A. 2012 (4 points)

One end of the rigid weightless rod is pivotally attached. To the other end by the thread thrown over the weightless block, mass m is hung. Two more masses of 20m and 12m are hung by the threads to the rod at the points that divide it into three equal parts (see Fig.). All the threads are inextensible and weightless. The rod is hold in a horizontal position and then it is released. Find the acceleration of all the weights immediately after the release of the rod.

Task 1.B. And diodes ... (2.5 points)

On a separate form you can find a graph of the current-voltage characteristics of one diode (dependence of the current through the diode on voltage on it). Five of same diodes are connected as shown in Figure. Plot a graph of the current in the circuit of the voltage source U, if the latter varies from 0 to 3 V.

To plot, use a separate form given to you.
Do not forget to return it!

Task 1.C. A flat lens (3.5 points)

Round transparent plane-parallel plate of the thickness h is made from the material which optical refractive index depends on the distance r to the central axis of the plate according to the law

(1)

where n0, beta are known positive constants. The plate is in the air whose refractive index n=1.
On the axis of the plate at a distance a (a<<h) the point light source S is places. Show that the plate is a kind of lens, that is, it forms an image of the source. Determine at what distance from the plate the image of the source occurs. What is the focal length of this lens?

Problem 2

Adventures of a piston (10 points)

An open cylindrical vessel of the height H=30.0 cm and cross-sectional area S=50.0 cm2 is filled by air under normal conditions, i.e. at the atmospheric pressure p0=1.01 x 10 5 Pa and the temperature T=273 K. The thin top heavy piston of mass M=50.0 kg is gently inserted into the vessel. The vessel wall and the piston are made of a material that conducts heat very poorly. Assume that air is an ideal diatomic gas with an average molar mass mu=29.0 g/mol, the acceleration of free fall is g= 9.80 m/s2, and the universal gas constant is equal to R= 8.31 J/(mol K). Heat capacity of the piston and the vessel, as well as the friction on the walls of the piston is completely ignored.

The piston is released. The process of transition to the final balance is done in two stages. In the first stage the piston oscillates. These gas processes cannot be considered in equilibrium. Because of the non-equilibrium, fluctuations of the piston are damped, i.e. mechanical energy is dissipated. Consider that half of the dissipated energy is transferred to the gas in the vessel and the other half to the atmosphere. At this stage it is also possible to neglect the thermal conductivity of the vessel and the piston. After the oscillations stop, the piston stops at some height H1.

The second stage is slow, i.e. during some period of time the piston moves and finally stops at the some height H2.

2.1 [0.5 points] What is the air pressure p1 in the vessel at the end of the first stage? Express the reply in terms of atmospheric pressure p0, adiabatic index gamma, and the parameter alpha. Find the numerical value of p1.
2.2 [1.5 points] What is the temperature T1 at the end of the first stage? Express the reply in terms of T0, gamma, and alpha. Find the numerical value of T1.
2.3 [0.5 points] Find the height H1. Express reply in terms of H, gamma, and alpha. Find the numerical value of H1.
2.4 [0.5 points] What is the air pressure p2 in the vessel at the end of the second stage? Express the reply in terms of p0 and alpha. Find the numerical value of p2.
2.5 [0.5 points] What is the temperature T2 at the end of the second stage?
2.6 [0.5 points] Find the height H2. Express the reply in terms of H and alpha. Find the numerical value of H2.
2.7 [2 points] Find the frequency of small oscillations omega around the equilibrium position of the piston, assuming that the process is quasi-static and adiabatic. Express the reply in terms of g, H, gamma, and alpha. Find the numerical value of omega.

At the end of the second stage, a big number of small holes are made in the bottom

of the vessel, with the total area of holes S0=5.00 10-4. The size of each hole is much smaller than the mean free path of molecules. After some time the piston starts to move with a constant velocity u.
It is known that the average number of molecules N hitting the unit surface area per unit time is equal to

(1)

where nu is the so called mean thermal velocity of the molecules, R is the universal gas constant. The average kinetic energy of translational motion of molecules falling into the holes is

(2)

where kB is the Boltzmann constant.

Considering that the flow of heat through the walls and piston is negligible, answer the following questions:

2.8 [1 point] The final air pressure under the piston is of the form p3=A f(alpha), where A is a constant that depends on p0, and f(alpha) is a function of alpha. Find A and f(alpha). Find the numerical value of p3.
2.9 [2 points] The final velocity of the piston is given by u = B g (alpha), where B is a constant depending on d, S, R, T0, and mu, while g(alpha) is a function of alpha. Find B and g(alpha). Find the numerical value of u.
2.10 [1 point] The final temperature of the gas under the piston is of the form T3=C h(alpha), where C is a constant that depends on T0, and h(alpha) is a function of alpha. Find C and h(alpha). Find the numerical value of T3.

Problem 3
Nuclear droplet (10 points)

In this task, we consider the main characteristics and conditions for the stability of atomic nuclei. Let the atomic nucleus contains A nucleons (A is atomic weight of elements), namely, Z protons (Z is element's number in the table of chemical elements) and N=A-Z neutrons. The expression for the total energy of the nucleus can be written as

(1)

where M is the mass of nucleus, mp is the mass of the free proton, mn is the mass of the free neutron, c is the speed of light, and Ep is the potential energy of the nucleons in the nucleus.
The potential energy of nucleon-nucleon interaction can be described by the following semi-empirical formula by Weizsäcker

(2)

where MeV MeV MeV MeV.

Weizsacker semiempirical formula corresponds to one of the simplest model of the atomic nucleus, the so-called spherical liquid drop model, which rely on the analogy between the nucleus and drop an ordinary liquid. The mass and charge of the nucleus assumed to be uniformly distributed inside a sphere of some radius, and the nucleon fluid is characterized by some parameter sigma, which is an analogue of the surface tension of the liquid.

The formula for potential energy Ep takes into account the following contributions:

• surface energy, which takes into account the surface tension of nuclear matter in the liquid drop model;
• the energy of the Coulomb repulsion of the protons within the nucleus;
• the exchange interaction energy, reflecting the trend towards the stability of nuclei at N=Z;
• direct dependence on the number of nucleons A due to nuclear forces.

Also in the derivation of this semiempirical formula Weizsäcker used experimentally established the following dependence of the radius of the atomic nucleus of the number of nucleons

(3)

where R0 is a constant.
Based on all the above, give answers to the following questions:

3.1 [2 points] Find the electrostatic energy EC of a sphere of radius R, uniformly charged with total charge Q. Express answer in terms of charge Q, the dielectric constant epsilon0, and radius R of the ball.
3.2 [1 point] Find the numerical value of the coefficient R0 in the formula (3).
3.3 [1 point] Find the numerical value of the density rho_m of nuclear matter.
3.4 [1 point] Find the numerical value of the surface tension sigma of liquid nucleons.

Suppose now that the nucleus is broken into two parts with atomic weights kA and (1-k)A respectively, where 0<k<1. We can assume that the nuclear charge and the number of neutrons are distributed between the fragments as the atomic weight.
3.5 [2 points] Nuclear fission becomes energetically favorable under the condition Z2/A > f(k). Find an expression for the function f(k) and plot it schematically.
3.6 [0.5 points] Wit the accuracy up to two significant figures, find the limiting value (Z2/A)0 at which the spontaneous fission is theoretically possible.
Under the condition of § 3.5, the nucleus can stay for a long time. For example, the half-life of Uranium-235 nucleus is equal to 713 million years. Consequently, an instantaneous fission is prevented by some energy barrier, which disappears at some critical value. In fact, the nucleus will be broken when a significant deviation in its shape from spherical occurs.

For simplicity, we assume that a spherical nucleus undergoes such deformations under which the surface is the surface of a prolate ellipsoid of rotation, which in Cartesian coordinates by the equation

(4)

where a is small, and b is big semi-axes of the ellipsoid, respectively.

The volume of a prolate spheroid is given by

(5)

and its surface area can be calculated by the formula

(6)

Let a spherical nucleus undergoes such a deformation that b=R(1+epsilon) and a=R(1-lambda), so that epsilon, lambda <<1, and R is initial radius of the nuclear droplet.

3.7 [0.5 points] Find the relation between epsilon and lambda.
Calculations show that the energy of the electrostatic interaction of protons of the deformed nucleus is about Ecdeformed = .
3.8 [2 points] Find the expression and the numerical value of (Z2/A)_critical.

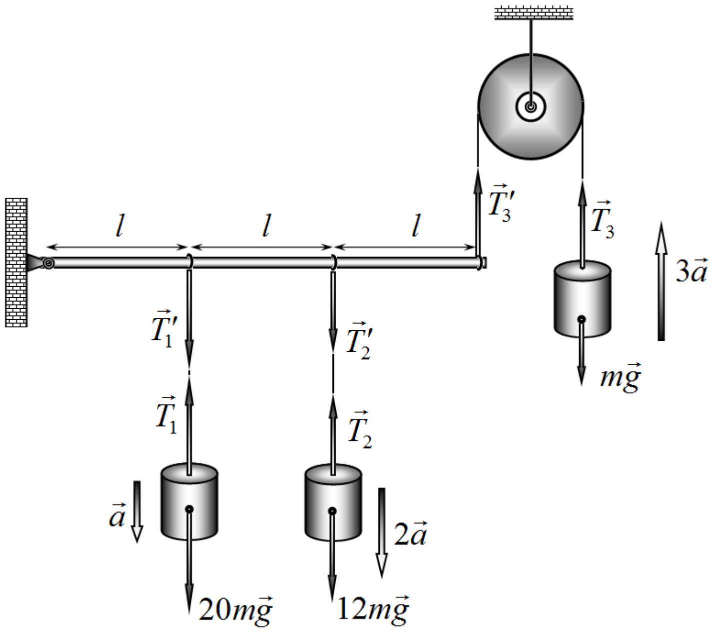
Known physical constants:
Elementary charge
The dielectric constant
The mass of the nucleon (proton or neutron) k
1eV in J

In addressing these tasks, you can use the formulas:

SOLUTIONS OF PROBLEMS OF THEORETICAL TOUR
Problem 1. (10 points)

Problem 1A. 2012

Forces acting on the system are shown in Fig. Since the hard core and non-tensile yarn, acceleration loads are as 1:2:3. The sum of the torques acting on a weightless rod is zero. These two conditions together with the equations of the 2nd law of Newton for all goods provide a system of equations



$$\begin{cases} 20ma = 20mg - T_1 \\ 12m \cdot 2a = 12mg - T_2 \\ m \cdot 3a = T_3 - mg \\ T_1 l + T_2 \cdot 2l = T_3 \cdot 3l \end{cases} \qquad (1)$$

from which we find

$$a = \frac{41}{77} g . \qquad (2)$$

However, this value of acceleration is obtained second load more free-fall acceleration. This means that the second thread of the load will not be tight, but its acceleration is equal to *g*. You can also show that the formal system of equations (1) that, for which the thread cannot be. Therefore, the thread to which the suspended second load, the rod does not work. Therefore, the system of equations (1) incorrectly describes the device under consideration. To calculate the acceleration of the rod and the other second load should be deleted.

Valid values for the acceleration of the first and third loads are from the following system of equations

$$\begin{cases} 20ma = 20mg - T_1 \\ m \cdot 3a = T_3 - mg \\ T_1 l = T_3 \cdot 3l \end{cases} . \qquad (3)$$

Finally, we obtain $a = \frac{17}{29} g$

$$a_1 = \frac{17}{29} g , \qquad a_2 = g , \qquad a_3 = \frac{51}{29} g . \qquad (4)$$

Grading scheme of Problem 1A.

№		points
1	Figure with all the forces	**0,4**
2	The system of equations (1)	**0,8**
3	The solution of (1) to speed up (2)	**0,4**
4	Exception 4 second load from a consideration of	**0,3**
5	Proof of exceptions (second acceleration load greater tensile strength filament is negative)	**0,7**
6	Acceleration second load is equal to g	**0,4**
7	The system of equations (3)	**0,6**
8	A solution of (3) for the acceleration (4)	**0,4**
	Total	**4,0**

Task 1.B. And diodes

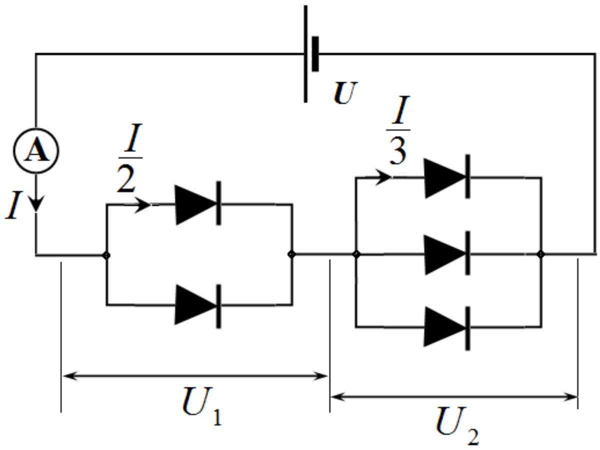
We denote the voltage on a pair of parallel-connected diodes by U_1 and on three of the diodes by U_2. The total current in the circuit I can be found in two ways:

- A double value of the current through one diode pair:

$$I = 2I_0(U_1); \tag{1}$$

- Three times the value as a force in one of the three diodes:

$$I = 3I_0(U_2). \tag{2}$$



We construct the graphs of $2I_0(U)$ and $3I_0(U)$. It's enough to “multiply” the graph of the $I_0(U)$ by appropriate factor, i.e. for several values of the voltage on the schedule to remove the corresponding values of the forces of currents, multiply them by 2 and 3 and apply the appropriate point on the graphs. When connected in series the total voltage is the sum of the stresses in some parts of the circuit, so

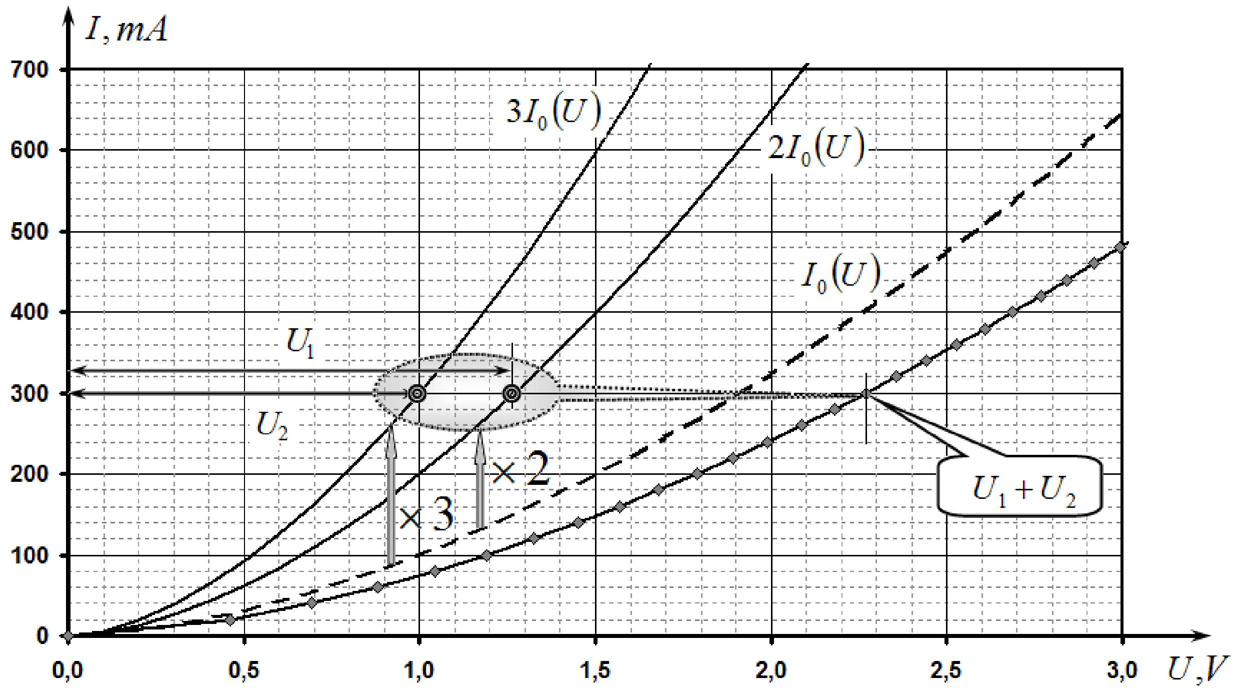
$$U_1 + U_2 = U. \tag{3}$$

Graphically, this condition corresponds to a “horizontally summing” graphs $2I_0(U)$ and $3I_0(U)$: for a given value of current and voltage values U_1 and U_2 are read, and their sum is then the value is applied to the chart.

Note that the formal solution of the problem can be written as (for the inverse functions):

$$U(I) = U_0\left(\frac{I}{2}\right) + U_0\left(\frac{I}{3}\right),$$

where $U_0(I)$ — the inverse of graphically given functions $I_0(U)$.



Grading scheme of Problem 1B

№	Contents of number scores	Points
1	In a parallel connection of power currents are added	**0,3**
2	Plotting functions $2I_0(U)$ and $3I_0(U)$	**0,6**
3	If you are connected in parallel are added voltage	**0,3**
4	"Horizontal" summation	**0,7**
5	are calculated for: • 3 points; • 6 points.	 **0,3** **0,6**
6	Alternatives to (realized ideas): • To approximate the dependence; • To solve the equations explicitly;	 **0,4** **0,4**
	Total	**2,5**

Tasks 1.C. Flat lens

Plate will form an image S' if the optical path length $l = SABS'$ for any ray of light, emerging from the source, and refract in plate, will be the same for all rays (tautochronism condition for lens).

Consider a beam incident on the plate at a distance from its axis. We assume that $r << a$, ie, we use the paraxial approximation. Distance $|SA|$ found using the Pythagorean theorem and make the approximation $r << a$, given that:

$$|SA| = \sqrt{a^2 + r^2} = a\sqrt{1 + \frac{r^2}{a^2}} \approx a\left(1 + \frac{1}{2}\frac{r^2}{a^2}\right), \tag{1}$$

Similarly, the distance is expressed $|BS'|$

$$|BS'| = \sqrt{b^2 + r^2} = b\sqrt{1 + \frac{r^2}{b^2}} \approx b\left(1 + \frac{1}{2}\frac{r^2}{b^2}\right). \tag{2}$$

Thus, the optical path length $SABS'$ is equal to

$$\begin{aligned} l = |SA| + n(r)h + |BS'| &= a\left(1 + \frac{1}{2}\frac{r^2}{a^2}\right) + n_0\left(1 - \beta r^2\right)h + b\left(1 + \frac{1}{2}\frac{r^2}{b^2}\right) = \\ &= a + n_0 h + b + \left(\frac{1}{2a} + \frac{1}{2b} - n_0\beta h\right)r^2 \end{aligned} \tag{3}$$

This value does not depend on r (i.e., the same for all rays) for the vanishing of the factor

$$\frac{1}{2a} + \frac{1}{2b} - n_0\beta h = 0, \tag{4}$$

which can be rewritten as

$$\frac{1}{a} + \frac{1}{b} = 2n_0\beta h. \tag{5}$$

This expression is identical in form with a thin lens formula

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{F}, \tag{6}$$

where F is focal length.

Comparing (5) and (6), we find the focal length of the plate

$$F = \frac{1}{2n_0\beta\, h}. \tag{7}$$

From formula (5) also found the distance from the plate to the image:

$$b = \frac{a}{2n_0\beta h a - 1}. \tag{8}$$

The alternative is a geometrical optics approximation.

This problem, in principle, can be solved in the framework of geometrical optics. The main stages (of the very complex solutions) are:

- Use the law of refraction by Snelius;
- The choice of an arbitrary beam and determine the angle of incidence on the plate;
- Angle of the beam after refraction at the front edge of plate;
- Obtaining a differential equation for the beam path inside the plate;
- Solution of this equation in the quadratic approximation;
- The angle at the exit of the plate (should be negative);
- The angle after refraction on the rear face;
- Determination of the distance to the intersection with the axis of the plate;
- Proof of the constancy of the distance for all the rays;
- Getting the lens formula;
- Write the formula for the focal length.

Grading scheme of Problem 1C

№		points
1	Formulation of basic idea: the constancy of the propagation time for all the paths	**1,5**
2	Using the quadratic approximation (for small angles)	**0,5**
3	Calculation of distances $\lvert SA\rvert$ and $\lvert BS'\rvert$ - The exact formula; - Expansion of the approximate formula;	**0,2** **0,4**
4	The optical path length (3)	**0,2**
5	A formula of a thin lens (4)	**0,3**
6	Focal length of lens (7)	**0,2**
7	Distance to the image (8)	**0,2**
	Total	**3,5**

Grading for geometrical optics approximation (alternative solution)

№		points
1	The Law of refraction	**0,1**
2	Two small-angle approximation (but quadratic)	**0,3**
3	The starting angle of the plate	**0,1**
4	The differential equation for the ray trajectory in a plate	**0,5**
5	The solution to the quadratic approximation	**0,5**
5	The value of the angle near the back edge	**0,2**
6	The value of the angle after refraction on the rear face	**0,1**
7	The point of intersection with the optical axis	**0,2**
8	Persistence of distance b for all the refracted rays	**0,1**
9	formula analogous to the thin lens	**0,2**
10	The formula for the focal length	**0,2**
	Total	**3,5**

Problem 2
Adventures of a piston (10 points)

2.1. [0.5 points] From the equilibrium condition of the piston, we find pressure of the gas

$$p_1 = p_0 + \frac{Mg}{S} = p_0(1+\alpha) = 1.99\times 10^5 \text{ Pa} \quad (1)$$

2.2 and 2.3. [2 points] In the first stage the gas is compressed and heats up to a certain temperature. Because the vessel wall and the piston are made of a material that conducts heat poorly, gas compression can be assumed to be adiabatic, but the process is not equilibrium and we cannot use the adiabatic equation. In the transition from the initial to the final state of the system piston+gas by external forces (gravity and atmospheric pressure) have made the work

$$A = Mg(H-H_1) + p_0S(H-H_1) = (Mg + p_0S)(H-H_1) \quad (2)$$

By hypothesis, only half of this work is to increase the internal energy of the gas

$$\Delta U = \frac{A}{2} \quad (3)$$

Where

$$\Delta U = \frac{\nu R}{\gamma-1}\left(T_1 - T_0\right) \quad (4)$$

Here, ν is the number of moles, R is the universal gas constant. We write the equation of state of ideal gas for the initial and final states

$$p_0SH = \nu RT_0 \quad (5)$$

$$\left(p_0 + \frac{Mg}{S}\right)SH_1 = \nu RT_1 \quad (6)$$

Solving system of equations (2) — (6), we obtain

$$T_1 = T_0\left(1+\frac{\gamma-1}{\gamma+1}\frac{Mg}{p_0S}\right) = T_0\left(1+\frac{\gamma-1}{\gamma+1}\alpha\right) = 317 \text{ K}, \quad (7)$$

$$H_1 = \frac{H}{(1+Mg/p_0S)}\left(1+\frac{\gamma-1}{\gamma+1}\frac{Mg}{p_0S}\right) = \frac{H}{(1+\alpha)}\left(1+\frac{\gamma-1}{\gamma+1}\alpha\right) = 17.7 \text{ cm}, \quad (8)$$

2.4. [0.5 points] As the piston continues to be in equilibrium, the pressure

$$p_2 = p_0 + \frac{Mg}{S} = p_0(1+\alpha) = 1.99\times 10^5 \text{ Pa}. \quad (9)$$

2.5. [0.5 points] After a sufficiently long period of time the gas temperature inside the vessel will be equal to the ambient temperature, i.e., becomes equal to

$$T_2 = T_0 = 273 \text{ K}. \quad (10)$$

2.6. [0.5 points] The height H_2 is found by (9) and (10), as well as the equation of state of gas

$$H_2 = \frac{p_0S}{p_0S+Mg}H = \frac{H}{1+\alpha} = 15.2 \text{ cm} \quad (11)$$

2.7. [2 points] Adiabatic equation of the form

$$pV^{\gamma} = const \quad (12)$$

we obtain

$$dp = -\gamma p \frac{dV}{V}. \tag{13}$$

Let piston has deviated from its equilibrium position at a small height x, then by (13) is equal to the pressure change

$$\delta p = -\gamma p_2 \frac{x}{H_2} = -\gamma \frac{\left(p_0 S + Mg\right)^2}{p_0 S^2 H} x. \tag{14}$$

The equation of motion of the piston can be written as

$$M\ddot{x} = -\delta p S = -\gamma \frac{\left(p_0 S + Mg\right)^2}{p_0 S H} x \tag{15}$$

whence we obtain the frequency of small oscillations

$$\omega = \left(p_0 S + Mg\right)\sqrt{\frac{\gamma}{p_0 SHM}} = (1+\alpha)\sqrt{\frac{\gamma g}{\alpha H}} = 13.5 \text{ Hz}. \tag{16}$$

2.8. [1 point] When moving with constant velocity piston continues to be in equilibrium, so the pressure

$$p_3 = p_0 + \frac{Mg}{S} = p_0(1+\alpha) = 1.99\times 10^5 \text{ Pa}, \tag{17}$$

that is

$$A = p_0, \quad f(\alpha) = 1+\alpha \tag{18}$$

2.9 and 2.10. [3 points] Suppose a vessel to establish certain temperature. There should be must be a balance in the number of particles and energy. The law of conservation of particles is given by

$$\frac{p_0 + \frac{Mg}{S}}{k_B T_3} uS = \frac{p_0 + \frac{Mg}{S}}{k_B T_3}\sqrt{\frac{8k_B T_3}{\pi m}} S_O - \frac{p_0}{k_B T_0}\sqrt{\frac{8k_B T_0}{\pi m}} S_O. \tag{19}$$

For the law of conservation of energy it is necessary to consider not only kinetic but also the rotational energy of each molecule. Therefore, the total energy carried by each molecule is

$$W_{tot} = \overline{W} + W_{rot} = 2k_B T + k_B T = 3k_B T \tag{20}$$

then the energy conservation law can be written as

$$(p_0 S + Mg)u = \frac{p_0 + \frac{Mg}{S}}{k_B T_3}\sqrt{\frac{8k_B T_3}{\pi m}} 3k_B T_3 S_O - \frac{p_0}{k_B T_0}\sqrt{\frac{8k_B T_0}{\pi m}} 3k_B T_0 S_O \tag{21}$$

Solving (18) and (19), we finally obtain

$$u = \frac{6S_O}{S}\sqrt{\frac{2RT_0}{\pi\mu}}\left((\alpha+1)\sqrt{4+2\alpha+\alpha^2} - 2 - 2\alpha - \alpha^2\right) = 1.91\times 10^{-3} \text{ m/s}, \tag{22}$$

that is

$$B = \frac{6S_O}{S}\sqrt{\frac{2RT_0}{\pi\mu}}, \quad g(\alpha) = (\alpha+1)\sqrt{4+2\alpha+\alpha^2} - 2 - 2\alpha - \alpha^2, \tag{23}$$

and temperature

$$T_3 = T_0\left(5 + 4\alpha + 2\alpha^2 - 2(\alpha+1)\sqrt{4+2\alpha+\alpha^2}\right) = 116 \text{ K}, \tag{24}$$

that is

$C = T_0$, $h(\alpha) = 5 + 4\alpha + 2\alpha^2 - 2(\alpha+1)\sqrt{4 + 2\alpha + \alpha^2}$. (25)

Grading scheme of Problem 2

N		points	
2.1	Formula (1)	0,25	**0,5**
	Numerical value of p_1	0,25	
2.2	Formula (2)	0,25	**1,5**
	Formula (3)	0,25	
	Formula (4)	0,25	
	Formulas (5) and (6)	0,25	
	Formula (7)	0,25	
	Numerical value of T_1	0,25	
2.3	Formula (8)	0,25	**0,5**
	Numerical value of H_1	0,25	
2.4	Formula (9)	0,25	**0,5**
	Numerical value of p_2	0,25	
2.5	Formula (10)	0,25	**0,5**
	Numerical value of T_2	0,25	
2.6	Formula (11)	0,25	**0,5**
	Numerical value of H_2	0,25	
2.7	Formula (12)	0,25	**2,0**
	Formula (13)	0,25	
	Formula (14)	0,5	
	Formula (15)	0,5	
	Formula (16)	0,25	
	Numerical value of ω	0,25	
2.8	Formula (18) for A	0,25	**1,0**
	Formula (18) for $f(\alpha)$	0,25	
	Formula (17)	0,25	
	Numerical value of p_3	0,25	
2.9	Formula (19)	0,25	**2,0**
	Formula (20)	0,5	
	Formula (21)	0,25	
	Formula (23) for B	0,25	
	Formula (23) for $g(\alpha)$	0,25	
	Formula (22)	0,25	
	Numerical value of u	0,25	
2.10	Formula for (25) for C	0,25	**1,0**
	Formula for (25) for $h(\alpha)$	0,25	
	Formula (24)	0,25	
	Numerical value of u	0,25	
Total			**10,0**

Problem 3
Nuclear droplet (10 points)

3.1 [2 points] We calculate the total electrostatic energy of the protons in the nucleus. Within the droplet model of the nuclear charge Ze is uniformly distributed inside a sphere of radius R, so that its bulk density is the same everywhere and equal to

$$\rho_q = \frac{3Q}{4\pi R^3}. \tag{1}$$

Using the Gauss theorem, we find the electric field inside and outside the ball

$$E(r)4\pi r^2 = \frac{1}{\varepsilon_0}\rho_q \frac{4\pi}{3} r^3 \tag{2}$$

$$E(r)4\pi r^2 = \frac{1}{\varepsilon_0}\rho_q \frac{4\pi}{3} R^3. \tag{3}$$

Hence we get

$$E(r) = \begin{cases} \dfrac{\rho_q r}{2\varepsilon_0}, & r \le R \\ \dfrac{\rho_q R^3}{2\varepsilon_0 r^2}, & r > R \end{cases}. \tag{4}$$

Full electrostatic energy given by the integral

$$E_C = \int_0^\infty w 4\pi r^2 dr = \int_0^\infty \frac{\varepsilon_0 E^2}{2} 4\pi r^2 dr = \frac{3Q^2}{20\pi\varepsilon_0 R}. \tag{5}$$

3.2 [1 point] From (5), $Q = Ze$ and $R(A) = R_0 A^{1/3}$ we see that the electrostatic energy corresponds to the third term in the Weizsacker semiempirical formula, so

$$a_3 \frac{Z^2}{A^{1/3}} = \frac{3Z^2 e}{20\pi\varepsilon_0 R_0 A^{1/3}} \tag{6}$$

whence

$$R_0 = \frac{3e}{20\pi\varepsilon_0 a_3} = 1.2\times 10^{-15} \text{ m.} \tag{7}$$

3.3 [1 point] The density of nuclear matter is given by

$$\rho_m = \frac{3Am}{4\pi R^3} = \frac{3m}{4\pi R_0^3} = 2.3\times 10^{17} \text{ kg/m}^3. \tag{8}$$

3.4 [1 point] The surface energy depends on surface tension

$$E_{sur} = \sigma S = 4\pi\sigma R^2 = 4\pi\sigma R_0^2 A^{2/3}. \tag{9}$$

We conclude that the surface energy corresponds to the second term of the semi-empirical formula Weizsäcker

$$4\pi\sigma R_0^2 A^{2/3} = e a_2 A^{2/3} \tag{10}$$

whence

$$\sigma = \frac{e a_2}{4\pi R_0^2} = 1.5\times 10^{17} \text{ N/m.} \tag{11}$$

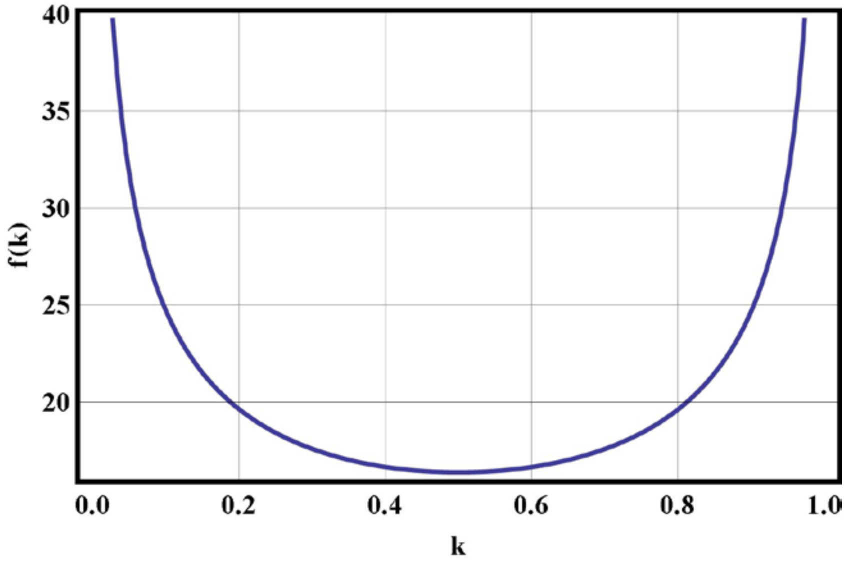
3.5 [2 points] Nuclear fission becomes energetically favorable only if the potential energy of the nuclei decreases, that is,

$$E_p(A,Z) - E_p(kA,kZ) - E_p((1-k)A,(1-k)Z) > 0 \qquad (12)$$

which yields

$$\frac{Z^2}{A} > f(k) = -\frac{a_2\left(1-k^{2/3}-(1-k)^{2/3}\right)}{a_3\left(1-k^{5/3}-(1-k)^{5/3}\right)}. \qquad (13)$$

Graph of the function $f(k)$ is presented below.



3.6 [0.5 points] The function $f(k)$ is symmetric with respect to the point $k = 0.50$, so at this point and the minimum, which corresponds to

$$(Z^2/A)_0 = 16. \qquad (14)$$

3.7 [0.5 points] Since the core is treated as a liquid, its volume should not change. Using the formula for the volume of an ellipsoid and the fact that $\varepsilon, \lambda \ll 1$ we obtain

$$V = \frac{4\pi}{3}R^3\left(1+\varepsilon-2\lambda\right) = \frac{4\pi}{3}R^3 \qquad (15)$$

whence

$$\varepsilon = 2\lambda. \qquad (16)$$

3.8 [2 points] Based on Taylor's formula for small deformations of the nucleus, taking into account (16) the surface area of the liquid increases by

$$\Delta S = \frac{32}{5}\pi R^2\lambda^2 = \frac{32}{5}\pi R_0^2 A^{2/3}\lambda^2 \qquad (17)$$

and a corresponding increase in surface energy is equal to

$$\Delta E_{surf} = \sigma\Delta S = \frac{32}{5}\pi\sigma R_0^2 A^{2/3}\lambda^2. \qquad (18)$$

Coulomb interaction energy of the protons is decreased by the

$$\Delta E_C = \frac{3Z^2e^2}{120\pi\varepsilon_0 R}\varepsilon(\varepsilon+\lambda) = \frac{3Z^2e^2}{20\pi\varepsilon_0 R_0 A^{1/3}}\lambda^2. \qquad (19)$$

Nucleus is unstable at the condition

$$\Delta E_C > \Delta E_{surf} \qquad (20)$$

whence

$$(Z^2/A)_{crtitcal} = \frac{128\pi^2\varepsilon_0\sigma R_0^3}{3e^2} = 37. \qquad (21)$$

Grading scheme of Problem 3

N		points	
3.1	Formula (1)	0.5	**2.0**
	Formula (1)	0.5	
	Formula (1)	0.5	
	Formula (1)	0.5	
3.2	Formula (6)	0.5	**1.0**
	Formula (7)	0.25	
	Numerical value of R_0	0.25	
3.3	Formula (8)	0.75	**1.0**
	Numerical value of ρ_m	0.25	
3.4	Formula (9)	0.25	**1.0**
	Formula (10)	0.25	
	Formula (11)	0.25	
	Numerical value of σ	0.25	
3.5	Formula (12)	0.5	**2.0**
	Formula (13)	0.5	
	Plot: X axis symbol indicated	0.25	
	Plot: Y axis symbol indicated	0.25	
	Plot: k shown from 0 to 1	0.25	
	Plot: appropriate values on plot	0.25	
3.6	Appropriate value of k	0.25	**0.5**
	Appropriate value of $(Z^2/A)_0$	0.25	
3.7	Formula (15)	0.25	**0.5**
	Formula (16)	0.25	
3.8	Formula (17)	0.5	**2.0**
	Formula (18)	0.25	
	Formula (19)	0.25	
	Formula (20)	0.5	
	Formula (21)	0.25	
	Numerical value of $(Z^2/A)_{crtitcal}$	0.25	
Total			**10.0**

EXPERIMENTAL COMPETITION

18 January, 2012

Please read the instructions first:

The Experimental competition consists of one problem. This part of the competition lasts 3 hours.

Please only use the pen that is provided to you.

You can use your own non-programmable calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.

You are provided with Writing sheet and additional papers. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the Writing sheets. Please use as little text as possible.

You should mostly use equations, numbers, figures and plots.
Use only the front side of Writing sheets. Write only inside the bordered area.

Fill the boxes at the top of each sheet of paper with your country (Country), your student code (Student Code), the question number (Question Number), the progressive number of each sheet (Page Number), and the total number of Writing sheets (Total Number of Pages). If you use some blank Writing sheets for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.

At the end of the exam, arrange all sheets for each problem in the following order:

Used Writing sheets in order;

- The sheets you do not wish to be evaluated
- Unused sheets and the printed question.
- Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper or equipment out of the room

Electric currents in volume [15 points]

Instruments and equipment: Vessel 250 ml, two steel spokes, multimeter, power supply (4.5 V battery), resistor with a resistance of several kΩ, two-pole switch, plastic tube for a cocktail, connecting wires, clean water, plastic cup, rubber, ruler, and adhesive tape.

You should be familiar with the formula for calculating the resistance of a thin cylindrical conductor,

$$R = \rho \frac{l}{S}, \qquad (1)$$

where ρ is the specific resistance of the conducting material, l is its length, S is its cross-sectional area.

However, when an electric current flows in a bulk system, trajectories of charged particles may be different, so the resistance of the medium depends on the nature of the distribution of electric currents. You have to measure the electric resistance of the water layer when the current flows between the spokes submerged in water.

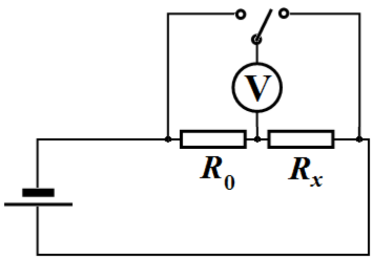
From time to time clean the spokes with rubber.
After each series of experiments, change water in the vessel.
Do not apply strong current through the water because this leads to appearance of a large number of ions in that can "spoil" the results of your experiments.

Straighten the spokes. In all the experiments try to maintain the spokes in parallel to each other.

Part 1. [0.5 points]

1.1 Measure the resistance of the given resistor R_0 by using the multimeter. Record the result of your measurement.

To measure the resistance of the water between the spokes, use an electric circuit shown in the figure on the right. Here, R_0 is the given resistor, R_x is the resistance of the water between the spokes to be measured.



1.2 For this circuit write down the formula which will be used by you to calculate the water resistance R_x between the spokes.

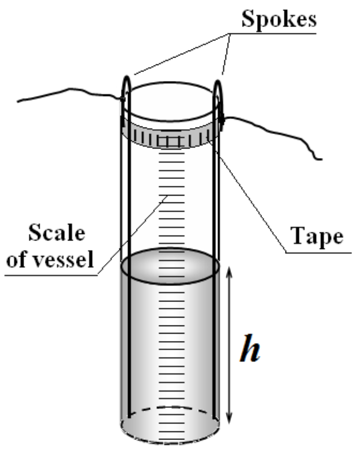
In the following, please use only the circuit above to obtain the resistance R_x!
In any case do not measure water resistance directly by the multimeter. Use multimeter only as a voltmeter.

Direct measurement of the resistance with a multimeter leads to a rapid discharge of its battery, and in addition, the voltage of the battery is rather high (9 V) to change the electrical properties of water.

Error analysis is not required in this Competition.

Part 2. [5 points]

Put the two spokes provided along the vessel walls keeping them in parallel at a maximum distance from each other. Bend the upper ends of the spokes over the edges of the vessel. Additionally fix them with strips of adhesive tape.
To measure the height of the water level use the scale engraved on the vessel. Determine in millimeters the grating period of the scale engraved on the vessel wall.



2.1 Measure dependence of the water resistance between the spokes as a function of the height of water poured into the vessel. Plot the obtained dependence in a graph.

2.2 Draw schematic streamlines of electric current in this case (make two draws: one in the plane of the spokes and the other in the plane perpendicular to the spokes).

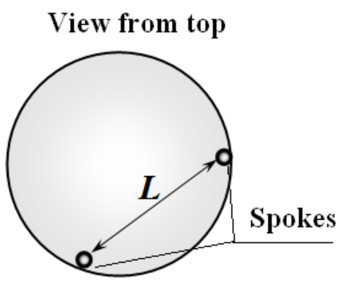
Streamlines of electric current are trajectories of charged particles.

2.3 On the basis of physical considerations about the nature of this current flow, make an assumption about the form of the obtained experimental dependence $R(h)$, and write it as a formula.

2.4 Using the linearization method, check the validity of the made assumption by plotting a graph in such coordinates that it becomes a straight line $y = ax + b$. Determine the numerical values of the parameters of the linearized dependence, as well as the unknown values of parameters entering the function $R(h)$.

Part 3. [3.5 points]

Fix one spoke close to the wall inside the vessel. Pour 200 ml of water into the vessel. Move the second spoke along the wall, changing the distance between the spokes. The depth of immersion of the second spoke should be maximal. Spokes should be positioned vertically and in parallel to each other.



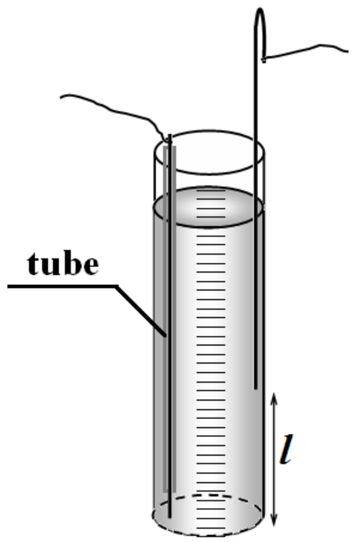
3.1 Measure the water resistance between the spokes as a function of the distance L between them. Plot the obtained dependence.

3.2 On the basis of physical considerations about the nature of this current flow, make an assumption about the form of the obtained experimental dependence $R(L)$, and write it as a formula.

3.3 Using the linearization method, check the validity of the made assumption by plotting a graph in such coordinates that it becomes a straight line $y = ax + b$. Determine the numerical values of the parameters of the linearized dependence, as well as unknown values of parameters entering the function $R(L)$.

Part 4. [4 points]

Put one spoke into the given plastic tube so that its lower end of a length of about 1 cm remains uncovered by the tube. Fill the vessel with water about to top end. The upper edge of the tube should be placed above the water level, and fixed to the spoke by the tape. Put the second uncovered spoke into the water keeping its lower end at different heights. Keep the spokes in parallel at a maximum distance from each other.



4.1 Measure the water resistance between the spokes as a function of the height l of the end of the second spoke. Plot the obtained dependence in a graph.

4.2 Draw a schematic streamlines of electric current in this case (in the plane of the spokes).

4.3 On the basis of physical considerations on the nature of current flow, make an assumption about the form of the obtained experimental dependence $R(l)$, and write it as a formula.

4.4 In the plot, specify the range of values of l in which the assumed formula for $R(l)$ is confirmed experimentally.

4.5 Determine the numerical values of the parameters entering the assumed formula for $R(l)$.

Part 5. [2 points]

5.1 On the basis of the above obtained experimental data (please choose which one to use) estimate the specific resistance of the water ρ.

SOLUTIONS FOR THE EXPERIMENTAL COMPETITION
Electric currents in volume

PART 1

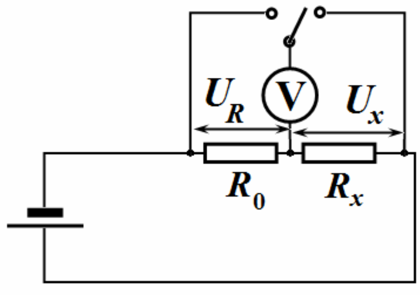
1.1 The resistance of the resistor provided is equal $R_0 = 2{,}0 \pm 0{,}1\,\text{k}\Omega$.

1.2 Since the resistorts are connected in series the same current flows through each of them, then the following relation holds

$$\frac{U_R}{R_0} = \frac{U_x}{R_x}$$

from which it follows that

$$R_x = R_0 \frac{U_x}{U_R}. \quad (1)$$



Thus, to measure an unknown resistance it is enough to measure the voltage drops on the unknown resistance and the resistor provided.

If the source voltage was stabilized, it would be sufficient to measure the voltage drop on just one of them.

PART 2

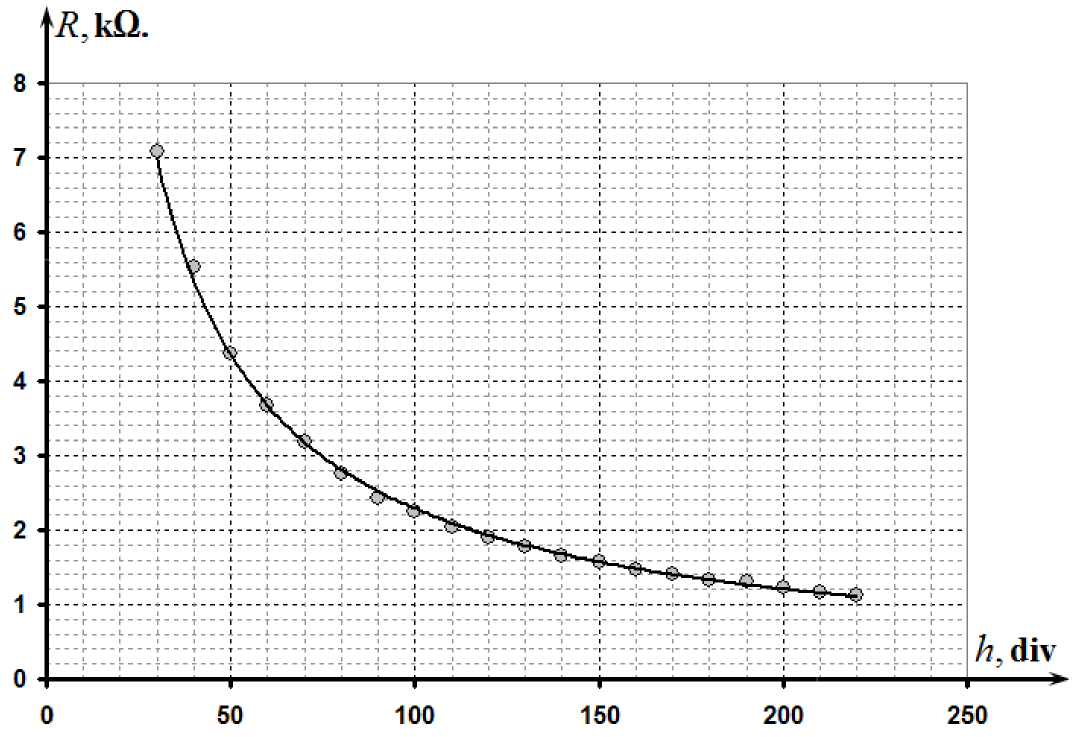
2.1 The results of measurement of the voltage drops against the height of the water poured into the vessel are presented in Table 1. This table also shows the calculated resistance of the water between the electrodes (sзokes).

Note that the height was measured by the scale of the measuring glass.

Table 1. Dependence of the resistivity on the height of the water level.

h, div.	$100/h$, div^{1}	U_x, V	U_R, V	R, kΩ	$1/R$, kΩ^{-1}
30	3,33	3,82	1,08	7,07	0,14
40	2,50	3,60	1,30	5,54	0,18
50	2,00	3,36	1,54	4,36	0,23
60	1,67	3,17	1,73	3,66	0,27
70	1,43	3,01	1,89	3,19	0,31
80	1,25	2,84	2,06	2,76	0,36
90	1,11	2,69	2,21	2,43	0,41
100	1,00	2,59	2,31	2,24	0,45
110	0,91	2,48	2,42	2,05	0,49
120	0,83	2,38	2,52	1,89	0,53
130	0,77	2,30	2,60	1,77	0,57
140	0,71	2,22	2,68	1,66	0,60
150	0,67	2,16	2,74	1,58	0,63
160	0,63	2,08	2,82	1,48	0,68
170	0,59	2,03	2,87	1,41	0,71
180	0,56	1,96	2,94	1,33	0,75
190	0,53	1,93	2,97	1,30	0,77
200	0,50	1,86	3,04	1,22	0,82
210	0,48	1,80	3,10	1,16	0,86
220	0,45	1,76	3,14	1,12	0,89

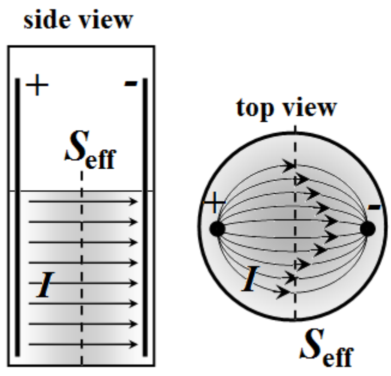
The graph of the obtained dependence is shown in the figure below.



Simple measurement can easily show that the volume $V_0 = 200\,\text{ml}$ corresponds to the height $h_0 = 170\,\text{mm}$.Therefore, the height of the water poured is calculated by the formula $h = V\dfrac{h_0}{V_0}$, i.e.the the division value of the scale is $\delta = 0{,}85\,\text{mm/ml}$.

2.2 The current distribution is schematically shown in the figure on the right.

The current flows between the lateral surfaces of the spokes, thus the height of water level determines an effective cross-sectional area.



2.3 It is therefore reasonable to assume that the resistance of water between the spokes is inversely proportional to the height of water level

$$R_{x0} = \frac{A}{h}, \qquad (2)$$

where A is a constant meaning the water resistance of the unit of height. Then, the measured resistance should be described by the formula

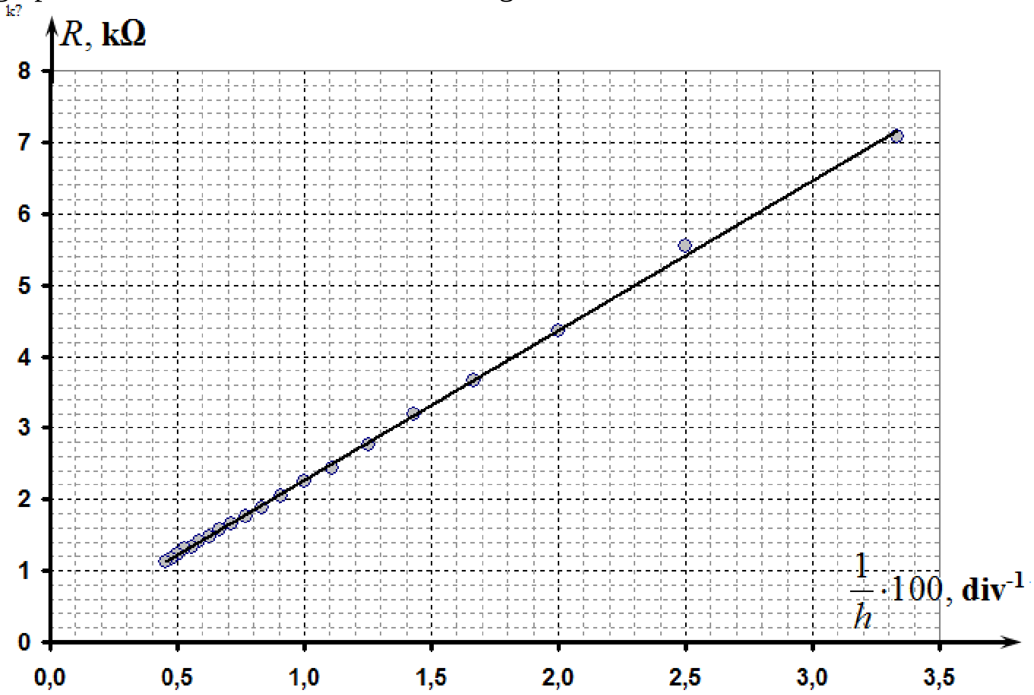
$$R_x = \frac{A}{h} + B, \qquad (3)$$

where B is a constant denoting the additional resistance (of contacts, of an oxide layer on the surface of the spokes, etc.).

2.4 To check the validity of formula (3) it is sufficient to plot the dependence of the resistance on the inversed height of the water column $1/h$. That is, the linear dependence should be observed for the following values:

$$y = R$$
$$x = \frac{1}{h}. \tag{4}$$

A graph of this function is shown in the figure below.



The parameters of this linear dependence, calculated by the mean square method

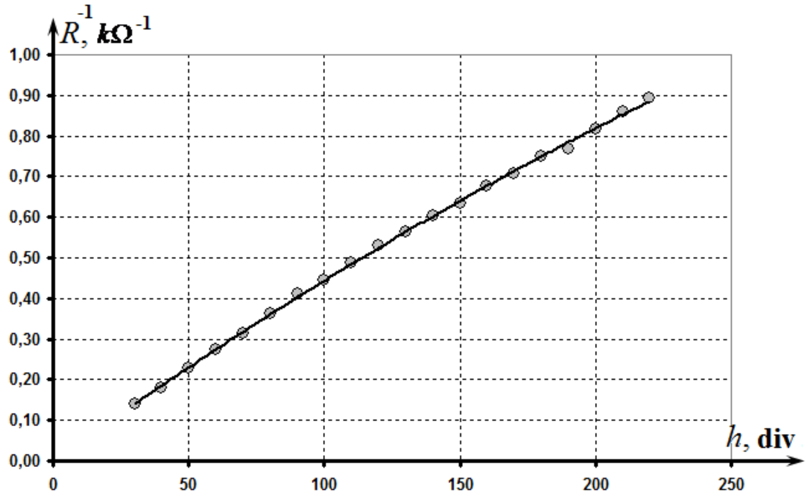
$$a = (210 \pm 3)\,\mathrm{k\Omega \cdot div}$$
$$b = (0{,}17 \pm 0{,}03)\,\mathrm{k\Omega} \tag{5}$$

To determine the parameters in relation (3) it is necessary to recalculate (5) from divisions of the scale to millimeters. Thus, we get

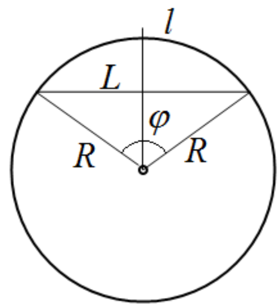
$k\Omega$

$$A = a \cdot \delta = (178 \pm 2)\,k\Omega \cdot mm$$
$$B = b = (0{,}17 \pm 0{,}03)\,k\Omega \quad . (6)$$

Note. Although it is possible to use the linearization of the type $\frac{1}{R} = \frac{h}{A}$, *but this leads to worse results, since it ignores the additional resistance of the circuit.*



PART 3



3.1 In order to measure the distance between the spokes it is easier to measure the length of the arc l between the spokes using the marks made on a strip of the adhesive tape. Then the distance between the spokes can be calculated using the geometric formula

$$L = D\sin\frac{l}{D}, \qquad (7)$$

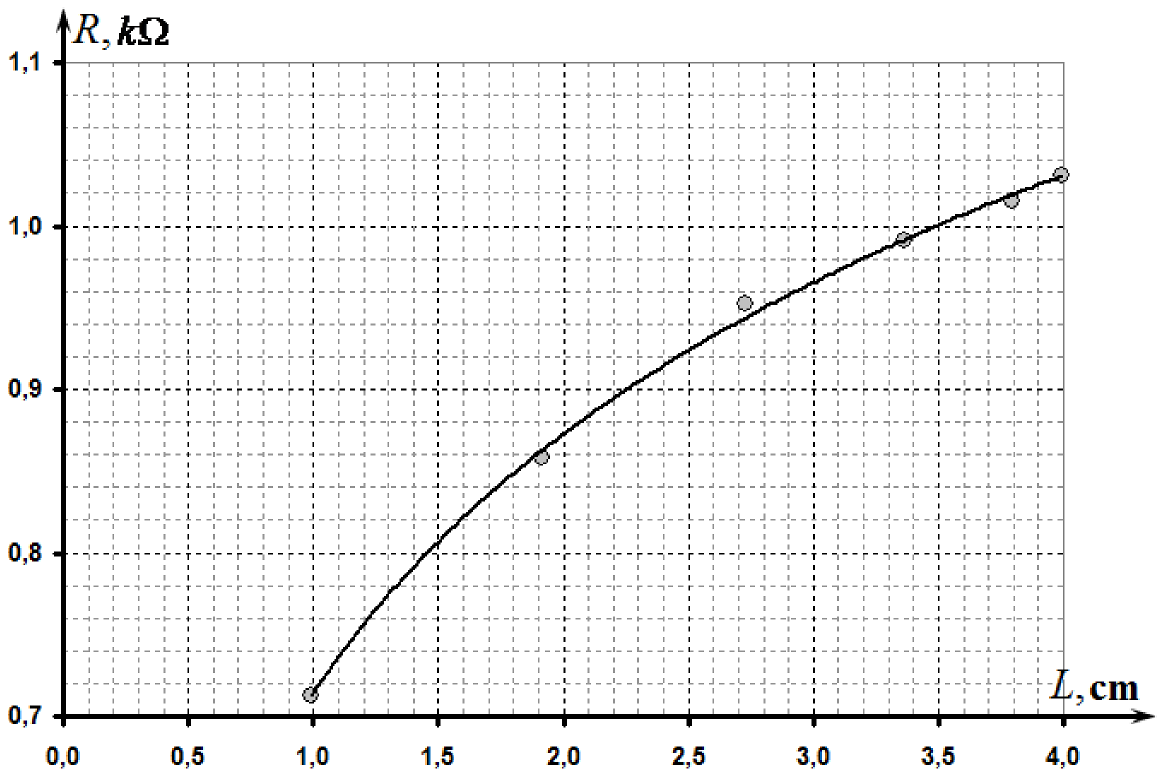
where $D = 40\text{mm}$ is the diameter of the measuring glass.

The measurement results of the water resistance on the distance between the spokes are shown in Table 2.

Table 2.

l, cm	L, cm	U_x, V	U_R, V	R, kΩ	$\ln L$
1	0,990	1,29	3,62	0,713	-0,010
2	1,918	1,45	3,38	0,858	0,651
3	2,727	1,59	3,34	0,952	1,003
4	3,366	1,62	3,27	0,991	1,214
5	3,796	1,64	3,23	1,015	1,334
6	3,990	1,65	3,2	1,031	1,384

The graph of the obtained dependence is presented in the figure below.



3.2 It is theoretically possible to show that the resistance of the medium between two long parallel electrodes in an infinite medium is given by

$$R = \frac{\rho}{\pi h}\ln\frac{L}{r_0}, \qquad (8)$$

where h is the length of the electrodes (spokes), r_0 is their radius.

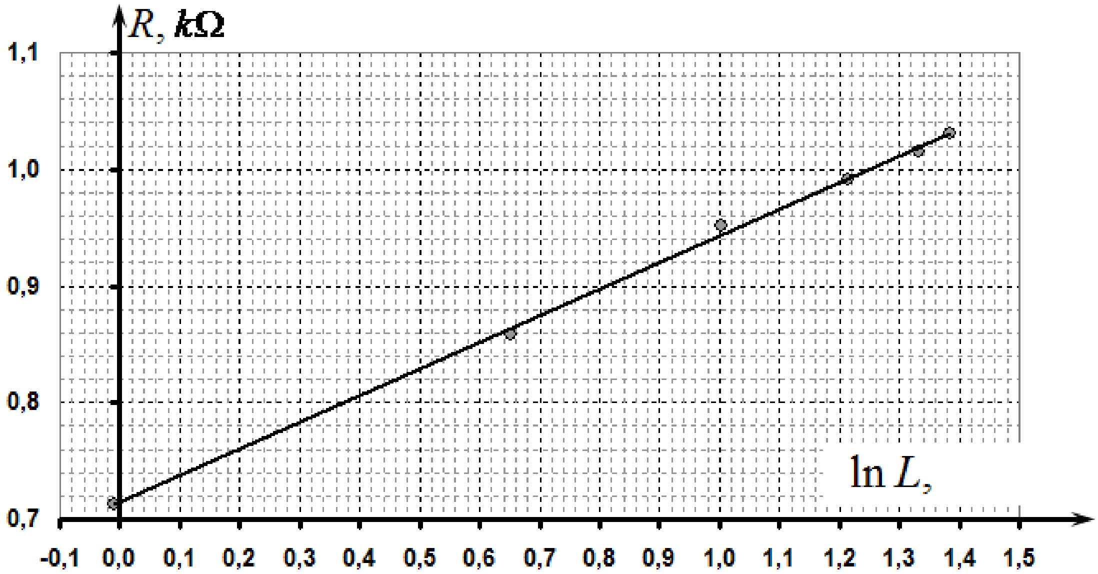
We can assume that in this case the water resistance between the electrodes depends linearly on the logarithm of the distance between them, that is,

$$R(L) = A\ln L + B\,. \tag{9}$$

3.3 To check the feasibility of (9) it is necessary to plot the dependence of the resistance on the logarithm of the distance $\ln L$. That is the linear dependence should be observed for the following values:

$$\begin{aligned} y &= R \\ x &= \ln L \end{aligned}\,. \tag{10}$$

This graph is shown in the figure below which confirms assumption (9).



The parameters of this linear dependence, calculated by the least square method, are found as follows

$$\begin{aligned} a &= (0,23 \pm 0,01)\,k\Omega \\ b &= (0,71 \pm 0,01)\,k\Omega \end{aligned} \tag{11}$$

It is obvious that the value of the parameter b depends on the unit of distance L. In this case, values in (11) correspond to the parameters in (9).

PART 4

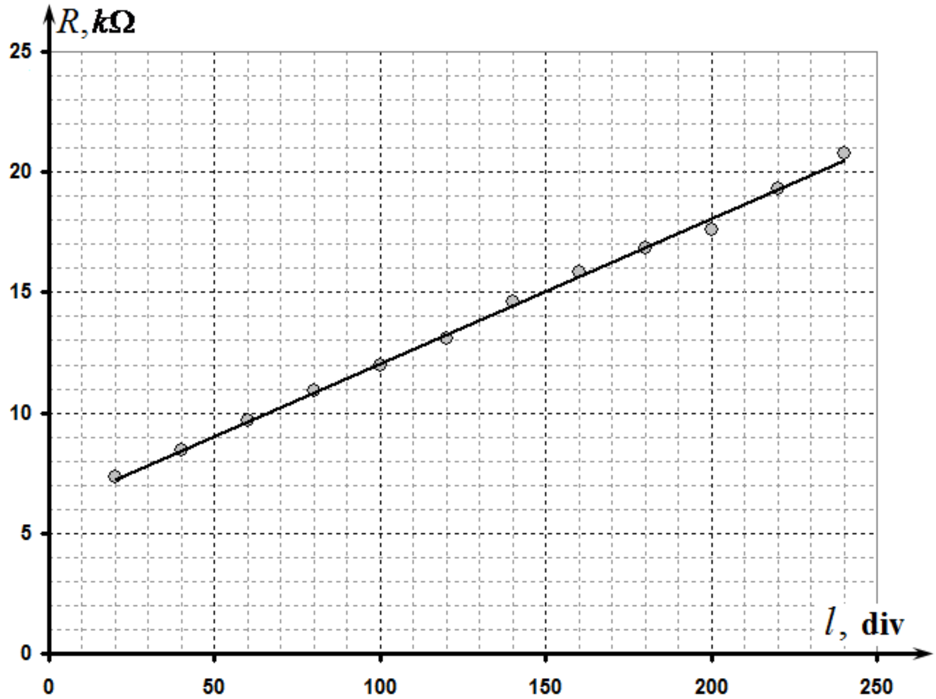
4.1 The results of the resistance measurements depending on the height of the second spoke in water are shown in Table 3. In this case, to measure the height one has to make use of the scale of the measuring glass, so as the units are milliliters.

Table 3

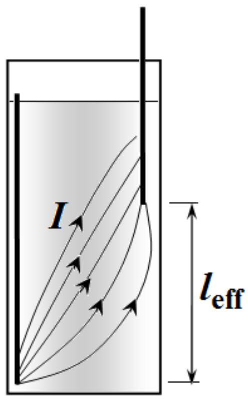
l,ml	U_x, V	U_R, V	R, kΩ
20	3,85	1,05	7,3
40	3,96	0,94	8,4
60	4,06	0,84	9,7
80	4,14	0,76	10,9
100	4,20	0,70	12,0
120	4,25	0,65	13,1
140	4,31	0,59	14,6

160	4,35	0,55	15,8
180	4,38	0,52	16,8
200	4,40	0,50	17,6
220	4,44	0,46	19,3
240	4,47	0,43	20,8

The graph of this dependence is shown in the figure below.



4.2 Approximate distribution of currents in this case is shown in the figure on the right.



4.3 In this case the distance between the spokes plays the role of the effective length of a conductor, so the water resistance between the spokes in this case is approximately linearly dependent on *l*, which is confirmed by the experimental data. Additional resistance may be due to a limitation of currents near the tips of the spokes.

Thus, this dependence is described by

$$R_x = al + b . \tag{12}$$

4.4 According to the obtained experimental data, the linear dependence holds at all distances *l*. Deviations from linearity are possible at small and large values of *l*. However, in the range of values $l \in [50, 150]$ the dependence is definetly linear.

The parameters in (12), calculated by the least square method, are found as

$$a' = (0,060 \pm 0,002)\,k\Omega\,/\,\text{div.}$$
$$b = (6,0 \pm 0,3)\,k\Omega \tag{13}$$

If the height *h* is measured in millimeters, then the valueof the coefficient *a* is

$$a = (0,071 \pm 0,002)\,\text{k}\Omega/\text{mm} . \tag{14}$$

Part 5

5.1 In order to estimate the specific water resistivity it is preferable to use data from Part 4.The linearity of the dependence obtained indicates that in the middle the stream lines are approximately parallel to the vessel walls. It is also possible to assume that in this region the current flows through the entire cross section of the vessel. Consequently, we can use the formula for the resistance presented in this problem.

The obtained value of the slope (14) is the resistance of one millimeter of the water column, which makes it possible to evaluate its specific electrical resistivity

$$a = \frac{\Delta R}{\Delta l} = \frac{\rho}{S} \quad \Rightarrow \quad \rho = aS = a\frac{V_0}{h_0}. \tag{7}$$

On substituting numerical values we get

$$\rho = a\frac{V_0}{h_0} = 0,071\frac{10^3\Omega}{10^{-3}\text{m}} \cdot \frac{200 \cdot 10^{-6}\text{m}^3}{170 \cdot 10^{-3}\text{m}} \approx 83\Omega \cdot \text{m} \tag{8}$$

Grading scheme for Experimental Competition

If the resistance measurements were carried out in an ohmmeter mode of the multimeter, all the points for the measurement results are reduced by two times!

In the table below italics indicates grading for alternative solutions.

N	Content	total	points
1.1	Resistance $R_0 = 2,0 \pm 0,1\text{k}\Omega$	0,2	0,2
1.2	Formula (1)	0,3	0,3
2.1	Measurements of the experimental data and plotting the graph (reasonable values have been obtained for the resistance in the range from 1 to 10 kΩ. Otherwise no points are given)	2,5	
	Measurements made in the range that include more than 180 divisions of the scale (max-min); in the range that includes more than 150 divisions; in the range that includes more than 100 divisions; less than 100 divisions;		0,75 (0,5) (0,25) (0)
	Number of experimental points: 10 and more; 7-9; 5-6; less than 5		0,75 (0,5) (0,25) (0)
	Resistance values are correctly calculated for each measured point		0,25
	Monotonically decreasing dependence is obtained; the resistance changes at least 5 times; resistance varies less than 5 times;		0,25 (0)
	Plot is made - size of plot is not less than 1/4 of sheet; - axes are denoted by units, digits are indicated; - all table data points are correctly drawn in a plot; - smooth line passing through the points is shown;		 0,1 0,1 0,2 0,1
2.2	Streamlines are shown in the figures	0,5	
	In the plane of the spokes: straight lines are perpendicular to the spokes (slight distortion might appear near the bottom		0,25

	and the top of the vessel);		
	In the perpendicular plane: convex symmetric lines between the electrodes filling most of the cross-section;		0,25
2.3	Form of dependence	0,8	
	- Inverse proportionality; (otherwise no points are given)		0,6
	- There is a constant component for the resistance;		0,2
2.4	Linearization and determination of parameters	1,2	
	Type of relationship: - Dependence $R(1/h)$; - Dependence $R^{-1}(h)$, or in a double logarithmic scale; - Other;		0,2 (0,1) (0)
	Plotting the graph of the linearized dependence - All points are plotted; - Smoothing line through the points is shown		0,2 0,2
	- Correct evaluation of the parameters of the linearized dependence $(200 \pm 30\%)$ kΩ·div step, or $(180 \pm 30\%)$ kΩ·mm - if the deviation is of 30% to 75% - the grade is twice less; - if there is a large deviation, no points are given) - By using Least Square Method; - By using plot (or by using all points); - By using two points;		0,4 (0,2) (0,1)
	Calculation of the height measured in units of length (mm or cm) - Measuring and calculation of the division value of the scale (correct) - Calculation of the slope (if all the previous calculations in units of length)		0,1 0,1 (0,1)
3.1	Measurements and ploting of the experimental data (reasonable values for the resistances in the range of 0.5 to 2 kΩ, otherwise no points are given)	1,5	
	Measurement of the distance between the spokes: - An arc of a circle with calculations; - Direct measurement by a ruler;		0,2 (0,1)
	- Measured in the range of 1 to 4 cm - (Otherwise nopoints are given);		0,3
	- Number of points − 6 or more; - Number of points − 4 - 5; - Less than 4		0,3 (0,2)
	An increasing convex relationship is obtained		0,2
	Plot is made - size of a plot is not less than 1/4 of sheet; - units of axes are stated, digits are indicated; - all table data points are correctly drawn in a plot; - smooth line passing through the points is shown;		0,1 0,1 0,2 0,1
3.2	Form of dependence	1	
	Logarithmic dependence There is constant contribution in the dependence;		0,7 0,3

	Some other reasonable convex increasing dependence		(0,3)
3.3	Linearization and determination of parameters	1	
	- Dependence $R(\ln L)$; - Other reasonable linearization in accordance with the formula 3.2		0,4 (0,2)
	Plotting the graph of the linearized dependence - All points are plotted; - Smoothing line is shown;		 0,1 0,1
	Parameters: (slope is in the range $a = (0,2 \pm 30\%)$ kΩ; - if the deviation of 30% to 75% the grading points are twice less; - if there is a large deviation, no points are given) - By using Least Square Method; - By using plot (or by using all points); - By using two points;		 0,4 (0,2) (0,1)
4.1	Measurements of the experimental data and plotting the graph (reasonable values have been obtained for the resistance in the range of 5 to 30 kΩ. Otherwise no points are given)	2,4	
	Measurements are made in the range of more than 180 divisions of the scale (max-min); in the range of more than 150 divisions; in the range of more than 100 divisions; less than 100 divisions;		0,75 (0,5) (0,25) (0)
	Number of experimental points: 10 and more; 7-9; 5-6; less than 5		0,75 (0,5) (0,25) (0)
	Resistance values are correctly calculated for each measured point		0,2
	Monotonically increasing dependence is obtained; there is linear part in the plot No linear part in the plot		0,3 (0,1)
	Plot is made - units of axes are stated, digits are indicated; - all table data points are correctly drawn in a plot; - smooth line passing through the points is shown;		 0,1 0,2 0,1
4.2	Sketch of streamlines	0,3	
	Lines - start from the open end of the spoke; - go straight up; - are distributed along the length of second spoke;		0,1 0,1 0,1
4.3	Form of the dependence	0,4	
	- there is term proportional to *l*; - there is a constant component; (other than linear types of dependence are not accepted);		0,2 0,2
4.4	Linearity interval	0,1	
	- interval is shown (deviations near the ends of interval measurement are allowed)		0,1
4.5	Parameters	0.8	

	- Correct evaluation of the linearized dependence: the slope ($(0,060 \pm 30\%)$ kΩ/div or $(0,07 \pm 30\%)$) kΩ/mm -if deviations are from 30% to 75% the grade pointa are twice less; - if there is a large deviation, no points are given) - By using Least Square Method; - By using plot (or by using all points); - By using two points;		0,6 (0,4) (0,2)
	- Constant component of the resistance: - In the range 4 - 10 kΩ - In the range of 2 - 4 kΩ or 10-12 kΩ; - Otherwise no grade points;		0,2 (0,1)
5.1	The calculation of the specific resistivity of water	2	
	- Dependence from Part 4 is taken; - Other reasonable dependence (with a correct formula for the resistance);		0,3 (0,1)
	Equation (7) is used to calculate the resistance by using: - The slope of the graph; - (using 1 - 2 points for calculation);		0,4 (0,2)
	The calculation of the specific resistance (80 Ω·m): - Values in the range 60-100 Ω·; - Values in the range 40-120 Ω·; - Values in the range 20 - 150 Ω·; - Otherwise zero grade points.		1,3 (1,0) (0,5)

THEORETICAL COMPETITION

January 15, 2013

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*.** Write only inside the boxed area.
6. Begin each question on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Question Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
    - Used ***Writing sheets*** in order;
    - The sheets you do not wish to be evaluated
    - Unused sheets and the printed question.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

Problem 1 (10 points)

This problem consists of three independent parts.

Problem 1.A Pulley (5,0 points)

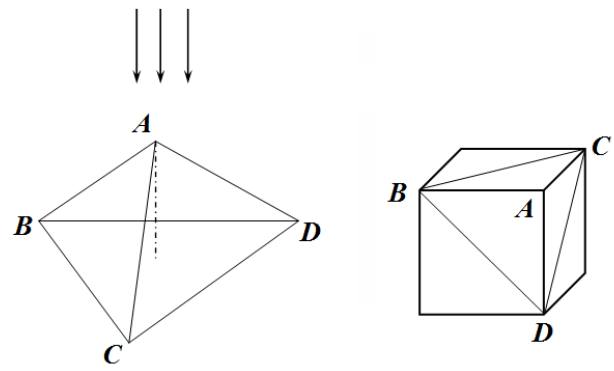
The rope of length L and mass m is put on the pulley of radius R. The rope begins to slide off the pulley without friction. Consider the time moment when the difference in height between the hanging ends of the rope is equal half of the rope length. For this time moment find the following:
1. acceleration of the rope;
2. the tension of the rope at the top of the pulley;
3. the point of the rope at which the tension is maximum (it is enough to find the angle between the radius to that point and the vertical).

Problem 1.B Assistant Vapor (2.0 points)

In the physical laboratory the thermometer reading was 20 ºC, and the barometer reading was 1 atm. The laboratory assistant Vapor pulled out a sturdy vessel of the closet, poured some water into it and sealed it with the lid. Then assistant Vapor slowly heated the vessel to the temperature 200 ºC. At this temperature, the pressure in the vessel turned out to be 2,88 atm. Find the temperature at which all the water evaporated. Justify your answer.

Problem 1.C Pyramid (3.0 points)

A regular triangular pyramid $ABCD$ is made of a transparent material with the refraction index $n = 1{,}6$. All angles at the top of the pyramid are right. The base of the pyramid forms a right triangle whose sides' lengths are equal $a = 2{,}0\ mm$. The pyramid can be thought of as "a corner cut off from the cube." The pyramid is homogeneously illuminated such that the light falls along its height, perpendicular to the base plane.

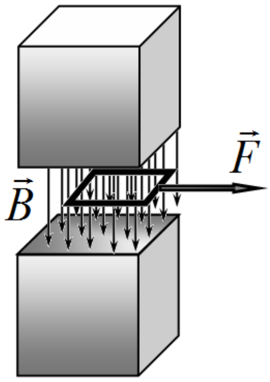


1. Draw an area at the base of the pyramid illuminated by the light refracted in the plane *ABC*.
2. A screen is placed in parallel to the pyramid base at a distance of $L = 10\ sm$ from it. Draw areas illuminated by the light refracted by the pyramid. Specify the position and sizes of these areas.

Problem 2 (10 points)

Square frame (10 points)

A square frame of side $a = 10.0\,sm$ is placed in a uniform magnetic field of $B = 1.00\times10^{-1}\,T$ generated by a permanent magnet. Completely neglect edge effects, assuming that the magnetic field between the poles of the magnet is uniform and is equal to zero outside of it. In all parts of the problem assume that at the initial time moment the frame of mass $m = 5.00\,g$ is at rest at the edge of the poles of the magnet, see figure.



1. **[0.2 points]** Redraw schematically the figure on the right to identify which of the poles of the magnet is the north N and which is the south S.

Part A

Let the frame be made of a conductive material with the resistance $R = 1.00\times10^{-2}$ ohm and a negligible inductance. At the time momnet $t = 0$ the frame is subject to the constant force $F = 1.00\times10^{-4}\,N$, which pulles the frame out of the magnetic field.

2. **[1.2 points]** Find an analytical dependence of the frame velocity $v(t)$ on time t assuming that the frame remains at all times between the poles of the magnet. Express your answer in tems of B, a, m, R and t.

3. **[0.8 points]** Write down an exact equation to determine the time interval t_0 at which the frame moves between the poles of the magnet. Express your answer in terms of B, a, m, R and t_0. Estimate t_0.

4. **[1.2 points]** Draw a graph of the frame velocity $v(t)$ on time t in the interval from 0 to 12*s*.

5. **[1.2 points]** Draw a graph of the frame current $I(t)$ on time t in the interval from 0 to 12*s*.

Part B

Let the frame be made of a superconducting material with the inductance of $L = 1.00\times10^{-1}\,H$, the same mass and sizes. At the time momnet $t = 0\,s$ the frame is subject to a constant force which pulles the frame out of the magnetic field.

6. **[1.8 points]** Find the minimum force $F_{\min}$ with which the frame can be pulled out of the magnet. Express your answer in terms of B, a, L and calculate its numerical value.

7. **[0.6 points]** At the assumptions of section 6 find the minimum time interval t_0 of the maximum departure of the frame out of the magnet. Express your answer in terms of m, B, a, L and calculate its numerical value.

8. **[1.0 points]** Draw a graph of the frame current $I(t)$ on time t in the interval from 0 to $4t_0$.

Part C

A square frame with sides a is made of a conductive material with the inductance L ,the resistance R and the mass m. At the time moment $t = 0$ the frame is given an initial velocity v_0 directed exactly as the force in the figure above. It is known that at the time moment when the frame almost leaves the magnet its velocity turns zero.

9. **[2.0 points]** Find an analytical expression for the frame current I_0 at the time moment when the frame velocity turns zero. Express your answer in terms of B, a, m, L, R and v_0.

Problem 3

Bohr's model of a hydrogen atom (10 points)

Consider a hydrogen atom, which consists of a proton, whose mass can be assumed virtually infinite, and an electron of mass $m_e = 9.11\times10^{-31}\,kg$. The proton has a positive charge of $e = 1.61\times10^{-19}$ Cl and the electron has a negative charge $-e$, so that the atom is totally neutral.

1. **[2.0 points]** It is known that most of the atoms are stable and can exist for a long time. This means that the electron being in the hydrogen atom cannot be at rest since the attraction force would inevitably make it fall onto the proton. Let the distance between the electron and the proton be equal to $r_0 = 5.00\times10^{-11}\,m$/ Assuming that the laws of classical physics are applicable, find the time t_1 that the initially motionless electron needs to fall onto the proton.

Hence, to guarantee the stability of the hydrogen atom, the electron must move around the proton like the planets revolve around the Sun. Let the electron orbit be circular and assume again that the laws of classical physics can still be applied.

2. **[1.0 points]** Find the electron velocity v as a function of the orbit radius r.
3. **[0.5 points]** Find the angular momentum L of the electron as a function of the orbit radius r.

Danish physicist Niels Bohr postulated that the angular momentum of the electron is an integer of the Planck constant $\hbar = 1.05\times10^{-34}$ Дж·с, that is $L = n\hbar$, where n is the main quantum number.

4. **[0.5 points]** Find the possible radii r_n of the electron orbits in the hydrogen atom.
5. **[0.5 points]** Calculate the numerical value of the minimum radius r_n of the electron orbit in the hydrogen atom.
6. **[1.0 points]** Find the possible values of the total energy E_n of the electron in the hydrogen atom.
7. **[0.5 points]** Calculate the numerical value of the minimum total energy E_1 of the electron in the hydrogen atom.

According to classical electrodynamics, if the electron is accelerated, it loses its energy due to electromagnetic radiation. The power of the radiation is given by

$$P = \frac{1}{6\pi\varepsilon_0}\frac{e^2a^2}{c^3},$$

where a is the electron acceleration, $\varepsilon_0 = 8.85\times10^{-12}$ *F/m* is the dielectric constant, $c = 3.00\times10^{8}$ m/s is the speed of light in vacuum.

Electromagnetic radiation would lead to the fact that in the Bohr model the electron would fall onto the proton. In the following assume that the electron orbit is almost circular.

8. **[1.5 points]** Assuming that at the time moment $t = 0$ the electron moves in the orbit with the radius r_1, find the dependence of the orbit radius on time t.
9. **[0.5 points]** Find and calculate the time τ_1 that is needed for the electron to fall from the orbit with the radius r_1 onto the proton.
10. **[2.0 points]** How many revolutions around the proton can the electron make during its fall for the time interval τ_1?

SOLUTIONS TO THE PROBLEMS OF THE THEORETICAL COMPETITION

Problem 1 (10 points)

Problem 1.A

Let l_1 and l_2 be the lengths of the hanging ends of the rope ($l_1<l_2$). Then $l_2 - l_1 = \frac{L}{2}$ and $l_1 + l_2 + \pi R = L$, from which $l_2 = \frac{3}{4}L - \frac{\pi}{2}R$.

1. Consider the rope after a very short time interval Δt from the moment when the height difference of the rope ends is $h = l_2 - l_1 = \frac{L}{2}$ to the moment when one end of the rope is displaced by the small interval Δx. Since there is no friction in the system, any increase in the kinetic energy of the rope should be equal to decrease in the potential energy. It could be easily noticed that the displacement of the rope along itself is equivalent to lowering of a small piece of the rope Δx by the height h:

$$\Delta\frac{mv^2}{2} = \frac{m}{L}\Delta x \cdot gh,$$

$$\frac{m2v\Delta v}{2} = \frac{m}{L}v\Delta t \cdot gh.$$

Hence,

$$a = \frac{\Delta v}{\Delta t} = \frac{h}{L}g = \frac{g}{2}.$$

2. Using similar approach as in section 1, an equation for the conservation of energy is written for the right hand part of the rope (from the top point) considered for a small time interval Δt. Let $l = l_2 + \frac{\pi}{2}R = \frac{3}{4}L$ be the length of that part, $M = m\frac{l}{L} = \frac{3}{4}m$ is its mass, T is the tension at the top point, $H = l_2 + R = \frac{3}{4}L - R\left(\frac{\pi}{2} - 1\right)$ is the difference in heights between the top point and the lowest points of the rope. Then:

$$\Delta\frac{Mv^2}{2} = \frac{m}{L}\Delta x \cdot gH - T\Delta x,$$

$$Mva = \frac{m}{L}v \cdot gH - Tv,$$

$$T = \frac{mgH}{L} - Ma = mg\left[\frac{3}{8} + \frac{R}{L}\left(1 - \frac{\pi}{2}\right)\right].$$

3. The second law of Newton for the small piece of the rope is written in projection to the tangent line:

$$\Delta m \cdot a = \Delta m \cdot g\sin\alpha + T_1 - T_2.$$

If that small piece is chosen at the point with the maximum tension, then $T_1 = T_2$, from which the following can be obtained:

$$\sin\alpha = \frac{a}{g} = \frac{1}{2}.$$

$$\alpha = 30^\circ$$

Grading scheme for Problem 1.A

№	Description	Points
1.	Correct value of the acceleration	1
2.	There is a correct reasoning, some justification for the final answer	1
3.	Correct method to find T_{top} is presented	1
4.	Exact answer for T_{top}	0,5
5.	Reasonable method to solve for α is presented	1
6.	Final answer for α	0,5
	Total	**5,0**

Problem 1.B Assistant Vapor

Air, closedinthevesselatthetemperatureof 200ºC,exertsthe following pressure:

$$1\ atm\cdot\frac{473\ K}{293\ K}=1{,}61\ atm$$

At the same temperature, the vapor pressure is$2{,}88-1{,}61=1{,}27\ atm$. Then, the vaporpressureatanytemperature$T$when all water is vaporized is found as:

$$P_{vap}=1{,}27\cdot\frac{T}{473}\ atm=\frac{T}{373}\ atm$$

Let$P(T)$ be the temperature dependence of the saturated vapor pressure. Then, all water isevaporated at the temperature, which is derived from the following equation:

$$P(T)=\frac{T}{373}\ atm,$$

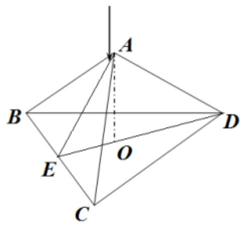
Solution of this equation could be obtained even without further knowledge of the function $P(T)$. It is well known, that at $P=1\ atm, T=373\ K=100\ ℃$. Thus, this checks the solution.

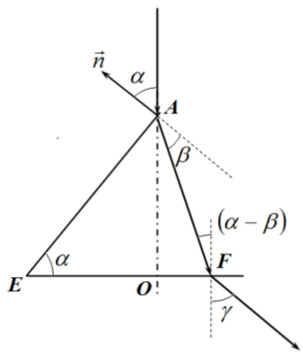
Marking scheme for Problem 1B:

№	Description	Points
1.	Correct value for air pressure at T=200℃	0,5
2.	Vapor pressure was found for any temperature T	0,5
3.	Conditionfor$T_{vap}$was written	0,5
4.	Correct final answer	0,5
Total		**2.0**

Problem 1.C Pyramid

1. Considera beam, which falls down on one of the sidesat the point, close to the top of the pyramid. The angle of incidence of the beam on the facet is equal to the angle between the lateral facet and the base of the pyramid $\angle AEO=\alpha$. It is easy to prove geometrically that $\cos\alpha=\frac{1}{\sqrt{3}}$, thus $\alpha=54{,}7°$.

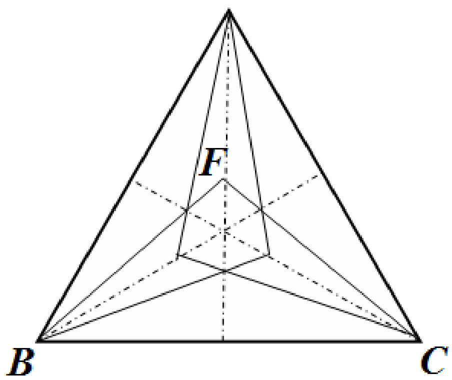




According tothe refraction law the angle β can be written as follows:

$$\sin\beta=\frac{\sin\alpha}{n}=\frac{1}{n}\sqrt{\frac{2}{3}}.\ \text{Thus, } \beta=24{,}6°.$$

Let us find the point F, where that beam falls on the base of the pyramid. The height of the lateral facet is $|AE|=\frac{a}{2}$, pyramid's height $|AO|=\frac{a}{2}\sin\alpha=\frac{a}{2}\sqrt{\frac{2}{3}}=0{,}41mm$.



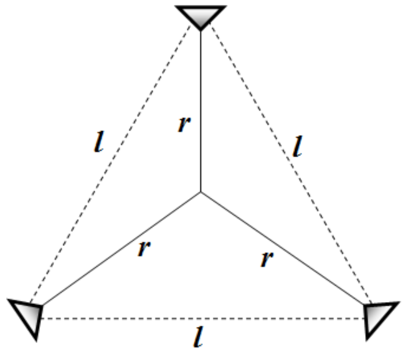
Then, $|OF|=|AO|tg(\alpha-\beta)=0{,}24\ mm$.Finally, the point sought is located at the distance $|DF|=|OD|-|OF|=\frac{a}{\sqrt{3}}-0{,}24mm=0{,}91mm$. The same point is located at a distance $|EO|=a\frac{\sqrt{3}}{2}-|DF|=0{,}82mm$

fromthemiddlepointofbase'sside Thus,the beam refracted by the facet BCF illuminates a triangle at the base of the pyramid. Symmetrical triangles are illuminated by the other two facets (see figure 2).

2. Let usfindanangle γ, of the beam after its refraction on the base of the pyramid. It follows fromFigure 1 andthe refractionlaw that:

$$\sin\gamma = n\sin(\alpha-\beta) = n(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = \sqrt{\frac{2}{3}}\sqrt{n^2-\frac{2}{3}} - \frac{\sqrt{2}}{3}. \quad (1)$$

Numericalvalueofthisangleis $\gamma = 41°$. As the beams, refracted by one facet are parallel to each other, they illuminate similar regions on the screen as they do at the base of the pyramid.Theonlydifferenceisthatthose triangular regions are displaced at the distance $r = Ltg\gamma = 8{,}6 cm$. Thus, threesmalltriangularregionsareseenonthescreen, which are located at the tops of the triangle of theside $l = r\sqrt{3} = 14{,}9 cm$.

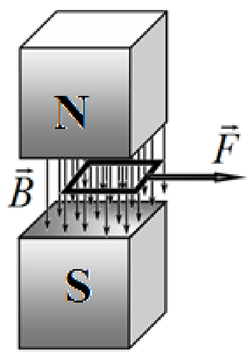


Marking scheme for Problem 1C:

№	Description	Points
1	Light refraction law	0.2
2	Incidence angle on the facet of the pyramid	0.4
3	Location of pointF	1.0
4	Illuminated regions at the base of the pyramid	0.4
5	Outlet angle at the base	0.4
6	Illuminated regions at the screen	0.6
Total		**3.0**

Problem 2 (10 points)
Square frame

1. In physics, it is agreed that the magnetic field lines begin at the north pole and ends at the South pole. Therefore, the drawing should look like this



2. Whentheframe is removedfromtheuniformmagneticfield,the induced emf can be found from the Faraday law

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(Bax\right) = Ba\text{v}(t)\,. \tag{1}$$

On the other hand Ohm's law is written as

$$\varepsilon = IR\,, \tag{2}$$

Thus, we get the relation between the current and the velocity

$$I(t) = \frac{Ba\text{v}(t)}{R}\,. \tag{3}$$

The frame is affectedbythe force that pullsthe frame back into the magnetic field.It isfoundfromAmpere'slaw

$$F_A(t) = BaI(t) = \frac{B^2a^2\text{v}(t)}{R}\,. \tag{4}$$

Thus, the equation of motion is written as

$$m\frac{dv(t)}{dt} = F - \frac{B^2a^2\text{v}(t)}{R}\,. \tag{5}$$

Solutionofequation (5) with the initial condition $\text{v}(0)=0$ is

$$\text{v}(t) = \frac{FR}{B^2a^2}\left[1-\exp\left(-\frac{B^2a^2}{mR}t\right)\right]. \tag{6}$$

3. Integrating (6) we get following relation

$$x(t) = \int_0^t \text{v}(t)dt = \frac{FR}{B^2a^2}\left[t + \frac{mR}{B^2a^2}\left(\exp\left(-\frac{B^2a^2}{mR}t\right)-1\right)\right]. \tag{7}$$

When the frame leaves the magnetic field

$$x(t_0) = a\,, \tag{8}$$

Thus, we get an equation for t_0

$$t_0 + \frac{mR}{B^2a^2}\left[\exp\left(-\frac{B^2a^2}{mR}t_0\right)-1\right] = \frac{B^2a^3}{FR}\,. \tag{9}$$

Equation (9) istranscendentalandcannotbesolvedanalytically. Toestimate t_0 wecansee from equation (6) thatforthe characteristic time $\tau \sim mR/B^2a^2$ the frame reaches the steady velocity $\text{v}_0 = FR/B^2a^2$. Weassumethatfrom 0 to τ the framemoveswiththe constantacceleration $w = F/m$, then it moves with the steady velocity v_0. Hence, we get an estimate

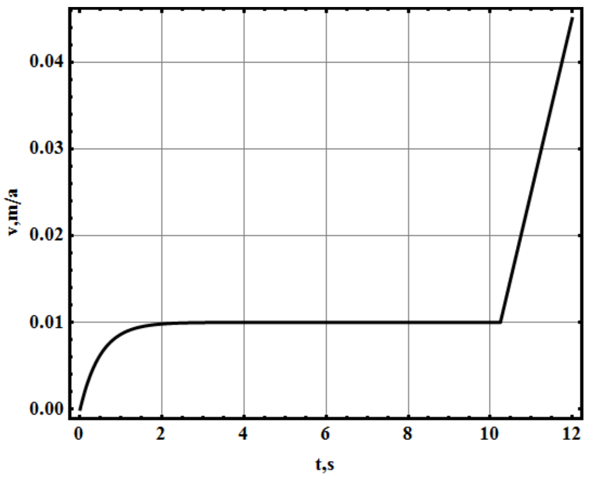
$$t_0 \sim \tau + \frac{a - \frac{wt^2}{2}}{\text{v}_0} = \frac{B^2a^3}{FR} + \frac{mR}{2B^2a^2} = 10.25\,\text{c}. \tag{10}$$

Note that numerical solution of equation (9) gives $t_0 \approx 10.5$ c.

4. Afterthe t_0 theframe continues its motion with the constant acceleration w. At the same time the speed should be a continuous function of time, so the time dependence is written as

$$\text{v}(t) = \begin{cases} \frac{FR}{B^2a^2}\left[1-\exp\left(-\frac{B^2a^2}{mR}t\right)\right], & t < t_0 \\ \frac{FR}{B^2a^2} + \frac{F}{m}(t-t_0), & t \ge t_0 \end{cases}. \tag{11}$$

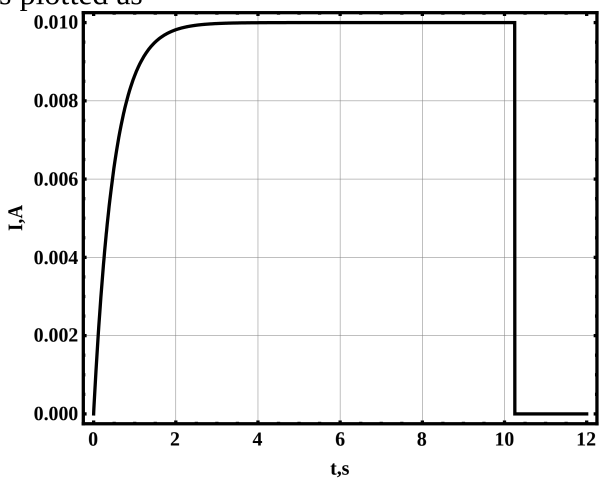
The corresponding graph is plotted as



5. Whilethe frameisbetweenthe magneticpoles, thecurrent is determined by the equations (3) and (6). After that, the frame current vanishesinstantaneously. Thus,

$$I(t)=\begin{cases}\frac{F}{Ba}\left[1-\exp\left(-\frac{B^2a^2}{mR}t\right)\right], & t<t_0\\ 0, & t\geq t_0\end{cases}. \tag{12}$$

The corresponding graph is plotted as



6. As in the previous part, when the frame is removed from the constant magnetic field,theemf (1) is induced.There is anotheremfappearing due to the self-induction of the superconductive frame

$$\varepsilon_L=-L\frac{dI}{dt}. \tag{13}$$

Since the resistance of the superconductive frame is zero, Ohm's law for the frame becomes

$$Bav(t)-L\frac{dI}{dt}=0. \tag{14}$$

Taking into accountthat $I=0$ when $x=0$ we get from equations (13) and (14)

$$I=\frac{Bax}{L}. \tag{15}$$

The corresponding force is given by

$$F_A=BaI=\frac{B^2a^2x}{L}. \tag{16}$$

Thus,the equation of motion is written as

$$m\frac{d^2x}{dt^2}=F-\frac{B^2a^2}{L}x. \tag{17}$$

Expression (17) isan equation ofsimpleharmonicoscillationswith the frequency

$$\omega = \frac{Ba}{\sqrt{mL}}, \tag{18}$$

that are performed near the new equilibrium position with the coordinate

$$x_0 = \frac{FL}{B^2 a^2}. \tag{19}$$

Obviously, theforce F is minimal when

$$x_0 = a/2, \tag{20}$$

whence

$$F_{\min} = \frac{B^2 a^3}{2L} = 5.00 \times 10^{-5}\ \text{H}. \tag{21}$$

7. From previoussection 6, the frame reaches the edge of the magnet for a half period of oscillations, thus

$$t_0 = \frac{\pi}{\omega} = \pi \frac{\sqrt{mL}}{Ba} = 7.02\ \text{c}. \tag{22}$$

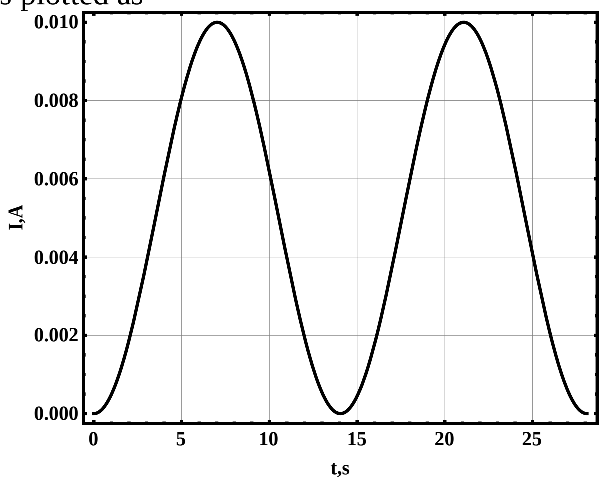
8. Solutionofequation (17) with the initial conditions $x(0)=0, x'(0)=0$ is written as

$$x(t) = \frac{F_{\min} L}{B^2 a^2}\left(1-\cos\omega t\right) = \frac{a}{2}\left(1-\cos\omega t\right). \tag{23}$$

According to equation (15) the frame current varies as

$$I(t) = \frac{Bax(t)}{L} = \frac{Ba^2}{2L}\left(1-\cos\omega t\right). \tag{24}$$

The corresponding graph is plotted as



9. Ohm's law for the frame is written as

$$Ba\text{v} - L\frac{dI}{dt} = IR, \tag{25}$$

and its equation of motion is as follows

$$m\frac{d\text{v}}{dt} = -BIa. \tag{26}$$

Equations (25) and (26) can be rewritten in finite differences as follows

$$Ba^2 - LI_0 = qR, \tag{27}$$

$$m\text{v}_0 = Bqa, \tag{28}$$

where q is the charge flown through the circuit.

Solving (27) and (28) together, we obtain

$$I_0 = \frac{B^2 a^3 - m\text{v}_0 R}{aBL}. \tag{29}$$

Marking scheme

№	Description	points	
1	Properly set northern N and southern S poles	0.2	**0.2**
2	Eq (1)	0.2	**1.2**
	Eq (2)	0.2	
	Eq (3)	0.2	
	Eq (4)	0.2	
	Eq (5)	0.2	
	Solution (6)	0.2	
3	Eq (7)	0.2	**0.8**
	Eq (8)	0.2	
	Eq (9)	0.2	
	Numerical value of t_0	0.2	
4	Eq (11)	0.2	**1.2**
	Graphof $v(t)$: axis signed and digitized	0.2	
	Graphof $v(t)$: there is a part with a constant velocity	0.2	
	Graphof $v(t)$: there is a part with constant acceleration	0.2	
	Graph of $v(t)$: continuous	0.2	
	Graphof $v(t)$: correct numerical values	0.2	
5	Eq (12)	0.2	**1.2**
	Graphof $I(t)$: axis signed and digitized	0.2	
	Graphof $I(t)$: there is a part with constant current	0.2	
	Graphof $I(t)$: there is a part with zero current	0.2	
	Graph of $I(t)$: discontinuous	0.2	
	Graphof $I(t)$: correct numerical values	0.2	
6	Eq (13)	0.2	**1.8**
	Eq (14)	0.2	
	Relationship (15)	0.2	
	Quasi-elastic force (16)	0.2	
	Eq of motion (17)	0.2	
	Equilibrium point (19)	0.2	
	Condition (20)	0.2	
	Eq (21) for $F_{\min}$	0.2	
	Numerical value $F_{\min}$	0.2	
7	Frequency (18)	0.2	**0.6**
	Eq (22)	0.2	
	Numerical value of t_0	0.2	
8	Eq (23)	0.2	**1.0**
	Eq (24)	0.2	
	Graphof $I(t)$: axis signed and digitized	0.2	
	Graphof $I(t)$: two periods of current oscillation	0.2	
	Graphof $I(t)$: correct value of amplitude	0.2	
9	Ohm's Law (25)	0.2	
	Eq of motion (26)	0.2	

	Eq (27)	0.7	**2.0**
	Eq (28)	0.7	
	Expression (29)	0.2	
Total			**10,0**

Problem 3 (10 pts)
Bohr model for hydrogen atom

1. We can use Kepler's third law to find the free fall time of the electron onto the proton. Consider a circular orbit of radius R, then the equation of motion of the electron can be written as

$$m_e\left(\frac{2\pi}{T}\right)^2 R = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2}, \quad (1)$$

where T is the period of revolution.
Thus,Kepler's third law for the electron is given by

$$\frac{a^3}{T^2} = \frac{1}{16\pi^3\varepsilon_0}\frac{e^2}{m_e}, \quad (2)$$

where a is the majorsemi-axisof an elliptic orbit.
Consider the free fall of the electron onto the proton as a motion along a very elongated ellipse with semi-axes $a = r_0/2$. Then,the free fall time of the electron equals to the half of this period

$$t_1 = \frac{T}{2} = \sqrt{\frac{\pi^3 m_e\varepsilon_0 r_0^3}{2e^2}} = 2.46\times10^{-17}\text{ c.} \quad (3)$$

2. To find the dependence of the electron velocity on the orbit radius we again use Newton's second law, which is now written in the form

$$m_e\frac{\mathrm{v}^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}, \quad (4)$$

which yields

$$\mathrm{v} = \sqrt{\frac{1}{4\pi\varepsilon_0}\frac{e^2}{m_e r}}, \quad (5)$$

3. The angular momentum of the electron is found from equation (5) as

$$L = m_e \mathrm{v} r = \sqrt{\frac{m_e r e^2}{4\pi\varepsilon_0}}. \quad (6)$$

4. Since, according to the Bohr model of the hydrogen atom the angular momentum of the electron is quantized, that is $L = n\hbar$, From equation (6) we can get

$$r_n = \frac{4\pi\varepsilon_0 n^2\hbar^2}{m_e e^2}. \quad (7)$$

5. Fromequation (7) weseethatthe minimalradiuscorresponds tothe quantumnumber $n=1$, hence

$$r_1 = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} = 5.19\times10^{-11}\text{ м.} \quad (8)$$

6. The total energy of the electron in the atom is the sum of the kinetic and potential energies.Takingintoaccountequation (5) wecanwritethetotalenergyofelectronin the following form

$$E = \frac{m_e v^2}{2} - \frac{e^2}{4\pi\varepsilon_0 r} = -\frac{e^2}{8\pi\varepsilon_0 r}. \quad (9)$$

Substituting the possible values of the orbit radii ofthe electron (7), we immediately obtain

$$E_n = -\frac{m_e e^4}{32\pi^2 \varepsilon_0^2 \hbar^2 n^2} . \quad (10)$$

7. Fromequation (10) weseethatthe minimaltotalenergycorrespondstothe quantumnumber $n=1$, hence

$$E_1 = -\frac{m_e e^4}{32\pi^2 \varepsilon_0^2 \hbar^2} = -2.24\times 10^{-18} \text{ Дж.} \quad (11)$$

8. The total energy of the electron (9) is spent on the emission of electromagnetic waves

$$\frac{dE}{dt} = -P . \quad (12)$$

From the equation (9) and (12) we get

$$r(t)^2 \frac{dr(t)}{dt} = -\frac{e^4}{12\pi^2 \varepsilon_0^2 m_e^2 c^3} . \quad (13)$$

By solving equation (13) with the initial condition $r(0) = r_1$, we get

$$r(t) = \sqrt[3]{r_1^3 - \frac{e^4}{4\pi^2 \varepsilon_0^2 m_e^2 c^3} t} . \quad (14)$$

9. The fallingtime τ_1 is found from the condition $r(t) = 0$. By substituting this into equation (14)

$$\tau_1 = \frac{4\pi^2 m_e^2 \varepsilon_0^2 c^3 r_1^3}{e^4} = \frac{256\pi^5 \varepsilon_0^5 c^3 \hbar^5}{m_e e^{10}} = 1.44\times 10^{-11} \text{ c.} \quad (15)$$

10. |For a short time interval dt the electron makes a turn by theangle $d\varphi$, defined as

$$d\varphi = \frac{\mathrm{v}}{r} dt . \quad (16)$$

Bysubstituting the velocity from equation (5) and dt from equation (13), we find

$$d\varphi = -\frac{6c^3 (\pi m_e \varepsilon_0)^{3/2}}{e^3} \sqrt{r} dr . \quad (17)$$

Total angle of rotation is found by the integration from r_1 to zero

$$\varphi = \int_{r_1}^{0} d\varphi = \int_{0}^{r_1} \frac{6c^3 (\pi m_e \varepsilon_0)^{3/2}}{e^3} \sqrt{r} dr = \frac{4c^3 (\pi m_e \varepsilon_0 r_1)^{3/2}}{e^3} = \frac{32\pi^3 \varepsilon_0^3 c^3 \hbar^3}{e^6} . \quad (18)$$

Thus, the total number of revolutions is equal to

$$N = \frac{\varphi}{2\pi} = \frac{16\pi^2 \varepsilon_0^3 c^3 \hbar^3}{e^6} = 1.96\times 10^5 . \quad (19)$$

Marking scheme

№	Description	points	
1	Eq (1)	0.5	**2.0**
	Eq(2)	0.5	
	Eq(3)	0.5	
	Correct numerical value of t_1	0.5	
2	Eq (4)	0.5	**1,0**
	Eq (5)	0.5	
3	Eq (6)	0.5	**0.5**
4	Eq (7)	0.5	**0.5**
5	Correct numerical value of r_1	0.5	**0.5**

6	Eq (9)	0.5	**1.0**
	Eq (10)	0.5	
7	Correct numerical value of E_1	0.5	**0.5**
8	The conservation law (12)	0.5	**1.5**
	Eq (13)	0.5	
	Eq (14)	0.5	
9	Correct numerical value τ_1	0.5	**0.5**
10	Eq (16)	0.5	**2.0**
	Eq(17)	0.5	
	Eq(18)	0.5	
	Correct numerical value N	0.5	
Total			**10,0**

EXPERIMENTAL COMPETITION

16 January, 2013

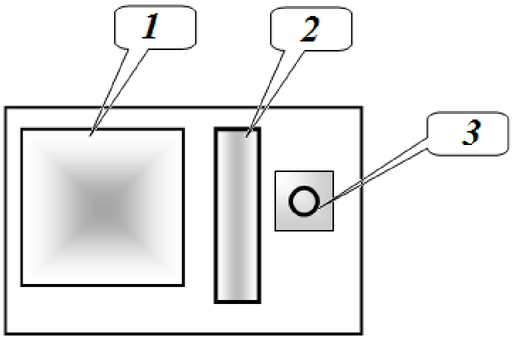
Please read the instructions first:

1. The Experimental competition consists of one problem. This part of the competition lasts 3 hours.
2. Please only use the pen that is provided to you.
3. You can use your own non-programmable calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet and additional papers***. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*.** Write only inside the bordered area.
6. Fill the boxes at the top of each sheet of paper with your country (***Country*)**, your student code (***Student Code***), the question number (***Question Number*)**, the progressive number of each sheet **(*Page Number***), and the total number of ***Writing sheets*** (***Total Number of Pages***). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
7. At the end of the exam, arrange all sheets for each problem in the following order:
    - Used ***Writing sheets*** in order;
    - The sheets you do not wish to be evaluated
    - Unused sheets and the printed question.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper or equipment out of the room

Advertising tricks [15 points]

Equipment: Laser, tripod with the clip, collecting lens on a stand, cardboard screen, graph paper, two guide rulers, measuring ruler, plate with three optical elements:
1. Fresnel mirror (square piece of foil with applied rings);
2. Glossy reflective stripe.
3. Plastic refractive element (orange piece of material covering the hole)



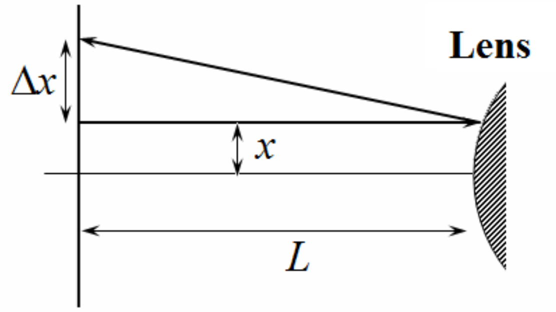
Turn on the laser only when you take measurements!

The wavelength of the laser beam is $\lambda = 680nm$.

Part 1.Lens

In this part of the problem the laser beam deflection angle from the optical axis should be considered small.

Place the lens on the stand between the guide rulers such that it could be moved in parallel to itself. Fix the laser in the tripod clip. Attach the cardboard screen to the laser. Direct the laser beam at the lens center such that the reflected beam is clearly visible on the screen. Ensure that the reflection is approximately backward. When the lens is displaced along the guide rulers the reflected spot moves across the screen. Note that the spot gets diffuse, that is why take measurements of the spot center.



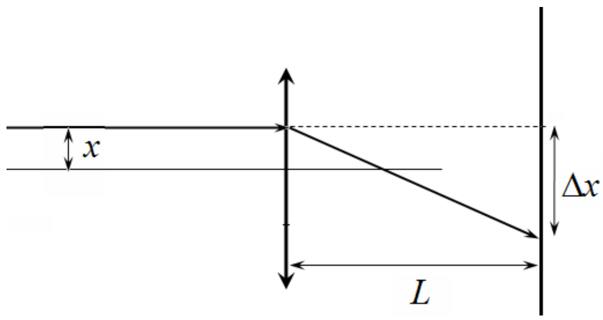
1.1 For both surfaces of the lens, measure the dependence of the displacement Δx of the reflected laser beam on the displacement x of the lens. Plot graphs of the corresponding dependences.

1.2 Using the data obtained, determine the radii R_1 and R_2 of the two surfaces of the lens.

1.3 Place the screen at a certain distance L behind the lens. Measure the dependence of the displacement Δx of the refracted laser beam on the lens displacement x. Plot the corresponding graph.

1.4. Using the experimental data obtained, determine the focal length of the lens.

1.5 Calculate the refractive index n of the lens material. Evaluate the experimental error of the refractive index.



Hint: You can use the following formula for the focal length F *of the lens*

$$\frac{1}{F} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

In the following parts of the problem estimation of errors is not required!

Part 2. Fresnel mirror

Using a piece of plasticine attach the plate with the optical elements to the lens, i.e. use the lens as a stand only.
Place the lens with the plate between the guide rulers such that the laser beam directly hits the Fresnel mirror. Place the laser with the screen at a distance of 15-20 cm from the mirror. Carefully adjust your installation such that the laser beam moves along the mirror diameter when the mirror itself is displaced. You should see the diffuse reflected spot which should move almost horizontally at the level of the incoming laser beam when the mirror is slightly displaced.
The Fresnel mirror is a set of concentric rings whose radii vary with the number k, counted from the center, according to the law

$$r_k = r_0 k^{\gamma} \tag{1}$$

In this part of the problem deflection angles cannot be considered small.

2.1 Measure the displacement of the center of the reflected spot Δx_1 as a function of the mirror displacement x. Plot the graph of the sine of the reflection angle as a function of the mirror displacement.
2.2 Estimate γ in formula (1).

Part 3. Glossy reflective stripe

Direct the laser beam onto the glossy reflective stripe 2. You can observe a few reflected beam on the screen.

3.1 Make a guess on the structure of the glossy reflective stripe that could explain the observed pattern of the reflected beams. Calculate the numerical values of the parameters of this structure.

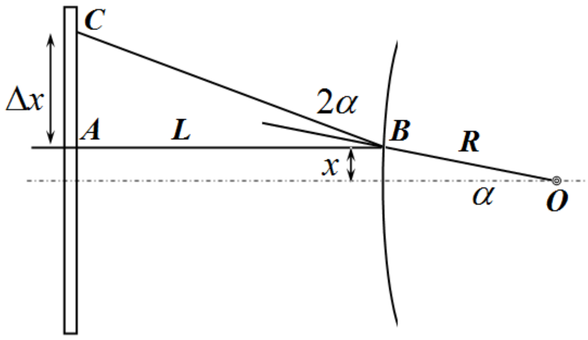
Part 4. Plastic refractive element

Direct the laser beam on the hole with the plastic orange element. This element has a three-dimensional structure. Place the screen behind this element, you should see a number of bright spots in the transmitted light.

4.1 Make a guess on the structure of this element that could explain the observed pattern of the reflected beams.
4.2 Estimate the refractive index of the material from which this optical element is made of.

SOLUTION FOR THE EXPERIMENTAL COMPETITION
Advertising tricks

Part 1. Lens



1.1-1.2 The figure on the right shows the path of the laser beam reflected from the lens surface. It is seen that the reflected beam makes the angle 2α to the lens axis, where

$$\alpha = \frac{x}{R}, \tag{1}$$

R is the radius of the lens surface and O is the center of curvature. Consequently, the displacement of the reflected beam on the screen is given by

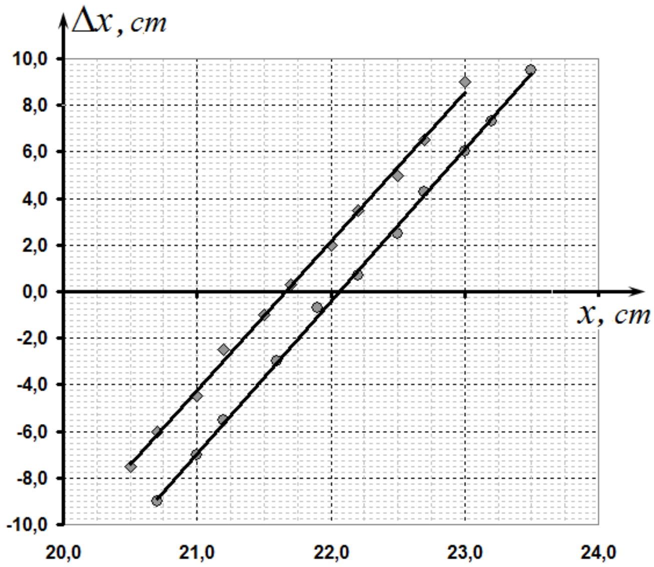
$$\Delta x = 2\alpha L = \frac{2L}{R} x. \tag{2}$$

Thus, the slope on the graph showing the laser beam displacement Δx aganst the lens displacement x determines the radius of the surface curvature. To find the slope it is not necessary to accurately determine the position of the surface center, i.e. the reference point can be chosen arbitrarily.

Table 1 shows the measured beam displacement for the two surfaces of the lens. The measurements have been made for $L = 22\,cm$. The graphs of the obtained dependences are also presented.

Table 1

Side 1		Side 2	
x, cm	$\Delta x, cm$	x, cm	$\Delta x, cm$
23,5	9,5	23,0	9,0
23,2	7,3	22,7	6,5
23,0	6,0	22,5	5,0
22,7	4,3	22,2	3,5
22,5	2,5	22,0	2,0
22,2	0,7	21,7	0,3
21,9	-0,7	21,5	-1,0
21,6	-3,0	21,2	-2,5
21,2	-5,5	21,0	-4,5
21,0	-7,0	20,7	-6,0
20,7	-9,0	20,5	-7,5



The slopes are preferably calculated by the least square method. The radii are calculated by the formula which follows from equation (2):

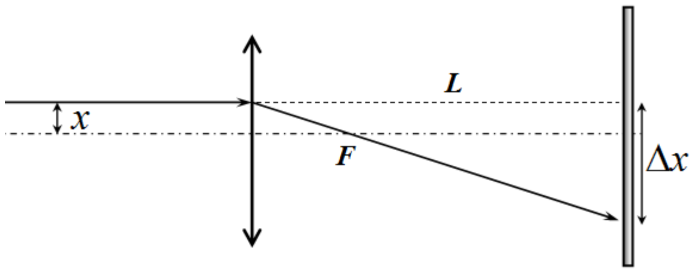
$$K = \frac{2L}{R} \quad \Rightarrow \quad R = \frac{2L}{K}. \tag{3}$$

Numerical calculations lead to the following values:

For the first side of the lens $\quad K_1 = 6{,}51 \pm 0{,}14$; $\quad R_1 = (6{,}76 \pm 0{,}14)\,cm$

For the second side of the lens $\quad K_2 = 6{,}4 \pm 0{,}2$; $\quad R_2 = (6{,}9 \pm 0{,}2)\,cm$

1.3-1.4 The figure on the right shows the path of the laser beam after refraction in the lens. It follows from the similarity of the triangles that $\frac{x}{F} = \frac{\Delta x}{L}$. Thereforeб the dependence of the laser beam displacement on the lens displacement is given by

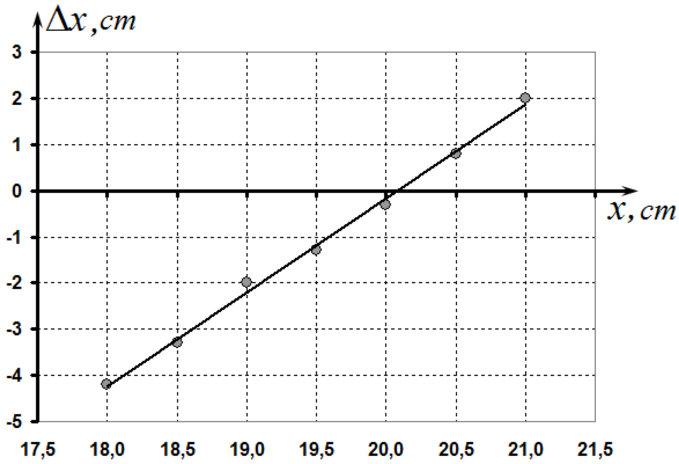


$$\Delta x = \frac{L}{F}x. \qquad (4)$$

Table 2 shows the results of epy measurements carried out for $L = 15\,cm$. The corresponding graph is plotted in the figure on the right.

Table 2

x, cm	Δx, cm
21,0	2,0
20,5	0,8
20,0	-0,3
19,5	-1,3
19,0	-2,0
18,5	-3,3
18,0	-4,2



The slope of the dependence is calculated by the least square method $K_3 = 2{,}03 \pm 0{,}10$. Accordingly, the focal length of the lens is found as

$$F = \frac{L}{K_3} = (7{,}4 \pm 0{,}4)\,cm. \qquad (5)$$

1.5 The refractive index of the lens can be calculated from the formula for the focal length

$$n = 1 + \frac{\frac{1}{F}}{\frac{1}{R_1} + \frac{1}{R_2}} = 1 + \frac{\frac{1}{7{,}4}}{\frac{1}{6{,}76} + \frac{1}{6{,}9}} \approx 1{,}5.$$

Experimental error turns out to be $\Delta n \approx 0{,}2$.

Part 2. Fresnel mirror

Fresnel mirror can be considered as a diffraction grating with the varying period. If the radii of the rings depends on the consecutive number from the center as

$$r_k = r_0 k^{\gamma}, \qquad (6)$$

then ,at large k the distance between the rings is found as

$$\Delta r = r_0\left((k+1)^{\gamma} - k^{\gamma}\right) = r_0 k^{\gamma}\left(\left(1 + \frac{1}{k}\right)^{\gamma} - 1\right) \approx \gamma r_0 k^{\gamma - 1}. \qquad (7)$$

This quantity can be treated as the grating period d. To process the experimental data it is necessary to express the grating period through the distance to the mirror center. It is derived from equation (6) that

$$k = \left(\frac{r}{r_0}\right)^{\frac{1}{\gamma}}. \quad (8)$$

Therefore, the grating period varies with the distance as

$$d = \gamma r_0 \left(\frac{r}{r_0}\right)^{\frac{\gamma-1}{\gamma}}. \quad (9)$$

Experimentally, this dependence can be obtained using the formula of the diffraction grating

$$d \sin\varphi = \lambda. \quad (10)$$

Therefore, the sine of the deflection angle depends on the mirror displacement as:

$$\sin\varphi = \frac{\lambda}{d} = Cr^{\frac{1-\gamma}{\gamma}}. \quad (11)$$

In the experiment it is possible to measure the dependence of the reflected beam displacement Δx on the distance r measured from the point of laser beam to the mirror center. Since the deflection angles in this case can not be considered small, the sine of the deflection angle φ should be calculated according to the formula

$$\sin\varphi = \frac{\Delta x}{\sqrt{(\Delta x)^2 + L^2}}. \quad (12)$$

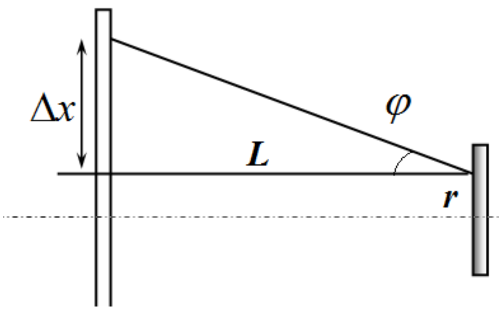
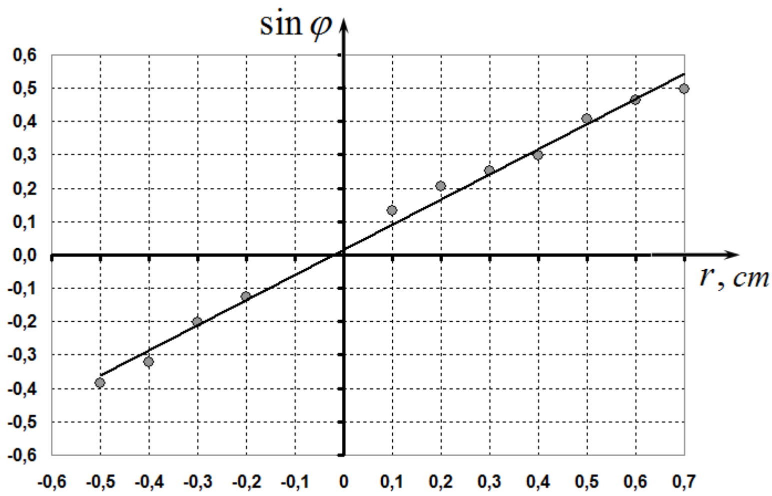


Table 3 shows the results of the measurements of the laser beam displacementon Δx the distance r. The same table shows the calculated values of the sine of the deflection angles. The graph on the right represents a plot of the sine of the deflection angle as a function of r.

Table 3

r, cm	$\Delta x, cm$	$\sin\varphi$
0,7	11,0	0,497
0,6	10,0	0,463
0,5	8,5	0,406
0,4	6,0	0,297
0,3	5,0	0,253
0,2	4,0	0,207
0,1	3,0	0,159
-0,2	-2,5	-0,127
-0,3	-4,0	-0,201
-0,4	-6,5	-0,321
-0,5	-8,0	-0,385



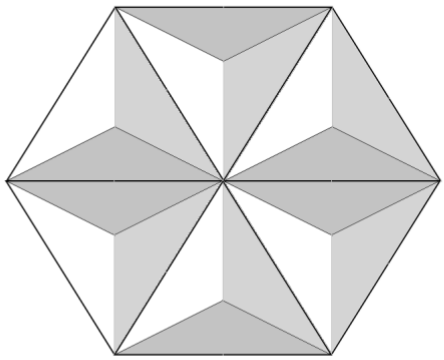
Since the obtained dependence is close to linear, in accordance with the general formula (11) the coefficient γ is found from the figure to be $0,5$.

Part 3. Glossy reflective stripe

In the reflected light at least five of the reflected beams are visible. One is at the center and four are at the corners of a square. Such a pattern can be explained by the two-dimensional diffraction grating. The diffraction angle is approximately measured to be equal 45 degrees. Hence, the grating period is found to be

$$d = \frac{\lambda}{\sin 45°} = \frac{680 \cdot 10^{-9} m}{\frac{\sqrt{2}}{2}} \approx 9,6 \cdot 10^{-7} m .$$

Part 4. Plastic refractive element



When the laser beam passes through the plastic element six spots are clearly seen on the screen. They are located at the vertices of a regular hexagon. Such a pettern is explained by the refraction of laser beams on the facets of the regular pyramids. Base of the pyramids fill the plane forming a hexagonal structure.
If the screen is placed at a distance $L = 6,5cm$, then the side length of the hexagon is equal $a = 4,5cm$. The same distance is measured from the center of the regular hexagon. These data allow us to calculate the sine of the deflection angle of laser beam

$$\sin\alpha = \frac{a}{\sqrt{a^2 + L^2}} = 0,57 .$$

In the problems of the theoretical competition the following formula for the deflection angle was derived

$$\sin\gamma = \sqrt{\frac{2}{3}}\sqrt{n^2 - \frac{2}{3}} - \frac{\sqrt{2}}{3},$$

which can be used to calculate the referaction index to be $n = 1,51$.

Grading scheme

Part 1. Lens		**(6)**
1.1	Measurements: - L is stated; - 7 or more data points; - *(4-6 data points);* - *(less than 4 data points);*	 0,2 0,7x2 *(0,3x2)* *0*
	Graphs: - axis ticked and digitized; - all the experimental points are plotted; - line smoothing is carried out;	 0,1 0,1x2 0,1x2
1.2	Calculation of the radii: - dependence (2); - Eq. (3); - slopes are calculated : - by using Least Square Method; - *(by using the graph);* - *(by using two points on the graph);* Calculation of radii: - range 6,5 -7,5 cm; - (*range 5-9 cm);* - *(out of range 5-9 см);*	 0,1 0,1 0,3x2 *(0,2x2)* *(0,1x2)* 0,2x2 *(0,1x2)* *(0x2)*
1.3	Measurements:	

	- L is sstated; - 5 or more data points; - *(3-4 data points);* - *(less than 3 data points);*	0,2 0,5 *(0,3)* *(0)*
	Graphs: - axis ticked and digitized; - all the experimental points are plotted; - line smoothing is carried out;	 0,1 0,1 0,1
1.4	Measurement of the focal length: - Eq. (4); slopes are calculated : - by using Least Square Method; - *(by using the graph);* - *(by using two points on the graph);* Measurement of the focal length: - range 7 -8 cm; - (*range 6-9 cm);* - *(out of range 6-9 cm);*	 0,2 0,3 *(0,2)* *(0,1)* 0,4 *(0,2)* *(0)*
1.5	Calculation of the refraction index: - Eq. to find index of refraction; - numerical value is in the range of 1,45-1,55; - *(range 1,35 – 1,65);* - *(out of the range 1,35 - 1,65);* - error estimation;	 0,1 0,5 *(0,2)* *(0)* 0,2
Part 2. Fresnel Mirror		**(4)**
2.1	Measurements: - L is stated; - 7 or more data points; - *(4-6 data points);* - *(less than 4 data points);*	 0,2 0,8 *(0,4)* *(0)*
	Eq for the sine of the angle; Values of sines are calculated ;	0,1 0,2
	Approximately linear relationship;	0,5
	Graphs: - axis ticked and digitized; - all the experimental points are plotted; - line smoothing is carried out;	 0,1 0,1 0,1
	Method to find γ: - the idea of a diffraction grating ; - dependence of the grating period on the distance to the center (9); - dependence of the sine of the deflecrtion angle on the distance to the center (11);	 0,3 0,4 0,4
	The value obtained for γ: - by using linear dependence on the sine of the angle; - by using other reasonable way; $\gamma = \frac{1}{2}$;	 0,3 *(0,2)* 0,5 *(0,2)*

	- *range 0,3-0,7;* - *out of range 0,3 – 0,7;*	*(0)*
Part 3. Glossy reflective stripe.		**2**
	- obtained more than 5 spots on the screen; - (*obtained 5 spots on the screen);* - *(obtained 3 spots on the screen);*	0,4 *(0,3)* *(0,2)*
	Structure of the diffraction grating: - two-dimensional; - (*onde-dimensional)*;	 0,5 *(0,3)*
	Deflection angles of the beams are measured; The formula to calculate grating period; Calculated grating period (1 μm); - error less than 10%; - *(in the range 30%);* - *(out of the range 30%);*	0,4 0,2 0,5 *(0,2)* *(0)*
Part 4. Plastic refractive element.		**3**
	- 6 spots on the screen are obtained;	0,5
	Structure - pyramid; Filling the plane (with a hexagonal structure);	0,5 0,5
	Eq to find the refraction index; Measured deflection angles of the beams are meausered; The numerical value of the refraction index: - numerical value in the range of 1,45-1,55; - *(in the range 1,35 – 1,65);* - *(out of range 1,35 - 1,65);*	0,5 0,5 0,5 *(0,3)* *(0)*

THEORETICAL COMPETITION

January 14, 2014

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*.** Write only inside the boxed area.
6. Begin each question on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Question Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
   - Used ***Writing sheets*** in order;
   - The sheets you do not wish to be evaluated
   - Unused sheets and the printed question.

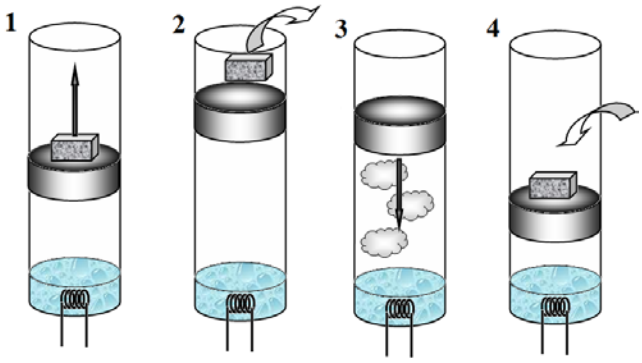
   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

Problem 1 (10 points)

This problem consists of three independent parts.

Problem 1A (3.0 points)

Steam engine consists of a vertical cylindrical vessel, in which a piston can move without friction. There is some water in the vessel with an electrical heater located inside. Steam engine cycle consists of four stages, as shown in the figure on the right:



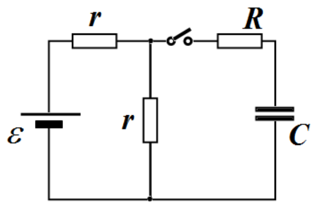
1. A load is placed on the piston, the heater is switched on making the water boiling and the vapor lifts up the piston with the load.
2. Once the piston has risen to some height, the load is quickly removed and the heater is immediately switched off.
3. The vapor under the piston cools down and condenses, the piston moves down slowly .
4. Once the piston has come down to some other height, the load is again put on the piston.

Draw a (P,V) diagram schematically showing the cycle of the steam engine and find its efficiency.

The atmospheric pressure is $P_0 = 1.0 \cdot 10^5 Pa$, the piston mass is $M = 2.0 kg$, the piston area is $S = 10 cм^2$, the load mass is $m = 1.0 kg$, and the free fall acceleration is $g = 9.8 m/s^2$. Assume that there is nothing under the piston except for the water vapor, and the dependence of the pressure of the saturated water vapor in the temperature range under consideration is approximated by the formula $P = at - b$, where $a = 4{,}85 kPa/K$, $b = 384 kPa$, t is the temperature in degrees Celsius.

Problem 1B (5.0 points)

In the circuit shown in the figure on the right, all the electrical components are ideal and their parameters are assumed to be given. Before switching on the key, the capacitor has been discharged. Find an amount of heat releasing in the resistor *R* after the key has been switched on.



Problem 1C (2.0 points)

Thin lens gives an image of the object, located perpendicular to its optical axis. The image size is 1 *cm*. If the distance from the object to the lens is increased by 5 *cm*, the image size remains 1 *cm*. Find the image size if the distance from the object to the lens is increased by another 5 *cm*.

Problem 2 Jet propulsion (10 points)

In a rocket engine thrust is created by the release of products of fuel combustion in the direction opposite to its motion. It is, of course, natural that the mass of the rocket decreases in the acceleration process. This idea was first proposed by the great Russian scientist K. Tsiolkovsky to implement the motion of objects in a vacuum, for example, in outer space. Nowadays space flights have become habitual. It is widely known that the space launching site, Baikonur, is situated on the territory of Kazakhstan. The first satellite and the first cosmonaut, Yu. Gagarin, were sent into space from Baikonur which is now a

complex of high-tech facilities intended to launch manned spacecraft into space, in particular, to the International Space Station.

Classical rocket

Let a rocket have an initial mass m_0 and let a fuel velocity relative to the rocket be constant and equal u. Assume that at the initial time moment the rocket is at rest in the laboratory frame of reference and no external force is present.

1. **[0.5 points]** Find the rocket velocity v as a function of its mass m. This formula is called after K. Tsiolkovsky. Express your answer in terms of m, m_0, u.

2. **[0.5 points]** An object of mass $m = 1000\,kg$ is required to be accelerated to the orbital velocity. Evaluate the initial rocket mass m_0, if the free fall acceleration is $g = 9.80\,m/c^2$ and the radius of the Earth is $R = 6400\,km$ and $u = 5{,}00\,km/s$.

Let a rocket move in the gravitational field of the Earth. The free fall acceleration g is assumed to be constant, whereas the fuel consumption $\mu(t) = -dm(t)/dt$ may depend on time.

3. **[0.75 points]** Write down the equation of motion of a rocket in Earth's gravitational field. This equation is called after I. Meshcherskij. Express your answer in terms of m, v, u, g, μ.

Assume in the following that the fuel exhaust velocity u is directed parallel to the free fall acceleration g, and the initial velocity of the rocket is zero.

4. **[0.75 points]** Find how the fuel consumption $\mu_{st}(t)$ should depend on time t in order for the rocket to hung motionless at some height. Express your answer in terms of m_0, u, g, t.

Assume now that the fuel consumption μ is also constant over time such that $\mu > \mu_{st}(t)$.

5. **[2.0 points]** In this case the rocket velocity dependence on time t can be represented as

$$\text{v}(t) = A_1 t + A_2 \ln(1 + A_3 t),$$

where A_1, A_2, A_3 are some constants.

Find A_1, A_2, A_3 and express them in terms of m_0, u, g, μ.

6. **[1.0 points]** Suppose that the initial mass of the rocket is equal m_0, and the final mass is to be m. Find the maximum height $H_{\max}$ that the rocket can reach and determine the corresponding optimum fuel consumption μ_{opt}. Express your answer in terms of m_0, m, u, g.

Relativistic rocket

In the previous part of the problem it has been assumed that the rocket moves with a nonrelativistic velocity. To implement interstellar travels it is necessary to accelerate the rocket to the speed close to that of light and, then, relativity effects cannot be ignored at the calculations.

To establish the characteristic features of the rocket motion in a relativistic case, we introduce the concept of the proper frame of reference. The proper frame of reference is an inertial frame of reference which moves with the speed of the rocket itself relative to the laboratory reference frame, i.e. it is the reference frame in which the rocket is at rest at any given time.

7. **[2.5 points]** Find the relation between the rocket acceleration in the proper reference frame a_p and its acceleration in the laboratory frame of reference a_r when the velocity of the rocket is v, and c stands for the speed of light. Express your answer in terms of a_p, a_r, v, c.

8. **[1.5 points]** Let the rocket be at rest at the initial time moment. Then, using the results of the previous question it can be shown that at any time moment the rocket mass in the proper reference frame is related to its speed in the laboratory reference frame as

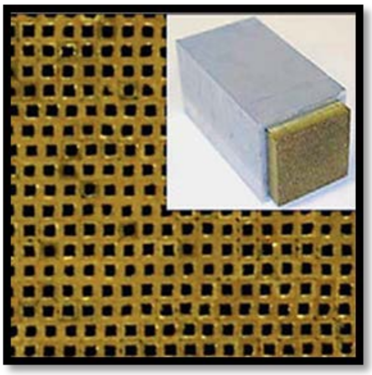
$$m = m_0 \left(\frac{1 - \text{v}/c}{1 + \text{v}/c} \right)^{\alpha}.$$

Find α and express it in terms of u, c.

9. **[0.25 points]** An object of mass $m = 1000\,kg$ is required to be accelerated to half the speed of light $v = 0.5c$ where the speed of light is $c = 3.00 \cdot 10^8\,m/s$. Evaluate the initial rocket mass together with the fuel m_0 and **write it down as a power of 10**, if the fuel exhaust velocity is $u = 5{,}00\,km/s$.

10. **[0.25 points]** It can be shown that from the practical point of view the best rocket is the one that exploits photons rather than hot gases produced at the fuel combustion. An object of mass $m = 1000\,kg$ is required to be accelerated to half the speed of light $v = 0.5c$. Evaluate the initial rocket mass together with the fuel m_0.

Problem 3 Metamaterials (10 points)



Metamaterials are composite materials whose properties are due not so much to the properties of its constituent elements but due to artificially tailored periodic structures. Metamaterials are synthesized in modern nanolaboratories by implantation different periodic structures with a variety of geometric shapes into the original natural material, which substantially modifies its physical properties. In a very rough approximation, those implants can be treated as artificially made atoms of extremely large size immersed into the original material. While synthesizing the metamaterial Ddveloper has the opportunity of varying various free parameters (structure sizes and constant or varying period between them, etc.).

In one nanolaboratory the metamaterial has been manufactured in the form of a wire of the length $L = 5.00cm$ and radius $R = 1.00mm$, whose conductivity depends on the distance from its axis according to the law $\sigma_0 = \beta r$. Physical properties of the wire have been experimentally measured and are presented in the following table:

PHYSICAL PROPERTY	NUMERICAL VALUE
Conductivity $\sigma_0 = \beta r$	$\beta = 1.00 \times 10^9 S/m^2$
Heat transfer coefficient	$\alpha = 20W/(m^2 \cdot K)$
Thermal conductivity coefficient	$\kappa = 0{,}01W/(m \cdot K)$
Young's modulus	$E = 1.00 \times 10^7 Pa$
Linear expansion coefficient	$\gamma = 1.00 \times 10^{-6} K^{-1}$

1. **[1.0 points]** Find an analytic formula for the total resistance R_0 of the wire, and calculate its numerical value.

An electric current $I = 1A$ is made to pass through the wire. It is known that the heat exchange with the environment obeys the Newton-Richman law,

$$P_{ext} = \alpha(T_s - T_0),$$

where P_{ext} stands for the power loss per unit surface of the wire with the surface temperature T_s, $T_0 = 293K$ denotes the ambient temperature and α is a constant, called the heat transfer coefficient.

2. **[1.0 points]** Find an analytic formula for the surface temperature T_s of the wire and calculate its numerical value.

The wire temperature varies with the depth due to the phenomenon known as thermal conductivity, which is described by the Fourier law

$$P = -\kappa S \frac{\Delta T}{\Delta x},$$

where P designates the power of the heat flow between the opposite faces of the parallelepiped with the square S, ΔT is the temperature difference between the faces of the parallelepiped situated at a distance Δx from each other, and κ is called the heat transfer coefficient.

3. **[2.5 points]** Find an analytic formula for the temperature $T_{\max}$ in the center of the wire, and calculate its numerical value.

4. **[0.5 points]** Find an analytic formula for the change δR_T of the wire radius due to its thermal expansion and calculate its numerical value.

Attention! *In all further calculations assume that the wire is infinitely long.*

5. **[0.5 points]** Find an analytic formula for the magnetic induction inside the wire as a function of the distance r from its axis.

6. **[1.0 points]** Find an analytic formula for the energy of the magnetic field inside the wire, and calculate its numerical value.

7. **[1.0 points]** The electric current causes an appearance of mechanical stress in the wire. Find an analytic formula for the pressure $p(r)$ inside the wire as a function of the distance r from its axis.

8. **[1.0 points]** Find an analytic formula for the mechanical stress energy W_σ of the wire, and calculate its numerical value.

9. **[1.0 points]** Find an analytic formula for the change δR_σ of the wire radius due to its mechanical stress, and calculate its numerical value.

10. **[0.5 points]** Find the value of the thermal expansion coefficient γ such that the total change of the wire radius would be zero when an electric current was passing through it.

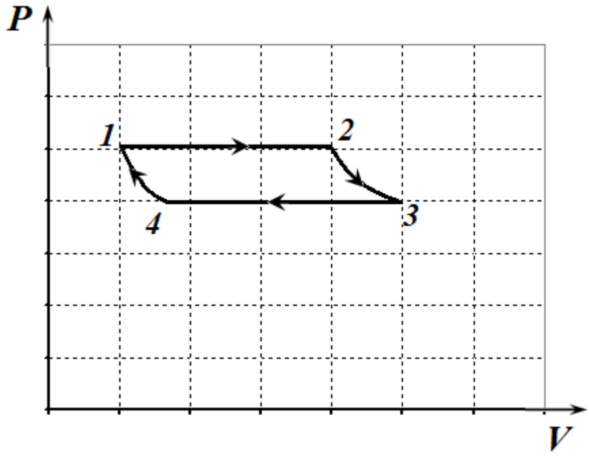
Help! *The value of the magnetic constant is* $\mu_0 = 4\pi \cdot 10^{-7}\,Гн/м$.

SOLUTIONS TO THE PROBLEMS OF THE THEORETICAL COMPETITION

Problem 1 (10 points)

Problem 1A (3 points)

In the first stage boiling occurs at a constant pressure, hence at a constant temperature. Likewise, in the third stage condensation takes place at a constant pressure and temperature. The second and the fourth stages can be considered as adiabatic. Schematic (***P,V***) diagram of the steam engine cycle is shown in the figure on the right. Since this cycle is composed of two isotherms and two adiabats, it is, thus, simply the Carnot cycle. Therefore, its efficiency is found as



$$\eta = \frac{T_1 - T_2}{T_1}, \qquad (1)$$

where T_1 is the boiling temperature in the first stage of the cycle, and T_2 is the condensation temperature in the third stage of the cycle.

The corresponding temperatures are found from the approximate formula for the saturated vapor pressure provided at the formulation of the problem.

The first stage of the cycle happens at constant pressure

$$P_1 = P_0 + \frac{(M+m)g}{S} \approx 1,3 \cdot 10^5 Pa\,. \qquad (2)$$

The vapor temperature is the temperature of the water boiling point and is equal to

$$t_1 = \frac{P_1 + b}{a} = \frac{130+384}{4,85} \approx 106°C = 379K\,. \qquad (3)$$

The temperature difference in formula (1) is conveniently evaluated as

$$T_1 - T_2 = \frac{P_1 - P_2}{a} = \frac{mg}{Sa} = \frac{20}{4,85} \approx 4,2\,K\,. \qquad (4)$$

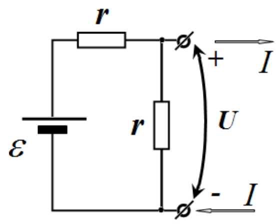
Thus, the efficiency of the steam engine is $\eta = \frac{T_1 - T_2}{T_1} = \frac{4,2}{379} = 1,1\%$.

Grading scheme

№	Content	Points
1.	1 and 3 stages are isobars and isotherms	0,25
2.	2 and 4 stages are adiabats	0,25
3.	The cycle is identified as a Carnot cycle	1,0
4.	Correct cycle diagram	0,75
5.	Formula (1) for the efficiency of the Carnot cycle	0,25
6.	Formulas (2) and (4)	0,25
7.	Correct numerical value for the efficiency	0,25
Total		**3.0**

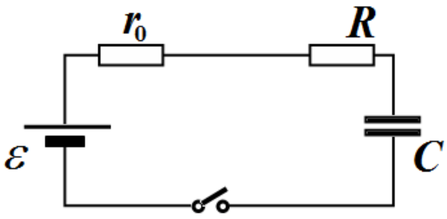
Problem 1B (5 points)

The first solution.



Consider the left part of the circuit. Its load characteristic (dependence of U against I)) is the straight line corresponding to an equivalent source with the parameters $\varepsilon_0 = \frac{\varepsilon}{2}$, $r_0 = \frac{r}{2}$.

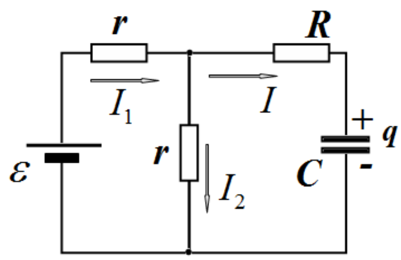
For the equivalent circuit



the total released heat is found as $Q_0 = \frac{C\varepsilon^2}{2}$. In the resistor R the released heat is obtained from the simple proportion as

$$Q = \frac{R}{R + r_0} Q_0 = \frac{R}{R + \frac{r}{2}} \cdot \frac{C\left(\frac{\varepsilon}{2}\right)^2}{2} = \frac{RC\varepsilon^2}{4(2R + r)}$$

The second solution.



The Kirchhoff set of equations has the following form

$$\begin{cases} \varepsilon = I_1 r + I_2 r \\ I_2 r = IR + \frac{q}{C} \\ I_1 = I + I_2 \\ I = \dot{q} \end{cases}$$

Eliminating I_1 и I_2, we obtain the relation

$$I\left(R + \frac{r}{2}\right) + \frac{q}{C} = \frac{\varepsilon}{2},$$

and multiplying it by I we get

$$I^2\left(R + \frac{r}{2}\right) = \frac{\varepsilon}{2} I - \frac{qI}{C} = \frac{\varepsilon}{2}\dot{q} - \frac{2q\dot{q}}{2C} = \frac{d}{dt}\left(\frac{\varepsilon q}{2} - \frac{q^2}{2C}\right).$$

Hence

$$I^2 R = \frac{R}{R + \frac{r}{2}} \frac{d}{dt}\left(\frac{\varepsilon q}{2} - \frac{q^2}{2C}\right),$$

and, thus,

$$\int_0^\infty I^2 R dt = \frac{R}{R + \frac{r}{2}} \left(\frac{\varepsilon q}{2} - \frac{q^2}{2C}\right)\Bigg|_{q(0)}^{q(\infty)}$$

On substituting $q(0) = 0$ and $q(\infty) = C\frac{\varepsilon}{2}$, we finally obtain

$$Q = \frac{RC\varepsilon^2}{4(R+2r)}.$$

Grading scheme

I. Direct solution

№	Content	Points
1.	There is a correct set of equations allowing to obtain the answer – 1.0; if there is an error in the set or it is not complete – 0.	1.0
2.	Correct expression for the current in the resistor *R* or/and for the charge of the capacitor	1.0
3.	Correct expression for the derivative of the square of the current in *R*, or correct dependence *I(t)* For manipulating errors – 1.0 points of 2.0 Propagation error is not accepted	3.0
4.	Correct formula for $Q = \int I^2 R dt$	1.0
Total		**5.0**

II. Solution with the equivalent source

Propagation errors are not accepted

№	Content	Points
1.	Idea: to change the left part of the circuit by an equivalent source and to apply the energy conservation law	1.0
2.	The parameters of the equivalent source 2.1 Two of the following statement are present: 1. No-load voltage Ɛ/2 2. Short circuit current Ɛ/*r* 3. Internal resistance is the parallel connection of two resistors *r* **or** the dependence *U*(*I*) is obtained 2.2 It is found that: $\mathcal{E}_0 = \mathcal{E}/2$, $r_0 = r/2$ (по 0.5 for each) (Justification of the equivalent circuit is not required)	1.0 1.0
3.	Total released heat in the equivalent circuit	1.0
4.	Correct answer	1.0
Total		**5.0**

Problem 1C (2 points)

It is known that the beam, passing through the focal point of the lens, goes parallel to the optical axis of the lens after refraction. Therefore, all the objects, shown in the figure, give the images of the same size, i.e. the lens magnification is inversely proportional to the distance from the object to the focal point.

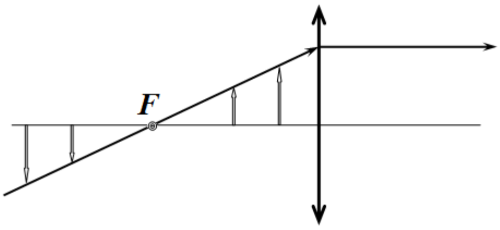


Figure 1

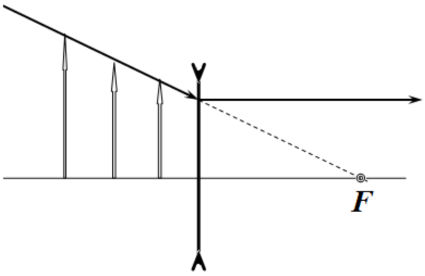


Figure 2

It is clearly seen from figure 2 that in the case of the diverging lens it is impossible to get the same image size at different positions of the object, so the lens is necessarily converging.

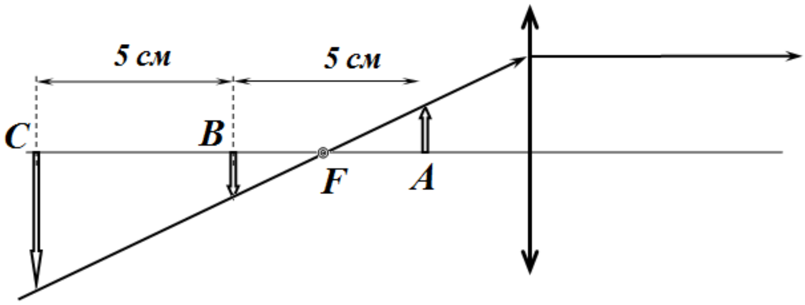


Figure 3

Positions A and B of the object, which are arranged symmetrically with respect to the focal point of the lens, result in the same image sizes (see figure 3). If the object is moved away by another 5 cm, then it will be located in the position C, in which the image of the same size would have been obtained by a three times larger object, so the image of the object is to be three times smaller. Answer: 1/3 cm.

Grading scheme

№	Content	Points
1.	Correct answer	1.0
2.	Correct justification of the correct answer	1.0
3.	There is an error in the application of the lens formulas	-0.5
Total		**2.0**

Problem 2 Jet propulsion (10 points)

1. Consider the rocket motion in the proper reference frame, i.e. the inertial reference frame which moves with the speed of the rocket itself relative to the laboratory reference frame. In the proper reference frame the rocket is always at rest at any given time. Let a rocket have mass m at the rime moment t and throw away some fuel of mass dm with the velocity u. As a result the rocket velocity changes by $d\text{v}$ and the conservation of the momentum can be written as

$$m d\text{v} - dmu = 0. \qquad (1)$$

In classical mechanics, the change in the rocket velocity in the laboratory reference frame must coincide with the change in rocket velocity in the proper reference frame by virtue of the Galilean transformations. Therefore, solving equation (1) with the initial condition $m = m_0$ at $v = 0$, we obtain the formula named after K. Tsiolkovsky

$$\mathrm{v} = u \ln\left(\frac{m_0}{m}\right). \qquad (2)$$

2. It is known that the orbital velocity at the Earth's surface is

$$v_1 = \sqrt{gR}\,, \qquad (3)$$

then from equation (2) the initial mass of the rocket is found as

$$m_0 = m \exp\left(\frac{v}{u}\right) = 4.87 \times 10^3\, kg. \qquad (4)$$

3. If an external force F is exerted on the rocket, then, in the proper reference frame the total momentum of the system does change, and equation (1) can be rewritten as

$$mdv - dmu = Fdt\,, \qquad (5)$$

or, using the notation $\mu = -dm/dt$, we obtain

$$m\frac{dv}{dt} = F - \mu u\,. \qquad (6)$$

By virtue of the relativity principle, this equation does not change its form in any inertial frame of reference and it is called after I. Meshcherskij.

On substituting $F = mg$, we finally obtain

$$m\frac{dv}{dt} = mg - \mu u\,. \qquad (7)$$

4. Since the rocket should hung motionlessly at some height, we assume that $v = 0$. Substituting $v = 0$ in equation (7) and differentiating it over time, we get

$$-\mu g = \frac{d\mu}{dt}u\,. \qquad (8)$$

Using the initial condition $\mu(0) = m_0 g / u$, we finally find

$$\mu(t) = \frac{m_0 g}{u}\exp\left(-\frac{gt}{u}\right)\,. \qquad (9)$$

5. Substituting $\mathrm{v}(t) = A_1 t + A_2 \ln(1 + A_3 t)$ and $m = m_0 - \mu t$ into equation (7), one gets

$$A_1 = -g\,, \qquad (10)$$

$$A_2 = -u\,, \qquad (11)$$

$$A_3 = -\frac{\mu}{m_0}\,. \qquad (12)$$

6. The rockets achieves its maximum velocity if the fuel burns out almost instantaneously, and, at the same time, the work done by the gravity force, turns out minimal. Thus, the optimal fuel consumption is

$$\mu_{opt} = \infty\,. \qquad (13)$$

Since the gravity force does not have time to affect the rocket velocity, it turns possible to use the Tsiolkovsky formula (2)

$$\mathrm{v} = u \ln\left(\frac{m_0}{m}\right). \qquad (14)$$

Hence, the maximum height of the rocket is

$$H_{\max} = \frac{u^2}{2g}\ln^2\left(\frac{m_0}{m}\right). \qquad (15)$$

7. Suppose that a particle moves with the velocity v' in the reference frame which, in turn, moves with the velocity v in the laboratory reference frame. Then, the particle velocity w in the laboratory reference frame is given by the relativistic formula

$$w = \frac{v + v'}{1 + \frac{vv'}{c^2}} . \tag{16}$$

Hence, we find the relationship between the velocity changes in corresponding reference frames as

$$dw = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{vv'}{c^2}\right)^2} dv' . \tag{17}$$

In accordance with the Lorentz transformations

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - v^2 / c^2}} \tag{18}$$

the time differences in two frames are related as

$$dt = dt' \frac{\left(1 + \frac{vv'}{c^2}\right)}{\sqrt{1 - v^2 / c^2}} . \tag{19}$$

Dividing equation (17) and (19) and assuming $v' = 0$, we finally obtain

$$a_r = \frac{dw}{dt} = \left(1 - \frac{v^2}{c^2}\right)^{3/2} \frac{dv'}{dt'} = \left(1 - \frac{v^2}{c^2}\right)^{3/2} a_p . \tag{20}$$

8. In the proper reference frame the rocket motion is classical, and its acceleration is given by

$$a_p = \frac{dv'}{dt'} = \frac{u}{m} \frac{dm}{dt'} . \tag{21}$$

Now we make use the transformation of acceleration (20) and time (19) for $v' = 0$ to obtain

$$\frac{dm}{dv} = \frac{m}{u(1 - v^2 / c^2)} . \tag{22}$$

Hence, we find that

$$\alpha = \frac{c}{2u} . \tag{23}$$

9. Evaluation gives rise

$$m_0 = m \left(\frac{1 + v / c}{1 - v / c} \right)^{c/2u} = 10^{28630}\, kg. \tag{24}$$

10. Evaluation gives rise

$$m_0 = m \left(\frac{1 + v / c}{1 - v / c} \right)^{c/2u} = 1730\, kg. \tag{25}$$

Grading scheme

№	**Content**	**Points**	
1	Formula (1)	0.25	**0.5**
	Formula (2)	0.25	
2	Formula (3)	0.25	**0.5**
	Correct numerical value in (4)	0.25	
3	Formula (5)	0.25	**0.75**
	Formula (6)	0.25	
	Formula (7)	0.25	

4	Formula (8)	0.25	**0.75**
	Initial condition $\mu(0)=m_0 g/u$	0.25	
	Formula (9)	0.25	
5	Equating the coefficients of the polynomial in time to zero	0.5	**2.0**
	Formula (10)	0.5	
	Formula (11)	0.5	
	Formula (12)	0.5	
6	Formula (13)	0.5	**1.0**
	Formula (14)	0.25	
	Formula (15)	0.25	
7	Formula (16)	0.25	**2.5**
	Formula (17)	0.5	
	Formula (18)	0.25	
	Formula (19)	0.5	
	Formula (20)	0.5	
8	Application of formula (19) to get equation (22)	0.5	**1.5**
	Formula (21)	0.25	
	Formula (22)	0.25	
	Formula (23)	0.5	
9	Correct numerical value in (25)	0.25	**0.25**
10	Correct numerical value in (26)	0.25	**0.25**
Total			**10,0**

Problem 3 Metamaterials (10 points)

1. Consider the conducting layer disposed radially at the interval $[r, r+dr]$. Its conductivity $d\rho$ is

$$d\rho = \sigma_0 \frac{dS}{L} = \beta r \frac{2\pi r dr}{L}, \tag{1}$$

and, hence, the total conductivity is given by

$$\rho = \int_0^R d\rho = \frac{2\pi\beta R^3}{3L}. \tag{2}$$

Thus, the resistance of the wire is found as

$$R_0 = \frac{1}{\rho} = \frac{3L}{2\pi\beta R^3} = 2.39\times10^{-2}\,Ohm. \tag{3}$$

2. The amount of heat generated in the wire per unit time is determined by Joule law

$$P_I = I^2 R_0. \tag{4}$$

In steady regime, the same amount of heat must be removed through the surface of the wire into the environment, therefore, according to the Newton-Richman law

$$P_I = 2\pi RLP_{ext} = 2\pi\alpha RL(T_s - T_0), \tag{5}$$

whence

$$T_s = T_0 + \frac{3I^2}{4\pi^2\alpha\beta R^4} = 297K. \tag{6}$$

3. Consider a cylinder of radius r. Let us find an amount of heat generated per unit time inside that cylinder. To do this, let us find the electric field strength in the wire. According to Ohm's law, the current density is

$$j = \sigma_0 E, \tag{7}$$

therefore, the total current can be written as

$$I = \int_0^r j2\pi r dr = E\int_0^r \sigma_0 2\pi r dr = \frac{2\pi R^3 \beta E}{3}. \quad (8)$$

Hence

$$E = \frac{3I}{2\pi\beta R^3}. \quad (9)$$

The electric power generated in the cylender is determined by the Joule law in differential form

$$P_r = \int_0^r \sigma_0 E^2 2\pi r L dr = \frac{3I^2 L r^3}{2\pi\beta R^6}. \quad (10)$$

It is evident that the power dissipated inside the cylinder must be taken away through the surface of the cylinder, thus,

$$P_r = P = -\kappa 2\pi r L \frac{dT}{dr}. \quad (11)$$

Solving differential equation (11), using (10) together with the initial condition

$$T(R) = T_s, \quad (12)$$

the following solution is obtained in the form

$$T(r) = T_0 + \frac{I^2(\alpha R^3 + 3\kappa R^2 - \alpha r^3)}{4\pi^2 \alpha\beta\kappa R^6}. \quad (13)$$

Thus, the temperature in the center of the wire is

$$T_{\max} = T_0 + \frac{I^2(\alpha R^3 + 3\kappa R^2)}{4\pi^2\alpha\beta\kappa R^6} = 299K. \quad (14)$$

4. The radius change of the wire is determined by the law of thermal expansion of solids and can be written as

$$\delta R_T = \int_0^R \gamma\left[T(r) - T_0\right]dr = \frac{3\gamma(\alpha R + 4\kappa)I^2}{16\pi^2\alpha\beta\kappa R^3} = 5.70\times 10^{-9} m. \quad (15)$$

5. The magnetic field induction is determined by the circulation theorem, which, in this case, is written as

$$B2\pi r = \int_0^r j2\pi r dr = E\int_0^r \sigma_0 2\pi r dr. \quad (16)$$

Using expression (9), we finally obtain

$$B(r) = \frac{\mu_0 I r^2}{2\pi R^3}. \quad (17)$$

6. The energy density of the magnetic field is given by

$$w_B(r) = \frac{B^2(r)}{2\mu_0}, \quad (18)$$

therefore, the energy of the magnetic field inside the wire

$$W_B = \int_0^R w_B(r) 2\pi r L dr = \frac{\mu_0 I^2 L}{24\pi} = 8.33\times 10^{-10} J. \quad (19)$$

7. Let us write the equilibrium condition for the wire layer of small width l and length L, disposed at the interval $r, r+dr$. The total Ampere force acting on this layer is written as

$$dF_A = jB(r)Lldr. \quad (20)$$

Hence, the pressure difference is obtained as

$$dp(r) = \frac{dF_A}{lL} = \frac{3\mu_0 I^2 r^3}{4\pi^2 R^6}dr. \quad (21)$$

Taking into consideration that the pressure at the wire pressure is zero, one gets

$$p(r)=\frac{3\mu_0 I^2(R^4-r^4)}{16\pi^2 R^6}. \tag{22}$$

8. As a result of the mechanical pressure the mechanical stress appears in the crystal lattice whose energy density is determined by the expression

$$w_\sigma=\frac{\sigma^2}{2E}=\frac{p^2(r)}{2E}, \tag{23}$$

thus, the total energy of mechanical deformations is found as

$$W_\sigma=\int_0^R w_\sigma 2\pi rLdr=\frac{3\mu_0^2 I^4 L}{320E\pi^3 R^2}=2.39\times10^{-18}\,J. \tag{24}$$

9. The radius change of the wire is determined by Hooke's law, which, in this case, can be written in the form

$$\varepsilon=\frac{\sigma}{E}=\frac{p(r)}{E}, \tag{25}$$

where ε is the relative change in radius.

Thus, the radius change due to mechanical stress is found as

$$\delta R_\sigma=\int_0^R \varepsilon dr=\frac{1}{E}\int_a^R p(r)dr=\frac{3\mu_0 I^2}{20\pi^2 ER}=1.91\times10^{-12}\,m. \tag{26}$$

10. Comparing expressions (15) and (25) we obtain

$$\gamma=\frac{4\mu_0\alpha\beta\kappa R^2}{5E(\alpha R+4\kappa)}=3.35\times10^{-10}\,K^{-1}. \tag{27}$$

Grading scheme

№	Content	Points	
1	Formula (1)	0.25	**1.0**
	Formula (2)	0.25	
	Formula (3)	0.25	
	Correct numerical numerical value in (3)	0.25	
2	Formula (4)	0.25	**1,0**
	Formula (5)	0.25	
	Formula (6)	0.25	
	Correct numerical numerical value in (6)	0.25	
3	Formula (7)	0.25	**2.5**
	Formula (8)	0.25	
	Formula (9)	0.25	
	Formula (10)	0.25	
	Formula (11)	0.25	
	Formula (12)	0.25	
	Formula (13)	0.5	
	Formula (14)	0.25	
	Correct numerical numerical value in (14)	0.25	
4	Formula (15)	0.25	**0.5**
	Correct numerical numerical value in (15)	0.25	
5	Formula (16)	0.25	**0.5**
	Formula (17)	0.25	
6	Formula (18)	0.5	**1.0**
	Formula (19)	0.25	

	Correct numerical numerical value in (19)	0.25	
7	Formula (20)	0.25	**1.0**
	Formula (21)	0.25	
	Formula (22)	0.5	
8	Formula (23)	0.5	**1.0**
	Formula (24)	0.25	
	Correct numerical numerical value in (24)	0.25	
9	Formula (25)	0.5	**1.0**
	Formula (26)	0.25	
	Correct numerical numerical value in (26)	0.25	
10	Formula (27)	0.25	**0.5**
	Correct numerical numerical value in (27)	0.25	
Total			**10,0**

EXPERIMENTAL COMPETITION

15 January, 2014

Please read the instructions first:

1. The Experimental competition consists of one problem. This part of the competition lasts 3 hours.
2. Please only use the pen that is provided to you.
3. You can use your own non-programmable calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet and additional papers***. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*.** Write only inside the bordered area.
6. Fill the boxes at the top of each sheet of paper with your country (***Country*)**, your student code (***Student Code***), the question number (***Question Number*)**, the progressive number of each sheet **(*Page Number***), and the total number of ***Writing sheets*** (***Total Number of Pages***). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
7. At the end of the exam, arrange all sheets for each problem in the following order:
    - Used ***Writing sheets*** in order;
    - The sheets you do not wish to be evaluated
    - Unused sheets and the printed question.

    Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper or equipment out of the room

Magnetic interactions (15 points)

Instruments and equipment: tripod, pendulum with a bead magnet, clay, ruler, magnetic beads, stopwatch, power supply (battery of the voltage 4.5V) , 6 Ohm rheostat, coil, multimeter, switch, connecting wires, nails, piece of chocolate.

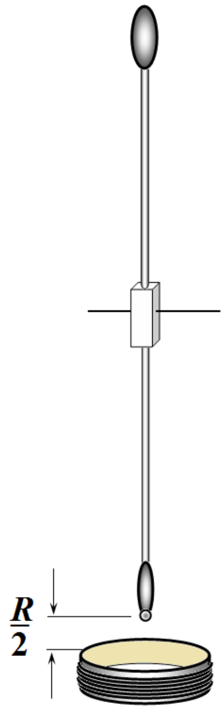
The pendulum consists of two wooden chopsticks stuck into the eraser. There are two pieces of clay fixed at the free ends of both chopsticks, the lower piece of clay contains a metal magnetized bead inside. Pendulum axis is simply a steel needle piercing the eraser.

Another magnetized bead is mounted on another piece of clay provided.

Attention! *Do not change the orientation of magnetized beads during the experiment!*

Part 1. Interaction with the magnetic field of a coil

Put the pendulum axis on the rack of the tripod. Make sure that the pendulum can make free oscillations without touching the tripod with the wooden chopsticks or the eraser. By changing the clay masses at the ends of the chopsticks make the oscillation period larger than 2 seconds. In the equilibrium position both chopsticks should be vertical.



1.1. Measure the oscillation period of the pendulum. Evaluate the corresponding experimental error.

Attention! In the subsequent parts error estimation is not required!

Place the coil under the pendulum so that the distance from the center of the coil to the magnetized bead in the pendulum to be approximately equal to half the radius of the coil.

Connect the coil to the power supply so that you could change and measure the strength of the current flowing through it.

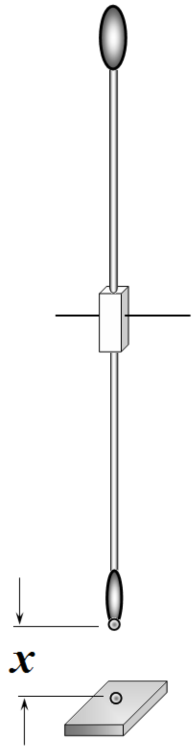
Attention! *Be sure to use the key, switch it on while doing measurements only, otherwise the battery will quickly discharge!*

1.2. Draw schematically an electric circuit that you have used for making measurements.

1.3. Measure the dependence of the oscillation period of the pendulum on the current in the coil. Plot the corresponding graph.

1.4. Based on the experimental data obtained prove that the force, acting on the magnetic bead, is proportional to the current in the coil. Justify your conclusion graphically.

Part 2. Pointlike interactions



Place the magnetized bead fixed on a piece of clay right under the pendulum. The beads should attract each other!

It can be assumed that the interaction force is central, i.e. it is directed along the line connecting the centers of the two beads. The magnitude of that force depends on the distance between the centers of the beads according to the law

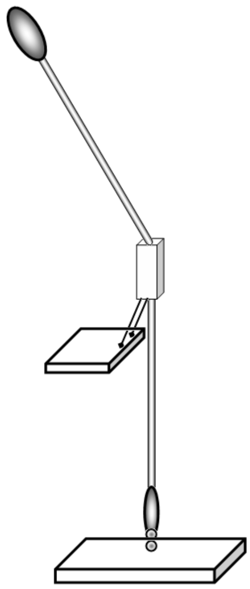
$$F = \frac{C}{r^{\gamma}} . \tag{1}$$

2.1. Put down the equation of motion of the pendulum in this case. Obtain the formula for the period of small oscillations.

2.2. Measure the dependence of the period of small oscillations on the distance between the centers of the beads. Plot the corresponding graph.

2.3. Using the experimental data obtained, evaluate the exponent γ in formula (1). Justify formula (1) graphically.

Part 3. Magnetic piece of chocolate



Let us reduce the damping by changing the design of the pendulum. Stick the upper chopstick into the eraser to make an angle of approximately $30°$. Stick two nails into the eraser to play the role of two legs on which the oscillations are performed. Attach two beads to the bottom of the pendulum. Make sure that the pendulum is steadily balanced on its legs while performing oscillations. To adjust the pendulum you can mount the ruler in the rack as a support.

Place a piece of chocolate under the pendulum, but do not take off its wrap! The lower end of the pendulum should move along the piece of chocolates at a distance of 1-2 *mm*.

It turns out that a piece of chocolate can affect the pendulum motion.

3.1. Establish experimentally which physical characteristics of the pendulum are affected by the piece of chocolate.

3.2. Take necessary measurements to confirm your assumption. Justify your answer graphically.

SOLUTION FOR THE EXPERIMENTAL COMPETITION

Magnetic interactions

Part 1. Interaction with the magnetic field of a coil

1.1. To measure the oscillation period it is necessary to measure the time of at least 10 oscillations several times. The following values for 10 oscillations are obtained:

$t_1 = 25,02s$

$t_2 = 25,06s$.

$t_3 = 24,92s$

Evaluation of the period from this data gives rise to

$$T = \frac{\langle t \rangle}{10} = 2,50s.$$

Experimental error is evaluated by the formula

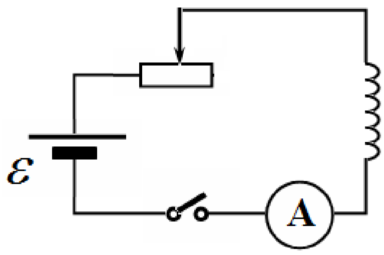
$$\Delta t = 2\sqrt{\frac{\sum_k \left(t_k - \langle t \rangle\right)^2}{n(n-1)}} = 0,08s,$$

and, therefore , the accuracy of the period is equal to $\Delta T = \Delta t / 10 = 0,008s$.

Finally, one can write

$$T = (2,50 \pm 0,01)s. \qquad (1)$$

1.2. The following circuit can be used for measurements (it is also acceptable for the rheostat to be used as a potentiometer).



1.3. Table 1 shows the dependence of the oscillation period of the pendulum againgst the current in the coil and below the corresponding graph is drawn.

Table 1

I, A	T, s	ν^2, s^{-2}	$(\nu^2 - \nu_0^2), s^{-2}$
0,00	2,655	0,142	0,000
0,80	2,062	0,235	0,093
0,37	2,350	0,181	0,039
-0,37	3,083	0,105	-0,037
-0,72	3,724	0,072	-0,070

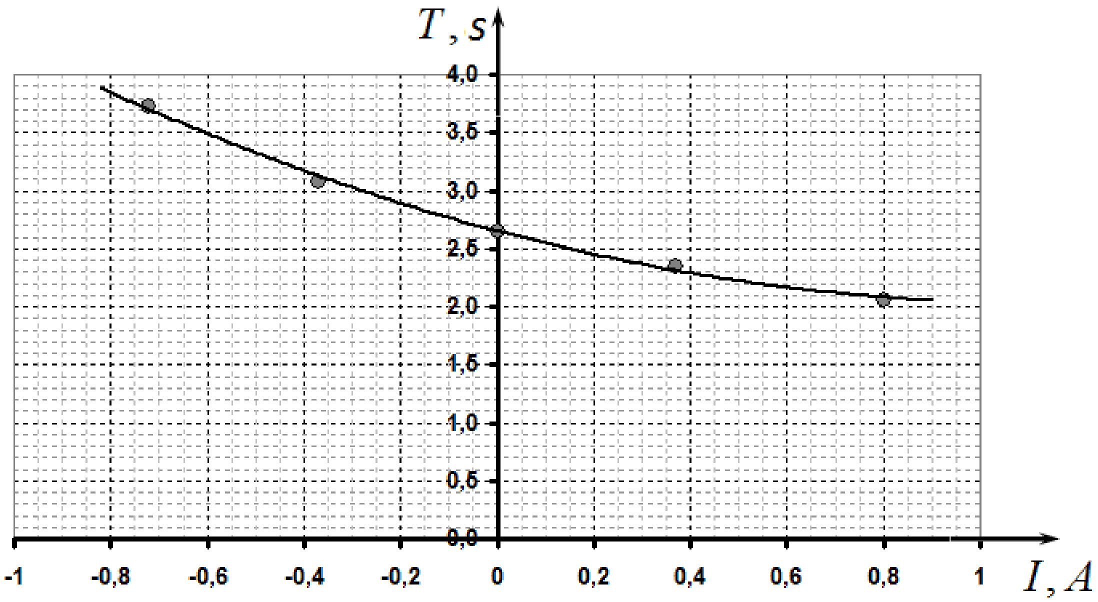


Figure 1. Dependence of the oscillation period of the pendulum on the current in the coil.

1.4. To describe the motion of the pendulum the following equation should be used

$$J\frac{d^2\varphi}{dt^2} = -mga\cdot\varphi - \mu I\varphi , \qquad (2)$$

where φ stands for the angle of deflection of the pendulum from the vertical, J denotes the moment of inertia of the pendulum about the axis of rotation, m is the pendulum mass, a designates the distance from the axis of rotation to the center of the pendulum mass, $\mu I\varphi$ refers to the moment of the force acting on a magnetized bead caused by the magnetic field of the coil. Equation (2) implies that the square of the oscillation frequency depends linearly on the current strength as:

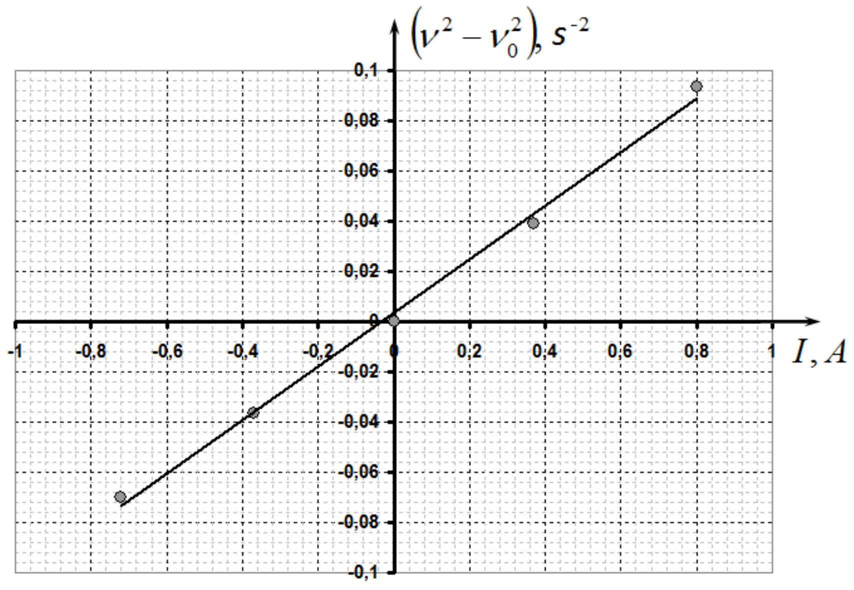
$$\nu^2 = \frac{1}{T^2} = \frac{mga + \mu I}{J}$$

It is clearly seen that the square of the oscillation frequency against the coil current is convinuently represented as

$$\nu^2 - \nu_0^2 = \frac{\mu I}{J},$$

where $\nu_0^2 = \dfrac{mga}{J}$ stands for the square of the oscillation frequency in the absence of the current in the coil.

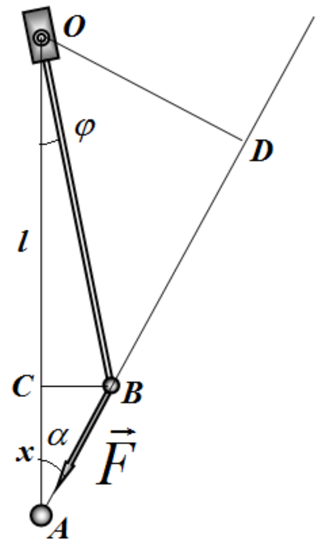
Thus, the linear dependence of the value $\left(\nu^2 - \nu_0^2\right)$ against the current strength proves the assertion of direct proportionality between the strength of the magnetic interaction and the current in the coil. The figure shows the corresponding graph which confirms the linearity assumption.



Part 2 . Pointlike interaction

2.1.
To write the equation of motion it is necessary to correctly calculate the torque of the interaction forces between magnetized beads. Since the force is central, the shoulder is a segment OD, and its length is

$$d = |OD| = (l + x) \cdot \alpha,$$



where l is the distance from the rotation axis to the bead in the pendulum, x stands for the distance between the beads in the equilibrium position. Hereinafter angles are assumed small. The angle α should be expressed through the angle φ of the pendulum deflection. For this purpose the following ratio can be used

$$|CB| = l\varphi = x\alpha,$$

which gives rise to

$$\alpha = \frac{l}{x}\varphi.$$

Thus, the motion of the pendulum is described by the equation

$$J\frac{d^2\varphi}{dt^2} = -mga\varphi - F\frac{l(l+x)}{x}\varphi. \quad (3)$$

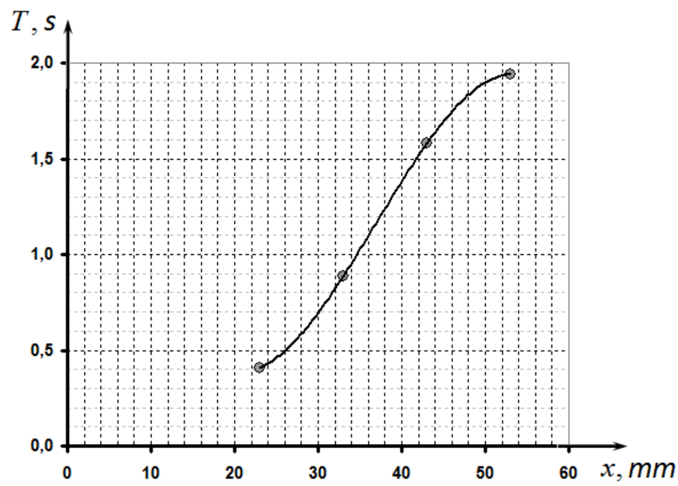
This equation implies that the formula for the oscillation period is given by

$$T = 2\pi\sqrt{\frac{J}{mga + F\dfrac{l(l+x)}{x}}}. \quad (4)$$

2.2. Measurement results of the time of 10 oscillations at different distances between the centers of the beads are given in Table 2 and are drawn in the graph below.

Table 2.

x, mm	t_1,s	t_2,s	t_3,s	T, s
23	4,27	4,09	3,90	0,409
33	8,72	9,03	8,83	0,886
43	15,96	15,73	15,68	1,579
53	19,20	19,64	19,36	1,940



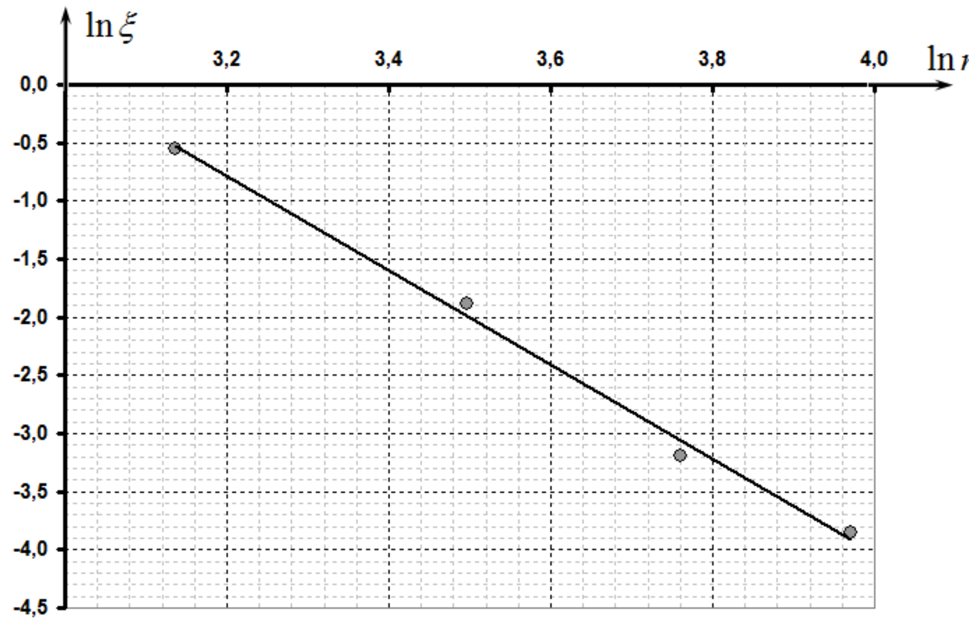
2.3. To determine the exponent it is necessary to express the strength of interaction in terms of measurable characteristics. It follows from the formula for the oscillation period that the change of the squared frequency is found as

$$\nu^2 - \nu_0^2 = F\frac{l(l+x)}{x}\frac{1}{J},$$

This means that the value

$$\xi = (\nu^2 - \nu_0^2)\frac{x}{l+x}$$

is proportional to the strength of the magnetic interaction $F = \frac{C}{r^{\gamma}}$. To determine the exponent it is necessary to plot the dependence of ξ on the distance ξ in a logarithmic scale. The slope coefficient in this graph provides the desired exponent.

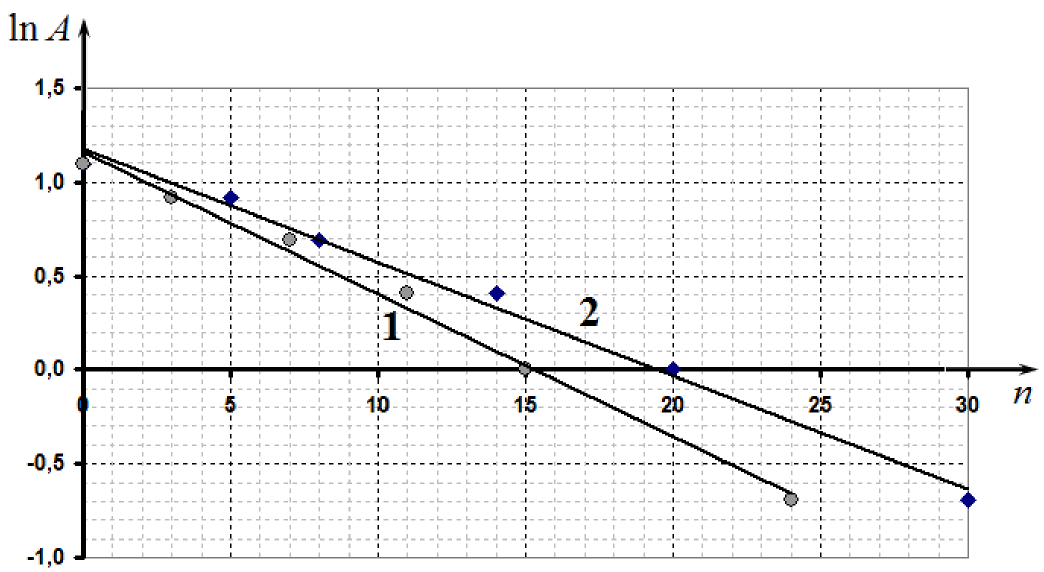


The figure shows the corresponding graph. It follows from this graph that the exponent is equal to $\gamma = 4$.

Part 3 . Magnetic chocolate

3.1 Chocolate does not affect the period of oscillation, but significantly increases the damping of oscillations. This occurs due to the occurrence of eddy currents in the foil.

3.2 In order to prove this, one can measure the dependence of the oscillation amplitude on the time (or, equivalently, on the number of oscillations in semi-logarithmic scale.
The graph below shows the corresponding data with (1) and without (2) chocolate. The graphs show an increase in the damping of oscillations in the presence of chocolate.



Grading scheme

№	Part of the problem	Total for the part	Points
1.1	Measurement of the period: - Period is larger than 2 seconds; - Not less than 3 measurements; - Each measurement includes at least 10 periods; - The average value is found; - Random error is evaluated;	0,5	 0,1 0,1 0,1 0,1 0,1
1.2	 Circuit diagram (all elements connected in series): - Source; - Coil; - Rheostat (two possible ways); - Key; - Ammeter;	0,5	 0,1 0,1 0,1 0,1 0,1
1.3	 Measurements (counted only if the period is measured in the range 1-5 s) - Measured at 7 (5, 3, less) values of the current; - Current flows in two directions; - Change in the period is not less than 50% (20% less); - Measured not less than 5 oscillations; Plotting: - Axis signed and digitized; - All the points of the table are plotted; - A smooth line is drawn;	2,0	 1*(0,5; 0,3; 0)* 0,4 0,2 *(0,1;0)* 0,1 0,1 0,1 0,1
1.4	 Linearization: - Dependence of the squared frequency on the current is linear; - all the points are included in evaluation; Plotting: - Axis signed and digitized; - All the points of the table are plotted; - A smooth straight line is drawn; Conclusions on the validity	1,0	 0,4 0,2 0,1 0,1 0,1 0,1
2.1	 The equation of motion: - General view (the dynamics of rotational motion); - Torque of the gravity force; - Torque of the magnetic interaction force; Formula for the period of oscillation	1,0	 0,3 0,2 0,3 0,2
2.2	Measurements of the oscillation period (counted only if the period is the range of 0.3-5 s) - Measurements for 7 (5.3 less) distances; - Change in the period is not less than 4 times (2 times, or less); Plotting: - Axis signed and digitized; - All the points of the table are plotted;	4,0	 2,5*(1,5; 1,0; 0)* 1,2*(0,5;0)* 0,1 0,1

	- A smooth line is drawn;		0,1
2.3	Determination of the exponent The correct linearization is found; The parameters of the linearized dependence are determined; Plotting: - Axis signed and digitized; - All the points of the table are plotted; - A smooth straight line is drawn; The slope lies in the range from 2 to 6; The exponent is found to be equal to 4;	4,0	 2,0 0,5 0,1 0,1 0,1 *(0,6)* 1,2
3.1	 Period is constant; Damping is increased; The reason is the eddy currents in the chocolate bar.	0,5	 0,2 0,2 0,1
3.2	 The dependence of the amplitude on the number of oscillations (it is equivalent to measure the number of oscillations for amplitude decrease in the specified limits) Plotting: - Axis signed and digitized; - All the points of the table are plotted; - A smooth line is drawn;	1,5	 1,2 0,1 0,1 0,1
	TOTAL	15	

THEORETICAL COMPETITION

January 13, 2015

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three Problems.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets.*** Write only inside the boxed area.
6. Begin each Problem solution on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Problem Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
   - Used ***Writing sheets*** in order;
   - The sheets you do not wish to be evaluated
   - Unused sheets and the printed question.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

Problem 1. Plotting (7 points)

For each part of this Problem, please, use the provided separate sheets of papers with figures in which you should do all necessary plotting. Plotting and its justification should be made on the same separate sheets.

Problem 1.1 (2.0 points)

Three small positively charged balls (whose charges are different) with masses $m, 2m, 3m$, are connected to each other by inextensible nonconductive threads such that the balls are located at the vertices of the equilateral triangle $A_1A_2A_3$ (see figure 1.1). When the threads connecting balls have been cut down (not simultaneously) all balls start to move on the same plane. In figure 1.1 the positions B_1 and B_2 of two balls are shown at some time moment. With the help of geometric constructions, indicate position B_3 of third ball at the same time moment.

Problem 1.2 (2.0 points)

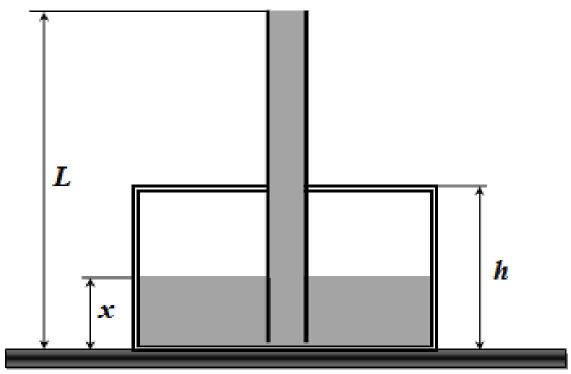
In figure 1.2, logarithmic scale is used to show straight lines of two processes performed by an ideal diatomic gas (P_0, V_0 are some constants). Plot a cyclic process lying in between those lines, which has maximum efficiency. In the same figure, extreme points of that cycle are indicated, with A_1 being the point of the minimum volume and A_3 being the point of the maximum volume of the gas. Find the efficiency of the cycle.

Problem 1.3 (3.0 points)

In figure 1.3, positions are shown of the point light source S and its image S' created by a thin lens. In the same figure, OO_1 is the main optical axis of the lens. Plot an image of the point source S_1 created by the same thin lens.

Problem 2. Vessel with water (7 points)

In the cylindrical vessel with the cross section area $S = 0.500\ m^2$ and height $h = 0.500\ m$ a tube of length $L = 2.00\ m$ with opened ends is inserted vertically through the hermetically sealed lid. The lower end of the tube is a bit above the bottom of the vessel. Water with the density $\rho = 1000\ kg/m^3$ is poured into the vessel as shown in the figure on the right. The cross section area of the tube is much smaller than the cross section area of the vessel and the vessel wall material conducts heat very well. Assume that the atmospheric pressure is $p_0 = 1.01 \cdot 10^5\ Pa$, the ambient temperature is $T_0 = 293\ K$ and the acceleration of gravity is $g = 9.80\ m/s^2$.



1. **[2.0 points]** Find the height of the water level in the vessel $x = x_0$ at the time moment when the tube is completely filled with water. Express your answer in terms of p_0, ρ, g, h, L, and find its numerical value.

Now, assume that the walls of the vessel and the tube are coated with a material which does not conduct heat at all. The air inside the vessel is then heated fast enough such that the water does not have enough time to warm up.

2. **[0.5 points]** Find air pressure inside the vessel $p(x)$ as a function of x. Express your answer in terms of p_0, ρ, g, L, x.
3. **[1.0 points]** Find air temperature inside the vessel $T(x)$ as a function of x. Express your answer in terms of p_0, ρ, g, L, x.
4. **[1.0 points]** Find the temperature T_m to which the air in the vessel must be heated in order to displace all the water from the vessel. Express your answer in terms of p_0, ρ, g, L, T_0 and find its numerical value.

5. **[2.5 points]** Find the amount of heat Q, which must be given to the air in the vessel in order to displace all the water from the vessel. Express your answer in terms of $p_0, \rho, g, h, L, S,$ and find its numerical value.

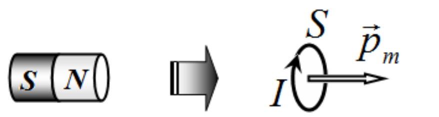
Problem 3. Delay and attenuation (16 points)

In this problem, do not take into account the finiteness of the propagation speed of the electromagnetic interaction.

Part 1: Magnet

1.1 Theoretical introduction

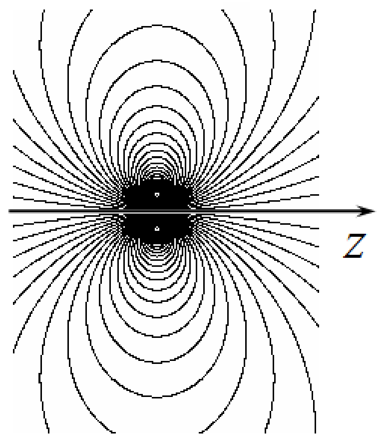
The magnetic field generated by a uniformly magnetized ferromagnetic cylinder (permanent magnet) is equivalent, at very large distances, to the field produced by a circular coil with a constant electric current.



The cylindrical magnet, as well as the coil with the current, are characterized by magnetic moment p_m, which is defined for the current loop as the product of the current and the area of the loop,

$$p_m = IS .$$

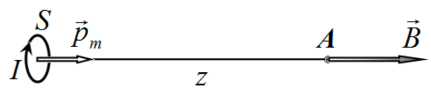
Such a source of magnetic field is also referred to as a *magnetic dipole*. The figure shows the magnetic field lines of such a dipole.



1.1.1. **[0.75 points]** Show that the magnetic field, B_z, on the axis of the dipole is determined at large distances by the formula

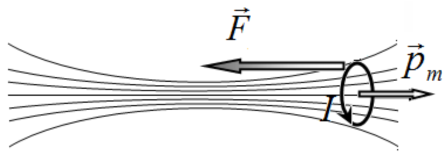
$$B_z = b\frac{p_m}{z^\beta},$$

where z is the coordinate measured along the axis of the dipole from its center. Find the values of the parameters b and β in the above formula.



1.1.2. **[1 point]** Let the coil with a current (i.e., magnetic dipole) with the magnetic moment p_m is influenced by an inhomogeneous axially symmetric field, the induction of which along the z-axis depends on z as function $B_z(z)$. Dipole axis coincides with the axis of symmetry of the field. Show that the force acting on the dipole from the magnetic field is given by

$$F_z = -p_m\frac{dB_z}{dz}.$$



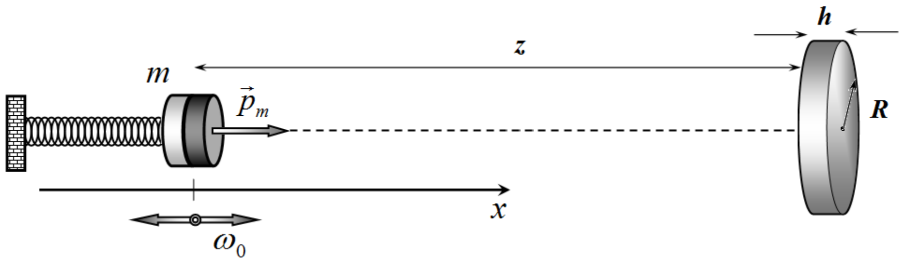
1.2 Oscillations of the magnet

Cylindrical magnet of mass m and magnetic moment p_m, is attached to the spring of stiffness k such that it can oscillate along the horizontal axis which is directed along the magnetic moment.

1.2.1. **[0.25 points]** Find the frequency ω_0 of free oscillations of the magnet in the absence of external fields.

At some distance z from the equilibrium position of the magnet a small metal disc is placed such that its axis coincides with the axis of the magnet. The disk has radius R and thickness h ($h \ll R \ll z$), the electrical resistivity of the disk material is ρ, and the magnetic permeability is put equal to $\mu = 1$. The magnet is moved from the equilibrium position and starts performing small oscillations described by function $x(t)$, where $x \ll z$.

1.2.2. **[2 points]** Find the force $F(x, v)$ exerted by the disc on the magnet as a function of its coordinate x and velocity v. Write down the equation of motion of the magnet.



1.2.3. **[0.75 points]** Find the relative change $\Delta\omega/\omega_0$ in the oscillation frequency of the magnet caused by the influence of the disc.

1.2.4. **[0.25 points]** Assuming that the attenuation is rather weak, obtain the characteristic attenuation time of the oscillations of the ball.

1.2.5. **[1.5 points]** Show that the loss of the mechanical energy of the magnet is equal to the amount of heat released in the disk for the same time period.

Mathematical tip

The equation of attenuating oscillations

$$d^2x/dt^2 + 2\beta dx/dt + \omega_0^2 x = 0$$

has the solution

$$x(t) = A\exp\left(-\frac{t}{\tau}\right)\cos(\omega t + \varphi),$$

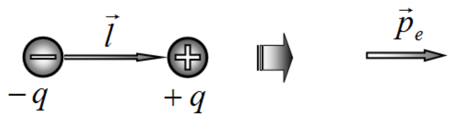
where $\omega = \sqrt{\omega_0^2 - \beta^2}$ is the frequency of attenuating oscillations, $\tau = 1/\beta$ is the characteristic attenuation time, and the parameters A, φ are determined by the initial conditions.

It is known that if $x \ll 1$ *the following approximate inequality holds* $(1+x)^\alpha \approx 1 + \alpha x$.

Part 2: Electric

2.1 Theoretical introduction

The system of two identical in magnitude and opposite in sign charges $(-q, +q)$, located at some fixed distance l from each other is called an *electric dipole* and is characterized by the dipole moment



$$p_e = ql.$$

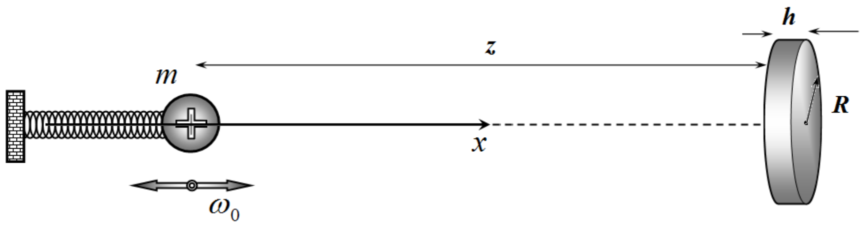
2.1.1. **[0.75 points]** The electric field generated by the dipole on its axis at the distance $z \gg l$, is defined by the formula

$$E = a\frac{p_e}{z^\alpha}.$$

Obtain the parameters a, α in this formula.

2.2 Fluctuations of charged ball

A small ball of mass m carrying the electric charge q is attached to a non-conductive spring of stiffness k and can oscillate along the horizontal axis x. At some distance z from the position of the equilibrium of the ball, a small metal perfectly conducting disk is fixed such that its axis coincides with the x axis. The disc has radius R and thickness h ($h \ll R \ll z$).



2.2.1. **[0.75 points]** Find the shift of the equilibrium position of the ball caused by the influence of the disc.

2.2.2. **[0.75 points]** Find the relative change in the oscillation frequency of the ball, $\Delta\omega/\omega_0$, caused by the influence of the disc.

Assume now that the electrical resistivity of the material of the disk is ρ (not zero).

2.2.3**. [1.5 points]** Obtain an equation describing the time variation of the induced dipole moment of the disc (i.e., the equation relating the dipole moment of the disk p and its rate of change over time dp/dt).
2.2.4. **[0.25 points]** Assuming that the disk is a capacitor whose plates are connected to each other by a resistor, obtain the characteristic time of the equivalent RC-circuit. Express your answer in terms of resistivity ρ of the material of the disk.

Assume in the following that the characteristic time obtained in 2.2.4 is much smaller than the oscillation period of the ball.
2.2.5. **[0.25 points]** Write down the relation between ω and ρ expressing the above stated assumption.

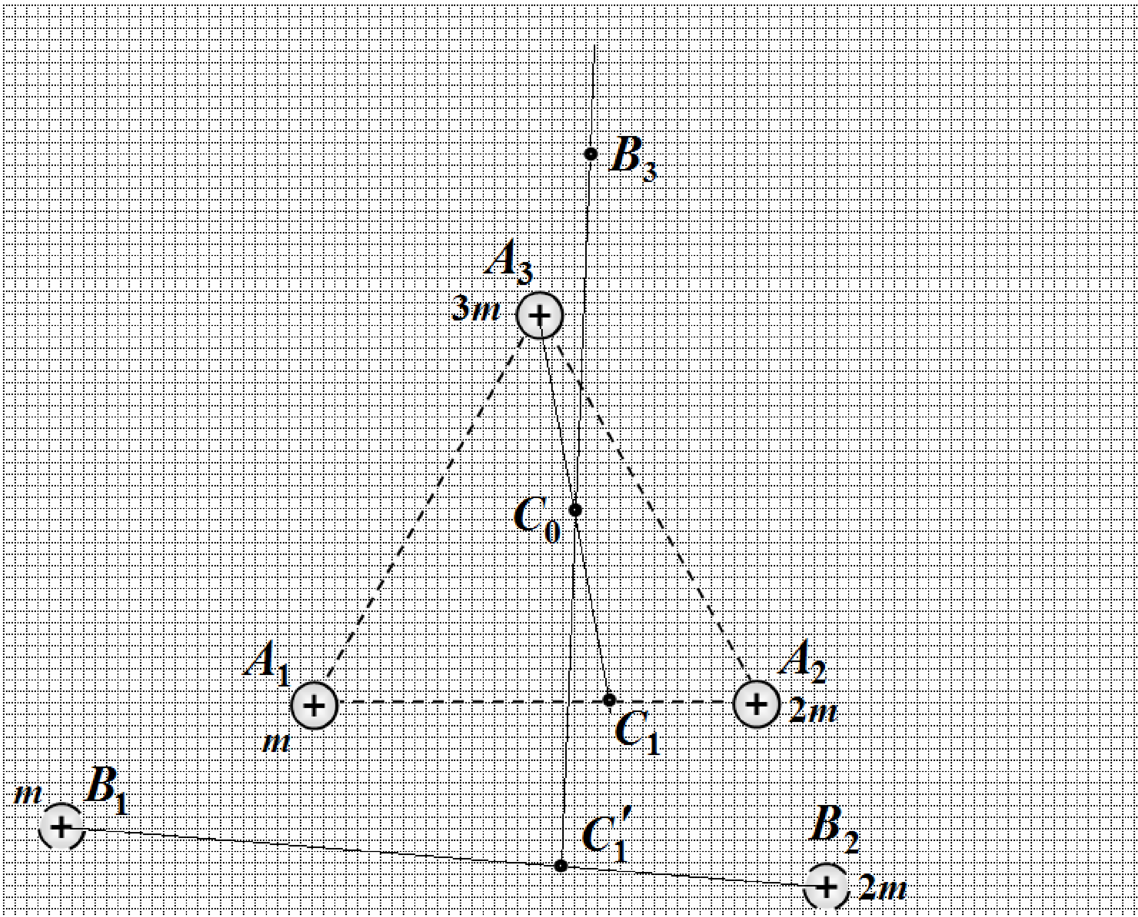
In the case of ideal conductivity of the disk the oscillations of the ball do not attenuate. At low resistivity of the material of the disc the attenuation of oscillations should also be small, and such oscillation can be approximately regarded as harmonic ones.
2.2.6. **[2 points]** Using this approximation and the equation derived in 2.2.3, obtain the expression of the dipole moment p of the disk via coordinate x and the speed v of the ball.
2.2.7. **[1.5 points]** Find expression for the force exerted on the ball by the disk. Write down the equation of motion of the ball.
2.2.8. **[0.25 points]** Find the characteristic attenuation time of the oscillations of the ball.
2.2.9. **[1.5 points]** Show that the loss of mechanical energy of the ball is equal to the heat generated in the disk for the same time period.

SOLUTIONS TO THE PROBLEMS OF THE THEORETICAL COMPETITION

Problem 1. Plotting (7 points)

Problem 1.1

The basic idea is that the center of mass of the whole system remains at rest! First of all, find the center of mass at the initial position: Divide the segment A_1A_2 by the ratio of 2:1 (point C_1); connect C_1 with point A_3 and cut segment C_1A_3 into halves (point C_0). The point C_0 is center of mass of the system. To find the position of the third ball one should do the following: divide the segment B_1B_2 by the ratio of 2:1 (point C_1'), write down straight line from to the point of the center of mass C_0, and draw the segment C_0B_3, whose length should be equal to the length of the segment $C_1'C_0$. The point B_3 is the position of third ball!



Marking scheme

1	The basic idea of the constancy of the center of mass is formulated	**1,0**
2	Found position of the center of mass - in circle 1; - in circle 2; - in circle 3;	 **0,5** ***(0,3)*** ***(0,2)***
3	Found position of the third ball: - in circle 1;	 **0,5**

	- in circle 2; - in circle 3;	***(0,3)*** ***(0,2)***
	Total	**2**

Problem 1.2

Straight lines, shown in figure 1.2, are isotherms since their slope is equal to -1. Consequently, their equations have the form

$$PV = const\,. \tag{1}$$

To obtain a cycle with the maximum efficiency it is necessary to build two adiabatic lines through the extreme points, i.e. Carnot cycle. Since the adiabatic equation has the form

$$PV^{\gamma} = const\,, \tag{2}$$

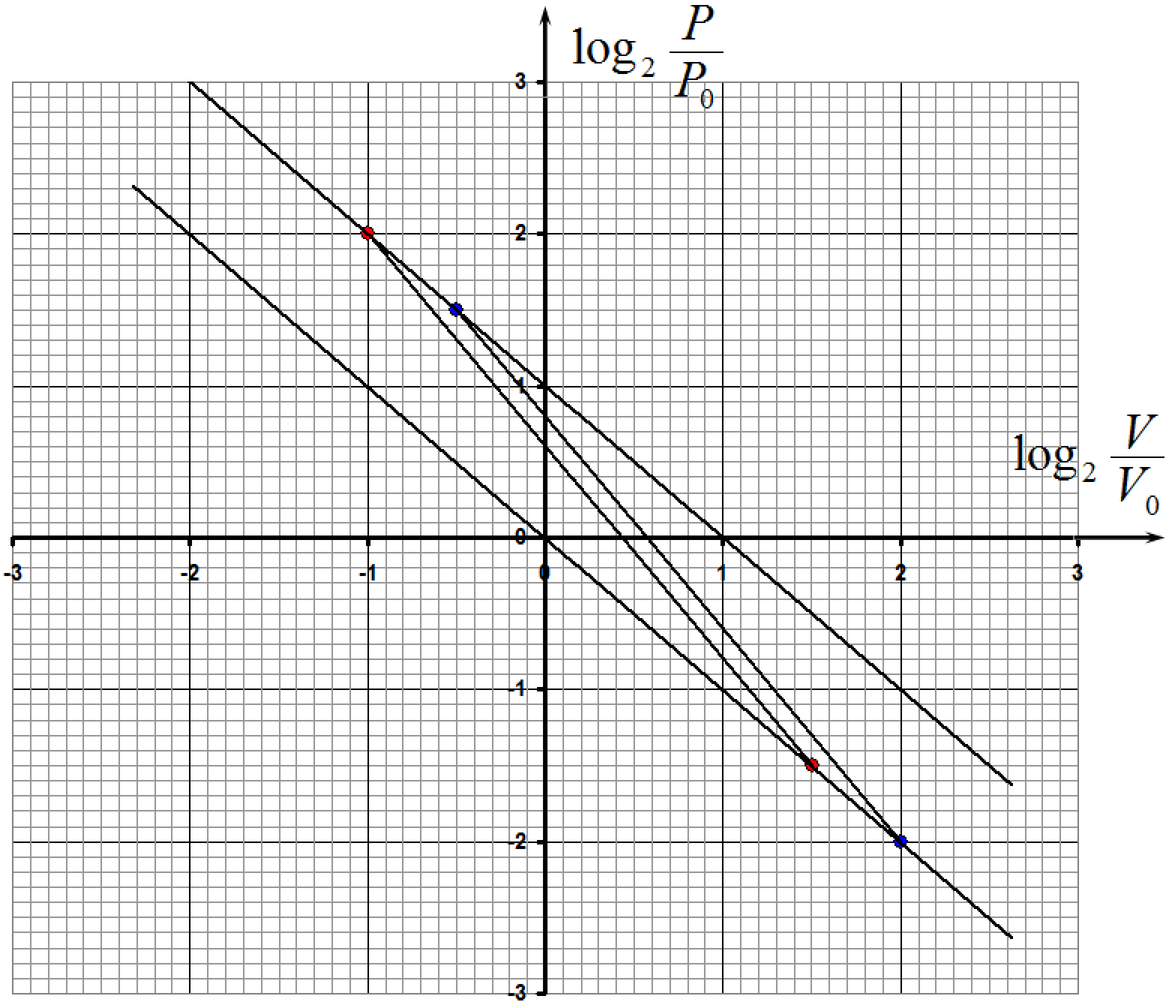
where $\gamma = \dfrac{C_p}{C_V} = \dfrac{7}{5} = 1{,}4$ is the adiabatic index of a diatomic gas.

In the logarithmic scale, the latter equation has the form

$$\log_2 P = const - 1{,}4\log_2 V\,. \tag{3}$$

The graphs of these functions are straight lines with the slope coefficient of -1.4. The constant in equation (1) is proportional to the absolute temperature. It follows from the shown graphs that the maximum temperature is 2 times greater than the minimal one, so that the cycle efficiency is

$$\eta = 1 - \frac{T_2}{T_1} = 50\%\,. \tag{4}$$

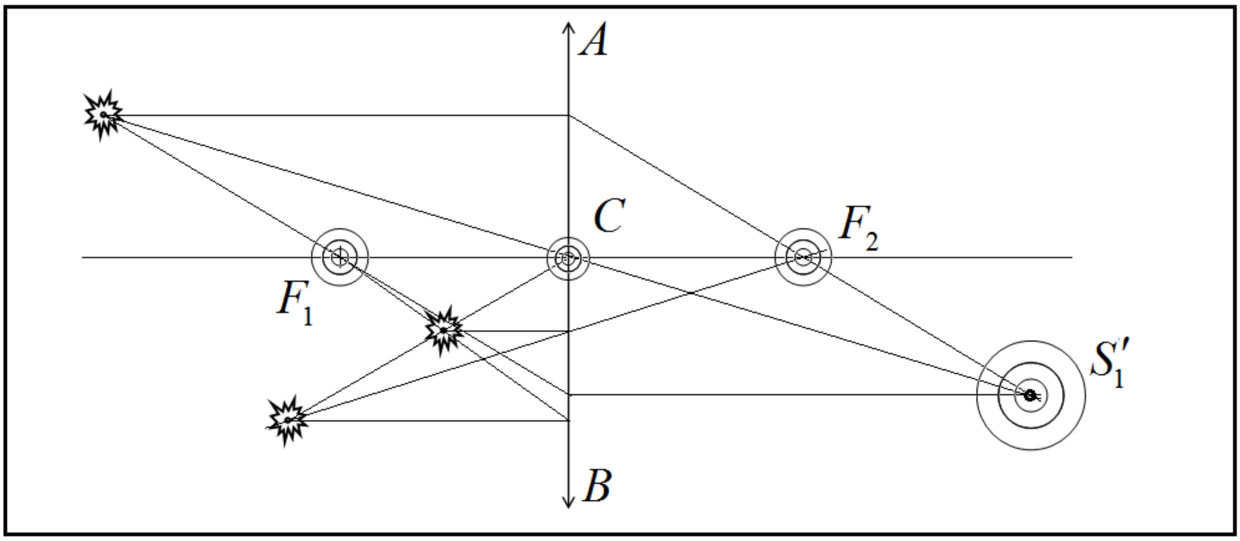


Marking scheme

1	It has been indicated that the straight lines are isotherms	**0,5**
2	Carnot cycle has been taken	**0,2**
3	Adiabatic equation has been written, including the formulation of it in logarithmic scale	**0,2** **0,2**
4	Straight lines have been properly plotted	**0,6**
5	The efficiency has been found	**0,3**
	Total	**2**

Problem 1.3

1. Plot straight line SS' to the intersection with the optical axis. This intersection point is the center of the lens.
2. To determine the positions of focuses, draw the line through point-source parallel to the optical axis, then draw the line through the point of intersection between the ray and the plane of the lens. Then plot the straight line through the point-image to the intersection with the main optical axis: this is the back focus of the lens. Similarly, we find the front focal point. Having focal points determined, finding the image of the second source is carried out in a conventional manner.



Marking scheme

1	The optical center of the lens has been found - in circle 1; - in circle 2; - in circle 3;	 **0,5** **(0,3)** **(0,2)**
2	Focal points of the lens are found and plotted - in circle 1; - in circle 2; - in circle 3;	 **1,0** **(0,6)** **(0,4)**
3	Image of the second source is found and plotted - in circle 1;	 **1,5**

	- in circle 2; - in circle 3;	**(1,0)** **(0,5)**
	Total	**3**

Problem 2. Vessel with water (7 points)

1. While pouring the water into the vessel the air is compressed and its pressure increases. At the moment when the tube is completely filled with water the air pressure inside the vessel is equal to

$$p = p_0 + \rho g(L - x_0). \tag{1}$$

Since the vessel wall highly conducts heat, the temperature of the air inside the vessel does not change, so the equations of state are

$$p_0 Sh = \nu RT_0, \tag{2}$$

$$pS(h - x_0) = \nu RT_0, \tag{3}$$

where S is the cross section area of the vessel, ν is number of moles of the air inside the vessel.

From Eq. (1) - (3) the following quadratic equation is obtained:

$$\rho g x_0^2 - [p_0 + \rho g(L + h)]x_0 + \rho g hL = 0, \tag{4}$$

which has the obvious solution:

$$x_0 = \frac{1}{2}\left[\frac{p_0}{\rho g} + L + h \pm \sqrt{\left(\frac{p_0}{\rho g} + L + h\right)^2 - 4hL}\right]. \tag{5}$$

Among two possible solutions (5), we should choose the one with the less value since it should be $x_0 = h$ at $p_0 = 0$ or $x_0 = 0$ at $L = 0$, that is,

$$x_0 = \frac{1}{2}\left[\frac{p_0}{\rho g} + L + h - \sqrt{\left(\frac{p_0}{\rho g} + L + h\right)^2 - 4hL}\right]. \tag{6}$$

Substituting the numerical values gives

$$x_0 = 7.86 \cdot 10^{-2}\ m. \tag{7}$$

2. The water is at equilibrium and, thus, the air pressure inside the vessel is found as a function of x as follows

$$p(x) = p_0 + \rho g(L - x). \tag{8}$$

3. Equation of state of an ideal gas for an arbitrary x is given by

$$p(x)S(h - x) = \nu RT(x), \tag{9}$$

which, together with Eq.(1), yields

$$T(x) = T_0\left(1 - \frac{x}{h}\right)\left(1 + \frac{\rho g(L-x)}{p_0}\right). \tag{10}$$

4. The temperature, at which the air displaces water out of the vessel, is determined by the condition $x = 0$, which, in accordance with (10), leads to

$$T_m = T_0\left(1 + \frac{\rho g L}{p_0}\right), \tag{11}$$

and the corresponding numerical value is evaluated as

$$T_m = 350\ K. \tag{12}$$

5. The change of the internal energy of the air is obtained as

$$\Delta U = \frac{5}{2}\nu R(T - T_0) = \frac{5}{2}\rho g L S h, \tag{13}$$

and the work done by the air to displace the water, is calculated as

$$A = \int_0^{x_0} p(x)S dx = \frac{1}{2}p_0 SL\left(1 + \frac{p_0}{2\rho g L} + \frac{\rho g L}{2p_0}\left[1 + \frac{2h}{L} - \frac{h^2}{L^2}\right]\right) -$$

$$- \frac{1}{4}p_0 SL\left(1 + \frac{\rho g(L-h)}{p_0}\right)\sqrt{\left(1 + \frac{h}{L} + \frac{p_0}{\rho g L}\right)^2 - \frac{4h}{L}}. \tag{14}$$

According to the first law of thermodynamics, the heat given to the air is found as

$$Q = \Delta U + A, \tag{15}$$

which, together with Eqs. (13) and (14), yields

$$Q = \frac{1}{2}p_0 SL\left(1 + \frac{p_0}{2\rho g L} + \frac{\rho g L}{2p_0}\left[1 + \frac{12h}{L} - \frac{h^2}{L^2}\right]\right) -$$

$$-\frac{1}{4}p_0SL\left(1+\frac{\rho g(L-h)}{p_0}\right)\sqrt{\left(1+\frac{h}{L}+\frac{p_0}{\rho gL}\right)^2-\frac{4h}{L}}. \quad (16)$$

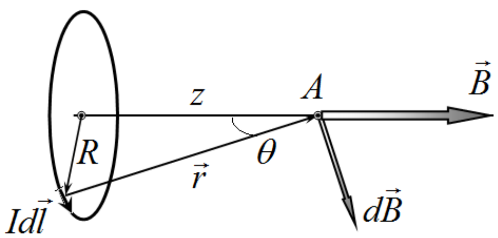
Substituting the numerical values gives

$$Q = 17.0\ kJ. \quad (17)$$

Marking scheme

№	Content	баллы	
1	Formula (1) $p = p_0 + \rho g(L - x_0)$	0,25	**2,0**
	Formula (2) $p_0Sh = \nu RT_0$	0,25	
	Formula (3) $pS(h - x_0) = \nu RT_0$	0,25	
	Formula (4) $\rho g x_0^2 - [p_0 + \rho g(L + h)]x_0 + \rho ghL = 0$	0,25	
	Formula (5) $x_0 = \frac{1}{2}\left[\frac{p_0}{\rho g} + L + h \pm \sqrt{\left(\frac{p_0}{\rho g} + L + h\right)^2 - 4hL}\right]$	0,25	
	Formula (6) $x_0 = \frac{1}{2}\left[\frac{p_0}{\rho g} + L + h - \sqrt{\left(\frac{p_0}{\rho g} + L + h\right)^2 - 4hL}\right]$	0,5	
	Formula (7) $x_0 = 7{,}86 \cdot 10^{-2}\ m$	0,25	
2	Formula (8) $p(x) = p_0 + \rho g(L - x)$	0,5	**0,5**
3	Formula (9) $p(x)S(h - x) = \nu RT(x)$	0,5	**1,0**
	Formula (10) $T(x) = T_0\left(1 - \frac{x}{h}\right)\left(1 + \frac{\rho g(L-x)}{p_0}\right)$	0,5	
4	Formula (11) $T_m = T_0\left(1 + \frac{\rho gL}{p_0}\right)$	0,5	**1,0**
	Formula (12) $T_m = 350\ K$	0,5	
5	Formula (13) $\Delta U = \frac{5}{2}\nu R(T - T_0) = \frac{5}{2}\rho gLSh$	0,5	**2,5**
	Formula (14) $A = \frac{1}{2}p_0SL\left(1 + \frac{p_0}{2\rho gL} + \frac{\rho gL}{2p_0}\left[1 + \frac{2h}{L} - \frac{h^2}{L^2}\right]\right) - -\frac{1}{4}p_0SL\left(1 + \frac{\rho g(L-h)}{p_0}\right)\sqrt{\left(1 + \frac{h}{L} + \frac{p_0}{\rho gL}\right)^2 - \frac{4h}{L}}.$	0,5	
	Formula (15) $Q = \Delta U + A$	0,5	
	Formula (16) $Q = \frac{1}{2}p_0SL\left(1 + \frac{p_0}{2\rho gL} + \frac{\rho gL}{2p_0}\left[1 + \frac{12h}{L} - \frac{h^2}{L^2}\right]\right) - -\frac{1}{4}p_0SL\left(1 + \frac{\rho g(L-h)}{p_0}\right)\sqrt{\left(1 + \frac{h}{L} + \frac{p_0}{\rho gL}\right)^2 - \frac{4h}{L}}.$	0,5	
	Formula (17) $Q = 17{,}0\ kJ$	0,5	
Total			**7,0**

Problem 3. Delay and attenuation (16 points)



1.1.1 It seems simpler to find the magnetic induction of the ring with the current using Biot-Savart law and the superposition principle. The z projection of the magnetic field vector generated by an arbitrary element of the ring is found as

$$dB_z = \frac{\mu_0}{4\pi}\frac{Idl}{r^2}\sin\theta \quad (1)$$

Using the geometry, one obtains

$$dB_z = \frac{\mu_0}{4\pi}\frac{Idl}{r^2}\frac{R}{r} = \frac{\mu_0}{4\pi}\frac{IdlR}{r^3} = \frac{\mu_0}{4\pi}\frac{IdlR}{\left(R^2+z^2\right)^{\frac{3}{2}}} \quad (2)$$

Summation over all elements of the ring is held elementary to eventually get

$$B_z = \frac{\mu_0}{4\pi}\frac{IR\cdot 2\pi R}{\left(R^2+z^2\right)^{\frac{3}{2}}} = \frac{\mu_0}{2\pi}\frac{I\cdot \pi R^2}{\left(R^2+z^2\right)^{\frac{3}{2}}} = \frac{\mu_0}{2\pi}\frac{p_m}{\left(R^2+z^2\right)^{\frac{3}{2}}}. \quad (3)$$

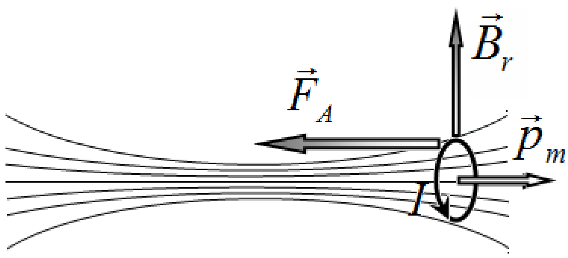
At $z >> R$ it simplifies to

$$dB_z = \frac{\mu_0}{2\pi}\frac{p_m}{\left(R^2+z^2\right)^{\frac{3}{2}}} \approx \frac{\mu_0}{2\pi}\frac{p_m}{z^3}, \quad (4)$$

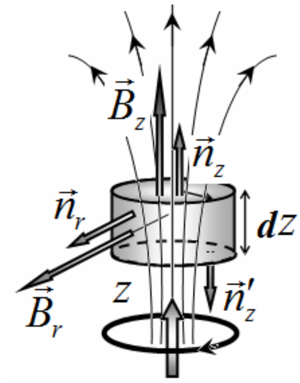
and finally,

$$b = \frac{\mu_0}{2\pi} \qquad \beta = 3 \quad (5)$$

1.1.2 It is obvious that the resulting Ampere force F_A is due to the radial component $\vec{B}_r$ of the magnetic field vector. At a short distance from the axis this component can be expressed in terms of the axial component of the field with the help of the magnetic flux theorem.



Selecting the surface shaped as a thin cylinder whose axis coincides with the axis of the field, then writing the expression for the magnetic flux through this surface and equating it to zero yields



$$B_z(z+dz)\cdot\pi r^2 - B_z(z)\cdot\pi r^2 + B_r\cdot 2\pi r dz = 0 \quad (6)$$

It is thus found from this equation that

$$B_r = -\frac{r}{2}\frac{dB_z}{dz} \quad (7)$$

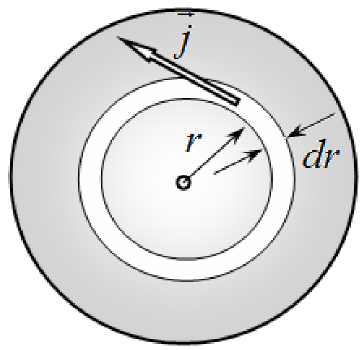
Using Ampere's law, it is easy to write the equation for the force acting on the ring as

$$F = IB_r 2\pi r = -I\frac{r}{2}\frac{dB_z}{dz}2\pi r = -p_m\frac{dB_z}{dz}. \quad (8)$$

1.2.1 The frequency of oscillations is determined by the well-known formula for the oscillation frequency of the spring pendulum

$$\omega_0 = \sqrt{\frac{k}{m}}. \quad (9)$$

1.2.2 The feedback mechanism from the disc to the magnet is described as follows: when the magnet moves the rotational electric field is induced in the disk which gives rise to eddy currents. The magnetic field of those eddy currents is a source of force acting on the magnet. For the calculation of the magnetic field generated by eddy currents in the disk, it should be noted that the size of the disk is small compared with the distance to the magnet, so the disk can also be considered as a magnetic dipole. Therefore, it suffices to find the induced magnetic moment of the disk, and then to use formula (4) for calculation of the induction field.



Let us consider a thin ring of radius r and thickness dr. The strength of the vortex electric field within this ring is found from Faraday's law of induction as:

$$2\pi r E = -\pi r^2\frac{dB_z}{dt} \quad\Rightarrow\quad E = -\frac{r}{2}\frac{dB_z}{dt}. \quad (10)$$

It is assumed here that within the entire disc one can neglect the variation of the axial component of the magnetic field generated by the magnet, which is also determined by formula (4). Thus, if the magnet coordinate is x, then the magnetic field at its position is obtained as

$$B_z = \frac{\mu_0}{2\pi}\frac{p_m}{(z-x)^3}. \tag{11}$$

Substituting this expression into (10) and calculating the derivative gives rise to

$$E = -\frac{r}{2}\frac{dB_z}{dt} = -\frac{r}{2}\cdot\frac{3\mu_0}{2\pi}\frac{p_m}{(z-x)^4}\frac{dx}{dt} \approx -\frac{3\mu_0}{4\pi}r\frac{p_m}{z^4}v \tag{12}$$

In the last formula it is taken into account that $x \ll z$.

The current density in the ring under consideration is determined by Ohm's law as

$$j = \frac{1}{\rho}E. \tag{13}$$

Since the current in the ring is $dI = jhdr$, the magnetic moment of the ring is equal to

$$dp'_m = \pi r^2 dI = \pi r^2 jdr\cdot h = -\pi r^2 dr\cdot h\frac{3\mu_0}{4\pi\rho}r\frac{p_m}{z^4}v = -\frac{3\mu_0}{4\rho}\frac{p_m}{z^4}hvr^3dr. \tag{14}$$

Integrating over the disk, one obtains its total magnetic moment as:

$$p'_m = -\frac{3\mu_0}{4\rho}\frac{p_m}{z^4}hv\int_0^R r^3dr = -\frac{3\mu_0}{16\rho}\frac{p_m}{z^4}hR^4v \tag{15}$$

To calculate the force acting on the magnet, formula (8) is applied wherein the magnitude of the magnetic field induction is calculated by formula (4). These substitutions lead to the expression

$$F = -p_m\frac{dB_z}{dz} = -p_m\frac{d}{dz}\left(\frac{\mu_0}{2\pi}\frac{p'_m}{z^3}\right) = \frac{3\mu_0}{2\pi}\frac{p_m p'_m}{z^4} = -\frac{3\mu_0}{2\pi}\frac{p_m}{z^4}\frac{3\mu_0}{16\rho}\frac{p_m}{z^4}hR^4v = -\frac{9}{32}\frac{\mu_0^2 p_m^2}{\pi\rho z^8}hR^4v. \tag{16}$$

The equation of the magnet motion with the influence of the disc has the form

$$mx'' = -kx - bx' \tag{17}$$

where $b = \frac{9}{32}\frac{\mu_0^2 p_m^2}{\pi\rho z^8}hR^4$ is the coefficient determined in formula (16).

1.2.3 Using the solution of the equation of attenuating oscillations provided in the problem formulation, one gets

$$\omega = \sqrt{\omega_0^2 - \beta^2} = \omega_0\left(1 - \frac{\beta^2}{\omega_0^2}\right)^{\frac{1}{2}} \approx \omega_0\left(1 - \frac{\beta^2}{2\omega_0^2}\right) \tag{18}$$

where $\beta = b/2m$. From this expression the relative frequency shift is found as

$$\frac{\Delta\omega}{\omega_0} = -\frac{\beta^2}{2\omega_0^2} = -\frac{1}{2km}\left(\frac{9}{64}\frac{\mu_0^2 p_m^2}{\pi\rho z^8}hR^4\right)^2$$

1.2.4 The characteristic attenuation time is thus equal to

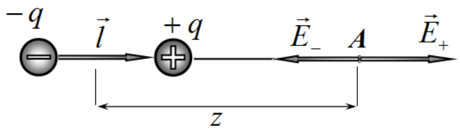
$$\tau = \frac{1}{\beta} = \frac{2m}{b} = \frac{64m\pi\rho z^8}{9\mu_0^2 p_m^2 hR^4}$$

1.2.5 According to Joule's law, the instantaneous power of heat loss in the disk is derived as

$$P = \int \rho j^2\, dV = \int_0^R \rho\left(\frac{3\mu_0 p_m v}{4\pi z^8\rho}r\right)^2 h\, 2\pi r\, dr = \frac{9h\mu_0^2 p_m^2 R^4}{32\pi z^8\rho}v^2$$

and is equal to the product of force (16) on the magnet speed.

Part 2: Electric



2.1.1 The electric field of the dipole is calculated according to the principle of superposition as

$$E = E_+ - E_- = \frac{q}{4\pi\varepsilon_0\left(z-\frac{l}{2}\right)^2} - \frac{q}{4\pi\varepsilon_0\left(z+\frac{l}{2}\right)^2} = \frac{q\cdot 2zl}{4\pi\varepsilon_0\left(z^2-\left(\frac{l}{2}\right)^2\right)^2} = \frac{zp_e}{2\pi\varepsilon_0\left(z^2-\left(\frac{l}{2}\right)^2\right)^2} \quad (19)$$

At rather large $z >> l$, it simplifies to

$$E = \frac{zp_e}{2\pi\varepsilon_0\left(z^2-\left(\frac{l}{2}\right)^2\right)^2} = \frac{1}{2\pi\varepsilon_0}\frac{p_e}{z^3} \quad (20)$$

Thus,

$$a = \frac{1}{2\pi\varepsilon_0}, \qquad \alpha = 3 \quad (21)$$

2.2.1 Let us derive an expression for the force exerted on the ball by the disc.

The electric field inside the conducting disc should be absent. The field $\vec{E}_0$ of the point charge q induces the surface charge densities $\pm\sigma$ on the disc surfaces, which generate an electric field $\vec{E}'$ equal in magnitude but opposite in direction to the field of the point charge. Given that the size of the disk is small compared to the charge separation, it is possible to neglect the variation electric field vector $\vec{E}_0$ at the site of the disc. Therefore, the field should be considered uniform, and the surface density of the induced charges should be treated constant.

The electric field strength of the point charge is determined by the formula $E_0 = \frac{q}{4\pi\varepsilon_0 z^2}$, and the electric field strength of the induced charges is related to their surface density as $E' = \frac{\sigma}{\varepsilon_0}$. Equating these two expression the surface density of the induced charges is found as $\sigma' = \frac{q}{4\pi z^2}$ and the magnitude of the induced charge on each side of the disc $q' = \sigma' S = \frac{qS}{4\pi z^2}$, where $S = \pi R^2$ stands for the disc area. Thus, the induced dipole moment of the disc is finally obtained as

$$p'_e = q'h = \frac{qSh}{4\pi z^2} = \frac{q}{4\pi z^2}V. \quad (22)$$

Therefore, the force acting on the ball is

$$F = qE = q\frac{1}{2\pi\varepsilon_0}\frac{p'_e}{z^3} = \frac{q^2}{8\pi^2\varepsilon_0 z^5}V \quad (23)$$

Since the ball moves, in the last formula z should be replaced by $(z-x)$. Given that $x << z$ the resulting expression for the force simplifies to

$$F = \frac{q^2V}{8\pi^2\varepsilon_0(z-x)^5} = \frac{q^2V}{8\pi^2\varepsilon_0 z^5}\left(1-\frac{x}{z}\right)^{-5} \approx \frac{q^2V}{8\pi^2\varepsilon_0 z^5} + 5\frac{q^2V}{8\pi^2\varepsilon_0 z^6}x \quad (24)$$

The equation of the ball motion in this case takes the form

$$mx'' = -kx + \frac{q^2V}{8\pi^2\varepsilon_0 z^5} + 5\frac{q^2V}{8\pi^2\varepsilon_0 z^6}x \tag{25}$$

The first term results in the additional displacement of the equilibrium position, which, within the approximations used, is equal to

$$k\Delta x = \frac{q^2V}{8\pi^2\varepsilon_0 z^5} \quad \Rightarrow \quad \Delta x = \frac{q^2V}{8\pi^2\varepsilon_0 z^5 k} \tag{26}$$

2.2.2 The second term in equation (24) determines the frequency shift of the oscillations as

$$\omega = \sqrt{\frac{k}{m} - \frac{5q^2V}{8\pi^2\varepsilon_0 z^6 m}} = \sqrt{\omega_0^2 - \frac{5q^2V}{8\pi^2\varepsilon_0 z^6 m}} \approx \omega_0\left(1 - \frac{5q^2V}{16\pi^2\varepsilon_0 z^6 k}\right) \tag{27}$$

which yields the relative frequency shift in the form

$$\frac{\Delta\omega}{\omega_0} = -\frac{5q^2V}{16\pi^2\varepsilon_0 z^6 k} \tag{28}$$

2.2.3 The surface charge density σ on the disk surface changes due to the current in the disc, which is determined by the electric field strength in the disk ($\dot{\sigma} = j = E/\rho$). The electric field in the disc is generated by the ball

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{(z-x)^2} \approx \frac{q}{4\pi\varepsilon_0 z^2}\left(1 + 2\frac{x}{z}\right)$$

and the oppositely directed field of the induced charges

$$E_2 = -\frac{\sigma}{\varepsilon_0}.$$

Ohm's law $E_1 + E_2 = \rho j$ can be thus written as

$$\frac{q}{4\pi\varepsilon_0 z^2}\left(1 + 2\frac{x}{z}\right) - \frac{\sigma}{\varepsilon_0} = \rho j$$

Taking into account the equalities $\dot{\sigma} = j$ and $p = \sigma S h = \sigma V$ ($V = \pi R^2 h$) he required equation is finally derived as

$$\varepsilon_0\rho\dot{p} + p = \frac{qV}{4\pi z^2}\left(1 + 2\frac{x}{z}\right).$$

2.2.4 $R = \rho\frac{h}{S}$, $C = \frac{\varepsilon_0 S}{h}$, $\tau = RC = \varepsilon_0\rho$.

2.2.5 $\frac{2\pi}{\omega} \gg \varepsilon_0\rho$ or $\varepsilon_0\rho\omega \ll 1$..

2.2.6 For harmonic oscillations, the ratio of the amplitude of $\varepsilon_0\rho\dot{p}$ to the amplitude of p is equal to $\varepsilon_0\rho\omega$,, i.e. it is very small. Therefore, the zero-order approximation in the equation

$$\varepsilon_0\rho\dot{p} + p = \frac{qV}{4\pi z^2}\left(1 + 2\frac{x}{z}\right)$$

the first member can be neglected to yield

$$p = \frac{qV}{4\pi z^2}\left(1 + 2\frac{x}{z}\right) \quad \text{and} \quad \dot{p} = \frac{qV}{2\pi z^3}v$$

The next approximation is obtained by substituting $\dot{p}$ into the initial equation

$$p = \frac{qV}{4\pi z^2}\left(1 + 2\frac{x}{z}\right) - \varepsilon_0\rho\dot{p} = \frac{qV}{4\pi z^2}\left(1 + 2\frac{x}{z}\right) - \varepsilon_0\rho\frac{qV}{2\pi z^3}v$$

Note that the solution using the vector diagram is also possible.

2.2.7 The force exerted on the ball by the disk is

$$F = \frac{p}{2\pi\varepsilon_0(z-x)^3}q = \frac{q}{2\pi\varepsilon_0 z^3}\left(1 + 3\frac{x}{z}\right)\left[\frac{qV}{4\pi z^2}\left(1 + 2\frac{x}{z}\right) - \varepsilon_0\rho\frac{qV}{2\pi z^3}v\right] =$$

$$= \frac{q^2R^2h}{8\pi\varepsilon_0 z^6}[(z + 5x) - 2\varepsilon_0\rho v]$$

The equation of motion of the ball is thus written as

$$mx'' = -kx + \frac{q^2R^2h}{8\pi\varepsilon_0 z^6}\left[(z+5x) - 2\varepsilon_0\rho v\right]$$

or

$$mx'' + \frac{q^2\rho R^2 h}{4\pi z^6}x' + \left[k - \frac{5q^2R^2h}{8\pi\varepsilon_0 z^6}\right]x = \frac{q^2R^2h}{8\pi\varepsilon_0 z^5}$$

2.2.8

$$b = \frac{q^2\rho R^2 h}{4\pi z^6}, \qquad \beta = \frac{b}{2m} = \frac{q^2\rho R^2 h}{8\pi z^6 m}, \qquad \tau = \frac{1}{\beta} = \frac{8\pi z^6 m}{q^2\rho R^2 h}.$$

2.2.9 According to Joule's law, the instant power heat production in the disk is found as

$$P = j^2\rho V = \left(\frac{qv}{2\pi z^3}\right)^2\rho V = \frac{q^2\rho R^2 h}{4\pi z^6}v^2 = bv^2$$

which is again equal to the power of force ($F = -bv$).

Marking scheme

№	Content	points	
1.1.1	The field is found on the axis	0,25	**1**
	Simplification uses z≫R	0,25	
	Correct *b*	0,25	
	Correct β	0,25	
1.1.2	Relation between B_r and B_z	0,25	**1**
	Radial component is found $B_r = -\frac{r}{2}\frac{dB_z}{dz}$ (the sign is important) If the minus sign in the above expression is absent	0,5 0.2	
	$F = -p_m\frac{dB_z}{dz}$ (the sign is unimportant)	0,25	
1.2.1	$\omega_0 = \sqrt{k/m}$	0,25	**0,25**
1.2.2	$E = -\frac{r}{2}\frac{dB_z}{dt}$ (the sign important)	0,75	**2**
	Ohm's law in differential form $j = E/\rho$	0,25	
	Magnetic moment $p'_m = \frac{3\mu_0}{16\rho}\frac{p_m}{z^4}hR^4v$	0,4	
	Force $F = -\frac{9}{32}\frac{\mu_0^2 p_m^2}{\pi\rho z^8}hR^4v = -bv$ If the minus sign in the above expression is absent Coefficient in the above expression is 21/32 instead of 9/32	0,4 0.1 0.2	
	ma = -kx-bv	0,2	
1.2.3	The frequency changes due to attenuation	0,3	**1**
	$\Delta\omega=\delta^2/2\omega_0$	0,3	
	δ=b/2m	0,2	
	Correct answer	0,2	
1.2.4	$\tau = \frac{64m\pi\rho z^8}{9\mu_0^2 p_m^2 hR^4}$	0,25	**0,25**
1.2.5	Energy losses in the disc are found	0,5	**1**
	The power of the friction force is found	0,4	
	They are equal	0,1	
2.1.1	The field of the dipole is found	0,25	**1**

	Simplification uses z≫R	0,25	
	Correct a	0,25	
	Correct α	0,25	
2.2.1	The constant component of the force is found $\frac{q^2V}{8\pi^2\varepsilon_0 z^5}$	0,5	**0,75**
	$\Delta x = \frac{q^2V}{8\pi^2\varepsilon_0 z^5 k}$	0,25	
2.2.2	The component of the force is found $\frac{5q^2V}{8\pi^2\varepsilon_0 z^6}x$	0,5	**0,75**
	$\frac{\Delta\omega}{\omega_0} = -\frac{5q^2V}{16\pi^2\varepsilon_0 z^6 k}$	0,25	
2.2.3	$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{(z-x)^2} \approx \frac{q}{4\pi\varepsilon_0 z^2}\left(1+2\frac{x}{z}\right)$	0,3	**1,5**
	$E_2 = -\frac{\sigma}{\varepsilon_0}$ If the minus sign in the above expression is absent	0,2 0.1	
	$E_1 + E_2 = \rho j$	0,2	
	$\dot{\sigma} = j$	0,2	
	$p = \sigma V$	0,2	
	$\varepsilon_0\rho\dot{p} + p = \frac{qV}{4\pi z^2}\left(1+2\frac{x}{z}\right)$	0,4	
2.2.4	$\tau = \varepsilon_0\rho$		**0,25**
2.2.5	$\varepsilon_0\rho\omega \ll 1$		**0,5**
2.2.6	$\varepsilon_0\rho\dot{p} \ll p$	0,5	**2**
	$\dot{p} = \frac{qV}{2\pi z^3}v$	1	
	$p = \frac{qV}{4\pi z^2}\left(1+2\frac{x}{z}\right) - \varepsilon_0\rho\frac{qV}{2\pi z^3}v$	0,5	
2.2.7	$F = \frac{p}{2\pi\varepsilon_0 z^3}q$	0,5	**1,5**
	$F = \frac{q^2R^2h}{8\pi\varepsilon_0 z^6}[(z+5x) - 2\varepsilon_0\rho v]$	0,5	
	$mx'' + \frac{q^2\rho R^2h}{4\pi z^6}x' + \left[k - \frac{5q^2R^2h}{8\pi\varepsilon_0 z^6}\right]x = \frac{q^2R^2h}{8\pi\varepsilon_0 z^5}$	0,5	
2.2.8	$\tau = \frac{8\pi z^6 m}{q^2\rho R^2 h}$		**0,25**
2.2.9	Energy losses in the disc are found	0,5	**1**
	The power of the friction force is determined	0,4	
	They are equal	0,1	
Total			**16**

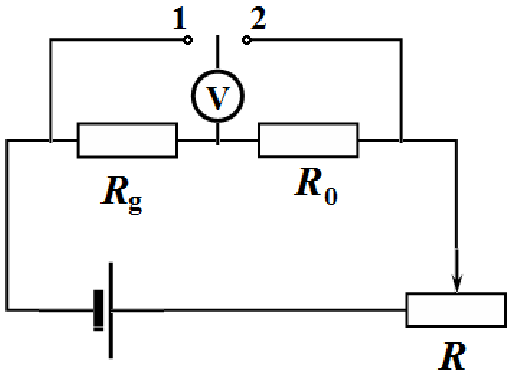
SOLUTION FOR THE EXPERIMENTAL COMPETITION

Resistance of graphite (15 points)

Part 1. The current-voltage characteristic of a graphite rod

1.1 Using the ohmmeter it is easy determine that the the sliding lead is ***b.***

1.2.1 To measure the current-voltage characteristics of the graphite rod, the traditional circuit, shown in the figure, can be used. When the voltmeter is thrown into position 1 the voltage across the graphite rod is measured, whereas in position 2 the measured voltage is the one across the resistor $R_0 = 1{,}0\,Ohm$. If the voltage is measured in volts, then the voltage across the resistor is numerically equal to the current strength in the circuit in amps.



It is possible to connect the variable resistor as a potentiometer, although in this case the maximum current in the circuit will be slightly less.

1.2.2 The results of measurements of the current-voltage characteristic of the graphite rod are presented in Table 1. This table also shows the power values calculated via

$$P = UI \tag{1}$$

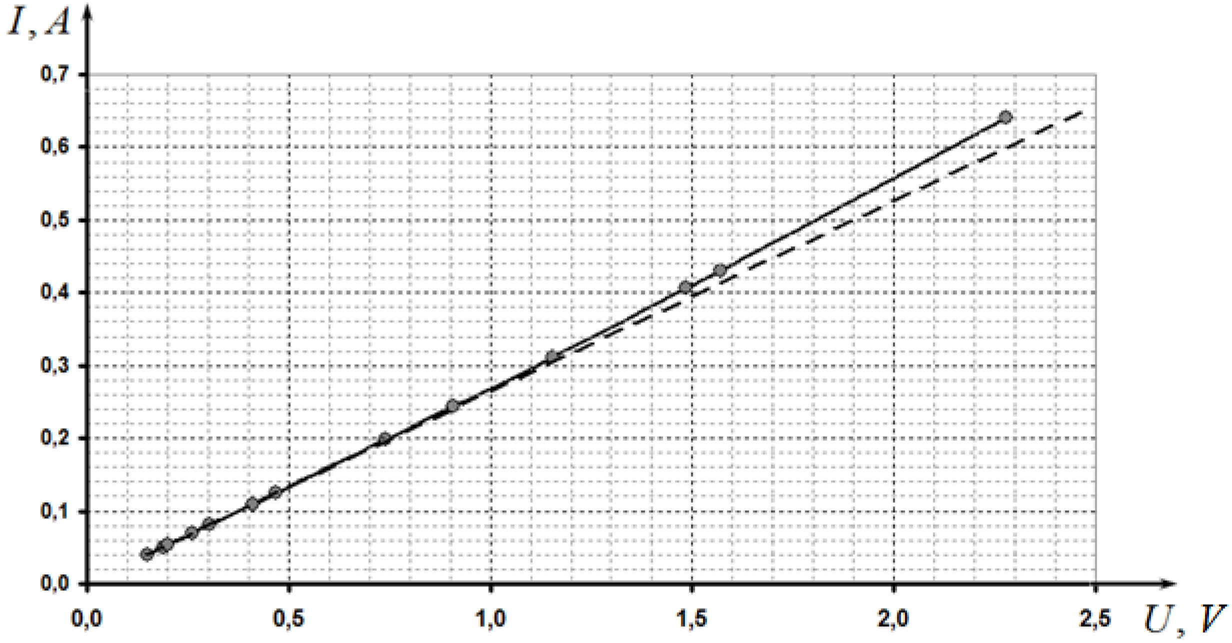
as well as the resistance of the graphite rod

$$R = \frac{U}{I}. \tag{2}$$

Table 1. Measurements made in the air

U, V	*I, A*		*P, W*	*R, Ohm*
0,150	0,040		0,0060	3,750
0,188	0,050		0,0094	3,760
0,202	0,054		0,0109	3,741
0,261	0,070		0,0183	3,729
0,304	0,081		0,0246	3,753
0,411	0,109		0,0448	3,771
0,470	0,124		0,0583	3,790
0,742	0,199		0,1477	3,729
0,907	0,243		0,2204	3,733
1,155	0,312		0,3604	3,702
1,486	0,406		0,6033	3,660
1,570	0,430		0,6751	3,651
2,280	0,640		1,4592	3,563

1.2.3 The current-voltage characteristic of the graphite rod is shown in the figure below.



It can be seen that the curve deviates somewhat up from the linear proportionality which is prescribed to decrease in the resistance of the grathite when the temperature increases.

Derivation of the theoretical formula.

In the steady state the condition of thermal equilibrium is writte as:

$$\frac{U^2}{R_0(1+\alpha\Delta T)} = \beta\Delta T\,. \tag{3}$$

Solving the quadratic equation for the temperature difference gives rise to

$$\frac{U^2}{R_0} = \beta\Delta T(1+\alpha\Delta T) \quad \Rightarrow \alpha\beta(\Delta T)^2 + \beta\Delta T - \frac{U^2}{R_0} = 0$$
$$\Delta T = \frac{-\beta \pm \sqrt{\beta^2 + 4\frac{U^2}{R_0}\alpha\beta}}{2\alpha\beta}\,. \tag{4}$$

From the data obtained it follows that the temperature coefficient of graphite resistance is negative. In addition, it can be shown that from the two roots of equation (3) the smaller one should be chosen (with + sign), since it corresponds to a stable thermal equilibrium. Therefore, the theoretical dependence has the form

$$I = \frac{U}{R_0(1+\alpha\Delta T)} = \frac{U}{R_0\left(1+\frac{-\beta+\sqrt{\beta^2+4\frac{U^2}{R_0}\alpha\beta}}{2\beta}\right)} \approx \frac{U}{R_0\left(1+\frac{U^2}{R_0}\frac{\alpha}{\beta}\right)} \approx \frac{U}{R_0}\left(1-\frac{U^2}{R_0}\frac{\alpha}{\beta}\right)\,. \tag{5}$$

The last two expressions are approximations valid for small α.

1.2.3 For a more accurate calculation of the resistance of the graphite rod at room temperature, only several data points at low voltages (less than 0.5V) should be taken at which the rod remains practically unheated. Then, the method of least squares must be employed to evaluate the slope, which is equal to the rod resistance.

Calculation of the obtained experimental data leads to the following result

$$R_0 = (3,78 \pm 0,03) Ohm.$$

To calculate the resistivity of use is the following formula

$$R = \rho \frac{4l}{\pi d^2} \quad \Rightarrow \quad \rho = \frac{\pi d^2}{4l} R = \frac{\pi \cdot (1,0 \cdot 10^{-3})^2}{4 \cdot 5,0 \cdot 10^{-2}} 3,78 = 5,93 \cdot 10^{-5} Onm \cdot m$$

Here $l = (5,0 \pm 0,2) sm$ is the length of the rod between the leads.

The calculation of experimental error is given by the formula

$$\Delta\rho = \rho \sqrt{\left(\frac{\Delta R}{R}\right)^2 + \left(2\frac{\Delta d}{d}\right)^2 + \left(\frac{\Delta l}{l}\right)^2} = 5,93 \cdot 10^{-5} \sqrt{\left(\frac{0,03}{3,78}\right)^2 + \left(2\frac{0,05}{1}\right)^2 + \left(\frac{0,2}{5}\right)^2} = 6 \cdot 10^{-6} Ohm \cdot m$$

The final result is written as

$$\rho = (5,9 \pm 0,6) \cdot 10^{-5} Ohm \cdot m \tag{6}$$

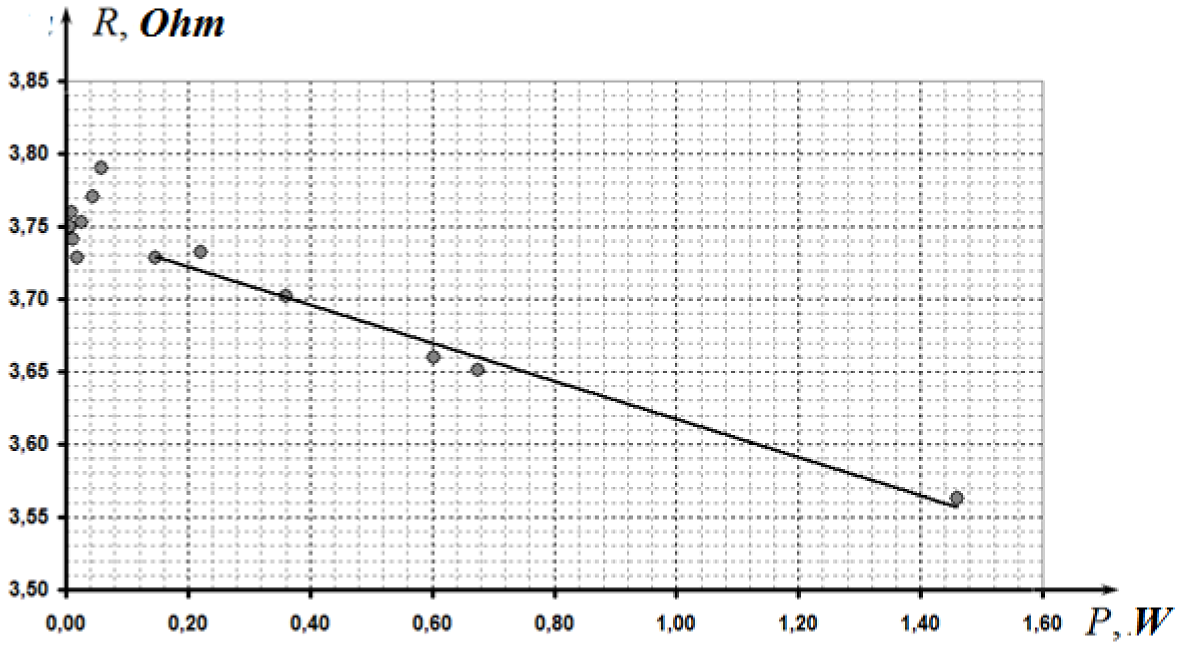
1.2.4 In the steady state the power, released when the current flows, is equal to the power of the heat losses:

$$P = \beta \Delta T \quad \Rightarrow \quad \Delta T = \frac{P}{\beta},$$

Therefore, the dependence of the resistance on the power takes the form

$$R_g = R_0(1 + \alpha \Delta T) = R_0(1 + \frac{\alpha}{\beta} P)). \tag{7}$$

1.2.5 The dependence of the resistance on the dissipated power is shown in the figure.



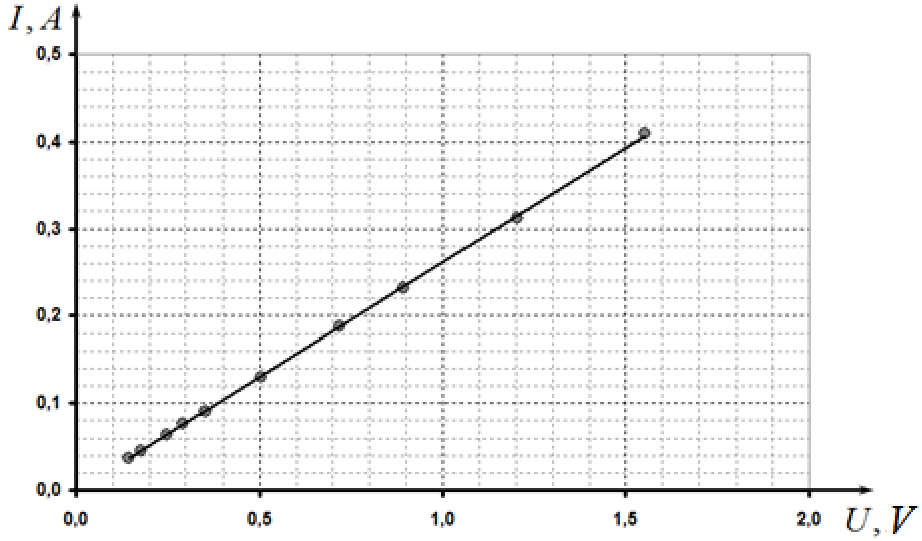
1.2.6 The linearity of this dependence is observed at powers larger than $0,2W$.

The coefficients of this dependence $R = aP + b$, calculated using the least square method, are found as $a = 0,13 \frac{Ohm}{W}$, $b = 3,75 Ohm$, consequently, the coefficient in formula (3) is evaluated as $\gamma = \frac{a}{b} = 0,035 W^{-1}$.

1.3.1 The results of measurements of the current-voltage characteristic of the rod, placed in the snow, are given in Table 2 and the corresponding graph is shown in the figure below.

Table 2

U, V	*I, A*
0,144	0,037
0,177	0,045
0,248	0,064
0,292	0,076
0,354	0,091
0,504	0,130
0,721	0,188
0,895	0,232
1,205	0,312
1,554	0,409



In this case there is also a weak nonlinearity with increasing power. Therefore, to calculate the resistance at zero temperature and only several initial points at low resistance should be used. Calculation for the first five points leads to the following value

$$R_g = (3,88 \pm 0,03)Ohm.$$

Since this value should obey the formula $R_g = R_0(1+\alpha\Delta T)$, the temperature coefficient of the resistance can be calculated as

$$\alpha = \frac{1}{\Delta T}\left(\frac{R_g}{R_0} - 1)\right) = -\frac{1}{20^\circ}\left(\frac{3,88}{3,78} - 1\right) = -1,3\cdot 10^{-3}K^{-1}.$$

The experimental error is mainly determined by the measurement error of resistance, thus it can be evaluated via the formula

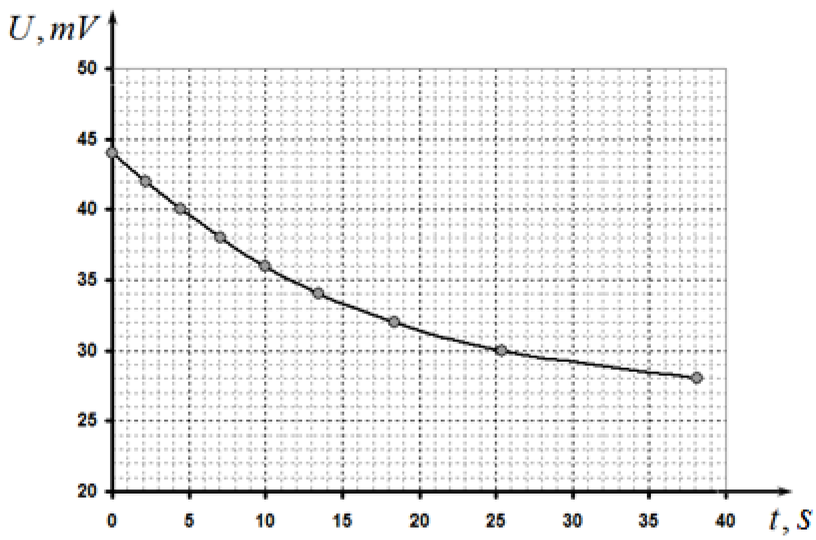
$$\Delta\alpha = \sqrt{\left(\frac{\Delta R_g}{\Delta T \cdot R_0}\right)^2 + \left(\frac{R_g \Delta R_0}{\Delta T \cdot R_0^2}\right)^2} \approx 6\cdot 10^{-4}K^{-1}..$$

Part 2. Cooling of the graphite rod

2.1 The results of measurements of time time needed to achieve the specified voltage are shown in Table 3 and the corresponding graph is shown n the figure below.

Table 3

U, mV	*t, s*	$\ln(U-\bar{U})$
44	0,00	3,045
42	2,22	2,944
40	4,45	2,833
38	7,03	2,708
36	9,99	2,565
34	13,50	2,398
32	18,42	2,197
30	25,38	1,946
28	38,14	1,609



Since the voltage is proportional to the measured voltage change and the resistance change is proportional to the change in temperature, the dependence measured coincides, up to an unimportant factor, with the temperature dependence on time.

The solution ot the equation

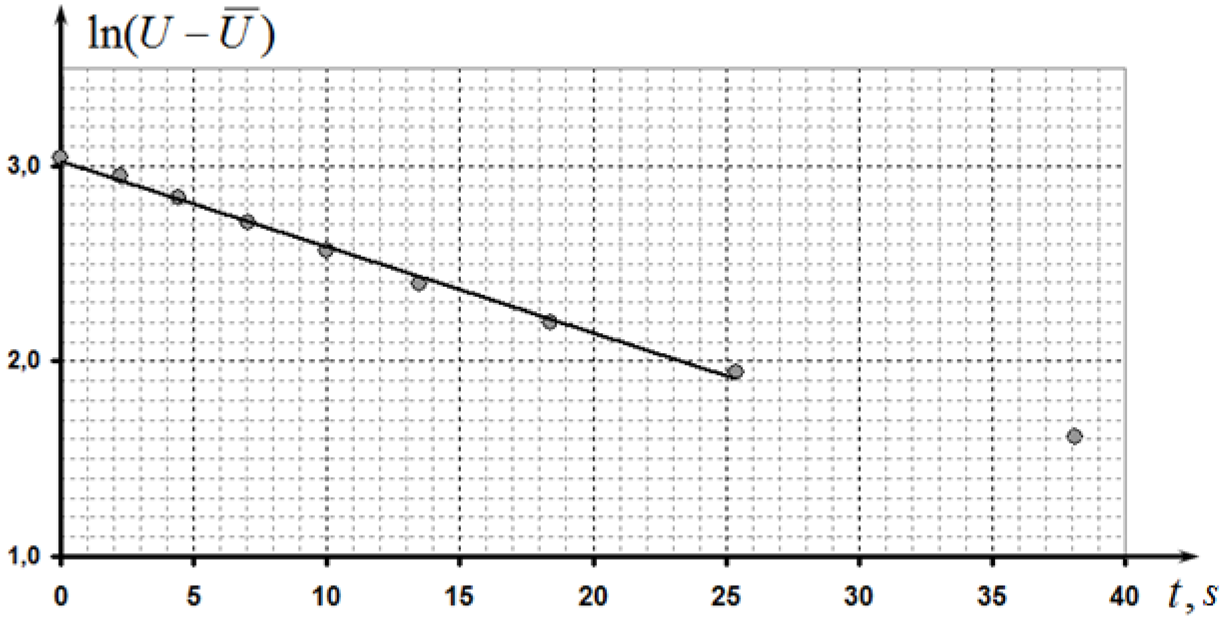
$$\frac{\Delta T}{\Delta t} = -\frac{1}{\tau}\left(T - \overline{T}\right)$$

is the exponential function

$$\left(T - \overline{T}\right) = \left(T - \overline{T}\right)_0 \exp\left(-\frac{t}{\tau}\right). \tag{8}$$

To determine the characteristic cooling time, the resulting dependence should be drawn in a semilogarithmic scale, $\ln(U - \overline{U})$ against the time. For numerical calculations it is necessary to measure the steady-state voltage value (achieved after waiting for a few minutes). In our measurements. Table 3 shows the results of calculations of logarithms.

The following figure shows the graph on the semilogarithmic scale.



The slope coefficient of the obtained almost linear dependence is found as $a = -0,044 s^{-1}$

Consequently, the characteristic time of thermal equilibration is equal to $\tau = -\frac{1}{a} \approx 23 s$.

Marking scheme

№	Content	points
1.1	Sliding laed is ***b***	**0,2**
1.2.1	Circuit: - the resistor anf the graphite rod are connected in series; - voltages are measured on the resistor and the graphite rod; - ability to vary the net electric current (the variable resistor connected in series, or as a voltage diivider); - the source is correctly connected;	**0,5** 0,2 0,1 0,1 0,1
1.2.2	marked only if the deviation from the table in the solution is less than 50% Measurements : - 10 points or more (*7-9 points; 5-6 points; less than 5 points*) - minimum voltage less than 0,2 V; - maximum voltage larger than 2 V; - deviation to the top from the linear dependence; Graph: - axes are anmed and ticked; - points in the Table correspond to the points in the graph; - smooth ine is drawn; Theoretical formula (thermal equilibrium equation, quadratic equation for temperature, smaller root is chosen, substitution into Ohm's law)	**3,0** 1,5(*0,8; 0,5;0*) 0,2 0,2 0,3 0,1 0,2 0,1 0,4
1.2.3	The resistance of the rod is calculated: - voltages not larger than 0,3 V are only used; - all points in the stated range are used for calcualtion (not less than 5); (*by 2points, by 1 point*) - the calculated value of the resistance is in the range 3,5-4,5 Ohm (*3,0 – 5,0 Ohm*); the length of the rod is measured (not larger than 5 sm); Fromula for ρ; ρ is calculated: in the range $\pm 20\%$ ($\pm 50\%$); Error is estimated (any method)	**1,1** 0,2 0,2 (*0,1; 0,05*) 0,1 (*0,05*) 0,1 0,1 0,3 (*0,1*) 0,1
1.2.4	 Equation for thermal equilibrium; Formula for $R(\Delta T)$	**0,2** 0,1 0,1
1.2.5	 Formula for the power; Formula for the resistance; Calculations for all points; Graph: - axes are named and ticked; - points in the Table correspond to the points in the graph; - smooth ine is drawn;	**0,7** 0,05 0,05 0,2 0,1 0,2 0,1
1.2.6	 The linear range is stated with the power larger than 0,2 W The slope is calculated by all points (by 2 points) Formula for calculation; -numerical value is in the range 0,025-0,045 W^{-1} (*0,01 – 0,06*)	**0,5** 0,1 0,2 (*0,1*) 0,2 (*0,1*)

1.3.1	marked only if the deviation from the table in the solution is less than 50% Measurements : - 10 points or more (*7-9 points; 5-6 points; less than 5 points*) - minimum voltage is less than 0,2 V; - maximum voltage is larger than 1,5 V; - almost linear dependence is obtained; - small deviation from the linear dependence to the top Graph: - axes are named and ticked; - points in the Table correspond to the points in the graph; - smooth ine is drawn;	**2,6** 1,5 (*0,8; 0,5*) 0,2 0,2 0,1 0,2 0,1 0,2 0,1
1.3.2	The resistance is calculated for the snow temperature: - points with voltage less than 0,5 V are only used; - calculation by all points (by 2 points, by 1 point); - the numerical value is in the range 3,5-4,5 Ohm (*3,0 – 5,0 Ohm*); - the resistance is larger than at room temperature; - formula for the temperature coefficient of resistance; - negative value; - numerical value is in the range $\pm 50\%$ ($\pm 75\%$); - error is estimated; - error is larger than 50%	**1,2** 0,2 0,2 (*0,1, 0,05*) 0,1 (*0,05*) 0,1 0,1 0,1 0,2(*0,1*) 0,1 0,1
2.1	 marked only if the deviation from the table in the solution is less than 50% Measurement: - not less than 7 points (*5-6 points; less than 5 points*) - decreasing dependence with convexity directed downward; The range of voltages is 1,5 larger; - there is a limiting value of the voltage;	**3** 2 (*1,5; 1*) 0,3 0,3 0,4
2.2	Graph: - axes are named and ticked; - points in the Table correspond to the points in the graph; - smooth ine is drawn;	**0,4** 0,1 0,2 0,1
2.3	Evaluation of the time equilibration: - by graph (the slope to the steady value); - by 1-2 points; - semilogarithmic linearization is applied; - numerical value is in thew range 20-30s (*15-40S, 10-45s*)	**1,6** *0,5* *0,2* 1,0 0,6 (*0,4; 0,2*)

EXPERIMENTAL COMPETITION

14 January, 2015

Please read the instructions first:

1. The Experimental competition consists of one problem. This part of the competition lasts 3 hours.
2. Please only use the pen that is provided to you.
3. You can use your own non-programmable calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet and additional papers***. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets.*** Write only inside the bordered area.
6. Fill the boxes at the top of each sheet of paper with your country (***Country*)**, your student code (***Student Code***), the question number (***Question Number*)**, the progressive number of each sheet **(*Page Number***), and the total number of ***Writing sheets*** (***Total Number of Pages***). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
7. At the end of the exam, arrange all sheets for each problem in the following order:
   - Used ***Writing sheets*** in order;
   - The sheets you do not wish to be evaluated
   - Unused sheets and the printed question.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper or equipment out of the room

Resistance of graphite (15 points)

Instruments and equipment: graphite rod, multimeter, power source (battery 4.5 V), fixed resistors 1.0 Ohm and 10 Ohm, variable resistor (use the sharp edge of the plastic ruler for its adjustment), connecting wires, stopwatch, vessel with snow.

Warning! Be careful, since the graphite rod is very fragile and may be easily broken. Connect it to a circuit with the clips called the "crocodile".

Graphite electrical resistance R_g depends on its temperature. One can approximate that this dependence is linear

$$R_g = R_0(1 + \alpha\Delta T),$$

wherein ΔT denotes the temperature difference between the room and the rod, R_0 is the resistance of the graphite rod at room temperature, α stands for the temperature coefficient of graphite resistance.

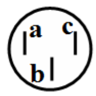
The room temperature will be announced during the experimental competition.

It can also be assumed that the power of the heat dissipated into the ambient air is proportional to the temperature difference between the rod and the air

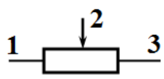
$$P = \beta\Delta T,$$

where β is the heat transfer coefficient.

Part 1. The current-voltage characteristic of a graphite rod



1.1 Use the multimeter (in this paragraph, use it in the mode of resistance measurements) to determine which of its pins corresponds to the sliding lead (indicated in the figure by 2).



1.2 Measure the dependence of the current through the graphite rod, which is placed in the air, on the voltage aceoss it.

For the measurements, use the multimeter in the mode of voltmeter since it works very unstable in the mode of ammeter. To measure current, use the resistor of 1.0 ohms. To change the current in a circuit use the variable resistor. Bear in mind that when the current flows through the rod it takes some time for the temperature to change. After changing the value of the variable resistor by turning its knob, you should wait at least 30-40 seconds before taking the readings.

1.2.1 Draw schematically an electrical circuit that you used for taking measurements.
1.2.2 Plot the graph of the dependence obtained. Derive a theoretical formula that describes this dependence.
1.2.3 Calculate the electric resistivity of graphite at room temperature. Estimate its experimental error. The relative error of each resistor is 5%. The rod diameter is $d = 1.0 \pm 0.05\ mm$. To measure the length the graph paper may be used.
1.2.4 Show theoretically that in the steady state the electrical resistance of the graphite rod depends linearly on the power P released in it due to the current flow

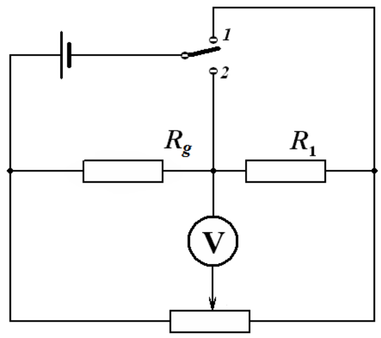
$$R_g = R_0(1 + \gamma P). \qquad (1)$$

1.2.5 Using the obtained experimental data, calculate the dependence of the resistance of the graphite rod on the power $R_g(P)$. Plot the corresponding graph.
1.2.6 Using the corresponding graph write down a range of powers for which formula (1) holds. Evaluate the value of γ for this particular range. Error estimation is not required.
1.3 Measure the dependence of the current through the graphite rod on the voltage across it $I(U)$, when the rod is immersed into the melting snow.

1.3.1 Plot the graph of the obtained dependence.
1.3.2 Evaluate the temperature coefficient of graphite resistance and estimate its experimental error $\Delta\alpha$.

Part 2. Cooling of the graphite rod

To measure small changes in resistance it is preferable to use a bridge circuit (see. Fig.). One of the wires connected to the battery can be placed in two positions:
1 – measurement mode;
2 – heating mode.



By turning the knob of the variable resistor, it can be ensured that the voltage across the voltmeter is equal to zero (or close to it). In this case the bridge is called balanced. Take resistor of 10 ohms as a resistor R_1.

If you change the resistance R_g of the graphite rod, the voltage across the voltmeter turns non zero. In this case it can be shown independently (you do not need to do that), that the voltage across the voltmeter is proportional to the change in the resistance of the graphite rod.

2.1 Without heating the rod, place the battery wire in the measurement mode (position 1). Use the knob of the variable resistor to make the multimeter readings close to zero (as close as possible).

Quickly switch the same battery wire in position 2, heat the rod by waiting for at least 1 minute. Then quickly switch the battery wire back in position 1 and measure the dependence of voltage on time.

2.2 Plot the dependence obtained.

2.3 If the rod is connected to a DC voltage source, but its temperature differs from the stationary one $\bar{T}$,, then the change of temperature T over time t is approximately governed by the equation

$$\frac{dT}{dt} = -\frac{1}{\tau}(T - \bar{T}).$$

Using the experimental data verify the validity of this equation. Calculate the value of the characteristic time τ of thermal equilibration. Error estimation is not required.

THEORETICAL COMPETITION

January 15, 2016

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*.** Write only inside the boxed area.
6. Begin each question on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Question Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
    - Used ***Writing sheets*** in order;
    - The sheets you do not wish to be evaluated
    - Unused sheets and the printed question.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

Problem 1 (10.0 points)

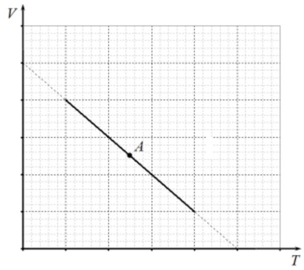
This problem consists of three independent parts.

Problem 1A (4.0 points)

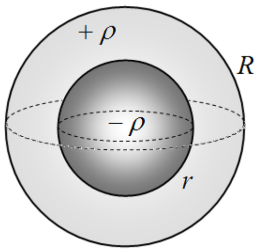
A homogeneous planet of radius R has no atmosphere and does not rotate. A stone is thrown from the surface of the planet at an angle α to the horizontal with the speed v_0, which is equal to the orbital velocity for the surface of the planet. Find the maximum height of ascent of the stone above the surface of the planet. At what range from its initial point, measured along the surface of the planet, the stone will hit the surface?

Problem 1B (3.0 points)

One mole of an ideal monatomic gas performs a process whose chart in *VT* coordinates completely lies on a straight line. Find the heat capacity of the gas at the point *A*, equidistant from the points of intersection of the process line with the coordinate axes.



Problem 1C (3.0 points)



Two spheres with the radii r and R ($r < R$) \with the common center divide the space into three domains. The interior of the small sphere is uniformly charged with the volume charge density $-\rho$, the domain in between the spheres is uniformly charged with the volume charge density $+\rho$, and there is no charge outside the larger sphere. Find the ratio of the radii R/r, at which the potential in the center of the symmetry of the system is equal to the potential at infinity.

Problem 2. Equilibrium in terms of potential energy (10.0 points)

One of the widely known principles of the general physics is that every system tends to decrease its potential energy, and the stable equilibrium position corresponds to the state with the minimum of its value.

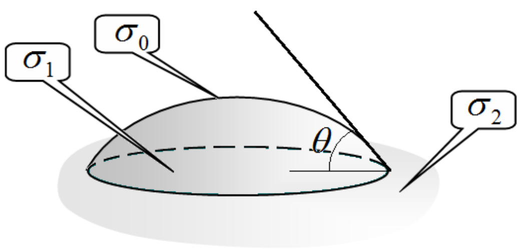
In this problem the interaction of the liquid with the solid surface is studied. To describe this interaction the following parameters are introduced:

σ_0 is the surface tension at the interface between the liquid and the air;

σ_1 is the surface tension at the interface between the liquid and solid;

σ_2 is the surface tension at the interface between the solid and the air;

θ is the contact angle (wetting angle).



The values σ_0, σ_1, σ_2 designate the surface energy per unit area of contact between media.

In all parts of the problem, use the following numerical values for water:

the surface tension $\sigma_0 = 0{,}072\,\dfrac{N}{m}$;

the contact angle $\theta = 20°$;

the mass density $\rho = 1{,}0 \cdot 10^3\,\dfrac{kg}{m^3}$;

the acceleration of gravity $g = 9{,}8\,\dfrac{m}{s^2}$.

1. Introduction (1.0 points)

1. **[1.0 points]** Prove that the change in the surface energy at the liquid-solid interface is found as

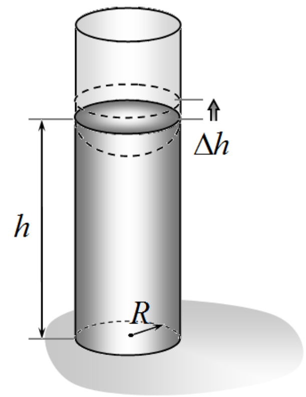
$$\Delta U_s = -\sigma_0 \cos\theta \ \Delta S\,, \qquad (1)$$

where ΔS stands for the change in the area of the contact between the liquid and solid.

2. Water in a vertical cylindrical tube (2.0 points)

The open tube with the inner radius $R = 1{,}0mm$ is lowered vertically so that its lower end touches the surface of the water.

Let a water level in the tube be at a certain height h, which does not necessarily correspond to its equilibrium value.



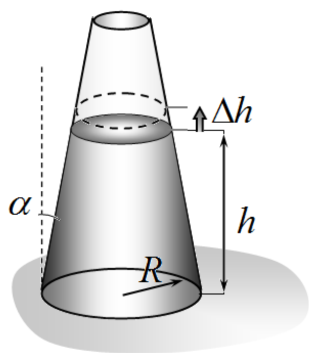
2.1 **[0.5 points]** Find the formula for the change in the surface energy of the system ΔU_S that corresponds to an additional small rise of water level Δh in the tube.
2.2 **[0.5 points]** Find the formula for the change in the potential energy ΔU_G of the liquid in the gravitational field that corresponds to an additional small rise of water level Δh in the tube.
2.3 **[1.0 points]** Using the principle of the minimum potential energy, find the formula for the height of the water in the tube h_0 in equilibrium position. Calculate its numerical value from the quantities provided above.

3. Water in a vertical conical tube (4.0 points)

A long conical tube is vertically lowered into the water so that its lower end touches the surface of the water. The inner radius of the tube at its lower base is equal to $R = 1{,}0mm$, and its inner radius at the upper base is close to zero. The tube walls make an angle α with the vertical.

Note: in the following neglect any change in the surface energy at the interface between the liquid and the air.

Let a water level in the tube be at a certain height h, which does not necessarily correspond to its equilibrium value.



3.1 **[0.5 points]** Find the formula for the change in the surface energy of the system ΔU_S that corresponds to an additional small rise of water level Δh in the tube.
3.2 **[0.5 points]** Find the formula for the change in the potential energy ΔU_G of the liquid in the gravitational field that corresponds to an additional small rise of water level Δh in the tube.
3.3 **[1.0 points]** Find the equation that determines the height of water in the tube in equilibrium position and rewrite it in terms of σ_0, θ, α and the value of h_0 found in 2.3.
3.4 **[1.0 points]** Let an angle be $\alpha = 1{,}0 \times 10^{-2}\ rad$. The tube is partially filled with water up to a certain level H. Find the dependence of the ultimate height of the water level in the tube as a function of H.
3.5 **[1.0 points]** Specify the range of angles α (providing its numerical values) at which the water completely fills the tube.

4. Outflow of water (3.0 points)

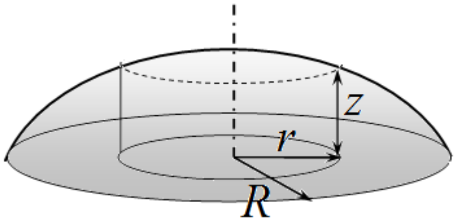
A bottle is completely filled with water, sealed tightly with the cork and turned upside down. Two identical round holes of radii R are drilled in the cork.

4.1 **[3.0 points]** At what minimum value of the hole radius the water will pour out of the bottle?

Mathematical tips

Rather small convex spherically shaped surface can be approximately described by a function

$$z = h\left(1 - \frac{r^2}{R^2}\right),$$



where R denotes the radius of the bulge with h being its height such that $h \ll R$.

Then, up to higher orders the area of the spherical part of the bulge is found as

$$S = \pi\left(R^2 + h^2\right),$$

and its potential energy in the gravitational field is derived as

$$U = \frac{\pi R^2 h^2}{6}\rho g .$$

Problem 3. Nonlinear capacitor (10.0 points)

The electrical circuit contains the voltage source U_0 \connected in series with the resistor $R = 1{,}00\ k\Omega$ and the nonlinear capacitor, whose capacitance depends on the voltage across it and is graphically shown in the figure below.

Note: *to solve this problem you may need to use some piece of squared paper provided under the graph,*

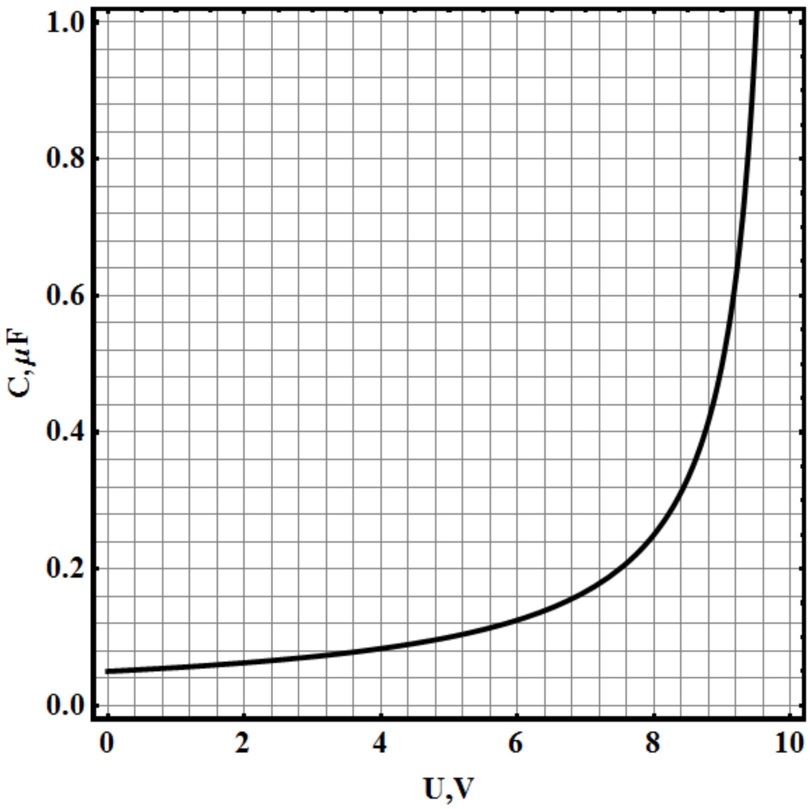
1. **[0.75 points]** Assume that $U_0 = 5{,}0\ V$. Find the charge of the capacitor, which appears on its plates after a sufficiently long period of time.

Suppose that at the initial time moment the charge of the capacitor is zero, and the voltage source provides $U_0 = 10$ V. It is seen from the graph that the capacitance of the capacitor at this voltage goes to infinity, i.e. $C(10\ V) = \infty$.

2. **[0.25 points]** How long does it take for a capacitor to be charged to the voltage $U_0 = 10$ V?
3. **[3.0 points]** Find the time moment t, when the charge of the capacitor is equal to $q = 4{,}0\ \mu Cl$.
4. **[0.5 points]** Find the time interval Δt at which the charge of the capacitor grows from $q_0 = 4{,}0\ \mu Cl$ to $q = 8{,}0\ \mu Cl$.
5. **[0.5 points]** Find the charge of the capacitor at the time moment $t_0 = 3{,}0\ ms$.

Let a source provide a constant voltage with a small portion of alternating voltage such that $U_0 = U + \delta U \sin \omega t = [5{,}000 + 0{,}100 \sin \omega t\,]$ V, where $\omega = 2500\ rad/s$. After a sufficiently long period of time electrical oscillations of voltage and current are set up in the circuit.

6. **[0.5 points]** What is the phase difference φ between the voltage oscillations across the capacitor and the resistor?
7. **[4.0 points]** Find the dependence of the electric current in the circuit $I(t)$ as a function of time.
8. **[0.5 points]** Find the voltage across the capacitor $U_C(t)$ as a function of time.

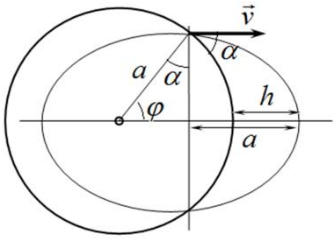

1.0
0.8
0.6
0.4
0.2
0.0
C, μF
0
2
4
6
8
10
U,V

SOLUTIONS TO THE PROBLEMS OF THE THEORETICAL COMPETITION

Attention

Problem 1 (10.0 points)

Problem 1A (4.0 points)



For Keplerian motion the total energy, as well as the period of rotation, depends only on the semi-major axis, which for stone's orbit coincides with the planet radius ($a = R$), and the starting point lies on the semi-minor axis because its distance to the focus (the center of the planet) is equal to a (see. fig.). From the figure one gets

$$R + h = a\sin\alpha + a,$$
$$h = R\sin\alpha.$$

This result can also be obtained by solving a set of equations which is written from the conservation of energy and angular momentum.

The flight range of the stone is found as $l = 2R\varphi = 2R(\frac{\pi}{2} - \alpha)$.

Grading scheme

№	Content	points
1.	The maximum height is found *Conservation of energy* *Conservation of angular momentum* *The set of equation is correctly solved* or Semi-major axis $a = R$ *is found* *Justification for semi-major axis* *The correct sketch is drawn* *Answer*	**2** 0,5 0,5 1 0,5 0,5 0,5 0,5
2.	The angle φ is obtained ($\varphi = \frac{\pi}{2} - \alpha$) Correct answer to the question	1 1
Total		**4.0**

Problem 1B (3.0 points)

According to the definition the heat capacity is written as

$$C = \frac{\delta Q}{\Delta T} = \frac{\Delta U + P\Delta V}{\Delta T} = C_V + P\frac{\Delta V}{\Delta T}. \quad (1)$$

Let the gas parameters at the point *A* be equal (P_0,V_0,T_0). Then, the equation of the process takes the following form

$$\frac{V}{V_0} + \frac{T}{T_0} = 2. \quad (2)$$

For small deviations one derives

$$\frac{\Delta V}{V_0} + \frac{\Delta T}{T_0} = 0, \quad (3)$$

$$\frac{\Delta V}{\Delta T} = -\frac{V_0}{T_0}. \quad (4)$$

With the aid of the equation of state $P_0V_0 = RT_0$, one finally gets

$$C = C_V + P\frac{\Delta V}{\Delta T} = C_V - P_0\frac{V_0}{T_0} = C_V - R = \frac{R}{2}. \quad (5)$$

Grading scheme

№	Content	points
1.	Expression for *C* that contains the derivative	1.0
2.	The derivative is found for the point *A*	1.0

3.	Correct answer ($C_V - R$ or $R/2$)	1.0
Total		**3.0**

Problem 1C (3.0 points)

It follows from the symmetry that the potential at the center of a uniformly charged volume of spherical shape with respect to the point at infinity is equal to

$$\varphi \sim k\frac{q}{R} \sim \alpha\frac{\rho R^3}{R} \sim \alpha\rho R^2, \qquad (1)$$

where α denotes a coefficient of proportionality which is the same for a sphere of any size.

Our system of charges can be represented as a superposition of a sphere of radius R with the volume charge density $+\rho$, and a sphere of radius r charged with the volume density -2ρ. Then, the potential at the center is found as

$$\alpha\rho R^2 + \alpha(-2\rho)r^2 = 0, \qquad (2)$$

where

$$\frac{R}{r} = \sqrt{2}. \qquad (3)$$

Grading scheme

№	Content	points
1.	The formula for the potential at the center $\varphi \sim \rho R^2$ Derivation of the potential at the center Wrong coefficient for the potential at the center Wrong derivation of the potential	1.0 1.0 - 0.5 0
2.	Superposition principle is used	1.0
3.	Correct answer	1.0
Total		**3.0**

Problem 2. Equilibrium in terms of potential energy (10.0 points)

1. Introduction (1.0 points)

1.1 **[1.0 points]** The change in the surface energy at the liquid-solid interface is found as

$$\Delta U_S = -(\sigma_2 - \sigma_1)\Delta S. \qquad (1)$$

Considering the small segment Δl of the boundary of the drop one can write the condition of its balance

$$(\sigma_2 - \sigma_1)\Delta l = \sigma_0 \Delta l \cos\theta. \qquad (2)$$

It follows from equations (1) and (2) that

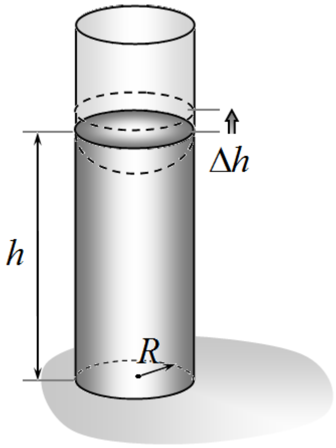
$$\Delta U_s = -\sigma_0 \cos\theta\ \Delta S. \qquad (3)$$

2. Water in a vertical cylindrical tube (2.0 points)

2.1 **[0.5 points]** The formula for the change in the surface energy of the system ΔU_S that corresponds to an additional small rise of water level Δh in the tube takes the form

$$\Delta U_S = -\sigma_0 \cos\theta \cdot \Delta S = -\sigma_0 \cos\theta \cdot 2\pi R \Delta h, \qquad (4)$$

where $\Delta S = 2\pi R \Delta h$ is the change of the contact area between the liquid and the inner surface of the tube.



2.2 **[0.5 points]** The formula for the change in the potential energy ΔU_G of the liquid in the gravitational field that corresponds to an additional small rise of water level Δh in the tube takes the form

$$\Delta U_G = \pi R^2 \Delta h \rho g h. \qquad (5)$$

It is taken into account that the liquid of the mass $\Delta m = \pi R^2 \Delta h \rho$ has risen to the height h.

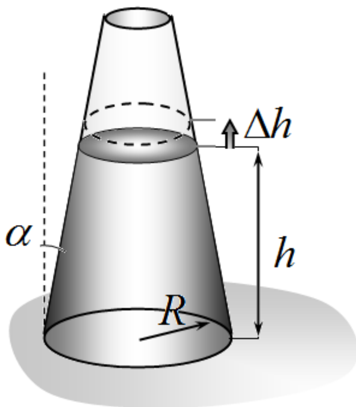
2.3 **[1.0 points]** If $|\Delta U_S|$ exceeds ΔU_G, the energy of the system decreases when the liquid has risen, and hence the liquid will continue to rise, otherwise liquid level will go down. At the position of equilibrium, the total change in energy should be equal to zero, and, thus,

$$\sigma_0 \cos\theta \cdot 2\pi R\Delta h = \pi R^2 \Delta h \rho g h \quad \Rightarrow \quad h_0 = \frac{2\sigma_0 \cos\theta}{\rho g R}. \tag{6}$$

Substitution of the numerical values leads to the following result

$$h_0 = \frac{2\sigma_0 \cos\theta}{\rho g R} = \frac{2 \cdot 0,072 \cdot \cos 20°}{1,0 \cdot 10^3 \cdot 9,8 \cdot 1,0 \cdot 10^{-3}} = 1,4 \cdot 10^{-2} m = 14mm. \tag{7}$$

3. Water in a vertical conical tube (4.0 points)



3.1 **[0.5 points]** The formula for the change in the surface energy of the system ΔU_S that corresponds to an additional small rise of water level Δh in the tube takes the form

$$\Delta U_S = -\sigma_0 \cos\theta \cdot \Delta S = -\sigma_0 \cos\theta \cdot 2\pi r \frac{\Delta h}{\cos\alpha}. \tag{8}$$

Here $r = R - h\,tg\alpha$ is the tube radius at the height h.

3.2 **[0.5 points]** The formula for the change in the potential energy ΔU_G of the liquid in the gravitational field that corresponds to an additional small rise of water level Δh in the tube takes the form

$$\Delta U_G = \pi r^2 \Delta h \rho g h. \tag{9}$$

3.3 **[1.0 points]** As above, the equilibrium position corresponds to the equality of the modules for the energy changes written here as

$$\sigma_0 \cos\theta \cdot 2\pi r \frac{\Delta h}{\cos\alpha} = \pi r^2 \Delta h \rho g h, \tag{10}$$

Substituting the expression for the radius of the tube at the height h, one gets the equation

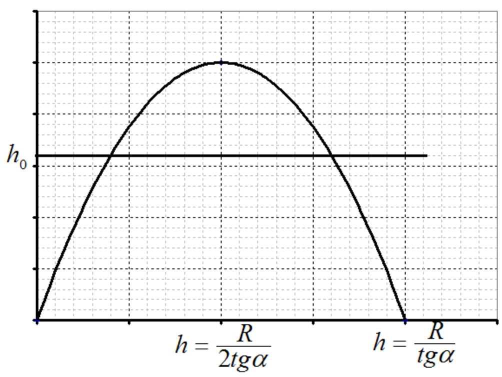
$$\frac{2\sigma_0 \cos\theta}{R - h\,tg\alpha} = \rho g h \cos\alpha, \tag{11}$$

in which the parameter h_0 is easily introduced as

$$\frac{2\sigma_0 \cos\theta}{\rho g R\left(1 - \frac{h}{R} tg\alpha\right)} = h\cos\alpha \quad \Rightarrow \quad \frac{h_0}{1 - \frac{h}{R} tg\alpha} = h\cos\alpha, \tag{12}$$

3.4 **[1.0 points]** The resulting equation is square with respect to h. Therefore it is necessary to analyze its roots, or condition of their absence. Let us rewrite equation (12) in the form

$$h_0 = h\cos\alpha\left(1 - \frac{h}{R} tg\alpha\right). \tag{13}$$



The quadratic function on the right side of this equation has zeros at $h = 0$ and $h = \frac{R}{tg\alpha}$,, and therefore, it reaches its maximum value of $\frac{R}{4tg\alpha}\cos\alpha$ at $h = \frac{R}{2tg\alpha}$. Consequently, equation (12) has no real roots at $h_0 > \frac{R}{4tg\alpha}\cos\alpha$. Otherwise, there are two roots. At the given parameters of the tube $\frac{R}{4tg\alpha}\cos\alpha = 25mm$, so there are two root corresponding to the two equilibrium positions. It is easy to show that the smaller root gives a stable equilibrium position, and the larger one is unstable and their numerical values are evaluated as

$$h = \frac{\cos\alpha \pm \sqrt{\cos^2\alpha - 4\dfrac{h_0}{R}\sin\alpha}}{2\dfrac{\sin\alpha}{R}} \quad \Rightarrow h_1 = 16,5mm,\ h_2 = 83,5mm$$

Thus, when $H < h_2$ the water level in the tube stops at the height h_1 and if the initial water level exceeds h_2 the water fills up the tube completely.

3.5 **[1.0 points]** The water fills up the tube at any initial value of H, if equation (12) has no roots at all. This condition is fulfilled when

$$h_0 > \frac{R}{4tg\alpha}\cos\alpha\ , \tag{14}$$

or

$$\sin\alpha > \frac{R}{4h_0} = 0,018\ .$$

4. Outflow of water (3.0 points)

4.1 **[3.0 points]** The water will start to pour out through the hole only if the water surface in the openings lose stability. This happens if the decrease of the potential energy in the gravitational field exceeds the increase in the absolute value of the surface energy. This condition is expressed by the inequality

$$2\sigma_0\pi h^2 < 2\frac{\pi R^2 h^2}{6}\rho g\ , \tag{15}$$

from which it follows that

$$R > \sqrt{\frac{6\sigma_0}{\rho g}} = 6,6mm \tag{16}$$

Grading scheme

№	Content	points	
1	The segment of the boundary is chosen	0,5	**1,0**
	Balance condition for the segment (2) $(\sigma_2 - \sigma_1)\Delta l = \sigma_0 \Delta l \cos\theta$	0,5	
2.1	Formula (4) $\Delta U_S = -\sigma_0\cos\theta\cdot\Delta S = -\sigma_0\cos\theta\cdot 2\pi R\Delta h$	0.5	**0.5**
2.2	Formula (5) $\Delta U_G = \pi R^2\Delta h\rho g h$	0,5	**0,5**
2.3	The equality for the change of energies (6) $\sigma_0\cos\theta\cdot 2\pi R\Delta h = \pi R^2\Delta h\rho g h \quad \Rightarrow \quad h_0 = \dfrac{2\sigma_0\cos\theta}{\rho g R}$	0,5	**1,0**
	Formula (7) $h_0 = \dfrac{2\sigma_0\cos\theta}{\rho g R}$	0,2	
	Correct numerical value (with significant digits of accuracy) $h_0 = 1,4\cdot 10^{-2}m = 14mm$	0.3	
3.1	Formula (8) $\Delta U_S = -\sigma_0\cos\theta\cdot\Delta S = -\sigma_0\cos\theta\cdot 2\pi r\dfrac{\Delta h}{\cos\alpha}$	0,5	**0,5**
3.2	Formula (9) $\Delta U_G = \pi r^2\Delta h\rho g h$	0,5	**0,5**
3.3	Equation (10) $\sigma_0\cos\theta\cdot 2\pi r\dfrac{\Delta h}{\cos\alpha} = \pi r^2\Delta h\rho g h$	0,5	**1.0**

	Equation (12) $\frac{2\sigma_0\cos\theta}{\rho g R\left(1-\frac{h}{R}tg\alpha\right)} = h\cos\alpha \quad \Rightarrow \quad \frac{h_0}{1-\frac{h}{R}tg\alpha} = h\cos\alpha$	0,5	
3.4	Solution of equation (12)	0,3	
	Analysis of the stability of the roots	0.5	**1,0**
	Correct result	0,2	
3.5	Condition for root absence $\sin\alpha > \frac{R}{4h_0}$	0,6	**1,0**
	Numerical value of the angle $\sin\alpha > 0,018$	0,4	
4.1	Basic idea: change in the potential energy must be greater than the change in the surface energy	1,5	
	Inequality (15) $2\sigma_0\pi h^2 < 2\frac{\pi R^2 h^2}{6}\rho g$	1,0	**3,0**
	Numerical value for the radius (with significant digits of accuracy) $R > \sqrt{\frac{6\sigma_0}{\rho g}} = 6,6mm$	0,5	
Total			**10,0**

Problem 3. Nonlinear capacitor (10,0 points)

1. **[0.75 points]** After a long period of time the electric current in the circuit turns zero and the capacitor will be fully charged, i.e.

$$I = 0. \tag{1}$$

The voltage provided by the source is thus drop on the capacitor whose capacitance at $U_0 = 5$ V is obtained from the graph

$$C = 0{,}10\ \mu F. \tag{2}$$

The charge of the capacitor is therefore found as

$$q = CU_0 = 0.50\ \mu Cl. \tag{3}$$

2. **[0.25 points]** Since the electric current in the circuit is finite and charging the capacitor to the voltage drop of $U_0 = 10$ V requires an infinite charge, the corresponding time is obtained as.

$$t = \infty. \tag{4}$$

3. **[3.0 points]** Let the capacitor be charged with the charge q and its capacitance is equal to C, then, since all elements are connected in series, one gets

$$U_0 = \frac{q}{C} + IR, \tag{5}$$

where the electric current is obtained as

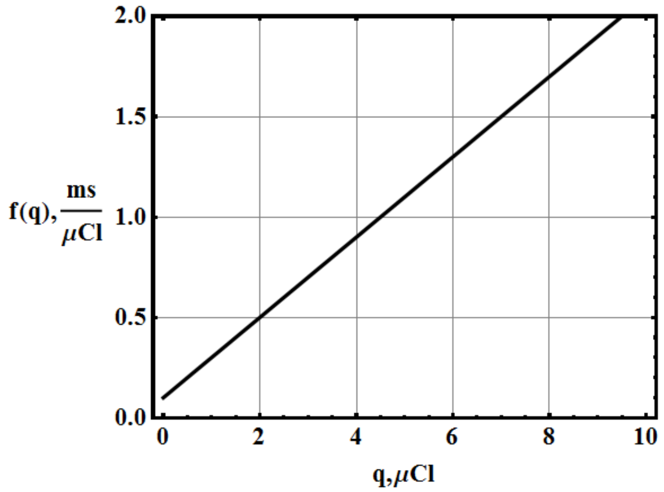
$$I = \frac{dq}{dt}. \tag{6}$$

Substituting (6) into (5) yields

$$dt = \frac{R}{U_0 - \frac{q}{C(q)}}dq = f(q)dq, \tag{7}$$

where $f(q) = R/\left(U_0 - \frac{q}{C(q)}\right)$ is a function of the charge of the capacitor.

The function $f(q)$ is easily retrieved from the provided graph of $C = C(U)$ and turns out linear as shown in the figure below.



The linear equation has the form

$$f(q) = a + bq, \tag{8}$$

$$a = 0.10\ \frac{\text{ms}}{\mu\text{Cl}}, \tag{9}$$

$$b = 0.20\frac{ms}{(\mu\text{Cl})^2}. \tag{10}$$

The time needed for a capacitor to be charge to $q = 4$ мкКл, is derived from equations (7) and (8) as

$$t = aq + \frac{1}{2}bq^2 = 2.0\,ms \tag{11}$$

4. **[0.5 points]** The time interval Δt, needed for the capacitor to increase its charge from $q_0 = 4$ μCl till $q = 8$ μCl is found as

$$t = (q - q_0)\left(a + \frac{1}{2}b(q + q_0)\right) = 5.2ms. \tag{12}$$

5. **[0.5 points]** Solving equation (11) one obtains that

$$q_{1/2} = \frac{-a \pm \sqrt{a^2 + 2bt}}{b}. \tag{13}$$

It is obvious that at the initial time moment $q(0) = 0$, that is why the plus sign must be taken in formula (13) and one finally gets that

$$q = \frac{\sqrt{a^2 + 2bt} - a}{b} = 5{,}0\ \mu\text{Cl}. \tag{14}$$

6. **[0.5 points]** For an ordinary capacitor its charge is proportional to the voltage drop across it, i.e.

$$q = CU, \tag{15}$$

and the electric current in the circuit is derived as

$$I = \frac{dq}{dt} = C\frac{dU}{dt} \sim \frac{dU}{dt}. \tag{16}$$

Since the capacitor and the resistor are connected in a series, then the electric current passing through them is the same, and, thus, the oscillation of voltage on the resistor is in phase with the oscillation of the current. Substituting $U \sim \sin\omega t$ gives rise to $I \sim \cos\omega t = \sin\left(\omega t - \frac{\pi}{2}\right)$, i.e. the phase difference between the oscillations of the voltage across the capacitor and the resistor is $\varphi = -\frac{\pi}{2}$.

The circuit contains the nonlinear capacitor but the proportionality in equation (16) stays the same, since the oscillation of the voltage is small compared with the constant voltage provided by the source, i.e.

$$\varphi = -\frac{\pi}{2}. \tag{17}$$

7. **[4.0 points]** The voltage drop has constant and alternating components. After a long period of time the constant component of the voltage will be dropped across the capacitor only, i.e.

$$U_C = 5{,}000\ V, \tag{18}$$

and the constant component of the voltage drop across the resistor will be equal to zero, i.e.

$$U_R = 0. \tag{19}$$

In our case the capacitance is voltage dependent, that is why equation (16) is rewritten in the form

$$I = \frac{dq}{dt} = C(U)\frac{dU}{dt} + U\frac{dC(U)}{dU}\frac{dU}{dt} = C_{\text{eff}}\frac{dU}{dt}, \tag{20}$$

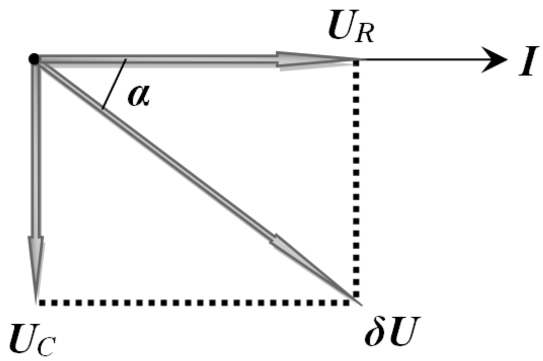
where the effective capacitance is found as

$$C_{\text{eff}} = C(U) + U\frac{dC(U)}{dU} = 0.200\ \mu F. \tag{21}$$

It is well known that the reactive resistance of the capacitor is equal to

$$X_{\text{C}} = \frac{1}{\omega C_{\text{eff}}}. \tag{22}$$

To calculate the electric current for the resistor and the capacitor connected in a series let us use the vector diagram shown in the figure below.



From this diagram the amplitude of the electric current is easily found as

$$I = \frac{\delta U}{\sqrt{R^2 + \frac{1}{\omega^2 C_{\text{eff}}^2}}} = 44.7\ \mu A. \tag{23}$$

and the corresponding phase shift α between the current and the voltage is obtained as

$$\alpha = \text{arctg}\left(\frac{U_C}{U_R}\right) = \text{arctg}\left(\frac{1}{\omega C_{\text{eff}} R}\right) = 1{,}11\ rad = 63{,}4°. \tag{24}$$

Finally, the dependence of the electric current on the time is derived as

$$I(t) = [44{,}7 \sin(2500t + 1{,}1)]\ \mu A. \tag{25}$$

8. **[0.5 points]** The amplitude of the voltage oscillation on the capacitor is obtained from the vector diagram as

$$U_C = \delta U \sin\alpha. \tag{26}$$

Finally, taking into account the constant component of the voltage drop on the capacitor one gets

$$U_{\text{C}}(t) = U_{\text{C}} + \delta U \sin\alpha \sin\left(\omega t - \frac{\pi}{2} + \alpha\right) = \\ = [5{,}000 + 0{,}089 \sin(2500t - 0{,}464)]\ V. \tag{27}$$

Grading scheme

№	Content	points	
1	Equation (1) $I = 0$	0.25	**0,75**
	Equation (2) $C = 0{,}10\ \mu F$	0.25	
	Equation (3) $q = CU_0 = 0.50\ \mu Cl$	0.25	
2	Уравнение (4) $t = \infty$.	0.25	**0,25**
3	Equation (5) $U_0 = \frac{q}{C} + IR$	0.25	
	Equation (6) $I = \frac{dq}{dt}$	0.25	
	Equation (7) $dt = \frac{R}{U_0 - \frac{q}{C(q)}} dq = f(q)dq$ with the function $f(q) =$	0.25	

	$R/\left(U_0 - \frac{q}{C(q)}\right)$		3.0
	Equation (8): it is found that $f(q) = a + bq$	1.25	
	Equation (9) $a = 0.10\ \frac{ms}{\mu Cl}$	0.25	
	Equation (10) $b = 0.20\frac{ms}{(\mu Cl)^2}$	0.25	
	Equation (11) $t = aq + \frac{1}{2}bq^2$	0.25	
	Numerical value in equation (11) $t = 2.0$ мс	0.25	
4	Equation (12) $t = (q - q_0)\left(a + \frac{1}{2}b(q + q_0)\right)$	0.25	0.5
	Numerical value in equation (12)	0.25	
5	Equation (14) $q = \frac{\sqrt{a^2+2bt}-a}{b}$	0.25	0.5
	Numerical value in equation (14)	0.25	
6	Equations (15), (16) or equivalent	0.25	0.5
	Equation (17) $\varphi = -\frac{\pi}{2}$	0.25	
7	Equation (18) $U_C = 5{,}000$ В	0,25	4.0
	Equation (20) $I = \frac{dq}{dt} = C(U)\frac{dU}{dt} + U\frac{dC(U)}{dU}\frac{dU}{dt} = C_{\text{eff}}\frac{dU}{dt}$	1,5	
	Equation (21): correct numerical value $C_{\text{eff}} = 0.200\ \mu F$	0.25	
	Equation (22) $X_{\mathrm{C}} = \frac{1}{\omega C_{\text{eff}}}$	0,25	
	Correct vector diagram or impedances	0,5	
	Equation (23) $I = \frac{\delta U}{\sqrt{R^2+\frac{1}{\omega^2 C_{\text{eff}}^2}}}$	0,25	
	Equation (23): correct numerical value $I = 44.7\ \mu A$	0,25	
	Equation (24) $\alpha = \text{arctg}\left(\frac{1}{\omega C_{\text{eff}} R}\right)$	0,25	
	Equation (24): correct numerical value $\alpha = 1{,}11\ rad = 63{,}4°$	0,25	
	Equation (25) $I(t) = [44{,}7\sin(2500t + 1{,}1)]\ \mu A$	0,25	
8	Equation (26) $U_C = \delta U \sin\alpha$	0.25	0.5
	Equation (27) $U_{\mathrm{C}}(t) = [5{,}000 + 0{,}089\sin(2500t - 0{,}464)]\ V$	0.25	
Total			**10,0**

EXPERIMENTAL COMPETITION

16 January, 2016

Please read the instructions first:

1. The Experimental competition consists of one problem. This part of the competition lasts 3 hours.
2. Please only use the pen that is provided to you.
3. You can use your own non-programmable calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet and additional papers***. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets.*** Write only inside the bordered area.
6. Fill the boxes at the top of each sheet of paper with your country (***Country*****),** your student code (***Student Code***), the question number (***Question Number*****)**, the progressive number of each sheet **(*****Page Number***), and the total number of ***Writing sheets*** (***Total Number of Pages***). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
7. At the end of the exam, arrange all sheets for each problem in the following order:
    - Used ***Writing sheets*** in order;
    - The sheets you do not wish to be evaluated
    - Unused sheets and the printed question.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper or equipment out of the room

Coal tablet (15.0 points)

In this experiment it is necessary for you to investigate the dependence of the electric resistance of activated coal tablet under mechanical stress.

Devices and the equipment: tablet of activated coal, two wooden rulers with electric contacts, a set of weights with hang-ons ($6\times100g$), multimeter, variable resistor with the maximum resistance of 100 Ohms, constant resistor with the resistance of 1.0 Ohm, source of the constant voltage of 9,0 V, connecting wires.

Teachers of this school have designed for you the mechanical part of the installation: one ruler is fixed, and the other is used to suspend weights. You should design and connect an electric circuit by yourself.

Part 1. Ohm's law (7.0 points)

1.1 Measure the dependence of the electric current passing through the coal tablet as a function the voltage across it. Measurements should be made for two values of the mechanical load, 200 and 400 g (per tablet). Plot the graphs of the corresponding dependences, specify whether it is possible to treat the resistance of the tablet as independent on the applied voltage. Calculate the resistance of the tablet for the above given mechanical stresses, estimate their errors.

Provide the sketch of the electric circuit that you used to make measurements. In this part of the experiment it is required to use a multimeter in the voltmeter mode only.

Part 2. Mechanical stress and resistance (5.0 points)

2.1 Make measurements for the dependence of the tablet resistance as a function of the mechanical stress on the tablet measured in grammes. Make measurements «in two directions»: increasing the stress and decreasing it.
2.2. Plot the graphs of the corresponding dependences.
2.3 Propose the simplest linearization of the obtained dependences. Plot the graphs of the linearized dependences.

Part 3. Designing scales (3.0 points)

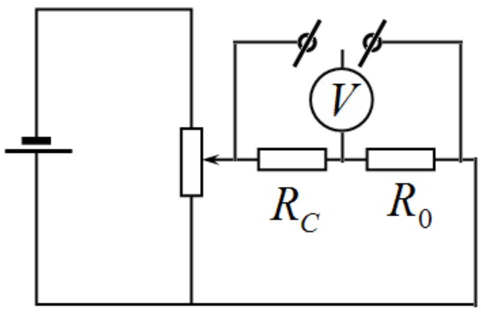
In this part of the experiment it is necessary for you to propose an installation for electronic scales in which the central role is played by the tablet investigated in the previous parts.
3.1 Propose an electric circuit in which the voltage on one of the elements is approximately linearly proportional to the total weight used as a load.
3.2 Plot the calibrating graph for your scales, i.e. the dependence of the voltage on the chosen element as a function of the load weight.

SOLUTION FOR THE EXPERIMENTAL COMPETITION

Coal tablet (15.0 points)

Part 1. Ohm's law (7.0 points)

To check whether Ohm's is in power in this caseб the circuit shown on the right should be connected such that the multimiter in voltmiter mode is used to measure the voltage across the tablet R_C and the resistor $R_0 = 1{,}0\,Ohm$. The latter is equal to the electric current.

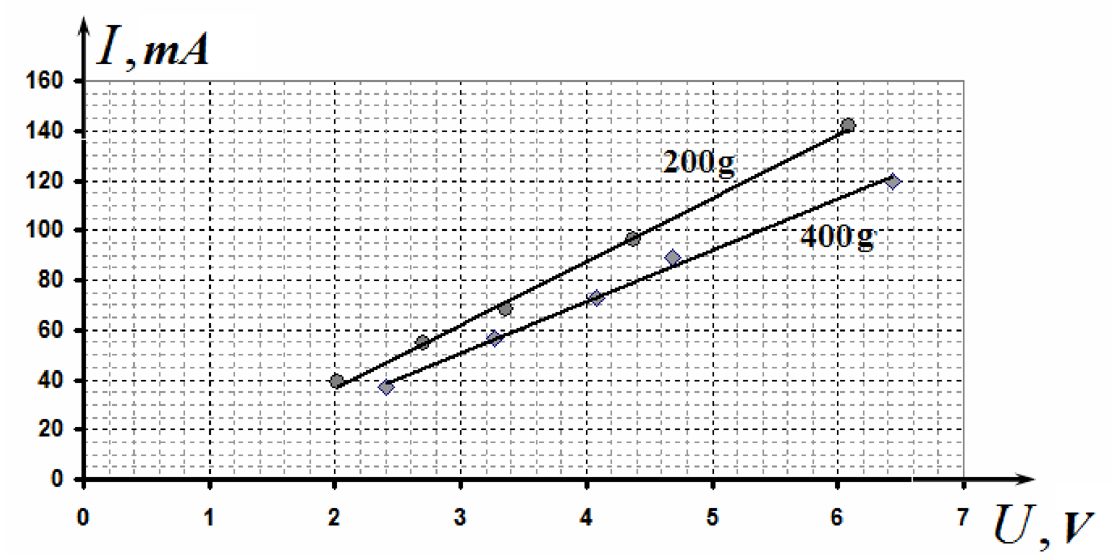


The results for two given weights are presented in table 1.

Table 1

$m = 200g$		$m = 400g$	
U, V	I, mA	U, V	I, mA
6,43	120,0	6,09	142,0
4,69	89,0	4,37	96,0
4,08	72,7	3,36	68,0
3,27	56,6	2,7	54,5
2,41	37,0	2,02	39,2

The graphs of the obtained dependences are shown below.



With a high degree of accuracy these dependences are linear which means that the resistance is independent of the voltage and Ohm's law holds.

To calculate the resistances it is required to use the leasrt square method. The result are summarized as follows

$$200\text{ г} - \ a = (21 \pm 2)\frac{mA}{V} \quad b = (-10 \pm 11)mA;$$

$$400\text{ г - } a = (25 \pm 2)\frac{mA}{V} \quad b = (-10 \pm 8)mA$$

Consequently, the corresponding resistances are found as

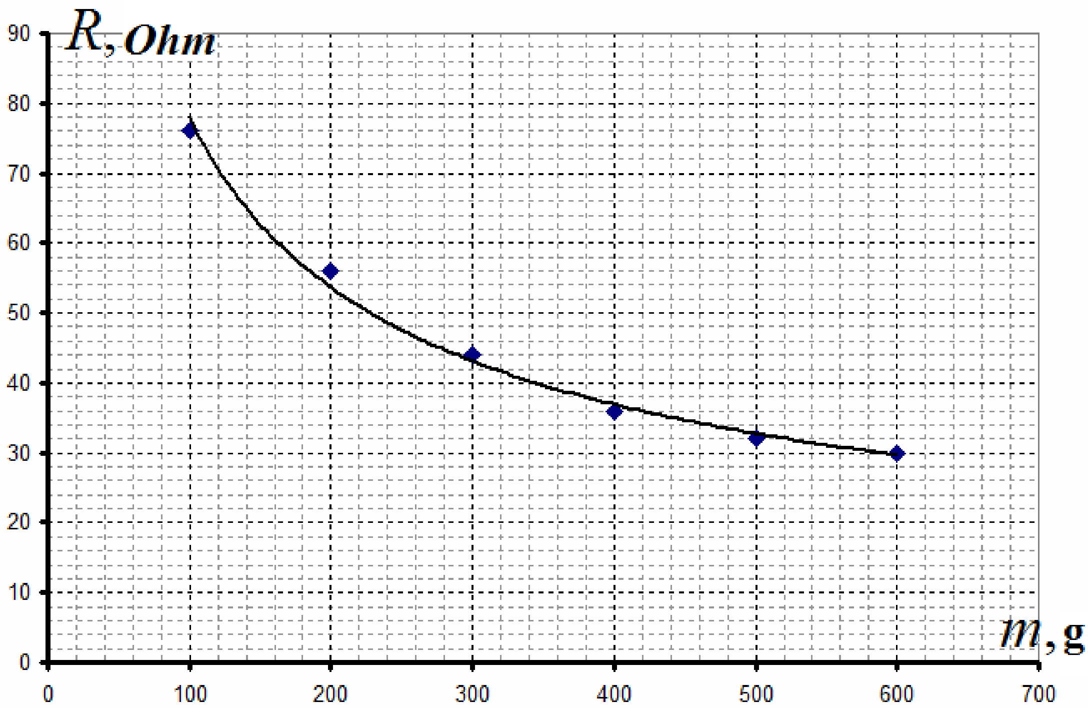
$$R_{200} = \frac{1}{a} = (47 \pm 5)Ohm \quad R_{400} = \frac{1}{a} = (40 \pm 3)Ohm$$

Part 2. Mechanical stress and resistance (5.0 points)

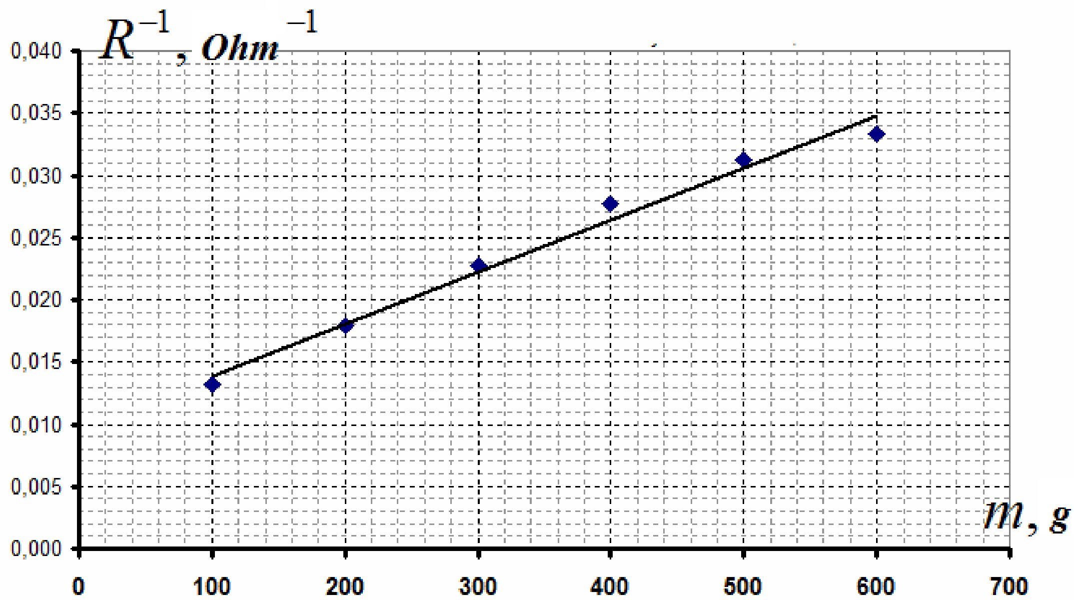
The resistance can be directly measured by the multimeter. And the results are shown in table 2.

Table 2.

m, g	R, Ohm	R^{-1}, Ohm^{-1}
100	76	0,0132
200	56	0,0179
300	44	0,0227
400	36	0,0278
500	32	0,0313
600	30	0,0333



It is seen from the graph above that the dependence is actually inversely proportional. Thus, the conductivity is approximately proportional to the mass of weights.

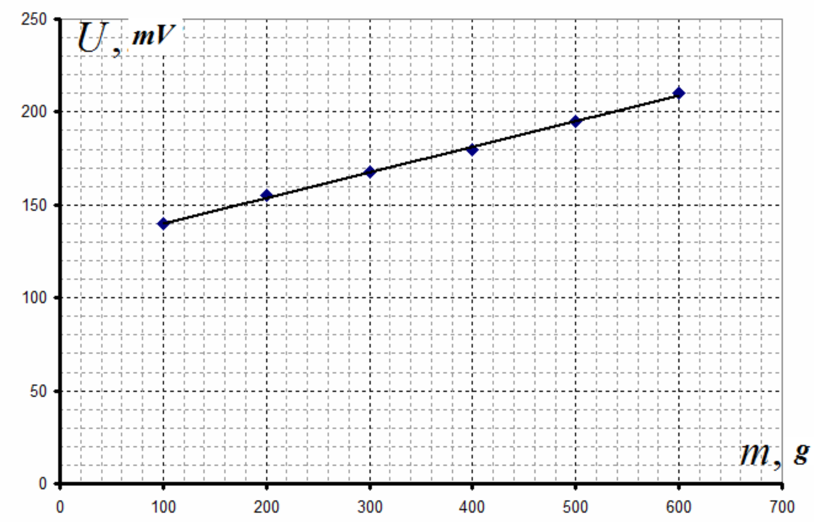


Part 3. Designing scales (3.0 points)

The obtained results demonstrate that the current in the circuit shown above is proportional to the mechanical stress on the tablet. That is why the circuit for electronic sales should be the same and the voltage must be measured across the resistor R_0. The "readings" of the scales as a function of the mechanical stress are presented in table 3 and in the corresponding graph.

Table 3.

m, g	U, mV
100	140
200	155
300	168
400	180
500	195
600	210



Grading scheme

	Content	Points	Total
3.1	The electrical circuit allows one to measure the electric current and the voltage.	1,0	1.0
	Setting up the mechanical stress according to lever rule	0,5	0,5
	Taking measurements: - no less than 5 points for each dependence (*3-4.less than 3*); - the voltage up to 6 V (up to 4 V, or less); Linear dependences are obtained	 2x1,0 (*2x0,5, 0*) 2x0.5 (2x0,25,0) 2x0,25	3,5
	Plotting the graph: Axis are labeled and ticked, All points are presnt in the graph; Smooth lines are drawn	 0,1 0.2 0,2	0,5
	Calculation of the resistances: (acceptable range – 30-60 Ohm) - using LSM; - graphically; - by two pints Errors are evaluated	 2x0,5 (2x0,4) (2x0,25) 2x0,25	1,5
2.1	Taking measuments: (the acceptable range 100-30 Ohm) - no less than 6 points; - the resistance decreases/increases at least twice; - repeated measurements are taken; - the obtained dependence is inversely proportional;	 1,5 1,0 0,5 0,5	3,5
2.2	Plotting the graph Axis are labeled and ticked, All points are presnt in the graph; Smooth lines are drawn	 0,1 0,2 0,2	0,5
2.3	Linearization is made	0,5	0,5
	The graph of the linearized dependence	0,5	0,5
3.1	The circuit is to measure the current	1,5	1,5
	Plotting the calibrating graph: no less than 5 points; Axis are labeled and ticked, All points are presnt in the graph; Smooth lines are drawn	 0,5 0,5 0,5	1,5
	Total	15	

THEORETICAL COMPETITION

January 14, 2017

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don’t have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets.*** Write only inside the boxed area.
6. Begin each question on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Question Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
   - Used ***Writing sheets*** in order;
   - The sheets you do not wish to be evaluated
   - Unused sheets and the printed question.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

Problem 1 (10.0 points)

This problem consists of three independent parts.

Problem 1A (3.0 points)

A new rapid-firing multibarrel machine gun is tested to provide $n = 100$ shots/s. The flight speed of a bullet is $u = 1000$ m/s, and its mass is equal to $m = 10$ g. The target is a sandbox of mass $M = 1000$ kg vertically suspended on a rope. Considering that the bullets are all stuck in the sandbox, evaluate the maximum deflection angle of the sandbox from the vertical after the shooting has started.

Problem 1B (4.0 points)

There is a bubble of radius R_1 somewhere in free space. With the help of an external ionizer the soap film is quickly charged up to some positive value, then over certain period of time the radius of the bubble ceases to change and becomes equal to $R_2 = 2R_1$. Find the electric charge q, which has been acquired by the soap film, if its heat capacity and heat conductivity are both negligible. The surface tension σ of the soap film does not depend on the temperature. The air is considered as an ideal diatomic gas.

Problem 1C (3.0 points)

Two identical source of coherent monochromatic waves with the wavelength λ are placed at the points S_1 and S_2 (see the provided separate sheet of paper for this subproblem). On the sheet of paper provided one wavelength corresponds to the size of the two squares. Receivers of waves are located at the points A_1 and A_2. Each source emits waves with the same intensity I_0 such that the change in the wave amplitude can be neglected with the distance from the sources. On the same sheet of paper for this subproblem plot those points in the highlighted oval area at which a third source should be placed to completely suppress the signals at the points A_1 and A_2. simulateously. What should be the intensity of the wave from the third source? All sources emit waves with the same phase and polarization perpendicular to the plane of the figure.

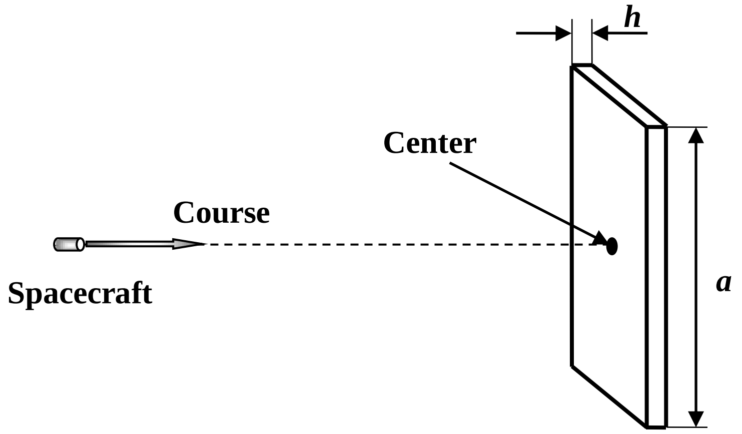
Attention! Make all the necessary drawings in the same sheet of paper provided for this subproblem, collect it together with the answer sheets **Writing sheets**, incorporating it into the overall numbering. Otherwise, your answer to this subproblem will not be evaluated!

Problem 2 (10.0 points)
Fantastic trip through the Universe

A reconnaissance spacecraft of the well developed civilization plows the universe. In this problem, you are asked to consider a few situations of that interstellar travel from the physical point of view. For all numerical calculations consider the gravitational constant known and equal to $G = 6.672 \times 10^{-11} m^3/(kg \cdot s^2)$.

1. Planets with strange shapes (3.9 points)

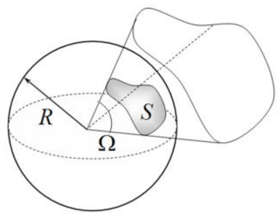
At some distance from the spacecraft the crew captain discovers the first planet that has a strange shape of a parallelepiped with a square base of side a and a very small thickness $h \ll a$. The captain gives the order to pursue a course to planet's center as shown in the figure below.



It has been revealed after turning off the engine that the spacecraft acceleration of the free fall g, that the planet provides at distances much greater h, remains proportional to the solid angle Ω, at which the planet is seen from the spacecraft, that is:

$$g = \alpha\Omega.$$

The solid angle Ω is a part of the space, which unites all the rays emanating from a given point (the vertex) and intersecting a surface (which is called a surface subtending the solid angle). The boundary of the solid angle is a certain conical surface.



The solid angle is measured by the ratio of the area S of the sphere centered at the vertex, which is cut by this solid angle, to the square of the sphere radius:

$$\Omega = \frac{S}{R^2}.$$

Solid angles are measured by abstract dimensionless quantities. The unit of the solid angle is a steradian, which, in SI units, is equal to the solid angle cutting out the surface area of R^2 from the sphere of radius R. The whole sphere corresponds to the solid angle of 4π steradian (full solid angle) from any vertex, situated inside the sphere, in particular, for the center of the sphere.

After landing on the planet surface and taking the soil samples, the scientists reported to the captain that the planet is composed of homogeneous material of the density $\rho_1 = 3000\, kg/m^3$ and the free fall acceleration near the geometric surface center of the planet remains almost constant and is equal to $g_1 = 9,81 \times 10^{-2} m/s^2$.

1.1 **[0.7 points]** Find and calculate the thickness of the planet h.
1.2 **[0.5 points]** Find and calculate the coefficient α.

After leaving the first planet, the captain and his crew meet another exotic planet that shapes a regular pyramid with a square base of the side $a = 10000 km$ and of the height $\frac{a}{2}$.

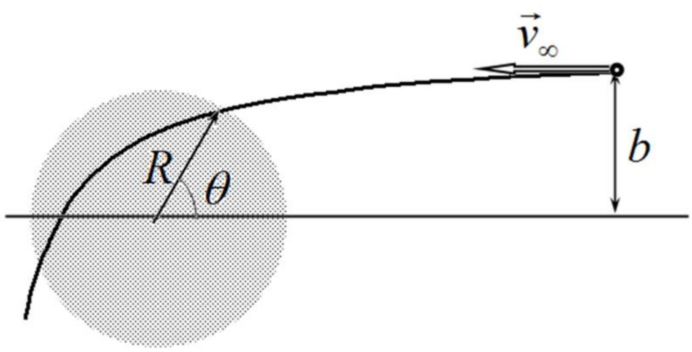
1.3 **[0.7 points]** Find and calculate the free fall acceleration g_2 measured at the top of the uniform pyramidal planet, if its density is equal to $\rho_2 = 4500 kg/m^3$.

The spacecraft has left the pyramidal planet from its top, starting with the characteristic parabolic velocity $v_1 = 3{,}45 km/s$. The next planet the spacecraft has met on its way is the one shaped like a perfect uniform cube with the side a. After accurate measurements, the captain and his crew have found that the density of the cubic planet is $\rho_3 = 5000 kg/m^3$.

1.4 **[2.0 points]** Find and calculate the parabolic velocity v_2 for the spacecraft to start from one of the vertices of the cubic planet.

2. Dusty cloud (6.1 points)

The spacecraft encounters a very large massive dust cloud of radius $R = 1{,}50 \times 10^7 km$ and of homogeneous density $\rho_4 = 50{,}0 kg/m^3$. The speed of the spacecraft at a large distance from the cloud reaches the value of $v_\infty = 100 km/s$, and the impact parameter measured from the cloud center is equal to $b = 1{,}50 \times 10^8 km$. The engine remains switched off.



2.1 **[2.5 points]** Find and calculate the coordinate of the spacecraft entry into the dust cloud, characterized by the angle θ.

2.2 **[2.0 points]** Find and calculate the minimum distance r_{min}, the spacecraft flies by from the cloud center. Resistance to the motion of the spacecraft caused by the cloud particles can be neglected.

Making sure that it is impossible to avoid a collision with the cloud, the captain of the spacecraft takes the decision to turn on the engine, thereby increasing the speed v_∞.

2.3 **[1.0 points]** Find and calculate the minimum speed $v_{\infty,min}$, at which the spacecraft passes safely by the dust cloud.

Successfully passing the obstacle, the captain and his crew have discovered that the particles of the dust clouds contain valuable elements.

2.4 **[0.6 points]** Find the minimum work A, which must be performed in order to gradually bring all the dust particles onto a very remote processing plant.

Problem 3 (10.0 points)
Resistance of a prism
1. Mathematical introduction (3,0 points)

By definition, it is believed that terms of the numerical sequence obey the recurrence relation if each successive term is expressed through the previous ones. For example, for a well known geometric progression we have

$$x_k = \lambda x_{k-1}, \tag{1}$$

where $k = 1, 2, 3, \ldots$, λ stands for a fixed number and zeroth term of the numerical sequence has some value of A, i.e. $x_0 = A$.

1.1 **[0.2 points]** Obtain an explicit formula for an arbitrary term of the sequence x_k, i.e. express it through the successive number k, the initial value A and λ.

Let us consider the number $\lambda = 2 + \sqrt{3}$. Taking its natural power of k, the result can be presented in the following form

$$\lambda^k = p_k + q_k\sqrt{3}, \tag{2}$$

where p_k, q_k denote some integer numbers.

1.2 **[0.4 points]** Find the recurrence relations, expressing the values of p_k, q_k through the previous values p_{k-1}, q_{k-1}. Find also the inverse relations, expressing p_{k-1}, q_{k-1} through p_k, q_k.

1.3 **[0.7 points]** Calculate the numerical values of the coefficients p_k, q_k for $k = 1, 2, 3, 4, 5$.

1.4 **[0.2 points]** Express the number $\lambda^{-k} = \left(2+\sqrt{3}\right)^{-k}$ in terms of p_k, q_k.

Let the terms of a certain numerical sequence obey the recurrence relation

$$x_{k+1} = 4x_k - x_{k-1}, \quad k = 1, \ldots, N-1, \tag{3}$$

where it is known that N is some integer number, $x_0 = A$ and $x_N = B$, A, B designate some values.

1.5 **[1.0 points]** Obtain an explicit formula for an arbitrary term x_k of the sequence (3), i.e. express it in terms of the number k and values A, B, N.

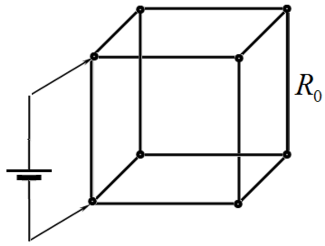
1.6 **[0.5 points]** Obtain an explicit formula for an arbitrary term x_k of the sequence (3) through p_k, q_k, found in.1.2-1.3.

Hint. The solution to the recurrence relation (3) must be sought in the form $x_k = C\lambda^k$, where C is a constant. Determine at what values of λ it is possible and construct an exact solution that satisfies all above stated conditions.

2. Wire frame in the shape of a prism (7.0 points)

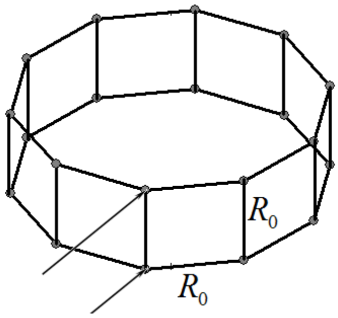
Problems are widely known in which you are asked to find the electrical resistance of a simple wire frame. An example of such a frame shaped in the form of a cube is shown below. Let the electric resistance of each edge be equal to R_0.

2.1 **[0.8 points]** Find the total resistance of the cube when the source is connected to the two adjacent cube vertices as shown on the right.

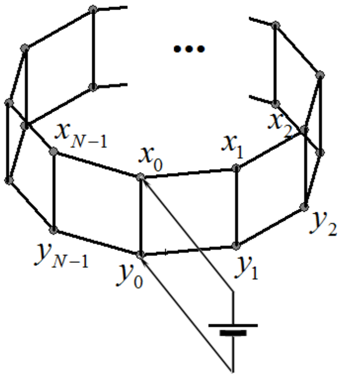


Let us now consider a more general case of the wire frame in the form of the regular prism with an arbitrary number N of side faces and determine its electrical resistance when the source is connected to the adjacent vertices of the side edge, as shown in the figure below. The resistance of each frame edge is equal to R_0.

For convenience, the vertices of the prism and their electric potentials on the upper and the bottom sides are consequently numbered and denoted, as shown in the figure below. DC voltage source is applied to zeroth vertices such that the source sets the potentials of the vertices equal to $x_0 = +\varphi_0$ and $y_0 = -\varphi_0$, respectively.



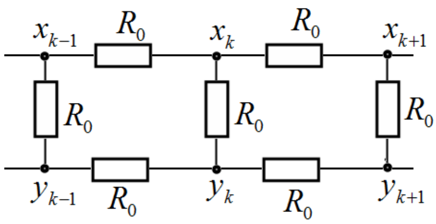
Wire frame in the shape of a prism



Numbering and denoting vertices' potentials

2.2 **[0.2 points]** Find the relation between the potentials x_k and y_k. Find the relation between the potentials x_k and x_{N-k}

Consider an arbitrary lateral edge, except zeroth ($k = 0$) and the last ($k = N-1$) ones. The corresponding circuit diagram is shown below.



2.3 **[1.0 points]** Find an recurrence relation for the potential x_k as expressed in terms of the neighboring vertices potentials for $k = 1,2...N-2$.

2.4 **[0.2 points]** Find boundary conditions at $k = 0$ and $k = N-1$ necessary for unambiguous determination of the potential x_k.

2.5 **[0.2 points]** Find explicit expressions for the potentials x_k and y_k for all possible numbers $k = 0,1,2...N-1$.

2.6 **[0.4 points]** Express the source current in terms of φ_0, R_0, N. Use the appropriate numbers p_k, q_k .obtained in the Mathematical introduction to this problem.

2.7 **[0.2 points]** Derive an explicit formula for the resistance R_N of the wire frame, expressed in terms of R_0, p_N, q_N.

2.8 **[1.0 points]** Find and tabulate the exact values of the frame resistances for $N = 1,2,3,4,5$.

2.9 **[0.5 points]** Draw the equivalent circuits for the exotic prisms with $N = 1$ and $N = 2$.

2.10 **[1.0 points]** Find the resistance R_∞ of the wire frame at $N \to \infty$.

2.11 **[1.5 points]** Find the minimum value of N at which the prism resistance deviation from R_∞ does not exceed 2%.

SOLUTIONS TO THE PROBLEMS OF THE THEORETICAL COMPETITION

Attention. Points in grading are not divided!

Problem 1 (10.0 points)

Problem 1A (3.0 points)

Suppose that during the time interval Δt the number of bullets hit the sandbox is equal to ΔN. Then, the momentum, transferred to the sandbox, is found as $\Delta p = \Delta Nmu$ which is equivalent to the action of a horizontal force

$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta Nmu}{\Delta t} = nmu\,. \tag{1}$$

For the deflection angle α this horizontal force F does work

$$A = Fl\sin\alpha\,. \tag{2}$$

Here l is the distance from the point of suspension to the center of mass.

The deflection angle is a maximum when all the work done is converted into the target potential energy equal to

$$W = Mgl(1-\cos\alpha)\,. \tag{3}$$

The energy conservation law $A = W$ yields the final answer

$$\alpha_{\max} = 2\,\mathrm{arctg}\left(\frac{F}{Mg}\right) = 2\,\mathrm{arctg}\left(\frac{nmu}{Mg}\right) = 0.2 \text{ rad} = 11.65^\circ\,. \tag{4}$$

Content	Points
Formula (1) $F = \frac{\Delta p}{\Delta t} = \frac{\Delta Nmu}{\Delta t} = nmu$	0,5
Formula (2) $A = Fl\sin\alpha$	0,5
Formula (3) $W = Mgl(1-\cos\alpha)$	0,5
Formula (4) $\alpha_{\max} = 2\,\mathrm{arctg}\left(\frac{F}{Mg}\right) = 2\,\mathrm{arctg}\left(\frac{nmu}{Mg}\right)$	1,0
Numerical value $\alpha_{\max} = 0.2 \text{ rad} = 11.65^\circ$	0,5
Total	**3,0**

Problem 1B (4.0 points)

Charge repulsion on the surface results in an increase of the bubble size. Due to inertia the bubble passes by the equilibrium position and oscillations occur. Due to internal friction of the gas the oscillations vanish, the bubble reaches a new equilibrium state such that the kinetic energy of the soap film is transferred to the internal energy of the gas, which means that the gas in this situation does not obey the adiabatic equation.

Let us make use of the law of energy conservation for the film-gas system of the form:

$$\frac{5}{2}P_1V_1 + \sigma 8\pi R_1^2 + \frac{kq^2}{2R_1} = \frac{5}{2}P_2V_2 + \sigma 8\pi R_2^2 + \frac{kq^2}{2R_2} \tag{1}$$

Taking into account the surface tension the initial pressure of the gas in the bubble is written as

$$p_1 = \frac{4\sigma}{R_1}\,. \tag{2}$$

The final pressure in view of the electrostatic repulsion force is found as (recall the well-known problem for the forces that attempt to tear out the charged sphere)

$$p_2 = \frac{4\sigma}{R_2} - \frac{q^2}{32\pi^2\varepsilon_0 R_2^4}. \qquad (3)$$

In our case

$$V_1 = 4\pi R_1^3/3,\ V_2 = 4\pi R_2^3/3. \qquad (4)$$

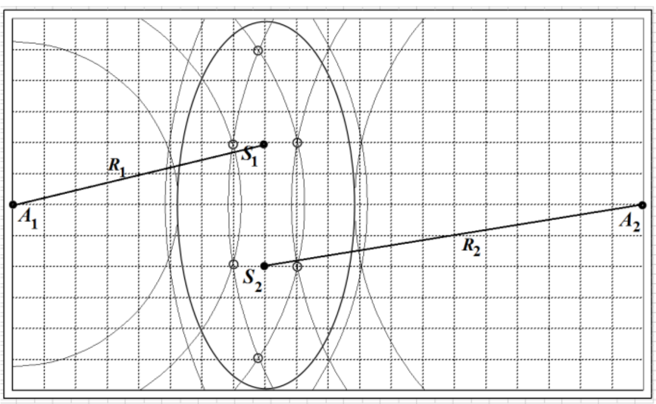
Under those conditions, the joint solution of equations (1) - (4) gives the answer

$$q = 32\pi\sqrt{\varepsilon_0\sigma R_1^3}. \qquad (5)$$

Content	Points
$P_{initial} = P_{surf}$	0.5
$P_{final} = P_{surf} - P_{electr}$	0.3
$P_{surf} = 4\sigma/R$	0.3
$P_{electr} = q^2/32\pi^2\varepsilon_0 R^4$	0.5
Conservation of energy instead of adiabatic process	0.5
$W_{surf} = 8\pi R^2\sigma$	0.3
$W_{electr} = q^2/8\pi\varepsilon_0 R$	0.5
$W_{gas} = (5/2)\nu RT = (5/2)PV$	0.4
Formula for the sphere volume	0.2
Correct answer	0.5
Total	**4.0**

Problem 1C (3.0 points)

The signal can be suppressed by the interference of waves. The waves coming from the sources S_1 and S_2 arrive at the receivers with the same phase, so the wave from the third source must arrive at receivers with the opposite phase than those from the sources S_1 and S_2. To assure this, the distance from the third source to the receivers must differ by the amount of $\frac{\lambda}{2} + m\lambda$, where $m = 0, \pm 1, \pm 2 \ldots$. To find the points that satisfy those conditions, it is necessary to plot two families of circles, one with the radii $R_1 + \frac{\lambda}{2} + m\lambda$ and with the center at the point A_1, and the other with the radii $R_2 + \frac{\lambda}{2} + m\lambda$ and with the center at the point A_2. The intersection points of those two families represent the points where the third source should be placed, they are marked by circles. The amplitude of waves from the third source must be 2 times greater than the amplitude of waves coming from sources S_1 and S_2, hence the wave intensity of the third source should be 4 times higher, i.e. $4I_0$.



Content	Points
Interference to suppress waves	0,5
Conditions for minima are used (waves out-of-phase);	0,2
Difference in distance must be integer number of half of the wavelength	0,3
Two families of circles are drawn	2×0,5
Intersection points are used	0,4
All 6 points are correctly stated in the highlighted area	6×0,1
Total	**3,0**

Problem 2. Fantastic trip through the Universe (10.0 points)

1. Planets with strange shapes (4.0 points)

1.1 **[0.7 points]** The easiest approach to the solution of the problem is the analogy between Coulomb force and Newton's law of gravitation:

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2} \text{ and } F = G\frac{m_1 m_2}{r_{12}^2}. \tag{1}$$

Further, it is a well known result from the Gauss theorem that the electric field strength of an infinite charged plane, with the surface density σ is found as

$$E = \frac{\sigma}{2\varepsilon_0}. \tag{2}$$

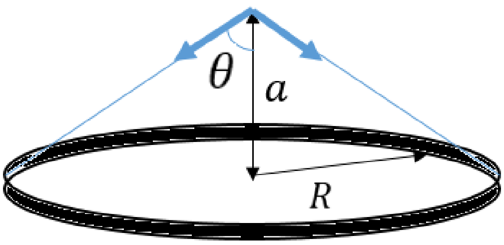
By analogy to the charged plane, the result for the planet is similarly obtained as:

$$g_1 = \frac{\rho_1 h}{2\cdot\left(1/_{4\pi G}\right)} = 2\pi G\rho_1 h, \tag{3}$$

$$h = \frac{g_1}{2\pi G\rho_1} = 78.0km. \tag{4}$$

***

The same result is easily achieved by cutting an infinite plane into thin rings and further integrating:



The attracting force of the ring of mass M, and of radius R at the distance a is written as:

$$F = G\frac{Mm}{R^2 + a^2}cos\theta = G\frac{Mm}{a^2}cos^3\theta.$$

We divide the plane of the height h into thin rings of thickness dr. Then the force of gravitycaused by the ring of the radius r is equal to

$$dF = G\frac{dMm}{a^2}cos^3\theta = G\frac{(\rho_1 h 2\pi r dr)m}{a^2}cos^3\theta.$$

It follows from the trigonometric considerations that $r = a\cdot tan\theta$, $dr = \frac{a}{cos^2\theta}d\theta$.

Substituting the above expression and integrating we find the total force F acting on the body of mass m:

$$F = 2\pi G\rho_1 hm\int_0^{\frac{\pi}{2}} sin\theta d\theta = 2\pi G\rho_1 hm.$$

This is identical to the answer obtained from the analogy with the electrostatic field.

1.2 **[0.5 points]** For an observer that is located close to the infinite plane, the solid angle is obviously equal to

$$\Omega_1 = \frac{4\pi}{2} = 2\pi, \tag{5}$$

and from the problem formulation we get

$$\alpha = \frac{g_1}{2\pi} \text{ or } \alpha = G\rho_1 h = 1.56\times 10^{-2}m/s^2. \tag{6}$$

1.3 **[0.7 points]** We divide the pyramid into thin layers of thickness Δh parallel to the base. All of these layers are visible from the top of the pyramid with the same solid angle Ω_2, which is equal to one sixth of the full solid angle (as if the observer was located inside the cubeat its center!):

$$\Omega_2 = \frac{1}{6}4\pi = \frac{2}{3}\pi. \tag{7}$$

The free fall acceleration of the single layer is found as

$$dg_2 = \frac{dF}{m} = \alpha\Omega_2 = \frac{2}{3}\pi G\rho_2\Delta h, \tag{8}$$

or after the summation over all the layers of the pyramid

$$g_2 = \frac{1}{3}\pi G\rho_2 a = 3.14\ m/s^2. \tag{9}$$

1.4 **[2.0 points]** Let the interaction energy between the spacecraft and the pyramidal planet at the time of take-off from its top be equal to U_1, and its speed be v_1. It follows from the law of the energy conservation for the parabolic velocity that:

$$\frac{mv_1^2}{2} - U_1 = 0. \tag{10}$$

Similarly, the law of the energy conservation for a spacecraft to start from the cubic planet is written as

$$\frac{mv_2^2}{2} - U_2 = 0, \tag{11}$$

where U_2 stands for the corresponding interaction energy with the cubic planet.
Let us show that there is a simple relationship between U_1 and U_2. To prove so, we consider the position of the spacecraft at the center of the cubic planet. On the one hand the position at the center of the cube is equivalent to finding the spacecraft at the tops of the six pyramids. Taking into account the change in the density of matter, the potential energy of the spacecraft at the center of the cube is obtained as

$$U_c = 6U_1\frac{\rho_3}{\rho_2}. \tag{12}$$

On the other hand the position of the spacecraft at the center of the cube is equivalent to being at the tops of the eight identical adjacent cubes with the side $\frac{a}{2}$. In general, the potential energy of the spacecraft in the field of the cubic planet is proportional to the square of its size since

$$U = G\sum\frac{m\rho_3\Delta V_i}{r_i} \sim Gm\rho_3 a^2. \tag{13}$$

Thus, for the cube of the half size, the interaction energy is 4 times less, which means that the potential energy of the spacecraft at the center of the cube is found as

$$U_c = 8\frac{U_2}{4} = 2U_2. \tag{14}$$

Equating the expressions (12) and (14) yield

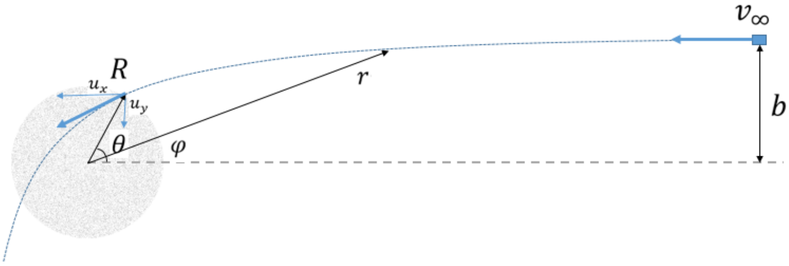
$$U_2 = 3U_1\frac{\rho_3}{\rho_2}. \tag{15}$$

Solving together equations (10), (11) and (15), we finally obtain

$$v_2 = \sqrt{\frac{3\rho_3}{\rho_2}}v_1 = 6{,}30\ km/s. \tag{16}$$

2. Dusty cloud (6.0 points)

2.1 **[2.5 points]** For this problem, we use a mixture of the polar and Cartesian coordinate systems as shown below.



The conservation of energy is written as:

$$\frac{mv_\infty^2}{2} = \frac{mu_x^2}{2} + \frac{mu_y^2}{2} - G\frac{Mm}{R}, \tag{17}$$

where $M = \frac{4}{3}\pi R^3\rho_4$ denotes the total mass of the cloud.

Change in the spacecraft momentum projection on the x-axis of the Cartesian coordinate system is given by

$$mu_x - mv_\infty = \int G\frac{Mm}{r^2}cos\varphi dt = \int G\frac{Mm}{r^2\dot{\varphi}}cos\varphi d\varphi. \quad (18)$$

The law of the angular momentum conservation for a system with the central force is written as

$$r^2\dot{\varphi} = v_\infty b. \quad (19)$$

Thus,

$$mu_x - mv_\infty = G\frac{Mm}{v_\infty b}\int_0^\theta cos\varphi d\varphi = G\frac{Mm}{v_\infty b}sin\theta. \quad (20)$$

Similarly for the y-axis projection:

$$mu_y - m\cdot 0 = G\frac{Mm}{v_\infty b}\int_0^\theta sin\varphi d\varphi = G\frac{Mm}{v_\infty b}(1-cos\theta). \quad (21)$$

To simplify further analysis the following dimensionless quantity is introduced

$$z = \frac{GM}{v_\infty^2 b}, \quad (22)$$

and then

$$u_x = (1+zsin\theta)v_\infty, \quad (23)$$

$$u_y = z(1-cos\theta)v_\infty. \quad (24)$$

Substitution of (23) and (24) into (17) gives rise to

$$1 = (1+zsin\theta)^2 + z^2(1-cos\theta)^2 - 2z\frac{b}{R}. \quad (25)$$

Solving this equation for θ, we find

$$\theta = \arcsin\frac{\frac{b}{R}-\frac{GM}{v_\infty^2 b}}{\sqrt{1+\left(\frac{GM}{v_\infty^2 b}\right)^2}} + \arcsin\frac{\frac{GM}{v_\infty^2 b}}{\sqrt{1+\left(\frac{GM}{v_\infty^2 b}\right)^2}}, \quad (26)$$

or

$$\theta = 2\arctan\frac{1-\sqrt{1+2\frac{GM}{v_\infty^2 b}\frac{b}{R}-\frac{b^2}{R^2}}}{\frac{b}{R}-2\frac{GM}{v_\infty^2 b}} = 0.789\, rad = 45.2°. \quad (27)$$

It should be noted that the angle θ, just as the total angle of deflection of the trajectory when moving through the dust cloud, can be obtained by integrating the equation obtained from the combination of the laws of conservation of energy and angular momentum written in the polar coordinates. Expressions are not presented here because the resulting integrals are quite cumbersome.

2.2 **[2.0 points]** To begin with we find the dependence of the potential energy of interaction between the cloud and the spacecraft at distances $r < R$ from its center. It is known that a spherical cloud layers, lying at a distance greater than r, does not affect the spacecraft, so the total active force is derived as

$$F(r) = -G\frac{\rho_4\cdot\frac{4}{3}\pi r^3}{r^2}m = -\frac{4}{3}\pi G\rho_4 mr, \quad (28)$$

and the corresponding potential energy is found in the form

$$U(r) = -\int F(r)dr = \frac{2}{3}\pi G\rho_4 mr^2 + C = G\frac{Mm}{2R^3}r^2 + C. \quad (29)$$

To determine the integration constant C, we recall that the potential energy must be a continuous at the point $r = R$, such that

$$G\frac{Mm}{2R^3}R^2 + C = -G\frac{Mm}{R}, \quad (30)$$

or finally for $r < R$

$$U(r) = \frac{GMm}{2R^3}r^2 - \frac{3GMm}{2R}. \quad (31)$$

At the time moment when the distance to the cloud center reaches its minimum value, the radial velocity turns zero. Then, from the laws of conservation of energy and angular momentum we have

$$\frac{mv_\infty^2}{2} = \frac{mv_0^2}{2} + \frac{GMm}{2R^3}r_{min}^2 - \frac{3GMm}{2R}, \quad (32)$$

$$v_0 r_{min} = v_\infty b, \quad (33)$$

which results in the following equation

$$1 = \frac{b^2}{r_{min}^2} + z\frac{r_{min}^2 b}{R^3} - 3z\frac{b}{R}, \tag{34}$$

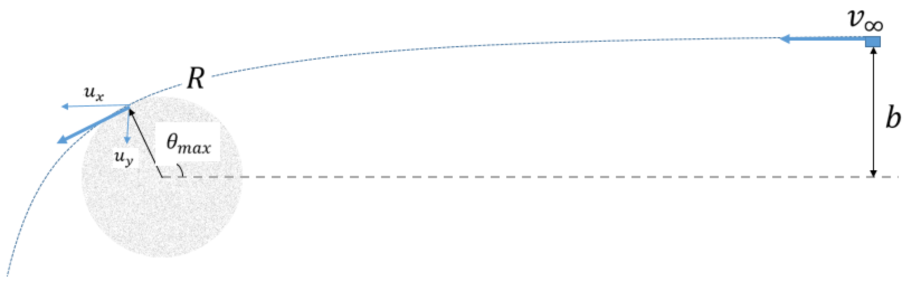
with the solution

$$r_{min} = \sqrt{\frac{(\frac{3zb}{R}+1) \pm \sqrt{(\frac{3zb}{R}+1)^2 - \frac{4zb^3}{R^3}}}{2\frac{zb}{R^3}}}. \tag{35}$$

The meaningful root is only the smallest one because there must be $r_{min} = 0$ at $b = 0$. Thus, we finally obtain

$$r_{min} = R\sqrt{\frac{\left(\frac{3GM}{v_\infty^2 R}+1\right) - \sqrt{(\frac{3GM}{v_\infty^2 R}+1)^2 - \frac{4b^2GM}{R^3 v_\infty^2}}}{\frac{2GM}{v_\infty^2 R}}} = 4.97 \times 10^9 m. \tag{36}$$

2.3 **[1.0 points]** Minimum velocity $v_{\infty,min}$, that allows the spacecraft to avoid a collision, corresponds to a situation when the spacecraft just touches the cloud as shown below.



In this case, the radial component of the velocity again turns zero, and the laws of conservation of energy and angular momentum can be written as:

$$\frac{mv_{\infty,min}^2}{2} = \frac{m0^2}{2} + \frac{mu_\tau^2}{2} - G\frac{Mm}{R}, \tag{37}$$

$$u_\tau R = v_\infty b, \tag{38}$$

which yields

$$v_{\infty,min} = \sqrt{\frac{2GM}{R\left(\frac{b^2}{R^2}-1\right)}} = 252\ km/s. \tag{39}$$

2.4 **[0.6 points]** Assume that the cloud is pulled apart at distances by small layers of thickness Δr so that the cloud always remains symmetrical. To remove a single thin layer at the moment when the cloud has a radius r, it is necessary to do the work

$$\Delta A = G\frac{(\rho_4\frac{4}{3}\pi r^3)(\rho_4 4\pi r^2 \Delta r)}{r} = \frac{16}{3}\pi^2 G\rho_4^2 r^4 \Delta r, \tag{40}$$

and to pull apart the whole cloud the following work must be done

$$A = \frac{16}{3}\pi^2 G\rho_4^2 \int_0^R r^4 \Delta r = \frac{16}{15}\pi^2 G\rho_4^2 R^5 = 1.33 \times 10^{45} J. \tag{41}$$

	Content	points	
1.1	The analogy between the Coulom law and the gravitation law of Newton (1): $F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}$ and $F = G\frac{m_1 m_2}{r_{12}^2}$.	0.2	0.7
	Formula (2) $E = \frac{\sigma}{2\varepsilon_0}$	0.2	
	Formula (4) $h = \frac{g_1}{2\pi G\rho_1}$	0.2	
	Numerical value of $h = 78.0km$	0.1	
1.2	Formula (5) $\Omega_1 = 2\pi$	0.2	0.5
	Formula (6) $\alpha = \frac{g_1}{2\pi}$ or $\alpha = G\rho_1 h$	0.2	
	Numerical value of $\alpha = 1.56 \times 10^{-2} m/s^2$	0.1	

1.3	Formula (7) $\Omega_2 = \frac{2}{3}\pi$	0.2	0.7
	Formula (8) $dg_2 = \frac{dF}{m} = \alpha\Omega_2 = \frac{2}{3}\pi G\rho_2\Delta h$	0.2	
	Formula (9) $g_2 = \frac{1}{3}\pi G\rho_2 a$	0.2	
	Numerical value of $g_2 = 3.14 m/s^2$	0.1	
1.4	Formulas /(10) and (11) $\frac{mv_1^2}{2} - U_1 = 0 \quad \frac{mv_2^2}{2} - U_2 = 0$	0.2	2.0
	Formula (12) $U_c = 6U_1\frac{\rho_3}{\rho_2}$	0.4	
	Formula (13) $U = G\sum\frac{m\rho_3\Delta V_i}{r_i} \sim Gm\rho_3 a^2$	0.4	
	Formula (14) $U_c = 8\frac{U_2}{4} = 2U_2$	0.2	
	Formula (15) $U_2 = 3U_1\frac{\rho_3}{\rho_2}$	0.4	
	Formula (16) $v_2 = \sqrt{\frac{3\rho_3}{\rho_2}}v_1$	0.3	
	Numerical value of $v_2 = 6{,}30 km/s$	0.1	
2.1	Formula (17) $\frac{mv_\infty^2}{2} = \frac{mu_x^2}{2} + \frac{mu_y^2}{2} - G\frac{Mm}{R}$	0.2	2.5
	Formula (18) $mu_x - mv_\infty = \int G\frac{Mm}{r^2}cos\varphi dt = \int G\frac{Mm}{r^2\dot{\varphi}}cos\varphi d\varphi$	0.4	
	Formula (19) $r^2\dot{\varphi} = v_\infty b$	0.2	
	Formula (20) $mu_x - mv_\infty = G\frac{Mm}{v_\infty b}sin\theta$	0.4	
	Formula (21) $mu_y = G\frac{Mm}{v_\infty b}(1 - cos\theta)$	0.4	
	Formula (23) or analogous $u_x = (1 + zsin\theta)v_\infty$	0.3	
	Formula (24) or analogous $u_y = z(1 - cos\theta)v_\infty$	0.3	
	Formula (26) or formula (27) $\theta = \arcsin\frac{\frac{b}{R} - \frac{GM}{v_\infty^2 b}}{\sqrt{1+\left(\frac{GM}{v_\infty^2 b}\right)^2}} + \arcsin\frac{\frac{GM}{v_\infty^2 b}}{\sqrt{1+\left(\frac{GM}{v_\infty^2 b}\right)^2}}$ or $\theta = 2\arctan\frac{1-\sqrt{1+2\frac{GM}{v_\infty^2 b}\frac{b}{R}-\frac{b^2}{R^2}}}{\frac{b}{R}-2\frac{GM}{v_\infty^2 b}}$	0.2	
	Numerical value of $\theta = 0{,}789\, rad = 45{,}2°$	0.1	
2.2	Formula (28) $F(r) = -G\frac{\rho_4\cdot\frac{4}{3}\pi r^3}{r^2}m = -\frac{4}{3}\pi G\rho_4 mr$	0.4	2.0
	Formula (29) $U(r) = \frac{2}{3}\pi G\rho_4 mr^2 + C = G\frac{Mm}{2R^3}r^2 + C$	0.3	
	Formula (30) $G\frac{Mm}{2R^3}R^2 + C = -G\frac{Mm}{R}$	0.4	
	Formula (32) $\frac{mv_\infty^2}{2} = \frac{mv_0^2}{2} + \frac{GMm}{2R^3}r_{min}^2 - \frac{3GMm}{2R}$,	0.2	
	Formula (33) $v_0 r_{min} = v_\infty b$	0.2	
	Formula (35) $r_{min} = \sqrt{\frac{(\frac{3zb}{R}+1)\pm\sqrt{(\frac{3zb}{R}+1)^2-\frac{4zb^3}{R^3}}}{2\frac{zb}{R^3}}}$	0.2	
	Correct root is chosen, formula (36)	0.2	
	Numerical value of $r_{min} = 4.97 \times 10^9 m$	0.1	
2.3	Formula (37) $\frac{mv_{\infty,min}^2}{2} = \frac{m0^2}{2} + \frac{mu_\tau^2}{2} - G\frac{Mm}{R}$	0.4	1.0

	Formula (38) $u_\tau R = v_\infty b$	0.3	
	Formula (39) $v_{\infty,min} = \sqrt{\frac{2GM}{R\left(\frac{b^2}{R^2}-1\right)}}$	0.2	
	Numerical value of $v_{\infty,min} = 252 km/s$	0.1	
2.4	Formula (40) $\Delta A = \frac{16}{3}\pi^2 G\rho_4^2 r^4 \Delta r$	0.3	**0.6**
	Formula (41) $A = \frac{16}{15}\pi^2 G\rho_4^2 R^5$	0.2	
	Numerical value of $A = 1.33 \times 10^{45} J$	0.1	
Total			**10.0**

Problem 3. Resistance of a prism (10.0 points)

1. Mathematical introduction (3.0 points)

1.1 **[0.2 points]** From the course of school mathematics it is known that geometrical progression terms are explicitly expressed as

$$x_k = A\lambda^k . \tag{1}$$

1.2 **[0.4 points]** Let us express λ^k recurrently in terms of λ^{k-1}:

$$\lambda^k = \lambda^{k-1} \cdot \lambda$$

and transform it as follows

$$\begin{gathered}\lambda^k = \left(p_k + q_k\sqrt{3}\right) = \left(p_{k-1} + q_{k-1}\sqrt{3}\right)\cdot\left(2+\sqrt{3}\right) = 2p_{k-1} + p_{k-1}\sqrt{3} + 2q_{k-1}\sqrt{3} + 3q_{k-1} = \\ = \left(2p_{k-1} + 3q_{k-1}\right) + \left(p_{k-1} + 2q_{k-1}\right)\sqrt{3}.\end{gathered} \tag{2}$$

This equality implies the required recurrence relations in the form

$$\begin{aligned} p_k &= 2p_{k-1} + 3q_{k-1} \\ q_k &= p_{k-1} + 2q_{k-1}. \end{aligned} \tag{3}$$

Inverse relations are obtained analogously

$$\begin{gathered}\lambda^{k-1} = p_{k-1} + q_{k-1} = \lambda^k \cdot \lambda^{-1} = \left(p_k + q_k\sqrt{3}\right)\cdot\left(2-\sqrt{3}\right) = \\ = \left(2p_k - 3q_k\right) + \left(2q_k - p_k\right)\sqrt{3},\end{gathered} \tag{4}$$

and, thus,

$$\begin{aligned} p_{k-1} &= 2p_k - 3q_k, \\ q_{k-1} &= 2q_k - p_k. \end{aligned} \tag{5}$$

1.3 **[0.7 points]** Calculation of the coefficients is much easier to carry out in series, given that $p_0 = 1, \quad q_0 = 0$. The results are shown in Table 1.

Table 1.

k	p_k	q_k
0	1	0
1	2	1
2	7	4
3	26	15
4	97	56
5	362	209

1.4 **[0.2 points]** Note that

$$\lambda^{-1} = \frac{1}{2+\sqrt{3}} = 2-\sqrt{3}, \tag{6}$$

therefore,

$$\lambda^{-k} = \left(2-\sqrt{3}\right)^k = p_k - q_k\sqrt{3}\,. \tag{7}$$

1.5 **[1.0 points]** Using the hint, we substitute $x_k = C\lambda^k$ into the recurrence relation and obtain the equation to determine λ in the form

$$\lambda^{k+1} = 4\lambda^k - \lambda^{k-1}\,. \tag{8}$$

After reduction the following quadratic equation is derived

$$\lambda^2 - 4\lambda + 1 = 0\,, \tag{9}$$

which has two solutions

$$\lambda_{1,2} = 2 \pm \sqrt{3}\,. \tag{10}$$

Consequently, the general solution to the recurrence relation (3) is explicitly written by

$$x_k = C_1\lambda_1^k + C_2\lambda_2^k\,, \tag{11}$$

where C_1, C_2 are arbitrary constants that are determined by the boundary conditions:

$$\begin{aligned} x_0 = A \quad &\Rightarrow \quad C_1 + C_2 = A \\ x_0 = B \quad &\Rightarrow \quad C_1\lambda_1^N + C_2\lambda_2^N = B. \end{aligned} \tag{12}$$

Solving the linear set of equation yields

$$\begin{cases} C_1 + C_2 = A, \\ C_1\lambda_1^N + C_2\lambda_2^N = B \end{cases} \Rightarrow \begin{cases} C_1 = \dfrac{B - A\lambda_2^N}{\lambda_1^N - \lambda_2^N}, \\ C_2 = \dfrac{A\lambda_1^N - B}{\lambda_1^N - \lambda_2^N}. \end{cases} \tag{13}$$

Substituting this solution into (11), it is possible to rewrite it in the following symmetrical form

$$\begin{aligned} x_k &= C_1\lambda_1^k + C_2\lambda_2^k = \frac{B - A\lambda_2^N}{\lambda_1^N - \lambda_2^N}\lambda_1^k + \frac{A\lambda_1^N - B}{\lambda_1^N - \lambda_2^N}\lambda_2^k = \\ &= \frac{A\lambda_1^N\lambda_2^k - B\lambda_2^k + B\lambda_1^k - A\lambda_2^N\lambda_1^k}{\lambda_1^N - \lambda_2^N} = \frac{A\left(\lambda_1^{N-k} - \lambda_2^{N-k}\right) + B\left(\lambda_1^k - \lambda_2^k\right)}{\lambda_1^N - \lambda_2^N}. \end{aligned} \tag{14}$$

The derivation of the last relation takes into account that according to the Vieta theorem $\lambda_2 = \lambda_1^{-1}$.

1.6 **[0.5 points]** In view of the above formulas for the $\lambda_{1,2}^k$, we find that

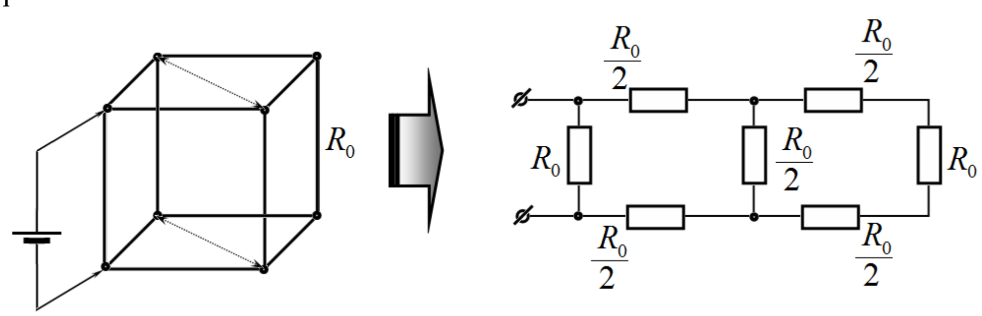
$$\lambda_1^k - \lambda_2^k = \lambda_1^k - \lambda_1^{-k} = \left(p_k + q_k\sqrt{3}\right) - \left(p_k - q_k\sqrt{3}\right) = 2q_k\sqrt{3}\,, \tag{15}$$

and, finally,

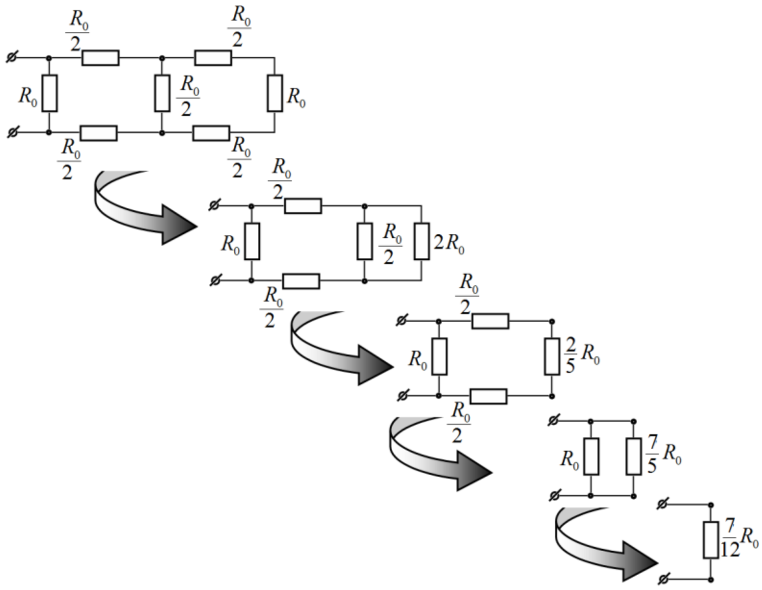
$$x_k = \frac{A\left(\lambda_1^{N-k} - \lambda_2^{N-k}\right) + B\left(\lambda_1^k - \lambda_2^k\right)}{\lambda_1^N - \lambda_2^N} = \frac{Aq_{N-k} + Bq_k}{q_N}\,. \tag{16}$$

2. Wire frame in the shape of a prism (7.0 points)

2.1 **[0.8 points]** If the vertices of the cube with the same potentials are connected, then, the following equivalent circuits are obtained



and easily calculated using the standard method as



Ultimately, the cube resistance for the given connection is found as

$$R = \frac{7}{12} R_0 . \tag{17}$$

2.2 **[0.2 points]** Visual symmetry of the circuit and of the initial conditions provides obvious relations

$$y_k = -x_k , \tag{18}$$

$$x_{N-k} = x_k . \tag{19}$$

2.3 **[1.0 points]** The algebraic sum of the currents entering a node is equal to zero, thus, using Ohm's law, the following equation is obtained for the node x_k

$$\frac{x_{k-1} - x_k}{R_0} + \frac{x_{k+1} - x_k}{R_0} + \frac{y_k - x_k}{R_0} = 0 . \tag{20}$$

Since $y_k = -x_k$, the recurrence relation holds

$$x_{k+1} - 4x_k + x_{k-1} = 0 . \tag{21}$$

2.4 **[0.2 points]** For an unambiguous determination of all values x_k, we need to explicitly specify two boundary conditions. One of those is the initial potential defined as

$$x_0 = \varphi_0 , \tag{22}$$

whereas the other follows from the symmetry condition (19), which is valid for any k, and, in particular, for $k = 0$ (despite the fact that the node with the number N does not exist in the circuit!)

$$x_N = x_0 . \tag{23}$$

2.5 **[0.2 points]** The recurrence relation (21) has been considered in the Mathematical introduction. Therefore, you can use the obtained solution (16) by setting:

$$x_k = \frac{A q_{N-k} + B q_k}{q_N} = \varphi_0 \frac{q_{N-k} + q_k}{q_N} . \tag{24}$$

2.6 **[0.4 points]** The current in the source circuit is found as the sum of the currents flowing from the node x_0:

$$I = \frac{x_0 - x_1}{R_0} + \frac{x_0 - x_{N-1}}{R_0} + \frac{x_0 - y_0}{R_0} = \frac{4x_0 - 2x_1}{R_0} . \tag{25}$$

Here it has been taken into account that $y_0 = -x_0$, $x_{N-1} = x_1$. Substituting the values for x_0, x_1, results in

$$I_0=\frac{4x_0-2x_1}{R_0}=\frac{2}{R_0}\left(2\phi_0-\phi_0\frac{q_{N-1}+q_1}{q_N}\right)=\frac{2\phi_0}{R_0}\left(2-\frac{q_{N-1}+1}{q_N}\right)=$$
$$=\frac{2\phi_0}{R_0}\frac{2q_N-q_{N-1}-1}{q_N}=\frac{2\phi_0}{R_0}\frac{2q_N-(2q_N-p_N)-1}{q_N}=\frac{2\phi_0}{R_0}\frac{p_N-1}{q_N}. \qquad (26)$$

At the last step the relation (5) has been used, $q_{N-1}=2q_N-p_N$.

2.7 **[0.2 points]** By formulation, the input voltage for the given circuit is

$$U_0=2\varphi_0, \qquad (27)$$

concequently, the resistance is found in the following elegant form

$$R_N=\frac{U_0}{I_0}=R_0\frac{q_N}{p_N-1}. \qquad (28)$$

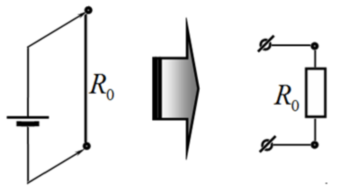
2.8 **[1.0 points]** Calculations are easily performed using numerical values in Table 1.

Table 2. Resistances of prisms.

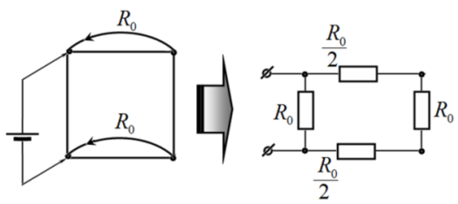
N	p_N	q_N	R_N
1	2	1	R_0
2	7	4	$R_0\frac{4}{7-1}=\frac{2}{3}R_0$
3	26	15	$R_0\frac{15}{26-1}=\frac{3}{4}R_0$
4	97	56	$R_0\frac{56}{97-1}=\frac{7}{12}R_0$
5	362	209	$R_0\frac{209}{362-1}=\frac{11}{19}R_0$

Note that for a cubic prism with $N=4$ the resistance coincides with that previously found in 2.1.

2.9 **[0.5 points]** For $N=1$ the circuit is obvious:



but for $N=2$ the prism should be additionally closed as:



In both cases the corresponding resistances coincide with the values shown in Table 2.

2.10 **[1.0 points]** The limit of the formula (28) can be found in various ways, for example, expressing

$$p_N=\frac{1}{2}\left(\lambda^N-\lambda^{-N}\right),\quad q_N=\frac{1}{2\sqrt{3}}\left(\lambda^N+\lambda^{-N}\right), \qquad (29)$$

where $\lambda=2+\sqrt{3}>1$.

Then,

$$R_\infty = \lim_{N\to\infty} R_N = R_0 \lim_{N\to\infty} \frac{q_N}{p_N - 1} = R_0 \lim_{N\to\infty} \frac{\frac{1}{2\sqrt{3}}\left(\lambda^N + \lambda^{-N}\right)}{\frac{1}{2}\left(\lambda^N - \lambda^{-N}\right) - 1} = \frac{R_0}{\sqrt{3}}. \tag{30}$$

2.11 **[1.5 points]** Evaluation gives ries to

$$\frac{R_\infty}{R_0} = \frac{1}{\sqrt{3}} \approx 0.577. \tag{31}$$

Then, we carry out the calculation of the relative error of the approximate expression for different values of N listed in Table 2.

Table 3.

N	R_N	$\frac{R_N}{R_0}$	$\varepsilon = \frac{R_\infty - R_N}{R_N}$
1	R_0	1.000	-0.423
2	$\frac{2}{3}R_0$	0.667	-0.134
3	$\frac{3}{4}R_0$	0.750	-0.038
4	$\frac{7}{12}R_0$	0.583	-0.010
5	$\frac{11}{19}R_0$	0.579	<-0.004

It is seen that already at $N = 4$ the relative error is 1%. Consequently, in this problem four is equal to infinity!

$$\infty \approx 4\,. \tag{32}$$

	Content	points	
1.1	Formula (1) $x_k = A\lambda^k$	0.2	**0.2**
1.2	Formula (3) $p_k = 2p_{k-1} + 3q_{k-1}$, $q_k = p_{k-1} + 2q_{k-1}$	0.2	**0.4**
	Formulas (5) $p_{k-1} = 2p_k - 3q_k$, $q_{k-1} = 2q_k - p_k$	0.2	
1.3	Correct initial values $p_0 = 1, \quad q_0 = 0$	0.2	**0.7**
	Correct values in Table 1. (see Table 1 below)	0.5	
1.4	Formula (7) $\lambda^{-k} = p_k - q_k\sqrt{3}$	0.2	**0.2**

Table 1.

k	p_k	q_k
0	1	0
1	2	1
2	7	4
3	26	15
4	97	56
5	362	209

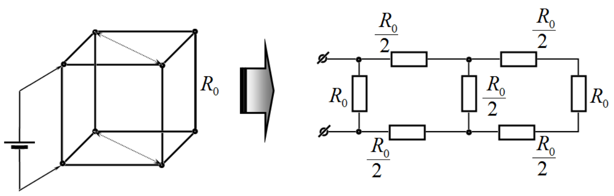
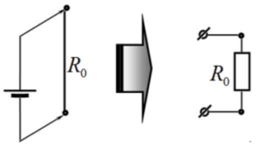
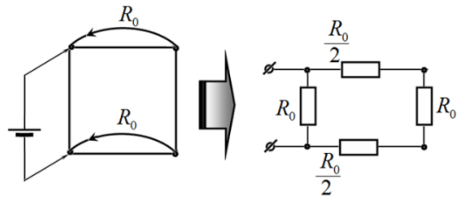
1.5	Formula (10) $\lambda_{1,2} = 2 \pm \sqrt{3}$	0.2	1.0
	Formula (11) $x_k = C_1\lambda_1^k + C_2\lambda_2^k$	0.2	
	Formula (12) $C_1 + C_2 = A$; $C_1\lambda_1^N + C_2\lambda_2^N = B$	0.2	
	Solution (13) $\begin{cases} C_1 = \dfrac{B - A\lambda_2^N}{\lambda_1^N - \lambda_2^N} \\ C_2 = \dfrac{A\lambda_1^N - B}{\lambda_1^N - \lambda_2^N} \end{cases}$	0.2	
	Formula (14) $x_k = \dfrac{A\left(\lambda_1^{N-k} - \lambda_2^{N-k}\right) + B\left(\lambda_1^k - \lambda_2^k\right)}{\lambda_1^N - \lambda_2^N}$	0.2	
1.6	Formula (16) $x_k = \dfrac{Aq_{N-k} + Bq_k}{q_N}$	0.5	0.5
2.1	Equivalent circuit 	0.3	0.8
	Formula (17) $R = \dfrac{7}{12}R_0$	0.5	
2.2	Formula (18) $y_k = -x_k$	0.1	0.2
	Formula (19) $x_{N-k} = x_k$	0.1	
2.3	Formula (20) $\dfrac{x_{k-1} - x_k}{R_0} + \dfrac{x_{k+1} - x_k}{R_0} + \dfrac{y_k - x_k}{R_0} = 0$	0.5	1.0
	Formula (21) $x_{k+1} - 4x_k + x_{k-1} = 0$	0.5	
2.4	Formula (22) $x_0 = \varphi_0$	0.1	0.2
	Formula (23) $x_N = x_0$	0.1	
2.5	Formula (24) $x_k = \dfrac{Aq_{N-k} + Bq_k}{q_N} = \varphi_0 \dfrac{q_{N-k} + q_k}{q_N}$	0.2	0.2
2.6	Formula (25) $I = \dfrac{4x_0 - 2x_1}{R_0}$	0.2	0.4
	Formula (26) $I_0 = \dfrac{2\varphi_0}{R_0}\dfrac{p_N - 1}{q_N}$	0.2	
2.7	Formula (27) $U_0 = 2\varphi_0$	0.1	0.2
	Formula (28) $R_N = \dfrac{U_0}{I_0} = R_0 \dfrac{q_N}{p_N - 1}$	0.1	
2.8	Correct values in Table 2.	1.0	1.0

Table 2. Resistances of prisms.

N	p_N	q_N	R_N
1	2	1	R_0

2	7	4	$R_0 \dfrac{4}{7-1} = \dfrac{2}{3} R_0$
3	26	15	$R_0 \dfrac{15}{26-1} = \dfrac{3}{4} R_0$
4	97	56	$R_0 \dfrac{56}{97-1} = \dfrac{7}{12} R_0$
5	362	209	$R_0 \dfrac{209}{362-1} = \dfrac{11}{19} R_0$

<table>
<tr><td rowspan="2">2.9</td><td>Equivalent circuit for $N=1$
</td><td>0.1</td><td rowspan="2">0.5</td></tr>
<tr><td>Equivalent circuit for $N=2$
</td><td>0.4</td></tr>
<tr><td rowspan="2">2.10</td><td>Formula (29) $p_N = \frac{1}{2}\left(\lambda^N - \lambda^{-N}\right), \quad q_N = \frac{1}{2\sqrt{3}}\left(\lambda^N + \lambda^{-N}\right)$</td><td>0.5</td><td rowspan="2">1.0</td></tr>
<tr><td>Formula (30) $R_\infty = \frac{R_0}{\sqrt{3}}$</td><td>0.5</td></tr>
<tr><td rowspan="3">2.11</td><td>Formula (31) $\frac{R_\infty}{R_0} \approx 0.577$</td><td>0.2</td><td rowspan="3">1.5</td></tr>
<tr><td>Correct values in Table 3.
Table 3.
<table>
<tr><td>N</td><td>R_N</td><td>$\frac{R_N}{R_0}$</td><td>$\varepsilon = \frac{R_\infty - R_N}{R_N}$</td></tr>
<tr><td>1</td><td>R_0</td><td>1.000</td><td>-0.423</td></tr>
<tr><td>2</td><td>$\frac{2}{3} R_0$</td><td>0.667</td><td>-0.134</td></tr>
<tr><td>3</td><td>$\frac{3}{4} R_0$</td><td>0.750</td><td>-0.038</td></tr>
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</table></td><td>1.0</td></tr>
<tr><td>Formula (32) $\infty \approx 4$</td><td>0.3</td></tr>
<tr><td>**Total**</td><td></td><td></td><td>**10.0**</td></tr>
</table>

EXPERIMENTAL COMPETITION

15 January, 2017

Please read the instructions first:

1. The Experimental competition consists of one problem. This part of the competition lasts 3 hours.
2. Please only use the pen that is provided to you.
3. You can use your own non-programmable calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet and additional papers***. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*.** Write only inside the bordered area.
6. Fill the boxes at the top of each sheet of paper with your country (***Country*),** your student code (***Student Code***), the question number (***Question Number*)**, the progressive number of each sheet **(*Page Number***), and the total number of ***Writing sheets*** (***Total Number of Pages***). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
7. At the end of the exam, arrange all sheets for each problem in the following order:
    - Used ***Writing sheets*** in order;
    - The sheets you do not wish to be evaluated
    - Unused sheets and the printed question.

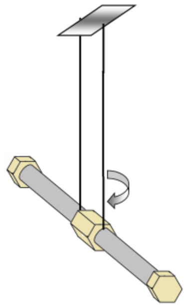
   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper or equipment out of the room

Torsion pendulum (15.0 points)

Instruments and equipment: two bolts and coupling nut, tripod with two holders or ring holders, thread, 2 wooden ruler of the length of 40 cm each, stopwatch, weight of 100 g, clay.

The torsion pendulum is two bolts rigidly joined by the nut and suspended on two threads,. Make sure that the pendulum makes only torsional oscillations in a horizontal plane. The free fall acceleration due to gravity is $g = 9.81\ m/s^2$.

Part 1. Free small oscillations (5.0 points)



1.1 Measure dependence of the period of small torsional oscillations of the pendulum on the length of threads in the range from 10 to 50 cm.
1.2 Derive the theoretical formula for the period of small torsional oscillations.
1.3 Prove graphically that the derived formula correctly describes the experimental data.
1.4 Calculate the radius of gyration of the pendulum and evaluate the experimental error.

Hint: The moment of inertia of a rigid body about a certain axis may be represented in the form $I = mR^2$, where m is the body mass and R stands for the radius of gyration.

In all following parts error analysis is not required!

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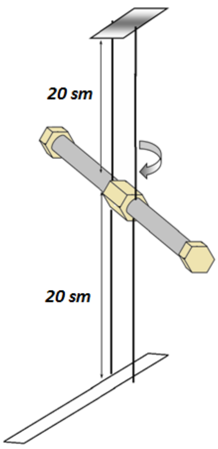
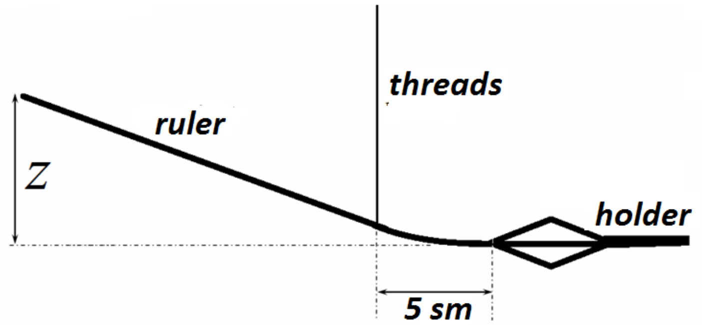
Part 2. Small oscillations with additional tension (5.0 points)



Tie another two threads from below the bolts. They must also be parallel and located at the same distance as the two upper threads. At the bottom tie them to a wooden ruler of length 40 cm. To do this, fix the ruler tight in the tripod holder. You can also place the ruler under the tripod platform. The threads must be tied to the ruler at a distance of 4-5 cm from the holder. The lengths of the threads at the top and bottom of the bolts must be the same and equal to about 20 cm. By raising or lowering the upper tripod holder, you can change the tension of the threads. Tension of threads is measured by the magnitude z of the ruler bending, as shown in the figure below.

2.1 Measure the dependence of the period of small torsional oscillations on the threads tension, i.e. on the ruler bending z.
2.2 Derive the theoretical dependence of the oscillation period on the threads tension.
2.3 Present the experimental data on the graph in such a way that it confirms the derived theoretical dependence.



Part 3. Twisting at large angles (5.0 points)

In this part the pendulum should be twisted at large angles, which are measured by the number of half-turns $N < 20$. The measurements should be carried out with a chosen moderate thread tension that you should exactly state.

3.1 Study the dependence of the time of the pendulum untwisting on the initial twisting angle N_0. Plot the graph of the resulting dependence.
3.2 It can be approximately assumed that the potential energy of elastic deformation depends on the twisting angle as

$$U = CN^{\gamma},$$

where C is a constant. Based on the experimental data, decide which range the exponent γ falls in: a) $\gamma < 2$; b) $\gamma = 2$; c) $\gamma > 2$.

If the pendulum is initially twisted to a certain angle N_0, the pendulum unwinds such that the threads turn in parallel again, and then due to inertia the pendulum is re-twisted to some smaller angle N_1.

3.3 Study the dependence of the re-twisting angle N_1 on the initial twisting angle N_0. The measurements must be taken at two different magnitudes of the threads tension (but you should state them). Plot the graphs of the obtained dependencies. Propose a simple formula to describe the resulting dependence. Evaluate the numerical values of the parameters in your dependence.

Adjusting the threads tension during torsional oscillations of the pendulum can provide some intake of energy. Place the tripod with the pendulum at the edge of the table. At the bottom the thread should cover the ruler, fixed in the holder or under the tripod platform. Suspend the weight of 100 g to the thread below the ruler.

3.4 Study the dependence of the re-twisting angle N_1 on the initial twisting angle N_0 in this case. Plot the graph of the obtained dependence. Propose a simple formula to describe the resulting dependence. Evaluate the numerical values of the parameters in your dependence.
3.5 Repeat the experiment described in section 3.4, but lift the suspended weight up at the stage of re-twisting and do not touch the weight at the stage of untwisting. Study the dependence of the re-twisting angle N_1 on the initial twisting angle N_0 in this case. Plot the graph of the obtained dependence in the same graph as in section 3.4. Propose a simple formula to describe the resulting dependence. Evaluate the numerical values of the parameters in your dependence.

SOLUTION TO THE EXPERIMENTAL COMPETITION

Torsion pendulum (15.0 points)

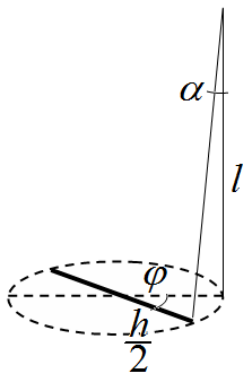
Part 1. Free small oscillations (5.0 points)

1.1 The measurement results showing the dependence of the oscillation period on the length of threads are shown in Table 1. For each threads length the measurement are taken 3 times for 10 oscillations. The oscillation period is calculated as an average of the measured time.

Table 1.

l, sm	t_1, s	t_2, s	t_3, s	T, s	T^2, s^2
61	48,19	48,02	48,34	4,82	23,22
49	43,57	43,31	43,81	4,36	18,98
39	38,56	38,34	38,37	3,84	14,76
28	33,1	32,94	32,78	3,29	10,85
19	26,44	27,03	26,62	2,67	7,13
10	18,91	19,35	19,22	1,92	3,67

1.2 By turning the bolts in a horizontal plane at a small angle φ the threads deviate from the vertical by a small angle α. The relationship between these angles are geometrically found in the form



$$\frac{h}{2}\varphi = l\alpha \quad \Rightarrow \quad \alpha = \frac{h}{2l}\varphi, \tag{1}$$

where l denotes the threads length and h stands for distance bewteen them.
Deviation from the vertical results in the following increase of the potential energy

$$\Delta U = mgl(1-\cos\alpha) \approx mgl\frac{\alpha^2}{2} = \frac{1}{2}mgl\left(\frac{h}{2l}\varphi\right)^2. \tag{2}$$

The equation of energy conservation for the torsional oscillations is written as

$$\frac{I\omega^2}{2} + \frac{1}{2}mgl\left(\frac{h}{2l}\varphi\right)^2 = E = const, \tag{3}$$

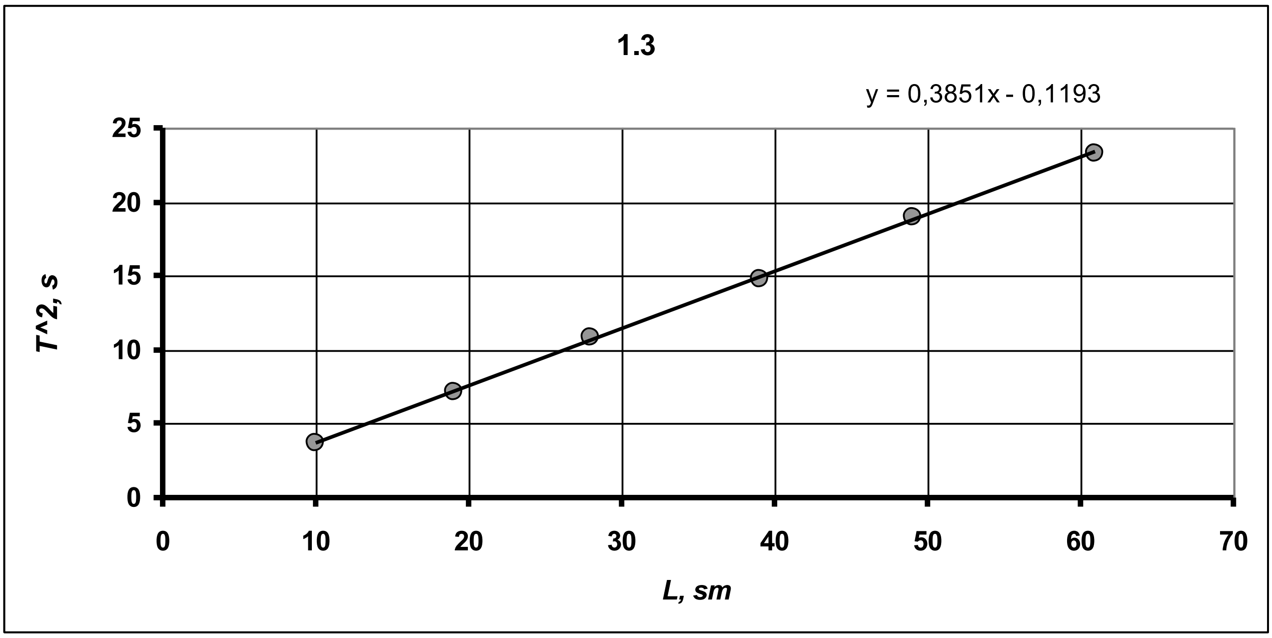
where I designates the moment of inertia of the pendulum.
The law of energy conservation (3) corresponds to the harmonic oscillations with the period

$$T = 4\pi\sqrt{\frac{I}{mg}\frac{l}{h^2}}. \tag{4}$$

1.3 To verify the resulting formula the graph must be plotted of the square of the period on the threads length (see Fig. 1.1). The obvious linear dependence proves the validity of the formula (4).
The coefficients of the linear dependence $T^2 = al + b$, calculated using the least square method, are obtained as

$$a = (0,385 \pm 0,009)\frac{s^2}{sm}, \quad b = -(0,12 \pm 0,3)s^2. \tag{5}$$

Since $\Delta b < b$ holds, the dependence is considered linear.



1.4 It follows from formula (4) that the period of oscillations can be expressed in terms of the radius of gyration as follows:

$$T = 4\pi\sqrt{\frac{I}{mg}\frac{l}{h^2}} = 4\pi\sqrt{\frac{mR^2}{mg}\frac{l}{h^2}} = \frac{4\pi R}{h\sqrt{g}}\sqrt{l}\ . \tag{6}$$

Therefore, the slope found in 1.3 allows one to calculate the radius of gyration as

$$a = \left(\frac{4\pi R}{h\sqrt{g}}\right)^2 \quad \Rightarrow \quad R = h\frac{\sqrt{ag}}{4\pi} = 4,33sm\ . \tag{7}$$

The distance between threads is measured as $h = (2,8 \pm 0,1)sm$.

The experimental error is calculated according to the formula

$$\Delta R = R\sqrt{\left(\frac{\Delta h}{h}\right)^2 + \left(\frac{1}{2}\frac{\Delta a}{a}\right)^2} = 4,33\sqrt{\left(\frac{0,1}{2,8}\right)^2 + \left(\frac{1}{2}\frac{0,009}{0,385}\right)^2} = 0,16sm\ . \tag{8}$$

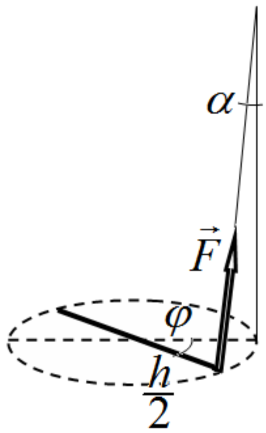
Part 2. Small oscillations with additional tension (5.0 points)

The results of measurements showing the dependence of the oscillation period on the therads tension are shown in Table 2.

Table 2.

z, mm	t_1, s	t_2, s	t_3, s	T, s	T^{-2}, s^{-2}
12	10,22	10,28	10,40	1,03	0,94
20	7,10	7,03	7,18	0,71	1,98
27	5,63	5,62	5,69	0,56	3,14
37	4,53	4,50	4,53	0,45	4,89
52	3,75	3,88	3,87	0,38	6,81

2.2 Torque of the restoring force dependes on the threads tension $\vec{F}$ and the twisting angle. Change in the tension force when twisting is the correction of the higher order.



Therefore, the rotational equation of motion in this case has the form

$$I\ddot{\varphi} = -kF\varphi\,. \tag{9}$$

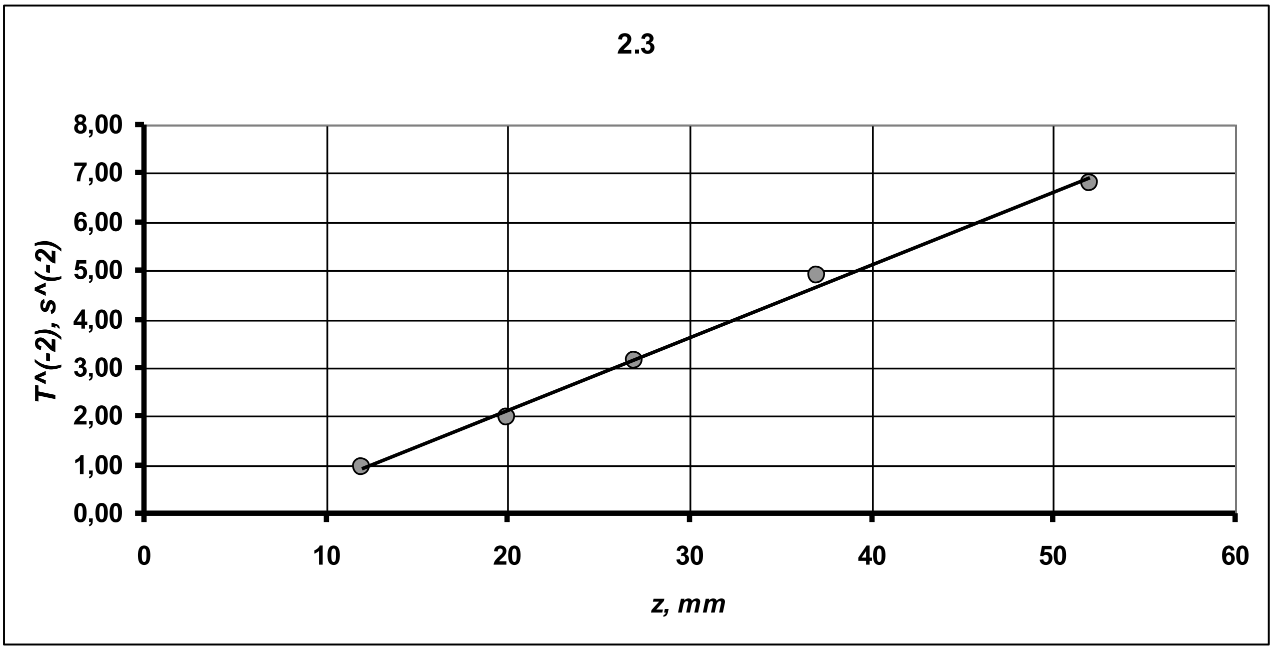
Consequently, the period of those oscillations is inversely proportional to the square root of the threads tension

$$T = \frac{C}{\sqrt{F}} = \frac{A}{\sqrt{z}}, \tag{10}$$

where C, A are some constant values.

2.3 To confirm this dependence the dependence $T^{-2}(z)$ should be plotted. Other ways of linearization in this case are less reliable because when the bending z is measured, the constant deviation is inevitable.

The graph of $T^{-2}(z)$ is shown in Fig. 2.3.



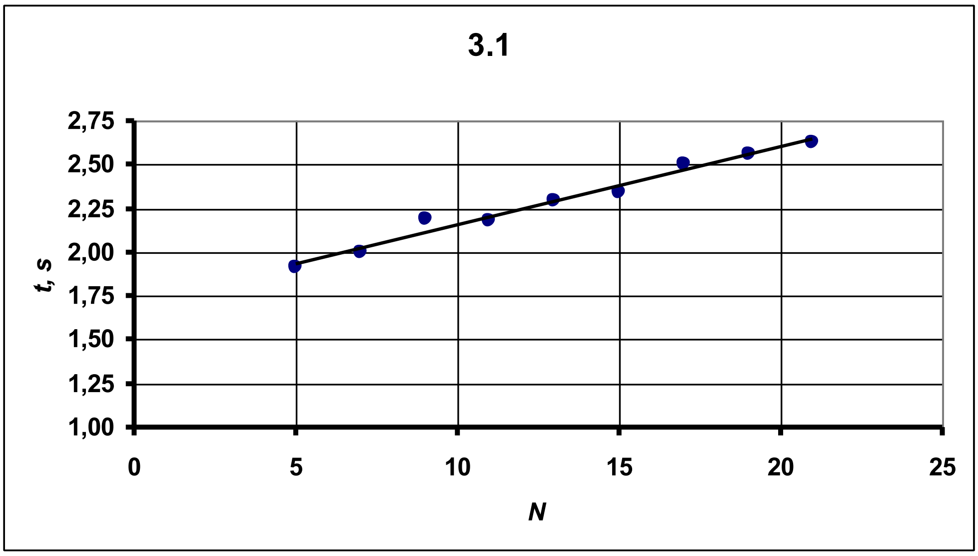
The resulting linear relationship confirms the theoretical conclusion that the period is a linear function of the threads tension.

Part 3. Twisting at large angles (5.0 points)

3.1 Dependence of the untwisting time on the twisting angle is shown in Table 3 and Fig. 3.1.

Table 3.

N	t, s
5	1,91
7	2,00
9	2,19
11	2,18
13	2,29
15	2,34
17	2,50
19	2,56
21	2,63

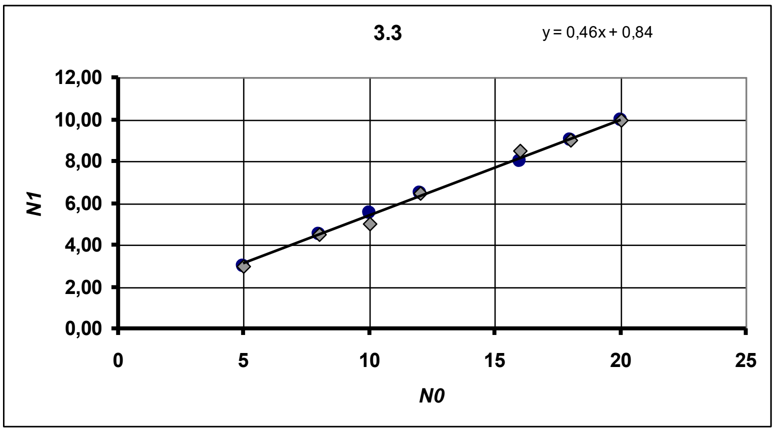


3.2 The untwisting time can be considered as a quarter of the oscillation period. Since the untwisting time increases with the "amplitude", this means that the potential energy increases slower than in the case of harmonic oscillations, i.e. . $\gamma < 2$.

3.3 The dependence of re-twisting half-turns N_1 on the initial twisting half-turns N_0 is shown in Table 4 and in graph 3.3. The resulting dependence is practically independent of the threads tension.

Table 4.

	$z = 35mm$	$z = 20mm$
N_0	N_1	N_1
20	10,0	10,0
18	9,0	9,0
16	8,0	8,5
12	6,5	6,5
10	5,5	5,0
8	4,5	4,5
5	3,0	3,0



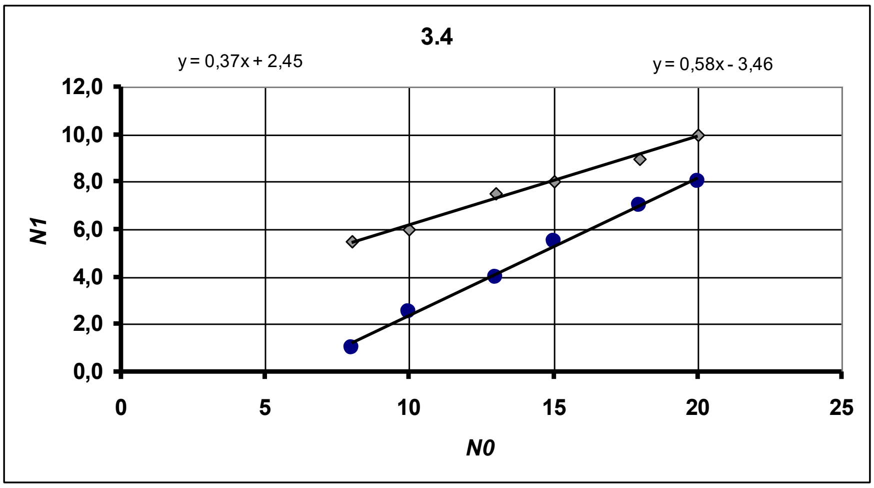
This dependence can be described by a linear function

$$N_1 = 0{,}46 N_0 + 0{,}84 . \tag{11}$$

3.4-3.5 The dependencies of half-turns N_1 on the initial twisting angle N_0 at lifting/unlifting the weight are summarized in Table 5 and in graph 3.4. These dependencies are linear. It is significant that during the ascent without the additional weight the corresponding values of N_1 lie significantly higher, indicating that the intake of energy appears during the weight unlifting. In addition, the last relationship can not be exactly considered proportional.

Table 5.

	С грузом	Без груза
N_0	N_1	N_1
20	8,0	10,0
18	7,0	9,0
15	5,5	8,0
13	4,0	7,5
10	2,5	6,0
8	1,0	5,5



These dependencies can be described by linear functions

$$\begin{aligned} N_1 &= 0{,}58 N_0 - 3{,}5 \\ N_1 &= 0{,}37 N_0 + 2{,}5 \end{aligned}. \qquad (12)$$

Marking scheme

	Content	Total	Points
	Part 1. Free small oscillations.	**5**	
1.1	***Marked only if the difference of measurements results from the official ones does not exceed 25%***	**2**	
	Number of different lengths of the pendulum: 5 or more (*3-4; less than 3*);		0,8*(0,4;0)*
	The periods are measured by 10 oscillations or more (*5-9; less than 5*);		0,3*(0,1;0)*
	The periods are calculated for all measurements;		0,2
	The range of length of the pendulum 40 *sm* or more (*30-40 sm, 20-30 sm; less than 20 см*)		0,7 *(0,5;0,3;0)*
1.2	Derivation of the theoretical formula: The exact formula is obtained $T = 4\pi\sqrt{\frac{I}{mg}\frac{l}{h^2}}$; *Only dependence $T = A\sqrt{l}$ is predicted (or wrong coefficient at $\sqrt{l}$)*	**0,5**	 0,5 *(0,2)*
1.3	***Marked only if the measurements in 1.1 have been marked!*** Correct linearization $T^2(l)$, $T(\sqrt{l})$; *Another linearization* $\ln T(\ln l)$ *Proved that the power is 1/2*	**1**	0,5 *(0,2)* *(0,3)*
	Graph plotting *(not linearized dependence is not marked):* - axis are named and ticked;		0,1

	- experimental data are drawn; - the line is drawn;		0,1 0,1
	The linear dependence is confirmed;		0,2
1.4	***Marked only if the measurements in 1.1 have been marked!*** The radius of gyration is calculated for all periods; *Only for 2 periods;* *Only for 1 period;*	**1,5**	 0,3 *(0,2)* *(0,1)*
	The slope is found for the linearized dependence using: Least square method; *From the graph;*		 0,2 *(0,1)*
	The distance between the threads is measured The accuracy is stated		0,1 0,1
	Formula for calculation of the radius of gyration		0,1
	Numerical value for the radius of gyration in the range: 4,2 – 4,4 *sm* (*4,0 – 4,6 sm; out of range*)		 0,4*(0,2;0)*
	Error evaluation: - error for the slope; - error for the distance between threads; - final error;		 0,1 0,1 0,1
	Part 2. Small oscillations with additional tension	**5**	
2.1	***Marked only if the difference of measurements results from the official ones does not exceed 50%***	**2**	
	Number of different threads tensions: 5 or more (*3-4; less than 3*);		 0,8*(0,4;0)*
	The periods are measured by 10 oscillations or more (*5-9; less than 5*);		 0,3*(0,1;0)*
	The periods are calculated for all measurements;		0,2
	The range of variation of the periods Not less than 2,5 times (*2,0 times, 1,5 times; less*)		0,7 *(0,5;0,3;0)*
2.2	Derivation of the theoretical formula: The dependence $T=\frac{A}{\sqrt{F}}$ is justified *threads tension is taken constant, the torque is proportional to* F*, the equation for oscillations)*; *Simply stated that* $T=\frac{A}{\sqrt{F}}$ *(no proof is provided)*	**1**	1,0 *(0,3; 0,3; 0,4)* *(0,2)*
2.3	***Marked only if the measurements in 2.1 have been marked!!*** Correct linearization $T^{-2}(z)$; *Another linearization* $T\left(\frac{1}{\sqrt{z}}\right)$ *The linearization* $\ln T(\ln l)$ *is used, Proved that the power is 1/2*	**2**	 1,0 *(0,5)* *(0,3+0,2)*
	Graph plotting *(not linearized dependence is not marked):* - axis are named and ticked; - experimental data are drawn; - the line is drawn;		 0,1 0,1 0,1
	The linear dependence is confirmed;		0,7
	Part 3. Twisting at large angles	**5**	
3.1	***Marked only if the difference of measurements results from the***	**1,0**	

	official ones does not exceed 50%		
	Number of different values of N: 5 or more (*3-4; less than 3*);		0,4 *(0,2;0)*
	The measurements are repeated 3 times;		0,1
	Growing dependence of $t(N)$ is obtained		0,2
	Graph plotting *(m**arked only if the measurements results have been marked**):* - axis are named and ticked; - experimental data are drawn; - the line is drawn;		 0,1 0,1 0,1
3.2	The power in potential energy $\gamma < 2$ Justification: U grows slowly than for harmonic oscillations	**0,3**	 0,2 0,1
3.3	***Marked only if the slope falls in the range 0,35-0,65***	**1,5**	
	Number of different values of N_0: 5 or more (*3-4; less than 3*);		0,5*(0,2;0)*
	The measurements are repeated 3 times or more;		0,1
	Growing linear dependence is obtained;		0,2
	Graph plotting *(m**arked only if the measurements results have been marked**):* - axis are named and ticked; - experimental data are drawn; - the line is drawn;		 0,1 0,1 0,1
	The linear function is proposed; Numerical values for the parameters are found;		0,1 0,2
	The slopes are equal for both threads tensions;		0,1
3.4	***Marked only if the slope falls in the range 0,25-0,65***	**1**	
	Number of different values of N_0: 5 or more (*3-4; less than 3*);		0,3*(0,1;0)*
	Growing linear dependence is obtained;		0,1
	Graph plotting *(m**arked only if the measurements results have been marked**):* - axis are named and ticked; - experimental data are drawn; - the line is drawn;		 0,1 0,1 0.1
	The linear function is proposed; Numerical values for the parameters are found;		0,1 0,2
3.5	***Marked only if the slope falls in the range 0,25-0,75 and there is a shift of line up!***	**1,2**	
	Number of different values of N_0: 5 or more (*3-4; less than 3*);		0,4*(0,2;0)*
	Growing linear dependence with the upper shift is obtained;		0,2+0,1
	Graph plotting *(m**arked only if the measurements results have been marked**):* - experimental data are drawn; - the line is drawn;		 0,1 0,1
	The linear function is proposed; Numerical values for the parameters are found;		0,1 0,2

SOLUTION TO THE EXPERIMENTAL COMPETITION

The Law of Archimedes (15.0 points)

Part 1. Installation parameters

1.1 A strip of millimeter paper is screwed onto the test-tube. We make marks on the strip, untwist it and obtain the lengths of 1, 2, 3 and 4 revolutions as

$l_1 = 63 mm$

$l_2 = 127 mm$

$l_3 = 191 mm$

$l_4 = 255 mm$

From these data we find that the length of one revolution is equal to $\langle l \rangle = (64{,}0 \pm 0.3) mm$

The diameter is then calculated by the formula $D = \frac{\langle l \rangle}{\pi} = 20{,}372 mm$, the unstrumental error is found as $\Delta D = D\frac{\Delta l}{\langle l \rangle} = 0{,}1 mm$ and the final result is written as

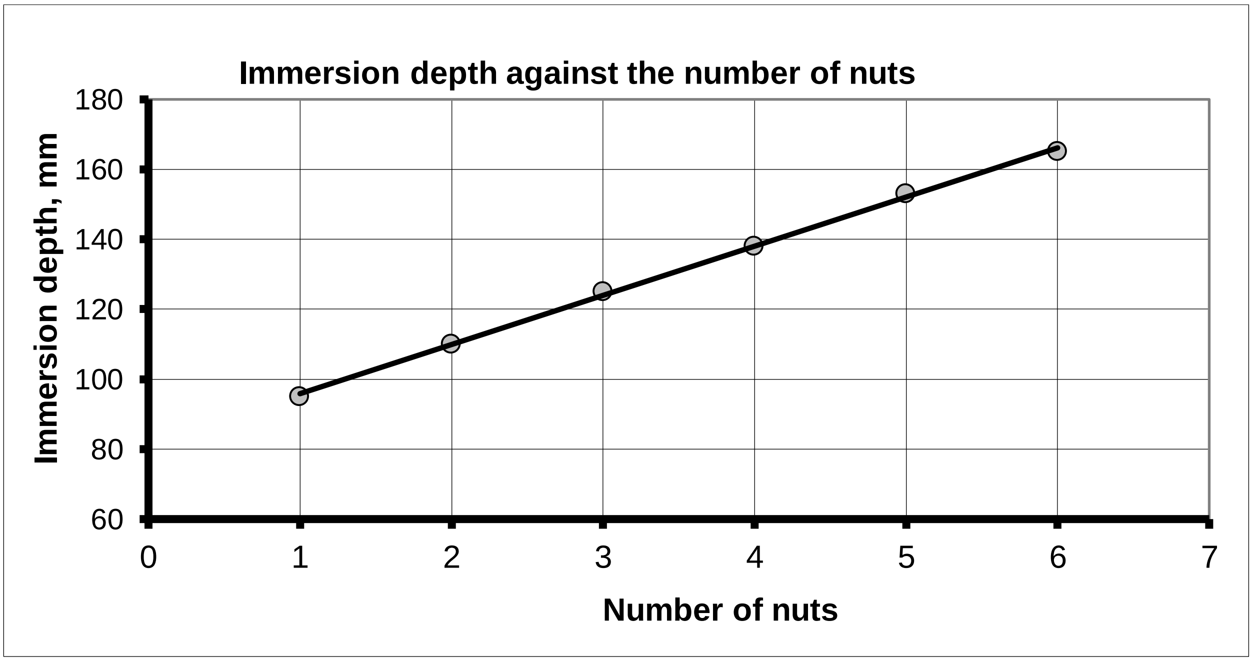
$$D = (20{,}4 \pm 0.1) mm.$$

1.2 The length of the test-tube is obtained as $L = (175 \pm 1) mm.$

1.3.1 – 1.3.2 . Dependence of the immersion depth of the tesr-tube on the number of nuts, placed in it, is shown in Table 1. First, the length x of the part of the test-tube, protruding above the water level, is measured. And then the immersion depth is calculated by the formula $h = L - x$.

Table 1

Number of nuts	x, mm	h, mm
1	80	95
2	65	110
3	50	125
4	37	138
5	22	153
6	10	165



The dependence obtained is linear and is described by the formula

$$h = an + b. \tag{1}$$

The parameters, calculated by the least square method, are equal

$$\begin{aligned} a &= (14{,}1 \pm 0{,}5)mm \\ b &= (81{,}8 \pm 1{,}8)mm \end{aligned}. \tag{2}$$

1.3.3 The theoretical formula for the resulting dependence follows from the equilibrium condition

$$(M + mn)g = \rho Shd \quad \Rightarrow \quad h = \frac{M + mn}{\rho S}. \tag{3}$$

where $S = \dfrac{\pi D^2}{4}$ stands for the cross-sectional area of the test tube.

From the comparison of expressions (3) and (1) it follows that

$$a = \frac{m}{\rho S} \quad \Rightarrow \quad m = \rho S a. \tag{4}$$

Numerical calculations lead to the following result

$$m = \rho \frac{\pi D^2}{4} a = 4{,}58 \cdot 10^{-3} kg = 4{,}58\ g. \tag{5}$$

The instrumental error in measuring the mass of the nut is calculated by the formula

$$\Delta m = m\sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(2\frac{\Delta D}{D}\right)^2} = 1{,}6 \cdot 10^{-4} kg. \tag{6}$$

The final weight of the nut is written as

$$m = (4{,}58 \pm 0{,}16)\,g. \tag{7}$$

The weight of the test-tube is calculated by the formula

$$b = \frac{M}{\rho S} \quad \Rightarrow \quad M = \rho S b = \rho \frac{\pi D^2}{4} b = 2{,}67 \cdot 10^{-2} kg = 26{,}7g.$$

The error in calculating the mass of the test-tube is found as

$$\Delta M = M\sqrt{\left(\frac{\Delta b}{b}\right)^2 + \left(2\frac{\Delta D}{D}\right)^2} = 0{,}6g.. \tag{8}$$

To simplify further calculations, we note that the ratio of the parameters of the linear dependence (2) is equal to the ratio of the mass of the test-tube and the nut:

$$n^* = \frac{M}{m} = \frac{b}{a} = 5{,}82. \tag{9}$$

Part 2. Oscillations of the test-tube

2.1 To simplify the calculations, the formula for the period of oscillations can be rewritten in the form

$$T_n = 2\pi\sqrt{\frac{h_0}{g}} = 2\pi\sqrt{\frac{an + b}{g}}. \tag{10}$$

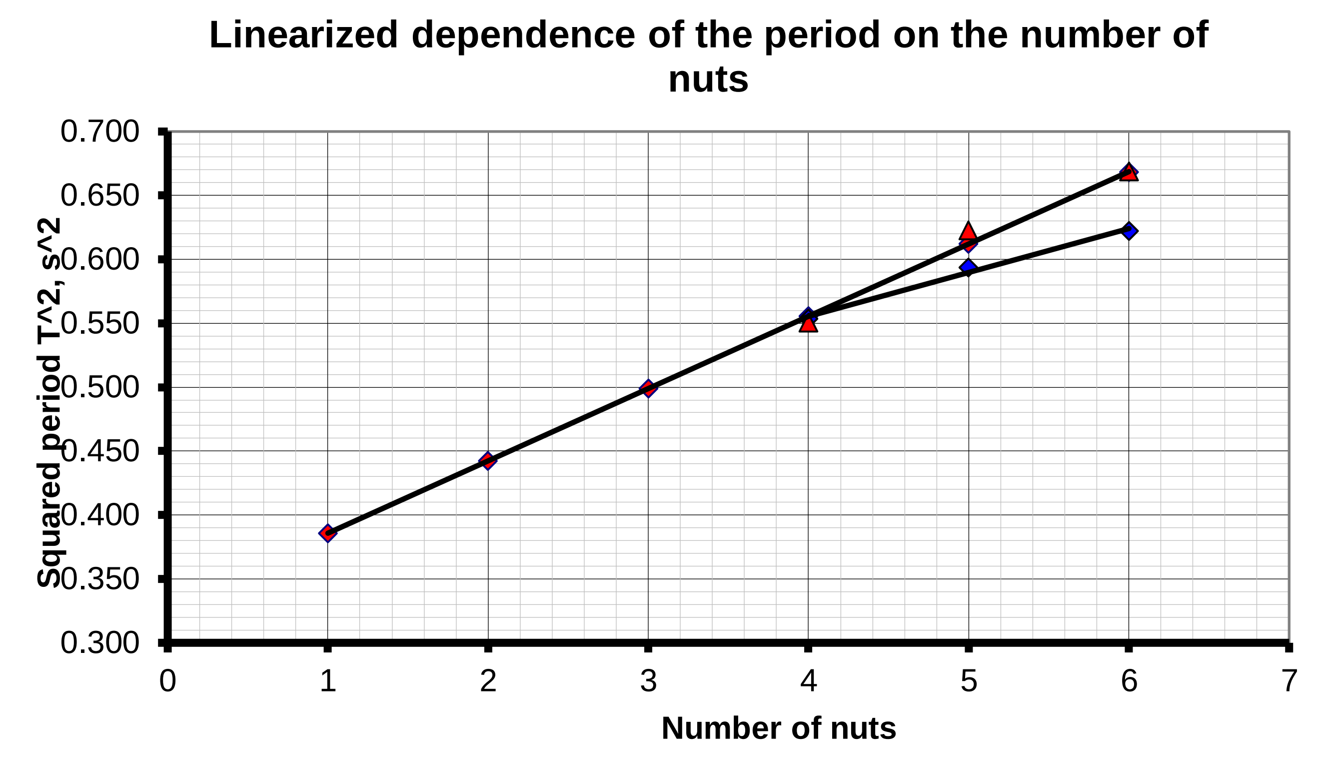
To linearize this dependence, it is necessary to plot and analyze the dependence of the squared period on the number of nuts $T^2(n)$. The results are summarized in Table 2.

Table 2.

Number of nuts	T, s	T^2, s^2
1	0,680	0,463
2	0,717	0,514
3	0,752	0,565

4	0,785	0,617
5	0,817	0,668
6	0,848	0,720

The graph of the dependence $T^2(n)$ is shown in the figure below.



2.2 The results of the measurements are given in tables
The random error in measuring the period is estimated from the following formula

$$\Delta t = 2\sqrt{\frac{\sum_k \left(t_k - \langle t \rangle\right)^2}{N(N-1)}} \quad ; \qquad \Delta T = \frac{\Delta t}{k}. \qquad (11)$$

Here t refers to the time needed to perform k periods of oscillations (in our case $k = 5$ and $k = 3$ respectively), $N = 10$ stands for the number of measurements.

Table 3. Oscillations in the wide vessel

Number of nuts	Number of periods k	Time t ,s	Period T , s	Averaged period $\langle T \rangle$, s	Error in the period ΔT	Squared period T^2, s^2
4	5	3,74	0,748	0,744	0,009	0,554
	5	3,64	0,728			
	5	3,77	0,754			
	5	3,71	0,742			
	5	3,74	0,748			
5	5	3,93	0,786	0,770	0,010	0,594
	5	3,81	0,762			
	5	3,89	0,778			
	5	3,83	0,766			
	5	3,80	0,760			

6	5	3,93	0,786	0,789	0,010	0,622
	5	3,93	0,786			
	5	4,04	0,808			
	5	3,93	0,786			
	5	3,89	0,778			

Table 3. Oscillations in the beaker

Number of nuts	Number of periods k	Time t ,s	Period T , s	Averaged period $\langle T \rangle$, s	Error in the period ΔT	Squared period T^2, s^2
4	3	2,21	0,74	0,742	0,014	0,551
	3	2,25	0,75			
	3	2,27	0,76			
	3	2,20	0,73			
	3	2,20	0,73			
5	3	2,38	0,79	0,789	0,015	0,623
	3	2,37	0,79			
	3	2,34	0,78			
	3	2,42	0,81			
	3	2,33	0,78			
6	2	1,61	0,81	0,818	0,036	0,669
	2	1,59	0,80			
	2	1,66	0,83			
	2	1,63	0,82			
	2	1,69	0,85			

2.4 What possible reasons can explain the deviation between experimental data and theoretical calculations?

Table 4

No.	Possible reasons	«Yes»	«No»
1	Measurement errors	X	
2	Oscillation damping		X
3	An increase in the effective mass of a moving test-tube due to water entraining		X
4	Change in pressure under the tube when it moves as compared to hydrostatic pressure	X	
5	Surface tension forces		X

Comments:
1. Of course, errors ifluence any result.
2.3 These reasons should lead to an increase in the period, and not to a decrease.
4. Apparently, the main reason, leading to a reduction in the period.
5. Too small forces.

Marking scheme

Part1. Installation parameters

№	Criteria	Total	Points
1.1	**Diamater measurement**	**0,9**	
	- sketch of the measurements: - rolling on the test-tube (2-3 revolutions; *1 revolution*); *- rolling the test-tube on the millimeter paper;* *- direct measurement of the diameter;*		 0,2 (*0,1*) (*0,1*) (*0,1*)
	Measurement results: - circumference in the range of 63-66 mm (*61-68 mm, out of range*)		0,2 *(0,1; 0)*
	Evaluation of the diameter: - formula: - numerical value (in accordance with the previous part)		 0,1 0,2 *(0,1; 0)*
	Instrumental errpr 0,25-0,35 mm (*larger*)		0,1 *(0)*
	Correctly rouded results*		0,1
1.2	**Measurement of the test-tube length**	**0,3**	
	- length in the range of 170-180 mm (*out of range*) - instrumental error 1 mm (*иное*)		0,1 *(0)* 0,1 *(0)*
	Correctly rounded result*		0,1
1.3.1	**Results of the immersion depth measurement**	**1,8**	
	Results differ from tabulated ± 2 mm (± 4 *mm, larger*)		1,2 *(0,6; 0)*
	Number of points* 6 (*3, less*)		0,6 *(0,3, 0)*
1.3.2	**Plotting the graph and calculating the parameters of the dependence** (marked only if 1.3.1 has been marked)	**1,0**	
	- axes are signed and ticked; - points are plotted in accordance with the table		0,1 0,2
	Parameters of the dependence: - form of dependence is a linear function - evaluation of the parameters; - errors of the parameters;		 0,1 2x0.2 2x0,2
1.3.3	**Calculation of masses of the nut and the test tube:** (marked only if 1.3.1 has been marked)	**2,0**	
	- formula of the theoretical dependence		0,4
	- formulas for calculating masses through the parameters of the linear dependence;		2x0,2
	- calculation of the mass of the nut: within 10% from the tabulated value (20%, larger)		0,4 *(0,2, 0)*
	- Nut mass error: errors in the slope and the diameter are taken into account (*only one contribution)*		0,2 *(0,1)*
	- calculation of the test-tube mass: within 10% from the tabulated value (20%, larger)		0,4 *(0,2, 0)*
	- error in the mass of the test-tube: errors in the shift and the diameter: erroes in the shift and in the diameter are taken into account (*only one contribution)*		0,2 *(0,1)*

* - marked only if the measurements are marked.

Part 2. Oscillations of the test-tube

№	Criteria	Total	Points
2.1	**Theoretical dependence**	**1,2**	
	- formula for the period $T(n)$ via measured parameters		0,2
	- periods are calculated		6x0,1
	- linearization $T^2(n)$ *(other)*		0,1(0)
	Plotting the graph: - axes are signed and ticked; - points are plotted in accordance with the table;		 0,1 0,2
2.2	Formula for evaluating the error in the period: - decrease of the random error with increasing the number of measurements; *- modulus of the average deviation from the mean value;*		 0,2 *(0,1)*
	Oscillations in the wide vessel	**3,0**	
	Results within the range $\pm 20\%$ ($\pm 30\%$, larger)		3x0,3 (0,2; 0)
	More than 7 measuments are taken (more than 4, less)*		3x0,3 (0.2; 0)
	Periods are calculated*		3x0,1
	Errors are calculated*		3x0,1
	Points are plotted in accordance with the table *		0.2
	Errors are stated in the graph*		0,2
	The periods of oscillations are found to be less than the theoretical one (more than 0,1 s)*		0,2
	Oscillations in the beaker	**3,3**	
	The results of the measurements within the range $\pm 20\%$ ($\pm 30\%$, larger)		3x0,3 (0,2; 0)
	More than 7 measuments are taken (more than 4, less)*		3x0,3 (0.2; 0)
	Periods are calculated*		3x0,1
	Errors are calculated*		3x0,1
	Points are plotted in accordance with the table *		0.2
	Errors are stated in the graph*		0,2
	The periods of oscillations are close to theoretical (the difference is not more than 0,2 s)*		0,3
	The periods of oscillations in different vessels are similar (differences not more than 0.2 s) *		0,2
2.4	Possible reasons	**1,5**	
	- each correct answer		5x0,3
	Total	**15**	

* - marked only if the measurements are marked.

THEORETICAL COMPETITION

January 12, 2018

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don’t have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets.*** Write only inside the boxed area.
6. Begin each question on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Question Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
   - Used ***Writing sheets*** in order;
   - The sheets you do not wish to be evaluated
   - Unused sheets and the printed question.

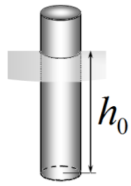
   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

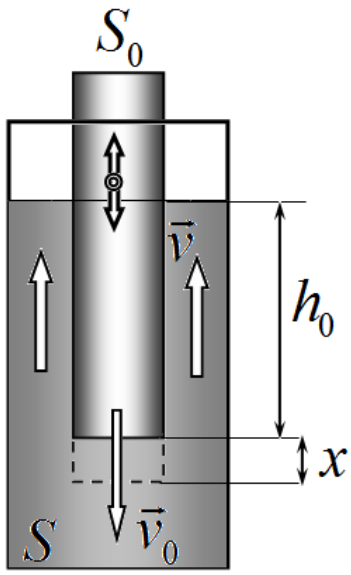
Problem 1 (10.0 points)

This problem consists of three independent parts.

Problem 1A (3.0 points)

A1 A narrow cylindrical test-tube with a displaced center of mass floats vertically in water in a very wide vessel. In equilibrium state, the test-tube is immersed into water to a depth h_0. The cross-sectional area of the tube is S_0. Determine the period of small vertical oscillations of the test-tube.





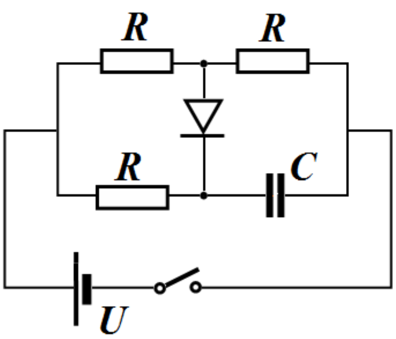
A2 The same test-tube is placed in a narrow cylindrical vessel with a cross-sectional area S filled with water. The test-tube makes small oscillations along the axis of the vessel.

A2.1 The test-tube sinks by some small value x. Express the change in the potential energy of the system through x, the initial depth of immersion h_0, cross-sectional areas S_0, S, the water density ρ and the acceleration of gravity g.

A2.2 Let the speed of the test-tube be v_0 near its equilibrium position. Express the kinetic energy of the system through the speed of the test tube v_0, the depth of immersion h_0, the cross-sectional areas S_0, S, and the water density ρ. Consider that in the gap between the test-tube and the walls of the vessel all liquid moves with the same speed v.

A2.3 Find the period of small oscillations of the test-tube in the narrow vessel.

Problem 1B (4.0 points)



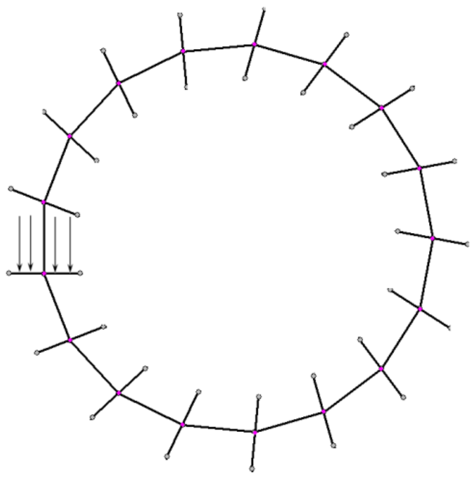
The circuit, shown in the figure on the left, consists of a capacitor with the capacitance C = 100 μF, an ideal diode, a constant voltage source U = 10.0 V, three identical resistors with the resistance R = 10.0 kΩ and a switch. At the initial moment, the capacitor is not charged and the switch is open. When the switch is shorted, the current through the diode goes for the time interval τ = 462 ms, and then stops.

1. Find the current through the diode immediately after shorting the switch.

2. Find the total charge that has flowed through the diode.

Problem 1C (3.0 points)

At the vertices of the regular 17-gon, there are 17 identical lenses. The optical centers of the lenses are located exactly at the vertices of the polygon, the planes of all lenses are perpendicular to one of the sides adjacent to the lens. The focal lengths of the lenses are all equal to $F = 10$ sm and coinciude with the length of the side of the 17-gon. One of the lenses is illuminated by a parallel light flux directed along its optical axis. It turns out that one of the rays has a closed trajectory. Determine the radius of the circle inscribed in this trajectory. Consider two cases: all of the lenses are collecting; all of the lenses are diverging. Consider all angles small such that $\sin\alpha \approx \tan\alpha \approx \alpha$.



Problem 2 (10.0 points)
Physics in the mountains

The atmosphere of a real planet, such as the Earth, has a rather complex structure in view of the great variety of processes and phenomena involved in its formation. In this problem we will consider two simple models of the lower layer of the atmosphere, called the troposphere, which extends to an altitude of 10-15 km above the Earth's surface. To understand the physics of some phenomena it is sufficient to consider the Earth's atmosphere consisting of a single-component diatomic gas with the molar mass $\mu_{air} = 28.9 \cdot 10^{-3}$ kg/mole.

Part 1. Isothermal atmosphere

In the atmosphere, the lowest near-surface layer has an almost constant temperature, as it is heated up by the surface of the Earth. Therefore, we assume in this part that the temperature of the atmosphere remains the same over its entire altitude and is equal to $T_0 = 293$ K, and the air pressure at the Earth's surface is $p_0 = 1.013 \cdot 10^5$ Pa. Assume that the acceleration of gravity $g = 9.81\ \mathrm{m/s^2}$ is independent of the altitude above the Earth's surface, since the total height of the atmosphere is much less than the Earth's radius $R_E = 6400$ km. The universal gas constant is $R = 8.31\ \mathrm{J/(mole \cdot K)}$.

1.1 Find and calculate the mass M of the Earth's atmosphere.
1.2 Find and calculate the air pressure p_H at the altitude of $H = 1500$ m above the Earth's surface.

From the physical point of view, the interesting question is how fast the atmosphere warms up with the change of day and night. From space observations, the so-called solar constant $\alpha = 1367\ \mathrm{W/m^2}$ is known, which is the total power of the solar radiation in the region of the Earth's orbit passing through a unit of a surface oriented perpendicular to its flow.

1.3 Estimate the amount of heat δQ needed to heat the atmosphere by $\Delta T = 1$ K.
1.4 Find and calculate the time interval τ for the Sun to shine in order to provide the Earth with the amount of heat δQ.

Part 2. Adiabatic atmosphere

The real troposphere is not isothermal at all and the air temperature decreases with altitude. Due to the constantly flowing convective processes, the troposphere can be considered practically adiabatic. Let the air temperature and pressure at the Earth's surface be $T_0 = 293$ K and $p_0 = 1.013 \cdot 10^5$ Pa, respectively. Consider the acceleration of gravity $g = 9.81\ \mathrm{m/s^2}$ be still independent of the altitude above the Earth's surface.

2.1 Find and calculate the air temperature T_H at the altitude of $H = 1500$ m above the Earth's surface.
2.2 Find and calculate the air pressure p_H at the altitude of $H = 1500$ m above the Earth's surface.

In the constructed model, the height of the Earth's troposphere is determined by reaching a certain critical temperature, at which other physical processes begin to play an important role.

2.3 Estimate the height difference ΔH_{atm} of the Earth's troposphere at day and night if the surface temperature changes within this period of time by $\Delta T_{dn} = 20$ K.

A mountain climber starts climbing on a fairly high mountain, at the foot of which the temperature and the air pressure are equal to $T_0 = 293$ K and $p_0 = 1.013 \cdot 10^5$ Pa. At the altitude of $H = 1500$ m, he decides to make a halt in order to boil some water and discovers that it boils faster than usual. He opens the handbook on physics available at the moment and finds that at temperature $T_1 = 373$ K, the saturated water vapor pressure is $p_1 = p_0 = 1.013 \cdot 10^5$ Pa, and at temperature $T_2 = 365$ K it is equal to $p_2 = 0.757 \cdot 10^5$ Pa.

2.4 Find and calculate the boiling temperature of water at the altitude of $H = 1500$ m.

After having resumed his climbing, the mountain climber discovers that snow appears at a certain altitude and special equipment is needed to proceed further.

2.5 Find and calculate the altitude h_0, at which the climber noticed the appearance of a snow cover on the mountain.

The climber recalled the conversation with the locals right before climbing, in which he was informed that the snow cover completely disappears from the mountain at temperatures $T > 310$ K at the foot of the mountain.

2.6 Find and calculate the height H_0 of the mountain on which the climber ascends.

Having risen even higher along the slope of the mountain to some altitude H', the climber notices the appearance of fog. Looking around, he realizes that there are no clouds and there is no wind. The climber knows that the molar mass of water is $\mu_{H_2O} = 18.0 \cdot 10^{-3}$ kg/mole, and according to the weather forecast, the relative humidity at the foot of the mountain was equal to $\varphi = 0{,}15$. In the handbook on physics, he finds a formula for the pressure of saturated water vapor in the temperature range $T \in (250{,}300)$K, which has the following form

$$\ln \frac{p_{vap}}{p_{vap0}} = a + b \ln \frac{T}{T_0},$$

where p_{vap} denotes the saturated vapor pressure at temperature T, p_{vap0} stands for the saturated vapor pressure at temperature T_0, $a = 3.63 \cdot 10^{-2}$, $b = 18.2$ are constants. In the calculations, consider that the water vapor is always in thermodynamic equilibrium with the surrounding air.

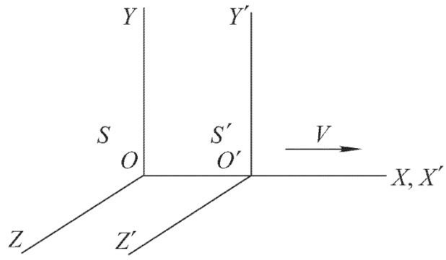
2.7 Find and calculate the altitude H'.

2.8 Find and calculate the minimum air humidity φ_{min} at the foot of the mountain such that the fog is still observed somewhere on the mountain slope.

Mathematical hint

You may need knowledge of the following integral $\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax + b|$.

Problem 3 (10.0 points)
Optics of moving media
Part 1. 4-dimensional vectors



Consider two inertial reference frames S and S' of which the second one moves at a speed V relative to the first one as shown in the figure on the right. It is assumed that the origins O and O' coincide at the initial time moment $t = t' = 0$ by the clocks of both reference frames. It is known that the Lorentz transformation of space-time coordinates (x', y', z', ct') of any event in the frame S' into the space-time coordinates (x, y, z, ct) of the same event in the frame S have the following form

$$x = \frac{x' + (V/c)ct'}{\sqrt{1 - V^2/c^2}}, \qquad y = y', \qquad z = z', \qquad ct = \frac{ct' + (V/c)x'}{\sqrt{1 - V^2/c^2}},$$

where $c = 2.9979$ m/s is the speed of light.

In the formulas of the Lorentz transformation the spatial coordinates and the time are intentionally brought to the same dimension as they together form components of the so-called 4-vector. It is known that all the components of 4-vectors are transformed in the same way at the transition from one inertial frame to the other. In particular, the momentum and the energy constitute the components of the 4-vector.

Suppose that an object moves in the reference frame S such that it has the total energy E and the momentum projections p_x, p_y, p_z on the coordinate axes OX, OY and OZ, respectively.

1.1 Write down the transformation of energy and momentum of the object from the reference frame S to the reference frame S'.

Suppose an object of the rest mass m moves in the reference frame S such that it has the total energy E and the momentum p. When converting its energy and momentum from one reference frame to another the value $E^2 - p^2c^2 = inv$ remains invariant.

1.2 Express inv in terms of m and c.

Part 2. Doppler effect and light aberration

Let a plane electromagnetic wave (EMW) propagate in XY-plane of the reference frame S so that it has the frequency ω and makes the angle φ with the axis OX.

2.1 Find the frequency ω' of EMW registered by an observer in the reference frame S'.
2.2 Find the angle φ' that EMW makes with the axis $O'X'$ in the reference frame S'.

Astronomical observations have shown that the position of a newly discovered massive star X on the celestial sphere (i.e. relative to very distant objects) does not remain constant throughout the year. It moves along an ellipse with axial ratio 0.900. Ecliptic latitude of a star is the angle between the direction to the star and the ecliptic plane, which can be assumed coincident with the plane of the Earth orbit around the Sun.

2.3 Find the ecliptic latitude δ of the star X and evaluate it in arc degrees.

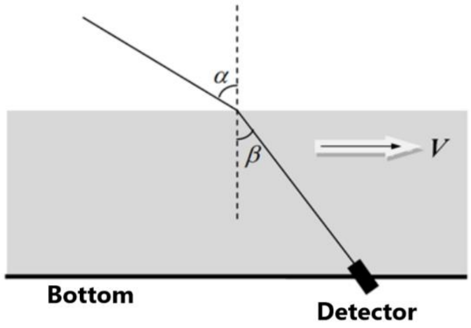
Observation of the emission spectrum of the star X has shown that all frequency in its spectrum are shifted to the red. The relative frequency deviation is $(\Delta\omega/\omega)_0 = 9.9945 \cdot 10^{-3}$. It has independently been established that the recession velocity of the star X from the Sun is equal to $v_x = \frac{1}{100}c$.

2.4 Find and evaluate the escape velocity v_{II} at the surface of the star X.

Part 3. Light in a moving medium

Consider the same two reference frames as in Part 1. Let an object move in the plane $X'Y'$ of the reference frame S' such that the projections of its velocity on the axis $O'X'$ and $O'Y'$ are equal $u_x{}'$ and $u_y{}'$, respectively.

3.1 Find the projections of the object velocity u_x on the axis OX and u_y on the axis OY in the reference frame S.



Consider a water flow moving at the speed of V relative to the bottom of the vessel. A plane EMW falls onto the water surface to make the angle α to the normal in the laboratory reference frame. A detector is fixed at the bottom of the vessel. The refractive index of water is known to be n.

If the water velocity $V \ll c$, the expression for the sine of the angle β at which the detector registers the direction of EMW propagation, takes the form:

$$\sin\beta = A_1 + B_1 V.$$

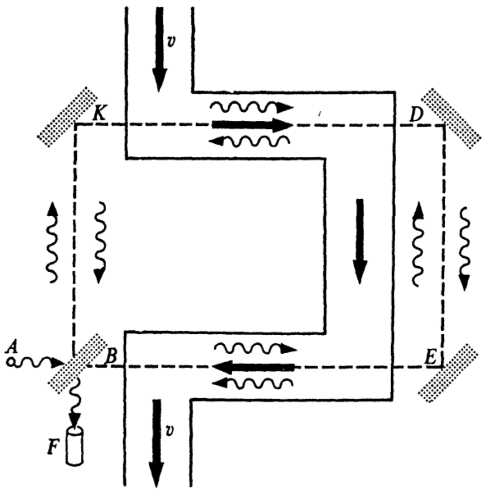
3.2 Find A_1, B_1 and express them in terms of α and n.

If the water velocity $V \ll c$, the expression for velocity v_m of the light propagation in the laboratory reference frame is found as

$$v_m = A_2 + B_2 V.$$

3.3 Find A_2, B_2 and express them in terms of β, n, c.

In 1860 A. Fizeau set up the following experiment. A monochromatic beam with the wavelength λ comes out of the source A and falls onto the semitransparent plate B at which it is divided into two coherent beams. The first beam, being reflected from the plate B, goes along the way $BKDEB$ (K, D and E are the mirrors), whereas the second beam, passing through the plate B, goes along the way $BEDKB$. The first beam, when returning to the plate B, is partially reflected from it and reaches the interferometer F. The second beam, when returning to the plate B, partially passes through it and reaches the interferometer F. Both interfering beams travels the same distance in the laboratory reference frame, including section BE and KD in which the water flows with the velocity v. The total distance covered by each beam in water in the laboratory reference frame is equal to $2L$.



3.4 Find the number of bands ΔN the interference pattern is shifted by when the liquid velocity changes from 0 to v and express it in terms of L, n, v, c, and λ.

In his own experiment A. Fizeau obtained $\Delta N = 0.230$ at $L = 1.49$ m, $v = 7.06$ m/s and $\lambda = 536$ nm.

3.5 Evaluate refractive index of water n in Fizeau's experiment.

Mathematical hint

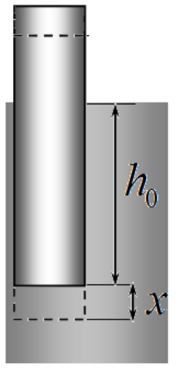
You may need to know the following approximate equality: $(1+x)^\alpha \approx 1 + \alpha x, \quad \text{at } x \ll 1.$

SOLUTIONS TO THE PROBLEMS OF THE THEORETICAL COMPETITION

Attention. Points in grading are not divided!

Problem 1 (10.0 points)

Problem A (3.0 points)



A1. When the test-tube is immersed to the depth x, it experiences the Archimedes' force and the force of gravity. Therefore, the equation of Newton's second law for the test-tube has the form

$$ma = mg - \rho S_0 (h_0 + x) g\ . \quad (1)$$

Here m is the mass of the test-tube and ρ stands for the water density.

In the equilibrium position, the following condition holds

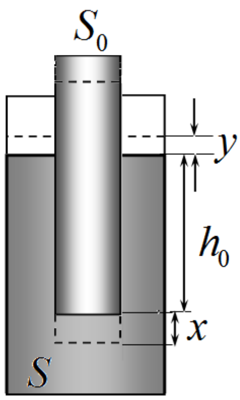
$$mg = \rho S_0 h_0 g\ . \quad (2)$$

It is thus immediately obtained that

$$a = -\frac{g}{h_0} x\ . \quad (3)$$

This is the equation of harmonic oscillations with the period

$$T = 2\pi \sqrt{\frac{h_0}{g}}\ . \quad (4)$$



A2.1 When the test-tube is lowered to the depth x, its potential energy is reduced by an amount

$$\Delta U_1 = -mgx\ . \quad (5)$$

If the test-tube is lowered to the depth x, the water level in the vessel rises to a height y that satisfies the condition (the condition of constancy of the water volume)

$$S_0 x = (S - S_0) y \quad \Rightarrow \quad y = \frac{S_0}{S - S_0} x\ . \quad (6)$$

Consequently, the water that was under the test tube rises above the original water level in the vessel. The mass of this water is found as

$$\Delta m = \rho S_0 x\ , \quad (7)$$

Its center of mass rises to a height

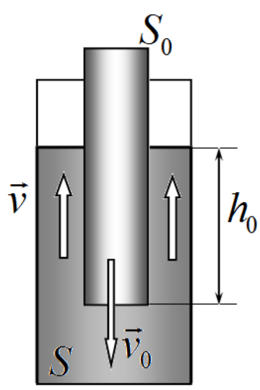
$$\Delta h_C = h_0 + \frac{1}{2}(x + y) = h_0 + \frac{1}{2}\left(x + \frac{S_0}{S - S_0} x\right) = h_0 + \frac{1}{2}\frac{S}{S - S_0} x\ . \quad (8)$$

The change in the potential energy of water is derived as

$$\Delta U_2 = \Delta m g \Delta h_C = \rho S_0 x g \left(h_0 + \frac{1}{2}\frac{S}{S - S_0} x\right). \quad (9)$$

The total change in the potential energy (with relation (2)) is finally obtained as:

$$\Delta U = \Delta U_1 + \Delta U_2 = \frac{1}{2}\frac{S_0 S}{S - S_0} \rho g x^2\ . \quad (10)$$



A2.2 If the tube drops with the velocity v_0, then the water between the walls and the test-tube rises at the speed of

$$v_0 S_0 = v(S - S_0) \quad \Rightarrow \quad v = \frac{S_0}{S - S_0} v_0\ . \quad (11)$$

The mass of rising water reads as

$$m_1 = \rho(S - S_0)h_0 \tag{12}$$

The total kinetic energy of the test-tube and the rising water is equal to

$$K = \frac{mv_0^2}{2} + \frac{m_1 v^2}{2} = \rho S_0 h_0 \frac{v_0^2}{2} + \rho(S - S_0)h_0 \frac{1}{2}\left(\frac{S_0}{S - S_0} v_0\right)^2 = \frac{1}{2}\frac{S_0 S}{S - S_0}\rho h_0 v_0^2 . \tag{13}$$

A2.3 The equation of the law of conservation of energy for the system under consideration is written as

$$\frac{1}{2}\frac{S_0 S}{S - S_0}\rho h_0 v_0^2 + \frac{1}{2}\frac{S_0 S}{S - S_0}\rho g x^2 = E = const . \tag{14}$$

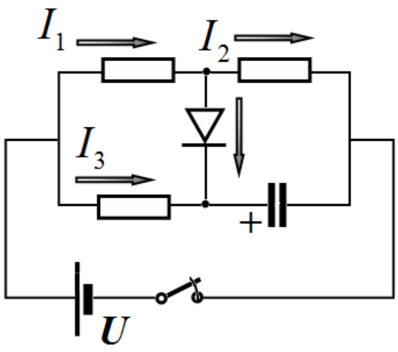
This equation is also an equation of harmonic oscillations with the same period

$$T = 2\pi\sqrt{\frac{h_0}{g}} . \tag{15}$$

Part	Content	Points	
A1	Formula (1) $ma = mg - \rho S_0(h_0 + x)g$	0,2	**0,8**
	Formula (2) $mg = \rho S_0 h_0 g$	0,2	
	Formula (3) $a = -\frac{g}{h_0}x$	0,2	
	Formula (4) $T = 2\pi\sqrt{\frac{h_0}{g}}$	0,2	
A2.1	Formula (5) $\Delta U_1 = -mgx$	0,2	**1,2**
	Formula (6) $S_0 x = (S - S_0)y \quad \Rightarrow \quad y = \frac{S_0}{S - S_0}x$	0,2	
	Formula (7) $\Delta m = \rho S_0 x$	0,2	
	Formula (8) $\Delta h_C = h_0 + \frac{1}{2}(x + y) = h_0 + \frac{1}{2}\left(x + \frac{S_0}{S - S_0}x\right) = h_0 + \frac{1}{2}\frac{S}{S - S_0}x$	0,2	
	Formula (9) $\Delta U_2 = \Delta mg\Delta h_C = \rho S_0 xg\left(h_0 + \frac{1}{2}\frac{S}{S - S_0}x\right)$	0,2	
	Formula (10) $\Delta U = \Delta U_1 + \Delta U_2 = \frac{1}{2}\frac{S_0 S}{S - S_0}\rho g x^2$	0,2	
A2.2	Formula (11) $v_0 S_0 = v(S - S_0) \quad \Rightarrow \quad v = \frac{S_0}{S - S_0}v_0$	0,2	**0,6**
	Formula (12) $m_1 = \rho(S - S_0)h_0$	0,2	
	Formula (13) $K = \frac{mv_0^2}{2} + \frac{m_1 v^2}{2} = \rho S_0 h_0 \frac{v_0^2}{2} + \rho(S - S_0)h_0 \frac{1}{2}\left(\frac{S_0}{S - S_0}v_0\right)^2 =$ $= \frac{1}{2}\frac{S_0 S}{S - S_0}\rho h_0 v_0^2$	0,2	
A2.3	Formula (14) $\frac{1}{2}\frac{S_0 S}{S - S_0}\rho h_0 v_0^2 + \frac{1}{2}\frac{S_0 S}{S - S_0}\rho g x^2 = E = const$	0,2	**0,4**

	Formula (15) $T = 2\pi\sqrt{\frac{h_0}{g}}$	0,2	
Total			**3,0**

Problem B (4.0 points)



Let I_k be the current through the resistor number k (see Fig.), q_k be the charge that has flowed through it up to the moment of closing the diode, q be the charge that has flowed through the diode, and Q be the charge of the capacitor.

Immediately after shortening the switch, the voltage across the capacitor is zero, the is true for the second resistor. Thus, $I_2 = 0$ and the answer to the first question is simply found as

$$I_0 = I_1(0) = U/R = 1 \text{ mA}. \quad (1)$$

В момент, когда ток через диод станет нулевым, токи через первый и второй резисторы будут одинаковы, поэтому будут одинаковы и напряжения на них: $U_1 = U_2 = U/2$. Такое же напряжение будет на конденсаторе и его заряд в этот момент: At the moment when the current through the diode becomes zero, the currents through the first and second resistors are equal, therefore, the voltages across them are also equal: $U_1 = U_2 = U/2$. The same voltage is across on the capacitor and its charge at this moment:

$$Q = CU/2. \quad (2)$$

Kirchhoff's rules give:

$$q_1 = q + q_2, \quad (3)$$

$$q_3 + q = Q. \quad (4)$$

$$I_1R = I_3R,$$

$$q_1 = q_3, \quad (5)$$

$$U = I_1R + I_2R. \quad (6)$$

Integrating the last equation in time from 0 to τ, we obtain:

$$U\tau = q_1R + q_2R. \quad (7)$$

Solving the obtained set of equations, we obtain the final answer as

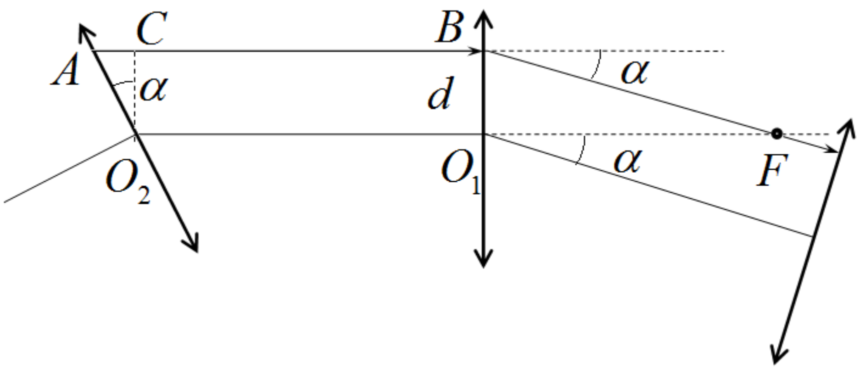
$$q = \frac{1}{3}CU\left(1 - \frac{\tau}{RC}\right) = 179 \text{ µCl}. \quad (8)$$

Content	Points
Formula (1) $I_0 = I_1(0) = U/R = 1$ mA	0.5
Numerical value $I_0 = 1$ mA	0.1
Formula (2) $Q = CU/2$	0.5
Formula (3) $q_1 = q + q_2$	0.5
Formula (4) $q_3 + q = Q$	0.5
Formula (5) $q_1 = q_3$	0.5
Formula (6) $U = I_1R + I_2R$	0.2
Formula (7) $U\tau = q_1R + q_2R$	0.5
Formula (8) $q = \frac{1}{3}CU\left(1 - \frac{\tau}{RC}\right)$	0.5
Numerical value $q = 179$ µCl	0.2
Total	**4.0**

Problem C (3.0 points)

Consider a ray AB passing parallel to one of the sides of the polygon. To describe a closed trajectory, it is necessary that, after refraction in the lens, the ray should run parallel to the next side. To do this, the ray must be deflected by an angle

$$\alpha = \frac{2\pi}{17}. \tag{1}$$

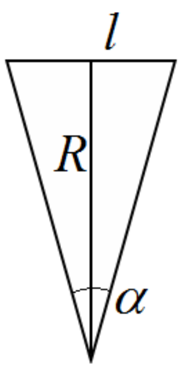


Since this ray is parallel to the optical axis, after the refraction it passes through the focus F. The required condition is satisfied by the ray moving at a distance

$$d = F\,tg\alpha \approx F\alpha \tag{2}$$

From the optical axes. Obviously, this ray propagates along the sides of the regular 17-gon, whose side length is equal to the length of the segment AB, or

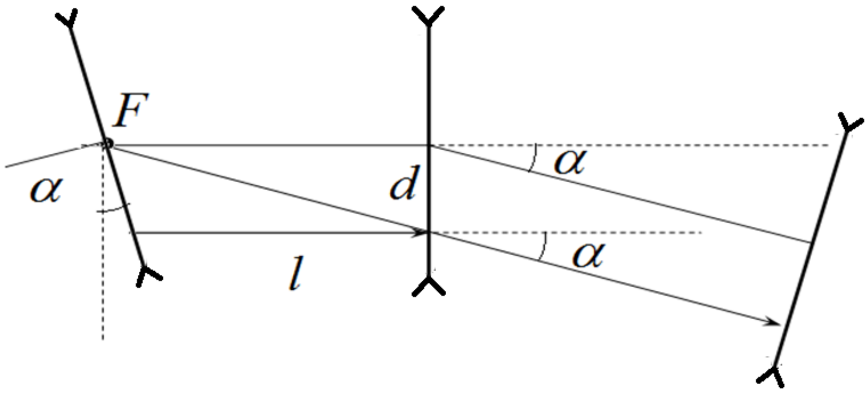
$$l = F + d\,tg\,\alpha = F(1+\alpha^2). \tag{3}$$



The radius of the circle, inscribed in this 17-gon, is finally found as

$$R = \frac{l}{2tg\frac{\alpha}{2}} = \frac{F\left(1+\alpha^2\right)}{\alpha} = 30{,}8\,sm. \tag{4}$$

For diverging lenses, the solution is similar, but we should only consider a ray that hits the lens below the optical axis.



In this case, the length of the side of the 17-gon, formed by the trajectory of the ray, is equal to

$$l = F - F\,tg^2\alpha = F\left(1-\alpha^2\right) \tag{5}$$

then, the radius of the inscribed circles found as

$$R = F\frac{1-\alpha^2}{\alpha} = 23{,}4\,см. \tag{6}$$

Content	Points
Formula (1) $\alpha = \dfrac{2\pi}{17}$	0,2
Formula (2) $d = F\alpha$	0,6
Formula (3) $l = F + d\,tg\,\alpha \approx F(1+\alpha^2)$	0,4
Formula (4) $R = \dfrac{l}{2tg\dfrac{\alpha}{2}} \approx \dfrac{F(1+\alpha^2)}{\alpha}$	0,4
α^2 is neglected	*(-0,2)*
Numerical value $R = 30{,}8\ sm$	0,2
Formula (5) $l = F - F\,tg^2\alpha \approx F(1-\alpha^2)$	0,6
Formula (6) $R \approx F\dfrac{1-\alpha^2}{\alpha}$	0,4
α^2 is neglected	*(-0,2)*
Numerical value $R = 23{,}4\,sm$	0,2
Total	**3,0**

Problem 2. Physics in the mountains (10,0 points)

Part 1. Isothermal atmosphere (3,2 points)

1.1 **[1,0 points]** The air pressure on the Earth's surface is caused by its gravity acting on the atmosphere, such that the equilibrium condition requires

$$p_0 S = Mg, \tag{1}$$

where

$$S = 4\pi R_E^2 \tag{2}$$

designates the Earth's surface.

From (1) and (2) one obtains

$$M = \frac{4\pi p_0 R_E^2}{g} = 5.32 \cdot 10^{18}\ \text{kg}. \tag{3}$$

1.2 **[1,0 points]** The pressure of the atmosphere varies with altitude due to the action of gravity on the gas. Let us consider the equilibrium of a layer of gas of thickness dh. The pressure difference dp at these altitudes must compensate for the gravitational forces of the gas layer of density ρ, which leads to the equation

$$dp = -\rho g dh. \tag{4}$$

On the other hand, from the equation of an ideal gas we find the relation between its density and pressure

$$\rho = \frac{\mu_{air} p}{RT_0}. \tag{5}$$

From expressions (4) and (5), we find that the pressure of the atmosphere at an altitude h is determined by the so-called barometric formula

$$p(h) = p_0 \exp\left(-\frac{\mu_{air} g}{RT_0} h\right) \tag{6}$$

and at the altitude of $H = 1500\ m$ it is equal to

$$p(H) = 85.0 \cdot 10^3\ \text{Pa}. \tag{7}$$

1.3 **[0,6 points]** In a homogeneous gravity field, the pressure of the atmosphere is determined by the mass of air above it, so the heating process can be considered isobaric, which means

$$\delta Q = \frac{M}{\mu_{air}} \frac{\gamma R}{\gamma - 1} \Delta T = 5.33 \cdot 10^{21}\ \text{J}, \tag{8}$$

where the adiabatic index of the diatomic gas is

$$\gamma = 7/5. \tag{9}$$

1.4 **[0,6 points]** For the time interval τ the amount of solar energy, absorbed by the Earth, is equal to

$$\delta Q = \alpha\pi R_E^2\tau \tag{10}$$

and the time interval sought is obtained as

$$\tau = \frac{M}{\alpha\pi R_E^2\mu_{air}}\frac{\gamma R\Delta T}{\gamma-1} = 30.3\cdot 10^3 \text{ s.} \tag{11}$$

Part 2. Adiabatic atmosphere (6,8 points)

2.1 **[1,2 points]** The temperature of the atmosphere does not remain constant with altitude, so equation (5) should be rewritten in the form

$$\rho = \frac{\mu_{air}p}{RT}. \tag{12}$$

Since the atmosphere is assumed adiabatic, one can write that

$$pT^{\frac{\gamma}{1-\gamma}} = const. \tag{13}$$

Solving together equations (4), (12) and (13) yields

$$\frac{dT}{dh} = -\frac{(\gamma-1)\mu_{air}g}{\gamma R} = -\beta = const. \tag{14}$$

Formula (14) proves that the temperature of the adiabatic atmosphere decreases with altitude as

$$T(h) = T_0 - \frac{(\gamma-1)\mu_{air}g}{\gamma R}h = T_0 - \beta h \tag{15}$$

and is found at $H = 1500\ m$ to be equal

$$T(H) = 278 \text{ К.} \tag{16}$$

2.2 **[0,4 points]** The pressure distribution over the altitude is determined by the adiabatic equation (13)

$$p(h) = p_0\left(\frac{T_0}{T(h)}\right)^{\frac{\gamma}{1-\gamma}} = p_0\left(\frac{T_0}{T_0-\beta h}\right)^{\frac{\gamma}{1-\gamma}} \tag{17}$$

and is found at $H = 1500\ m$ to be equal

$$p(H) = 84.6\cdot 10^3\ Pa. \tag{18}$$

2.3 **[0,8 points]** Since the temperature of the upper part of the troposphere is fixed, it follows from (15) that its height is determined by the condition

$$T(h) = T_0 - \beta h = const. \tag{19}$$

Thus, the change in the height of the troposphere at daytime and nighttime is derived as

$$\Delta H_{atm} = \frac{\gamma R\Delta T_{dn}}{(\gamma-1)\mu_{air}g} = 2{,}05\cdot 10^3 \text{ m.} \tag{20}$$

2.4 **[0,6 points]** In the stated range of temperatures and pressures, one can approximate the saturated water vapor pressure by a linear function of the form

$$p(T) = p_1 + \frac{p_2-p_1}{T_2-T_1}(T-T_1). \tag{21}$$

The boiling of the liquid begins when the saturated vapor pressure is equalized with the external pressure of the atmosphere, which allows an intensive vaporization process to occur in the emerging bubbles. Equating expressions (18) and (21) gives rise to

$$T_{boil} = 368 \text{ К.} \tag{22}$$

2.5 **[0,8 points]** The melting point of ice varies little with the external pressure, so snow appears when the temperature reaches 0 °C, i.e.

$$T_{melt} = 273 \text{ К.} \tag{23}$$

Consequently, using formula (15), we determine the altitude at which the snow cover appears as

$$h_0 = \frac{\gamma R(T_0 - T_{melt})}{(\gamma-1)\mu_{air}g} = 2.05\cdot 10^3 \text{ m.} \tag{24}$$

2.6 **[0,4 points]** If the air at the foot of the mountain is quite hot, then the temperature over the entire mountain slope cannot fall to zero degrees Celsius. Then, formula (24) provides the height of the mountain to be

$$H_0 = \frac{\gamma R(T - T_{melt})}{(\gamma-1)\mu_{air}g} = 3.78\cdot 10^3 \text{ m.} \tag{25}$$

2.7 **[2,0 points]** Since the water vapor is in thermodynamic equilibrium with the surrounding air, their temperatures are equal at all altitudes. The equilibrium condition for the vapor is written analogously to (4) as

$$dp_{vap} = -\rho_{vap} g dh, \tag{26}$$

and its density is obtained from the ideal gas equation of state in the following form

$$\rho_{vap} = \frac{\mu_{H_2O} p_{vap}}{RT}, \tag{27}$$

in which the temperature dependence on the altitude is governed by formula (15).

By formulation, the pressure of unsaturated water vapor at the foot of the mountain reads as

$$p_{vap}(0) = \varphi p_{vap0}, \tag{28}$$

whereas the saturated vapor pressure at the altitude H' is denoted as

$$p_{vap}(h) = p_{vap}. \tag{29}$$

Integrating equation (25) with the aid of (26) and (15) and initial conditions (28) and (29), it is found that

$$\ln \frac{p_{vap}}{p_{vap0}} = \ln \varphi + \frac{\mu_{H_2O} g}{\beta R} \ln \frac{T}{T_0}. \tag{30}$$

On the other hand, it is known from the handbook that

$$\ln \frac{P_{vap}}{P_{vap0}} = a + b \ln \frac{T}{T_0}, \tag{31}$$

and solving it together with (30) provides the following temperature at the altitude H'

$$T(H') = T_0 \exp\left(\frac{a - \ln \varphi}{\frac{\mu_{H_2O} g}{\beta R} - b}\right). \tag{32}$$

Then, the altitude itself is delivered by formula (15) as

$$H' = \frac{T_0 - T(H')}{\beta} = \frac{T_0}{\beta}\left(1 - \exp\left(\frac{a - \ln \varphi}{\frac{\mu_{H_2O} g}{\beta R} - b}\right)\right) = 2.55 \cdot 10^3 \text{ m}. \tag{33}$$

2.8 **[0,6 points]** For the fog to be absent on the mountain, one has to put in formula (33)

$$H' = H_0, \tag{34}$$

from which we obtain the desired expression for the air humidity

$$\varphi_{min} = \left(1 - \frac{\beta H_0}{T_0}\right)^{b - \frac{\mu_{H_2O} g}{\beta R}} \exp a = 0.119. \tag{35}$$

	Content	Points	
1.1	Formula (1) $p_0 S = Mg$	0,4	**1,0**
	Formula (2) $S = 4\pi R_E^2$	0,2	
	Formula (3) $M = \frac{4\pi p_0 R_E^2}{g}$	0,2	
	Correct numerical value $M = 5.32 \cdot 10^{18}$ kg	0,2	
1.2	Formula (4) $dp = -\rho g dh$	0,2	**1,0**
	Formula (5) $\rho = \frac{\mu_{air} p}{RT_0}$	0,2	
	Formula (6) $p(h) = p_0 \exp\left(-\frac{\mu_{air} g}{RT_0} h\right)$	0,4	
	Correct numerical value $p(H) = 85.0 \cdot 10^3$ Pa	0,2	
1.3	Formula (8) $\delta Q = \frac{M}{\mu_{air}} \frac{\gamma R}{\gamma - 1} \Delta T$	0,2	**0,6**
	Correct numerical value $\delta Q = 5.33 \cdot 10^{21}$ J	0,2	
	Formula (9) $\gamma = 7/5$ or equivalent $C_P = 7/2R$	0,2	
1.4	Formula (10) $\delta Q = \alpha \pi R_E^2 \tau$	0,2	**0,6**
	Formula (11) $\tau = \frac{M}{\alpha \pi R_E^2 \mu_{air}} \frac{\gamma R \Delta T}{\gamma - 1}$	0,2	
	Correct numerical value $\tau = 30.3 \cdot 10^3$ s	0,2	

2.1	Formula (12) $\rho = \frac{\mu_{air}p}{RT}$	0,2	1,2
	Formula (13) $pT^{\frac{\gamma}{1-\gamma}} = const$	0,2	
	Formula (14) $\frac{dT}{dh} = -\frac{(\gamma-1)\mu_{air}g}{\gamma R} = -\beta = const$	0,4	
	Formula (15) $T(h) = T_0 - \frac{(\gamma-1)\mu_{air}g}{\gamma R}h = T_0 - \beta h$	0,2	
	Correct numerical value $T(H) = 278$ K	0,2	
2.2	Formula (17) $p(h) = p_0\left(\frac{T_0}{T(h)}\right)^{\frac{\gamma}{1-\gamma}} = p_0\left(\frac{T_0}{T_0-\beta h}\right)^{\frac{\gamma}{1-\gamma}}$	0,2	0,4
	Correct numerical value $p(H) = 84.6 \cdot 10^3$ Pa	0,2	
2.3	Formula (19) $H_{atm} = \frac{\gamma R T_0}{(\gamma-1)\mu_{air}g}$	0,4	0,8
	Formula (20) $\Delta H_{atm} = \frac{\gamma R \Delta T_{dn}}{(\gamma-1)\mu_{air}g}$	0,2	
	Correct numerical value $\Delta H_{atm} = 2{,}05 \cdot 10^3\ m$	0,2	
2.4	Formula (21) $p(T) = p_1 + \frac{p_2-p_1}{T_2-T_1}(T-T_1)$	0,4	0,6
	Correct numerical value $T_{boil} = 368$ K	0,2	
2.5	Formula (23) $T_{melt} = 273$ K.	0,2	0,8
	Formula (24) $h_0 = \frac{\gamma R(T_0 - T_{melt})}{(\gamma-1)\mu_{air}g}$	0,4	
	Correct numerical value $h_0 = 2.05 \cdot 10^3 m$	0,2	
2.6	Formula (25) $H_0 = \frac{\gamma R(T - T_{melt})}{(\gamma-1)\mu_{air}g}$	0,2	0,4
	Correct numerical value $H_0 = 3.78 \cdot 10^3\ m$	0,2	
2.7	Formula (26) $dp_{vap} = -\rho_{vap}g\,dh$	0,2	2,0
	Formula (27) $\rho_{vap} = \frac{\mu_{H_2O}p_{vap}}{RT}$	0,2	
	Formula (28) $p_{vap}(0) = \varphi p_{vap0}$	0,2	
	Formula (30) $\ln\frac{p_{vap}}{p_{vap0}} = \ln\varphi + \frac{\mu_{H_2O}g}{\beta R}\ln\frac{T}{T_0}$	0,6	
	Formula (32) $T(H') = T_0 \exp\left(\frac{a-\ln\varphi}{\frac{\mu_{H_2O}g}{\beta R}-b}\right)$	0,2	
	Formula (33) $H' = \frac{T_0 - T(H')}{\beta} = \frac{T_0}{\beta}\left(1-\exp\left(\frac{a-\ln\varphi}{\frac{\mu_{H_2O}g}{\beta R}-b}\right)\right)$	0,4	
	Correct numerical value $H' = 2.55 \cdot 10^3\ m$	0,2	
2.8	Formula (34) $H' = H_0$	0,2	0,6
	Formula (35) $\varphi_{max} = \left(1-\frac{\beta H_0}{T_0}\right)^{b-\frac{\mu_{H_2O}g\beta}{R}} \exp a$	0,2	
	Correct numerical value $\varphi_{max} = 0.119$	0,2	
Total			**10,0**

Problem 3. Optics of moving media (10.0 points)

Part 1. 4-dimensional vectors (1,4 points)

1.1 **[0,8 points]** To bring the momentum and the energy to the same unit it is sufficient to divide the energy by the speed of light or to multiply the momentum by the speed of light. Moreover, by virtue of the principle of relativity, it is necessary to make the substitution $V \to -V$. Thus, one gets

$$p_x' = \frac{p_x - (V/c)(E/c)}{\sqrt{1-V^2/c^2}}, \tag{1}$$

$$p_y' = p_y, \tag{2}$$

$$p_z' = p_z, \tag{3}$$

$$E'/c = \frac{E/c - (V/c)p_x}{\sqrt{1 - V^2/c^2}}. \tag{4}$$

1.2 **[0,6 points]** In any inertial frame of reference the expression for the momentum is written as

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}, \tag{5}$$

and the expression for the total energy has the form

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}. \tag{6}$$

This implies that the invariant sought is equal to

$$inv = E^2 - p^2c^2 = m^2c^4. \tag{7}$$

Part 2. Doppler effect and light aberration (4,6 points)

2.1 **[1,0 points]** Since the rest mass of photons is zero, it follows from (7) that the momentum and energy of a photon are related as follows

$$p = \frac{E}{c}. \tag{8}$$

It is known that the photon energy is given by the Planck formula as

$$E = \hbar\omega. \tag{9}$$

The photon momentum projections on the coordinate axes are written as

$$p_x = p\cos\varphi, \tag{10}$$

$$p_y = p\sin\varphi, \tag{11}$$

and on substituting into (B1.4), one finds

$$\omega' = \omega\frac{1 - V\cos\varphi/c}{\sqrt{1 - V^2/c^2}}. \tag{12}$$

This is the well known formula for the relativistic Doppler effect.

2.2 **[0,4 points]** It follows from (2), 8) and (9) that

$$\frac{\hbar\omega'}{c}\sin\varphi' = \frac{\hbar\omega}{c}\sin\varphi. \tag{13}$$

Using (12), it is merely found that

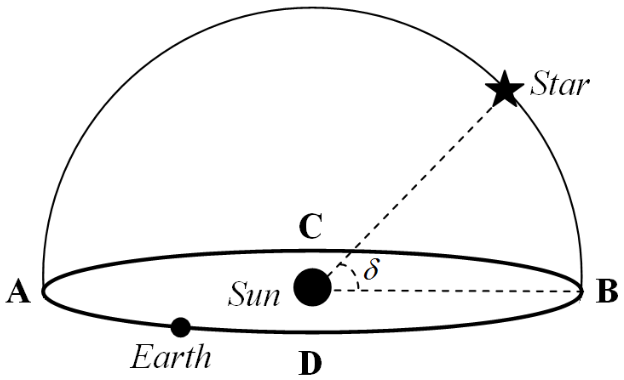
$$\sin\varphi' = \frac{\sqrt{1 - V^2/c^2}\sin\varphi}{1 - V\cos\varphi/c}. \tag{14}$$

Expression (14) is a classical formula for the light aberration.

2.3 **[1,0 points]** The position of the star on the celestial sphere varies throughout the year due to the orbital motion of the Earth around the Sun and the aberration of light which is schematically shown in the figure on the right. Since the speed of Earth's orbital motion is much less than the speed of light, it follows from (14) that the aberration angle is equal to

$$\delta\varphi = \varphi' - \varphi \approx \frac{V}{c}\sin\varphi, \tag{15}$$

where φ denotes the angle between V and the direction towards the star.



The figure shows that the angle φ varies periodically from a minimum value δ at the point D, reaches the value of $\pi/2$ at the point B, has a maximum value of $\pi-\delta$ at point C, and finally becomes equal to $\pi/2$ at point A. Hence, one can infer that the star apparent position on the celestial sphere moves along an ellipse with angular dimensions of the semi-axes

$$a_1 = \frac{V}{c} \tag{16}$$

and

$$a_2 = \frac{V}{c}\sin\delta\,. \tag{17}$$

It is found from the given data that

$$\delta = \arcsin\left(\frac{a_2}{a_1}\right) = 64.2^\circ\,. \tag{18}$$

2.4 **[2,2 points]** According to formula (12) for the Doppler effect the relative frequency shift at $\varphi = 0$ is found to be

$$\left(\frac{\Delta\omega}{\omega}\right)_D = 1-\sqrt{\frac{1-v_X/c}{1+v_X/c}} \approx 9.95\times10^{-3}\,. \tag{19}$$

This shows that the Doppler effect cannot fully explain the red shift in the spectrum of the star. It is natural to assume that when the light leaves the surface of the star the photon frequency decreases due to the gravitational redshift.

The gravitational mass is found from the principle of equivalence as

$$m_{ph} = \frac{\hbar\omega}{c^2}, \tag{20}$$

and the gravity force, acting on the photon at a distance r from the star, is equal, according to the Newton law, to

$$F = G\frac{m_{ph}M}{r^2}\,. \tag{21}$$

The energy conservation law for the motion of the photon can be written as

$$\hbar d\omega = -Fdr\,. \tag{22}$$

Thus,

$$\frac{d\omega}{\omega} = -\frac{GM}{c^2}\frac{dr}{r^2}\,. \tag{23}$$

On integrating (B4.5) in the range of the stellar radius R до ∞ leads to the following equation

$$\ln\left(\frac{\omega}{\omega_0}\right) = -\frac{GM}{c^2R}\,, \tag{24}$$

where ω_0 and ω stand for the frequencies of the photon on the stellar surface and at infinite distance from it, respectively.

Hence, the frequency of the photon at infinite distance from the star is obtained as

$$\omega = \omega_0 \exp\left(-\frac{GM}{c^2 R}\right) = \omega_0 \exp\left(-\frac{v_{II}^2}{2c^2}\right), \tag{25}$$

where the escape velocity is determined by the classical expression

$$v_{II} = \sqrt{\frac{2GM}{R}}\,. \tag{26}$$

Combining formulas (19) and (25) yields

$$\left(\frac{\Delta\omega}{\omega}\right)_0 = 1 - \exp\left(-\frac{v_{II}^2}{2c^2}\right)\sqrt{\frac{1 - v_X / c}{1 + v_X / c}}\,. \tag{27}$$

On substituting numerical values, one gets

$$v_{II} = \sqrt{2\ln\left(\frac{\sqrt{\frac{1 - v_X / c}{1 + v_X / c}}}{1 - \left(\frac{\Delta\omega}{\omega}\right)_0}\right)}\,c = 2.83 \cdot 10^6 \text{ m/s}\,. \tag{28}$$

Part C. Light in a moving medium (4,0 points)

3.1 **[1,1 points]** By definition, the projections of the object velocity in the reference frame S' are defined as expressions

$$u_x' = \frac{dx'}{dt'}, \tag{29}$$

$$u_y' = \frac{dy'}{dt'}. \tag{30}$$

The same projections in the reference frame S are given by

$$u_x = \frac{dx}{dt}, \tag{31}$$

$$u_y = \frac{dy}{dt}. \tag{32}$$

The Lorentz transformations can be rewritten in the form of finite differences as

$$dx = \frac{dx' + Vdt'}{\sqrt{1 - V^2 / c^2}}, \tag{33}$$

$$dy = dy', \tag{34}$$

$$dt = \frac{dt' + dx'V / c^2}{\sqrt{1 - V^2 / c^2}}. \tag{35}$$

On dividing term by term the left and right hand sides of (33)-(35) and using (29)-(32) yields

$$u_x = \frac{u_x' + V}{1 + \frac{u_x' V}{c^2}}, \tag{36}$$

$$u_y = \frac{\sqrt{1 - V^2 / c^2}}{1 + \frac{u_x' V}{c^2}} u_y'. \tag{37}$$

3.2 **[1,4 points]** Let us sit in the reference frame associated with the water. According to formula (14) the light aberration appears in this reference frame, whereby making the angle α' of incidence of a plane wave on the water surface equal to

$$\cos\alpha' = \frac{\sqrt{1-V^2/c^2}\cos\alpha}{1-V\sin\alpha/c} \approx \cos\alpha\left(1+V\sin\alpha/c\right) \qquad \text{or}$$

$$\sin\alpha' = \frac{\sin\alpha - V/c}{1-V\sin\alpha/c} \approx \sin\alpha - V\cos\alpha^2/c. \tag{38}$$

In the reference frame associated with the water flow, the refraction law has a usual form

$$\sin\alpha' = n\sin\beta', \tag{39}$$

and the speed of light propagation is

$$v_{ph} = \frac{c}{n}. \tag{40}$$

Going back to the laboratory reference frame with the aid of (36) and (37) one finds

$$v_m \sin\beta = \frac{v_{ph}\sin\beta' + V}{1+\frac{v_{ph}V\sin\beta'}{c^2}} \approx v_{ph}\sin\beta' + V, \tag{41}$$

$$v_m \cos\beta = \frac{\sqrt{1-V^2/c^2}}{1+\frac{v_{ph}V\sin\beta'}{c^2}} v_{ph}\cos\beta' \approx v_{ph}\cos\beta'. \tag{42}$$

Using (38)-(42), it is finally obtained that

$$\sin\beta \approx \frac{1}{n}\sin\alpha - \frac{n^2+\cos 2\alpha}{n}\frac{V}{c}, \tag{43}$$

making

$$A_1 = \frac{1}{n}\sin\alpha, \tag{44}$$

$$B_1 = -\frac{n^2+\cos 2\alpha}{n}. \tag{45}$$

3.3 **[0,4 points]** Again using (38)-(42) yields

$$v_m \approx \frac{c}{n} + V\left(1-\frac{1}{n^2}\right)\sin\beta. \tag{46}$$

whereby

$$A_2 = \frac{c}{n}, \tag{47}$$

$$B_2 = \left(1-\frac{1}{n^2}\right)\sin\beta. \tag{48}$$

3.4 **[0,9 points]** When the light propagates in the direction of the water flow, the angle β in formula (48) should be taken $\pi/2$ and the corresponding speed is found as

$$v_+ = \frac{c}{n} + V\left(1-\frac{1}{n^2}\right), \tag{49}$$

when the light propagates in the direction opposite to the water flow, the corresponding speed is obtained as

$$v_- = \frac{c}{n} - V\left(1-\frac{1}{n^2}\right). \tag{50}$$

Since the total path covered by the two light beams in water is $2L$, the difference in their propagation time Δt is equal to

$$\Delta t = \frac{2L}{v_-} - \frac{2L}{v_+} \approx \frac{4Lv\left(n^2-1\right)}{c^2}, \tag{51}$$

and the corresponding path difference is derived as follows

$$\Delta l = c\Delta t = \frac{4Lv\left(n^2-1\right)}{c}. \quad (52)$$

Thus, the interference pattern is shifted by the number of bands equal to

$$\Delta N = \frac{\Delta l}{\lambda} = \frac{4Lv\left(n^2-1\right)}{c\lambda}. \quad (53)$$

3.5 **[0,2 points]** Using formula (53) the water refraction index is found to be

$$n = \sqrt{1+\frac{c\lambda\Delta N}{4Lv}} = 1.37. \quad (54)$$

Part	Content	Points	
1.1	Formula (1) $p_x' = \frac{p_x-(V/c)(E/c)}{\sqrt{1-V^2/c^2}}$	0,2	**0,8**
	Formula (2) $p_y' = p_y$	0,2	
	Formula (3) $p_z' = p_z$	0,2	
	Formula (4) $E'/c = \frac{E/c-(V/c)p_x}{\sqrt{1-V^2/c^2}}$	0,2	
1.2	Formula (5) $p = \frac{mv}{\sqrt{1-v^2/c^2}}$	0,2	**0,6**
	Formula (6) $E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$	0,2	
	Formula (7) $inv = E^2 - p^2c^2 = m^2c^4$	0,2	
2.1	Formula (8) $p = \frac{E}{c}$	0,2	**1,0**
	Formula (9) $E = h\omega$	0,2	
	Formula (10) $p_x = p\cos\varphi$	0,2	
	Formula (11) $p_y = p\sin\varphi$	0,2	
	Formula (12) $\omega' = \omega\frac{1-V\cos\varphi/c}{\sqrt{1-V^2/c^2}}$	0,2	
2.2	Formula (13) $\frac{h\omega'}{c}\sin\varphi' = \frac{h\omega}{c}\sin\varphi$	0,2	**0,4**
	Formula (14) $\sin\varphi' = \frac{\sqrt{1-V^2/c^2}\sin\varphi}{1-V\cos\varphi/c}$	0,2	
2.3	Formula (15) $\delta\varphi = \varphi' - \varphi \approx \frac{V}{c}\sin\varphi$	0,2	**1,0**
	Formula (16) $a_1 = \frac{V}{c}$	0,2	
	Formula (17) $a_2 = \frac{V}{c}\sin\delta$	0,2	
	Formula (18) $\delta = \arcsin\left(\frac{a_2}{a_1}\right)$	0,2	
	Numerical value $\delta = 64.2°$	0,2	

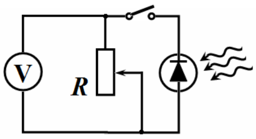
2.4	Formula (19) $\left(\frac{\Delta\omega}{\omega}\right)_D = 1-\sqrt{\frac{1-v_X/c}{1+v_X/c}} \approx 9.95\times10^{-3}$	0,2	2,2
	Formula (20) $m_{ph} = \frac{\hbar\omega}{c^2}$	0,2	
	Formula (21) $F = G\frac{m_{ph}M}{r^2}$	0,2	
	Formula (22) $\hbar d\omega = -Fdr$	0,2	
	Formula (23) $\frac{d\omega}{\omega} = -\frac{GM}{c^2}\frac{dr}{r^2}$	0,2	
	Formula (24) $\ln\left(\frac{\omega}{\omega_0}\right) = -\frac{GM}{c^2R}$	0,2	
	Formula (25) $\omega = \omega_0\exp\left(-\frac{GM}{c^2R}\right) = \omega_0\exp\left(-\frac{v_{II}^2}{2c^2}\right)$	0,2	
	Formula (26) $v_{II} = \sqrt{\frac{2GM}{R}}$	0,2	
	Formula (27) $\left(\frac{\Delta\omega}{\omega}\right)_0 = \left[1-\sqrt{\frac{1-v_X/c}{1+v_X/c}}\right]\exp\left(-\frac{v_{II}^2}{2c^2}\right)$	0,2	
	Formula (28) $v_{II} = \sqrt{2\ln\left(\frac{1-\sqrt{\frac{1-v_X/c}{1+v_X/c}}}{\left(\frac{\Delta\omega}{\omega}\right)_0}\right)}\,c$	0,2	
	Numerical value $v_{II} = 7.108\times10^{-4}c = 21.31\text{km/s}$	0,2	
3.1	Formula (29) $u_x' = \frac{dx'}{dt'}$	0,1	1,1
	Formula (30) $u_y' = \frac{dy'}{dt'}$	0,1	
	Formula (31) $u_x = \frac{dx}{dt}$	0,1	
	Formula (32) $u_y = \frac{dy}{dt}$	0,1	
	Formula (33) $dx = \frac{dx'+Vdt'}{\sqrt{1-V^2/c^2}}$	0,1	
	Formula (34) $dy = dy'$	0,1	
	Formula (35) $dt = \frac{dt'+dx'V/c^2}{\sqrt{1-V^2/c^2}}$	0,1	
	Formula (36) $u_x = \frac{u_x'+V}{1+\frac{u_x'V}{c^2}}$	0,2	
	Formula (37) $u_y = \frac{\sqrt{1-V^2/c^2}}{1+\frac{u_x'V}{c^2}}u_y'$	0,2	

3.2	Formula (38) $\cos\alpha' = \dfrac{\sqrt{1-V^2/c^2}\cos\alpha}{1-V\sin\alpha/c} \approx \cos\alpha\left(1+V\sin\alpha/c\right)$ or $\sin\alpha' = \dfrac{\sin\alpha - V/c}{1-V\sin\alpha/c} \approx \sin\alpha - V\cos\alpha^2/c$	0,2	**1,4**
	Formula (39) $\sin\alpha' = n\sin\beta'$	0,2	
	Formula (40) $v_{ph} = \dfrac{c}{n}$	0,2	
	Formula (41) $v_m \sin\beta = \dfrac{v_{ph}\sin\beta' + V}{1+\dfrac{v_{ph}V\sin\beta'}{c^2}} \approx v_{ph}\sin\beta' + V$	0,2	
	Formula (42) $v_m \cos\beta = \dfrac{\sqrt{1-V^2/c^2}}{1+\dfrac{v_{ph}V\sin\beta'}{c^2}} v_{ph}\cos\beta' \approx v_{ph}\cos\beta'$	0,2	
	Formula (44) $A_1 = \dfrac{1}{n}\sin\alpha$	0,2	
	Formula (45) $B_1 = -\dfrac{n^2+\cos 2\alpha}{n}$	0,2	
3.3	Formula (47) $A_2 = \dfrac{c}{n}$	0,2	**0,4**
	Formula (48) $B_2 = \left(1-\dfrac{1}{n^2}\right)\sin\beta$	0,2	
3.4	Formula (49) $v_+ = \dfrac{c}{n} + V\left(1-\dfrac{1}{n^2}\right)$	0,2	**0,9**
	Formula (50) $v_- = \dfrac{c}{n} - V\left(1-\dfrac{1}{n^2}\right)$	0,2	
	Formula (51) $\Delta t = \dfrac{2L}{v_-} - \dfrac{2L}{v_+} \approx \dfrac{4Lv\left(n^2-1\right)}{c^2}$	0,2	
	Formula (52) $\Delta l = c\Delta t = \dfrac{4Lv\left(n^2-1\right)}{c}$	0,2	
	Formula (53) $\Delta N = \dfrac{\Delta l}{\lambda} = \dfrac{4Lv\left(n^2-1\right)}{c\lambda}$	0,1	
3.5	Formula (54) $n = \sqrt{1+\dfrac{c\lambda\Delta N}{4Lv}}$	0,1	**0,2**
	Numerical value $n = \sqrt{1+\dfrac{c\lambda\Delta N}{4Lv}} = 1.37$	0,1	
Total			**10,0**

SOLUTION TO THE EXPERIMENTAL COMPETITION

Absorption of light (15,0 балла)

Part 1. Studying a photodetector



1.1 The working measurement circuit is shown in the figure on the right. When the switch is shortened, the multimeter should be switched to the voltage measurement mode, i.e. the voltage across the resistor is measured. When the switch is open, the multimeter should be switched to the ohmmeter mode, the resistance is measured.

1.2 – 1.5 The results of the measurement of the voltage across the resistor as a function of its resistance are shown in Table 1. It also presents the results of calculations of the current using the formula

$$I = \frac{U}{R} \tag{1}$$

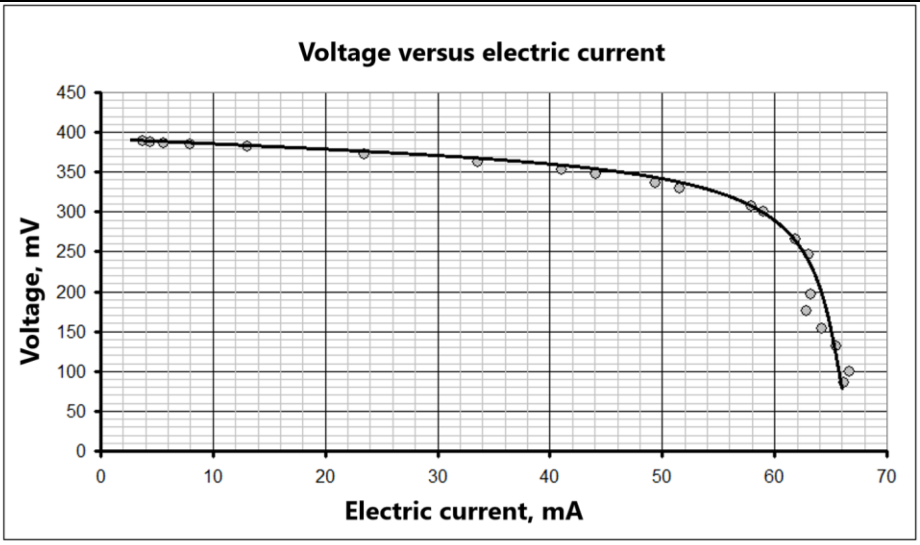
and the heat power generated in the resistor and calculated via the expression

$$P = UI = I^2 R = \frac{U^2}{R} \tag{2}$$

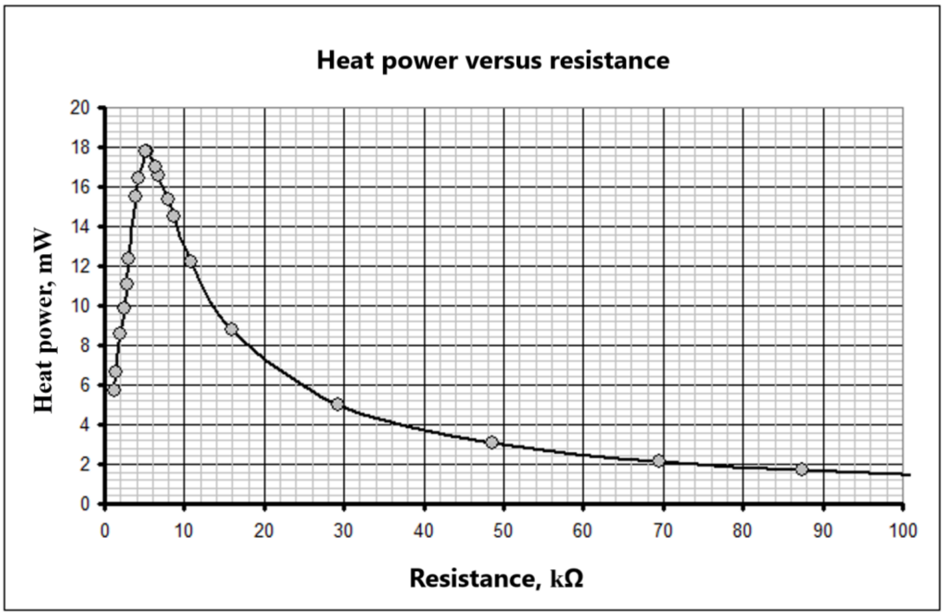
Table 1

R, $k\Omega$	U, mV	I, μA	P, μW
104	389	3,74	1,46
87,4	388	4,44	1,72
69,5	387	5,57	2,15
48,5	385	7,94	3,06
29,3	382	13,04	4,98
15,9	373	23,46	8,75
10,8	363	33,61	12,20
8,6	353	41,05	14,49
7,9	348	44,05	15,33
6,8	336	49,41	16,60
6,4	330	51,56	17,02
5,3	307	57,92	17,78
5,1	301	59,02	17,76
4,3	266	61,86	16,45
3,9	246	63,08	15,52
3,1	196	63,23	12,39
2,8	176	62,86	11,06
2,4	154	64,17	9,88
2	131	65,50	8,58
1,5	100	66,67	6,67
1,3	86	66,15	5,69

The graph of the voltage across the resistor versus its current is shown in the figure below.



The graph of the heat power in the resistor versus its resistance is shown in the figure below.



To determine the position of the maximum, it is necessary to take additional measurements in the range of 0 to 10 kΩ. According to the measurement results, it turns out that the maximum heat power generated in the resistor is reached at

$$R = 5,4k\Omega\,. \tag{3}$$

Part 2. Absorption of laser radiation

2.1 Since the transmittances of the same type of light filters are equal, and the laser radiation is monochromatic, the dependence of the transmitted light intensity on the number of filters has the form of a geometric progression

$$I_n = k^n I_0\,. \tag{4}$$

2.2 The results of measurements of the dependence of the light intensity on the number of filters are given in Table 2. The measurements have been carried out at the resistance of 3.3 kΩ.

Table 2

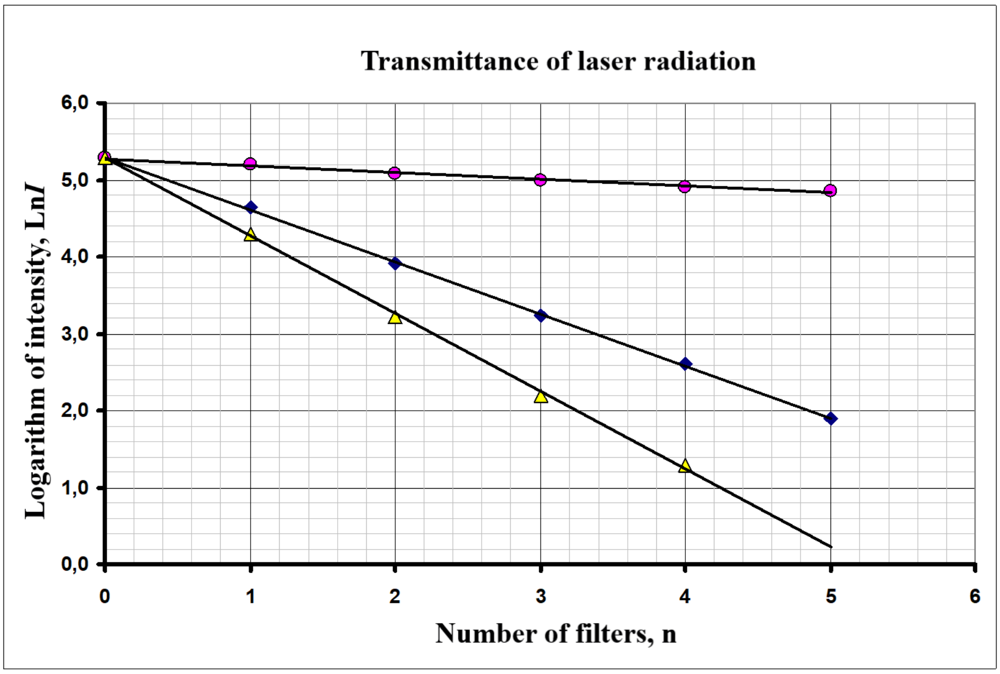
	grey filter	yellow filter	blue filter		grey filter	yellow filter	blue filter
n	*U, mV*	*U, mV*	*U, mV*	*n*	*Ln U*	*Ln U*	*Ln U*
0	196	197	197	0	5,28	5,28	5,28
1	104	180	74	1	4,64	5,19	4,30
2	50	161	24,9	2	3,91	5,08	3,21

3	25,4	148	9	3	3,23	5,00	2,20
4	13,6	135	3,6	4	2,61	4,91	1,28
5	6,6	128		5	1,89	4,85	

To verify formula (4) it is convenient to present it in the semi-log scale as

$$\ln I_n = \ln I_0 - n \ln k \,. \qquad (5)$$

At such a scale, the dependence of $\ln I_n$ on the number n of light filters should be linear. The measurements and calculations confirm this conclusion, i.e. formula (4) correctly describes the intensity of the transmitted light. The figure below shows the results obtained.



2.3 As it follows from formula (5), the slope of the dependence is equal to the logarithm of the transmittance

$$a = \ln k \,, \qquad (6)$$

therefore, the transmittance is calculated by the formula

$$k = \exp(a) \,, \qquad (7)$$

and its error is determined by the expression

$$\Delta k = \exp(a) \cdot \Delta a \,, \qquad (8)$$

Table 3 shows the results of calculations (using the least squares method) of the slope coefficients a, their errors Δa, the transmittances k and their errors Δk for all three types of light filters.

Table 3

	grey filter	yellow filter	blue filter
a	-0,678	-0,089	-1,011
Δa	0,014	0,008	0,032
k	0,508	0,915	0,364
Δk	0,007	0,007	0,012

2.4 The transmittance of a pair of filters for the laser monochromatic radiation is equal to the product of the transmittances of each filter:

$$k_{1,2} = k_1 \cdot k_2 \qquad (9)$$

Table 4 shows the measured and calculated transmittances for all pairs. As it follows from the data presented, there is a good agreement between these results, i.e. formula (9) is experimentally verified.

Table 4

Pair of filters		measured		calculated
blue + yellow		0,342		0,333
blue + grey		0,187		0,185
yellow + grey		0,460		0,464

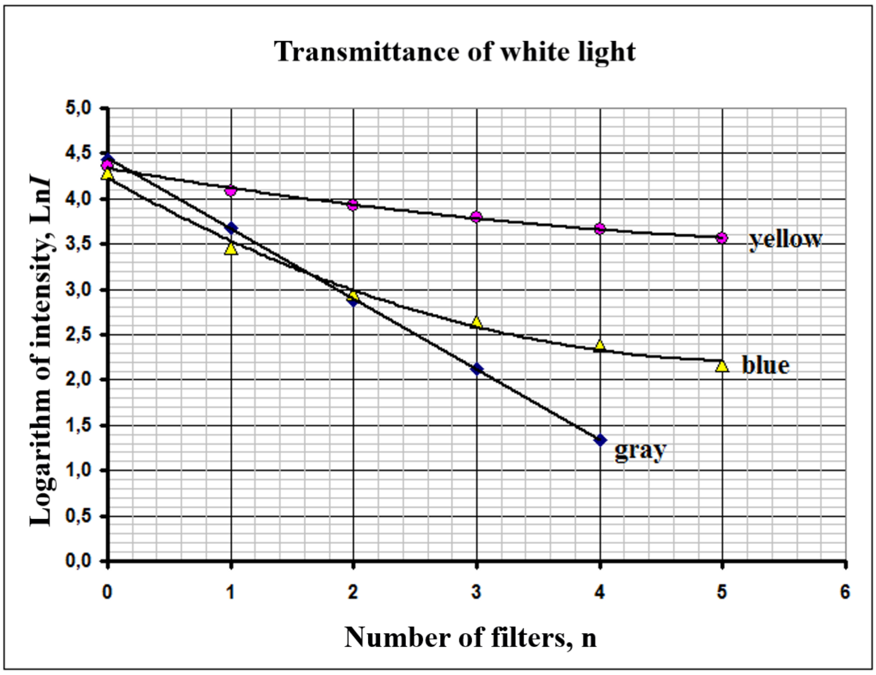
Part 3. Absorption of white light

3.1 The results of measurements of the light intensity transmitted through the filters are shown in Table 5. (the measurements have been made with the resistance of 10.1 kΩ).

Table 5

	grey filter	yellow filter	blue filter		grey filter	yellow filter	blue filter
n	*U, mV*	*U, mV*	*U, mV*	*n*	*Ln U*	*Ln U*	*Ln U*
0	84,8	77,8	72,6	0	4,44	4,35	4,28
1	39,4	59,4	31,7	1	3,67	4,08	3,46
2	17,9	50,8	19,2	2	2,88	3,93	2,95
3	8,4	44,5	14	3	2,13	3,80	2,64
4	3,8	39,2	10,8	4	1,34	3,67	2,38
5		35	8,7	5		3,56	2,16

The graphs of these dependencies are shown in the figure below.



3.2 It can be seen from the graph that formula (5) is correct for grey filters, but not for blue ones. The main reason for the violation of the obtained law is that the transmittance significantly depends on the wavelength of incident light. So, for grey filters, the transmittance varies little in the visible range of wavelength and formula (5) is applicable within the experimental error. For blue filters, though, the transmittance varies noticeably, such that formula (5) turns inapplicable.

3.3 The measured transmittances for the three light filters provided are given in Table 6.

Table 6

grey filter	0,450
blue filter	0,431
yellow filter	0,759

Table 7 shows the measured and calculated transmittances of pairs of light filters. It can be seen that for the white source the transmittance of a pair of filters is not equal to the product of transmittances of each filter, which is prescribed to the dependence of transmittances on the wavelength.

Table 7

Pair of filters		measured		calculated
blue + yellow		0,271		0,327
blue + grey		0,188		0,194
yellow + grey		0,362		0,342

Marking scheme

Part	Content	Total for part	Points
Part 1. Studying a photodetector		**6,0**	
1.1	Switch in the right place	**0,2**	0,2
1.2	Measurements $U(R)$: ***Marked only if data are within 50% from the provided in the official solution:*** - maximumal voltage no more than 350 mV; - maximumal resistance no more than 90 kΩ; - minimal resistamce less than 2 kΩ; - number of points 15 or more *(10-14, 7- 9, less)*; - not less than 5 points in the range of 0-10 kΩ; - monotonically dcreasing dependence; - units stated correctly (kΩ. mV);	**3,5**	 0,2 0,2 0,2 2(*1;0,5;0*) 0,5 0,2 0,1+0,1
1.3	Load characteristic: (***marked only if 1.2 had been marked)*** - formula to calculate electric current; - current calculated for all experimental points; - unit of current stated (μA); - correct qualitative behavior (slow decrease followed by sharp decline); Plotting a graph: - axis named and ticked; - all points correctly drawn; - smooth curve drawn;	**0,9**	 0,1 0,2 0,1 0,2 0,1 0,1 0.1
1.4	Heat power versus resistance: (***marked only if 1.2 had been marked)*** - formula to calculate heat power; - heat power calculated for all experimental points; - correct unit of heat power (μW); - correct qualitative behavior (maximum in the rage of 0-10 kΩ followed by slow decrease); Plotting a graph: - axis named and ticked; - all points correctly drawn; - smooth curve drawn;	**0,9**	 0,1 0,2 0,1 0.2 0,1 0,1 0,1
1.5	Maximum found In the range of 5-6 kΩ (*4-7* kΩ, *out of*)	**0,5**	0,5*(0,3;0)*
Part 2. Absorption of laser radiation		**5,5**	
2.1	Formula $I_n = k^n I_0$	**0,2**	0,2
2.2	Transmittance measurements	**2,6**	

	Marked only if data are within 50% from the provided in the official solution:		
	- Measurements in the range of 0-200 mV;		0,1
	- Initial measurement without filters;		0,1x3=0,3
	- measurements with 5 (4) filters;		0,3x3=0,9
	- decreasing dependence obtained;		0,1x3=0.3
	Linearization		
	- Correct semi-log scale;		0,2
	- logarithns calculated for all points;		0,2
	Plotting a graph **(marked only if measurements had been marked)**::		
	- axis named and ticked;		0,1
	- all points correctly drawn;		0,1
	- straight line drawn;		0.1
	Linear dependence obtained;		0,3
2.3	Formula for the transmittance $k = \exp(a)$	**1,4**	0,1
	Formula for the experimental error $\Delta k = \exp(a) \cdot \Delta a$;		0,1
	Calculations for all points (LSM, averaging);		0,3
	(Calculations for 2 points only);		*0,1*
	Transmittances found:		
	Gray in the range of 0,45-0,55 (*0,4 -0,6; out of*)		0,2*(0,1;0)*
	Yellow in the range of 0,85-0,95 (*0,8-0,98; out of*)		0,2*(0,1;0)*
	Blue in the range of 0,30-0,40 *(0,25-0,45; out of*)		0,2*(0,1;0)*
	Experimntal errors found		0,1x3=0,3
2.4	Measurement of transmittance	**1,3**	
	Gray+yellow in the range of 0,30-0,40 (*0,25-0,45; out of)*		0,2*(0,1;0)*
	Blue+gray in the range of 0,15-0,25 (*0,10-0,30; out of)*		0,2*(0,1;0)*
	Gray +yellow in the range of 0,40-0,50 *(0,35 – 0,55; out of);*		0,2*(0,1;0)*
	Formula $k_{1,2} = k_1 \cdot k_2$;		0,1
	Products found;		0,1x3=0,3
	Experimental results agree with the theory;		0.1x3=0,3
Part 3. Absorption of white light		**3,5**	
3.1	Transmittance measurements	**1,9**	
	Marked only if data are within 50% from the provided in the official solution:		
	- Measurements in the range of 0-100 mV;		0,1
	- Initial measurement without filters;		0,1x3=0,3
	- measurements with 5 (4) filters;		0,2x3=0,6
	- logarithns calculated for all points;		0,3
	Plotting a graph **(marked only if measurements are marked)**:		
	- axis named and ticked;		0,1
	- all points correctly drawn;		0,1
	- smooth curve drawn;		0,1
	Linear dependence for gray filter;		0,1
	Nonlinear dependence for blue filter		0,2
3.2	Theroretical formula not confirmed	**0,3**	0,1
	Reason: transmittance depends on the wavelength		0,2
3.3	Transmittance measurements	**1,3**	
	Gray filter in the range of 0,4-0,5 (0,35-0,55; out of)		0,2*(0,1;0)*
	Blue filter in the range of 0,4-0,5 (0,35-0,55; out of)		0,2*(0,1;0)*
	Yellow filter in the range of 0,7 – 0,8 (0,65 – 0,85; out of)		0,2*(0,1;0)*
	Blue+yellow filters in the range of 0,22-0,3 (0,18 -3,5;out of)		0,2*(0,1;0)*
	Blue+gray filters in the range of 0,14-0,23 (0,1 -0,27;out of)		0,2*(0,1;0)*

	Yellow+gray filters in the range of 0,3 – 0,4 (0,25 – 0,45; out of) Experimental and theoretical transmittances do not coincide		0,2*(0,1;0)* 0,1
	TOTAL	**15**	

EXPERIMENTAL COMPETITION

12 January, 2019

Please read the instructions first:

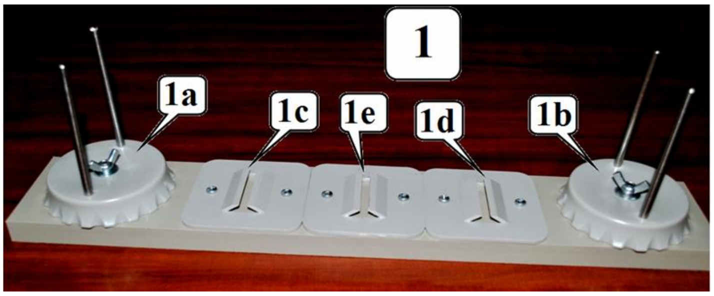
1. The Experimental competition consists of one problem. This part of the competition lasts 3 hours.
2. Please only use the pen that is provided to you.
3. You can use your own non-programmable calculator for numerical calculations. If you don’t have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet and additional papers***. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*.** Write only inside the bordered area.
6. Fill the boxes at the top of each sheet of paper with your country (***Country*),** your student code (***Student Code***), the question number (***Question Number***), the progressive number of each sheet **(*Page Number***), and the total number of ***Writing sheets*** (***Total Number of Pages***). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
7. At the end of the exam, arrange all sheets for each problem in the following order:
   - Used ***Writing sheets*** in order.
   - The sheets you do not wish to be evaluated.
   - Unused sheets.
   - The printed problems.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper or equipment out of the room

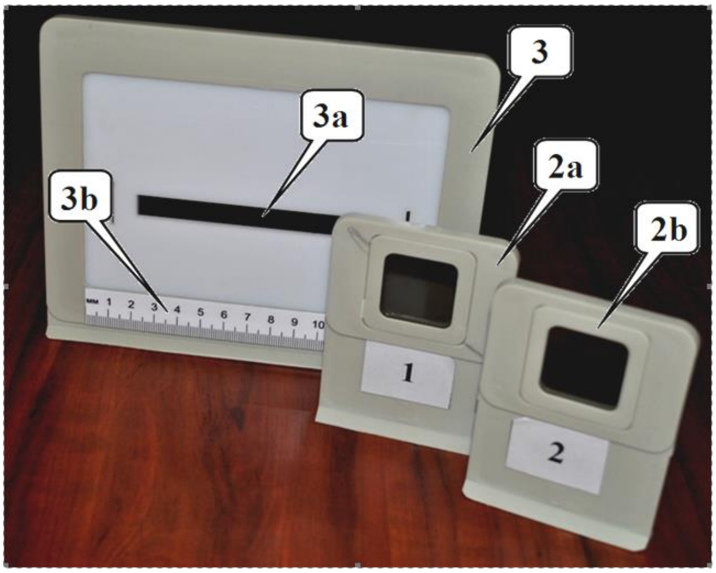
Absorption of light (15.0 points)

Experimental equipment

1. Optical bench with holders:
1a – support for a light source with the screw;
1b – support for a photodetector with the screw;
1c, 1d, 1e – holders;



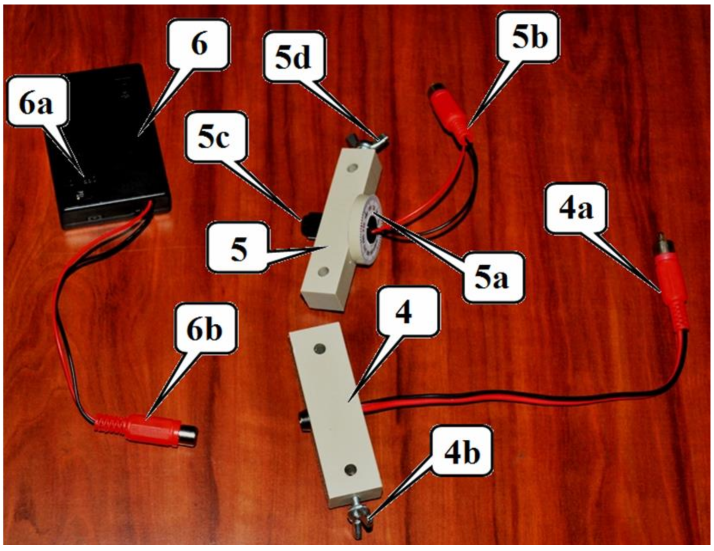
2a – neutral filter on the stand;
2b, 3, 3a, 3b – not used in this experiment;



Light sources:
4 – light emitting diode (LED):
4a – leads to a power supply;
4b – fixing screw;
5 – laser:
5a – ring for turning the laser with the scale;
5b – leads to a power supply;
5c –screw for the beam width adjustment on the front side;
5d – fixing screw;
6 – power supply for light sources:
6a – switch;
6b – leads to a light source.

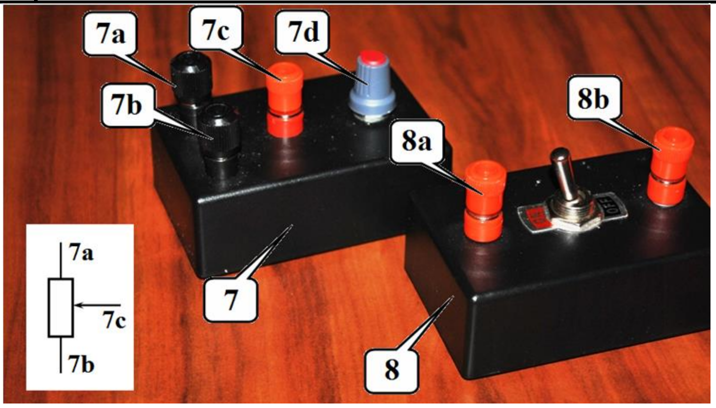
Keep the source operating only while making measurements!

Do not point the laser beam in your or anyone's eyes, it is very dangerous!



7 – variable resistor 100 kΩ:
7a, 7b, 7c – terminals for connection to a circuit;
7d – knob for changing the resistance;

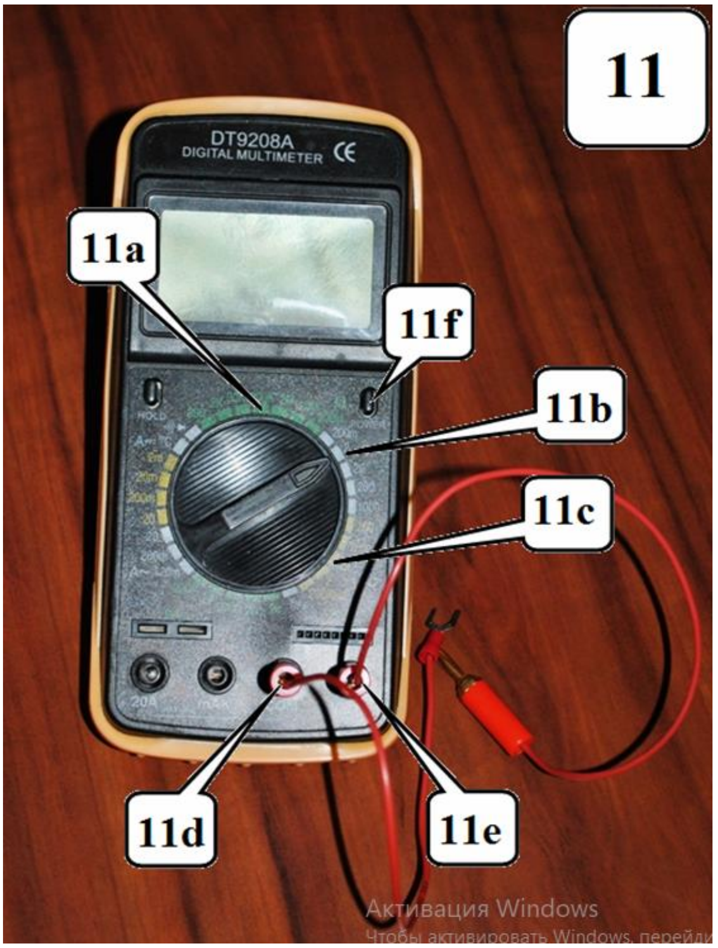
8 – switch:
8a, 8b – terminals for connection to a circuit;



11 – multimeter;
11a – register to measure resistance (100 kΩ);
11b – register to measure DC voltage (2V);
11d, 11e – connectors for test leads;
11f –power on/off.

If the display multimeter is in a "sleep" mode - double-press power on/off!

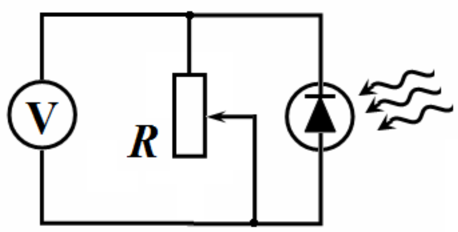
When measuring the resistance with the multimeter, it must be disconnected with a power supply!



A set of film filters: 5 grey filters, 5 blue filters, 5 yellow filters.
Clothes-peg.

Part 1. Studying a photodetector

To measure the light intensity the photodiode is used hereinafter. The emf, produced when the light falls on the photodiode, depends in a complex way on the intensity of the incident light. Therefore, the circuit, shown in the figure on the right, is used to measure the light intensity. The voltage measured by a multimeter depends on both the intensity of the incident light and the resistance of the variable resistor.



Install the photodetector and the laser on the optical bench. Direct the light beam exactly at the photodetector. Install a neutral filter in one of the holders. Rotating the laser ring, make sure that the voltage across the resistor (at its maximum resistance) is at least 350 mV.

1.1 Modify the circuit shown above by adding the switch, so that you can measure the resistance of the resistor and the voltage across it with the single multimeter provided. Draw schematically the electric circuit that you propose.

1.2 Using your electric circuit from 1.1, measure the dependence of the voltage across the resistor on its resistance.

1.3 Using the data from 1.2, draw a graph of the voltage across the resistor as a function of its current.

1.4 Draw a graph of the heat power generated in the resistor as a function of its resistance.

1.5 Determine the resistance of the resistor at which the heat power turns maximal. Make additional measurements, if necessary.

Part 2. Absorption of laser radiation

In this part of the experiment, make all measurements with the resistance of the variable resistor approximately equal to 3 kΩ. Write down the value of the resistance at which you have taken your measurements. Install the neutral filter on the optical bench between the laser and the photodetector. Turning the laser ring, make sure that the maximum voltage across the resistor is approximately 180-190 mV. Measure in the range of the multimeter 200mV. In this mode, the voltage on the multimeter is directly proportional to the intensity of the incident light. Hold the light filters with the clothes-peg and hold them directly near the photodetector.

The intensity of the light I transmitted through the filter is proportional to the intensity of the incident light I_0:

$$I = kI_0 \tag{1}$$

The transmittance k does not depend on the intensity of the incident light but may depend on its wavelength.

2.1 Using formula (1), obtain the dependence of the intensity of transmitted light through n identical light filters on the number n of light filters.

2.2 Measure the dependence of the light intensity I_n on the number of filters n for all the filters provided (gray, yellow and blue). Draw graphs of the dependencies obtained in scales that allow you to verify the formula obtained in Section 2.1.

2.3 Calculate the transmittance of all filters. Estimate the experimental errors of their values.

2.4 Measure the transmittance for pairs of different filters: gray + blue, gray + yellow; blue + yellow. Theoretically calculate the transmittance of each pair of filters.

Part 3. Absorption of white light

Replace the laser with the LED which is a source of white light. Remove the neutral filter. Direct the radiation of the LED straight at the photodetector. Set the resistance of the resistor to approximately 10 kΩ. Write down the value of the resistance at which you have taken your measurements.

3.1 Measure the dependence of the light intensity on the number of filters for all types of filters provided (gray, yellow and blue). Draw graphs of these dependencies in the same scale as in Part 2.

3.2 Verify whether the formula obtained in paragraph 2.1 is fulfilled for the white light source. Specify a main cause leading to possible violations of that relation.

3.3 Measure the transmittance for pairs of different filters: gray + blue, gray + yellow; blue + yellow. Using the data obtained in the whole experiment, verify whether the transmittance of a pair of filters is equal to the product of the transmittances of each filter.

THEORETICAL COMPETITION

January 11, 2019

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don’t have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*.** Write only inside the boxed area.
6. Begin each question on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Question Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
    - Used ***Writing sheets*** in order.
    - The sheets you do not wish to be evaluated.
    - Unused sheets.
    - The printed problems.

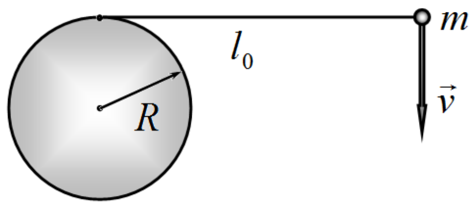
   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

Problem 1 (10.0 points)

This problem consists of three independent parts.

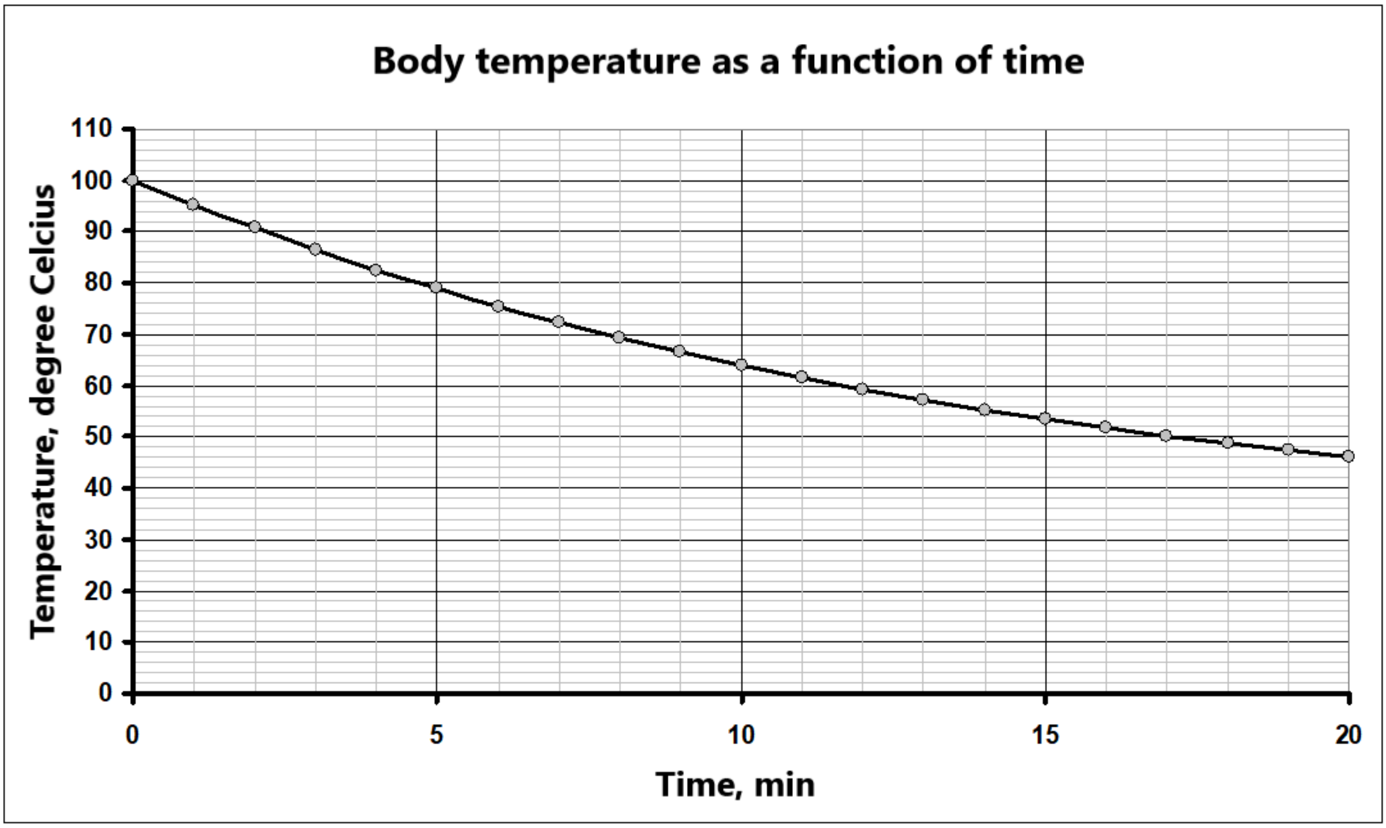
Problem 1A (4.0 points)

A vertical cylinder of radius R is fixed on a smooth horizontal surface. A thread is tightly wound on the cylinder, the free end of which is l_0 long and attached to a small puck of mass m. The puck is initially given a horizontal velocity v, which is perpendicular to the thread (see figure). How long will it take for the thread to be torn up if the maximum tension force it can withstand is T.



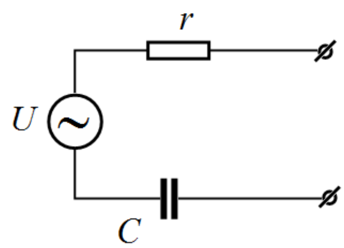
Problem 1B (3.0 points)

A body cools in the air so that the rate of heat transfer is proportional to the temperature difference between the body and the air. The graph below shows the dependence of the body temperature on time. Find the air temperature.



Problem 1C (3.0 points)

The equivalent circuit of a real source of alternating voltage of frequency $\omega = 1.00 \cdot 10^3$ s^{-1} consists of an ideal voltage source with the amplitude $U = 15.0$ V, of a resistor with the resistance $r = 2019$ Ohm and of a capacitor with the capacitance $C = 100$ μF. Different circuits of resistors, capacitors and coils can be connected to the source as a load. Propose a load circuit, which assures maximum of the generated heat output in the load itself. Draw schematically your load circuit and evaluate parameters of its elements. If you have found several solutions, provide the simplest one. Find also the maximum power, which can be generated in a load.



Problem 2. Conductors in an electric field (10.0 points)

When a conductor is placed in a constant external electric field, electric charges appear on its surface. This phenomenon is called an electrostatic induction, and the charges themselves are then called induced. This phenomenon is explained by a large number of free charge carriers in the conductor, usually electrons, which can move freely inside.

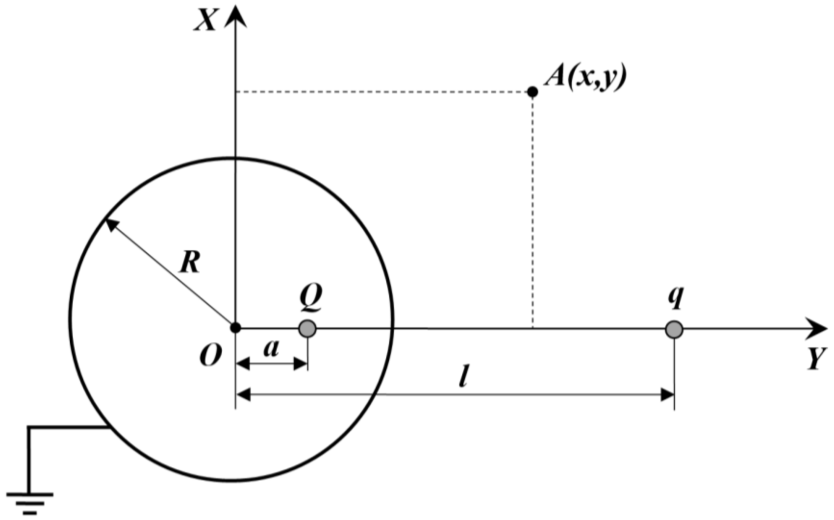
The distribution of the induced charges for a conductor of an arbitrary shape can be quite complex, but the following statements hold:

1) The electric field strength inside the conductor is exactly zero;
2) The electric field strength near the surface of the conductor is directed along the surface normal;
3) The induced charges are only located on the conductor surface;
4) All of the conductor points have the same electric potential.

In this problem, we consider several methods for calculating electric fields in the presence of conductors and apply them to a specific physical situation. Consider the vacuum permittivity ε_0 known.

Conductive ball and point charge

A conducting ball of radius R is grounded and a point-like charge q is placed at a distance l from it. In this case, in accordance with the foregoing, induced charges appear on the ball surface, which distort the electric field of the point-like charge q. The image method is applicable to this situation, whose essence is described as follows. The electric field outside the ball can be represented as a superposition of the field of the point-like charge q and the field of some fictitious point-like charge Q located somewhere inside the ball at a distance a from its center and on the line connecting the point-like charge q to the ball center. To evaluate the total electric field, we make use of the Cartesian coordinate system on the plane shown in the figure below. This is quite enough, since the system has axial symmetry.



2.1 Calculate the electric field potential at an arbitrary point A with coordinates (x, y) lying outside the ball. Express your answer in terms of $q, Q, l, a, x, y, \varepsilon_0$.

2.2 Using the above expression, evaluate the electric field potential on the ball surface. Express your answer in terms of $q, Q, l, a, x, R, \varepsilon_0$.

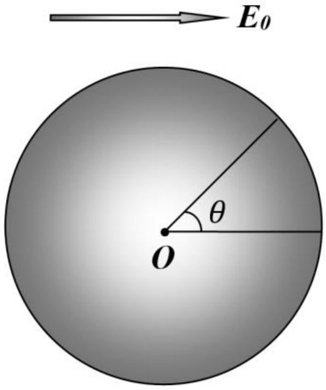
2.3 Using your answer from 2.2, find the charge Q and the distance a, expressing them in terms of q, l, R.

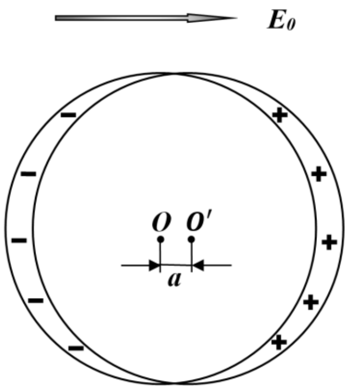
2.4 Calculate the work A that needs to be done on the point-like charge q in order to very slowly move it to infinity. Express your answer in terms of q, l, R, ε_0.

2.5 Find the interaction energy W of the induced charges with one another and express it in terms of q, l, R, ε_0.

Conductive ball in a uniform electric field

A conducting ball of radius R is placed in an external uniform electric field of strength E_0. In this case, the field of induced electric charges can be represented as a superposition of the field of two fictitious uniformly charged balls of the same radius R located at a very small distance $a \ll R$ from each other. The net charge of these two balls is , of course, zero due to the charge conservation law, so one of them can be assumed to be charged with the bulk charge density ρ, whereas the other is to be charged with the bulk charge density $-\rho$.

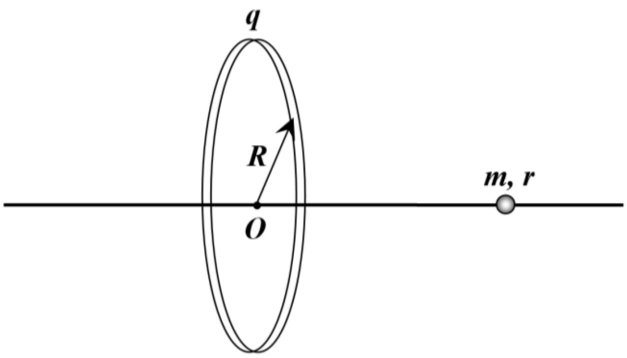




2.6 Calculate the electric field strength E_ρ inside a uniformly charged ball with the bulk charge density ρ at a distance r from its center. Express your answer in terms of ρ, r, ε_0.
2.7 Calculate the electric field strength at the intersection of two fictitious balls mentioned above. Express your answer in terms of ρ, a, ε_0.
2.8 Find the surface density of the induced charges σ as a function of the angle θ shown in the figure above. Express your answer in terms of $E_0, \theta, \varepsilon_0$.
2.9 Find the electric field E outside the ball and just near its surface point, characterized by the angle θ. Express your answer in terms of E_0, θ.

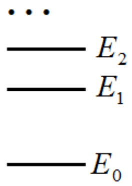
Conductive ball and charged ring

A thin ring of radius R is charged uniformly along its length by a charge q. An uncharged small conducting ball of radius r and mass m can slide without friction along the long non-conducting needle, coinciding with the axis of the ring.



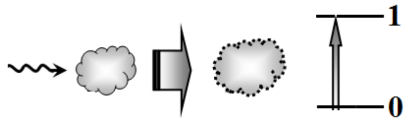
2.10 Calculate the frequency ω of small oscillations of the ball near its equilibrium position on the needle. Express your answer in terms of $q, R, r, m, \varepsilon_0$.
2.11 The ball is initially located at the ring center and is at rest. Find the work A, which must be done to very slowly push the ball along the needle to infinity. Express your answer in terms of q, R, r, ε_0.

Problem 3. Laser (10.0 points)



According to quantum theory, a molecule can only be found in certain states characterized by discrete energy values $E_0, E_1, E_2, \dots$. These states are represented by horizontal segments on the vertical energy scale and are numbered in order of increasing energy, starting from 0. In the absence of external influence, the molecule is found in the state with the number 0 and the lowest possible energy E_0. This state is then called a ground state, and the rest are then called excited states. The molecule can be driven from one energy state to another by absorbing or emitting light quanta, i.e. photons. The intensity of the light flux is characterized everywhere below by the density of the photon flux I, i.e. the number of photons passing perpendicularly through the unit area per unit of time, the dimension of this quantity is obviously equal to $[I] = \mathrm{m}^{-2} \cdot \mathrm{s}^{-1}$.

For further consideration, it is necessary to take into account the following processes that take place when the light flux passes through a medium.



Absorption. If a molecule is found in the ground state 0, then it can absorb a photon and be raised to the excited state 1. Such a transition is possible if the photon energy is equal to the difference between the energies of the excited and ground states, $h\nu_{01} = E_1 - E_0$, where ν_{01} is the photon frequency and $h = 6.63 \cdot 10^{-34}\,\mathrm{J \cdot s}$ stands for the Planck constant. If N_0 molecules are found in the ground state 0, then the number of molecules dN that have absorbed photons and have been elevated to the excited state in a very short time interval dt is equal to

$$dN = I\sigma_{01}N_0 dt\,, \tag{1}$$

where the quantity σ_{01} is called the absorption cross section and is determined by the properties of the molecules.

Stimulated emission. If a molecule is found in the excited state 1, then under the influence of a photon with the frequency ν_{01}, it can be forced to move to the ground state 0 with the emission of another photon that is completely identical to the first one and propagates in the same direction, has the same energy and polarization. The number of such transitions dN for a very small time interval dt is described by the formula similar to formula (1) as

$$dN = I\sigma_{10}N_1 dt\,, \tag{2}$$

where N_1 denotes the number of molecules in the excited state 1 with energy E_1, and σ_{10} signifies the cross section of the stimulated emission.

Spontaneous emission. A molecule in the excited state can spontaneously drop to its ground state with the emission of a photon. In contrast to the stimulated emission, the direction of emission and the polarization of the photon are both random, and the energy can slightly vary, i.e. the spontaneous emission cannot contribute to an increase in the light flux. The number of spontaneous transitions dN from the excited state 1 to the ground state 0 for a very small time interval dt is written as

$$dN = AN_1 dt = \frac{1}{\tau} N_1 dt\,, \tag{3}$$

where N_1 still designates the number of molecules in the excited state 1 with energy E_1, A is the transition probability or the Einstein coefficient, and its inverse value $\tau = A^{-1}$ is referred to as the lifetime of the excited state.

To describe each state k of the molecule, it is convenient to use not the total number N_k of molecules in the state, but its reduced value divided by the total number of molecules N in the medium

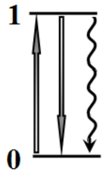
$$n_k = \frac{N_k}{N}\,, \tag{4}$$

This value is called the population of the state. For populations, the normalizing condition is satisfied: the sum of the populations of all the states of the molecules is equal to unity, i.e.

$$n_0 + n_1 + n_2 + ... = 1. \tag{5}$$

If the number of molecules in a certain excited state exceeds the number of molecules in the ground state, then this state of the medium is called the population inversion, and the emitted radiation may prevail over absorption, which results in an increase in the intensity of the light flux propagating in such a medium. This phenomenon is used in optical quantum light generators, called lasers. The population inversion is conventionally created using an external energy source, called pumping. In this problem, we consider the operation of an optically pumped laser, when the population inversion is generated by an external light flux. Unlike this pumping light flux, the laser light flux is monochromatic, coherent, polarized and narrowly directed.

Population inversion: two-level system



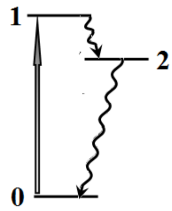
Consider a medium that is subject to monochromatic pumping light flux I_0. The incident light flux leads to transitions of molecules only between two states, i.e. the ground state 0 and the excited state 1. The absorption σ_{01} and stimulated emission σ_{10} cross sections are equal, i.e. $\sigma_{10} = \sigma_{01} = \sigma$ and the lifetime in the excited state is found to be τ.

3.1 Write down an equation describing the change in the population of the excited state 1 over time, i.e. express the derivative dn_1/dt in terms of n_1, I_0, σ and τ.

3.2 Find the population of the excited state and the difference of the populations between the excited and ground states under steady conditions as functions of the pumping light flux I_0. Express your answer in terms of the parameter $I_0\sigma\tau$.

3.3 Is it possible to achieve an amplification of the laser light flux in this case?

Population inversion: three-level system



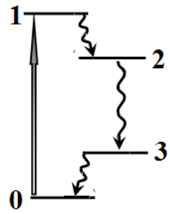
Let three states of the molecule be involved in possible transitions: the ground state 0 and the two excited states 1, 2. Under the action of the pumping light flux I_0, the molecule can be raised from the ground state 0 to the first excited state 1. The absorption cross section of this transition is equal to σ. As a result of intramolecular relaxation, the molecule that has resided in state 1 almost instantly drops to the lower energy state 2, whose lifetime is τ. In this case, assume that the stimulated emission is completely absent.

3.4 Write down an equation describing the change in the population n_2 of the excited state 2 over time.

3.5 Find the population $\bar{n}_2$ of the excited state 2 and the population difference $(\bar{n}_2 - \bar{n}_0)$ of the excited and the ground states under steady conditions as functions of the pumping light flux I_0. Express your answer in terms of the parameter $I_0\sigma\tau$.

3.6 At what minimum value of the parameter $I_0\sigma\tau$ is it possible to amplify the laser light flux with the frequency equal to the frequency of the transition $2 \to 0$?

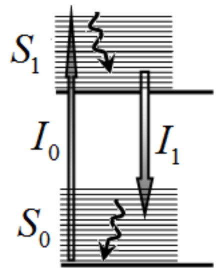
Population inversion: four-level system



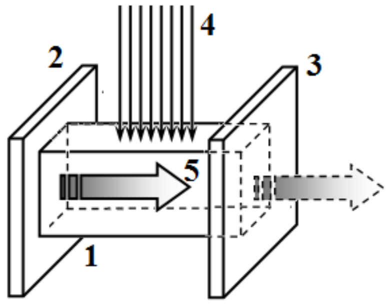
Let four states of the molecule be involved in possible transitions. Under the action of the pumping light flux I_0, the molecule can be raised from the ground state 0 to the first excited state 1. The absorption cross section of this transition is equal to σ. As a result of intramolecular relaxation, the molecule that has resided in state 1 almost instantly drops to the lower energy state 2, whose lifetime is τ. From this state, the molecule undergoes a transition to the intermediate state 3, which results in the emission of a photon. In this case, assume that the stimulated emission is completely absent. As a result of intramolecular relaxation, the molecule that has fallen into state 3 almost instantly drops to the ground energy state 0.

3.7 At what minimum value of the parameter $I_0\sigma\tau$ is it possible to amplify the laser light flux with the frequency equal to the frequency of the transition $2 \to 3$?

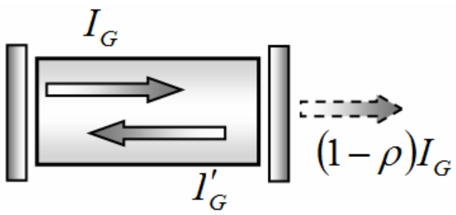
Resonator



The four-level system is usually implemented in dye solutions. Dyes consist of complex molecules with multiple energy states. Therefore, the possible energy states are grouped into bands: the ground state band S_0 contains an almost continuous spectrum of sub-levels, and the same is true for the first excited state band S_1. Thus, there are two bands of possible states. The absorption of the pumping light flux I_0 leads to transitions from the sublevels of the ground state band S_0 to different sublevels of the excited state band S_1. Transitions between the sub-levels of state 1 occur almost instantaneously, so the stimulated emission of the laser light flux I_1 occurs at lower frequencies, and the stimulated emission due to the pumping light flux I_0 can be neglected. In the approximation described, it is sufficient to know the population of the ground and excited state bands. Rhodamine 6G is used as a dye, for which: the absorption cross section in the transition $S_0 \to S_1$ is $\sigma_A = 3,90 \cdot 10^{-16} cm^2$; the stimulated emission cross section $S_1 \to S_0$ is $\sigma_E = 2,20 \cdot 10^{-16} cm^2$; the lifetime of the molecule in the state S_1 is $\tau = 4,20 \cdot 10^{-9} s$.



To generate light, cuvette 1 with the solution of rhodamine 6G is placed between two parallel mirrors 2 and 3, thus forming a resonator. The solution is excited by a uniform pumping light flux 4 of intensity I_0, whose frequency strictly corresponds to the maximum absorption of the solution. The pumping light flux 4 is directed perpendicular to the axis of the resonator and fully illuminates the entire cell. The intensity of this flux, of course, decreases as it passes through the solution; however, for carrying out our estimations, consider I_0 constant in the entire bulk of the resonator, assuming it to be averaged over the solution volume. The laser light flux 5 generated in the resonator propagates along the resonator axis, and its amplification occurs due to multiple reflections from the resonator mirrors. Consider mirror 2 fully reflective, and the second mirror 3 translucent with the reflectance ρ. The absorption of light in the mirrors, the solvent-body, as well as the scattering of light and other losses can be completely ignored. The resonator has the following parameters: the cuvette length is $l = 3,00\ cm$; the rhodamine 6G concentration is $\gamma = 1.30 \cdot 10^{16} cm^{-3}$; the reflection coefficient of the translucent mirror is $\rho = 0,90$; the refractive index of the solution of rhodamine 6G is $r = 1,50$. The speed of light is denoted as $c = 3,00 \cdot 10^{10} cm/s$.



For a simplified description of the laser light flux propagating along the resonator axis, one can consider the intensity of the light fluxes averaged over the length of the resonator. We denote the average intensity of the laser light flux propagating to the translucent mirror as I_G, and the intensity of the laser light flux propagating in the opposite direction as I'_G. Since the transmittance of mirror 3 is small, then we can assume that the average intensities of these fluxes be approximately equal $I_G \approx I'_G$.

3.8 Let a laser light flux I_G be created in the resonator. Show that in the absence of absorption and stimulated emission, the change in the intensity of the laser flux over time is described by the equation

$$\frac{dI_G}{dt} = -\frac{1}{T} I_G, \tag{6}$$

where T stands for the so-called photon lifetime in the resonator. Express the parameter T in terms of the parameters of the resonator. Calculate its numerical value.

3.9 Show that in the absence of the photon losses through mirror 3, the laser light flux variation over time obeys the following equation

$$\frac{dI_G}{dt} = KnI_G, \tag{7}$$

where n designates the population of the excited state of rhodamine 6G, and K is the resonator gain. Express the resonator gain K in terms of the parameters of the resonator and the stimulated emission cross section σ_E of rhodamine 6G. Calculate its numerical value.

Stationary generation mode

In this part, we assume that the pumping light flux is constant and does not depend on time. In the stationary mode, all quantities remain constant: the population of the excited state and the laser light flux. Assume that the population of the excited state is low, i.e. $n << 1$.

3.10 Write down a set of equations describing the change in the population $\frac{dn}{dt}$ of the excited state and the laser light flux $\frac{dI_G}{dt}$ in the resonator.

3.11 Obtain the formula and calculate a minimum (threshold) value n_{th} of the population of the excited state at which the amplification (generation) of laser light in the resonator occurs. Express it in terms of the parameters of the resonator K, T .

3.12 Derive the formula and calculate a minimum (threshold) value of the pumping light flux $I_{0,th}$ at which the laser light amplification in the resonator starts. Let the wavelength of the pumping light flux be $\lambda = 520nm$. Calculate the pumping flux in energetic units of W / cm^2 .

3.13 Find the laser flux at the output of the resonator as a function of the pumping light flux I_0, and express it in terms of the ratio $\eta = I_0 / I_{0,th}$, which is called the threshold overrun, and the characteristics of molecules. Draw a graph of the laser flux at the output of the resonator as a function η .

3.14 Find the quantum output $f = N_E / N_A$, i.e. the ratio of the number of photons N_E leaving the resonator per unit of time to the number of photons N_A absorbed in the resonator per the same unit of time as a function of the parameter η .

Mathematical hint for the theoretical competition

You may need to know the following integrals:

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b|.$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \textit{ where } n \textit{ is an integer number}$$

SOLUTIONS TO THE PROBLEMS OF THE THEORETICAL COMPETITION

Attention. Points in grading are not divided!

Problem 1 (10.0 points)

Problem 1A (4.0 points)

Since the thread is inextensible and under stress, then the speed of the puck is always perpendicular to the thread. Therefore, the tension force of the thread does not perform any work on the puck and its speed remains constant by modulus

$$v = const. \tag{1}$$

The puck moves along its trajectory with the curvature radius equal to the length l of the unwound thread, therefore, the condition for the thread to be torn up is found from Newton's second law as

$$T = m\frac{v^2}{l}. \tag{2}$$

The length of the thread changes as a result of winding on the cylinder according to

$$dl = -Rd\alpha, \tag{3}$$

where

$$d\alpha = \omega dt, \tag{4}$$

and the angular velocity of the thread rotation is obtained as follows

$$\omega = \frac{v}{l}. \tag{5}$$

It follows from equations (3)-(5) that

$$ldl = -Rvdt, \tag{6}$$

and its integration entails

$$l^2 - l_0^2 = -2Rvt. \tag{7}$$

Substituting formula (1) into (7), the time moment sought is finally found as

$$t = \frac{l_0^2 - \left(\frac{mv^2}{T}\right)^2}{2Rv} = \frac{l_0^2T^2 - m^2v^4}{2RvT^2}. \tag{8}$$

Content	Points
The puck speed remains unchanged	1
$T = m\frac{v^2}{l}$	0.5
$ldl = -Rvdt$	1
$l^2 - l_0^2 = -2Rvt$	0.5
$t = \frac{l_0^2T^2 - m^2v^4}{2RvT^2}$	1
Total	**4.0**

Problem 1B (3.0 points)

Possible solution. The power of the heat transfer from the body to the air is proportional to the difference between the body T and the air T_x temperatures with the factor α, i.e.

$$P = \alpha(T - T_x), \tag{1}$$

as a result, the body with the heat capacity C cools down by the temperature dT over time period dt, which obeys the heat balance equation

$$CdT = -Pdt. \tag{2}$$

Equations (1) and (2) with the initial condition $T = T_0$ have a solution

$$T(t) = T_x + (T_0 - T_x)e^{-\beta t}, \tag{3}$$

where $\beta = \alpha / C$ is a constant.

Let the body be cooled from the temperature T_0 to the temperature T_1 for a certain time interval, then it follows from (3) that

$$(T_1 - T_0) = \gamma (T_0 - T_x), \tag{4}$$

where γ is a constant.

Over the following same time interval, this difference will also change in γ times

$$(T_2 - T_1) = \gamma (T_1 - T_x). \tag{5}$$

Equations (4) and (5) result in the relation

$$\frac{(T_0 - T_x)}{(T_1 - T_0)} = \frac{(T_1 - T_x)}{(T_2 - T_0)}, \tag{6}$$

which has the following solution

$$T_x = \frac{T_0 T_2 - T_1^2}{(T_0 + T_2) - 2T_1}. \tag{7}$$

It is obtained from the graph provided: the initial temperature $T_0 = 373\text{K}$, in 10 minutes the temperature is equal to $T_1 = 337K$, and in 20 minutes it reaches the value of $T_2 = 319K$. Substituting these data into equation (7), the air temperature is finally calculated as

$$T_x = 301K = 28°C. \tag{8}$$

Content	Points
Correct method for determining the air temperature	1.5
The air temperature lies in the interval $T_x = 27.5 - 28.5°\text{C}$	1.5
The air temperature lies in the interval $T_x = 27.0 - 29.0°\text{C}$	(1.0)
The air temperature lies in the interval $T_x = 26.5 - 29.5°\text{C}$	(0.5)
Out of the above intervals	0
Total	**3.0**

Problem 1C (3.0 points)

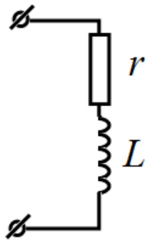
Let R be the active component of the load (the real part of the impedance), and X be the reactive component of the entire circuit (the imaginary part of the total impedance). Then the current amplitude is found as

$$I = \frac{U}{\sqrt{(r+R)^2 + X^2}}.$$

The average thermal power in the load reads as

$$P = \frac{1}{2} I^2 R = \frac{U^2 R}{2[(r+R)^2 + X^2]}.$$

It is seen that the maximum power is achieved at $X = 0$, i.e. there should be no phase shift in the circuit. The remaining expression has a maximum at $R = r$.



The phase shift would be zero if a coil was connected in series with the capacitor such that $\frac{1}{\omega C} = \omega L$, and, thus, $L = \frac{1}{\omega^2 C} = 1.00 \cdot 10^{-2}\ Hn$.

It turns out that the simplest load must consist of the resistor with the resistance of 2019 Ohms and the coil with the inductance of $1.00 \cdot 10^{-2}\ Hn$.

The maximum power is obtained as

$$P_{max} = \frac{1}{2}\frac{U^2}{4r} = \frac{U^2}{8r} = 13.9 \text{ mW}.$$

Content	Points
The phase shift is zero	1

Without justification	*(0,5)*
The inductance of the coil $L = \frac{1}{\omega^2 C}$	0,7
Correct numerical value $L = 10^{-2}$ Hn	0,3
Maximum power at $R = r$	0,5
The maximum power itself $P_{max} = \frac{U^2}{8r}$	0,3
Correct numerical value $P_{max} = 14$ mW	0,2
Total	**3,0**

Problem 2. Conductors in an electric field (10,0 points)

Conductive ball and point charge

2.1 The electric potential of the point-like charge q is equal to

$$\varphi_1 = \frac{q}{4\pi\varepsilon_0\sqrt{(l-x)^2+y^2}}, \qquad (1)$$

whereas the electric potential of the fictitious point-like charge Q is found to be

$$\varphi_2 = \frac{Q}{4\pi\varepsilon_0\sqrt{(x-a)^2+y^2}}. \qquad (2)$$

According to the principle of superposition, the full potential is just a sum of equations (1) and (2)

$$\varphi = \varphi_1 + \varphi_2 = \frac{q}{4\pi\varepsilon_0\sqrt{(l-x)^2+y^2}} + \frac{Q}{4\pi\varepsilon_0\sqrt{(x-a)^2+y^2}}. \qquad (3)$$

2.2 The equation of the circle corresponding to the surface of the ball is written as

$$x^2 + y^2 = R^2. \qquad (4)$$

Eliminating y with the help of relation (4) and substituting it into formula (3) yield

$$\varphi = \frac{q}{4\pi\varepsilon_0\sqrt{l^2-2lx+R^2}} + \frac{Q}{4\pi\varepsilon_0\sqrt{a^2-2ax+R^2}}. \qquad (5)$$

2.3 The potential of the ball is zero, since it is grounded, i.e.

$$\varphi = 0. \qquad (6)$$

Equating expression (5) to zero, it can be rewritten in the form

$$\frac{Q}{q} = -\frac{\sqrt{a^2-2ax+R^2}}{\sqrt{l^2-2lx+R^2}} = \beta = const < 0. \qquad (7)$$

Raising equation (7) in the square, one gets the following equation

$$2x(l\beta^2 - a) + a^2 + R^2 - \beta^2(l^2 + R^2) = 0. \qquad (8)$$

Equation (8) should be satisfied for all $x \in (-R, R)$, and this is possible only if the coefficient at the linear term x and the free term are separately equal to zero, i.e.

$$l\beta^2 - a = 0, \qquad (9)$$

$$a^2 + R^2 - \beta^2(l^2 + R^2) = 0. \qquad (10)$$

Solving the set of equations (9) and (10), the following two solutions are obtained

$$a = l, \quad \beta = -1, \qquad (11)$$

$$a = \frac{R^2}{l}, \quad \beta = -\frac{R}{l}. \qquad (12)$$

Only solution (12) is nonzero, so we finally get

$$Q = -q\frac{R}{l}, \qquad (13)$$

$$a = \frac{R^2}{l}. \qquad (14)$$

2.4 The force acting on the point-like charge reads as

$$F = \frac{q^2 Rl}{4\pi\varepsilon_0(l^2-R^2)^2}, \qquad (15)$$

and, therefore, the work sought is found by integrating as

$$A = \int_l^\infty F dl = \frac{q^2 R}{8\pi\varepsilon_0(l^2-R^2)}. \qquad (16)$$

2.5 Let the point-like charge be slowly moved from the initial position to infinity such that the resulting current strength in the ball is negligibly small and the release of Joule heat can be omitted. Let W_q be the energy of the point-like charge q, W_Q be the sought interaction energy of induced charges, W_{Qq} be the interaction energy of the point charge q with the induced charges, which is simply obtained as

$$W_{Qq} = -\frac{q^2 R}{4\pi\varepsilon_0(l^2 - R^2)}. \tag{17}$$

When the charge is removed to infinity, the law of energy conservation must be satisfied, which in this case has the form

$$W_q + W_Q + W_{Qq} + A = W_q. \tag{18}$$

The set of equations (16)-(18) finally provides the following result

$$W_Q = \frac{q^2 R}{8\pi\varepsilon_0(l^2 - R^2)}. \tag{19}$$

Conductive ball in a uniform electric field

2.6 To find the electric field inside a uniformly charged ball, the Gauss theorem is written for a spherical volume of radius $r < R$. The charge inside this volume is easily derived as

$$q = \frac{4}{3}\pi r^3 \rho, \tag{20}$$

and the electric field flux is found to be

$$\Phi_E = 4\pi r^2 E. \tag{21}$$

According to the Gauss theorem

$$\Phi_E = \frac{q}{\varepsilon_0}, \tag{22}$$

which ultimately entails

$$\vec{E} = \frac{\rho}{3\varepsilon_0}\vec{r}. \tag{23}$$

The last expression takes into account that the electric field strength vector is collinear to the vector $\vec{r}$.

2.7 Now consider the two fictitious balls with the bulk charge densities of opposite signs and evaluate the electric field in the domain of their intersection. Take an arbitrary point inside this domain and draw the radii of the vectors from the centers of the balls, denoting them $\overrightarrow{r_+}$ and $\overrightarrow{r_-}$, respectively. Then, applying formula (23) for each ball results in

$$\overrightarrow{E_+} = \frac{\rho}{3\varepsilon_0}\overrightarrow{r_+}, \tag{24}$$

$$\overrightarrow{E_-} = -\frac{\rho}{3\varepsilon_0}\overrightarrow{r_-}. \tag{25}$$

The net electric field is found with the help of the superposition principle as

$$\vec{E} = \frac{\rho}{3\varepsilon_0}(\overrightarrow{r_+} - \overrightarrow{r_-}) = \frac{\rho}{3\varepsilon_0}\vec{a}, \tag{26}$$

where $\vec{a}$ stands for the vector, drawn from the center of the negatively charged ball to the center of the positively charged ball.

2.8 The field strength inside the conducting ball must be zero. The induced charges create, according to formula (26), a uniform electric field, which must completely compensate for the external electric field, whence we obtain that

$$\rho a = 3\varepsilon_0 E_0. \tag{27}$$

The charges of the fictitious balls are fully compensated with the exception of a thin layer near their surfaces, which can be considered a surface charge. The layer thickness δ depends on the angle θ and, due to the smallness of a, is equal to

$$\delta = a\cos\theta. \tag{28}$$

Hence, the magnitude of the surface charge near the angle θ is equal to

$$\sigma = \frac{\rho V}{S} = \frac{\rho S\delta}{S} = \rho\delta. \tag{29}$$

It immediately follows from equations (27)-(29) that

$$\sigma = 3\varepsilon_0 E_0 \cos\theta. \tag{30}$$

2.9 Consider a thin cylinder near the surface of the conductor and apply the Gauss theorem to it. Since the field inside the conductor is absent, and is directed normally just outside of it, then according to the Gauss theorem

$$ES = \frac{\sigma S}{\varepsilon_0}, \tag{31}$$

which yields

$$E = 3E_0 \cos\theta. \tag{32}$$

Conductive ball and charged ring

2.10 The conducting ball is very small, so that the electric field of the ring E in its vicinity can be considered almost uniform. It has been shown in the previous part of this problem that its polarization can be represented as two fictitious balls of opposite charge. These two balls behave in an external field as a dipole with the moment

$$\vec{p} = q\vec{a}, \tag{33}$$

where

$$q = \rho \frac{4}{3}\pi r^3. \tag{34}$$

Using (27), formulas (33) and (34) produce

$$\vec{p} = 4\pi r^3 \varepsilon_0 \vec{E}, \tag{35}$$

Let us evaluate the electric field of the ring E in the vicinity of the ball as a function of its distance z to the center. Obviously, the ring field is directed along the needle. Dividing the ring into small parts that carry an electric charge Δq_i the projection of their field on the direction of the needle has the form

$$\Delta E_z = \frac{\Delta q_i \cos\alpha}{4\pi\varepsilon_0 (z^2+R^2)^2}. \tag{36}$$

Taking into account

$$\cos\alpha = \frac{z}{(z^2+R^2)^{1/2}} \tag{37}$$

and summing over all elements of the ring, one gets

$$E(z) = \frac{qz}{4\pi\varepsilon_0 (z^2+R^2)^{3/2}}. \tag{38}$$

The force acting on the dipole is obtained as

$$F = qE(z+a) - qE(z) = qa\frac{dE}{dz} = p\frac{dE}{dz}. \tag{39}$$

Substituting formulas (35) and (38) into (39) gives rise to

$$F = \frac{q^2 r^3 z(R^2 - 2z^2)}{4\pi\varepsilon_0 (z^2+R^2)^4}. \tag{40}$$

It follows from expression (40) that there are three equilibrium positions, which are determined by the points

$$z_1 = 0, \tag{41}$$

$$z_{2,3} = \pm\frac{R}{\sqrt{2}}. \tag{42}$$

A simple analysis proves that the equilibrium position (41) is unstable, and the symmetric positions (42) are both stable.

Near the position of the stable equilibrium, expression (40) for the force simplifies to

$$F = -\frac{8q^2 r^3 x}{81\pi\varepsilon_0 R^6}, \tag{43}$$

where

$$x = z - \frac{R}{\sqrt{2}} \ll R. \tag{44}$$

Newton's equation for the motion of the ball along the needle at small deviations x has the form

$$m\ddot{x} + \frac{8q^2 r^3}{81\pi\varepsilon_0 R^6}x = 0, \tag{45}$$

which is a harmonic equation with the frequency

$$\omega = \sqrt{\frac{8q^2 r^3}{81\pi\varepsilon_0 m R^6}}. \tag{46}$$

2.11 There is no need to integrate formula (40). In the initial position, the conducting ball is not polarized and in the final state it is also not polarized, since at zero and at infinity separations the electric field of the ring vanishes. Therefore, it is immediately inferred from the law of energy conservation that

$$A = 0. \tag{47}$$

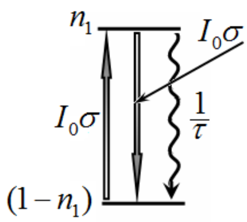
It is natural that integrating expression (40) from zero to infinity gives the same answer.

Part	Content	Points	
2.1	Formula (1) $\varphi_1 = \frac{q}{4\pi\varepsilon_0\sqrt{(l-x)^2+y^2}}$	0,2	0,6
	Formula (2) $\varphi_2 = \frac{Q}{4\pi\varepsilon_0\sqrt{(x-a)^2+y^2}}$	0,2	
	Formula (3) $\varphi = \varphi_1 + \varphi_2 = \frac{q}{4\pi\varepsilon_0\sqrt{(l-x)^2+y^2}} + \frac{Q}{4\pi\varepsilon_0\sqrt{(x-a)^2+y^2}}$	0,2	
2.2	Formula (4) $x^2 + y^2 = R^2$	0,2	0,4
	Formula (5) $\varphi = \frac{q}{4\pi\varepsilon_0\sqrt{l^2-2lx+R^2}} + \frac{Q}{4\pi\varepsilon_0\sqrt{a^2-2ax+R^2}}$	0,2	
2.3	Formula (6) $\varphi = 0$	0,2	1,8
	Formula (7) $\frac{Q}{q} = -\frac{\sqrt{a^2-2ax+R^2}}{\sqrt{l^2-2lx+R^2}} = \beta = const < 0$	0,2	
	Formula (8) $2x(l\beta^2 - a) + a^2 + R^2 - \beta^2(l^2 + R^2) = 0$	0,2	
	Formula (9) $l\beta^2 - a = 0$	0,2	
	Formula (10) $a^2 + R^2 - \beta^2(l^2 + R^2) = 0$	0,2	
	Formula (11) $a = l, \quad \beta = -1$	0,2	
	Formula (12) $a = \frac{R^2}{l}, \quad \beta = -\frac{R}{l}$	0,2	
	Formula (13) $Q = -q\frac{R}{l}$	0,2	
	Formula (14) $a = \frac{R^2}{l}$	0,2	
2.4	Formula (15) $F = \frac{q^2Rl}{4\pi\varepsilon_0(l^2-R^2)^2}$	0,2	0,4
	Formula (16) $A = \int_l^\infty Fdl = \frac{q^2R}{8\pi\varepsilon_0(l^2-R^2)}$	0,2	
2.5	Formula (17) $W_{Qq} = -\frac{q^2R}{4\pi\varepsilon_0(l^2-R^2)}$	0,2	0,6
	Formula (18) $W_q + W_Q + W_{Qq} + A = W_q$	0,1	
	Formula (19) $W_Q = \frac{q^2R}{8\pi\varepsilon_0(l^2-R^2)}$	0,3	
2.6	Formula (20) $q = \frac{4}{3}\pi r^3\rho$	0,1	0,4
	Formula (21) $\Phi_E = 4\pi r^2 E$	0,1	
	Formula (22) $\Phi_E = \frac{q}{\varepsilon_0}$	0,1	
	Formula (23) $\vec{E} = \frac{\rho}{3\varepsilon_0}\vec{r}$	0,1	
2.7	Formula (24) $\overrightarrow{E_+} = \frac{\rho}{3\varepsilon_0}\overrightarrow{r_+}$	0,1	0,4
	Formula (25) $\overrightarrow{E_-} = -\frac{\rho}{3\varepsilon_0}\overrightarrow{r_-}$	0,1	
	Formula (26) $\vec{E} = \frac{\rho}{3\varepsilon_0}(\overrightarrow{r_+} - \overrightarrow{r_-}) = \frac{\rho}{3\varepsilon_0}\vec{a}$	0,2	
2.8	Formula (27) $\rho a = 3\varepsilon_0 E_0$	0,2	0,8
	Formula (28) $\delta = a\cos\theta$	0,2	
	Formula (29) $\sigma = \frac{\rho V}{S} = \frac{\rho S\delta}{S} = \rho\delta$	0,2	
	Formula (30) $\sigma = 3\varepsilon_0 E_0 \cos\theta$	0,2	
2.9	Formula (31) $ES = \frac{\sigma S}{\varepsilon_0}$	0,2	0,4

	Formula (32) $E = 3E_0 \cos\theta$	0,2	
2.10	Formula (33) $\vec{p} = q\vec{a}$	0,4	3,8
	Formula (34) $q = \rho\frac{4}{3}\pi r^3$	0,2	
	Formula (35) $\vec{p} = 4\pi r^3 \varepsilon_0 \vec{E}$	0,4	
	Formula (36) $\Delta E_z = \frac{\Delta q_i \cos\alpha}{4\pi\varepsilon_0(z^2+R^2)^2}$	0,2	
	Formula (37) $\cos\alpha = \frac{z}{(z^2+R^2)^{1/2}}$	0,2	
	Formula (38) $E(z) = \frac{qz}{4\pi\varepsilon_0(z^2+R^2)^{3/2}}$	0,4	
	Formula (39) $F = qE(z+a) - qE(z) = qa\frac{dE}{dz} = p\frac{dE}{dz}$	0,4	
	Formula (40) $F = \frac{q^2 r^3 z(R^2-2z^2)}{4\pi\varepsilon_0(z^2+R^2)^4}$	0,2	
	Formula (41) $z_1 = 0$	0,2	
	Formula (42) $z_{2,3} = \pm\frac{R}{\sqrt{2}}$	0,2	
	Formula (43) $F = -\frac{8q^2r^3x}{81\pi\varepsilon_0 R^6}$	0,4	
	Formula (44) $x = z - \frac{R}{\sqrt{2}} \ll R.$	0,2	
	Formula (45) $\ddot{x} + \frac{8q^2r^3}{81\pi\varepsilon_0 R^6}x = 0$	0,2	
	Formula (46) $\omega = \sqrt{\frac{8q^2r^3}{81\pi\varepsilon_0 m R^6}}$	0,2	
2.11	Formula (47) $A = 0$	0.4	0,4
	Formal integral of formula (40) without the correct answer	(0.1)	
Total			**10,0**

Problem 3. Laser (10.0 points)

Population inversion: two-level system



3.1 The figure on the right shows a diagram of possible transitions and their probabilities. If the population of the excited state is equal to n_1, then the population of the ground state is equal to $(1-n_1)$, since the molecule can only be in one of two states.

The balance equation describing the change in the population directly follows from the drawn diagram as

$$\frac{dn_1}{dt} = -\frac{1}{\tau}n_1 - I_0\sigma n_1 + I_0\sigma(1-n_1). \tag{1}$$

3.2 In the stationary mode $dn_1/dt = 0$, then it follows from equation (1) that the population of the excited state is given by the formula

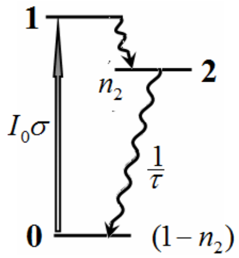
$$\bar{n}_1 = \frac{I_0\sigma\tau}{1+2I_0\sigma\tau}. \tag{2}$$

Accordingly, the difference in the populations of the excited and ground states is equal to

$$\Delta\bar{n} = \bar{n}_1 - (1-\bar{n}_1) = 2\frac{I_0\sigma\tau}{1+2I_0\sigma\tau} - 1 = -\frac{1}{1+2I_0\sigma\tau}. \tag{3}$$

3.3 Even with the intensity of the pumping light flux tending to infinity, the population inversion in the two-level system cannot be achieved, therefore, the laser light flux cannot be amplified in this system.

Population inversion: three-level system



3.4 In this system, there are no forced transitions "down", so the balance equation for the population of state 2 is written as:

$$\frac{dn_2}{dt} = -\frac{n_2}{\tau} + I_0\sigma(1-n_2). \qquad (4)$$

Here, it is taken into account that the molecule can only be in two states: the excited state 2, or the ground state 0.

3.5 In the stationary mode $dn_2/dt = 0$, therefore, as it follows from equation (4), the population of the excited state is derived as

$$\bar{n}_2 = \frac{I_0\sigma}{\frac{1}{\tau} + I_0\sigma} = \frac{I_0\sigma\tau}{1+I_0\sigma\tau}. \qquad (5)$$

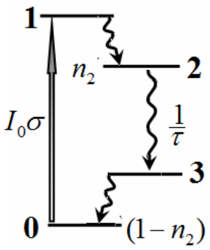
The difference between the populations of the excited and ground states is found by the formula

$$\Delta n = \bar{n}_2 - \bar{n}_0 = \bar{n}_2 - (1-\bar{n}_2) = \frac{I_0\sigma\tau - 1}{1+I_0\sigma\tau}. \qquad (6)$$

3.6 Laser light amplification is possible when the population inversion is reached, i.e. $\Delta n > 0$. It follows from formula (6) that this is possible when

$$I_0\sigma\tau > 1. \qquad (7)$$

Population inversion: four-level system



3.7 In the four-level system, the balance equation for the population of state 2 coincides with equation (4), and the stationary value of the population of this state is also described by formula (5). The essential difference of this system is that from state 2 the transition is undertaken to intermediate state 3, whose population is practically equal to 0. Therefore, in this system the population difference is equal to

$$\Delta n = \bar{n}_2 = \frac{I_0\sigma\tau}{1+I_0\sigma\tau}, \qquad (8)$$

and the population inversion between states 2 and 3 is achieved with practically arbitrary value of the parameter

$$I_0\sigma\tau > 0. \qquad (9)$$

Resonator

3.8 The change in the number dN of photons in the resonator is due only to their output through the translucent mirror. For a short period of time dt, the number of photons that leave the resonator through the mirror is found to be

$$dN_{out} = (1-\rho) I_G S dt = -dN. \qquad (10)$$

where S stands for the cross section area of the resonator.

The intensity of the laser light flux I_G can be expressed in terms of the average density of photons $\frac{N}{Sl}$ in the resonator and the speed of their propagation $\frac{c}{r}$ in the form

$$I_G = \frac{1}{2}\frac{N}{Sl}\frac{c}{r}. \qquad (11)$$

The factor $1/2$ takes into account that the laser light in the resonator propagates in two opposite directions. Expressing the number of photons in the resonator through the intensity of the generation flux

$$N = \frac{2rSl}{c} I_G \qquad (12)$$

and substituting it into equation (10), one gets

$$dI_G = -\frac{c}{2rSl}(1-\rho) I_G S dt = -(1-\rho)\frac{c}{2rl} I_G dt\,. \tag{13}$$

This equation has the required form

$$\frac{dI_G}{dt} = -\frac{c(1-\rho)}{2rl} I_G = -\frac{1}{T} I_G\,, \tag{14}$$

where the photon lifetime in the resonator is determined by the formula

$$T = \frac{2rl}{c(1-\rho)} = 3{,}00 \cdot 10^{-9}\, s\,. \tag{15}$$

3.9 Consider the change in the number of photons in the presence of the stimulated emission and the absence of losses through the mirror. In accordance with the definition of the stimulated emission cross section, the number of generated photons can be described by the equation

$$dN = 2I_G \sigma_E n\gamma V dt = 2I_G \sigma_E n\gamma S l dt\,. \tag{16}$$

Here $n\gamma V$ denotes the number of dye molecules in the resonator being in the excited state, and $V = Sl$ is the resonator volume.

Substituting the expression for the number of photons in the resonator (12) into the last equation, the desired equation is finally obtained

$$\frac{dI_G}{dt} = \frac{\gamma c \sigma_E}{r} n I_G = K n I_G\,, \tag{17}$$

with the resonator gain

$$K = \frac{\gamma c \sigma_E}{r} = 5{,}72 \cdot 10^{10}\, s^{-1}\,. \tag{18}$$

Stationary generation mode

3.10 To describe the dynamics of the intensity of the laser light flux, it is necessary to combine equations (14) and (17):

$$\frac{dI_G}{dt} = K n I_G - \frac{1}{T} I_G\,. \tag{19}$$

The population of the excited state is described by the balance equation

$$\frac{dn}{dt} = I_0 \sigma_A (1-n) - \frac{1}{\tau} n - 2 I_G \sigma_E n\,, \tag{20}$$

which takes into account the absorption of the pumping light flux, spontaneous and stimulated emissions from the excited state.

3.11 To initiate the laser light amplification, it is necessary that the derivative in equation (19) should be greater than zero, therefore the threshold value of the population of the excited state is equal to

$$n_{th} = \frac{1}{KT} = 5{,}83 \cdot 10^{-3} \ll 1\,. \tag{21}$$

3.12 To derive the threshold value of the intensity of the pumping light flux, we make use of equation (20) in the absence of the laser light flux $I_G = 0$, whence we get

$$I_{0,th} = \frac{n_{th}}{\tau \sigma_A (1-n_{th})} \approx \frac{n_{th}}{\tau \sigma_A} = 3{,}58 \cdot 10^{21} cm^{-2} \cdot s^{-1}\,. \tag{22}$$

To find the pumping energy flux, the calculated flux (22) must be multiplied by the energy of one quantum

$$\varepsilon = \frac{hc}{\lambda} = 3{,}83 \cdot 10^{-19} J\,, \tag{23}$$

therefore, the pumping energy intensity is obtained as

$$I_E = \varepsilon I_{0,th} = 1{,}37\cdot 10^3 \frac{W}{cm^2}. \tag{24}$$

3.13 In the stationary mode, the time derivatives in equations (19) and (20) vanish. Equation (19) then yields

$$\bar{n} = \frac{1}{KT}, \tag{25}$$

and it is found from equation (20) that

$$I_G = \frac{I_0\sigma_A - \frac{1}{\tau}\bar{n}}{2\sigma_E \bar{n}}. \tag{26}$$

Expressing the intensity of the pumping light flux through its threshold value

$$I_0 = \eta I_{0,th} = \eta \frac{\bar{n}}{\tau\sigma_A} \tag{27}$$

and substituting it into formula (23), one obtains

$$I_G = \frac{\eta \frac{\bar{n}}{\tau\sigma_A}\sigma_A - \frac{1}{\tau}\bar{n}}{2\sigma_E\bar{n}} = \frac{\eta-1}{2\tau\sigma_E}. \tag{28}$$

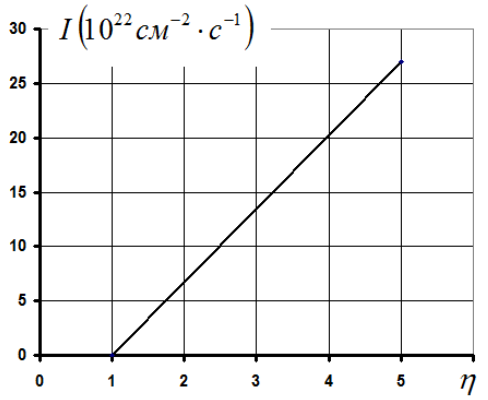
At the output of the resonator, the laser light intensity is equal to

$$I = (1-\rho) I_G = \frac{(1-\rho)}{2\tau\sigma_E}(\eta-1) = E(\eta-1), \tag{29}$$

in which the constant factor is introduced as

$$E = \frac{1-\rho}{2\tau\sigma_E} = 5{,}41\cdot 10^{22} cm^{-2}\cdot s^{-1}. \tag{30}$$

The graph of relation (29) is a straight line, as shown in the figure below.



3.14 On the one hand, the number of light quanta absorbed in the resonator per unit time is calculated by the formula

$$N_A = \eta I_{0,th}\sigma_A \gamma S l. \tag{31}$$

On the other hand, the number of quanta leaving the resonator per unit time is

$$N_E = E(\eta-1)S. \tag{32}$$

Thus, the quantum output turns out to be equal

$$f = \frac{N_E}{N_A} = \frac{E(\eta-1)}{(I_0)_{tr}\sigma_A(\gamma l)}. \tag{33}$$

The substitution of all parameters included in this formula leads to the final result

$$f = \frac{\eta-1}{\eta}. \tag{34}$$

Part	Content	Points	
3.1	Equation (1): $\frac{dn_1}{dt}=-\frac{1}{\tau}n_1-I_0\sigma n_1+I_0\sigma\left(1-n_1\right)$	0,3	0,3
3.2	Formula (2): $\bar{n}_1=\frac{I_0\sigma\tau}{1+2I_0\sigma\tau}$	0,2	0,3
	Formula (3): $\Delta\bar{n}=\bar{n}_1-\left(1-\bar{n}_1\right)=2\frac{I_0\sigma\tau}{1+2I_0\sigma\tau}-1=-\frac{1}{1+2I_0\sigma\tau}$	0,1	
3.3	Answer: «no»	0,2	0,2
3.4	Equation (4): $\frac{dn_2}{dt}=-\frac{n_2}{\tau}+I_0\sigma\left(1-n_2\right)$	0,2	0,2
3.5	Formula (5): $\bar{n}_2=\frac{I_0\sigma}{\frac{1}{\tau}+I_0\sigma}=\frac{I_0\sigma\tau}{1+I_0\sigma\tau}$	0,1	0,2
	Formula (6): $\Delta n=\bar{n}_2-\bar{n}_0=\bar{n}_2-\left(1-\bar{n}_2\right)=\frac{I_0\sigma\tau-1}{1+I_0\sigma\tau}$	0,1	
3.6	Inequality (7): $I_0\sigma\tau>1$	0,3	0,3
3.7	Formula (5) is again used	0,1	0,5
	Formula (8): $\Delta n=\bar{n}_2=\frac{I_0\sigma\tau}{1+I_0\sigma\tau}$	0,1	
	Inequality (9): $I_0\sigma\tau>0$	0,3	
3.8	Formula (10): $dN_{out}=\left(1-\rho\right)I_G S dt=-dN$	0,3	1,5
	Formula (11): $I_G=\frac{1}{2}\frac{N}{Sl}\frac{c}{r}$	0,5	
	Formula (15): $T=\frac{2rl}{c\left(1-\rho\right)}$	0,4	
	Numerical value: $T=3,00\cdot10^{-9}s$	0,3	
3.9	Formula (16): $dN=2I_G\sigma_E n\gamma V dt=2I_G\sigma_E n\gamma S l dt$	0,6	1,5
	Formula (18): $K=\frac{\gamma c\sigma_E}{r}$	0,5	
	Numerical value: $K=5,72\cdot10^{10}s^{-1}$	0,4	
3.10	Equation (19): $\frac{dI_G}{dt}=KnI_G-\frac{1}{T}I_G$	0,2	0,5
	Equation (20): $\frac{dn}{dt}=I_0\sigma_A\left(1-n\right)-\frac{1}{\tau}n-2I_G\sigma_E n$	0,3	
3.11	Derivative should be positive;	0,1	0,5
	Formula (21): $n_{th}=\frac{1}{KT}$	0,2	
	Numerical value: $n_{th}=5,83\cdot10^{-3}$	0,2	
3.12	The intensity of the laser light flux: $I_G=0$	0,1	1,0

	Formula (22): $I_{0,th} = \frac{n_{th}}{\tau\sigma_A(1-n_{th})} \approx \frac{n_{th}}{\tau\sigma_A}$	0,3	
	Numerical value: $I_{0,th} = 3,58 \cdot 10^{21} cm^{-2} \cdot s^{-1}$	0,3	
	Formula (23): $\varepsilon = \frac{hc}{\lambda}$	0,1	
	Formula (24): $I_E = \varepsilon I_{0,th}$	0,1	
	Numerical value: $I_E = 1,37 \cdot 10^3 \frac{W}{cm^2}$	0,1	
3.13	Derivatives turn zero	0,1	**2,0**
	Formula (25): $\bar{n} = \frac{1}{KT}$	0,2	
	Formula (26): $I_G = \frac{I_0\sigma_A - \frac{1}{\tau}\bar{n}}{2\sigma_E\bar{n}}$	0,2	
	Formula (27): $I_0 = \eta I_{0,th} = \eta\frac{\bar{n}}{\tau\sigma_A}$	0,2	
	Formula (28): $I_G = \frac{\eta\frac{\bar{n}}{\tau\sigma_A}\sigma_A - \frac{1}{\tau}\bar{n}}{2\sigma_E\bar{n}} = \frac{\eta-1}{2\tau\sigma_E}$	0,3	
	Formula (30): $E = \frac{1-\rho}{2\tau\sigma_E}$	0,2	
	Numerical value: $E = 5,41 \cdot 10^{22} см^{-2} \cdot c^{-1}$	0,3	
	Drawing graph: axis are named and ticked	0,1	
	Drawing graph: straight line	0,2	
	Drawing graph: straight line passes through 1	0,2	
3.14	Formula (31): $N_A = \eta I_{0,th}\sigma_A\gamma Sl$	0,4	**1,0**
	Formula (32): $N_E = E(\eta-1)S$	0,3	
	Formula (34): $f = \frac{\eta-1}{\eta}$	0,3	
Total			**10,0**

THEORETICAL COMPETITION

January 10, 2020

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*.** Write only inside the boxed area.
6. Begin each question on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Question Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
    - Used ***Writing sheets*** in order.
    - The sheets you do not wish to be evaluated.
    - Unused sheets.
    - The printed problems.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

Problem 1 (10.0 points)

This problem consists of three independent parts.

Problem 1.1 (4.0 points)

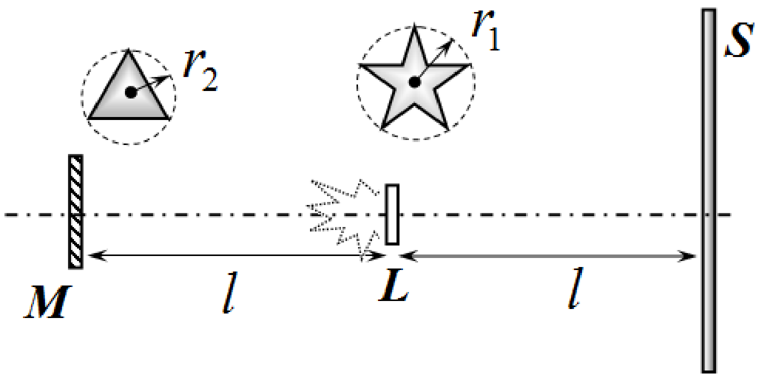
A mathematical pendulum is located on the equator of the Earth. Denote the period of its oscillations at midday T_1, and at midnight T_2. Find the relative difference of these periods $\varepsilon = \frac{T_2 - T_1}{T_1}$. Use the following approximations and numerical values: Earth is an ideal ball of radius $r_1 = 6{,}4 \cdot 10^3\ km$; the Earth's orbit is a circle of radius $r_2 = 1{,}5 \cdot 10^8\ km$ with the center located in the center of the Sun; the axis of rotation of the Earth is perpendicular to the plane of the Earth's orbit; the influence of the Moon and other planets is neglected; acceleration of gravity at the Earth's pole is $g_0 = 9{,}8\ m/s$; the period of revolution of the Earth around its axis is 1 day; the period of revolution of the Earth around the Sun is 1 year.

Problem 1.2 (3.0 points)

The space between two concentric well conductive spheres of radii a and $b > a$ is filled with a substance whose specific resistance depends only on the distance to the spheres' common center. An electric current of strength I flows in between the spheres, such that the bulk density of Joule heat losses in the substance is the same at all points and is equal to w. Determine the average density of the space charge ρ_Q accumulated in the volume of the substance over a sufficiently long time of the electric current flow.

Problem 1.3 (3.0 points)

At equal distances $l = 50\ sm$ in between the screen S and the flat mirror M, a flat matte source L emitting white light only towards the mirror is placed. The planes of the screen, mirror and source are all parallel to one other. The source has the shape of a five-pointed star inscribed in a circle of radius r_1, whereas the mirror has a regular triangle shape inscribed in a circle of radius r_2. The source and mirror centers lie on the same axis perpendicular to the screen plane. Draw a schematic image of the source on the screen, keeping its orientation in accordance with the figure below. Estimate the sizes of all elements of the image.



Consider only two specific cases:

1.3.1 $r_1 = 1{,}0\ mm$ and $r_2 = 10\ mm$;

1.3.2 $r_1 = 10\ mm$ and $r_2 = 0{,}1\ mm$.

Problem 2. Phase states and phase transitions (10.0 points)

At a given pressure, the transition from one phase state of matter to another always occurs at a strictly defined temperature, and the transition itself is called a phase transition. For example, ice at atmospheric pressure melts at 0 °C, so that when the heat is supplied, the temperature of the mixture of ice and water remains unchanged until all the ice turns into water.

In all the subtasks proposed below, consider that the specific volume of the liquid phase is negligible compared to the specific volume of saturated vapor, which can be considered an ideal gas. Assume as well that the heat capacity of liquid water is independent of temperature.

Useful physical constants

Gas constant $R = 8{,}31\, J/(mol \cdot K)$;
molar mass of air $\mu_{air} = 29{,}0\ g/mol$;
acceleration of gravity $g = 9{,}81\ m/s^2$.
Normal conditions:
pressure $P_0 = 1\ atm = 760\ mm\ Hg = 101325\ Pa$;
temperature $T_0 = 273{,}15\ K = 0$ °C.

Properties of water (H_2O)

Molar mass of water $\mu_w = 18{,}0\ g/mol$;
water density $\rho_w = 1{,}00\ g/sm^3$;
ice density $\rho_i\ =\ 0{,}920\ g/sm^3$;
melting point of ice at normal pressure $t_m = 0{,}00$ °C;
boiling point of water at normal pressure $t_b = 100{,}0$ °C;
specific heat of water $c_w = 4{,}20\, J/(g \cdot K)$;
specific heat of melting ice $q_i = 334\, J/g$;
specific heat of water vaporization (at 100 °C) $r_w = 2259\, J/g$;
Poisson's adiabatic exponent for water vapor $\gamma = C_P/C_V = 4/3$.

Specific heat of phase transition

If a transition from one phase state to another is associated with the release or absorption of a certain amount of heat, called the transition heat, then such a transition is called the first-order phase transition. In this case, the transition heat q for a unit mass is called the specific heat of the phase transition (melting, evaporation, sublimation).

Since the phase transition occurs at constant pressure, according to the first law of thermodynamics, the heat q is spent on changing the internal energy u and on performing the work A against constant external pressure:

$$q\ =\ u_2 - u_1\ +\ A,$$

where u_1, u_2 stand for the specific internal energies of the first and second phases, respectively.

During melting (crystallization), due to a small difference in the densities of the liquid and solid phases, the volume change as a result of the phase transition is small, therefore, the work A can be neglected in comparison with the change in internal energy.

2.1 Evaluate how much of the evaporation heat of water at $t_b = 100$ °C is spent on changing the internal energy. Express your answer in %.
2.2 Evaluate the specific heat of water vaporization at room temperature $t\ =\ 20{,}0$ °C.

In the following, the specific heat of vaporization of all liquids is considered to be temperature independent.

The Clausius–Clapeyron relation

When the pressure changes, the temperature of the first-order phase transition changes as well, i.e. the phase transition occurs at a strictly defined dependence $P(T)$ between the pressure P and the temperature T of the matter under investigation. This dependence, depicted on the (T,P)-plane, is called the (T,P)-phase diagram, and the P(T) curve itself is called the phase equilibrium curve. The Clapeyron-Clausius relation gives the slope of the phase equilibrium curve P(T) in the following form:

$$\frac{dP}{dT} = \frac{q}{T(v_2 - v_1)},$$

where q denotes the specific heat of transition from phase 1 with the specific volume v_1 to phase 2 with the specific volume v_2.

2.3 Assuming that the pressure of the saturated water vapor at the temperature $t_b = 100\ ℃$ is known, obtain an explicit dependence of the pressure of the saturated water vapor on its temperature $P(T)$.

2.4 Evaluate the boiling point of water at the highest peak in Kazakhstan – Khan-Tengri mountain. The height of the Khan-Tengri mountain peak is $h \approx 7000\ m$ above the sea level. The altitude air temperature should be considered constant and equal to $t_0 = 0\ ℃$.

2.5 At what pressure (in atmospheres) the ice melts at the temperature of $t = -\ 1{,}00\ ℃$?

2.6 It is known that ice crystals begin to break down if a force is applied along any direction of the crystal to create a pressure $P > P_{cr} \sim 1000\ atm$. Therefore, snow in frosty weather "crunches" when walking. Estimate the maximum air temperature t_{max} at which the snow still "crunches" when walking.

2.7 One mole of the saturated water vapor occupies a vessel and has the temperature of $t_b = 100\ ^0$C. The vapor heats up and at the same time its volume changes such that it remains saturated at all times. Find the molar heat capacity of vapor in such a process.

Border boiling

Border boiling is boiling at the interface between two immiscible liquids. The border boiling point may vary significantly from the volume boiling points of each liquid.

Tetrachloromethane or hydrogen tetrachloride is a heavy (density $\rho = 1{,}60\ g/sm^3$) transparent liquid with a molar mass $\mu = 153{,}8\ g/mol$. Under normal atmospheric pressure, carbon tetrachloride boils at a temperature of $t = 76{,}65\ ℃$, while it practically does not dissolve in water. A vessel with a volume of $V = 100\ ml$ is half-filled with the carbon tetrachloride, and the same (by volume) amount of water is poured over it. In this case, a clear water-carbon tetrachloride border is formed. When the vessel is uniformly heated in a water bath, the border boiling at the liquid interface begins at the temperature of $t^* = 66{,}0\ ℃$, which is significantly lower than the volume boiling temperature of each liquid.

2.8 Calculate the specific heat of evaporation of carbon tetrachloride, if it is known that the pressure of the saturated water vapor at the border boiling point is $P_w(t^*) = 196\ mm\ Hg$.

2.9 Find the mass of liquid remaining in the vessel by the time the other liquid is completely boiled away at such border boiling.

Consider another pair of immiscible liquids, water and fluoroketone.

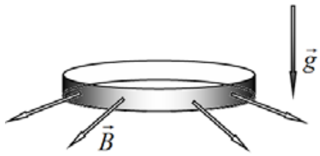
Fluoroketone, sometimes called "dry water", is used to extinguish fires in libraries, museums, and offices because it does not wet paper. It is a heavy (density $\rho = 1{,}72\ g/sm^3$) transparent liquid with a molar mass $\mu = 316\ g/mol$, which practically does not dissolve in water. The boiling point of fluoroketone at atmospheric pressure is $t_f = 49{,}2\ ℃$, its specific heat of vaporization is $r = 95{,}0\ J/g$. If water is poured over the fluoroketone into the vessel, a clear water-fluoroketone border is also formed.

2.10 Estimate the boiling point t_x of liquids at the water-fluoroketone border if the saturated vapor pressure of water is known at the volume boiling point of fluoroketone to be $P_w(t_f) = 89{,}0\ mm\ Hg$.

Problem 3. Ring in a magnetic field (10.0 points)

Uniformly charged ring

A very thin ring of mass m and radius r is uniformly charged along its length with a charge q. At the initial moment of time, the ring rests horizontally and is released without a push. The subsequent motion of the ring appears in the vertical gravitational field of the Earth, characterized by the acceleration of gravity g and in the horizontal radial magnetic field of induction B. Neglect air resistance and assume that the plane of the ring remains horizontal at all times.



3.1 Find the maximum velocity of the ring center of mass v_{max} for the entire time of motion.

3.2 Find the time interval Δt elapsed from the start of the ring motion to its first reaching of the maximum velocity of the center of mass.

3.3 Find the maximum height h_{max} at which the ring center of mass falls over the entire time of motion.

Conductive ring

A very thin ring of mass m and radius r is made of a conductive material with a resistivity ρ and a cross section area $s \ll r^2$. At the initial time moment $t = 0$ the ring rests horizontally and is released without a push. The subsequent motion of the ring appears in the vertical gravitational field of the Earth, characterized by the acceleration of gravity g and in the horizontal radial magnetic field of induction B. Neglect air resistance and assume that the plane of the ring remains horizontal at all times.

3.4 Find the steady-state velocity v_0 of the ring center of mass after a sufficiently large period of time having passed.

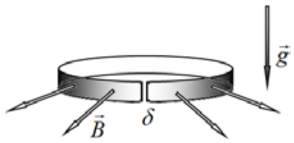
3.5 The dependence of the current strength $I(t)$ in the ring on time t has the following form

$$I(t) = A_1 + B_1 \exp(\gamma_1 t).$$

Find the constants A_1, B_1 and γ_1.

Conductive ring with a cut

A very thin ring of mass m and radius r is made of a conductive material with a resistivity ρ and a cross section area s. A cut with a width $\delta \ll \sqrt{s} \ll r$ was made along the radius of the ring. At the initial time moment $t = 0$ the ring rests horizontally and is released without a push. The subsequent motion of the ring appears in the vertical gravitational field of the Earth, characterized by the acceleration of gravity g and in the horizontal radial magnetic field of induction B. Neglect air resistance and assume that the plane of the ring remains horizontal at all times.



3.6 Find the steady-state acceleration a_0 of the ring center of mass after a sufficiently large period of time having passed.

3.7 The dependence of the current strength $I(t)$ in the ring on time t has the following form

$$I(t) = A_2 + B_2 \exp(\gamma_2 t).$$

Find the constants A_2, B_2 and γ_2.

Mathematical hints for the theoretical problems

The following integrals may be useful:

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b|.$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \text{ where } n \text{ is integer}$$

$$(1+x)^\gamma \approx 1 + \gamma x + \frac{\gamma(\gamma-1)}{2}x^2, \textit{ for } x \ll 1 \textit{ and any } \gamma$$

SOLUTIONS TO THE PROBLEMS OF THE THEORETICAL COMPETITION

Attention. Points in grading are not divided!

Problem 1 (10.0 points)

Problem 1.1 (4.0 points)

The oscillation period of a mathematical pendulum is determined by the formula

$$T = 2\pi\sqrt{\frac{l}{g}}, \tag{1}$$

where g stands for the acceleration of gravity at a given time of day.

The difference in the periods of oscillation of the pendulum at midday and midnight is due to the influence of the Sun: gravitational attraction and centrifugal force due to the Earth’s motion around the Sun. Using formula (1) for the period of pendulum oscillation the relative change in the periods can be represented as

$$\varepsilon = \frac{T_2-T_1}{T_1} = \sqrt{\frac{g_1}{g_2}} - 1, \tag{2}$$

where g_1, g_2 denotes the acceleration of gravity at midday and midnight, respectively.

The directions of the Earth's rotation around its own axis and around the Sun coincide, as shown in Figure 1. The directions of action of gravitational and centrifugal forces are different at midday and midnight, as shown in Figure 2.

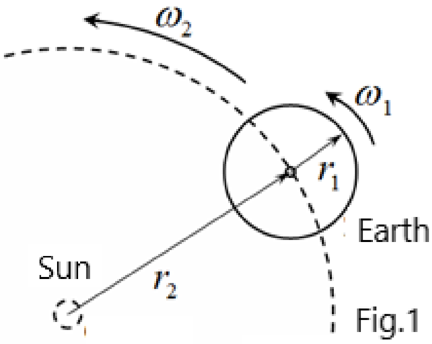


Fig.1

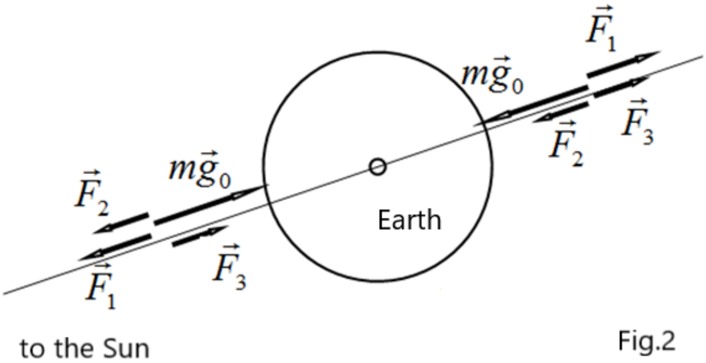


Fig.2

In Figure 2: mg_0 is the force of gravitational attraction to the Earth; F_1 is the centrifugal force due to the rotation of the Earth around its own axis; F_2 is the force of gravitational attraction to the Sun; F_3 is the centrifugal force due to the motion of the Earth around the Sun.

Then the acceleration of gravity, taking into account the influence of the Sun, are determined by the expressions:

At midday:

$$g_1 = g_0 - \omega_1^2 r_1 - G\frac{M}{(r_2-r_1)^2} + \omega_2^2 r_2. \tag{3}$$

At midnight:

$$g_2 = g_0 - \omega_1^2 r_1 + G\frac{M}{(r_2+r_1)^2} - \omega_2^2 r_2. \tag{4}$$

In the above formulas M designates the mass of the Sun and G signifies the gravitational constant.

To simplify the obtained expressions, we use the equation describing the motion of the Earth around the Sun in the following form

$$G\frac{M}{r_2^2} = \omega_2^2 r_2. \tag{5}$$

Given this relation, the acceleration difference is represented as

$$\Delta g = g_1 - g_2 = \omega_2^2 r_2\left(2 - \left(1 - \frac{r_1}{r_2}\right)^{-2} - \left(1 + \frac{r_1}{r_2}\right)^{-2}\right). \tag{6}$$

Note that in this case, in order to obtain a nonzero result in the power series expansions, it is necessary to keep the second order terms, i.e. $(1+x)^{-2} \approx 1 - 2x + 3x^2$, such that:

$$\Delta g = -6\omega_2^2 r_2 \left(\frac{r_1}{r_2}\right)^2. \tag{7}$$

Thus, the relative change in the periods of oscillations due to the influence of the Sun is equal

$$\varepsilon \approx \frac{\Delta g}{2g_2} \approx \frac{\Delta g}{2g_0}, \tag{8}$$

so that the final relation is derived as

$$\varepsilon = -3\frac{\omega_2^2 r_2}{g_0}\left(\frac{r_1}{r_2}\right)^2 \approx -3{,}3 \cdot 10^{-12}. \tag{9}$$

Content	**Points**
Formula (1): $T = 2\pi\sqrt{\frac{l}{g}}$	0,2
Formula (2): $\varepsilon = \frac{T_2 - T_1}{T_1} = \sqrt{\frac{g_1}{g_2}} - 1$	0,2
Earth’s gravity is accounted for	0,2
Sun's gravity is accounted for	0,2
Centrifugal force due to the Earth motion around the Sun is accounted for	0,2
Centrifugal force due th the Earth rotation is accounted for	0,2
Formula (3): $g_1 = g_0 - \omega_1^2 r_1 - G\frac{M}{(r_2 - r_1)^2} + \omega_2^2 r_2$	0,4
Formula (4): $g_2 = g_0 - \omega_1^2 r_1 + G\frac{M}{(r_2 + r_1)^2} - \omega_2^2 r_2$	0,4
Formula (5): $G\frac{M}{r_2^2} = \omega_2^2 r_2$	0,3
Formula (6): $\Delta g = g_1 - g_2 = \omega_2^2 r_2\left(2 - \left(1 - \frac{r_1}{r_2}\right)^{-2} - \left(1 + \frac{r_1}{r_2}\right)^{-2}\right)$	0,3
Formula (7): $\Delta g = -6\omega_2^2 r_2\left(\frac{r_1}{r_2}\right)^2$	0,4
Formula (8): $\varepsilon \approx \frac{\Delta g}{2g_2} \approx \frac{\Delta g}{2g_0}$	0,3
Formula (9): $\varepsilon = -3\frac{\omega_2^2 r_2}{g_0}\left(\frac{r_1}{r_2}\right)^2$	0,3
Numerical value in formula (9): $\varepsilon \approx -3{,}3 \cdot 10^{-12}$	0,4
Total	**4,0**

Problem 1.2 (3.0 points)

Consider a conductor with a resistivity ρ, length l and a cross section area S in which the current I flows. According to the Joule-Lenz law, the heat power dissipated in a conductor per unit of time is equal to

$$W = I^2 R, \tag{1}$$

where the current density is defined as

$$j = \frac{I}{S}, \tag{2}$$

and the resistance is found by the formula

$$R = \rho\frac{l}{S}. \tag{3}$$

It follows from formulas (1)-(3) that the heat power per unit volume is determined by the expression

$$w = \frac{W}{Sl} = \rho j^2. \tag{4}$$

On the other hand, Ohm's law is written as

$$U = IR, \tag{5}$$

in which the voltage across the conductor is expressed in terms of the field strength E in the form

$$U = El. \tag{6}$$

Hence, equation (5), taking into account (2), (3) and (6), is written in the following differential form

$$j = \frac{1}{\rho}E, \quad (7)$$

Thus, according to the Joule-Lenz law, the heat power dissipated per unit of volume of the substance is

$$w = \rho(r)j(r)^2, \quad (8)$$

where the current density is determined by the expression

$$j(r) = \frac{I}{4\pi r^2}, \quad (9)$$

with $\rho(r)$ denotes the dependence of the resistivity on the distance r to the common center of spheres.

On the other hand, Ohm's law (7) is written in the differential form as

$$j(r) = \frac{1}{\rho(r)}E(r), \quad (10)$$

where $E(r)$ stands for the electric field strength in the substance.

It follows from relations (8)-(10) that the electric field strength has the form

$$E(r) = \frac{w}{j(r)} = \frac{4\pi w}{I}r^2. \quad (11)$$

To determine the charge inside the conducting substance, we use the Gauss theorem for the closed volume, which is practically enclosed between spheres of radii a and b

$$E(b)4\pi b^2 - E(a)4\pi a^2 = \frac{Q}{\varepsilon_0}. \quad (12)$$

where Q symbolizes the total charge inside the conductive substance.

Since the volume of the substance enclosed between the two spheres is equal to

$$V = \frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3, \quad (13)$$

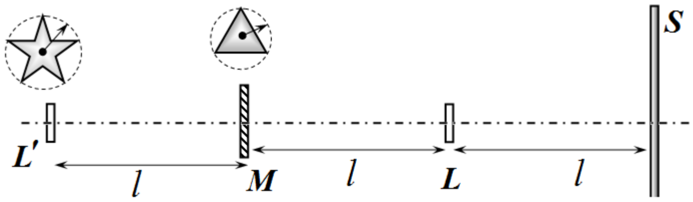
Then, the average charge density in the conducting substance is obtained as

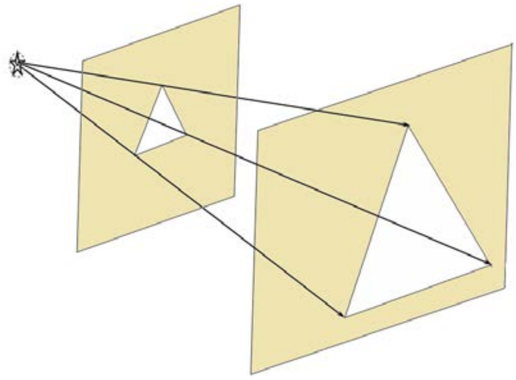
$$\rho_Q = \frac{Q}{V} = \frac{12\pi\varepsilon_0 w}{I}\left(\frac{b^4-a^4}{b^3-a^3}\right). \quad (14)$$

Content	**Points**
Formula (1): $W = I^2R$	0,2
Formula (2): $j = \frac{I}{S}$	0,2
Formula (3): $R = \rho\frac{l}{S}$	0,2
Formula (4): $w = \frac{W}{Sl} = \rho j^2$	0,2
Formula (5): $U = IR$	0,2
Formula (6): $U = El$	0,2
Formula (7): $j = \frac{1}{\rho}E$	0,2
Formula (8): $w = \rho(r)j(r)^2$	0,2
Formula (9): $j(r) = \frac{I}{4\pi r^2}$	0,2
Formula (10): $j(r) = \frac{1}{\rho(r)}E(r)$	0,2
Formula (11): $E(r) = \frac{w}{j(r)} = \frac{4\pi w}{I}r^2$	0,2
Formula (12): $E(b)4\pi b^2 - E(a)4\pi a^2 = \frac{Q}{\varepsilon_0}$	0,3
Formula (13): $V = \frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3$	0,2
Formula (14): $\rho_Q = \frac{12\pi\varepsilon_0 w}{I}\left(\frac{b^4-a^4}{b^3-a^3}\right)$	0,3
Total	**3,0**

Problem 1.3 (3.0 points)

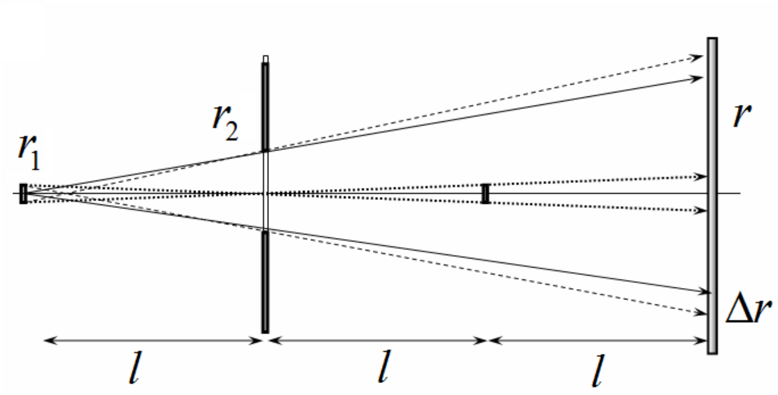
To analyze the image on the screen, it is more convenient to build first the image L' of the source in the mirror. This image is located at the distance l from the mirror and has the same dimensions as the real source.





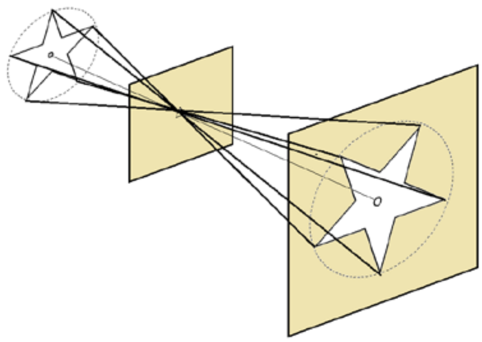
1.3.1 In this case, the source size is much smaller than the size of the mirror. As a first approximation, the source can be considered point-like. Therefore, the illuminated area on the screen has the form of a regular triangle repeating the shape of the mirror (see. fig.).

It follows from simple geometric constructions that the size of the triangle is 3 times the size of the mirror, i.e. a triangle on the screen can be inscribed in a circle of radius $r = 3r_2 = 30\ mm$.



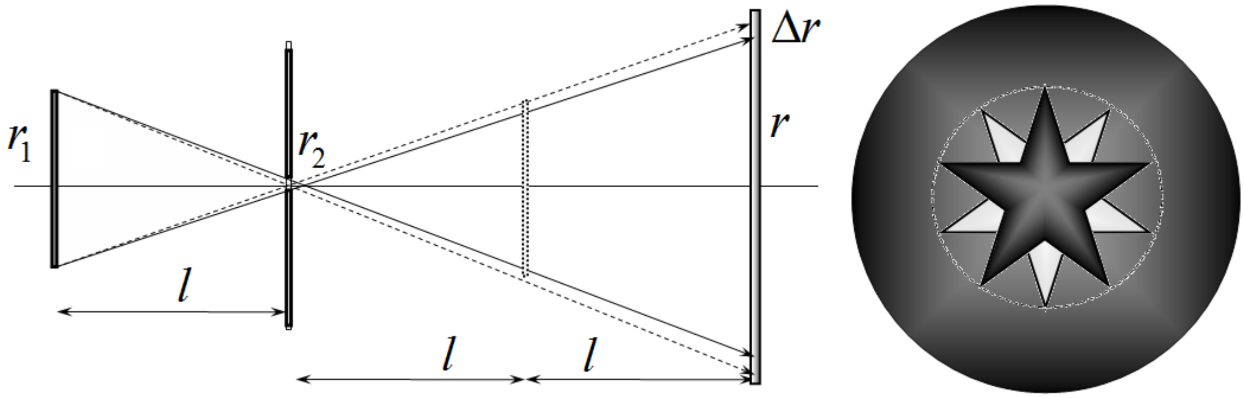
Since the source has, albeit small, but finite dimensions, the image of the triangle is to be slightly blurry, i.e. bordered by a semi-illuminated strip (border). The width of this strip is approximately equal to $\Delta r \approx 3r_1 = 3\ mm$. It can be imagined that each source point gives an image in the form of a triangle, these images are displaced relative to each other by the twice displacement of the source points.

In the center of the triangle there should be a blurred shadow from the source (shadow and semi shadow) whose radius is $r_S \approx 2r_1 = 2\ mm$.



1.3.2 In this case, the size of the source is much larger than the size of the mirror, which in the first approximation can be considered as a very small "point" hole that forms an inverted image of the source. Such an effect is used in a pinhole camera, which also forms an inverted image.

It follows from geometric constructions that a star can be inscribed in a circle of radius $r = 2r_1 = 20\ mm$. The final dimensions of the source lead to slight blurring of the image with the width of the semi-illuminated strip (border) approximately equal to $\Delta r = 2r_2 = 0.2\ mm$. Further, it should be noted that the real source creates a shadow on the screen in the form of the same five-pointed star and of the same size! However, this shadow is not inverted. Therefore, only part of the bright star is closed, as shown in the figure. Thus, only five irregular quadrangles remain illuminated on the screen.



	Content	Points	
1.3.1	The rays are correctly constructed (the image of the source, or the correct reflection of the rays);	0.3	**1.3**
	Image grading: the main part is an inverted triangle; (*if not, then the rest in this paragraph is not counted*); Triangle size - numerical value (side or radius); There is a semi-illuminated border; Border width; There is a blurred shadow in the center; The size of the shadow (partial shade) is the radius in the range of 1-2 mm;	0,3 0,1 0,2 0,1 0,2 0,1	
1.3.2	The rays are correctly constructed (the image of the source, or the correct reflection of the rays);	0,2	**1.7**
	Image grading The main part is an inverted star; (*if not, then the rest in this paragraph is not counted*); The radius of the star (numerical value); There is a border; Estimation of the border thickness; There is a shadow from the source; Shadow is not an inverted star; The size of the shadow coincides with the size of the inverted star; Illuminated areas - 5 quadrangles;	 0,4 0,2 0,1 0,2 0,2 0,1 0,2 0,1	
	Total		**3,0**

Problem 2. Phase states and phase transitions (10,0 points)
Specific heat of phase transition

2.1 The work of steam against constant external pressure during evaporation of a unit water mass is found as

$$A = P(v_2 - v_1). \qquad (1)$$

Since $v_2 \gg v_1$, we can neglect the specific volume of liquid v_1 in comparison with the specific volume of vapor v_2. Then, considering water vapor as an ideal gas with the equation of state

$$Pv = \frac{RT}{\mu_w} \qquad (2)$$

the work sought is obtained as

$$A = P(v_2 - v_1) \approx \frac{RT_b}{\mu_w}. \qquad (3)$$

Thus, the ratio of work to the total heat of evaporation at $T = 373$ К is determined by the expression

$$\frac{A}{r_{\text{в}}} = \frac{RT_b}{\mu_w r_w}, \tag{4}$$

and the rest of the heat goes to the increase of the internal energy of the system $\Delta u = r_w - A$, i.e.

$$\frac{\Delta u}{r_w} = 1 - \frac{RT_b}{\mu_w r_w} = 92{,}4\ \%. \tag{5}$$

2.2 The evaporation of one mole of water at a temperature T consumes heat

$$\mu_w r(T) = U_2(T) - U_1(T) + PV_2 = U_2(T) - U_1(T) + RT. \tag{6}$$

A similar expression for the temperature $T_b = 373$ К has the form

$$\mu_w r_w = U_2(T_b) - U_1(T_b) + RT_b\,. \tag{7}$$

Subtracting equation (7) from equation (6), we obtain for the change in the molar heat of evaporation

$$\mu_w \Delta r = \Delta U_2 - \Delta U_1 + R\Delta T = C_P \Delta T - \mu_w c_w \Delta T = \mu_w \left(\frac{C_P}{\mu_w} - c_w\right)\Delta T, \tag{8}$$

where $\Delta T = T - T_b$ and $\Delta r = r(T) - r_w$.

Given that for water vapor, the molar heat capacity at constant pressure is

$$C_P = 4R, \tag{9}$$

we obtain the specific heat of water evaporation

$$r(T) = r_w - \left(c_w - \frac{4R}{\mu_w}\right)(T - T_b) = 2447\ J/g. \tag{10}$$

It is interesting to note that the heat of evaporation is increased by $\Delta r / r_w \approx 8\ \%$.

The Clausius–Clapeyron relation

2.3 Neglecting the specific volume of water compared to the volume of vapor, we apply the Clapeyron-Clausius equation to the vaporization in the form

$$\frac{dP}{dT} = \frac{r}{Tv} = \frac{\mu_w r_w}{RT^2} P \tag{11}$$

or

$$\frac{dP}{P} = \frac{\mu_w r_w dT}{RT^2}. \tag{12}$$

Integrating this expression at $r = r_w = const$ gives rise to

$$P = P_0 \exp\left(\frac{\mu_w r_w}{R}\left(\frac{1}{T_b} - \frac{1}{T}\right)\right). \tag{13}$$

2.4 As it follows from equation (13) the explicit dependence of the boiling point of water on external pressure has the form

$$T = \frac{T_b}{1 - \frac{RT_b}{\mu_w r_w}\ln\frac{P}{P_0}}\,. \tag{14}$$

According to the barometric formula for an isothermal atmosphere, we have

$$P = P_0 \exp\left(-\frac{\mu_{air} g h}{RT_0}\right). \tag{15}$$

Substituting this expression into formula (14), we obtain the dependence of the boiling temperature on height and the numerical value of the boiling temperature of water at the altitude of $h = 7\ km$

$$T = \frac{T_b}{1 + \frac{T_b \mu_{air} g h}{T_0\ \mu_w\ r_w}} = 349{,}6\ \text{К} = 76{,}6\ ℃. \tag{16}$$

2.5 It follows from the Clapeyron-Clausius relation the following holds at the vicinity of 0 ℃

$$\frac{dP}{dT} = \frac{r_w}{T_0\left(\frac{1}{\rho_w} - \frac{1}{\rho_i}\right)}. \tag{17}$$

Therefore, we obtain that in order to lower the melting temperature of ice by 1^0C, the pressure should be increased by

$$\Delta P = \frac{dP}{dT}\Delta T = 139\ atm, \tag{18}$$

so that the pressure should be equal to $P = 140\ atm$.

2.6 In order for ice crystals to break when walking, and not to melt under the influence of pressure P_{cr}, the outdoor temperature should be lower than

$$t_{max} = \frac{P_{cr}}{(dP/dT)} \approx -7{,}21\,^{\circ}\mathrm{C}\,, \tag{19}$$

in which the derivative (dP/dT) is determined by formula (17).

2.7 Since for one mole of vapor $PV = RT$, then

$$d(PV) = PdV + VdP = RdT, \tag{20}$$

thus, the elementary work of the vapor when changing its volume is derived as

$$PdV = RdT - VdP. \tag{21}$$

From the first law of thermodynamics it follows that the heat supplied to the vapor has the form

$$\delta Q = dU + \delta A = C_V dT + RdT - VdP = C_P dT - VdP\,. \tag{22}$$

Given that from the Clapeyron-Clausius relation $dP/dT = r_w \mu_w/(T_b V)$,, we obtain the heat capacity of the vapor

$$C = \frac{\delta Q}{dT} = C_P - \frac{VdP}{dT} = C_P - \frac{\mu_w r_w}{T_b} = -75{,}7\,J/(K \cdot mol). \tag{23}$$

Thus, the heat from the vapor must be removed so that it does not overheat as a result of expansion. It is interesting to note that the specific heat in this process turned out to be almost equal to the specific heat of water with a minus sign $c = C_P/\mu_w - r_w/T_b = -4{,}21\,J/(g \cdot K)$.

Border boiling

2.8 A liquid boils when bubbles are formed inside such that the pressure of its saturated vapor reaches the atmospheric pressure P_0. At the liquids border, the total vapor pressure in the bubbles formed upon boiling is the sum of the partial pressures of the saturated vapor of carbon tetrachloride and water at t*

$$P_0 = P(t^*) + P_w(t^*). \tag{24}$$

It follows that the saturated vapor pressure of carbon tetrachloride at a boiling point is found as

$$P^* = P(t^*) = P_0 - P_w(t^*). \tag{25}$$

From the Clapeyron-Clausius relation for carbon tetrachloride it follows that

$$\frac{dP}{P} = \frac{\mu r dT}{RT^2}. \tag{26}$$

After integrating from the lower bound $T = t + 273{,}15 = 349{,}8\,K$ to the higher bound $T^* = t^* + 273{,}15 = 339{,}15\,K$ results in the following formula

$$ln(P_0/P^*) = r\mu\Delta T/RTT^*, \tag{27}$$

which means the heat of vaporization of carbon tetrachloride is obtained as

$$r = \frac{RTT^* \ln(P_0/P^*)}{\mu(T - T^*)} \approx 180\ \mathrm{J/g}. \tag{28}$$

For reference: the experimental value is $r = 195\,J/g$.

2.9 The ratio of evaporation rates from the border layer is obviously equal to the ratio of the masses of vapor of tetrachlomethane and water in the bubbles formed during boiling, which, in turn, is equal to the ratio of the densities of the vapors found as

$$\frac{m}{m_w} = \frac{\rho}{\rho_w} = \frac{P^*\mu}{P_w(t^*)\mu_w} \approx 25. \tag{29}$$

Thus, carbon tetrachloride evaporates 25 times faster (by weight) than water. This means that by the time of evaporation of carbon tetrachloride, the amount of water that finally evaporates is written as

$$\Delta m = \frac{\rho V}{2}\frac{m_w}{m} = 3{,}25\,g\,. \tag{30}$$

Accordingly, the amount of water remaining after evaporation of all carbon tetrachloride is derived as

$$M_w = \rho_w V/2 - \Delta m = 46{,}7\,g. \tag{31}$$

2.10 Let border boiling occur at a certain temperature t_x, then the saturated vapor pressure of fluoroketone P and the saturated vapor pressure of water P_w at this temperature should equal the external atmospheric pressure, i.e.

$$P_0 = P(t_x) + P_w(t_x). \tag{32}$$

Thus, the saturated vapor pressure of fluoroketone at the border boiling point decreases by the value of the saturated vapor pressure of water at this temperature

$$P(t_x) = P_0 - P_w(t_x). \tag{33}$$

From the Clapeyron – Clausius equation (in the approximation of small liquid volume and vapor ideality) it follows that the slope of the phase equilibrium line $P(T)$ at the volume boiling point of fluoroketone reads as

$$\alpha_f = \frac{\mathrm{d}P}{\mathrm{d}T} = \frac{r\mu P_0}{RT_f^2}. \tag{34}$$

For water at the same temperature, a similar derivative is more than 6 times less

$$\alpha_{\mathrm{w}} = \frac{\mathrm{d}P}{\mathrm{d}T} = \frac{\mu_w r_w P_{\mathrm{w}}(t_f)}{RT_f^2}. \tag{35}$$

Since $\alpha_f/\alpha_{\mathrm{w}} \approx 6{,}30$, the decrease in pressure and, correspondingly, in the boiling point are both small relative to the same values for fluoroketone, therefore, we can use the linear approximation near t_f

$$P_0 - P(t_x) = \alpha_f \Delta T = P_{\mathrm{w}}(t_x) = P_{\mathrm{w}}(t_f) - \alpha_{\mathrm{w}}\Delta T, \tag{36}$$

where $\Delta T = T_f - T_x$, wherefrom the lowering of the boiling point is found as

$$\Delta T = \frac{P_{\mathrm{w}}(t_f)}{(\alpha_f + \alpha_{\mathrm{w}})}. \tag{37}$$

Finally, the temperature for the border boiling is obtained as

$$t_x = t_f - \Delta T = 46{,}3\ ℃. \tag{38}$$

For reference: the experimental value is $t_x = (46 \pm 1)$ ℃.

	Content	Points	
2.1	Formula (1): $A = P(v_2 - v_1)$	0,2	**1,0**
	Formula (2): $Pv = \frac{RT}{\mu_w}$	0,2	
	Formula (4): $\frac{A}{r_{в}} = \frac{RT_b}{\mu_w r_w}$	0,2	
	Formula (5): $\frac{\Delta u}{r_w} = 1 - \frac{RT_b}{\mu_w r_w}$	0,2	
	Numerical value in formula (5): 92,4 %	0,2	
2.2	Formula (6): $\mu_w r(T) = U_2 - U_1 + PV_2 = U_2 - U_1 + RT$	0,2	**1,0**
	Formula (7): $\mu_w r_w = U_2(T_b) - U_1(T_b) + RT_b$	0,2	
	Formula (9): $C_P = 4R$	0,2	
	Formula (10): $r(T) = r_w - \left(c_w - \frac{4R}{\mu_w}\right)(T - T_b)$	0,2	
	Numerical value in formula (10): $2447\ J/g$	0,2	
2.3	Formula (11): $\frac{dP}{dT} = \frac{r}{Tv} = \frac{\mu_w r_w}{RT^2}P$	0,2	**0,4**
	Formula (13): $P = P_0 \exp\left(\frac{\mu_w r_w}{R}\left(\frac{1}{T_b} - \frac{1}{T}\right)\right)$	0,2	
2.4	Formula (14): $T = \frac{T_b}{1 - \frac{RT_b}{\mu_w r_w}\ln\frac{P}{P_0}}$	0,2	**1,0**
	Formula (15): $P = P_0 \exp\left(-\frac{\mu_{air} g h}{RT_0}\right)$	0,4	
	Formula (16): $T = \frac{T_b}{1 + \frac{T_b \mu_{air} g h}{T_0\ \mu_w\ r_w}}$	0,2	
	Numerical value in formula (16): 76,6 ℃	0,2	
2.5	Formula (17): $\frac{dP}{dT} = \frac{q_i}{T_0\left(\frac{1}{\rho_w} - \frac{1}{\rho_i}\right)}$	0,2	**0,6**

	Formula (18): $\Delta P = \frac{dP}{dT}\Delta T$	0,2	
	Numerical value in formula (17): $\Delta P = 139\ atm$ or $P = 140\ atm$	0,2	
2.6	Formula (19): $t_{max} = \frac{P_{cr}}{(dP/dT)}$	0,4	**0,6**
	Numerical value in formula (19): $t_{max} \approx -7{,}21$ ℃	0,2	
2.7	Formula (21): $PdV = RdT - VdP$	0,2	**1,0**
	Formula (22): $\delta Q = dU + \delta A = C_V dT + RdT - VdP = C_P dT - VdP$	0,2	
	Formula (23): $C = \frac{\delta Q}{dT} = C_P - \frac{VdP}{dT} = C_P - \frac{\mu_w r_w}{T_b}$	0,4	
	Numerical value in formula (23): $C = -75{,}7\ J/(K \cdot mol)$	0,2	
2.8	Formula (24): $P_0 = P(t^*) + P_w(t^*)$	0,4	**1,2**
	Formula (26): $\frac{dP}{P} = \frac{\mu r dT}{RT^2}$	0,2	
	Formula (27): $\ln(P_0/P^*) = r\mu\Delta T/RTT^*$	0,2	
	Formula (28): $r = \frac{RTT^* \ln(P_0/P^*)}{\mu(T-T^*)}$	0,2	
	Numerical value in formula (28): $r \approx 180\ J/g$.	0,2	
2.9	Formula (29): $\frac{m}{m_w} = \frac{\rho}{\rho_w} = \frac{P^*\mu}{P_w(t^*)\mu_w}$	0,4	**1,0**
	Formula (30): $\Delta m = \frac{\rho V}{2}\frac{m_w}{m}$	0,2	
	Formula (31): $M_w = \rho_w V/2 - \Delta m$	0,2	
	Numerical value in formula (31): $M_w = 46{,}7\ g$	0,2	
2.10	Formula (32): $P_0 = P(t_x) + P_w(t_x)$	0.2	**2,2**
	Formula (34): $\alpha_f = \frac{dP}{dT} = \frac{r\mu P_0}{RT_f^2}$	0,3	
	Formula (35): $\alpha_w = \frac{dP}{dT} = \frac{r_w \mu_w P_w(t_f)}{RT_f^2}$	0,3	
	Estimation: $\alpha_f/\alpha_w \approx 6{,}30$	0,3	
	Formula (36): $P_0 - P(t_x) = \alpha_f \Delta T = P_w(t_x) = P_w(t_f) - \alpha_w \Delta T$	0,3	
	Formula (37): $\Delta T = \frac{P_w(t_f)}{(\alpha_f + \alpha_w)}$	0,4	
	Formula (38): $t_x = t_f - \Delta T$	0,2	
	Numerical value in formula (38): $t_x = 46{,}3$ ℃	0,2	
Total			**10,0**

Problem 3. Ring in a magnetic field (10.0 points)
Uniformly charged ring

3.1 Under the action of gravity, the center of mass of the ring acquires a velocity v directed vertically downward. In this case, the Lorentz force F_{L1} arises, leading to the rotation of the ring around its own axis with an angular velocity ω, which in turn leads to the appearance of the vertical component of the Lorentz force F_{L2} directed against gravity, regardless of the sign of the ring charge.

The equation of motion of the ring center of mass has the form

$$m\frac{dv}{dt} = mg - F_{L2}, \qquad (1)$$

where the Lorentz force is written as

$$F_{L1} = qv_{rot}B, \qquad (2)$$

and the linear speed of ring rotation

$$v_{rot} = \omega r. \qquad (3)$$

Thus, the equation of motion of the ring center of mass is finally derived as

$$m\frac{dv}{dt} = mg - q\omega rB. \tag{4}$$

The equation of rotational motion of the ring is written as

$$I\frac{d\omega}{dt} = M_{L1}, \tag{5}$$

where the torque of the Lorentz force moment F_{L1} is determined by the expression

$$M_{L1} = qvBr, \tag{6}$$

and the moment of inertia of the ring is equal to

$$I = mr^2. \tag{7}$$

Let h be the vertical displacement of the ring center of mass, then its speed is

$$v = \frac{dh}{dt}. \tag{8}$$

Putting together (5), (6) and (8) and integrating over time, taking into account the initial condition $\omega = 0$ at $h = 0$, we obtain the relation

$$I\omega = qBrh. \tag{9}$$

At the moment when the speed of the ring center of mass is maximum, the total force on the right side of equation (4) vanishes, which leads to the expression

$$mg = qBr\omega_0. \tag{10}$$

Using relation (9) for this particular moment in time

$$I\omega_0 = qBrh_0, \tag{11}$$

we apply the law of energy conservation in the following form

$$mgh_0 = \frac{mv_{max}^2}{2} + \frac{I\omega_0^2}{2}. \tag{12}$$

Solving equations (10)-(12) together with the expression for the moment of inertia (7), we find the maximum velocity of the ring center of mass in the following form

$$v_{max} = \frac{mg}{qB}. \tag{13}$$

3.2 Substituting relation (9) into the equation (4) of motion of the ring center of mass and using (8), we obtain the equation of harmonic oscillations

$$m\frac{d^2h}{dt^2} = mg - \frac{(qB)^2}{m}h \tag{14}$$

with the frequency

$$\omega_L = \frac{qB}{m}. \tag{15}$$

The time sought is quarter of the period of oscillation, i.e.

$$\Delta t = \frac{\pi}{2\omega_L} = \frac{\pi m}{2qB}. \tag{16}$$

3.3 The initial velocity of the ring center of mass is zero and reaches its maximum at the moment of passage of the equilibrium position, therefore, the maximum height h_{max} by which the ring center of mass descends is obtained as

$$h_{max} = 2h_0 = \frac{2gm^2}{q^2B^2}. \tag{17}$$

Conductive ring

3.4 Under the action of gravity, the ring center of mass acquires a velocity v directed vertically downward. In this case, an induction current I arises in the ring as a result of the action of a magnetic field, which leads to the appearance of a vertical Lorentz force F_L directed against gravity.

The equation of motion of the ring center of mass has the form

$$m\frac{dv}{dt} = mg - F_L, \tag{18}$$

and the Lorentz force is determined by the expression

$$F_L = BIL, \tag{19}$$

with the ring length

$$L = 2\pi r. \tag{20}$$

When moving in a magnetic field, the following electromotive force arises in the ring

$$\mathcal{E} = \frac{d\Phi}{dt} = BLv, \tag{21}$$

which, according to Ohm's law, leads to the appearance of the induction current of strength

$$\mathcal{E} = IR, \tag{22}$$

where the ring resistance is

$$R = \rho\frac{L}{s}. \tag{23}$$

In the steady-state fall mode of the ring center, its velocity $v = v_0$ remains unchanged, then from (18)-(23) we obtain

$$v_0 = \frac{mg\rho}{2\pi r s B^2}. \tag{24}$$

3.5 Expressing the velocity from (21), (22) and substituting it into equation (18), as well as using (19), we obtain the differential equation

$$\frac{mR}{BL}\frac{dI}{dt} = mg - BLI, \tag{25}$$

with the initial condition

$$I(0) = 0. \tag{26}$$

The solution to equation (25) when (26) is satisfied is the function

$$I(t) = \frac{mg}{2\pi r B}\left[1 - \exp\left(-\frac{2\pi r s B^2}{m\rho}t\right)\right]. \tag{27}$$

Whence,

$$A_1 = \frac{mg}{2\pi r B}, \tag{28}$$

$$B_1 = -\frac{mg}{2\pi r B}, \tag{29}$$

$$\gamma_1 = -\frac{2\pi r s B^2}{m\rho}. \tag{30}$$

Conductive ring with a cut

3.6 The equation of motion of the ring center of mass of the is still described by equations (18)-(20), and an electromotive force (21) is also generated in the ring. However, in this case, charges of the opposite sign accumulate at the cut edges; therefore, instead of Ohm's law (22), we have

$$\mathcal{E} - \frac{q}{C} = IR, \tag{31}$$

where

$$C = \frac{\varepsilon_0 S}{\delta}. \tag{32}$$

Since the cut edges are charged by induction current, then

$$I = \frac{dq}{dt}. \tag{33}$$

In the steady state, the acceleration of the ring center of mass is constant, so according to the equation of motion (18) and (19), the current strength is also constant. Differentiating (31) with (33) and (21) taken into account, we finally obtain the steady-state acceleration

$$a_0 = \frac{g}{\left(1+\frac{B^2(2\pi r)^2\varepsilon_0 S}{m\delta}\right)}. \tag{34}$$

3.7 Differentiating (31) with (21) and (33) taken into account, we obtain

$$BL\frac{dv}{dt} = \frac{I}{C} + R\frac{dI}{dt}. \tag{35}$$

Dividing this equation by the equation of motion (18) and substituting (19), we obtain the differential equation for the current in the ring

$$R\frac{dI}{dt} = gBL - \left(\frac{1}{C} + \frac{B^2L^2}{m}\right)I \tag{36}$$

with the initial condition

$$I(0) = 0. \tag{37}$$

The solution of equation (36) with (37) is the function

$$I(t) = \frac{2\pi r g \varepsilon_0 s B}{\delta\left(1+\frac{B^2(2\pi r)^2\varepsilon_0 s}{m\delta}\right)}\left[1 - \exp\left(-\left(1 + \frac{B^2(2\pi r)^2\varepsilon_0 s}{m\delta}\right)\frac{\delta}{2\pi r\rho\varepsilon_0}t\right)\right]. \tag{38}$$

Hence,

$$A_2 = \frac{2\pi r g\varepsilon_0 s B}{\delta\left(1+\frac{B^2(2\pi r)^2\varepsilon_0 s}{m\delta}\right)}, \tag{39}$$

$$B_2 = -\frac{2\pi r g \varepsilon_0 s B}{\delta\left(1+\frac{B^2(2\pi r)^2\varepsilon_0 s}{m\delta}\right)}, \tag{40}$$

$$\gamma_2 = -\left(1+\frac{B^2(2\pi r)^2\varepsilon_0 s}{m\delta}\right)\frac{\delta}{2\pi r\rho\varepsilon_0}. \tag{41}$$

	Content	Points	
3.1	Equation (1): $m\frac{dv}{dt} = mg - F_{L2}$	0,3	**3,2**
	Formula(2): $F_{L1} = qv_{rot}B$	0,2	
	Formula (3): $v_{rot} = \omega r$	0,2	
	Equation (4): $m\frac{dv}{dt} = mg - q\omega rB$	0,2	
	Equation (5): $I\frac{d\omega}{dt} = M_{L1}$	0,3	
	Formula(6): $M_{L1} = qvBr$	0,2	
	Formula (7): $I = mr^2$	0,2	
	Formula (8): $v = \frac{dh}{dt}$	0,2	
	Formula (9): $I\omega = qBrh$	0,4	
	Formula (10): $mg = qBr\omega_0$	0,4	
	Formula (12): $mgh_0 = \frac{mv_{max}^2}{2} + \frac{I\omega_0^2}{2}$	0,3	
	Formula (13): $v_{max} = \frac{mg}{qB}$	0,3	
3.2	Equation (14): $m\frac{d^2h}{dt^2} = mg - \frac{(qB)^2}{m}h$	0,3	**0,8**
	Formula (15): $\omega_L = \frac{qB}{m}$	0,2	
	Formula(16): $\Delta t = \frac{\pi}{2\omega_L} = \frac{\pi m}{2qB}$	0,3	
3.3	Formula(17): $h_{max} = 2h_0 = \frac{2gm^2}{q^2B^2}$	0,2	**0,2**
3.4	Equation (18): $m\frac{dv}{dt} = mg - F_L$	0,3	**1,8**
	Formula (19): $F_L = BIL$	0,2	
	Formula (20): $L = 2\pi r$	0,2	
	Formula (21): $\mathcal{E} = \frac{d\Phi}{dt} = BLv$	0,3	
	Formula (22): $\mathcal{E} = IR$	0,3	
	Formula (23): $R = \rho\frac{L}{s}$	0,2	
	Formula (24): $v_0 = \frac{mg\rho}{2\pi r s B^2}$	0,3	
3.5	Equation (25): $\frac{mR}{BL}\frac{dI}{dt} = mg - BLI$	0,2	**1,0**
	Condition (26): $I(0) = 0$	0,2	
	Formula (28): $A_1 = \frac{mg}{2\pi rB}$	0,2	
	Formula (29): $B_1 = -\frac{mg}{2\pi rB}$	0,2	
	Formula (30): $\gamma_1 = -\frac{2\pi r s B^2}{m\rho}$	0,2	
3.6	Equation (31): $\mathcal{E} - \frac{q}{C} = IR$	0,3	**1,0**
	Formula (32): $C = \frac{\varepsilon_0 S}{\delta}$	0,2	
	Formula (33): $I = \frac{dq}{dt}$	0,2	

	Formula (34): $a_0 = \frac{g}{\left(1+\frac{B^2(2\pi r)^2\varepsilon_0 s}{m\delta}\right)}$	0,3	
3.7	Equation (35): $BL\frac{dv}{dt} = \frac{I}{C} + R\frac{dI}{dt}$	0,5	**2,0**
	Equation (36): $R\frac{dI}{dt} = gBL - \left(\frac{1}{C} + \frac{B^2L^2}{m}\right)I$	0,5	
	Condition (37): $I(0) = 0$	0,1	
	Formula (39): $A_2 = \frac{2\pi r g\varepsilon_0 sB}{\delta\left(1+\frac{B^2(2\pi r)^2\varepsilon_0 s}{m\delta}\right)}$	0,3	
	Formula (40): $B_2 = -\frac{2\pi r g\varepsilon_0 sB}{\delta\left(1+\frac{B^2(2\pi r)^2\varepsilon_0 s}{m\delta}\right)}$	0,3	
	Formula (41): $\gamma_2 = -\left(1 + \frac{B^2(2\pi r)^2\varepsilon_0 s}{m\delta}\right)\frac{\delta}{2\pi r\rho\varepsilon_0}$	0,3	
Total			**10,0**

EXPERIMENTAL COMPETITION

11 January, 2020

Please read the instructions first:

1. The Experimental competition consists of one problem. This part of the competition lasts 3 hours.
2. Please only use the pen that is provided to you.
3. You can use your own non-programmable calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet and additional papers***. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*. Write only inside the bordered area.
6. Fill the boxes at the top of each sheet of paper with your country (***Country*), your student code (*Student Code***), the question number (***Question Number***), the progressive number of each sheet **(*Page Number***), and the total number of ***Writing sheets*** (***Total Number of Pages***). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
7. At the end of the exam, arrange all sheets for each problem in the following order:
   - Used ***Writing sheets*** in order.
   - The sheets you do not wish to be evaluated.
   - Unused sheets.
   - The printed problems.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper or equipment out of the room

Maxwell's Disc (15,0 points)

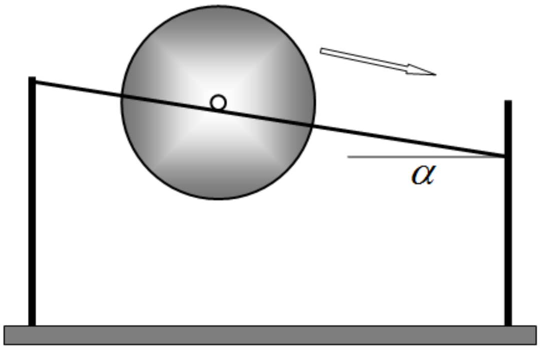
In this experiment, you are asked to design all the experimental setups by yourself, connect threads and mark them up if necessary. You can further fasten the discs with pieces of plasticine.

Instruments and equipment: Maxwell's disk (two disks mounted on a wooden stick), two tripods, threads, a wooden stick, stopwatch, ruler, 2 loads of 100 g each, a piece of plasticine, adhesive tape.

Do not calculate experimental errors if you are not asked to do so.

Part 1. Rolling down

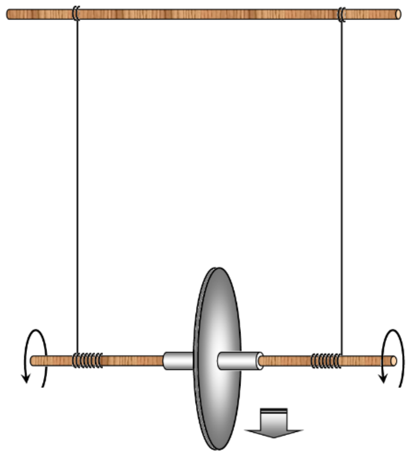
Pull two threads between the two tripods, so that they are parallel to each other and make up some angle α to the horizontal. Maxwell's disc should slide without slipping on these threads, resting on the wooden parts of the stick.



1.1 Derive the formula for the acceleration with which the axis of the disk rolls, depending on the angle of inclination α. Assume the disc is homogeneous and neglect the mass of the wooden stick.
1.2 Investigate experimentally the law of motion, i.e. the dependence of the coordinate of the axis of the disk $x(t)$ along the thread on time t, when it rolls along the threads. Draw a graph of the resulting dependence. Prove that the motion of the disc axis is approximately uniformly accelerated. Calculate the disk axis acceleration. Indicate at what angle you have taken your measurements.
1.3 Measure the dependence of the acceleration of the disk axis on the angle of inclination of the threads α. Draw a graph of the resulting dependence.
1.4 On the same graph plot the theoretical dependence of the acceleration on the angle of inclination α of the threads. Indicate the results of additional measurements that you have used to calculate the accelerations.

Part 2. Moving down

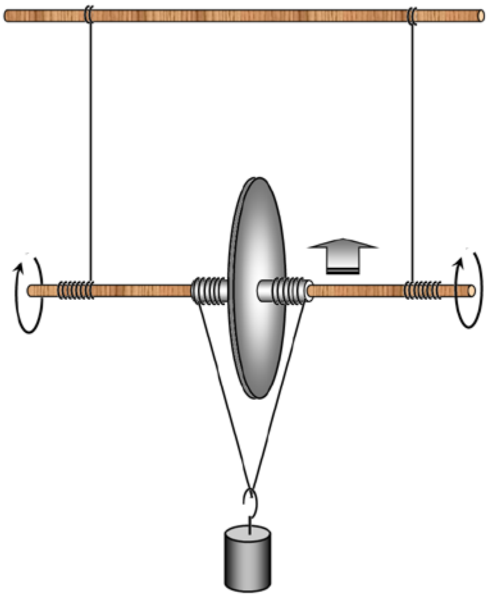
Hang the disc on two threads as shown. Fix the threads on the stick with the adhesive tape to prevent them from slipping. Wind the threads onto the sticks and release the disc.



2.1 Investigate the law of motion of the disk axis when it moves down. Draw a graph of the resulting dependence.
2.2 According to the measurement data, calculate the acceleration at which the axis of the disc moves down. Estimate the experimental error of the obtained acceleration.
2.3 Derive the theoretical formula for the acceleration of the disc axis. Calculate the theoretical value of the acceleration with which the axis of the disc moves down.

Part 3. Moving up

If two additional threads are wound on the stick and a load is suspended on them, then with proper winding of the threads, the disk starts to move up.



3.1 Design the experimental setup such that the disk moves up spinning around. Show schematically in the figure how the threads should be wound on the stick.
3.2 Measure the dependence of the acceleration of the disk axis for different masses of suspended loads.

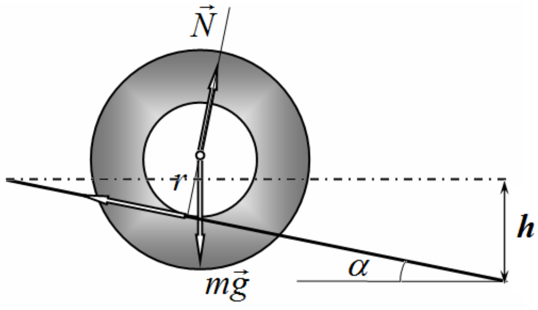
SOLUTION TO THE EXPERIMENTAL COMPETITION

Maxwell's disk (15,0 points)

Part 1. Rolling down

1.1 The basic equation of the dynamics of rotational motion relative to the point of contact of the stick and thread is written as

$$\left(\frac{mR^2}{2}+mr^2\right)\frac{a}{r}=mgr\sin\alpha\,, \qquad (1)$$



where R stands for the disc radius and r denotes the stick radius.

The formula for the disc axis acceleration is thus obtained as

$$a=\frac{g}{1+\frac{1}{2}\left(\frac{R}{r}\right)^2}\sin\alpha. \qquad (2)$$

1.2 To change and measure the angle that the threads make with the horizontal, it is necessary to shift up one of the holders of the tripod to the height h. Therefore, the sine of the inclination angle is found as

$$\sin\alpha=\frac{h}{L}, \qquad (3)$$

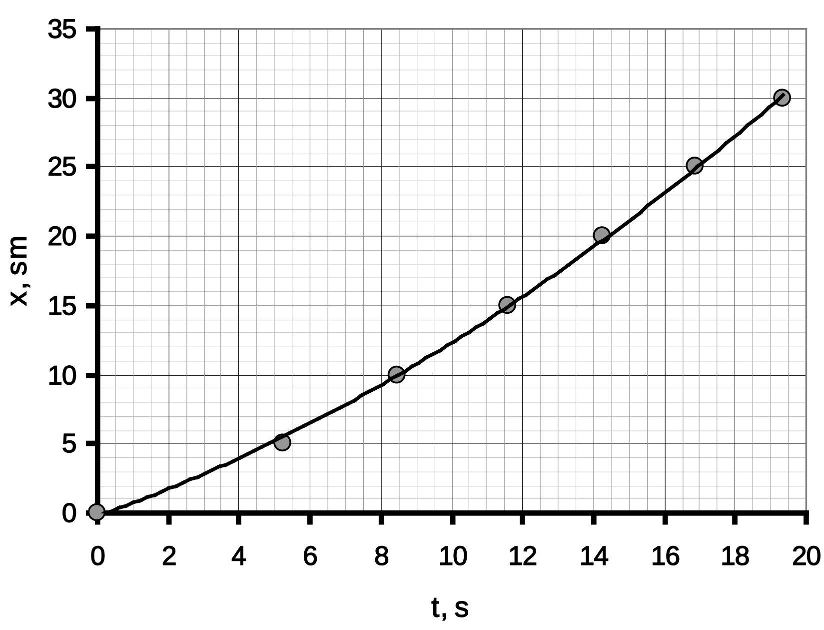
where L designates the threads' length (in our experiment $L=70sm$).

In this part, the measurements have been taken for $h=3,0sm$. It is necessary to put marks on the threads at regular intervals and fix the travel time with a stopwatch. The difficulty is that in practice it is difficult to release the disk from the first mark without an initial push, therefore the first mark is used as a reference point, but the disk speed is thus not zero! The time dependence of the disc axis position is shown in Table 1 and in Fig. 1.

Table 1.

x. см	*t, c*	*x/t*
0	0	
5	5,24	0,95
10	8,44	1,18
15	11,57	1,30
20	14,25	1,40
25	16,90	1,48
30	19,34	1,55

Fig. 1



The law of uniform acceleration has the form

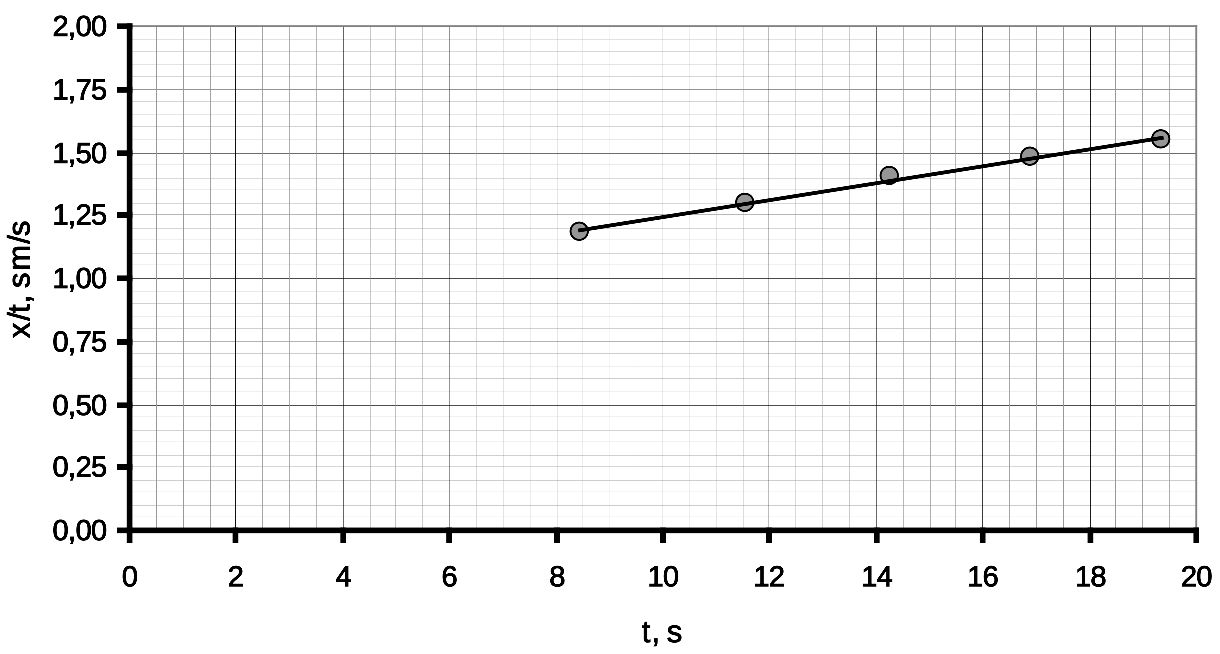
$$x(t)=v_0t+\frac{at^2}{2}. \qquad (4)$$

Various methods can be used for linearization, but the following is preferred:

$$\frac{x}{t} = v_0 + \frac{a}{2}t. \tag{5}$$

Fig. 2 shows a plot of the value $\frac{x}{t}$ (actually, average speed) versus time t.

Fig. 2



The approximate linearity of this dependence proves that the experimental law of motion can be described by function (4).

The acceleration of the disk axis is equal to twice the value of the slope of this graph and it is evaluated by the method of least squares to be equal to

$$a = 0{,}081 \frac{sm}{s^2}. \tag{6}$$

1.3 Similar measurements have to be carried out for other values of the inclination angles of the threads. The corresponding results are shown in Table 2.

Table 2

h=5 sm			h=7 sm			h=9 sm			h=11 sm		
x. sm	*t, s*	*x/t*	*x. sm*	*t, s*	*x/t*	*x. sm*	*t, s*	*x/t*	*x. sm*	*t, s*	*x/t*
0			0			0			0		
5	4,66	1,07	5	3,53	1,42	5	3,30	1,52	5	3,38	1,48
10	7,51	1,33	10	5,81	1,72	10	5,24	1,91	10	5,32	1,88
15	9,95	1,51	15	7,94	1,89	15	7,14	2,10	15	6,92	2,17
20	12,28	1,63	20	9,70	2,06	20	8,73	2,29	20	8,36	2,39
25	14,23	1,76	25	11,26	2,22	25	10,10	2,48	25	9,45	2,65
30	16,10	1,86	30	12,84	2,34	30	11,46	2,62	30	10,89	2,75

Figure 3 shows the graphs of the dependences $\frac{x}{t}$ on time t which are used to calculate the experimental values of the accelerations.

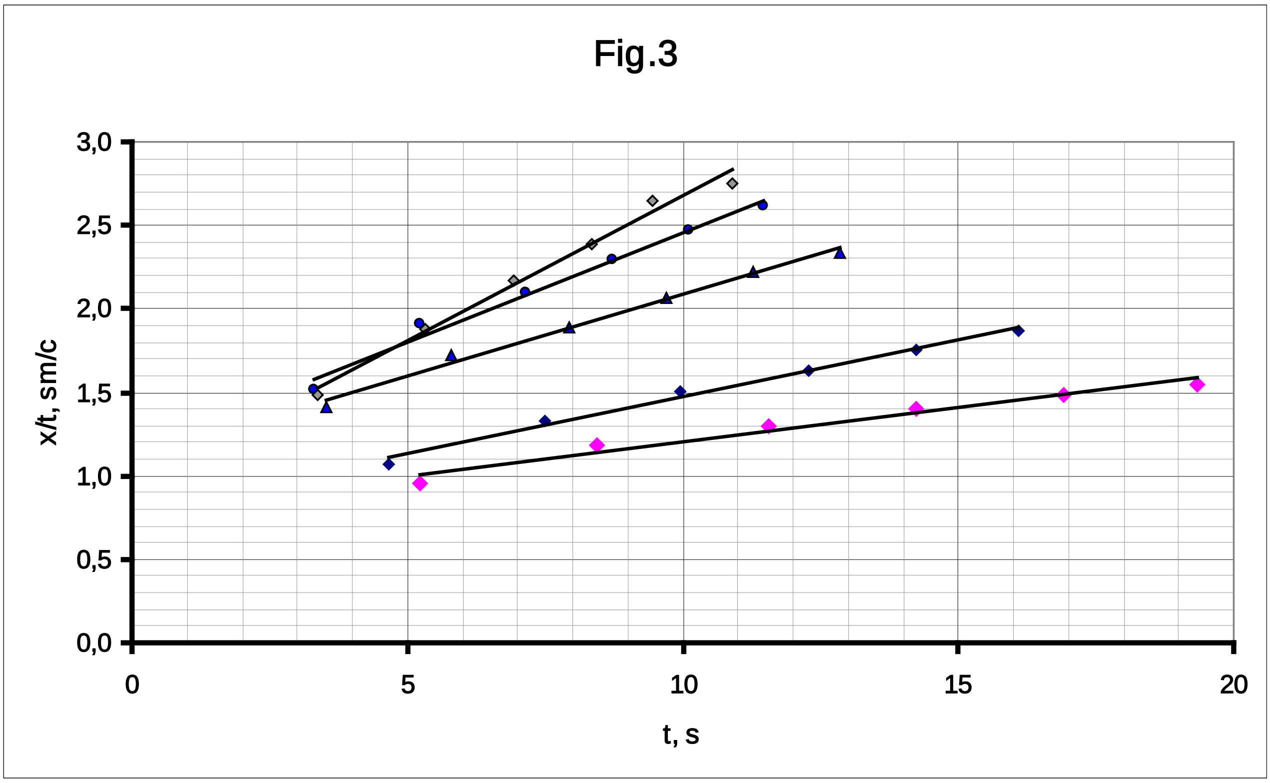
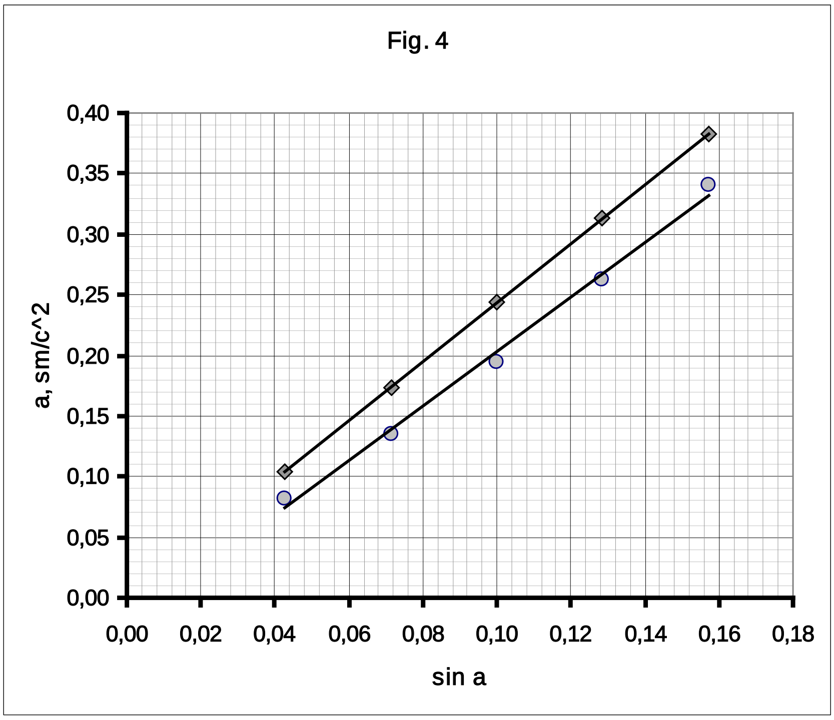


Table 3 shows the accelerations calculated by the slope coefficients of the graphs and corresponding theoretical values evaluated via formula (2). In the calculations, the following measured values are taken: the disk diameter $D = 85mm$ and the stick diameter $d = 3,0mm$. Fig. 4 shows graphs of the dependences of the accelerations on the angle of threads inclination.

Table 3

h , sm	a, theor. sm/s^2	a, exper. sm/s^2	$\sin\alpha$
3	0,104	0,081	0,043
5	0,174	0,135	0,071
7	0,244	0,195	0,100
9	0,313	0,262	0,129
11	0,383	0,340	0,157

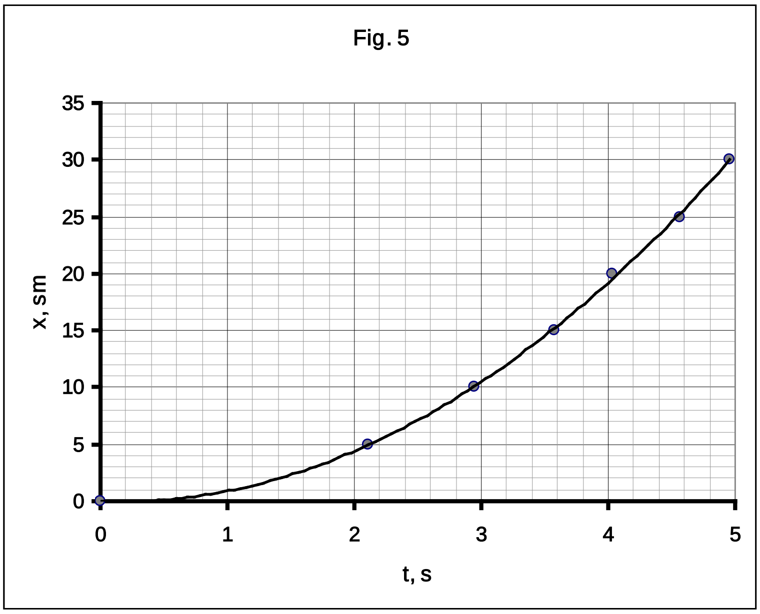


Part 2. Moving down

2.1 In this case, the travel time is small, therefore, measurements should be carried out for each coordinate repeatedly. Table 4 lists the results of the time measurements needed for the disc to travel distance x, averaged over 3 measurements. Fig. 5 demonstrates a graph of the obtained dependence.

Table 4.

x, sm	t, s	x/t	t^2
0	0		0
5	2,11	2,37	4,45
10	2,95	3,39	8,70
15	3,58	4,19	12,82
20	4,03	4,96	16,24
25	4,56	5,48	20,79
30	4,96	6,05	24,60

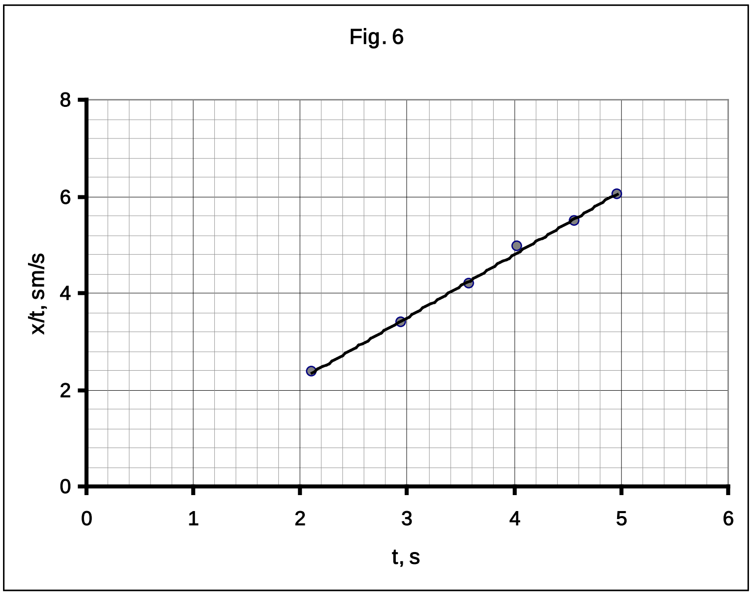


To calculate the acceleration, we use the previous methodology: we draw the dependence $\frac{x}{t}$ on t (Fig. 6) and find the parameters of the linearized dependence.

The least squares calculations give the following values of the coefficients of the dependence $\frac{x}{t} = Kt + b$:

$$K = (1{,}30 \pm 0{,}07)\frac{sm}{s^2}$$

$$b = (-0{,}40 \pm 0{,}3)\frac{sm}{s}$$



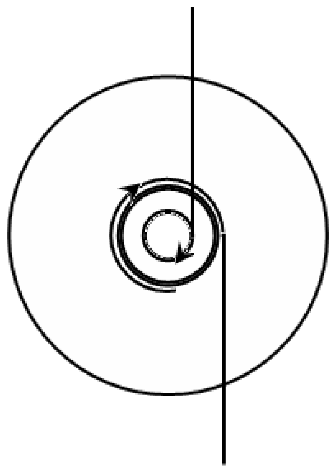
Then, the disc axis acceleration is found as

$$a = (2{,}60 \pm 0{,}13)\frac{sm}{s^2}. \quad (7)$$

2.3 The formula for the acceleration is automatically obtained from formula (2), in which $\sin\alpha = 1$:

$$a = \frac{g}{1 + \frac{1}{2}\left(\frac{r}{R}\right)^2}. \quad (8)$$

The calculation using this formula gives the value of $a = 2{,}44\frac{sm}{s^2}$.



Part 3. Moving up

3.1 Threads should be wound such that when the threads tied to the load are untwisted, the threads attached to the Maxwell disk are twisted. It is also obvious that the threads with the load should be wound on a part of the stick with a larger radius.

3.2 In this case, the beginning of the motion of the disk axis is easily recorded, so you can simply measure the rise time to a fixed height and calculate the acceleration according to the formula:

$$H = \frac{at^2}{2} \quad \Rightarrow \quad a = \frac{2H}{t^2}. \tag{9}$$

During the measurements, the following data shown in Table 5 (at $H = 12sm$) have been obtained.

Table 5

n	t_1, s	t_2, s	t_3, s	$\langle t \rangle$, s	$a\ \frac{sm}{s^2}$
1	2,07	2,13	2,23	2,14	5,22
2	1,6	1,56	1,63	1,60	9,41

Marking scheme

	Content	**Total points for parts**	**Points**
Part 1. Rolling down		**8**	
1.1	Formula (2) $a = \dfrac{g}{1+\frac{1}{2}\left(\frac{r}{R}\right)^2}\sin\alpha$	**0,3**	0.3
1.2	Experimental setup is properly designed; appropriate data are obtained by order of magnitude	**0,5**	0,5
	The value of $\sin\alpha$ is stated: Measurement method; Numerical value lies in the range of 0,04 – 0,2;	**0,2**	 0,1 0,1
	The dependence measurement Marked only if $\sin\alpha$ is in the range above, Data deviation from the official solution is within 25%: - range of coordinate measurement exceeds 20 sm (*exceeds 10 sm, less than 10 sm*); - number of data points 5 and more (*4, less than 4*); - the nonlinear dependence is obtained which is close to parabolic;	**1,2**	 0,5(*0,3;0*) 0,5(*0,3;0*) 0,2
	Graph of $x(t)$ (*marked only if the corresponding data have been marked*): - axes are named and ticked; - all data points are in the graph; - smooth curve is plotted;	**0,3**	 0,1 0,1 0.1
	Analysis of the uniform acceleration model: - the law of motion $x(t) = v_0 t + \frac{at^2}{2}$; - *Without the initial velocity;*	**0,2**	 0.2 *(0)*
	The analysis methodology: - linearization $\frac{x}{t} = v_0 + \frac{a}{2}t$; - *calculation of velocities* $v(t)$; - *linearization* $x(t^2)$; - *acceleration calculation by 2-3 points;*	**1,0**	 1 *(0,5)* *(0.2)* *(0.1)*
	Graph of the linearized dependence	**0.3**	

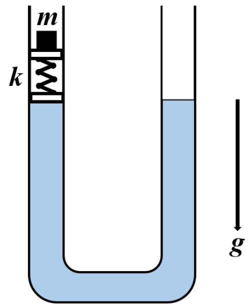
	(*marked only if corresponding data have been marked*): - axes are named and ticked; - all data points are in the graph; - smooth curve is plotted;		 0,1 0,1 0.1
	Calculation of the acceleration (not marked if there is no unit); - according to the linearized dependence (LSM, graph); - by 2-3 points; - by 1 point;	**0,4**	 0,4 *(0,2)* *(0,1)*
1.3	Acceleration dependence on the angle Marked only if data deviation from the official solution is within 25%: Sine of the angle is within the range 0,04 – 0,2 : - number of angles taken is 4 or more (*3, less than 3*); - number of data points in each dependence is 5 or more (3-4; less than 3);	**1,4**	 0,2 0,8*(0,5;0)* 0,4(0,2;0)
	Acceleration calculation (for each point but no more than 4): - linearization; - *by 1-3 points*; Calculation of $\sin\alpha$;	**1,0**	 0,2x4 (0,1x4) 0.2
	Graph of the angle dependence of the acceleration (*marked only if corresponding data points have been marked*): - axes are named and ticked; - all data points are in the graph; - smooth curve is plotted;	**0,3**	 0.1 0,1 0.1
1.4	Disc and stick radii are measured; Correct formula is used for calculations; Linear dependence is drawn; Experimental data lie systematically below the theoretical curve;	**0,9**	0,2 0,1 0,1 0,5
Part 2. Moving down		**4**	
	Experimental setup is properly designed; appropriate data are obtained by order of magnitude	**0,5**	0,5
2.1	Experimental data Marked only if data deviation from the official solution is within 25%: - the range of the coordinate measurements exceeds 20 sm (exceeds 10 sm; less than 10); - number of data points is 5 or more (3-4; less than 3); - average is done by 3 or more repeatitions; - the dependence close to parabolic is obtained;	**1,5**	 0,5(*0,3;0*) 0,5(*0,3;0*) 0,3 0,2
	Graph of $x(t)$ (marked only if corresponding data have been marked): - axes are named and ticked; - all data points are in the graph; - smooth curve is plotted;	**0,3**	 0,1 0,1 0.1
2.2	Acceleration calculations (marked only if corresponding data have been marked): - linearization is used (allowed to use $x(t^2)$; - acceleration is calculated using the linearized depemdence (by 1-2 points); - experimental error is calculated;	**1,3**	 0,5 0,5 0,3
2.3	Correct formula for the acceleration is derived; Numerical value is correctly evaluated;	**0.4**	0,2 0.2
Part 3. Moving up		**3**	

3.1	Correct schematic figure for wounding the threads (threads are on one side of the stick);		1
3.2	Time of moving up is measured (within the range of 0,7 – 3,0 s)		0,5x2
	Accelerations are calculated (within 50% deviation from the official solution)		0,5x2
	Total	**15**	

THEORETICAL COMPETITION
January 8, 2021

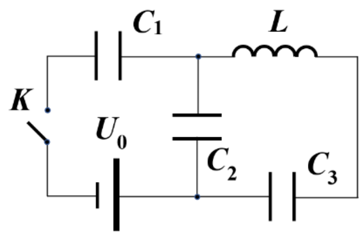
Problem 1 (10.0 points)
This problem consists of three independent parts.

Problem 1.1 (4.0 points)
Water of mass density $\rho = 1.00 g/sm^3$ is poured into a vertical U-shaped tube of small constant cross-section $s = 8.00 sm^2$, such that the total length of water in both legs is $l = 50.0 sm$. Two pistons are placed into one leg of the tube with the spring of stiffness $k = 1.00 N/m$ in between. The pistons are watertight and can slide along the tube without friction. At the initial moment, a weight of mass $m = 10.0 g$ is placed on the upper piston. Determine the possible frequencies of small harmonic vibrations of the system near a new equilibrium position. The mass of the pistons and the spring can be neglected, the acceleration of gravity is equal to $g = 9.80 m/s^2$. Consider water as an ideal incompressible liquid.



Problem 1.2 (3.0 points)
Some insects, such as water striders, are capable of moving freely on the water surface, because their legs are densely covered with non-wetting hairs. To understand why this turns possible, consider the following model problem. A square plate with the side $a = 10.0 sm$ and thickness $h = 2.00 mm$ is carefully laid on the water surface. The density of the plate material is equal to $\rho = 1.10 g/sm^3$, the density of water is $\rho_0 = 1.00 g/sm^3$, and its surface tension constitutes $\sigma = 7.30 \cdot 10^{-2} N/m$. Find the maximum mass m of a weight that can be put on the plate so that it still does not sink. Assume that the weight does not deform the plate, the acceleration of free fall is $g = 9.80 m/s^2$.

Problem 1.3 (3.0 points)
The electrical circuit shown schematically in the figure consists of three capacitors with capacities C_1, C_2 and C_3, a coil of inductance L and a source of constant voltage U_0. At the initial moment of time, the capacitors are not charged, and the current in the coil is zero. The switch K is shorted. Find the maximum current I_{max} through the coil and determine the minimum voltage U_{min} across the capacitor C_2. Assume that the resistance of the connecting wires is rather small.



Problem 2. Thermodynamics of one-component plasma (10.0 points)

Plasma, considered a fourth state of matter, is an ionized gas containing electrons, ions and neutral particles. In a plasma, particle concentrations and temperatures vary over a very wide range, so that a great variety of physical effects can play an essential role. Therefore, at present, a great deal of plasma models have been developed, and this problem deals with one of them, which is called a one-component plasma model. Namely, let us consider a fully ionized plasma with no neutral particles present, which consists of positively charged deuterium nuclei moving against a neutralizing uniformly charged background formed by electrons. This model is a very good approximation for ultrahigh-pressure plasmas occurring at the center of white dwarfs and giant planets like Jupiter. Let the charge and the mass of deuterium nuclei be $e = 1.602\cdot10^{-19}\,Cl$ and $m = 3.44\cdot10^{-24}\,g$ respectively, and their concentration be $n = 1{,}62\cdot10^{27}\,sm^{-3}$ at temperature $T = 1.76\cdot10^{4}\,K$. Under these conditions, an essential role is played by the interaction between deuterium nuclei, which are located at the sites of a cubic lattice, whose two-dimensional projection is shown in Figure 2.1. Plasma must preserve its neutrality, therefore, each cube with a nucleus located in its center is neutral and referred to as a unit cell. The field produced by each cubic cell is rather complicated, and instead the system is formally reduced to spherical cells, whose two-dimensional projection is depicted in Figure 2.2. The justification of such a replacement is not obvious and depends on the type of problems under investigation.

In numerical calculations, consider the following known: Boltzmann's constant $k_B = 1.38\cdot10^{-23}\,J/K$, the vacuum permittivity $\varepsilon_0 = 8{,}85\cdot10^{-12}\,F/m$.

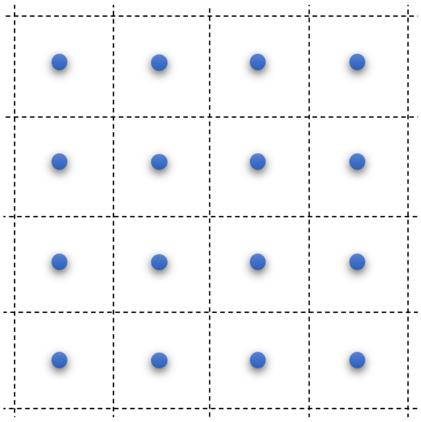


Figure 2.1. Two-dimensional projection of a one-component plasma model with cubic cells.

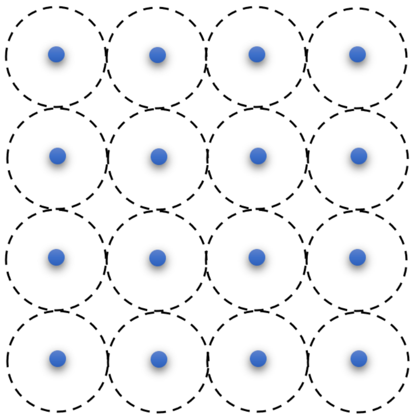


Figure 2.2. Two-dimensional projection of a one-component plasma model with spherical cells.

2.1 Calculate the smallest distance a between neighboring nuclei.

2.2 Show that interaction energy between nuclei plays a significant role under the given conditions. To do this, estimate the ratio Γ of the interaction energy of neighboring nuclei to their thermal energy. Neglect the presence of a neutralizing background.

2.3 Calculate the bulk charge density ρ of a spherical cell in a one-component plasma model.

2.4 Calculate the potential difference between two points of a spherical cell located at distances $a/2$ and $a/4$ respectively.

2.5 Calculate the frequency of small oscillations ω_p of a nucleus near the equilibrium position in a spherical cell.

2.6 At the given plasma temperature, estimate the rms amplitude A of oscillations of nuclei near their equilibrium position.

2.7 The internal energy U of a one-component plasma, containing N spherical cells in the volume V, has the form

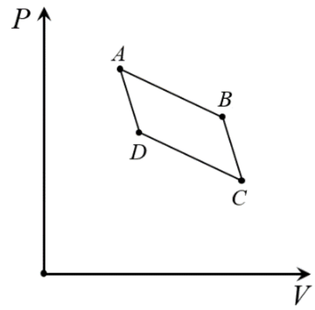
$$U = \alpha_1 N + \alpha_2 \frac{N^{4/3}}{V^{1/3}}.$$

Determine constants α_1 and α_2.

The plasma state of matter is a promising working body for the controlled nuclear fusion. The main problem in the implementation of nuclear fusion is to overcome the so-called Coulomb barrier, which is the Coulomb repulsion between positively charged nuclei. Note that the presence of the neutralizing background of nuclei results in lowering of the Coulomb barrier, since the repulsive force between nuclei decreases. Consider the process of two fusing cells, which occurs as follows. Two cells fuse into one spherical cell with the same bulk density of the neutralizing background, and a new nucleus appears in its center, formed by the fusion of two initial nuclei.

2.8 Calculate the Coulomb barrier lowering for the fusion of two deuterium nuclei cells under given conditions.

The expression for the internal energy of a one-component plasma in the vell model in 2.7 above is interesting in that it explicitly depends on the volume, which is characteristic for nonideal systems. Let a thermodynamic state of the system, whose composition remains unchanged, be depicted by a dot on the pressure (P) – volume (V) diagram. In this diagram consider a process consisting of two isotherms AB and CD, as well as two adiabats BC and AD. Variations in volumes, temperatures and pressures in this process may be considered so small that the quadrilateral $ABCD$ can be assumed a parallelogram.



2.9 Using the above cycle, express the derivative $(\partial U/\partial V)_T$ of the internal energy with respect to volume at a fixed temperature in terms of the derivative $(\partial P/\partial T)_V$ of the pressure with respect to temperature at a fixed volume as well as the temperature T and pressure P of the system.

2.10 The pressure P of a one-component plasma, containing N spherical cells in the volume V, has the form

$$P = \beta_1 \frac{N}{V} + \beta_2 \left(\frac{N}{V}\right)^{\beta_3}.$$

Determine constants β_1, β_2 and β_3. Calculate the numerical value of the pressure for the plasma parameters given in the problem statement.

Problem 3. Optical waveguide (10.0 points)

At present, various waveguides are widely used to transmit energy and information. The propagation of electromagnetic waves in waveguides differs significantly from the propagation of waves in free space, and in this problem you are asked to describe the propagation of electromagnetic waves in the simplest plane waveguide.

Description of waves

A plane monochromatic electromagnetic wave propagating along the axis Ox is described by the formula

$$\vec{E}(t,x) = \vec{E}_0 \cos(\omega t - kx + \varphi). \qquad (1)$$

Here $\vec{E}, \vec{E}_0$ denote the electric field strength of the wave and its amplitude, respectively, the magnitude of k is called the wavenumber, ω refers to the circular wave frequency, φ signifies the initial phase, and the expression in the cosine is called the wave phase.

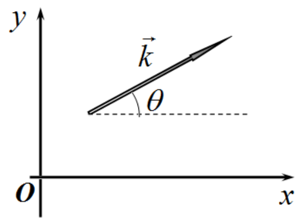
3.1 Express the wavenumber k in terms of the wavelength λ and the period of oscillation T in terms of the angular frequency ω.
3.2 Express the velocity c of wave propagation in terms of k and ω.

In a more general case, a monochromatic plane wave is described by the function

$$\vec{E}(t,\vec{r}) = \vec{E}_0 \cos\left(\omega t - \vec{k}\cdot\vec{r} + \varphi\right). \qquad (2)$$

In this expression, $\vec{r}$ stands for the radius vector of an arbitrary point in space, $\vec{k}$ designates the wave vector equal in magnitude to the wavenumber and pointing to the direction of wave propagation.

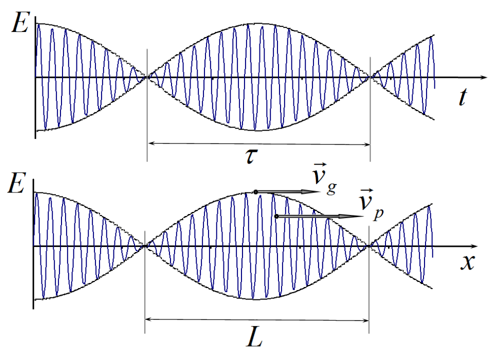
Let the wave vector $\vec{k}$ lie in the plane Oxy and be directed at an angle θ to the axis Ox, as shown in the figure to the right.



3.3 Draw schematically a family of equidistant wave surfaces, which are surfaces of equal wave phases, for a plane wave described by function (2).
3.4 Write down explicitly the expression for the electric field strength of wave (2) as a function of coordinates $\vec{E}(t,x,y)$.

An ideal monochromatic wave is infinite in time and space, therefore, it cannot carry information. To transmit information, it is necessary to use either separate pulses (restricted in time) or variate the wave amplitude (wave modulation). In these cases, the wave ceases to be monochromatic, and can be represented as a sum (superposition) of monochromatic waves.

Consider a wave that is the sum of two waves propagating along the axis Ox: the first one having the angular frequency ω_0 and wavenumber k_0; the second wave frequency is $\omega_0 + \Delta\omega$, $\Delta\omega \ll \omega_0$, with the wavenumber $k_0 + \Delta k, \Delta k \ll k_0$. Note that, in the general case, the wavenumber is a certain function of frequency $k(\omega)$.



3.5 Show that the sum of these two monochromatic waves is a modulated wave consisting of separate wave packets. Write down the formula describing the slow variation of the amplitude $A_0(x,t)$ of the resulting wave in space and time (it is called an envelope).
3.6 Determine the time duration of an individual wave packet τ. Write down the relationship between the duration τ and the frequency difference $\Delta\nu = \Delta\omega / 2\pi$.
3.7 Determine the spatial length of the wave packet L.

It turns out that the speed of the wave surface of constant phase v_p, which is called the phase velocity, differs from the speed of the wave packet v_g, which is called the group velocity. The speed of the envelope maximum can be considered a group velocity.

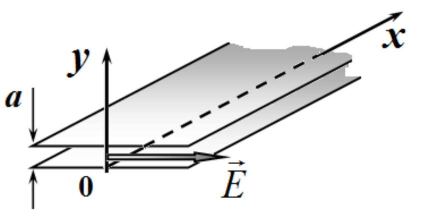
3.8 Find the phase velocity v_p of the considered modulated wave and express it in terms of $\omega, k, \Delta\omega, \Delta k$.

3.9 Find the group velocity v_g of the considered modulated wave and express it in terms of $\omega, k, \Delta\omega, \Delta k$.

3.10 Establish a relationship between phase v_p and group v_g velocities for electromagnetic waves in a vacuum.

Plane waveguide

In this part, consider the propagation of electromagnetic waves in a plane waveguide. The waveguide is formed by two infinite parallel conductive plates located at a distance a from each other. Assume a vacuum in between the plates.



Under investigation are electromagnetic waves, whose electric field strength vectors are directed parallel to the plates (they are called TE waves). Let us introduce a coordinate system, whose Ox axis lies in one of the plates, and whose Oy axis is directed perpendicular to the plates.

A wave propagating along the axis Ox in between the plates is described by the function

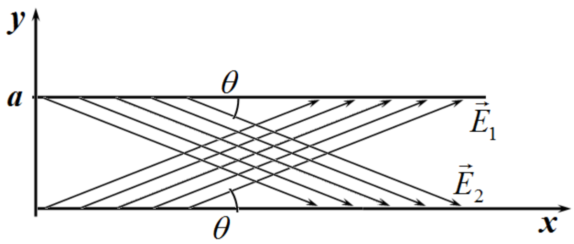
$$E(t,x,y) = E_0 \cos(\omega t - k_x x)\sin(k_y y), \qquad (3)$$

where ω is the known circular frequency of the wave. For this wave to propagate in the waveguide without energy losses, the electric field strength on the plates must be zero.

3.11 Find the values k_y at which the wave can propagate in the waveguide without energy losses.

A set of possible values k_y is discrete and characterized by some integer m. Waves corresponding to different values of this number are called modes (types of possible waves).

3.12 Show that the wave described by function (3) can be represented as a superposition of two plane waves $E_1(t,x,y)$ and $E_2(t,x,y)$ with the wave numbers k_0, propagating symmetrically at angles $\pm\theta$ to the plates, see figure below.



3.13 Express the values k_x, k_y in terms of the wavenumber k_0 and angle θ.

3.14 Determine the possible angles θ_m at which the wave can propagate in the waveguide without energy losses. Express the values of these angles in terms of the distance a between the plates and the wavelength λ in vacuum.

3.15 Determine the phase velocities v_p of the waves of each mode. Express these velocities in terms of the wave frequency ω and the speed of light c in a vacuum.

Short pulses with a carrier frequency ω_0 are supplied to the waveguide input. Since these pulses have a finite time duration τ, they cannot be treated as a monochromatic wave, but instead contain a set of monochromatic components in a certain frequency range $\Delta\omega \ll \omega_0$. These input pulses form a set of pulses in each of the possible waveguide modes.

3.16 Determine the speed of the pulse propagation in the mode number m.

3.17 At what minimum distance X from the waveguide input the number of pulses is to be doubled if $a/\lambda = 1.2$. Express your answer in terms of the speed of light c and the pulse duration τ.

To avoid the appearance of "extra" pulses at the information transmission process, waveguides are used to operate in a single-mode regime, in which only one mode can propagate.

3.18 Find ratios a/λ at which only one mode can propagate in the waveguide.

Mathematical hints for the theoretical problems

The following formulas may be useful:

$\int x^n dx = \frac{x^{n+1}}{n+1}$, where n is an integer;

$(1+x)^\gamma \approx 1 + \gamma x + \frac{\gamma(\gamma-1)}{2}x^2$, *for* $x \ll 1$ *and any* γ.

SOLUTIONS TO THE PROBLEMS OF THE THEORETICAL COMPETITION

Attention. Points in grading are not divided!

Problem 1 (10.0 points)

Problem 1.1 (4.0 points)

Let x be the spring compression, and y be the change in the water level in the tube leg in which the pistons are located. When water moves in a tube, an inertial force acts on the body, which is equal to

$$F = -m\ddot{y}, \tag{1}$$

so that the equation of the weight motion is written as

$$m\ddot{x} = -kx + mg - m\ddot{y}. \tag{2}$$

The equation of motion of water in the tube has the form

$$\rho sl\ddot{y} = -2\rho sgy + kx. \tag{3}$$

The new equilibrium position is determined by the conditions $x = x_0 = const$ and $y = y_0 = const$, so that substitution into equations (2) and (3) gives rise to

$$x_0 = \frac{mg}{k}, \tag{4}$$

$$y_0 = \frac{kx_0}{2\rho gs} = \frac{m}{2\rho s}. \tag{5}$$

According to the problem statement, it is said that the system performs harmonic oscillations about the new equilibrium position, therefore, a solution to equations (2) and (3) is sought in the following form

$$x = x_0 + A\cos\omega t, \tag{6}$$

$$y = y_0 + B\cos\omega t, \tag{7}$$

and after substitution we obtain the following set of equations

$$A(\omega_1^2 - \omega^2) = B\omega^2, \tag{8}$$

$$A\omega_3^2 = B(\omega_2^2 - \omega^2), \tag{9}$$

where $\omega_1^2 = \frac{k}{m}$, $\omega_2^2 = \frac{2g}{l}$, $\omega_3^2 = \frac{k}{\rho sl}$.

After dividing equations (8) and (9), we obtain a quadratic equation for a possible oscillation frequencies

$$(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2) = \omega^2\omega_3^2, \tag{10}$$

which admits the solution

$$\omega_{1,2}^2 = \frac{\omega_1^2 + \omega_2^2 + \omega_3^2 \pm \sqrt{(\omega_1^2 + \omega_2^2 + \omega_3^2)^2 - 4\omega_1^2\omega_2^2}}{2}. \tag{11}$$

Note that both roots are always positive and give the following possible frequencies of harmonic oscillations

$$\omega_1 = \sqrt{\frac{\omega_1^2 + \omega_2^2 + \omega_3^2 - \sqrt{(\omega_1^2 + \omega_2^2 + \omega_3^2)^2 - 4\omega_1^2\omega_2^2}}{2}} = 5.19s^{-1}, \tag{12}$$

$$\omega_2 = \sqrt{\frac{\omega_1^2 + \omega_2^2 + \omega_3^2 + \sqrt{(\omega_1^2 + \omega_2^2 + \omega_3^2)^2 - 4\omega_1^2\omega_2^2}}{2}} = 12.07s^{-1}. \tag{13}$$

In reality, the motion of the system is represented by the addition of harmonic oscillations with frequencies (12) and (13).

Content	Points
Formula (1): $F = -m\ddot{y}$	0.3
Formula (2): $m\ddot{x} = -kx + mg - m\ddot{y}$	0.3

Formula (3): $\rho sl\ddot{y} = -2\rho sgy + kx$	0.3
Formula (4): $x_0 = \dfrac{mg}{k}$	0.3
Formula (5): $y_0 = \dfrac{kx_0}{2\rho gs} = \dfrac{m}{2\rho s}$	0.3
Formula (6): $x = x_0 + A\cos\omega t$	0.3
Formula (7): $y = y_0 + B\cos\omega t$	0.3
Formula (8): $A(\omega_1^2 - \omega^2) = B\omega^2$	0.2
Formula (9): $A\omega_3^2 = B(\omega_2^2 - \omega^2)$	0.2
Formula (10): $(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2) = \omega^2\omega_3^2$	0.4
Formula (11): $\omega_{1,2}^2 = \dfrac{\omega_1^2 + \omega_2^2 + \omega_3^2 \pm \sqrt{(\omega_1^2 + \omega_2^2 + \omega_3^2)^2 - 4\omega_1^2\omega_2^2}}{2}$	0.3
Formula (12): $\omega_1 = \sqrt{\dfrac{\omega_1^2 + \omega_2^2 + \omega_3^2 - \sqrt{(\omega_1^2 + \omega_2^2 + \omega_3^2)^2 - 4\omega_1^2\omega_2^2}}{2}}$	0.2
Numerical value in formula (12): $\omega_1 = 5.19 s^{-1}$	0.2
Formula (13): $\omega_2 = \sqrt{\dfrac{\omega_1^2 + \omega_2^2 + \omega_3^2 + \sqrt{(\omega_1^2 + \omega_2^2 + \omega_3^2)^2 - 4\omega_1^2\omega_2^2}}{2}}$	0.2
Numerical value in formula (13): $\omega_2 = 12.07 s^{-1}$	0.2
Total	**4.0**

Problem 1.2 (3.0 points)

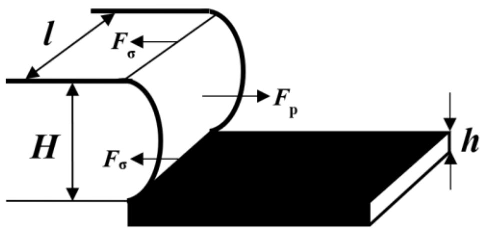
The plate has volume

$$V = a^2 h, \tag{1}$$

and it is subject to the gravity force

$$F_p = \rho V g. \tag{2}$$

At the line of contact between the plate and water, a difference in water levels occurs, as shown in the figure below.



As a result, on the lower surface of the plate with the area

$$S = a^2 \tag{3}$$

differential pressure applies

$$\Delta p = \rho_0 g(H + h), \tag{4}$$

which results in a vertically upward force

$$F = \Delta p S. \tag{5}$$

To determine the value of H, we select a certain volume of water with the width l near its contact line with the plate. It is subject to the surface tension force equal to

$$F_\sigma = 2\sigma l, \tag{6}$$

as well as the force due to the pressure of the liquid column

$$F_{\bar{p}} = \bar{p}\Delta S, \tag{7}$$

where the average pressure ia found as

$$\bar{p} = \frac{1}{2}\rho_0 gH \tag{8}$$

together with the cross-sectional area

$$\Delta S = Hl\,. \tag{9}$$

From the water equilibrium condition

$$F_\sigma = F_{\bar{p}} \tag{10}$$

it follows that the height difference is obtained as

$$H = 2\sqrt{\frac{\sigma}{\rho_0 g}}\,. \tag{11}$$

The additional weight on the plate is acted upon by the gravity force

$$F_m = mg\,, \tag{12}$$

and equilibrium condition

$$F_p + F_m = F \tag{13}$$

the mass of the weight is finally derived as

$$m = (\rho_0 - \rho)a^2 h + 2a^2\sqrt{\frac{\sigma\rho_0}{g}} = 52.6g\,. \tag{14}$$

Content	Points
Formula (1): $V = a^2 h$	0.2
Formula (2): $F_p = \rho V g$	0.2
Formula (3): $S = a^2$	0.2
Formula (4): $\Delta p = \rho_0 g(H+h)$	0.2
Formula (5): $F = \Delta p S$	0.2
Formula (6): $F_\sigma = 2\sigma l$	0.2
Formula (7): $F_{\bar{p}} = \bar{p}\Delta S$	0.2
Formula (8): $\bar{p} = \frac{1}{2}\rho_0 gH$	0.2
Formula (9): $\Delta S = Hl$	0.2
Formula (10): $F_\sigma = F_{\bar{p}}$	0.2
Formula (11): $H = 2\sqrt{\frac{\sigma}{\rho_0 g}}$	0.2
Formula (12): $F_m = mg$	0.2
Formula (13): $F_p + F_m = F$	0.2
Formula (14): $m = (\rho_0 - \rho)a^2 h + 2a^2\sqrt{\frac{\sigma\rho_0}{g}}$	0.2
Numerical value in formula (14): $m = 52.6g$	0.2
Total	**3.0**

Problem 1.3 (3.0 points)

The current through the coil cannot change instantly and immediately after the key K is shorted it remains equal to zero. At the same time, since the resistance of the connecting wires is very small, the capacitors C_1 and C_2 are almost instantly charged up to charges q_{10} and q_{20} respectively, whereas the capacitor C_3 remains uncharged

$$q_{30} = 0\,, \tag{1}$$

since it can only be charged through the coil. Note that Joule heat is generated in the connecting wires.

Thus, at the initial moment of time, the capacitors C_1 and C_2 are connected in series to a constant voltage source U_0 and their charges are equal

$$q_{10} = q_{20}\,, \tag{2}$$

and the corresponding voltages add up, so that

$$\frac{q_{10}}{C_1} + \frac{q_{20}}{C_2} = U_0\,. \tag{3}$$

Thus, we find from equations (2) and (3) that

$$q_{10} = q_{20} = \frac{C_1 C_2}{C_1 + C_2} U_0\,. \tag{4}$$

The total energy of the system immediately after the key K shortening turns out to be

$$W_0 = \frac{q_{10}^2}{2C_1} + \frac{q_{20}^2}{2C_2} = \frac{C_1 C_2 U_0^2}{2(C_1 + C_2)}\,. \tag{5}$$

After charging the capacitors C_1 and C_2, the current through the coil starts to increase and harmonic oscillations are generated in the system, at which Joule losses can already be neglected, since the resistance of the connecting wires is very small.

Note that at that moment in time when the current in the coil is maximum, the voltage across it is zero and the capacitors C_2 and C_3 turn out to be connected in parallel. For such a connection of capacitors, the following relations for charges are satisfied

$$q_1 = q_2 + q_3\,. \tag{6}$$

$$\frac{q_2}{C_2} = \frac{q_3}{C_3}\,. \tag{7}$$

$$\frac{q_1}{C_1} + \frac{q_2}{C_2} = U_0\,. \tag{8}$$

Solving together the set of equations (6)-(8), we find the charges of the capacitors

$$q_1 = \frac{C_1(C_2 + C_3)}{C_1 + C_2 + C_3} U_0\,. \tag{9}$$

$$q_2 = \frac{C_1 C_2}{C_1 + C_2 + C_3} U_0\,. \tag{10}$$

$$q_3 = \frac{C_1 C_3}{C_1 + C_2 + C_3} U_0\,, \tag{11}$$

and the energy of the system in this state is obviously equal to

$$W = \frac{C_1(C_2 + C_3) U_0^2}{2(C_1 + C_2 + C_3)} + \frac{L I_{\max}^2}{2}\,. \tag{12}$$

In this case, the work of the source is found as

$$A = (q_1 - q_{10}) U_0\,, \tag{13}$$

and the energy conservation law is written in the following form

$$W_0 + A = W\,, \tag{14}$$

which provides the maximum currect

$$I_{\max} = \sqrt{\frac{C_3}{(C_1 + C_2)(C_1 + C_2 + C_3)L}}\, C_1 U_0\,. \tag{15}$$

Finding the minimum voltage $U_{\min}$ across the capacitor C_2 is a slightly more difficult task that has a rather simple solution. It is obvious that harmonic oscillations occur in the system, at which the potential energy is constantly transformed into kinetic energy and backwards. For the presented electrical circuit, the role of the kinetic energy is played by the energy of the coil. Therefore, when the current through the coil is zero, then the system is in its large deviation from equilibrium, while the voltage across the capacitor C_2 is

$$U_{20} = \frac{q_{20}}{C_2} = \frac{C_1}{C_1 + C_2} U_0 . \tag{16}$$

Note that the zero coil current corresponds to the initial moment when the key K is just shorted.

After a quarter of a period has passed, the current in the coil becomes maximum and the system passes the equilibrium position, whereas the voltage across the capacitor C_2 drops to the value

$$U_2 = \frac{q_2}{C_2} = \frac{C_1}{C_1 + C_2 + C_3} U_0 , \tag{17}$$

that is, it falls by $U_{20} - U_2$. After another quarter of the period, the voltage across the capacitor will further drop by the same amount, which is, at the same time, equal to $U_2 - U_{\min}$, so the minimum voltage is ultimately obtained as

$$U_{\min} = 2U_2 - U_{20} = \frac{C_1(C_1 + C_2 - C_3)}{(C_1 + C_2)(C_1 + C_2 + C_3)} U_0 . \tag{18}$$

Content	Points
Formula (1): $q_{30} = 0$	0.2
Formula (2): $q_{10} = q_{20}$	0.2
Formula (3): $\frac{q_{10}}{C_1} + \frac{q_{20}}{C_2} = U_0$	0.2
Formula (4): $q_{10} = q_{20} = \frac{C_1 C_2}{C_1 + C_2} U_0$	0.2
Formula (5): $W_0 = \frac{q_{10}^2}{2C_1} + \frac{q_{20}^2}{2C_2} = \frac{C_1 C_2 U_0^2}{2(C_1 + C_2)}$	0.2
Formula (6): $q_1 = q_2 + q_3$	0.2
Formula (7): $\frac{q_2}{C_2} = \frac{q_3}{C_3}$	0.2
Formula (8): $\frac{q_1}{C_1} + \frac{q_2}{C_2} = U_0$	0.2
Formula (10): $q_2 = \frac{C_1 C_2}{C_1 + C_2 + C_3} U_0$	0.2
Formula (12): $W = \frac{C_1(C_2 + C_3) U_0^2}{2(C_1 + C_2 + C_3)} + \frac{L I_{\max}^2}{2}$	0.2
Formula (13): $A = (q_1 - q_{10}) U_0$	0.2
Formula (14): $W_0 + A = W$	0.2
Formula (15): $I_{\max} = \sqrt{\frac{C_3}{(C_1 + C_2)(C_1 + C_2 + C_3) L}} C_1 U_0$	0.2
Formula (18): $U_{\min} = 2U_2 - U_{20}$	0.2
Formula (18): $U_{\min} = \frac{C_1(C_1 + C_2 - C_3)}{(C_1 + C_2)(C_1 + C_2 + C_3)} U_0$	0.2
Total	**3.0**

Problem 2. Thermodynamics of one-component plasma (10.0 points)

2.1 The smallest distance between neighboring deuterium nuclei coincides with the edge of the cube, and since there is 1 nucleus per cube, their concentration is

$$n = \frac{1}{a^3}, \tag{1}$$

therefore

$$a = \frac{1}{\sqrt[3]{n}} = 8.51 \cdot 10^{-12} \, м \tag{2}$$

2.2 The electrostatic energy of interaction of two nuclei located at the distance a from each other is found as

$$W_p = \frac{e^2}{4\pi\varepsilon_0 a}, \tag{3}$$

and their thermal energy is evaluated by the formula

$$E_T = k_B T, \tag{4}$$

whence the sought ratio is obtained in the form

$$\Gamma = \frac{W_p}{E_T} = \frac{e^2}{4\pi\varepsilon_0 a k_B T} = 111. \tag{5}$$

2.3 In general, the spherical cell is neutral, and its radius is equal to

$$R = a/2 \tag{6}$$

with the corresponding volume

$$V = \frac{4}{3}\pi R^3, \tag{7}$$

therefore, the bulk charge density is expressed as

$$\rho = -\frac{e}{V} = -\frac{6e}{\pi a^3} = -\frac{6}{\pi} ne = -4.95 \cdot 10^{14} \, Cl/m^3. \tag{8}$$

2.4 Let us apply Gauss's theorem

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0} \tag{9}$$

to the sphere of radius r centered at the location of the nucleus. The flux of the electric field strength E through this sphere, due to symmetry, is delivered by

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = E 4\pi r^2, \tag{10}$$

and the net charge Q inside the sphere is obtained as

$$Q = e + \rho \frac{4}{3}\pi r^3. \tag{11}$$

It follows from equations (9)-(11) that

$$E = \frac{e}{4\pi\varepsilon_0 r^2} + \frac{\rho r}{3\varepsilon_0}, \tag{12}$$

and the sought potential difference is determined by the expression

$$\varphi(a/4) - \varphi(a/2) = -\int_{a/2}^{a/4} E dr, \tag{13}$$

which finally yields

$$\varphi(a/4) - \varphi(a/2) = \frac{5e}{16\pi\varepsilon_0 a} = 211V. \tag{14}$$

2.5 The second term on the right-hand side of expression (12) determines the strength of the electric field created by the uniform charge distribution; therefore, the equation of motion of the nucleus projected onto the radial direction has the form

$$m\ddot{r} = \frac{e\rho}{3\varepsilon_0} r\,, \tag{15}$$

which is an equation of harmonic oscillations with a frequency

$$\omega_p = \sqrt{-\frac{e\rho}{3m_p\varepsilon_0}} = \sqrt{\frac{2e^2}{\pi m_p \varepsilon_0 a^3}} = \sqrt{\frac{2ne^2}{\pi m_p \varepsilon_0}} = 2.94\cdot10^{16} s^{-1}\,. \tag{16}$$

2.6 At a fixed temperature, the mean square thermal velocity of the nucleus is derived as

$$v = \sqrt{\frac{k_B T}{m_p}}\,, \tag{17}$$

and the corresponding amplitude of deviation from the equilibrium position is determined as

$$A = \frac{v}{\omega_p} = 2{,}85\cdot10^{-13} m\,. \tag{18}$$

It can be seen that the condition $A \ll a$ holds, i.e. the deuterium nuclei do indeed perform small oscillations near their equilibrium positions.

2.7 The internal energy of the system consists of the thermal energy of the thermal motiont of nuclei and the electrostatic energy of each cell. In turn, the electrostatic energy of each cell consists of the interaction energy of nuclei with the surrounding electron neutralizing background and the energy of the background itself.

Let us divide the cell into spherical layers and consider the layer located at the distance r from the cell center and having the thickness dr. Its charge is obtained as

$$dq = \rho 4\pi r^2 dr\,, \tag{19}$$

and the corresponding interaction energy with the nucleus is

$$W_1 = \int_0^R \frac{e\,dq}{4\pi\varepsilon_0 r} = -\frac{3e^2}{4\pi\varepsilon_0 a}\,. \tag{20}$$

The energy density of the electric field is found by the formula

$$w = \frac{1}{2}\varepsilon_0 E^2\,, \tag{21}$$

and since the electric field strength of the uniform background is determined by the second term in expression (12) and outside the sphere has the form like that of a point-дшлу charge, which formally coincides with the first term of expression (12),, then the electrostatic energy of the uniform background is evaluated as follows

$$W_2 = \frac{1}{2}\varepsilon_0 \int_0^R \left(\frac{\rho r}{3\varepsilon_0}\right)^2 4\pi r^2 dr + \frac{1}{2}\varepsilon_0 \int_R^\infty \left(\frac{e}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr = \frac{3e^2}{10\pi\varepsilon_0 a}\,. \tag{22}$$

Thus, the total electrostatic energy of a single cell is written as

$$W = W_1 + W_2 = -\frac{9e^2}{20\pi\varepsilon_0 a} \tag{23}$$

and is equal to the work that must be done to create it.

As shown above, the nucleus in the cell center is a three-dimensional harmonic oscillator, so its thermal chaotic energy is determined as

$$E = 3Nk_B T\,, \tag{24}$$

and hence the internal energy of N cells has the form

$$U = E + NW = 3Nk_B T - \frac{9e^2}{20\pi\varepsilon_0}\frac{N^{4/3}}{V^{1/3}}\,. \tag{25}$$

Thus, the sought constants are found as

$$\alpha_1 = 3k_B T\,, \tag{26}$$

$$\alpha_2 = -\frac{9e^2}{20\pi\varepsilon_0}\,. \tag{27}$$

2.8 In the absence of the neutralizing backgrounds of two nuclei, their fusion corresponds to the bare Coulomb barrier. The presence of neutralizing backgrounds leads to a decrease in the Coulomb barrier, which is obviously determined by the interaction of nuclei with their backgrounds and the self-energy of the backgrounds, i.e. by expression (23). In this case, the thermal energy of nuclei remains small in comparison with the lowering of the Coulomb barrier.

Each of the two cells before fusing has the electrostatic energy

$$W = -\frac{9e^2 n^{1/3}}{20\pi\varepsilon_0}. \tag{28}$$

After fusion, a new cell is formed with the volume

$$V' = 2V \tag{29}$$

with a helium nucleus in the center having an electric charge

$$e' = 2e. \tag{30}$$

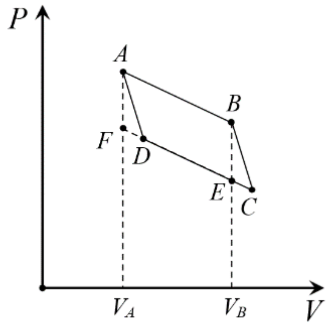
In accordance with the general formula, the electrostatic energy of the formed cell is derived as

$$W' = -2^{5/3}\frac{9e^2 n^{1/3}}{20\pi\varepsilon_0}, \tag{31}$$

whence the following expression for the Coulomb barrier lowering is obtained

$$\delta U_c = 2W - W' = \frac{(2^{2/3}-1)9e^2 n^{1/3}}{10\pi\varepsilon_0} = 5.72\cdot 10^{-17}\,J. \tag{32}$$

2.9 The circular process $ABCD$ is the Carnot cycle. Let us denote the temperature on the isotherm AB as T_{AB}, and on the isotherm CD as T_{CD}, while they differ very little from each other, so that $T_{AB} \approx T_{CD} \approx T$ and $T_{AB} - T_{CD} \ll T$. The work A done in the cycle is equal to the area of the parallelogram $ABCD$, which is, in turn, equal to the area of the parallelogram $ABEF$. Since $AF = (\partial P/\partial T)_V (T_{AB} - T_{CD})$, the work in the cycle is derived as



$$A = \left(\frac{\partial P}{\partial T}\right)_V \left(T_{AB} - T_{CD}\right)\left(V_B - V_A\right). \tag{33}$$

In the process AB, the temperature is constant, so the change in internal energy is expressed as

$$U_B - U_A = \left(\frac{\partial U}{\partial V}\right)_T \left(V_B - V_A\right), \tag{34}$$

and the supplied amount of heat according to the first law of thermodynamics takes the form

$$Q = U_B - U_A + P(V_B - V_A). \tag{35}$$

Since the process $ABCD$ is a Carnot cycle, its efficiency is written as

$$\frac{A}{Q} = \frac{T_{AB} - T_{CD}}{T_{AB}}, \tag{36}$$

and combining equations (33) - (36), we obtain the required relation

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P. \tag{37}$$

2.10 Substituting formula (25) into equation (37), we obtain the first-order differential equation

$$T\left(\frac{\partial P}{\partial T}\right)_V - P = \frac{3e^2}{20\pi\varepsilon_0}\frac{N^{4/3}}{V^{4/3}}, \tag{38}$$

whose solution takes the form

$$P(T,V) = C(V)T - \frac{3e^2}{20\pi\varepsilon_0}\frac{N^{4/3}}{V^{4/3}}, \tag{39}$$

where $C(V)$ refers to some constant, which, in principle, can depend on the volume of the system.

In the absence of interaction between the nuclei, the pressure of the system should be reduced to the pressure of an ideal gas

$$P(T,V)\,|_{e\to 0}=\frac{Nk_BT}{V}\,, \tag{40}$$

and we immediately find

$$P(T,V)=\frac{Nk_BT}{V}-\frac{3e^2}{20\pi\varepsilon_0}\frac{N^{4/3}}{V^{4/3}}\,. \tag{41}$$

Thus, the sought constants are obtained as

$$\beta_1=k_BT\,, \tag{42}$$

$$\beta_2=-\frac{3e^2}{20\pi\varepsilon_0}\,., \tag{43}$$

$$\beta_3=\frac{4}{3}\,. \tag{44}$$

Substituting the numerical values, we obtain the numerical value for the pressure

$$P=-2{,}59\cdot 10^{16}\,\Pi a\,. \tag{45}$$

The pressure turns out to be negative! In fact, the pressure of the entire system includes the pressure of the electronic component and is definitely positive.

	Content	Points	
2.1	Formula (2): $a=\frac{1}{\sqrt[3]{n}}$	0.2	**0.4**
	Numerical value in formula (2): $a=8.51\cdot 10^{-12}\,m$	0.2	
2.2	Formula (3): $W_p=\frac{e^2}{4\pi\varepsilon_0 a}$	0.2	**0.8**
	Formula (4): $E_T=k_BT$	0.2	
	Formula (5): $\Gamma=\frac{e^2}{4\pi\varepsilon_0 a k_B T}$	0.2	
	Numerical value in formula (5): $\Gamma=111$	0.2	
2.3	Formula (6): $R=a/2$	0.1	**0.6**
	Formula (7): $V=\frac{4}{3}\pi R^3$	0.1	
	Formula (8): $\rho=-\frac{6}{\pi}ne$	0.2	
	Numerical value in formula (8): $\rho=-4.95\cdot 10^{14}\,Cl/m^3$	0.2	
2.4	Formula (9): $\oint_S \mathbf{E}\cdot d\mathbf{S}=\frac{Q}{\varepsilon_0}$	0.2	**1.4**
	Formula (10): $\oint_S \mathbf{E}\cdot d\mathbf{S}=E4\pi r^2$	0.2	
	Formula (11): $Q=e+\rho\frac{4}{3}\pi r^3$	0.2	
	Formula (12): $E=\frac{e}{4\pi\varepsilon_0 r^2}+\frac{\rho r}{3\varepsilon_0}$	0.2	
	Formula (13): $\varphi(a/4)-\varphi(a/2)=-\int_{a/2}^{a/4} E dr$	0.2	
	Formula (14): $\varphi(a/4)-\varphi(a/2)=\frac{5e}{16\pi\varepsilon_0 a}$	0.2	

	Numerical value in formula (14): $\varphi(a/4)-\varphi(a/2)=211V$	0.2	
2.5	Formula (15): $m\ddot{r}=\dfrac{e\rho}{3\varepsilon_0}r$	0.2	0.6
	Formula (16): $\omega_p=\sqrt{\dfrac{2ne^2}{\pi m_p\varepsilon_0}}$	0.2	
	Numerical value in formula (16): $\omega_p=2.94\cdot10^{16}s^{-1}$	0.2	
2.6	Formula (17): $v=\sqrt{\dfrac{k_BT}{m_p}}$	0.2	0.6
	Formula (18): $A=\dfrac{v}{\omega_p}$	0.2	
	Numerical value in formula (18): $A=2.85\cdot10^{-13}m$	0.2	
2.7	Formula (19): $dq=\rho 4\pi r^2dr$	0.2	1.8
	Formula (20): $W_1=-\dfrac{3e^2}{4\pi\varepsilon_0 a}$	0.2	
	Formula (21): $w=\dfrac{1}{2}\varepsilon_0E^2$	0.2	
	Formula (22): $W_2=\dfrac{3e^2}{10\pi\varepsilon_0 a}$	0.2	
	Formula (23): $W=-\dfrac{9e^2}{20\pi\varepsilon_0 a}$	0.2	
	Formula (24): $E=3Nk_BT$	0.2	
	Formula (25): $U=E+NW=3Nk_BT-\dfrac{9e^2}{20\pi\varepsilon_0}\dfrac{N^{4/3}}{V^{1/3}}$	0.2	
	Formula (26): $\alpha_1=3k_BT$	0.2	
	Formula (27): $\alpha_2=-\dfrac{9e^2}{20\pi\varepsilon_0}$	0.2	
2.8	Formula (28): $W=-\dfrac{9e^2n^{1/3}}{20\pi\varepsilon_0}$	0.2	1.2
	Formula (29): $V'=2V$	0.2	
	Formula (30): $e'=2e$	0.2	
	Formula (31): $W'=-2^{5/3}\dfrac{9e^2n^{1/3}}{20\pi\varepsilon_0}$	0.2	
	Formula (32): $\delta U_c=\dfrac{(2^{2/3}-1)9e^2n^{1/3}}{10\pi\varepsilon_0}$	0.2	
	Numerical value in formula (32): $\delta U_c=5.72\cdot10^{-17}J$	0.2	
2.9	Formula (33): $A=\left(\dfrac{\partial P}{\partial T}\right)_V\left(T_{AB}-T_{CD}\right)\left(V_B-V_A\right)$	0.2	1.0
	Formula (34): $U_B-U_A=\left(\dfrac{\partial U}{\partial V}\right)_T\left(V_B-V_A\right)$	0.2	
	Formula (35): $Q=U_B-U_A+P(V_B-V_A)$	0.2	

	Formula (36): $\frac{A}{Q}=\frac{T_{AB}-T_{CD}}{T_{AB}}$	0.2	
	Formula (37): $\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$	0.2	
2.10	Formula (38): $T\left(\frac{\partial P}{\partial T}\right)_V - P = \frac{3e^2}{20\pi\varepsilon_0}\frac{N^{4/3}}{V^{4/3}}$	0.2	1.6
	Formula (39): $P(T,V)=C(V)T-\frac{3e^2}{20\pi\varepsilon_0}\frac{N^{4/3}}{V^{4/3}}$	0.2	
	Formula (40): $P(T,V)\|_{e\to 0}=\frac{Nk_BT}{V}$	0.2	
	Formula (41): $P(T,V)=\frac{Nk_BT}{V}-\frac{3e^2}{20\pi\varepsilon_0}\frac{N^{4/3}}{V^{4/3}}$	0.2	
	Formula (42): $\beta_1 = k_BT$	0.2	
	Formula (43): $\beta_2 = -\frac{3e^2}{20\pi\varepsilon_0}$	0.2	
	Formula (44): $\beta_3 = \frac{4}{3}$	0.2	
	Numerical value in formula (45): $P=-2.59\cdot 10^{16}\,Pa$	0.2	
Total			**10.0**

Problem 3. Optical waveguide (10.0 points)

Description of waves

3.1 The function

$$\vec{E}(t,x)=\vec{E}_0\cos(\omega t - kx+\varphi), \tag{1}$$

describing a wave at a fixed moment in time $t=t_0$ gives the distribution of the electric field strength in space. When the coordinate is changed by the wavelength λ, the argument of the cosine must change to 2π, therefore

$$(\omega t_0 - k(x+\lambda)+\varphi)-(\omega t_0 - kx+\varphi)=2\pi\ . \tag{2}$$

It follows from this relation that

$$k=\frac{2\pi}{\lambda}\ . \tag{3}$$

Fixing a point in space $x=x_0$ and reasoning similarly, we can write

$$(\omega(t+T)-kx_0+\varphi)-(\omega t - kx_0+\varphi)=2\pi\ , \tag{4}$$

which yields

$$\omega=\frac{2\pi}{T}\ . \tag{5}$$

3.2 The speed of propagation of a monochromatic wave is the speed of motion of a certain wave surface of constant phase. This surface satisfies the equation

$$\omega t - kx+\varphi = const\ . \tag{6}$$

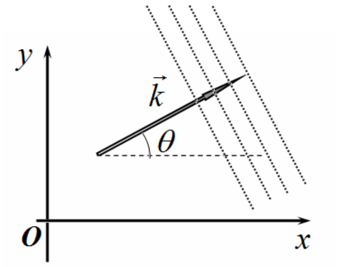
It follows from this relation that the wave propagation speed is

$$c=\frac{dx}{dt}=\frac{\omega}{k}\ . \tag{7}$$

3.3 The surface of the constant phase at a fixed time instant satisfies the equation

$$\vec{k}\cdot\vec{r}=const\ , \tag{8}$$

and this is a family of planes perpendicular to the wave vector.



3.4 Expanding the scalar product, we obtain the wave equation in the coordinate representation:

$$E = E_0' \cos(\omega t - kx\cos\theta - ky\sin\theta + \varphi) \qquad (9)$$

3.5 Since the superposition principle is valid for the electric field strength, we can write for a composite wave

$$E = E_0 \cos(\omega_0 t - k_0 x) + E_0 \cos((\omega_0 + \Delta\omega)t - (k_0 + \Delta k)x) =$$
$$= 2E_0 \cos\left(\left(\omega_0 + \frac{\Delta\omega}{2}\right)t - \left(k_0 + \frac{\Delta k}{2}\right)x\right)\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right) \quad . \qquad (10)$$

Taking into account that $\Delta\omega << \omega_0$, and, consequently, $\Delta k << k_0$, we rewrite this expression as:

$$E = A_0(x,t)\cos(\omega_0 t - k_0 x). \qquad (11)$$

Here the following notation is used

$$A_0(x,t) = 2E_0 \cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right) \qquad (12)$$

for slowly varying wave amplitude.

3.6 To determine the time duration of the packet, it should be taken into account that when passing from one "zero" of the cosine to the next, the argument of the cosine changes to π, therefore

$$\frac{\Delta\omega}{2}\tau = \pi \quad \Rightarrow \quad \tau = \frac{2\pi}{\Delta\omega}. \qquad (13)$$

Taking into account that $\Delta\omega = 2\pi\Delta\nu$, we obtain from expression (13) the relationship between the packet duration and its spectral width as

$$\tau\Delta\nu = 1. \qquad (14)$$

3.7 For a similar reasoning, it is not difficult to find that

$$\frac{\Delta k}{2}L = \pi \quad \Rightarrow \quad L = \frac{2\pi}{\Delta k}. \qquad (15)$$

3.8 The phase velocity can be found as the velocity of motion of the wave surface of constant phase. It is derived from function (11) that this surface satisfies the condition

$$(\omega_0 t - k_0 x) = const, \qquad (16)$$

which results in the phase velocity

$$v_p = \frac{\omega_0}{k_0}. \qquad (17)$$

3.9 To determine the group velocity, we write down the condition that the wave amplitude, for example, is maximum

$$\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x = 0. \qquad (18)$$

It is concluded from this expression that the group velocity is given by the formula

$$v_g = \frac{\Delta\omega}{\Delta k}. \qquad (19)$$

3.10 For electromagnetic waves in vacuum, the relation $\lambda\nu = c$ is fulfilled, which validates

$$\omega = kc, \qquad (20)$$

and it is finally obtained that

$$v_p = \frac{\omega}{k} = c, \qquad (21)$$

$$v_g = \frac{\Delta\omega}{\Delta k} = c = v_p, \tag{22}$$

i.e. both the phase and group speeds are equal to the speed of light c in a vacuum.

Plane waveguide

3.11 The function proposed in the problem statement describes the field in a plane waveguide

$$E(t,x,y) = E_0 \cos(\omega t - k_x x)\sin(k_y y) \tag{23}$$

and satisfies one boundary condition: at $y=0$ the electric field strength $E=0$. Therefore, one should choose such values of k_y so that the second boundary condition is fulfilled: at $y=a$ the field strength should also vanish. This condition is satisfied when

$$\sin k_y a = 0 \quad \Rightarrow \quad k_y a = m\pi \quad \Rightarrow \quad k_y = m\frac{\pi}{a}. \tag{24}$$

In the expressions above m stands for a positive integer, $m = 1, 2, 3...$

3.12 Let us write the equations of symmetric waves

$$E_1 = E_0' \cos\left(\omega t - k_0 x\cos\theta + k_0 y \sin\theta + \varphi\right), \tag{25}$$

$$E_2 = E_0' \cos\left(\omega t - k_0 x\cos\theta - k_0 y \sin\theta - \varphi\right), \tag{26}$$

where $k_0 = \dfrac{\omega}{c}$ is the wavenumber for waves, propagating at an angle $\pm\theta$ to the planes of the waveguide, and summing them up yileds

$$E = E_1 + E_2 = 2E_0' \cos\left(\omega t - k_0 x\cos\theta\right)\cos\left(k_0 y\sin\theta + \varphi\right), \tag{27}$$

with the following relation

$$E_0' = E_0 / 2. \tag{28}$$

Note that there should be $\varphi = -\pi/2$.

3.13 Comparison of the obtained formulas (23) and (27) implies that they coincide if

$$k_x = k_0 \cos\theta, \tag{29}$$

$$k_y = k_0 \sin\theta. \tag{30}$$

3.14 Comparing the values of k_y in formulas (24) and (30), we find

$$k_y = k_0 \sin\theta_m = m\frac{\pi}{a} \quad \Rightarrow \quad \sin\theta_m = m\frac{\pi}{ak_0}. \tag{31}$$

The wavenumber of the considered waves in vacuum is related to the wavelength by

$$k_0 = \frac{2\pi}{\lambda}, \tag{32}$$

then the values of the possible angles are given by the formula

$$\sin\theta_m = m\frac{\pi}{ak_0} = m\frac{\lambda}{2a}. \tag{33}$$

3.15 Equation (23) implies that the phase velocity of wave propagation in the waveguide is given by the formula

$$v_p = \frac{\omega}{k_x} = \frac{\omega}{k_0\cos\theta}. \tag{34}$$

Expressing the value of the cosine of the angle in terms of its sine, which is determined by formula (33), we obtain

$$v_p = \frac{\omega}{k_0\cos\theta} = \frac{\omega}{k_0\sqrt{1-\sin^2\theta}} = \frac{\omega}{k_0\sqrt{1-\left(m\dfrac{\lambda}{2a}\right)^2}}. \tag{35}$$

Finally, using formulas relating frequencies, wavelengths and the speed of light in a vacuum $\frac{\omega}{k_0} = c$, $\lambda = \frac{2\pi}{\omega}c$, it is derived that

$$v_p = \frac{\omega}{k_0 \cos\theta} = \frac{\omega}{k_0\sqrt{1-\sin^2\theta}} = \frac{c}{\sqrt{1-\left(m\frac{\pi c}{\omega a}\right)^2}} . \tag{36}$$

Thus, it turns out that the phase velocity of propagation of an undamped wave in the waveguide is greater than the speed of light in vacuum.

3.16 The propagation velocity of a pulse is the group velocity, therefore, it is determined by formula (19), in which the ratio of the increments can be replaced by the derivative

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1} . \tag{37}$$

To evaluate the velocity using this formula, it is necessary to obtain explicitly the dependence of the wavenumber on the frequency, $k(\omega)$. To do so, we use the general formula for the phase velocity $v_p = \frac{\omega}{k}$ and obtain

$$k = \frac{\omega}{v_p} = \frac{\omega}{c}\sqrt{1-\left(m\frac{\pi c}{\omega a}\right)^2} = \frac{1}{c}\sqrt{\omega^2-\left(m\frac{\pi c}{a}\right)^2} , \tag{38}$$

and the pulse propagation velocity is written as

$$v = v_g = \left(\frac{dk}{d\omega}\right)^{-1} = \left(\frac{1}{c}\frac{\omega}{\sqrt{\omega^2-\left(m\frac{\pi c}{a}\right)^2}}\right)^{-1} = c\sqrt{1-\left(m\frac{\pi c}{a\omega}\right)^2} . \tag{39}$$

As follows from this formula, the group velocity is naturally less than the speed of light in a vacuum. Also, it should be indicated that this speed is equal to $c\cos\theta$, which is quite obvious.

3.17 Let us turn to formula (33) and substitute the given ratio $a/\lambda = 1.2$

$$\sin\theta_m = m\frac{\lambda}{2a} \approx 0{,}42m . \tag{40}$$

Since the sine of any argument does not exceed unity, it follows from the obtained expression that only two modes with $m=1$ and $m=2$ can propagate in a given waveguide, and, in other words, the input pulse generate two pulses of these modes in the waveguide. The propagation velocities of pulses in these modes differ markedly. First of all, let us express these velocities in terms of a given ratio a/λ in the form

$$v = c\sqrt{1-\left(m\frac{\pi c}{a\omega}\right)^2} = c\sqrt{1-\left(m\frac{\lambda}{2a}\right)^2} . \tag{41}$$

At the waveguide input, pulses in both modes are excited simultaneously, but since they move at different speeds, as the distance traveled increases, they diverge in time. The number of pulses doubles when pulses in different modes diverge for a time exceeding the pulse duration, hence, the minimum distance X can be found from the condition

$$\frac{X}{v_2} - \frac{X}{v_1} = \tau , \tag{42}$$

which leads to the final answer

$$X = \frac{\tau}{\frac{1}{v_2} - \frac{1}{v_1}} = \frac{c\tau}{\frac{1}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}} - \frac{1}{\sqrt{1-\left(2\frac{\lambda}{2a}\right)^2}}} \approx 1{,}4c\tau\,. \tag{43}$$

3.18 For a waveguide to operate in a single-mode regime, it is necessary to satisfy the following condition

$$\sin\theta_2 = 2\frac{\lambda}{2a} > 1\,, \tag{44}$$

which yields the inequality

$$\frac{a}{\lambda} < 1\,. \tag{45}$$

	Content	Points	
3.1	Formula (2): $(\omega t_0 - k(x+\lambda)+\varphi)-(\omega t_0 - kx+\varphi)=2\pi$	0.2	**0.8**
	Formula (3): $k=\frac{2\pi}{\lambda}$	0.2	
	Formula (4): $(\omega(t+T)-kx_0+\varphi)-(\omega t - kx_0+\varphi)=2\pi$	0.2	
	Formula (5): $\omega=\frac{2\pi}{T}$	0.2	
3.2	Формула (6): $\omega t - kx+\varphi = const$	0.2	**0.4**
	Formula (7): $v=\frac{dx}{dt}=\frac{\omega}{k}$	0.2	
3.3	A family of planes, perpendicular to the wave vector..	0.2	**0.2**
3.4	Formula (9): $E=E_0'\cos(\omega t - kx\cos\theta - ky\sin\theta+\varphi)$	0.2	**0.2**
3.5	Formula (10): $E=2E_0\cos\left(\left(\omega_0+\frac{\Delta\omega}{2}\right)t-\left(k_0+\frac{\Delta k}{2}\right)x\right)\cos\left(\frac{\Delta\omega}{2}t-\frac{\Delta k}{2}x\right)$	0.2	**0.4**
	Formula (12): $A_0(x,t)=2E_0\cos\left(\frac{\Delta\omega}{2}t-\frac{\Delta k}{2}x\right)$	0.2	
3.6	Formula (13): $\frac{\Delta\omega}{2}\tau=\pi \quad\Rightarrow\quad \tau=\frac{2\pi}{\Delta\omega}$	0.2	**0.4**
	Formula (14): $\tau\Delta\nu=1$	0.2	
3.7	Formula (15): $\frac{\Delta k}{2}L=\pi \quad\Rightarrow\quad L=\frac{2\pi}{\Delta k}$	0.2	**0.2**
3.8	Formula (16): $(\omega_0 t - k_0 x)=const$	0.2	**0.4**
	Formula (17): $v_p=\frac{\omega_0}{k_0}$	0.2	
3.9	Formula (18): $\frac{\Delta\omega}{2}t-\frac{\Delta k}{2}x=0$	0.2	**0.4**
	Formula (19): $v_g=\frac{\Delta\omega}{\Delta k}$	0.2	
3.10	Formula (20): $\omega=kc$	0.2	**0.6**
	Formula (21): $v_p=\frac{\omega}{k}=c$	0.2	
	Formula (22): $v_g=\frac{\Delta\omega}{\Delta k}=c=v_p$	0.2	

3.11	Formula (24): $k_y = m\dfrac{\pi}{a}$	0.2	**0.2**
3.12	Formula (25): $E_1 = E_0' \cos\left(\omega t - k_0 x\cos\theta + k_0 y\sin\theta + \varphi\right)$	0.2	**0.8**
	Formula (26): $E_2 = E_0' \cos\left(\omega t - k_0 x\cos\theta - k_0 y\sin\theta - \varphi\right)$	0.2	
	Formula (28): $E_0' = E_0 / 2$	0.2	
	Condition: $\varphi = -\pi / 2$	0.2	
3.13	Formula (29): $k_x = k_0 \cos\theta$	0.2	**0.4**
	Formula (30): $k_y = k_0 \sin\theta$	0.2	
3.14	Formula (31): $k_y = k_0 \sin\theta_m = m\dfrac{\pi}{a} \quad \Rightarrow \quad \sin\theta_m = m\dfrac{\pi}{ak_0}$	0.3	**0.6**
	Formula (33): $\sin\theta_m = m\dfrac{\pi}{ak_0} = m\dfrac{\lambda}{2a}$	0.3	
3.15	Formula (34): $v_p = \dfrac{\omega}{k_x} = \dfrac{\omega}{k_0\cos\theta}$	0.3	**0.6**
	Formula (36): $v_p = \dfrac{c}{\sqrt{1-\left(m\dfrac{\pi c}{\omega a}\right)^2}}$	0.3	
3.16	Formula (37): $v_g = \dfrac{\Delta\omega}{\Delta k} = \dfrac{d\omega}{dk} = \left(\dfrac{dk}{d\omega}\right)^{-1}$	0.3	**1.0**
	Formula (38): $k = \dfrac{\omega}{v_p} = \dfrac{\omega}{c}\sqrt{1-\left(m\dfrac{\pi c}{\omega a}\right)^2} = \dfrac{1}{c}\sqrt{\omega^2 - \left(m\dfrac{\pi c}{a}\right)^2}$	0.4	
	Formula (39): $v_g = c\sqrt{1-\left(m\dfrac{\pi c}{a\omega}\right)^2}$	0.3	
3.17	Формула (40): $\sin\theta_m \approx 0,42m$	0.2	**1.8**
	Possible modes with $m=1$ and $m=2$	0.4	
	Formula (41): $v = c\sqrt{1-\left(m\dfrac{\lambda}{2a}\right)^2}$	0.4	
	Formula (42): $\dfrac{X}{v_2} - \dfrac{X}{v_1} = \tau$	0.4	
	Formula (43): $X \approx 1,4c\tau$	0.4	
3.18	Formula (44): $\sin\theta_2 = 2\dfrac{\lambda}{2a} > 1$	0.3	**0.6**
	Formula (45): $\dfrac{a}{\lambda} < 1$	0.3	
Total			**10.0**

January 9, 2021

COMPUTER EXPERIMENT:

A mathematical pendulum or what angle can be considered rather small ...

At its core, physics is an experimental science and this is definitely its strength. However, without comprehending a large number of experimental facts, physics would degenerate into a description of a huge amount of phenomena and processes. This is how the physical laws and the corresponding models came to life in the remote past by ignoring insignificant features of the subject under consideration. In recent decades, rapid progress has been witnessed in such a field as computer modeling or, as it has become common to say, a computer experiment. The point is that the developed physical models can be directly implemented on a computer in the form of a computational process and the regularities of interest can be thoroughly investigated in their pure forms. In this competition, you are asked to carry out such a computer experiment for a well-known system of a mathematical pendulum.

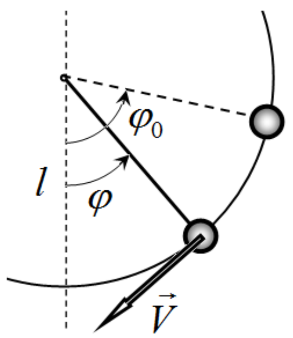
A formula is well known for the period of oscillation of a mathematical pendulum of length l subject to a uniform gravity field of the Earth, characterized by the acceleration of gravity g. However, this formula is only applicable for rather small deflection angles. The main question that you have to answer when doing this computational experiment is: ***"What angular deflection can be considered small?"***

In the educational literature on laboratory experiments, you can find an indication that the maximum deflection angle should not exceed $1°$, $2°$, $5°$, etc. You must answer the above question on the basis of this computer experiment! Namely, you are asked to study the dependence of the oscillation period of a mathematical pendulum on its amplitude, which is the maximum angular deflection from the vertical.

The order of conducting and processing the results of a computer experiment does not differ much from the order of a real, full-scale experiment. Therefore, the parts of this problem directly correspond to the main stages of a real physical experiment.

1. Constructing a theoretical model.

Consider a mathematical pendulum, which is a small massive ball suspended on an inextensible thread of length l. The pendulum is subject to the gravity field with the free fall acceleration g. In the following neglect the air resistance completely.



1.1 Write down a formula for the period T of small oscillations of a mathematical pendulum.

Assume that at the initial moment of time $t_0 = 0$ the angle of the thread deflection from the vertical is φ_0, and the initial velocity of the ball is equal to zero. The ball moves along an arc of a circle, therefore, its position is determined by the angle of the thread deflection from the vertical φ, and the rate of change of this angle in time is determined by the angular velocity $\omega = \dfrac{d\varphi}{dt}$.

1.2 Obtain an exact formula for the dependence of the angular velocity of the pendulum on the deflection angle $\omega(\varphi)$ for a given angular amplitude φ_0 and known values of l, g.

The motion of the pendulum is symmetrical with respect to the vertical, therefore, to calculate the period of oscillation, it is sufficient to evaluate the time t_1 of its motion from the maximum to zero deflection.

1.3 Write down an exact expression for calculating the time t_1 from the known dependence of the angular velocity on the deflection angle $\omega(\varphi)$.

1.4 Express a period of oscillations T in terms of time t_1.

In a computer experiment, when performing calculations, real dimensional quantities are rarely used, since they can have very different orders of magnitude and are extremely inconvenient. Usually, all quantities are made dimensionless or reduced with the aid of some values characteristic for a given problem. For example, in our study, the characteristic time is the period of oscillations, so it is convenient to introduce the dimensionless time τ, which is determined by the following formula:

$$\tau = t\sqrt{\frac{g}{l}}\,.$$

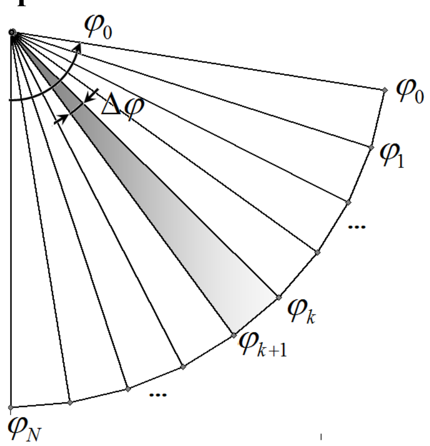
1.5 Write down a formula relating the angular velocity in dimensionless units $\tilde{\omega} = \frac{d\varphi}{d\tau}$ to the previously defined angular velocity ω.

1.6 Determine a period $\tilde{T}$ of small oscillations of the mathematical pendulum in the dimensionless units of time.

1.7 Determine a dependence of the angular velocity $\tilde{\omega}$ on the deflection angle φ: $\tilde{\omega}(\varphi)$.

ATTENTION! In what follows, the introduced dimensionless quantities are used everywhere: time τ, period $\tilde{T}$ and angular velocity $\tilde{\omega}$, which are respectively denoted as t, T *and* ω.

2. Designing an experimental setup, planning an experiment.



In a computer experiment, this stage corresponds to the development of a calculation algorithm. In this case, the main idea of numerical (computer) calculations is to divide the trajectory of motion into small sections, in which the motion is described approximately.

We divide the interval of motion from $\varphi = \varphi_0$ to $\varphi = 0$ into N equal intervals of width $\Delta\varphi$. Let us denote the splitting points as φ_k, $k = 0, 1, ...N$ and the angular velocities at these points as ω_k. The main approximation used in further calculations is that at each interval from φ_k to φ_{k+1} the motion of the pendulum is considered uniformly accelerated. It is natural to expect that with an increase in the number of partition intervals N, the calculation accuracy should grow.

Within the framework of the approximation made, it is straightforward to find the time of the pendulum motion in the interval from φ_0 to 0. For a given amplitude φ_0 and the number of partition intervals N, the calculation algorithm is revealed in the sequence of answers to the following questions.

2.1 Determine the partition interval $\Delta\varphi$.

2.2 Determine the coordinates of the splitting points φ_k.

2.3 Express the angular velocity ω_k at the point φ_k at an arbitrary initial angle of deflection φ_0. Write down this formula for a particular case of $\varphi_0 = \frac{\pi}{2}$.

2.4 Determine the travel time Δt_k for the k-th interval from φ_{k-1} to φ_k.

2.5 Find an expression for the time t_k it takes the ball to reach the angle φ_k. To simplify matters, express it in terms of the travel time t_{k-1} to the previous value of the angle φ_{k-1}.

2.6 Put down a formula for the oscillation period T_N for a given split into intervals.

3. Trial experiment, estimation of errors.

At this stage, it is necessary to make sure that the installation is operational, which in this case means the possibility of performing calculations according to the algorithm developed above, and to assess whether the required accuracy of results is achieved.

As noted earlier, calculation errors depend on the number of partition intervals N. In this task, you have to carry out calculations not on a computer, but "manually" using your calculator. A growth of N reduces the error of calculations, but increases the time of their execution. Therefore, it is important to choose its optimal value, i.e. the minimum value at which the required accuracy is achieved. At this stage, carry out all calculations at $\varphi_0 = \frac{\pi}{2}$.

ATTENTION! Hereinafter, calculations should be carried out with an accuracy of 4 decimal digits. To save time, carefully think over the entire sequence of calculations: use previously calculated values, define necessary constants that are present in the formulas (so as not to recalculate them several times), write down results of intermediate calculations in the most convenient form.

3.1 Calculate the travel times t_k for the points with angles φ_k for $N = 1, 2, 4, 8, 16, 32$. Find the approximate values of the periods of oscillation T_N, calculated for a given N. The results should be complied in Table 1.

3.2 Plot Graph 1 of the law of motion $\varphi(t)$ of the pendulum for a quarter of the period based on the results of calculations at $N = 16$.

3.3 On the same Graph 1, plot the law of motion $\varphi(t)$, assuming that the oscillations are small. The results of calculations of the law of motion should be presented in Table 2.

As an estimate of the relative error in calculating the oscillation period when dividing into N intervals, we use the following value

$$\varepsilon_N = \frac{T_N - T_{32}}{T_{32}},$$

where T_{32} stands for the period calculated at $N = 32$, which is closest to the true value.

The dependence of the relative calculation error ε_N on the number of partition intervals N is described by the approximate formula

$$\varepsilon_N = \frac{C}{N^\gamma},$$

where C and γ are some constants.

3.4 Calculate the relative errors ε_N in determining the periods. The results must be presented in Table 3.

3.5 Prove in Graph 2 the applicability of the above formula for the relative error and find the values of the parameters C and γ.

3.6 Determine the minimum value $N_{\min}$ at which the relative error in calculating the period does not exceed 0.2%.

In further calculations, use only the found value $N_{\min}$ for the number of partition intervals.

4. Experiment: the dependence of the period on the amplitude.

At this stage of the computer experiment, we determine the dependence of the oscillation period of the mathematical pendulum on the amplitude, $T(\varphi_0)$, which is described by the formula

$$T(\varphi_0) = T_0\left(a + \frac{\varphi_0^2}{b}\right),$$

where T_0 designates the period of small oscillations of the pendulum, a, b are constant values.

4.1 Calculate the periods of oscillation of the mathematical pendulum for the following set of amplitudes φ_0 : $15°, 30°, 45°, 60°, 75°$ and $90°$, which you have already determined.

4.2 Prove in Graph 3 the applicability of the above formula for the dependence of the oscillation period of the pendulum on its amplitude.

4.3 Determine the values of parameters a, b.

Let the error in measuring the oscillation period of the pendulum in a real experiment be approximately equal to 5%.

4.4 Determine at what angles φ_0, expressed in degrees, the oscillations of the mathematical pendulum can be considered small.

SOLUTION
COMPUTER EXPERIMENT:
A mathematical pendulum or what angle can be considered rather small ...

1. Constructing a theoretical model.

1.1 The formula for the period of small oscillations of a mathematical pendulum has the form

$$T = 2\pi\sqrt{\frac{l}{g}}. \qquad (1)$$

1.2 From the law of conservation of mechanical energy for the ball of the pendulum (the zero level of the potential energy is taken at the suspension point)

$$\frac{ml^2\omega^2}{2} = mgl(\cos\varphi - \cos\varphi_0), \qquad (2)$$

the following formula is derived for the angular velocity of the pendulum

$$\omega = \sqrt{\frac{2g}{l}\left(\cos\varphi - \cos\varphi_0\right)}. \qquad (3)$$

1.3 Let us divide the entire section of motion from φ_0 to zero into infinitely small intervals $d\varphi$. The time dt it takes for the pendulum to pass this interval is found as

$$dt = \frac{d\varphi}{\omega} = \frac{d\varphi}{\sqrt{\frac{2g}{l}\left(\cos\varphi - \cos\varphi_0\right)}}. \qquad (4)$$

Then the time of motion t_1 is obtained as the sum of small intervals, which finally reduces to a simple integration

$$t_1 = \int_0^{\phi_0} dt = \int_0^{\phi_0} \frac{d\varphi}{\sqrt{\frac{2g}{l}\left(\cos\varphi - \cos\varphi_0\right)}}. \qquad (5)$$

1.4 The oscillation period is 4 times longer than the found time t_1, viz.

$$T = 4t_1. \qquad (6)$$

1.5 The angular velocity in dimensionless units is expressed as follows

$$\tilde{\omega} = \frac{d\varphi}{d\tau} = \frac{d\varphi}{\sqrt{\frac{g}{l}}dt} = \sqrt{\frac{l}{g}}\,\omega. \qquad (7)$$

1.6 The period of small oscillations in dimensionless units reads as

$$\tilde{T} = \sqrt{\frac{g}{l}}\,T = 2\pi. \qquad (8)$$

1.7 The dependence of the angular velocity $\tilde{\omega}$ on the deflection angle φ has the form

$$\tilde{\omega}(\varphi) = \sqrt{2(\cos\varphi - \cos\varphi_0)}. \qquad (9)$$

2. Designing an experimental setup, planning an experiment.

2.1 The partition interval is equal to

$$\Delta\varphi = \frac{\varphi_0}{N}. \qquad (10)$$

2.2 First, you should set the "zeroth" angle of deflection, and the coordinates of the rest of the splitting points are given by the formula

$$\varphi_k = \varphi_{k-1} - \Delta\varphi. \qquad (11)$$

2.3 The angular velocity ω_k at the point φ_k is described by the formula

$$\omega_k = \sqrt{2(\cos\varphi_k - \cos\varphi_0)}. \qquad (12)$$

In the particular case of $\varphi_0 = \frac{\pi}{2}$ this formula further simplifies to

$$\omega_k = \sqrt{2\cos\varphi_k}\,. \tag{12a}$$

2.4 In the recommended approximation of uniformly accelerated motion, the average speed at the selected interval is equal to the arithmetic mean of the angular velocities at the ends of the interval, i.e.

$$\langle\omega\rangle = \frac{1}{2}(\omega_{k-1} + \omega_k),$$

Therefore, the travel time Δt_k of the k'th interval from φ_{k-1} to φ_k is obtained as

$$\Delta t_k = \frac{2\Delta\varphi}{\omega_{k-1} + \omega_k}\,. \tag{13}$$

2.5 The time t_k to reach the angle φ_k is found via

$$t_k = t_{k-1} + \Delta t_k \tag{14}$$

at the initial condition $t_0 = 0$.

2.6 The oscillation period T_N when dividing into N intervals is eventually written as

$$T_N = 4t_N\,. \tag{15}$$

3. Trial experiment, estimation of errors.

3.1 The results of calculating the angular velocities, times and periods of oscillations for the indicated values of the number of partition intervals are shown in Table 1.

Table 1. Calculation of periods of oscillations with different numbers of partition intervals.

***N*=**	**32**				***N*=**	**16**			
$\Delta\varphi$	0,0491				$\Delta\varphi$	0,0982			
k	φ	ω	Δt_k	t_k	***k***	φ	ω	Δt_k	t_k
0	1,5708	0,0000		0,0000	0	1,5708	0,0000		0,0000
1	1,5217	0,3133	0,3134	0,3134	1	1,4726	0,4428	0,4435	0,4435
2	1,4726	0,4428	0,1299	0,4432	2	1,3744	0,6246	0,1840	0,6274
3	1,4235	0,5417	0,0997	0,5430	3	1,2763	0,7620	0,1416	0,7690
4	1,3744	0,6246	0,0842	0,6271	4	1,1781	0,8749	0,1200	0,8890
5	1,3254	0,6971	0,0743	0,7014	5	1,0799	0,9710	0,1064	0,9954
6	1,2763	0,7620	0,0673	0,7687	6	0,9817	1,0541	0,0970	1,0923
7	1,2272	0,8208	0,0620	0,8307	7	0,8836	1,1264	0,0900	1,1824
8	1,1781	0,8749	0,0579	0,8886	8	0,7854	1,1892	0,0848	1,2672
9	1,1290	0,9247	0,0546	0,9432	9	0,6872	1,2434	0,0807	1,3479
10	1,0799	0,9710	0,0518	0,9950	10	0,5890	1,2896	0,0775	1,4254
11	1,0308	1,0140	0,0495	1,0444	11	0,4909	1,3281	0,0750	1,5004
12	0,9817	1,0541	0,0475	1,0919	12	0,3927	1,3593	0,0731	1,5735
13	0,9327	1,0915	0,0458	1,1377	13	0,2945	1,3834	0,0716	1,6451
14	0,8836	1,1264	0,0443	1,1819	14	0,1963	1,4006	0,0705	1,7156
15	0,8345	1,1589	0,0430	1,2249	15	0,0982	1,4108	0,0698	1,7854
16	0,7854	1,1892	0,0418	1,2667	16	0,0000	1,4142	0,0695	1,8549
17	0,7363	1,2173	0,0408	1,3075					
18	0,6872	1,2434	0,0399	1,3474					
19	0,6381	1,2674	0,0391	1,3865					
20	0,5890	1,2896	0,0384	1,4249					
21	0,5400	1,3098	0,0378	1,4626					
22	0,4909	1,3281	0,0372	1,4999					
23	0,4418	1,3446	0,0367	1,5366					

24	0,3927	1,3593	0,0363	1,5729					
25	0,3436	1,3723	0,0359	1,6088					
26	0,2945	1,3834	0,0356	1,6445					
27	0,2454	1,3929	0,0354	1,6798					
28	0,1963	1,4006	0,0351	1,7150					
29	0,1473	1,4065	0,0350	1,7500					
30	0,0982	1,4108	0,0348	1,7848					
31	0,0491	1,4134	0,0348	1,8196					
32	0,0000	1,4142	0,0347	1,8543					
			T_N=	**7,4171**				T_N=	**7,4197**

N=	**8**				N=	**4**			
$\Delta\varphi$	0,1963				$\Delta\varphi$	0,3927			
k	φ	ω	Δt_k	t_k	k	φ	ω	Δt_k	t_k
0	1,5708	0,0000		0,0000	0	1,5708	0,0000		0,0000
1	1,3744	0,6246	0,6287	0,6287	1	1,1781	0,8749	0,8977	0,8977
2	1,1781	0,8749	0,2619	0,8906	2	0,7854	1,1892	0,3805	1,2783
3	0,9817	1,0541	0,2036	1,0941	3	0,3927	1,3593	0,3082	1,5864
4	0,7854	1,1892	0,1751	1,2692	4	0,0000	1,4142	0,2832	1,8696
5	0,5890	1,2896	0,1584	1,4276					
6	0,3927	1,3593	0,1483	1,5759					
7	0,1963	1,4006	0,1423	1,7182					
8	0,0000	1,4142	0,1395	1,8577					
			T_N=	**7,4307**				T_N=	**7,4785**

N=	**2**				N=	**1**			
$\Delta\varphi$	0,7854				$\Delta\varphi$	1,5708			
k	φ	ω	Δt_k	t_k	k	φ	ω	Δt_k	t_k
0	1,5708	0,0000		0,0000	0	1,5708	0,0000		0,0000
1	0,7854	1,1892	1,3209	1,3209	1	0,0000	1,4142	2,2214	2,2214
2	0,0000	1,4142	0,6034	1,9242					
			T_N=	**7,6969**				T_N=	**8,8858**

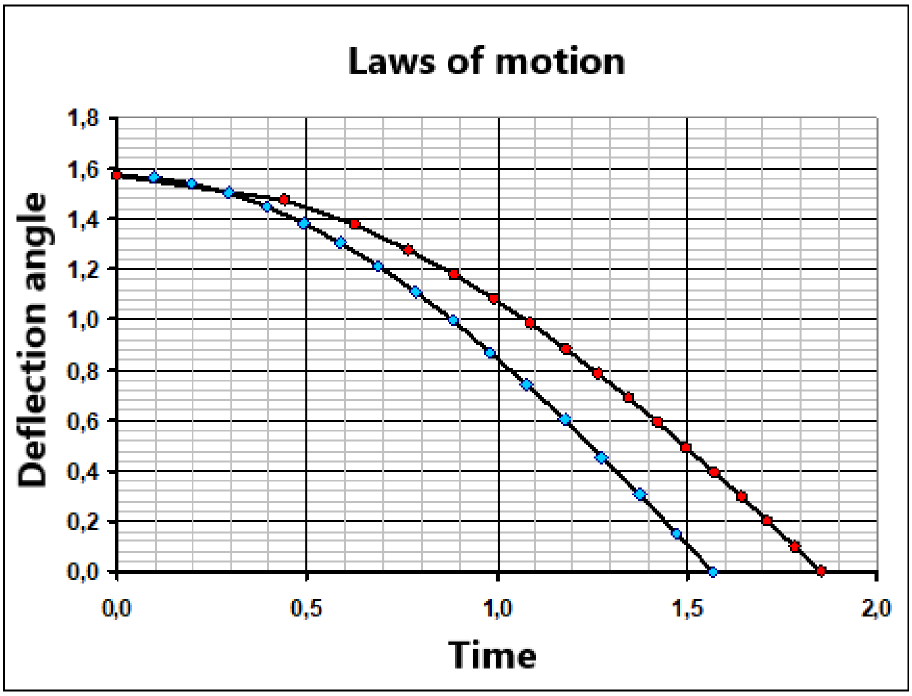
3.2 – 3.3 To calculate the graph points of the law of motion in the approximation of small oscillations, it is necessary to use the formula

$$\varphi(t) = \frac{\pi}{2}\cos t\,. \tag{16}$$

The calculation results for this law are presented in Table 2 and in the graph. There is also a line in the graph representing the calculated law of motion (16). It is interesting to note that in the first case, the time values are set, and the corresponding deflection angles are calculated; and in the second one, on the contrary, the deflection angles are set and corresponding times are calculated.

Table 2.

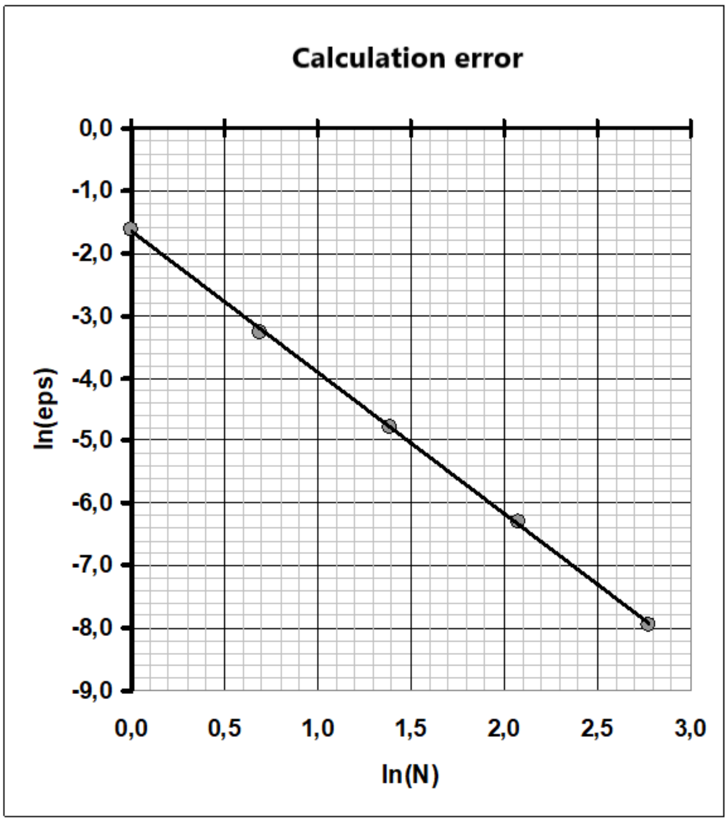
k	t	φ
0	0,0000	1,5708
1	0,0982	1,5632
2	0,1963	1,5406
3	0,2945	1,5032
4	0,3927	1,4512
5	0,4909	1,3853
6	0,5890	1,3061
7	0,6872	1,2142
8	0,7854	1,1107
9	0,8836	0,9965
10	0,9817	0,8727
11	1,0799	0,7405
12	1,1781	0,6011
13	1,2763	0,4560
14	1,3744	0,3064
15	1,4726	0,1540
16	1,5708	0,0000



3.4 - 3.5 The results of calculating the errors ε_N of the oscillation periods for different numbers N of partition intervals are shown in Table 3.

Table 3. Calculation errors.

N	T	ε_N	$\ln N$	$\ln \varepsilon_N$
1	8,8858	1,98E-01	0,0000	-1,6195
2	7,6969	3,77E-02	0,6931	-3,2774
4	7,4785	8,27E-03	1,3863	-4,7952
8	7,4307	1,83E-03	2,0794	-6,3026
16	7,4197	3,50E-04	2,7726	-7,9584
32	7,4171	0,00E+00		



To determine the parameters of the dependence $\varepsilon_N = \dfrac{C}{N^\gamma}$, it must be represented on a double logarithmic scale as

$$\ln \varepsilon_N = \ln C - \gamma \ln N . \qquad (17)$$

The figure on the right shows a graph of this dependence, drawn according to the data in Table 3. The linearity of the obtained dependence clearly proves the applicability of the formula for the dependence of the calculation error on the number N.

The parameters of this linear relationship, calculated using the least squares method, are equal: the slope coefficient $a \approx -2{,}3$ and the shift $b \approx -1{,}65$. Therefore, the sought parameters of the dependence are found as

$$\begin{aligned} \gamma &\approx -a = 2{,}3 \\ C &= \exp(b) \approx 0{,}19 \end{aligned} . \qquad (18)$$

3.6 It is easy to find from formula (17) that the number of partition intervals required to achieve the error $\varepsilon = 0{,}002$ is expressed as

$$N_{\min} = \left(\frac{C}{\varepsilon}\right)^{1/\gamma} \approx 8. \tag{19}$$

So, all further calculations should be carried out at $N = N_{\min} = 8$.

4. Experiment: the dependence of the period on the amplitude.

4.1 The results of calculating the periods of oscillations for various amplitudes of oscillations are shown in Table 4.

Table 4. Calculation of periods of oscillations.

φ_0	0,2618		15°		φ_0	0,5236		30°	
N=	8				N=	8			
$\Delta\varphi$	0,0327				$\Delta\varphi$	0,0654			
k	φ	ω	Δt_k	t_k	***k***	φ	ω	Δt_k	t_k
0	0,2618	0,0000		0,0000	0	0,5236	0,0000		0,0000
1	0,2291	0,1261	0,5190	0,5190	1	0,4581	0,2484	0,5270	0,5270
2	0,1963	0,1724	0,2193	0,7383	2	0,3927	0,3402	0,2224	0,7494
3	0,1636	0,2036	0,1741	0,9124	3	0,3272	0,4023	0,1763	0,9257
4	0,1309	0,2259	0,1524	1,0648	4	0,2618	0,4470	0,1541	1,0799
5	0,0982	0,2419	0,1399	1,2047	5	0,1963	0,4791	0,1413	1,2212
6	0,0654	0,2527	0,1323	1,3370	6	0,1309	0,5008	0,1336	1,3548
7	0,0327	0,2590	0,1279	1,4649	7	0,0654	0,5135	0,1291	1,4839
8	0,0000	0,2611	0,1259	1,5908	8	0,0000	0,5176	0,1269	1,6108
			T=	6,3630				T=	6,4432
			T/To=	1,0127				T/To=	1,0255

φ_0	0,7854		45°		φ_0	1,0472		60°	
N=	8				N=	8			
$\Delta\varphi$	0,0982				$\Delta\varphi$	0,1309			
k	φ	ω	Δt_k	t_k	***k***	φ	ω	Δt_k	t_k
0	0,7854	0,0000		0,0000	0	1,0472	0,0000		0,0000
1	0,6872	0,3631	0,5408	0,5408	1	0,9163	0,4664	0,5613	0,5613
2	0,5890	0,4987	0,2278	0,7687	2	0,7854	0,6436	0,2359	0,7972
3	0,4909	0,5913	0,1801	0,9488	3	0,6545	0,7660	0,1857	0,9829
4	0,3927	0,6584	0,1571	1,1059	4	0,5236	0,8556	0,1614	1,1444
5	0,2945	0,7069	0,1438	1,2497	5	0,3927	0,9207	0,1474	1,2917
6	0,1963	0,7398	0,1357	1,3855	6	0,2618	0,9653	0,1388	1,4306
7	0,0982	0,7590	0,1310	1,5165	7	0,1309	0,9914	0,1338	1,5643
8	0,0000	0,7654	0,1288	1,6453	8	0,0000	1,0000	0,1315	1,6958
			T=	6,5810				T=	6,7832
			T/To=	1,0474				T/To=	1,0796

φ_0	1,3090		75°		φ_0	1,5708		90°	
N=	8				N=	8			
$\Delta\varphi$	0,1636				$\Delta\varphi$	0,1963			
k	φ	ω	Δt_k	t_k	***k***	φ	ω	Δt_k	t_k
0	1,3090	0,0000		0,0000	0	1,5708	0,0000		0,0000
1	1,1454	0,5548	0,5899	0,5899	1	1,3744	0,6246	0,6287	0,6287
2	0,9817	0,7704	0,2469	0,8368	2	1,1781	0,8749	0,2619	0,8906
3	0,8181	0,9217	0,1934	1,0302	3	0,9817	1,0541	0,2036	1,0941
4	0,6545	1,0340	0,1673	1,1976	4	0,7854	1,1892	0,1751	1,2692

5	0,4909	1,1163	0,1522	1,3497	5	0,5890	1,2896	0,1584	1,4276
6	0,3272	1,1731	0,1429	1,4927	6	0,3927	1,3593	0,1483	1,5759
7	0,1636	1,2065	0,1375	1,6302	7	0,1963	1,4006	0,1423	1,7182
8	0,0000	1,2175	0,1350	1,7652	8	0,0000	1,4142	0,1395	1,8577
			T=	7,0608				T=	7,4307
			T/To=	1,1238				T/To=	1,1826

Below is a summary table for calculations of the oscillation period for different amplitudes.

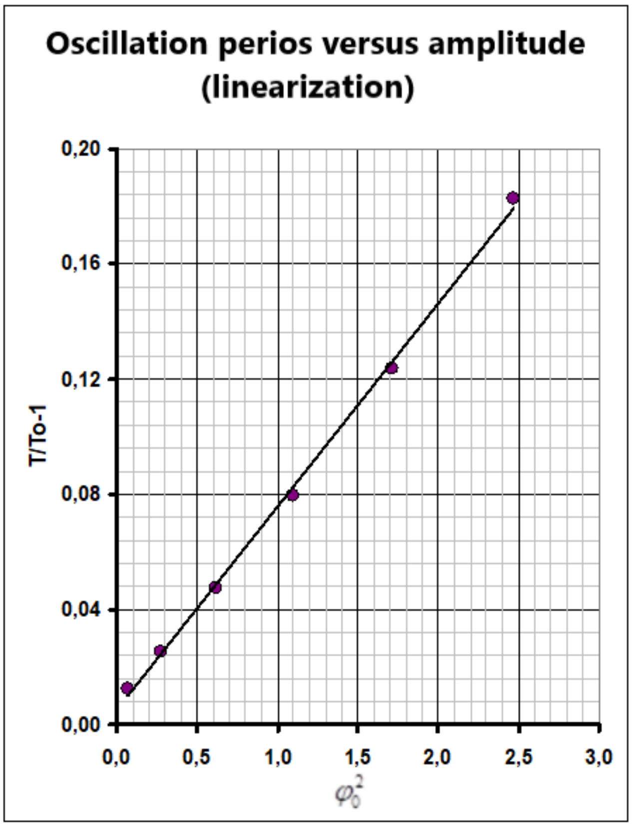
Table 5.

$\varphi_0,^\circ$	φ_0	T	$\dfrac{T}{T_0}$	φ_0^2	$\dfrac{T}{T_0}-1$
15	0,2618	6,3630	1,0127	0,0685	0,0127
30	0,5236	6,4432	1,0255	0,2742	0,0255
45	0,7854	6,5810	1,0474	0,6169	0,0474
60	1,0472	7,0608	1,0796	1,0966	0,0796
75	1,3090	7,0608	1,1238	1,7135	0,1238
90	1,5708	7,4307	1,1826	2,4674	0,1826

4.2 – 4.3 For small oscillations, the formula for the oscillation period

$$T(\varphi_0)=T_0\left(a+\frac{\varphi_0^2}{b}\right) \tag{20}$$

must coincide with formula (1), whence it follows that the parameter $a=1$. To check the applicability of formula (20) to the description of the calculation results, it is necessary to draw a graph of the dependence of the value $\left(\dfrac{T}{T_0}-1\right)$ on the square of the amplitude φ_0^2, which is shown in the figure on the right.



The slope coefficient of the obtained dependence is $0,0706$, therefore, the parameter b appearing in formula (20) is approximately equal to

$$b \approx 14. \tag{21}$$

4.4 With an acceptable error of a real experiment is 5%, the deviation of the period from the period of small oscillations would not be noticeable if the inequality holds

$$\frac{\varphi_0^2}{b}<0.05, \tag{22}$$

whence it follows that the angles can be considered rather small for $\varphi_0<45^\circ$.

	Content	Total for each part	Points
	1. Constructing a theoretical model.	**1.5**	
1.1	Formula (1): $T=2\pi\sqrt{\dfrac{l}{g}}$	**0.1**	0.1

1.2	Law of motion - conservation of energy (2); - formula (3): $\omega=\sqrt{\frac{2g}{l}\left(\cos\varphi-\cos\varphi_0\right)}$	**0.4**	 0.1 0.3
1.3	Integration of the law of motion - formula (4): $dt=\frac{d\varphi}{\omega}$ - formula (5): $t_1=\int_0^{\phi_0}\frac{d\varphi}{\sqrt{\frac{2g}{l}\left(\cos\varphi-\cos\varphi_0\right)}}$	**0.3**	 0.1 0.2
1.4	Period (6): $T=4t_1$	**0.1**	0.1
1.5	Formula (7): $\tilde{\omega}=\sqrt{\frac{l}{g}}\omega$	**0.2**	0.2
1.6	Formula (8): $\tilde{T}=2\pi$	**0.2**	0.2
1.7	Formula (9): $\tilde{\omega}(\varphi)=\sqrt{2(\cos\varphi-\cos\varphi_0)}$	**0.2**	0.2
2. Designing an experimental setup, planning an experiment.		**1.5**	
2.1	Formula (10): $\Delta\varphi=\frac{\varphi_0}{N}$	**0.1**	0.1
2.2	Formula (11): $\varphi_k=\varphi_{k-1}-\Delta\varphi$	**0.1**	0.1
2.3	Formula (12): $\omega_k=\sqrt{\cos\varphi_k-\cos\varphi_0}$	**0.1**	0.1
2.4	Uniformly accelerated motion Main idea $\langle\omega\rangle=\frac{1}{2}\left(\omega_{k-1}+\omega_k\right)$ Formula (13): $\Delta t_k=\frac{2\Delta\varphi}{\omega_{k-1}+\omega_k}$	**1.0**	 0.5 0.5
2.5	Formula (14): $t_k=t_{k-1}+\Delta t_k$	**0.1**	0.1
2.6	Formula (15): $T_N=4t_N$	**0.1**	0.1
3. Trial experiment, estimation of errors.		**9.0**	
3.1	*Graded only if the periods obtained differ from those in this official solution lees than.* Periods are correctly calculated for N=32 N=16 N=8 N=4 N=2 N=1	**4.2**	 1.2 1.0 0.8 0.6 0.4 0.2
3.2	Graph: All points are plotted in accordance with the table; Smooth line is drawn;	**0.5**	 0.3 0.2
3.3	Graph Law of motion is obtained, table 2 All points are plotted in accordance with the table Smooth line is drawn;	**1.0**	 0.5 0.3 0.2
3.4	Errors are correctly evaluated, table 3	**0.5**	0.5
3.5	Graph Double logarithm scale is used Linearized dependence is drawn	**2.3**	 0.5 0.5

	Linear dependence is obtained The power is found as $\gamma \approx 2{,}3 \pm 0{,}1$ The coefficient is found as $C \approx 0{,}19 \pm 0{,}1$		0.3 0.5 0.5
3.6	Concluded that $N = 8 \pm 1$	**0.5**	0.5
4. Experiment: the dependence of the period on the amplitude.		**8.0**	
4.1	*Graded only if the periods obtained differ from those in this official solution lees than 0,02* For each period obtained One period is added from part 3	**4.2**	 0.8 0.2
4.2	Graph of the linearized dependence Correct linearization is used $T(\varphi_0^2)$ Graph is plotted Linearized dependence is obtained	**1.5**	 0.5 0.5 0.5
4.3	Parameters of the linearized dependence Parameter $a = 1$ (only exact value is accepted) Parameter b in the range 13-16	**1.3**	 0.3 1.0
4.4	Small angle is estimated as $\varphi < 45°$	**1.0**	1.0
	TOTAL	**20.0**	

SOLUTIONS TO THE PROBLEMS OF THE THEORETICAL COMPETITION

Attention. Points in grading are not divided!

Problem 1 (10.0 points)

Problem 1.1 (4.0 points)

At the initial moment of time, the ball rotates as a whole around the point of its contact with the table. Let the ball rotate through a certain angle α, then the change in the potential energy of the center of mass of the ball is written as

$$E_p = mgR(1-\cos\alpha), \tag{1}$$

and it turns into kinetic energy

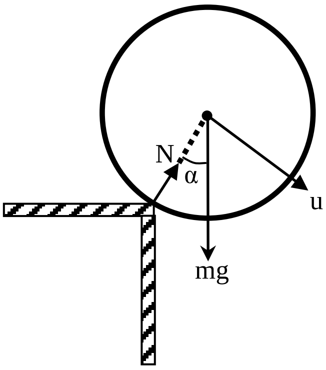
$$E_k = \frac{7}{10}mu^2, \tag{2}$$

where u is the speed of the center of mass of the ball.

According to the law of conservation of energy, we get

$$E_p = E_k. \tag{3}$$

At further motion, the ball is separated from the table. The equation of motion of the center of mass of the ball (Newton's second law) in the projection on the radial direction has the form



$$m\frac{u^2}{R} = mg\cos\alpha - N, \tag{4}$$

where N stands for the normal reaction force of the table, and the friction force is not shown in the figure.

The condition for the separation of the ball from the table is defined as

$$N = 0. \tag{5}$$

Solving jointly equation (1)-(5), we find the separation angle and the speed of the ball at this moment

$$\cos\alpha = \frac{10}{17}, \tag{6}$$

$$u = \sqrt{\frac{10}{17}gR}. \tag{7}$$

The further motion of the ball is the free fall of its center of mass in the Earth's gravitational field. The initial horizontal and vertical velocities are respectively equal to

$$v_x = u\cos\alpha. \tag{8}$$

$$v_y = u\sin\alpha, \tag{9}$$

The flight range is determined by the formulas of uniformly accelerated motion in the earth's gravity field as

$$L = R\sin\alpha + v_x t. \tag{10}$$

$$H - R(1-\cos\alpha) = v_y t + \frac{gt^2}{2}, \tag{11}$$

where t denotes the free flight time.

Eliminating time t from equations (10) and (11), we find

$$L = \frac{567\sqrt{21}+20\sqrt{68305}}{4913}R \approx 1.6R. \tag{12}$$

Content	Points
Formula (1): $E_p = mgR(1-\cos\alpha)$	0.3

Formula (2): $E_k = \frac{7}{10}mu^2$	0.2
Formula (3): $E_p = E_k$	0.2
Formula (4): $m\frac{u^2}{R} = mg\cos\alpha - N$	0.3
Formula (5): $N = 0$	0.4
Formula (6): $\cos\alpha = \frac{10}{17}$	0.4
Formula (7): $u = \sqrt{\frac{10}{17}gR}$	0.4
Formula (8): $v_x = u\cos\alpha$	0.2
Formula (9): $v_y = u\sin\alpha$	0.2
Formula (10): $L = R\sin\alpha + v_x t$	0.4
Formula (11): $H - R(1-\cos\alpha) = v_y t + \frac{gt^2}{2}$	0.4
Formula (12): $L = \frac{567\sqrt{21} + 20\sqrt{68305}}{4913}R \approx 1.6R$	0.6
Total	**4.0**

Problem 1.2 (3.0 points)

The work dA done by the gas when its volume changes by dV reads as

$$dA = pdV\ , \tag{1}$$

where p denotes the gas pressure.

The change in the internal energy dU of one mole of an ideal monatomic gas is associated with a change in its temperature dT by the relation

$$dU = \frac{3}{2}RdT\ . \tag{2}$$

According to the formulation of the problem, the following relation holds

$$\eta = \frac{dA}{dU} = const\ , \tag{3}$$

which, along with the ideal gas equation

$$pV = RT\ , \tag{4}$$

leads to the following relation

$$\frac{2}{3\eta}\frac{dV}{V} = \frac{dT}{T}\ . \tag{5}$$

Equation (5) is easily integrated and reduced to the form

$$\frac{T}{T} = \left(\frac{V}{V_0}\right)^{\frac{2}{3\eta}}\ . \tag{6}$$

In the initial state, the ideal gas equation gives

$$p_0V_0 = RT_0\ , \tag{7}$$

whereas in the final state

$$\frac{p_0}{2}4V_0 = RT\ , \tag{8}$$

and, therefore, the temperature of the gas in the final state is obtained as

$$T = 2T_0\ . \tag{9}$$

From equations (6) and (9) it is easy to find the coefficient

$$\eta = \frac{4}{3}. \tag{10}$$

The total work of the gas in the process is determined by the integral of equation (1) and is equal to

$$A = \int_{V_0}^{4V_0} pdV = 2p_0V_0 = 2.0\times10^5 \text{ J}, \tag{11}$$

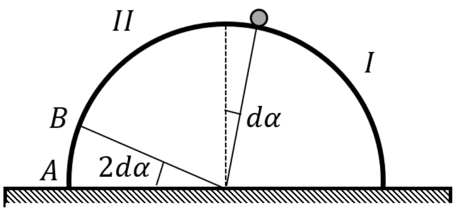
Note: The process described in this problem is polytropic, i.e. it occurs at a constant heat capacity. Indeed, since the work done by the gas is a fixed part of the change in the internal energy, this means that the heat capacity of the gas remains constant throughout the process. In this case, the polytropic equation $pV^n = const$ is valid under the chosen conditions of the problem with $n = 1/2$, and the work of the gas, obviously, does not depend on its type, whether it is a monatomic or polyatomic gas.

Content	Points
Formula (1): $dA = pdV$	0.2
Formula (2): $dU = \frac{3}{2}RdT$	0.2
Formula (3): $\eta = \frac{dA}{dU} = const$	0.2
Formula (4): $pV = RT$	0.2
Formula (5): $\frac{2}{3\eta}\frac{dV}{V} = \frac{dT}{T}$	0.2
Formula (6): $\frac{T}{T} = \left(\frac{V}{V_0}\right)^{\frac{2}{3\eta}}$	0.4
Formula (7): $p_0V_0 = RT_0$	0.2
Formula (8): $\frac{p_0}{2}4V_0 = RT$	0.2
Formula (9): $T = 2T_0$	0.2
Formula (10): $\eta = \frac{4}{3}$	0.4
Formula (11): $A = 2p_0V_0$	0.4
Numerical value in formula (11): $A = 2.0\times10^5$ J	0.2
Total	**3.0**

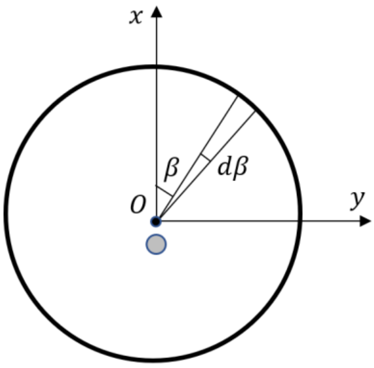
Problem 1.3 (3.0 points)

To study the problem of the stability of the equilibrium position, consider a situation in which the ball deviates from the top position by a very small angle $d\alpha$ and determine the forces acting on it.

The first force is electrostatic, but to study the equilibrium we need only its component directed tangentially to the surface of the hemisphere. The idea of its calculation is based on the fact that in the projection onto the radial direction, the electrostatic forces are compensated from two symmetrical regions of the hemisphere I and II with respect to the new ball position, so that the only uncompensated force is due to the segment AB of the hemisphere, cut off by an inclined plane passing at an angle $2d\alpha$. The left figure below shows the corresponding section in the vertical plane.



Side view



Top view

Let us consider a part of the sphere segment (see the right figure above, which shows the top view), cut off additionally by the angles β amd $\beta + d\beta$, such that its area is found as

$$dS = 2R\cos\beta R d\beta d\alpha \, , \tag{1}$$

with the electric charge being equal to

$$dq = -\sigma dS \, . \tag{2}$$

In the Cartesian coordinate system, whose origin coincides with the top of the hemisphere, and the axis is directed vertically downwards, the radius vector, directed from the point where the ball is located to the selected part of the sphere segment, is determined by the coordinates

$$\vec{r} = (R\cos\beta, R\sin\beta, R) \, , \tag{3}$$

and hence the vector of the desired force is derived as

$$\vec{F} = -\frac{Qdq}{4\pi\varepsilon_0 r^3}\vec{r} \, . \tag{4}$$

This force has the following projection on the tangential direction

$$F_Q = -\frac{Qdq}{4\pi\varepsilon_0(\sqrt{2}R)^3} R\cos\beta \, . \tag{5}$$

therefore, integration over β from $-\pi/2$ to $\pi/2$ provides the total module of the electrostatic force from the entire segment in the form

$$F_Q = \frac{Q\sigma}{8\sqrt{2}\pi\varepsilon_0} d\alpha \, , \tag{6}$$

The second force acting on the ball is the force of gravity, whose projection on the tangential direction is obtained as

$$F_g = mgd\alpha \, . \tag{7}$$

The minimum charge of the ball is determined by the equality of forces

$$F_g = F_Q \, , \tag{8}$$

which leads to the final answer

$$Q = \frac{8\sqrt{2}\pi\varepsilon_0 mg}{\sigma} \, . \tag{9}$$

Obviously, for larger charges the equilibrium position is stable.

Content	Points
Formula (1): $dS = 2R\cos\beta R d\beta d\alpha$	0.3
Formula (2): $dq = \sigma dS$	0.3
Formula (3): $\vec{r} = (R\cos\beta, R\sin\beta, R)$	0.2
Formula (4): $\vec{F} = -\frac{Qdq}{4\pi\varepsilon_0 r^3}\vec{r}$	0.2

Formula (5): $F_Q = \dfrac{Qdq}{4\pi\varepsilon_0(\sqrt{2}R)^3}R\cos\beta$	0.3
Formula (6): $F_Q = \dfrac{Q\sigma}{8\sqrt{2}\pi\varepsilon_0}d\alpha$	0.5
Formula (7): $F_g = mgd\alpha$	0.2
Formula (8): $F_g = F_Q$	0.5
Formula (9): $Q = \dfrac{8\sqrt{2}\pi\varepsilon_0 mg}{\sigma}$	0.5
Total	**3.0**

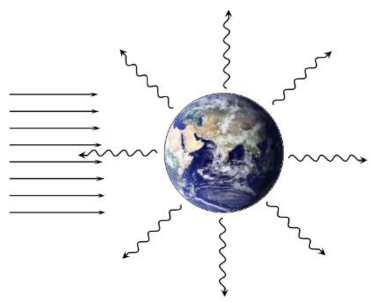
Problem 2. Greenhouse effect (10.0 points)

Atmosphere without greenhouse effect

2.1 Direct calculation by Wien's formula gives the following result

$$\lambda_{\max S} = \frac{b}{T_S} = 0.446\,\mu\text{m}. \tag{1}$$

2.2 In the steady state, the power of solar radiation incident on the Earth is equal to the power of the thermal radiation of the Earth. When writing the energy balance equation, it must be taken into account that the Sun illuminates the Earth from one side, and the Earth radiates in all directions, i.e.



$$W \cdot \pi R^2 = \sigma T_0^4 \cdot 4\pi R^2. \tag{2}$$

It follows from this relation that

$$T_0 = \sqrt[4]{\frac{W}{4\sigma}} = 280.3\text{ K}, \tag{3}$$

and the same temperature in degrees Celsius is equal to

$$t_0 = 7.15\text{ °C}. \tag{4}$$

2.3 According to the Wien's formula, we find that at the given temperature, the maximum radiation corresponds to the wavelength

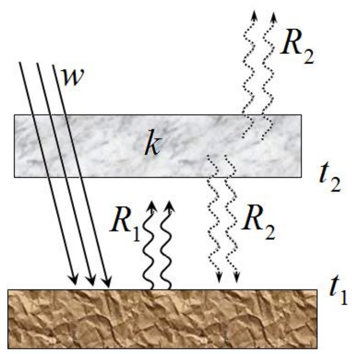
$$\lambda_{\max E} = \frac{b}{T_E} = 10.3\,\mu\text{m}. \tag{5}$$

2.4 The same geometric relationships that lead to equation (2) allow one to conclude that the power of solar radiation per unit area of the Earth's surface is found as

$$w = \frac{W \cdot \pi R^2}{4\pi R^2} = \frac{W}{4} = 350\text{ W/m}^2. \tag{6}$$

Various atmosphere models

2.5 We introduce the following notation:



t_1 (or T_1 in the Kelvin scale) – the emperature of the Earth's surface and the lower layer of the atmosphere immediately adjacent to it; t_2 (or T_2) – the temperature of the upper layer of the atmosphere; w – the flux density of solar radiation, i.e. the energy incident on a unit area of the Earth's surface per unit time (or irradiated); R_1 – the thermal radiation power per unit area of the Earth; R_2 – the thermal radiation power per unit area of the atmospheric layer; the radiation fluxes of this layer towards the Earth and into outer space are equal.

The energy balance equation for a unit area of the Earth's surface has the following form

$$w + R_2 = R_1. \tag{7}$$

A similar equation for the upper layer of the atmosphere gives rise to

$$KR_1 = 2R_2. \tag{8}$$

Using the laws of thermal radiation, energy fluxes can be expressed in terms of the temperatures of the radiating surfaces as follows

$$R_1 = \sigma T_1^4, \tag{9}$$

$$R_2 = K\sigma T_2^4. \tag{10}$$

Therefore, taking into account formulas (2) and (3), we obtain from expressions (7)-(10) the temperature of the Earth's surface in the form

$$T_1 = \frac{T_0}{\sqrt[4]{1-\frac{K}{2}}}. \tag{11}$$

Maximum greenhouse effect

2.6 For the maximum greenhouse effect $K=1$, therefore, it is obtained for this model

$$T_1 = T_0\sqrt[4]{2} = 333.3\text{ K} = 60.2\ ^\circ\text{C}. \tag{12}$$

Thus, the maximum increase in temperature due to the greenhouse effect on the "black earth" is equal to

$$\Delta t_1 = 53.0\ ^\circ\text{C}. \tag{13}$$

Water greenhouse effect

2.7 The Earth as a black body irradiates the energy

$$W_0 = \int_0^\infty r_0(\lambda, T_1)\,d\lambda, \tag{14}$$

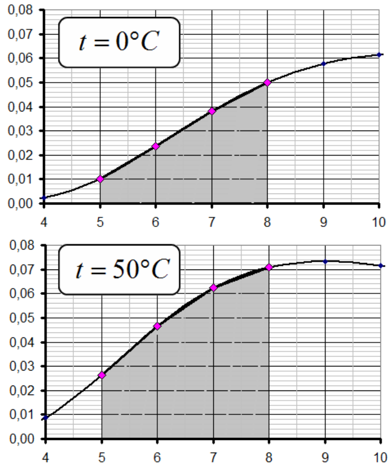
The absorbed energy can be expressed in terms of the spectral absorption coefficient and the spectral density of the Earth's radiation as follows

$$W_A = \int_0^\infty k(\lambda) r_0(\lambda, T_1)\,d\lambda, \tag{15}$$

then the total absorption coefficient of terrestrial radiation by theupper layer of the atmosphere is calculated by the formula

$$K = \frac{W_A}{W_0} = \frac{\int_0^\infty k(\lambda) r_0(\lambda, T_1)\,d\lambda}{\int_0^\infty r_0(\lambda, T_1)\,d\lambda} = \frac{\sigma T_1^4 \int_0^\infty k(\lambda)\varphi(\lambda, T_1)\,d\lambda}{\sigma T_1^4 \int_0^\infty \varphi(\lambda, T_1)\,d\lambda} = \int_0^\infty k(\lambda)\varphi(\lambda, T_1)\,d\lambda. \tag{16}$$

2.8 Since in the indicated wavelength range from 5.0 to 8.0 μm the water vapor absorbs all incident radiation, the total absorption coefficient is equal to the fraction of radiation energy falling into this interval. This fraction of energy is evalulated as the areas under the graphs given in the problem introduction.



The calculations carried out for 4 points gives the following values for the absorption coefficients

$$t_1 = 0\ ^\circ\text{C}: \qquad K_0 = 0.092, \tag{17}$$

$$t_1 = 50\ ^\circ\mathrm{C}: \qquad K_{50} = 0.158. \tag{18}$$

2.9 It follows from the proposed relationship $K(t_1) = K_0(1+\alpha t_1)$ that

$$K_0 = 0.092, \tag{19}$$

$$\alpha = \frac{1}{t_{50}}\left(\frac{K_{50}}{K_0} - 1\right) = 0.014\ \mathrm{K}^{-1}. \tag{20}$$

2.10 At the temperature of $t_1 = 5{,}4°C$, the absorption coefficient of the upper layer of the atmosphere is found as

$$K(t_0) = K_0(1+\alpha t_0) = 0.101. \tag{21}$$

Since the absorption coefficient is rather small, formula (12) for the steady temperature can be simplified to

$$T_1 = \frac{T_0}{\sqrt[4]{1-\frac{K}{2}}} \approx T_0\left(1+\frac{K}{8}\right), \tag{22}$$

and the rise in temperature is obrained as

$$\Delta t_1 = T_0\frac{K(t_0)}{8} = 3.55\ ^\circ\mathrm{C}. \tag{23}$$

2.11 To accurately answer the question, it is necessary to solve the nonlinear equation

$$T_1 = \frac{T_0}{\sqrt[4]{1-\frac{K(T_1)}{2}}}. \tag{24}$$

However, the relative change in the absolute temperature is small, so we represent the sought temperature in the form

$$T_1 = T_0 + \Delta t, \tag{25}$$

from which we find the value of the temperature change in view of the condition $\Delta t \ll T_0$

$$\Delta t = \frac{T_0\frac{K_0(1+\alpha t_0)}{8}}{1 - T_0\frac{\alpha K_0}{8}} = \frac{\Delta t_1}{1 - T_0\frac{\alpha K_0}{8}} \approx 3.73\ ^\circ\mathrm{C}. \tag{26}$$

Amplification of the greenhouse effect by carbon dioxide

2.12 Let us calculate the absorption coefficient due to carbon dioxide. To make estimates, we can assume that the air temperature differs slightly from $0°C$. To do this, we take into account that: 1) in the range from 2.5 to 3.0 μm, the energy of the thermal radiation of the Earth is negligible; 2) in the range from 6.5 μm to 7.0 μm all radiation is absorbed by water vapor; 3) in the range from 16 μm to 18 μm, the fraction of radiation energy is equal to $\Phi = 0.08$ (calculated according to the graph for $t = 0°C$) . Therefore, the additional absorption coefficient due to the presence of carbon dioxide is found as

$$K_2 = 0.04. \tag{27}$$

Since the absorption of carbon dioxide and water vapor lie in different spectral ranges, the total absorption coefficient is equal to the sum of the absorption coefficients of water and carbon dioxide. Then the change in the steady-state surface temperature (taking into account absorption by carbon dioxide) increases by the value

$$\Delta t_1 = T_0\frac{K_2}{8} \approx 1.4\ ^\circ\mathrm{C}. \tag{28}$$

2.13 To calculate the absorption coefficient with increased concentration, we use the obvious reasoning: in the presence of several absorbing layers, the total transmission is equal to the product of the transmission coefficients of individual layers, therefore

$$1 - k_1 = (1-k_0)^2. \tag{29}$$

Hence it follows that if the concentration is doubled, the spectral absorption coefficient is expected to increase from 0.50 to

$$k_1 = 2k_0 - k_0^2 = 0.75\,. \qquad (30)$$

Therefore, the total absorption coefficient becomes equal to

$$K_2 = k\Phi = 0.06\,. \qquad (31)$$

i.e. increases by $\Delta K_2 = 0.02$. Therefore, the additional rise in temperature is finally obtained as

$$\Delta t_1' = T_0 \frac{\Delta K_2}{8} \approx 0.7\ ^\circ\text{C}\,. \qquad (32)$$

	Content	**Points**	
2.1	Formula (1): $\lambda_{\max S} = \dfrac{b}{T_S}$	0.1	**0.2**
	Numerical value in formula (1): $\lambda_{\max S} = 0.446\,\mu m$	0.1	
2.2	Formula (2): $W \cdot \pi R^2 = \sigma T_0^4 \cdot 4\pi R^2$	0.4	**1.0**
	Formula (3): $T_0 = \sqrt[4]{\dfrac{W}{4\sigma}}$	0.2	
	Numerical value in formula (3): $T_0 = 280.3$ K	0.2	
	Numerical value in formula (4): $t_0 = 7.15\ ^\circ$C	0.2	
2.3	Formula (5): $\lambda_{\max E} = \dfrac{b}{T_E}$	0.1	**0.2**
	Numerical value in formula (5): $\lambda_{\max E} = 10{,}3\,\mu$m	0.1	
2.4	Formula (6): $w = \dfrac{W}{4}$	0.1	**0.2**
	Numerical value in formula (6): $w = 350$ W/m^2	0.1	
2.5	Formula (7): $w + R_2 = R_1$	0.2	**1.2**
	Formula (8): $KR_1 = 2R_2$	0.2	
	Formula (9): $R_1 = \sigma T_1^4$	0.2	
	Formula (10): $R_2 = K\sigma T_2^4$	0.2	
	Formula (11): $T_1 = \dfrac{T_0}{\sqrt[4]{1 - \dfrac{K}{2}}}$	0.4	
2.6	Direct use of $K = 1$	0.1	**0.5**
	Formula (12): $T_1 = T_0 \sqrt[4]{2}$	0.2	
	Numerical value in formula (13): $\Delta t_1 = 53.0\ ^\circ$C	0.2	
2.7	Formula (14): $W_0 = \int_0^\infty r_0(\lambda, T_1)\, d\lambda$	0.2	**0.8**
	Formula (15): $W_A = \int_0^\infty k(\lambda) r_0(\lambda, T_1)\, d\lambda$	0.2	
	Formula (16): $K = \int_0^\infty k(\lambda) \varphi(\lambda, T_1)\, d\lambda$	0.4	
2.8	Numerical value in (17): $t_1 = 0\ ^\circ$C: $K_0 = 0.092$	0.6	**1.2**
	Numerical value in (18): $t_1 = 50\ ^\circ$C: $K_{50} = 0.158$	0.6	

2.9	Numerical value in (19): $K_0 = 0.092$	0.2	**0.4**
	Numerical value in (20): $\alpha = 0.031\ \mathrm{K}^{-1}$	0.2	
2.10	Numerical value in (21): $K(t_0) = 0.0757$	0.4	**0.8**
	Numerical value in (23): $\Delta t_1 = 2.65$ °C	0.4	
2.11	Formula (24): $T_1 = \dfrac{T_0}{\sqrt[4]{1-\dfrac{K(T_1)}{2}}}$	0.2	**1.0**
	Formula (25): $T_1 = T_0 + \Delta t$ at $\Delta t \ll T_0$	0.2	
	Formula (26): $\Delta t = \dfrac{T_0\dfrac{K_0(1+\alpha t_0)}{8}}{1-T_0\dfrac{\alpha K_0}{8}} = \dfrac{\Delta t_1}{1-T_0\dfrac{\alpha K_0}{8}}$	0.4	
	Numerical value in formula (26): $\Delta t \approx 2.84$ °C	0.2	
2.12	Numerical value in (27): $K_2 = 0.04$	0.5	**1.0**
	Numerical value in (28): $\Delta t_1 \approx 1.4$ °C	0.5	
2.13	Formula (29): $1-k_1 = (1-k_0)^2$	0.5	**1.5**
	Formula (30): $k_1 = 2k_0 - k_0^2$	0.2	
	Numerical value in formula (31): $K_2 = k\Phi = 0.06$	0.4	
	Numerical value in (32): $\Delta t_1' \approx 0.7$ °C	0.4	
Total			**10.0**

Problem 3. Corpuscular interpretation of light pressure (10.0 points)

Introduction

3.1 Let the concentration of photons with the energy ε in the incident radiation be equal to n, then the wave intensity is determined by the relation

$$I_0 = c\varepsilon n, \tag{1}$$

where c stands for the speed of light.

The number of photons ΔN falling on the area element ΔS at the angle φ per unit of time is written as

$$\Delta N = cn\Delta t\Delta S\cos\varphi. \tag{2}$$

The number of absorbed photons per unit of time is found as follows

$$\Delta N_a = (1-R)\Delta N, \tag{3}$$

whereas the number of reflected ones

$$\Delta N_r = R\Delta N. \tag{4}$$

The normal component of the momentum, transferred by one photon to the area element upon absorption, is equal to

$$\Delta p_a = \frac{\varepsilon}{c}\cos\varphi, \tag{5}$$

and the same value at reflection is put down as

$$\Delta p_r = 2\frac{\varepsilon}{c}\cos\varphi. \tag{6}$$

The total momentum transferred to the area element is determined by the expression

$$\Delta p = \Delta N_a\Delta p_a + \Delta N_r\Delta p_r, \tag{7}$$

and the pressure sought is calculated by the formula

$$p_s = \frac{\Delta p}{\Delta S \Delta t} = \frac{I_0}{c}(1+R)\cos^2\varphi . \quad (8)$$

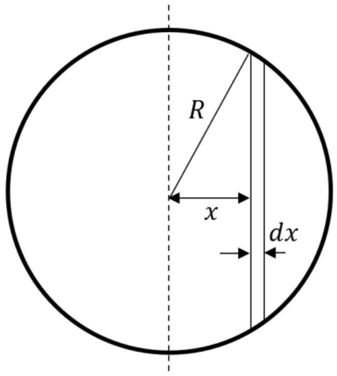
3.2 At normal incidence $\varphi = 0$ and at complete absorption $R = 0$, we obtain

$$p_s = \frac{I_s}{c} = 4.70 \cdot 10^{-6}\,\text{Pa} . \quad (9)$$

and, accordingly, at total reflection $R = 1$

$$p_s = \frac{2I_s}{c} = 9.40 \cdot 10^{-6}\,\text{Pa} . \quad (10)$$

3.3 Consider a section of the spherical surface perpendicular to the incident light flux. For the mirror part of the surface, which completely reflects light, the mechanical torque is equal to zero, since the transmitted momentum is directed strictly along the radius of the sphere.



Let us consider a strip in the section located from the center of the sphere at distances from x to $x + dx$. The selected part of the completely absorbing surface has the area

$$dS = 2\sqrt{R^2 - x^2}\,dx , \quad (11)$$

and the number of absorbed photons per unit time is equal to

$$\Delta N_a = \frac{I_s}{\varepsilon} dS , \quad (12)$$

each of which has the momentum

$$\Delta p_a = \frac{\varepsilon}{c} . \quad (13)$$

The force shoulder is

$$l = x , \quad (14)$$

therefore, the torque of forces acting on the selected area is obtained as

$$dM = \Delta N_a \Delta p_a l = \frac{2I_s}{c}\sqrt{R^2 - x^2}\,x dx , \quad (15)$$

and the total torque of forces is determined by the integral

$$M = \int_0^R dM = \frac{2I_s R^3}{3c} = 3.13 \cdot 10^{-6}\,\text{N} \cdot \text{m} . \quad (16)$$

Space station with the mirror sail

3.4 At the initial rest point of the station of mass m with the sail of area S, located at the distance R_0 from the Sun of mass M_S, the gravitational force is exactly balanced by the light pressure force, which leads to the equation

$$G\frac{M_S m}{R_0^2} = \frac{2n_0\varepsilon}{c} S , \quad (17)$$

where G refers to the gravitational constant, n_0 is the concentration of photons of solar radiation with energy ε at the location of the station.

Due to the spherically symmetric expansion, the photon concentration changes with the distance r from the Sun according to the law

$$n(r) = n_0\left(\frac{R_0}{r}\right)^2. \tag{18}$$

The initial momentum of photons before the collision with the sail is written as

$$p_0 = \frac{\varepsilon}{c}, \tag{19}$$

whereas the final one constitutes

$$p = \frac{\varepsilon}{c}\frac{c-V}{c+V}. \tag{20}$$

This relationship is easily obtained from the kinematics and is actually the formula for the Doppler effect. In addition, the momentum of a photon after reflection from the sail mirror can be easily obtained from the laws of conservation of momentum and energy by considering an absolutely elastic collision of a photon with a moving massive mirror.

Thus, the change in the momentum of the photon is transferred to the mirror and is equal to

$$\Delta p = p - p_0 = \frac{2\varepsilon}{c+V}, \tag{21}$$

and the number of photons falling per unit time Δt on the sail is derived as

$$\frac{\Delta N}{\Delta t} = n(r)S(c-V). \tag{22}$$

Hence, the force acting on the station due to the solar radiation is determined by the expression

$$f = \Delta p\frac{\Delta N}{\Delta t} = 2n_0\varepsilon S\left(\frac{R_0}{r}\right)^2\frac{c-V}{c+V} = G\frac{M_S m}{r^2}\frac{c-V}{c+V}. \tag{23}$$

The station is also subject to the force of gravitational attraction from the Sun

$$f_g = G\frac{M_S m}{r^2}. \tag{24}$$

which means that the motion of the station in the radial direction is described by Newton's second law in the form

$$m\frac{dV}{dt} = f - f_g = -2G\frac{M_S m}{r^2}\frac{V}{c+V}. \tag{25}$$

Bearing in mind that for a small displacement

$$dr = Vdt, \tag{26}$$

we obtain from expression (25) the differential equation

$$(c+V)dV = -2GM_S\frac{dr}{r^2}, \tag{27}$$

which is easily integrated and, if the station stops, gives rise to

$$cV_0 + \frac{1}{2}V_0^2 = 2GM_S\left(\frac{1}{R_0} - \frac{1}{R}\right). \tag{28}$$

Solving equation (28), we find the distance sought as

$$R = \frac{R_0}{1 - \dfrac{(cV_0 + \frac{1}{2}V_0^2)R_0}{2GM_S}}, \tag{29}$$

which, under the condition of the Earth's orbital motion

$$GM_S = V_E^2 r_E, \tag{30}$$

as well as the relation $V \ll c$, yields the final answer of the form

$$R = \frac{R_0}{1 - \dfrac{cV_0R_0}{2V_E^2 r_E}} = 9.93\cdot 10^{10}\,\text{m}. \tag{31}$$

3.5 It follows from formula (31) that the station is able to fly away to infinity $R \to \infty$ only if the denominator of the expression becomes zero, which results in

$$V_{\min} = \frac{2V_E^2 r_E}{cR_0} = 18.1 \text{ m/s} . \qquad (32)$$

Poynting-Robertson effect

3.6 The mass of the dust particle is determined by the expression

$$m = \rho \frac{4}{3}\pi a^3 , \qquad (33)$$

and its cross-sectional area is

$$S = \pi a^2 . \qquad (34)$$

Let us determine the effective force acting on the particle as a result of light absorption. To reduce it to the pressure of light, let us move to the frame of reference associated with the dust particle. In this frame of reference, the particle is affected by the pressure of light, calculated by formula (9), but its direction does not coincide with the radial one due to the aberration of light, namely, it makes a small angle V / c with it. Thus, in the tangential direction of the particle trajectory, a force appears due to the absorption of photons, equal to

$$F = -V \frac{I_S}{c^2} S , \qquad (35)$$

which creates a torque about the center of attraction found as

$$M = -FR. \qquad (36)$$

Since the trajectory of the dust particle is almost circular, its velocity can be written as

$$V = \sqrt{\frac{GM_S}{R}} , \qquad (37)$$

and the angular momentum relative to the attracting center

$$L = mVR . \qquad (38)$$

Collecting equations (33)-(38) together, we write

$$\frac{dL}{dt} = M , \qquad (39)$$

whence we finally find the time sought in the following form

$$\tau = \frac{2\mu\rho a c^2}{3I_S} = 1.27 \cdot 10^8 \text{ s} . \qquad (40)$$

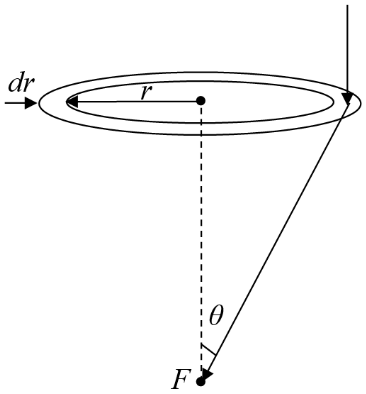
At the derivation, change in the intensity of solar radiation with distance is neglected, since the radius of the orbit decreases only slightly and the corresponding corrections are of higher order of smallness.

Note: A consistent explanation of the Poynting-Robertson effect is based on the following interpretation. In the reference frame associated with the particle, it absorbs the solar radiation, which propagates at a small angle to the radial direction, and then reradiates the accumulated energy isotropically in all directions. In the reference frame associated with the Sun, the primary radiation of the Sun propagates in the radial direction, and the reradiation of the particle itself is no longer isotropic. In the first case, the appearance of the braking force moment is explained by the aberration of solar radiation, whereas in the second case, by the Doppler effect for the reradiation of the particle itself.

Laser tweezer

3.7 Let us calculate the force acting on the first converging lens, which is equal to the total change in the momentum of photons incident on the lens per unit time. Obviously, the momentum changes due to the refraction of light in the glass, since its direction changes, but not the module.

Consider all the rays passing through the ring on the lens, located from its center at distances from r to $r + dr$.



The area of this ring is written as

$$dS = 2\pi r dr\ . \tag{41}$$

The change in the longitudinal momentum of photons passing through the given ring per unit time is equal to

$$dp_{\parallel} = \frac{I}{c}(1-\cos\theta)dS\ , \tag{42}$$

where the angle of refraction is found as folows

$$\sin\theta = \frac{r}{F}\ , \tag{43}$$

since all rays converge at the focus of the lens.

Integrating the resulting expression over the entire surface of the lens, we obtain

$$f_{\parallel} = \int_0^R dp_{\parallel} = \frac{\pi I}{c}\left(R^2 - \frac{2}{3F}\left[F^3 - \left(F^2 - R^2\right)^{3/2}\right]\right) \approx \frac{\pi I R^4}{4cF^2} = 2.64\cdot 10^{-17}\ \text{N}\ . \tag{44}$$

Since the foci of the lens L and the particle M coincide, when leaving the "lens-particle" system, the light beam propagates again parallel to the optical axis, and, therefore, as a result of refraction on the particle M, the photon momentum is restored. Consequently, the force acting on the particle M is equal in magnitude to $f_{\parallel}$, but is directed towards the converging lens. This force draws the particle into the laser radiation field. This is the principle of operation of the "laser tweezer".

3.8 Consider all the rays passing through the element of the semiring on the lens, located from its center at distances from r to $r+dr$, and also cut off by azimuth angles from β to $\beta + d\beta$. The area of this semicircle element is derived as

$$dS = r dr d\beta\ . \tag{45}$$

The change in the transverse momentum of photons passing through the given ring per unit time is equal to

$$dp_{\perp} = \frac{I}{c}\sin\theta\sin\beta dS\ , \tag{46}$$

and integration over the entire surface of the half of the lens, taking into account formula (43), leads to the expression

$$f_{\perp} = \int dp_{\perp} = \frac{I}{cF}\int_0^{\pi}\int_0^{R} r^2 dr \sin\beta d\beta = \frac{2IR^3}{3cF} = 2.24\cdot 10^{-16}\ \text{N}\ . \tag{47}$$

	Content	Points	
3.1	Formula (1): $I_0 = c\varepsilon n$	0.1	**0.8**
	Formula (2): $\Delta N = cn\Delta t\Delta S\cos\varphi$	0.1	
	Formula (3): $\Delta N_a = (1-R)\Delta N$	0.1	
	Formula (4): $\Delta N_r = R\Delta N$	0.1	
	Formula (5): $\Delta p_a = \frac{\varepsilon}{c}\cos\varphi$	0.1	

	Formula (6): $\Delta p_r = 2\dfrac{\varepsilon}{c}\cos\varphi$	0.1	
	Formula (7): $\Delta p = \Delta N_a \Delta p_a + \Delta N_r \Delta p_r$	0.1	
	Formula (8): $p_s = \dfrac{I_0}{c}(1+R)\cos^2\varphi$	0.1	
3.2	Formula (9): $p_s = \dfrac{I_s}{c}$	0.1	**0.4**
	Numerical value in formula (9): $p_s = 4.70\cdot 10^{-6}\,\text{Pa}$	0.1	
	Formula (10): $p_s = \dfrac{2I_s}{c}$	0.1	
	Numerical value in formula (10): $p_s = 9.40\cdot 10^{-6}\,\text{Pa}$	0.1	
3.3	Moment of forces on the mirror part of the sphere $M = 0$	0.1	**1.0**
	Formula (11): $dS = 2\sqrt{R^2 - x^2}\,dx$	0.1	
	Formula (12): $\Delta N_a = \dfrac{I_s}{\varepsilon}dS$	0.1	
	Formula (13): $\Delta p_a = \dfrac{\varepsilon}{c}$	0.1	
	Formula (14): $l = x$	0.1	
	Formula (15): $dM = \dfrac{2I_s}{c}\sqrt{R^2 - x^2}\,dx$	0.1	
	Formula (16): $M = \dfrac{2I_s R^3}{3c}$	0.2	
	Numerical value in formula (16): $M = 3.13\cdot 10^{-6}\,\text{N}\cdot\text{m}$	0.2	
3.4	Formula (17): $G\dfrac{M_S m}{R_0^2} = \dfrac{2n_0\varepsilon}{c}S$	0.4	**3.6**
	Formula (18): $n(r) = n_0\left(\dfrac{R_0}{r}\right)^2$	0.2	
	Formula (19): $p_0 = \dfrac{\varepsilon}{c}$	0.1	
	Formula (20): $p = \dfrac{\varepsilon}{c}\dfrac{c-V}{c+V}$	0.2	
	Formula (21): $\Delta p = p - p_0 = \dfrac{2\varepsilon}{c+V}$	0.1	
	Formula (22): $\dfrac{\Delta N}{\Delta t} = n(r)S(c-V)$	0.2	
	Formula (23): $f = G\dfrac{M_S m}{r^2}\dfrac{c-V}{c+V}$	0.2	
	Formula (24): $f_g = G\dfrac{M_S m}{r^2}$	0.2	
	Formula (25): $m\dfrac{dV}{dt} = -2G\dfrac{M_S m}{r^2}\dfrac{V}{c+V}$	0.4	
	Formula (26): $dr = Vdt$	0.2	
	Formula (27): $(c+V)dV = -2GM_S\dfrac{dr}{r^2}$	0.2	

	Formula (28): $cV_0 + \frac{1}{2}V_0^2 = 2GM_S\left(\frac{1}{R_0} - \frac{1}{R}\right)$	0.2	
	Formula (29): $R = \dfrac{R_0}{1 - \dfrac{(cV_0 + \frac{1}{2}V_0^2)R_0}{2GM_S}}$	0.2	
	Formula (30): $GM_S = V_E^2 r_E$	0.2	
	Formula (31): $R = \dfrac{R_0}{1 - \dfrac{cV_0R_0}{2V_E^2 r_E}}$	0.4	
	Numerical value in formula (31): $R = 9.93\cdot 10^{10}\,\text{m}$	0.2	
3.5	Formula (32): $V_{\min} = \dfrac{2V_E^2 r_E}{cR_0}$	0.2	**0.4**
	Numerical value in formula (32): $V_{\min} = 18.1$ m/s	0.2	
3.6	Formula (33): $m = \rho\frac{4}{3}\pi a^3$	0.1	**2.0**
	Formula (34): $S = \pi a^2$	0.1	
	Formula (35): $F = -V\dfrac{I_S}{c^2}S$	0.4	
	Formula (36): $M = FR$	0.2	
	Formula (37): $V = \sqrt{\dfrac{GM_S}{R}}$	0.2	
	Formula (38): $L = mVR$	0.2	
	Formula (39): $\dfrac{dL}{dt} = M$	0.2	
	Formula (40): $\tau = \dfrac{2\mu\rho a c^2}{3I_S}$	0.4	
	Numerical value in formula (40): $\tau = 1.27\cdot 10^8$ s	0.2	
3.7	Formula (41): $dS = 2\pi r dr$	0.1	**1.0**
	Formula (42): $dp_{\parallel} = \dfrac{I}{c}(1-\cos\theta)dS$	0.2	
	Formula (43): $\sin\theta = \dfrac{r}{F}$	0.2	
	Formula (44): $f_{\parallel} = \dfrac{\pi I}{c}\left(R^2 - \dfrac{2}{3F}\left[F^3 - \left(F^2 - R^2\right)^{3/2}\right]\right) \approx \dfrac{\pi I R^4}{4cF^2}$	0.3	
	Numerical value in formula (44): $f_{\parallel} = 2.64\cdot 10^{-17}$ N	0.2	
3.8	Formula (45): $dS = r dr d\beta$	0.1	**0.8**
	Formula (46): $dp_{\perp} = \dfrac{I}{c}\sin\theta\sin\beta dS$	0.2	
	Formula (47): $f_{\perp} = \dfrac{2IR^3}{3cF}$	0.3	
	Numerical value in formula (47): $f_{\perp} = 2.24\cdot 10^{-16}$ N	0.2	
Total			**10.0**

THEORETICAL COMPETITION
February 16, 2022

Please read this first:

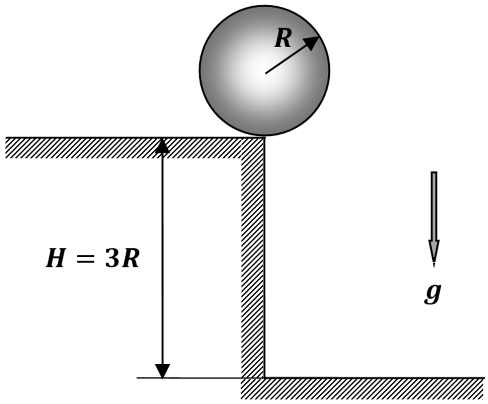
1. The duration of the theoretical competition is 4 hours. There are three problems.
2. You can use your own calculator for numerical calculations.
3. You are provided with ***Writing sheet*** and additional white sheets of paper. You can use the additional sheets of paper for drafts of your solutions, but these sheets will not be graded. Your final solutions should be written on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures, and plots.
4. Use only the front side of ***Writing sheets*.** Write only inside the boxed areas.
5. Start putting down your solution to each problem on a new ***Writing sheet***.
6. Fill in the boxes at the top of each ***Writing sheet*** with your country (**Country),** your student code (**Student Code**), problem number (**Question Number)**, the progressive number of each ***Writing sheet*** **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be graded, put a large X across the entire sheet and do not include it in your numbering.

Problem 1 (10.0 points)

This problem consists of three independent parts.

Problem 1.1 (4.0 points)

A ball of radius R rests on the edge of a horizontal table of height $h = 3R$. The ball starts to fall without a significant initial push, and there is no slipping with the edge during the entire time of motion. Find at what distance l from the base of the table the ball first touches the floor. The free fall acceleration is equal to g, air resistance is negligible. It is known that the kinetic energy of a ball rolling without slipping is equal to $E = \frac{7}{10}mv^2$, where v denotes the speed of the ball center with m being its mass.

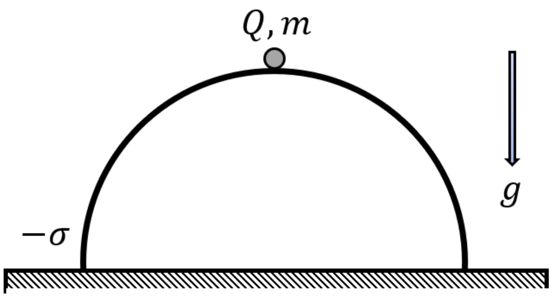


Problem 1.2 (3.0 points)

A quasi-static process is carried out with one mole of an ideal monatomic gas, as a result of which its initial volume $V_0 = 1\ m^3$ increases four times, and the initial pressure $p_0 = 10^5$ Pa decreases two times. It is known that for each small section of the quasi-static process, the ratio of work to the change in internal energy is a constant value. Find the total work A done by the gas in this process.

Problem 1.3 (3.0 points)

A thin-walled dielectric hemisphere, negatively charged with the surface density $-\sigma$, is placed on the horizontal table. A point-like ball of mass m is carefully placed on its top. Determine the minimum positive charge Q of the ball, such that it is still in a state of stable equilibrium at the top of the hemisphere. The charges between the hemisphere and the ball are not redistributed. The free fall acceleration is g.



Problem 2. Greenhouse effect (10.0 points)

Introduction

Any heated body, whose temperature T is above absolute zero, radiates electromagnetic waves. The spectrum of this radiation, called thermal, depends on the optical properties of the body surface and its temperature. Despite the fact that the radiation of each body is its individual characteristic, general laws are well known that describe the thermal radiation.

Kirchhoff's law. In a state of thermodynamic equilibrium, the ratio of the emissivity of a body $r(\lambda, T)$ to its absorptivity $k(\lambda, T)$ is a universal function $r_0(\lambda, T)$ that does not depend on individual characteristics

$$\frac{r(\lambda,T)}{k(\lambda,T)} = r_0(\lambda, T).$$

The value $r(\lambda, T)\Delta\lambda$ has the meaning of the energy emitted per unit area per unit time in a narrow wavelength range from λ to $\lambda + \Delta\lambda$. The absorptivity of the body $k(\lambda, T)$ is a dimensionless absorption coefficient equal to the ratio of the radiation energy absorbed by the body to the total radiation energy incident on the body surface, if the wavelengths of the incident radiation lie in a narrow wavelength range from λ to $\lambda + \Delta\lambda$. If the body completely absorbs all the incident electromagnetic radiation $k(\lambda, T) = 1$, then such a body is called an absolute black body.

Wien's displacement law. The wavelength λ_{max},, at which the function r$r_0(\lambda, T)$ has a maximum, is related to the absolute temperature by the relation

$$\lambda_{max} T = b,$$

where $b = 2.898 \cdot 10^{-3}$ m/K is called the Wien constant.

Stefan-Boltzmann law. The total emissivity of a black body over all wavelengths is described by the formula

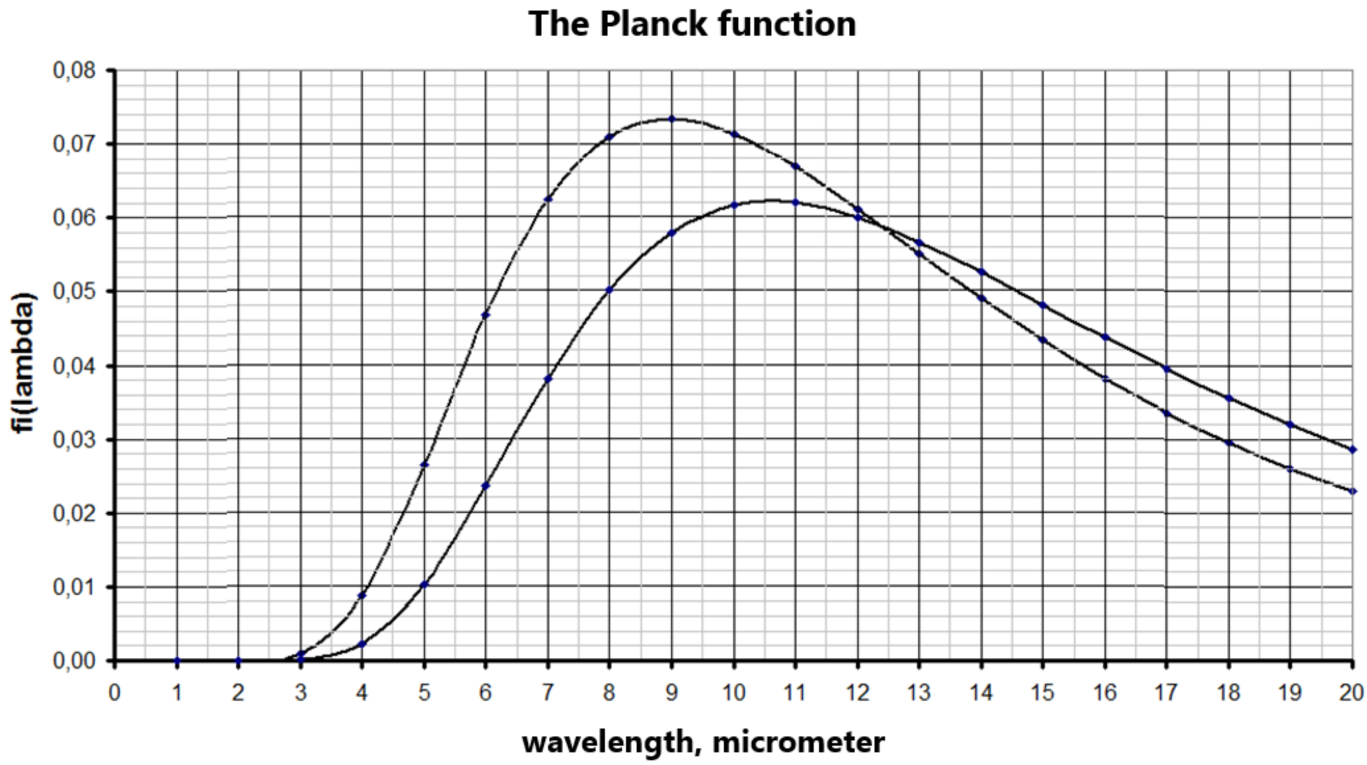
$$R(T) = \int_0^{\infty} r_0(\lambda, T) d\lambda = \sigma T^4,$$

where $\sigma = 5.670 \cdot 10^{-8}\ W/(m^2 \cdot K^4)$ stands for the Stefan-Boltzmann constant.

Using the Stefan-Boltzmann law, the formula for $r_0(\lambda, T)$ can be represented as

$$r_0(\lambda, T) = \sigma T^4 \varphi(\lambda, T).$$

Here, the Planck function $\varphi(\lambda, T)$ describes the energy distribution in the black body radiation spectrum; the value $\varphi(\lambda, T)\Delta\lambda$ is equal to the fraction of the thermal radiation energy per narrow spectral interval from λ to $\lambda + \Delta\lambda$. The total area under the graph of the function $\varphi(\lambda, T)$ is equal to unity. In this problem, it is recommended to use the graphs of this function, shown in the figure below and plotted at temperatures $t =$ 0°C and $t = 50$°C.



Model of the Earth and its atmosphere

Climate change associated with an increase in the average temperature of the atmosphere is now an established scientific fact. It is believed that the main cause of the global warming is the greenhouse effect. Solar radiation, whose main part lies in the visible region of the spectrum, passes almost completely through the atmosphere, and then is absorbed by the earth's surface. On the contrary, the thermal radiation of the earth's surface, which lies mainly in the infrared region of the spectrum, is significantly absorbed by certain atmospheric gases, mainly water vapor and carbon dioxide. In this problem, the simplest model of the greenhouse effect is considered and some numerical estimates are made of its influence on the atmospheric temperature.

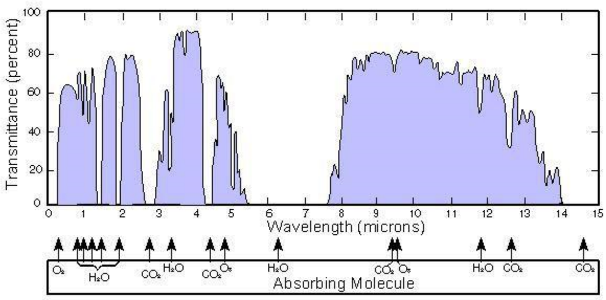
The surface of the Earth is assumed to be an absolutely black body of a spherical shape, covered with a layer of the atmosphere, whose thickness is much less than the Earth radius. Conventionally, the atmosphere is divided into two parts: 1) the lower layer, directly adjacent to the earth's surface and having the same temperature as the earth's surface; 2) the upper (greenhouse) layer, capable of absorbing thermal radiation coming from the earth's surface. It is assumed that the transfer of energy between the Sun, the earth's surface and the atmosphere is carried out only through radiation, and the temperatures of the earth's surface and atmospheric layers are the same at all their points and do not depend on time, for example, on the time of day or year season. In the following, the solar constant $W = 1.40 \cdot 10^3$ W/m^2 is considered known, which is the power of solar radiation incident on the unit area of the Earth, oriented perpendicular to the incident light.

Atmosphere without the greenhouse effect

2.1 Evaluate the wavelength $\lambda_{max\,S}$, which corresponds to the maximum in the thermal radiation of the Sun, if the surface temperature of the Sun is approximately equal to $T_S = 6500$ K.
2.2 Neglecting the absorption of the atmosphere and considering the earth's surface as an absolutely black body, evaluate the steady-state temperature of the Earth's surface T_0, and also determine this temperature t_0 in the Celsius scale. This temperature is called below as the "black earth" temperature.
2.3 Calculate the wavelength $\lambda_{max\,E}$, which corresponds to the maximum in the radiation of the "black earth".
2.4 Calculate the power of solar radiation w per unit area of the Earth's surface.

Various atmosphere models

In reality, the upper layer of the atmosphere effectively absorbs electromagnetic radiation of certain wavelengths in the infrared region of the spectrum. For example, the figure on the right shows the real dependence of the atmospheric transmission as a function of the wavelength. As can be seen, this dependence is rather complicated, therefore, in the problem below, several simplified models are considered.



Let K be the total absorption coefficient of the upper layer of the atmosphere for the thermal radiation of the Earth, i.e. the ratio of the energy of the thermal radiation absorbed by the upper layer of the atmosphere to the total energy incident on the upper layer of atmosphere from the earth's surface.

2.5 Obtain the formula for the steady-state temperature of the Earth's surface T_1 and express it in terms of the "black earth" temperature T_0 and the absorption coefficient K.

Maximum greenhouse effect

Let the upper layer of the atmosphere completely transmit the solar radiation and completely absorb the thermal radiation of the Earth.

2.6 Calculate how much the temperature of the Earth's surface $\Delta t_1 = T_1 - T_0$ increases compared to the temperature of the "black earth" due to the maximum greenhouse effect.

Water greenhouse effect

Let the spectral absorption coefficient $k(\lambda)$ of the upper layer of the atmosphere be a known function of the wavelength λ of the incident radiation and be independent of its temperature.

2.7 Express the total absorption coefficient of the upper atmosphere K in terms of $k(\lambda)$ and the Planck distribution function $\varphi(\lambda, T_1)$,, where T_1 denotes the temperature of the Earth's surface.

Assume that the absorption in the upper layer of the atmosphere is completely due to water vapor. Approximately, it can be considered that water vapor completely absorbs radiation whose wavelengths lie in the range from 5.00 to 8.00 μm, whereas the rest is completely transmitted.

2.8 Using the plots of the Planck distribution function given in the introduction section of this problem, calculate the numerical values of the total absorption coefficient K of the upper atmosphere for two values of the Earth's surface temperatures $t_1 = 0°\text{C}$ and $t_1 = 50°\text{C}$.

In the above specified temperature range from $t_1 = 0°\text{C}$ to $t_1 = 50°\text{C}$, the dependence of the total absorption coefficient on the ground temperature t_1 is approximately described by a linear function of the temperature itself: $K(t_1) = K_0(1 + \alpha t_1)$,, where K_0, α are some constants.

2.9 Calculate the numerical values of the parameters K_0 and α.

In the following, assume that the temperature changes under question are small, so formulas of approximate calculus can be used.

2.10 Neglecting the dependence of the absorption coefficient of the atmosphere on the temperature and assuming it to be equal to the absorption coefficient at the temperature of the "black Earth" T_0, calculate the change in the temperature of the Earth's surface $\Delta t_1 = T_1 - T_0$.

2.11 Calculate the change in the temperature of the Earth's surface $\Delta t_1 = T_1 - T_0 \ll T_0$ if the dependence of the atmospheric absorption coefficient on the temperature is described by the linear function of the earth's temperature as defined above.

Amplification of the greenhouse effect by carbon dioxide

Let us take into account the effect of the absorption by carbon dioxide present in the atmosphere. At the current concentration of carbon dioxide in the atmosphere (approximately 0.05%), it can be considered that carbon dioxide completely absorbs the radiation of the Earth in the wavelength ranges from 2.50 to 3.00 μm and from 6.50 to 7.00 μm, and in the range from 16.0 to 18.0 μm the absorption coefficient equals 0.500. For other wavelengths, the absorption of radiation by carbon dioxide can be neglected.

2.12 Estimate how much, as compared to the water greenhouse effect model, the temperature of the Earth's surface increases due to the absorption of radiation by carbon dioxide.

2.13 Estimate how much, as compared to 2.12, the temperature of the Earth's surface increases if the concentration of carbon dioxide in the atmosphere increases by $\eta = 2.00$ times as compared to its current concentration.

Problem 3. Corpuscular interpretation of light pressure (10.0 points)

Introduction

Electromagnetic waves, reflected from the interfaces between media or absorbed by them, exert mechanical pressure, whose corpuscular interpretation is the subject of this problem. The main postulate of the corpuscular theory of electromagnetic radiation states that electromagnetic radiation, in particular light, is a beam of particles called photons with an energy determined by Planck's formula. In what follows, take the speed of light be equal to $c = 2.98 \cdot 10^8$ m/s.

3.1 Let a parallel beam of light with intensity I_0 fall on a flat surface at an angle φ with the normal, and the coefficient of reflection from the surface is $R = I_r/I_0$, where I_r stands for the intensity of the reflected light. Find the pressure of light p_s exerted on the surface, provided that the reflection coefficient does not depend on the angle of incidence.

3.2 Calculate the pressure p_s of the solar radiation, whose intensity is equal to $I_s = 1400$ W/m^2. Assume that light is incident perpendicular to: (a) the completely absorbing earth's surface; b) the completely reflective (mirror) surface.

3.3 A ball with the radius $R = 1.00$ m is illuminated by a wide parallel beam of sunlight with intensity $I_s = 1400$ W/m^2. One half of the ball is made of a material that completely reflects light, while the other half absorbs it completely. Both halves are symmetrically illuminated by the beam. Calculate the moment of light pressure forces acting on the ball about its axis of symmetry, which is perpendicular to the beam and lies in the plane dividing the ball into the mirror and absorbing halves.

Space station with the mirror sail

The space station rests far from the planets at a distance of $R_0 = 5.00 \cdot 10^7$ km from the Sun, held by a solar sail (fully reflective mirror) oriented perpendicular to the sun's rays. At certain time moment, the station's engines turn on for a short time and give it an initial speed $V_0 = 9.00$ m/s in the direction from the Sun. Completely ignore the influence of the solar wind, which is a beam of ionized particles, mostly protons, helium nuclei and some others. Consider it known that the Earth moves around the Sun in a circular orbit of the radius $r_E = 1.50 \cdot 10^8$ km with the speed of $V_E = 30.0$ km/s.

3.4 Calculate the maximum distance R from the Sun to the station.

3.5 Determine the minimum value of the speed $V_0 = V_{min}$, at which the station will be able to fly away from the Sun

Poynting-Robertson effect

In astrophysics, the physical process is well known, such that in the solar system solid dust particles slowly fall onto the Sun along a spiral trajectory whose shape is very close to circular. Let us consider a similar spherical dust particle of radius $a = 1.00$ mm and density $\rho = 3.00 \cdot 10^3$ kg/m^3, which rotates around the Sun at such a distance from it that light with intensity $I_s = 1400$ W/m^2 falls on the dust particle. Under given conditions, the radial component of the pressure of light on the particle can be neglected in comparison with the gravity of the Sun. Consider that dust the particle completely absorbs the incident radiation.

3.6 Evaluate the characteristic time τ, during which the distance to the Sun R decreases by the relative value $\mu = \frac{\delta R}{R} = 10^{-4}$.

Laser tweezer

In 2018, the Nobel Prize in physics was awarded to A. Ashkin for the creation of "laser tweezer", a device that allows one to hold and move transparent microscopic objects with the help of light. In one of the devices of such a "laser tweezer", a parallel beam of light from a laser passes through a converging lens L and hits a microparticle M, which can also be considered a converging lens. Point F is the common focus of L and M (see figure 3.1 below). The light intensity in the beam is $I = 1.00\ \mu W/sm^2$, the beam radius is $R = 1.00$ cm, the focal length of the lens L is $F = 10.0$ cm. Ignore completely the absorption and reflection of light.

3.7 Calculate the force acting on the microparticle.

To create a force acting on the particle in the transverse direction of the beam, the left half of the lens L is covered by a diaphragm (see figure 3.2 below).

3.8 Calculate the force acting on the microparticle in the transverse direction of the beam.

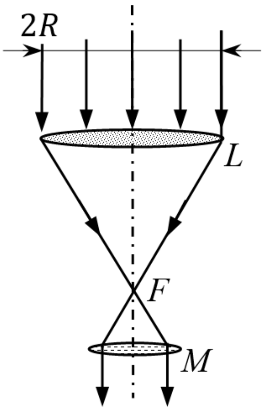


Figure 3.1

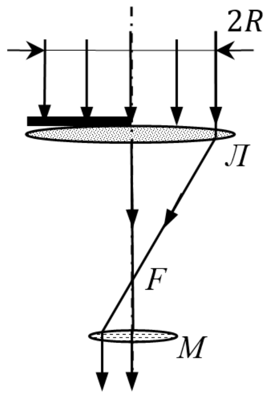


Figure 3.2

Mathematical hints for the theoretical problems

The following formulas may be useful:

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$ is a fixed number, C refers to an arbitrary constant

$\int \frac{dx}{x} = ln\ ln\ |x| + C$, where C stands for an arbitrary constant

$(1+x)^\gamma \approx 1 + \gamma x + \frac{\gamma(\gamma-1)}{2}x^2$, for $|x| \ll 1$ and any value of γ

$ln\ ln\ (1+x) \approx x$, for $|x| \ll 1$

February 17, 2022

Please read this first:

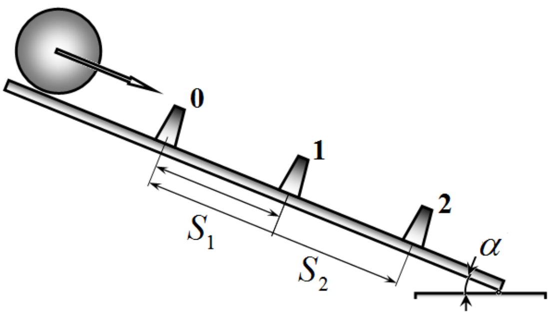
1. The duration of the experimental competition is 4 hours. There are two problems.
2. You can use your own calculator for numerical calculations.
3. You are provided with ***Writing sheet*** and additional white sheets of paper. You can use the additional sheets of paper for drafts of your solutions, but these sheets will not be graded. Your final solutions should be written on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures, and plots.
4. Use only the front side of ***Writing sheets*. ** Write only inside the boxed areas.
5. Start putting down your solution to each problem on a new ***Writing sheet***.
6. Graphs must be drawn on ***Writing sheets*** with the graph paper. All drawings must be done with a pen, not a pencil!
7. Fill in the boxes at the top of each ***Writing sheet*** with your country (**Country),** your student code (**Student Code**), problem number (**Question Number)**, the progressive number of each ***Writing sheet*** **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be graded, put a large X across the entire sheet and do not include it in your numbering.

Dry friction
Experiment 1: Sliding friction

To measure the coefficient of sliding friction, the rolling of a solid homogeneous cylinder on an inclined plane is studied at various angles of inclination of the plane to the horizon. We assume that the force of sliding friction is described by the well-known Coulomb-Amonton law:

$$F = \mu_s N \,, \tag{1}$$

where N stands for the normal reaction force and μ_s denotes the coefficient of sliding friction.



Consider the experimental setup to be a wide flat steel plate, whose angle of inclination to the horizon can be arbitrarily varied. The error in setting the angle of inclination of the plane to the horizon is $\Delta\alpha = 0.2°$.

The solid homogeneous steel cylinder can roll down the plate with its axis remaining horizontal all the time. To measure the acceleration, three optical sensors (0, 1, 2) are fixed on the plate. Each sensor consists of a light source and a photodetector, which are mounted on the identical stands. The solid homogeneous cylinder is placed on the plate and released. The cylinder moves down between these stands and blocks the light beam so that at the moment of light interruption, an electrical impulse is generated that controls an electronic stopwatch (not shown in the figure). When the cylinder passes sensor 0, a stopwatch is started, when the cylinder passes sensors 1 and 2, the times of these passages are recorded.

Thus, the following time intervals are measured in the experiment:

t_1 – cylinder motion time interval from sensor 0 to sensor 1;

t_2 – cylinder motion time interval from sensor 0 to sensor 2.

These time intervals are recorded with 4 significant digits. The instrumental error in measuring time intervals t_1 and t_2 is equal $\Delta t = 2\cdot 10^{-4} s$.

The distances between the sensors are measured as:

from sensor 0 to sensor 1 $S_1 = (50.0 \pm 0.2)sm$;

from sensor 0 to sensor 2 $S_2 = (100.0 \pm 0.2)sm$.

When making numerical calculations, assume that the free fall acceleration is equal $g = 9.81 m/s^2$.

The results of measuring time intervals t_1 and t_2 are given in Table 1 for different angles of inclination of the plate to the horizon.

Table 1. Time intervals of the cylinder motion.

$\alpha°$	t_1, s	t_2, s
20	0,4546	0,7187
25	0,3936	0,6290
30	0,3462	0,5589
35	0,3229	0,5211
40	0,3358	0,5283
45	0,3084	0,4911
50	0,2682	0,4347
55	0,2816	0,4432
60	0,2600	0,4113
65	0,2461	0,3908
70	0,2308	0,3675
75	0,2218	0,3542

Theoretical part

1.1 Derive formulas for the acceleration of the cylinder axis in two cases: A) the motion of the cylinder occurs without slipping – acceleration a_1; B) when moving along the plate, the cylinder slips – acceleration a_2. Express your answers in terms of α, g, μ_s.

1.2 Express the maximum angle $\alpha = \alpha_{cr}$ of inclination of the plate, at which the motion of the cylinder still occurs without slipping, in terms of the coefficient of friction μ_s.

Processing of measurement data

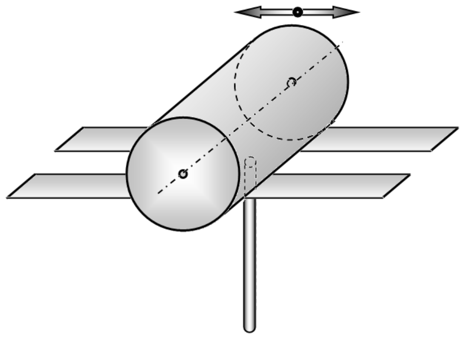
1.3 Using the measurement results given in Table 1, calculate the acceleration with which the cylinder axis moved for each angle of inclination of the plate. Put down the formula for the acceleration, according to which the calculations are carried out. Draw a graph of the acceleration versus the angle and with its aid find an approximate value of the critical α_{cr}.

1.4 Carry out the linearization of the obtained dependence, i.e. find such values $X(\alpha, a)$ and $Y(\alpha, a)$ so that the dependence $Y(X)$ turns linear for the both cases when the cylinder moves without slipping and with slipping. Draw a linearized dependence $Y(X)$ for all experimental points.

1.5 Using the linearized relationship $Y(X)$, calculate the coefficient of friction μ_s between the cylinder and the plate. Estimate the error $\Delta\mu_s$ of the obtained value. Put down the formulas used in your calculations.

1.6 Using the linearized relationship $Y(X)$, calculate the value of the critical angle α_{cr}, at which the cylinder starts to slip. Estimate the error of the obtained value. Put down the formulas used in your calculations.

Experiment 2: Rolling friction



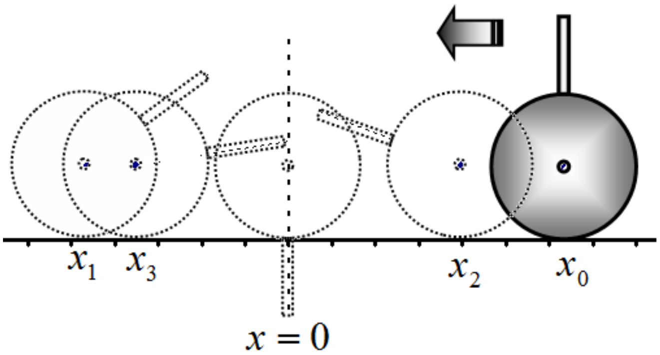
In reality, even in the absence of slipping between the bodies, there are frictional forces called rolling friction forces. In this experiment, the following setup is used to study the rolling friction. A small rod is rigidly attached to the side surface of a massive solid cylinder, whose imaginary prolongation crosses the axis of the cylinder. The cylinder is located on the two horizontal plates so that it can roll over them without slipping. In this case, the rod lies always in the gap between the plates.

Let us use the following notation: the radius of the cylinder R, the mass of the cylinder M, the mass of the attached rod m, which can be considered significantly less than the mass of the cylinder, the distance from the center of mass of the rod to the axis of the cylinder l.

Usually, the formula for the rolling friction force is written as

$$F = \frac{\kappa}{R} N,$$

where N is the normal reaction force, R stands for the radius of the rolling body, and κ refers to the rolling friction coefficient. In this part of the problem, it is necessary to determine the dimensionless value $\mu_r = \frac{\kappa}{R}$, which for brevity is called the coefficient of rolling friction. Typically, the coefficient of rolling friction is much less in magnitude than the coefficient of sliding friction studied above.



During the experiment, the cylinder is placed into its initial position in such a way that the rod is directed vertically upwards, after which the cylinder is released without any significant push. The cylinder starts to roll on the plates, performing damped oscillations due to the action of the rolling friction force. In this case, the coordinates of successive extreme positions of the cylinder (stoppage points) are written down: x_0 – initial position, $x_1, x_2, x_3...$ – coordinates of successive stoppage points. The origin of coordinates $x = 0$ corresponds to the point where the rod is directed vertically downwards. Table 2 shows the experimental values of the coordinates of the stoppage points, whose measurement error is found as $\Delta x = 0,2\,sm$.

Table 2. Coordinates of the stoppage points.

k	x_k, sm
0	15,8
1	-11,6
2	10,1
3	-9,0
4	7,7
5	-7,0
6	5,7
7	-5,3
8	4,7
9	-3,9
10	2,8

To determine the setup parameters, the period of small oscillations of the cylinder near the equilibrium position (point $x = 0$) is measured. For this, the times of five oscillations of the cylinder with the rod are measured several times. The results of these measurements are shown in Table 3. The instrumental error of the time measurement is $\Delta t = 0,02 s$

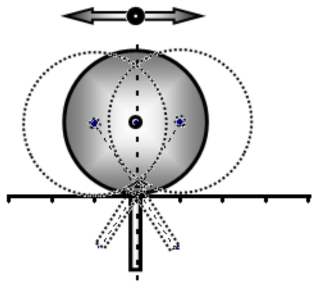


Table 3. The times of five oscillations

k	t_5, s
1	7,39
2	7,21
3	7,26
4	7,47
5	7,44

Theoretical part

2.1 Derive the formula for the period of the small fluctuations described right above. Express the oscillation period in terms of the setup parameters M, m, R, l and the free fall acceleration g.

2.2 Derive an equation relating the coordinates of two successive cylinder stoppage points x_k, x_{k-1} in the described cylinder rolling experiment. This equation, in addition to the coordinates, may include the setup parameters M, m, R, l, the free fall acceleration g and the coefficient of rolling friction μ_r.

2.3 Express the coordinate of the k'th cylinder stoppage point in terms of the initial coordinate x_0 and the coordinates of all previous stoppage points $x_1, x_2, \ldots x_{k-1}$. This equation, in addition to the coordinates of the stoppage points, should include only the free fall acceleration g and the period of small oscillations T.

Processing of measurement data

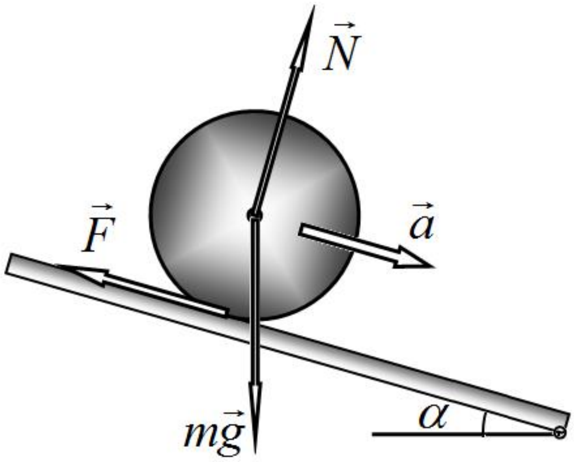
2.4 Using the measurement results given in Table 3, calculate the period T of small oscillations of the cylinder. Estimate the measurement error ΔT of this quantity.

2.5 Propose such values $Y(x_k)$ and $X(x_0, x_1, \ldots x_k)$ such that the dependence $Y(X)$ turns linear and allows you to calculate the coefficient of rolling friction μ_r. Draw a graph of the linearized relationship $Y(X)$.

2.6 Using the linearized relationship $Y(X)$, calculate the coefficient of rolling friction μ_r. Estimate the error $\Delta\mu_r$ of the obtained value.

SOLUTION
Dry friction
Experiment 1: Sliding friction
Theoretical part

1.1 The formulas for the accelerations of the cylinder axis can be obtained in various ways, but they are all based on the use of Newton's second law. When a cylinder rolls down an inclined plate, it is subject to the following forces: $m\vec{g}$ – gravity due to the Earth, $\vec{N}$ – normal reaction force, $\vec{F}$ – friction force.



A) In this case, the friction force does not perform any work, so we can write down the equations of the law for the conservation of mechanical energy as follows

$$\frac{3}{4}mR^2V_c^2 = mgS\sin\alpha\,, \tag{1}$$

which takes into account the energy of rotation of the cylinder around its own axis.

To determine the acceleration, we calculate the time derivative of this equation to obtain

$$\frac{3}{4}mR^2\cdot 2V_C a = mg\sin\alpha\cdot V_C\,. \tag{2}$$

Here $V_C = \dfrac{dS}{dt}$ stands for the speed of the cylinder axis, and $a = \dfrac{dV_C}{dt}$ refers to its acceleration.

It follows from equation (2) that when rolling without slipping, the acceleration of the cylinder axis is described by the formula

$$a_1 = \frac{2}{3}g\sin\alpha\,. \tag{3}$$

An alternative way to derive this formula is to use the equation for the rotational motion of the cylinder.

B) If the cylinder slips during its downward motion, then the friction force is determined by the formula

$$F = \mu_s N = \mu_s mg\cos\alpha\,, \tag{4}$$

then the equation of Newton's second law in projection onto an inclined plane has the form

$$ma = mg\sin\alpha - \mu_s mg\cos\alpha\,, \tag{5}$$

It follows from this equation that the acceleration of the cylinder in this case is equal to

$$a_2 = g(\sin\alpha - \mu_s\cos\alpha), \tag{6}$$

1.2 It is obvious that the motion without slipping occur at angles of inclination of the plate that are less than some critical value α_{cr}, whose magnitude can be found in various ways. For example, it can be found by equating the accelerations described by formulas (3) and (6).

Here is another method for obtaining the critical angle. From equation (2) of Newton's law, we obtain

$$ma = mg\sin\alpha - F\,.$$

Let us express the value of the static friction force, taking into account the found acceleration, as

$$F = mg\sin\alpha - ma = \frac{1}{3}mg\sin\alpha$$

and assume that this force does not exceed the force of sliding friction

$$F < \mu_s N = \mu_s mg\cos\alpha\,.$$

It is derived from the last inequality that

$$\frac{1}{3}mg\sin\alpha < \mu_s mg\cos\alpha \quad \Rightarrow \quad tg\,\alpha < 3\mu_s\,. \tag{7}$$

Thus, the value of the critical angle is determined by the formula

$$\alpha_{cr} = arctg\left(3\mu_s\right). \tag{8}$$

Processing of measurement data

1.3 To calculate accelerations according to the measurement results given in the problem formulation, we write expressions for the distances traveled by the balls

$$\begin{cases} S_1 = V_0 t_1 + \dfrac{at_1^2}{2} \\ S_2 = V_0 t_2 + \dfrac{at_2^2}{2} \end{cases} \Rightarrow \quad \frac{S_2}{t_2} - \frac{S_1}{t_1} = \frac{a}{2}\left(t_2 - t_1\right). \tag{9}$$

The formula for calculating accelerations is then found as:

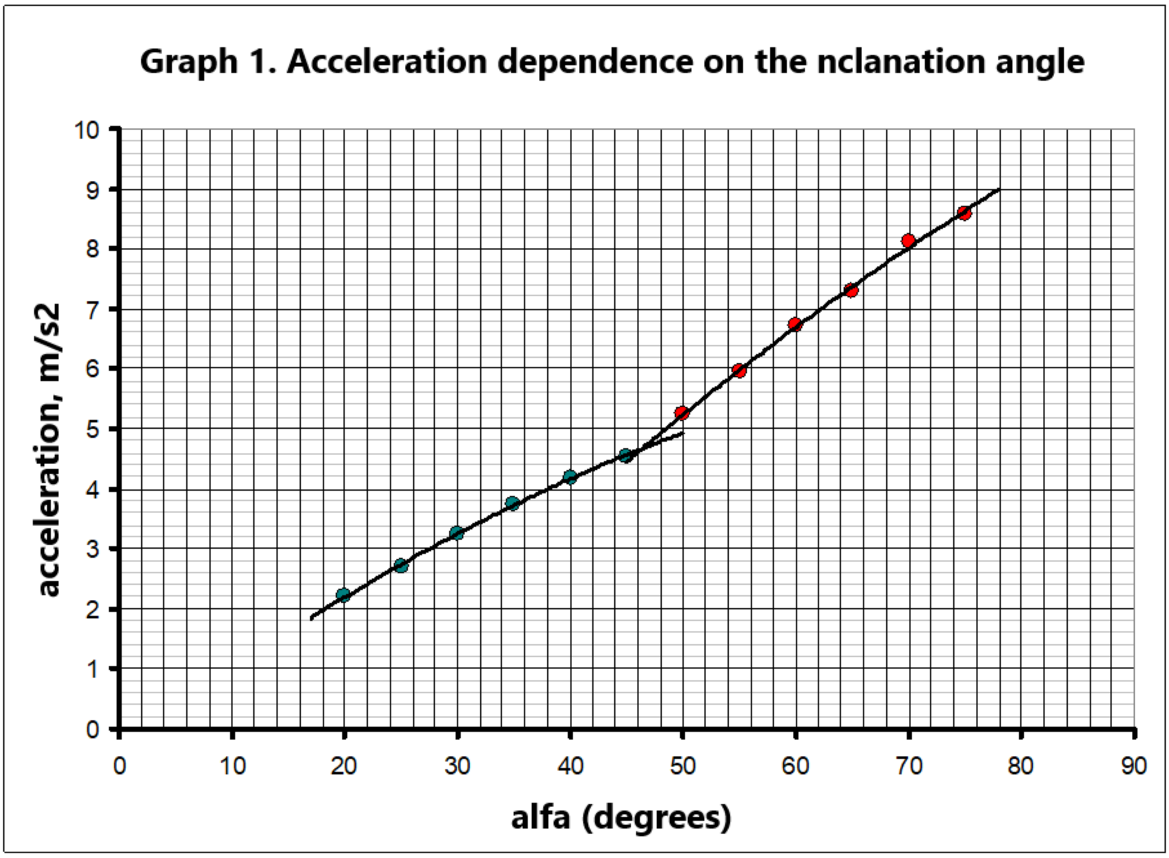
$$a = \frac{2\left(\dfrac{S_2}{t_2} - \dfrac{S_1}{t_1}\right)}{t_2 - t_1}. \tag{10}$$

The results of acceleration calculations according to formula (10) are shown in Table 1.

Table 1. Calculation of accelerations.

$\alpha°$	t_1, s	t_2, s	$a, \frac{m}{s^2}$
20	0.4546	0.7187	2.208
25	0.3936	0.6290	2.715
30	0.3462	0.5589	3.244
35	0.3229	0.5211	3.739
40	0.3358	0.5283	4.196
45	0.3084	0.4911	4.543
50	0.2682	0.4347	5.239
55	0.2816	0.4432	5.950
60	0.2600	0.4113	6.718
65	0.2461	0.3908	7.286
70	0.2308	0.3675	8.116
75	0.2218	0.3542	8.595

Based on these data, Graph 1 of the dependence of acceleration on the angle of inclination of the plate is drawn.



It is clearly seen in the graph that it consists of two different branches: at small angles, the motion occurs without slipping (acceleration is described by formula (3)), whereas at large angles, the cylinder slips, so the acceleration is described by formula (6). The abscissa of the intersection point of these graphs is the critical angle, which is seen to be approximately equal to

$$\alpha_{cr} \approx 46^\circ .$$

Note: In principle, it is not needed to specify the free fall acceleration, which can be determined from the behavior of the acceleration at small angles. However, this is not required in this problem.

The linearization of the obtained dependence can be carried out in two alternative ways, which are approximately equivalent. Let us consider these methods in detail.

Solution 1

1.4 As a variable Y we take the combination

$$Y = \frac{a}{g\cos\alpha}, \tag{11}$$

then it follows from the formulas for accelerations:

$$\begin{cases} a_1 = \dfrac{2}{3} g\sin\alpha \\ a_2 = g(\sin\alpha - \mu_s\cos\alpha) \end{cases} \Rightarrow \begin{cases} \dfrac{a_1}{g\cos\alpha} = \dfrac{2}{3} tg\ \alpha \\ \dfrac{a_2}{g\cos\alpha} = tg\ \alpha - \mu_s \end{cases},$$

i.e. the new variable Y is a linear function of $X = tg\,\alpha$, such that

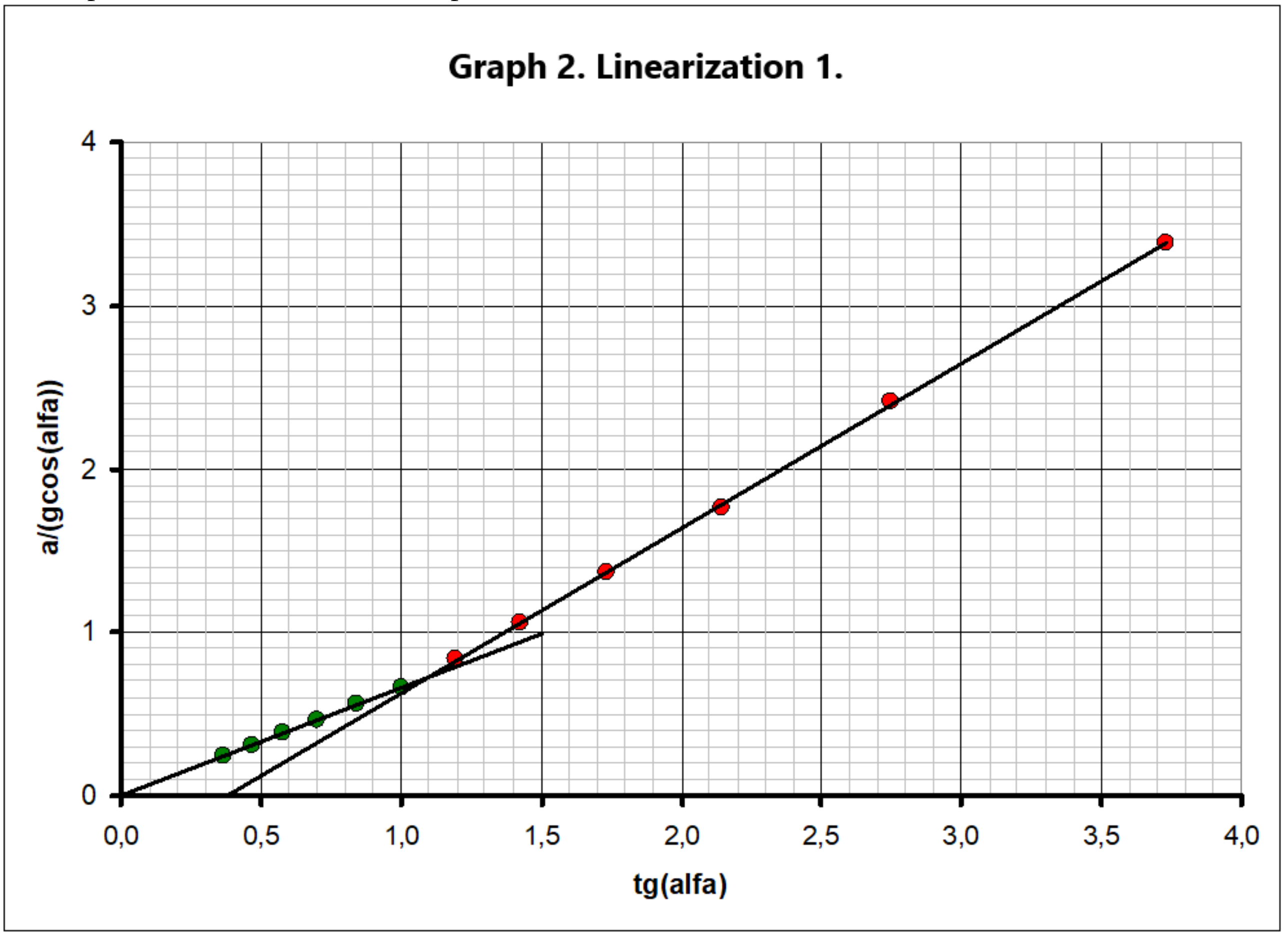
$$\begin{cases} Y = \dfrac{2}{3} X, \quad \alpha < \alpha_{cr} \\ Y = X - \mu_s, \quad \alpha > \alpha_{cr} \end{cases}. \tag{12}$$

Calculations of the chosen variables are provided in Table 2. In the table the highlighted values correspond to the motion without slipping.

Table 2. Linearization 1.

$\alpha°$	$a, \frac{m}{s^2}$	$X = tg\,\alpha$	$Y = \frac{a}{g\cos\alpha}$
20	2.208	0.3640	0.2395
25	2.715	0.4663	0.3053
30	3.244	0.5774	0.3818
35	3.739	0.7002	0.4653
40	4.196	0.8391	0.5584
45	4.543	1.0000	0.6549
50	5.239	1.1918	0.8308
55	5.950	1.4281	1.0574
60	6.718	1.7321	1.3697
65	7.286	2.1445	1.7575
70	8.116	2.7475	2.4188
75	8.595	3.7321	3.3851

Graph 2 shows the linearized dependencies.



1.5 Indeed, both dependences turn out to be linear. We represent these dependencies in the form

$$Y = cX + b. \tag{13}$$

The calculation of the coefficients of these dependencies by the least squares gives the following results:

At the motion

without slipping: $c_1 = 0.659 \pm 0.008$, $b_1 = 0.0004 \pm 0.005$, with slipping: $\begin{array}{l} c_2 = 1.011 \pm 0.009 \\ b_2 = -0.38 \pm 0.02 \end{array}.$

It follows from formula (12) that the coefficient of sliding friction is equal to

$$\mu_s = -b_2 = 0.38 \pm 0.02. \quad (14)$$

Note that the calculated random error (of the order of 10%) significantly exceeds the relative errors of direct measurements of both distances between sensors and motion times. Therefore, the latter can be ignored.

1.6 The value of the critical angle can be found by equating the two linear relations above:

$$\frac{2}{3}X = X - \mu_s \quad \Rightarrow \quad X_{cr} = tg\,\alpha_{cr} = 3\mu_s, \quad (15)$$

or

$$\alpha_{cr} = arctg\left(3\mu_s\right) = 0.855 = 49^\circ$$

The error of this value is equal to

$$\Delta a_{cr} = \frac{3\Delta\mu_S}{\left(3\mu_S\right)^2 + 1} = 0.03,$$

and we finally get

$$\alpha_{cr} = 0.86 \pm 0.03 = 49^\circ \pm 2^\circ. \quad (16)$$

Solution 2

1.4 As a variable Y we take the combination

$$Y = \frac{a}{g\sin\alpha}, \quad (17)$$

then it follows from the formulas for accelerations:

$$\begin{cases} a_1 = \frac{2}{3} g\sin\alpha \\ a_2 = g\left(\sin\alpha - \mu_s\cos\alpha\right) \end{cases} \Rightarrow \begin{cases} \frac{a_1}{g\sin\alpha} = \frac{2}{3} \\ \frac{a_2}{g\sin\alpha} = 1 - \mu_s ctg\,\alpha \end{cases},$$

i.e. the new variable Y is a linear function of $X = ctg\,\alpha$, such that

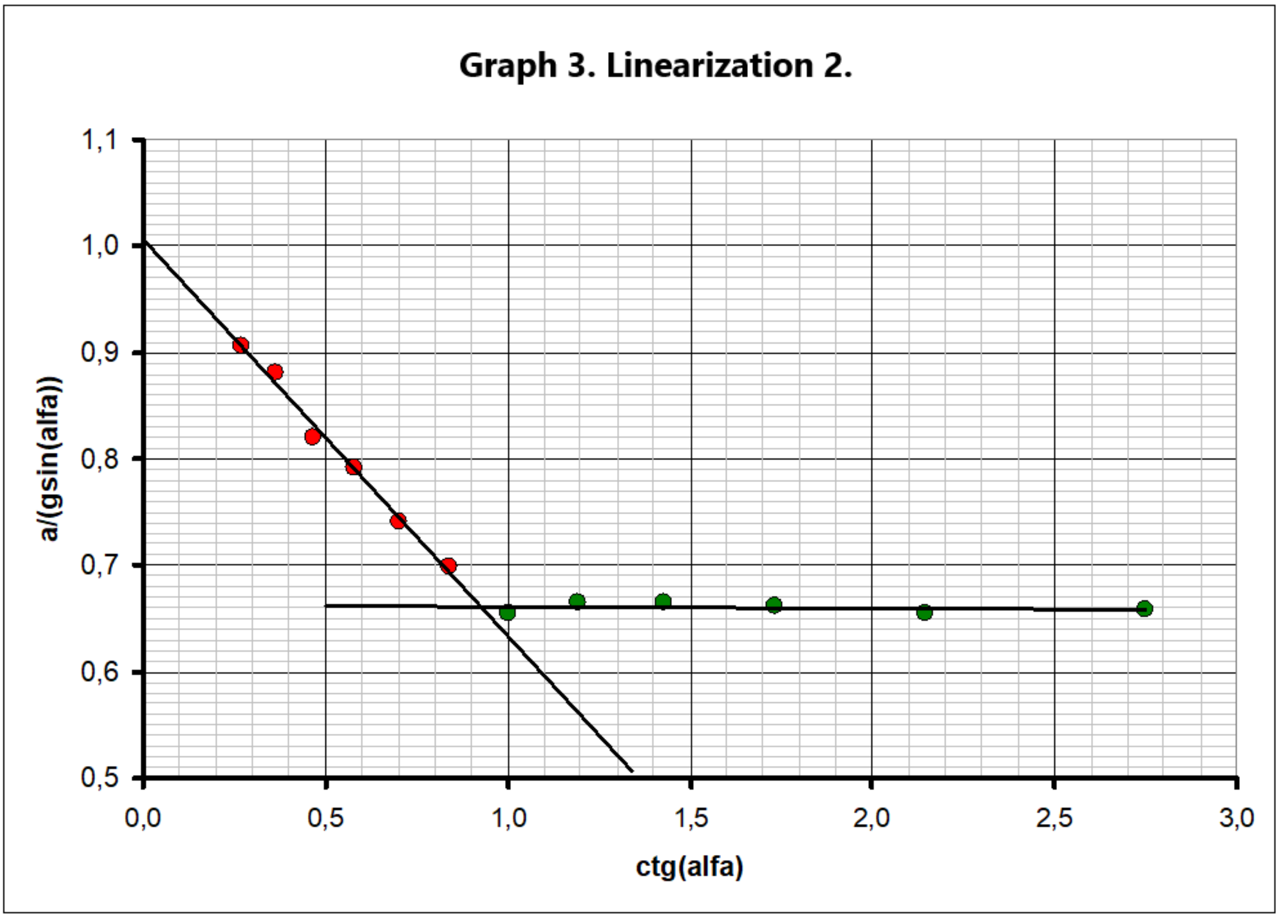
$$\begin{cases} Y = \frac{2}{3}, & \alpha < \alpha_{cr} \\ Y = 1 - \mu_s X, & \alpha > \alpha_{cr} \end{cases}. \quad (18)$$

Calculations of the chosen variables are provided in Table 3. In the table the highlighted values correspond to the motion without slipping.

Table 3. Linearization 2.

$\alpha°$	$a, \frac{m}{s^2}$	$X = ctg\,\alpha$	$Y = \frac{a}{g\sin\alpha}$
20	2.208	2.7475	0.6580
25	2.715	2.1445	0.6548
30	3.244	1.7321	0.6613
35	3.739	1.4281	0.6645
40	4.196	1.1918	0.6655
45	4.543	1.0000	0.6549
50	5.239	0.8391	0.6972
55	5.950	0.7002	0.7404
60	6.718	0.5774	0.7908
65	7.286	0.4663	0.8195
70	8.116	0.3640	0.8804
75	8.595	0.2679	0.9070

Graph 3 shows the linearized dependencies.



1.5 Indeed, both dependences turn out to be linear. We represent these dependencies in the form

$$Y = cX + b. \tag{19}$$

The calculation of the coefficients of these dependencies by the least squares gives the following results:

At the motion

without slipping: $\begin{matrix} c_1 = -0.002 \pm 0.003 \\ b_1 = 0.663 \pm 0.006 \end{matrix}$,

with slipping: $\begin{matrix} c_2 = -0.38 \pm 0.02 \\ b_2 = 1.01 \pm 0.01 \end{matrix}$.

It follows from formula (18) that the coefficient of sliding friction is equal to

$$\mu_s = -c_2 = 0.38 \pm 0.02. \tag{20}$$

In this case, the errors of direct measurements can also be ignored.

1.6 The calculation of the critical angle is carried out similarly and we obtain

$$\alpha_{cr} = arctg(3\mu_s) = 0.84 = 48°$$

The error of this value is equal to

$$\Delta a_{cr} = \frac{3\Delta\mu_S}{(3\mu_S)^2 + 1} = 0.02,$$

and we finally get

$$\alpha_{cr} = 0,84 \pm 0,01 = 48° \pm 1°. \tag{21}$$

Experiment 2: Rolling friction

Theoretical part

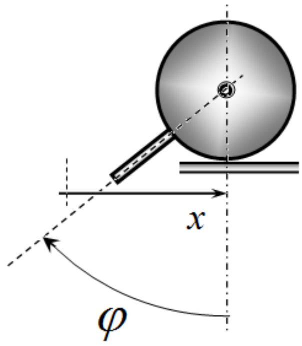
2.1 When the axis of the cylinder is displaced by the distance x, the axis of the rod deviates by the angle

$$\varphi = \frac{x}{R}. \tag{22}$$

To calculate the period of small oscillations, we write the equation for the law of conservation of energy

$$\frac{3}{4}MV^2 + mgl(1-\cos\varphi) = mgl(1-\cos\varphi_0). \tag{23}$$

Hereinafter, we neglect the kinetic energy of the rod motion, since its mass is small.



On the other hand, the potential energy of the massive cylinder remains constant, so the change in the potential energy of the system is the change in the potential energy of the rod. For small oscillations, the approximate formula for the cosine of a small angle $\cos\varphi \approx 1 - \frac{\varphi^2}{2}$ should be used, which leads to a simplification of equation (23) to:

$$\frac{3}{4}MV^2 + mgl\frac{\varphi^2}{2} = mgl\frac{\varphi_0^2}{2}.$$

Using relation (22), we obtain the equation

$$\frac{3}{4}MV^2 + \frac{mgl}{R^2}\frac{x^2}{2} = \frac{mgl}{R^2}\frac{x_0^2}{2}, \tag{24}$$

which is the equation of harmonic oscillations with the period of

$$T = 2\pi\sqrt{\frac{3MR^2}{2mgl}}. \tag{25}$$

2.2 At the stoppage points, the kinetic energy of the cylinder with the rod is zero, so the change in the potential energy when moving from one stoppage point to the next is equal to the work of the friction force. Therefore, the following relation is valid:

$$mgl(1-\cos\varphi_k) = mgl(1-\cos\varphi_{k-1}) - \mu_r Mg|x_k - x_{k-1}|. \tag{26}$$

Applying relation (22) between the angle of rotation and the displacement of the cylinder, we obtain the required relation

$$\left(1-\cos\frac{x_k}{R}\right) = \left(1-\cos\frac{x_{k-1}}{R}\right) - \mu_r\frac{M}{ml}|x_k - x_{k-1}|. \tag{27}$$

2.3 Let us sum up relations, similar to (27), for all intervals of motion from the initial position to the k 'th stoppage point, which finally yields

$$\left(1-\cos\frac{x_k}{R}\right) = \left(1-\cos\frac{x_0}{R}\right) - \mu_r\frac{M}{ml}S_k, \tag{28}$$

where S_k is the path the cylinder passes to the k 'th stoppage point

$$S_k = |x_0 - x_1| + |x_2 - x_1| + \ldots + |x_k - x_{k-1}| = \sum_{j=1}^{k}|x_j - x_{j-1}|. \tag{29}$$

In the initial position, the rod is directed vertically, i.e. $\varphi_0 = \pi$, therefore $x_0 = \pi R$, and one can write down

$$R = \frac{x_0}{\pi}. \tag{30}$$

From the formula for the oscillation period, one can also express:

$$\frac{M}{ml} = \frac{T^2}{4\pi^2}\frac{2g}{3R^2} = \frac{T^2 g}{6\pi^2 R^2} = \frac{T^2 g}{6x_0^2}.$$

We substitute these values into equation (28), which yields

$$1-\cos\left(\pi\frac{x_k}{x_0}\right)=2-\mu_r\frac{T^2g}{6x_0^2}S_k\,. \tag{31}$$

Thus, the value $Y_k=1-\cos\left(\pi\frac{x_k}{x_0}\right)$ linearly depends on the path S_k. The slope of dependence coefficient contains the rolling friction coefficient sought as well as other known values. The unity in expression (31), of course, can be omitted. But the value Y, up to a constant factor, is equal to the potential energy, therefore, in the accepted definition, the dependence $Y(S)$ is more preferable.

Processing of measurement data

2.4 The oscillation period is calculated in the traditional way.

We calculate the average value of the time of 5 oscillations: $\langle t_5\rangle = 7.354\ s$;

We also calculate the random error of this value: $\Delta t_5 = 2\sqrt{\dfrac{\sum_i\left(t_i-\langle t\rangle\right)^2}{n(n-1)}}=0.11\,s$.

Then the oscillation period is obtained as

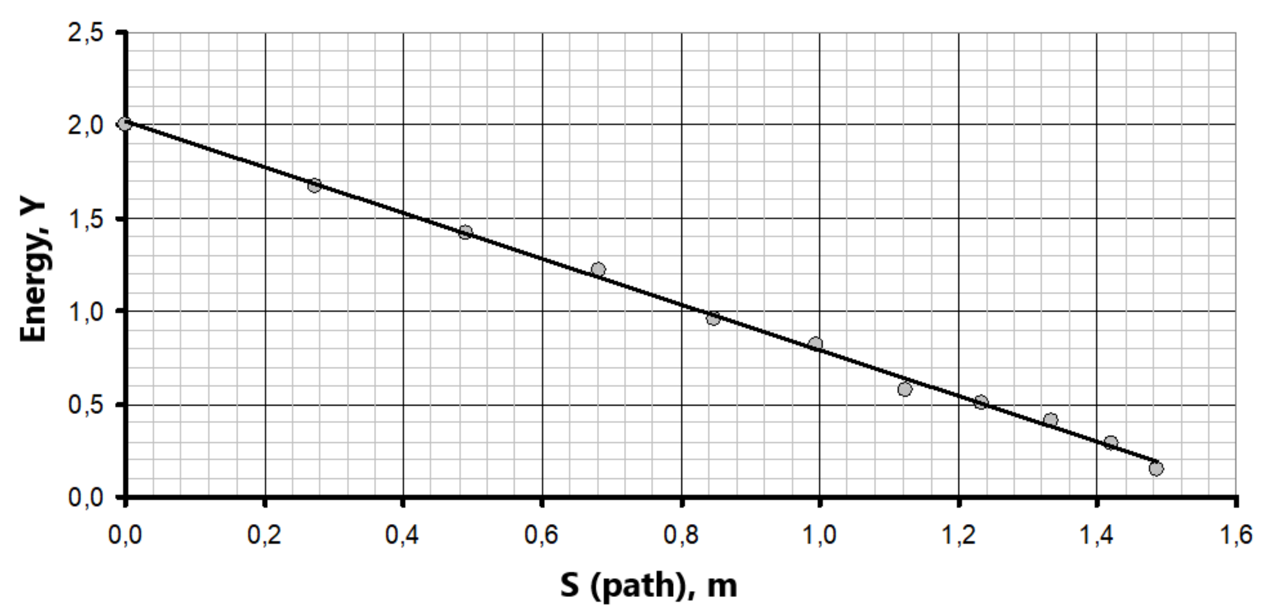
$$T=(1.47\pm 0.02)\ s\,. \tag{32}$$

2.5 The results of calculations of the quantities, appearing in formula (31), are shown in Table 4.

Table 4. Linearization.

k	x_k, cm	$\Delta S_k=\lvert x_k-x_{k-1}\rvert$, m	S_k, m	$Y_k=1-\cos\left(\pi\frac{x_k}{x_0}\right)$
0	15.8		0	2.000
1	-11.6	0.274	0.274	1.671
2	10.1	0.217	0.491	1.424
3	-9.0	0.191	0.682	1.217
4	7.7	0.167	0.849	0.960
5	-7.0	0.147	0.996	0.822
6	5.7	0.127	1.123	0.576
7	-5.3	0.110	1.233	0.506
8	4.7	0.100	1.333	0.406
9	-3.9	0.086	1.419	0.286
10	2.8	0.067	1.486	0.151

The dependence $Y(S)$ is shown in the following figure.



Graph 4. Energy as a function of the path

2.6 The resulting dependence is linear, which confirms the theoretical model used. The slope coefficient of this graph, calculated by the least squares, is equal to

$$C = (1.23 \pm 0.02)\, m^{-1}.$$

The theoretical formula for this coefficient makes it possible to express the value of the rolling friction coefficient:

$$C = \mu_r \frac{T^2 g}{6x_0^2} \quad \Rightarrow \quad \mu_r = C\frac{6x_0^2}{T^2 g} = 8.69\cdot 10^{-3}. \tag{33}$$

To calculate the error of this value, we make use of the formula for the error of indirect measurements:

$$\Delta\mu_r = \mu_r\sqrt{\left(\frac{\Delta C}{\langle C\rangle}\right)^2 + \left(2\frac{\Delta T}{\langle T\rangle}\right)^2 + \left(2\frac{\Delta x_0}{\langle x_0\rangle}\right)^2} = \\ = 8.69\cdot 10^{-3}\sqrt{\left(\frac{0.02}{1.23}\right)^2 + \left(2\frac{0.02}{1.47}\right)^2 + \left(2\frac{0.2}{15.8}\right)^2} = 3.8\cdot 10^{-4}. \tag{34}$$

The final result is written as

$$\mu_r = (8.7 \pm 0.4)\cdot 10^{-3}. \tag{35}$$

Part	Content	Points	Total
Experiment 1: Sliding friction			**10.0**
Theoretical part			**1.5**
1.1	Motion without slipping:		**1.0**
	equation of motion;	0.2	
	formula (3);	0.3	
	Motion with slipping:		
	equation of motion;	0.2	
	formula (6);	0.3	
1.2	Critical angle value:		**0.5**
	the condition for the start of slippinge: the boundary value of the static friction force; equality of accelerations (3) and (6);	0.2	

	relation between the critical angle and the friction coefficient: formula (7) or (8);	0.3	
Processing of measurement data			**8.5**
1.3	Calculation of accelerations:		**2.5**
	formula (10) for acceleration;	0.2	
	all accelerations are calculated (graded if the formula for calculating acceleration is graded); 0.1 for each point, acceptable error of acceleration calculation is ± 0.01;	1.2	
	Plotting a graph (graded if the results of acceleration calculations are graded):		
	the axes are named and ticked;	0.1	
	all points are drawn;	0.1	
	two smoothing curved lines are drawn;	0.2	
	two dependences are obtained: two different intersecting curves are visible in the graph;	0.3	
	Critical angle value;		
	determination method: the point of intersection of two curves;	0.2	
	numerical value: acceptable error $\pm 1°$;	0.2	
1.4	Linearization of the dependence:		**2.8**
	New variables: any reasonable: $\left(\frac{a}{\sin\alpha}, ctg\alpha\right), \left(\frac{a}{\cos\alpha}, tg\alpha\right)$, and equivalent, leading to linear dependencies;	1.0	
	Numerical calculations: graded if the choice of new variables X, Y is graded; 0.1 for each correctly calculated point, acceptable error ± 0.02;	1.2	
	Plotting a graph (graded if the results of the calculations of the variables X, Y are graded):		
	the axes are named and ticked;	0.1	
	all points are drawn;	0.1	
	two straight lines are visible;	0.2	
	two straight lines are drawn;	0.2	
1.5	Calculation of the coefficient of sliding friction:		**1.9**
	coefficients of the linear dependences are calculated: 0.2 for each coefficient (graphically for all points, LSM); 0.1 – by two points; the use of theoretical values is not graded;	0.8	
	formula for the coefficient of friction through the coefficients of linear dependencies;	0.2	
	numerical value of the coefficient of sliding friction: graded if the formula for the coefficient of friction is graded, the acceptable error ± 0.05;	0.4	
	formula for the error: only through the coefficients of the linear dependencies;	0.2	
	numerical value of the error: graded if the formula is graded;	0.2	
	correct rounding;	0.1	
1.6	Critical angle value:		**1.3**
	calculation method: analytical calculation of the intersection point of two lines, the use of numerical theoretical values of the coefficients is acceptable;	0.5	

	numerical value of the critical angle: graded if the method of determination is graded;	0.2	
	formula for the error: through the errors of the coefficients of the straight lines;	0.3	
	numerical value of the error: graded if the formula is graded;	0.2	
	correct rounding;	0.1	
Experiment 2: Rolling friction			**10.0**
Theoretical part			**2.5**
2.1	Derivation of the formula for the period of small oscillations:		**0.5**
	initial equation: either dynamic or energetic;	0.2	
	approximation of small oscillations;	0.1	
	formula (25) for the period: incorrect numerical factor (-0.1);	0.2	
2.2	Recursive relation:		**1.0**
	use of the law of conservation of energy;	0.3	
	expression for the potential energy in terms of deflection angle;	0.2	
	relation between the angle and coordinate;	0.1	
	final form of the recursive relation: formula (27) or equivalent;	0.4	
2.3	Expression for the stoppage point coordinates:		**1.0**
	expression for the energy change through the path traveled;	0.2	
	determination of the disk radius through the initial coordinate;	0.2	
	use of the formula for the period of small oscillations;	0.2	
	final relation: formula (31) or equivalent;	0.4	
Processing of measurement data			**7.5**
2.4	Calculation of the period:		**1.0**
	Formula for the period;	0.2	
	numerical value of the period, acceptable error $\pm 0{,}01$;	0.2	
	formula for the error;	0.2	
	numerical value of the error;	0.2	
	the dimension is indicated;	0.1	
	correct rounding;	0.1	
2.5	Linearization:		**3.2**
	choice of variables (equivalent variables are acceptable if expressed in terms of the measured known values);	1.6	
	calculated for all points (graded if the choice of variables is graded); 0.1 for each point, acceptable error ± 0.02;	1.1	
	Plotting a graph (graded if the calculation of points is graded):		
	the axes are named and ticked;	0.1	
	all points are drawn;	0.2	
	smoothing straight line is drawn;	0.2	
2.6	Calculation of rolling friction coefficient:		**3.3**
	Calculation method of the friction coefficient: according to the slope coefficient – 0.6; by 1-2 points – 0.3;	0.6	
	slope coefficient found: according to LSM – 0.5; graphically – 0.4; by two points – 0.2; acceptable error $\pm 0{,}05$;	0.5	
	error calculation method;	0.2	
	numerical value of the error;	0.2	
	formula for calculating the rolling friction (only through the slope	0.3	

	factor);		
	numerical value of the rolling friction coefficient (graded if the formula is graded); the error propagation rule does not apply to numerical values!	0.5	
	formula for the error, the errors of all quantities included in the formula are taken into account;	0.6	
	numerical value of the error;	0.2	
	correct rounding;	0.2	
	TOTAL		**20.0**

THEORETICAL COMPETITION

February 2, 2023

Please read this first:

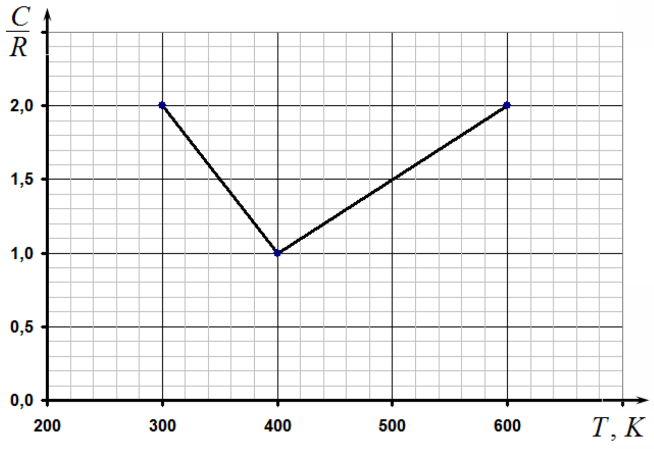
1. The duration of the theoretical competition is 4 hours. There are three problems.
2. You can use your own calculator for numerical calculations.
3. You are provided with ***Writing sheet*** and additional white sheets of paper. You can use the additional sheets of paper for drafts of your solutions, but these sheets will not be graded. Your final solutions should be written on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures, and plots.
4. Use only the front side of ***Writing sheets*.** Write only inside the boxed areas.
5. Start putting down your solution to each problem on a new ***Writing sheet***.
6. Fill in the boxes at the top of each ***Writing sheet*** with your country (**Country),** your student code (**Student Code**), problem number (**Question Number)**, the progressive number of each ***Writing sheet*** **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be graded, put a large X across the entire sheet and do not include it in your numbering.

Problem 1 (10.0 points)

This problem consists of three independent parts.

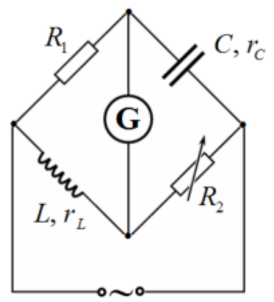
Problem 1.1 (4.0 points)

One mole of an ideal monatomic gas, located in the cylinder under the piston, is quasi-statically heated from the temperature $T_1 = 300\text{ K}$ to $T_2 = 600\text{ K}$, changing its volume in such a way that the dependence of the heat capacity C of the gas on its temperature has the form shown in the figure below. Find the work done on the gas from the state in which the graph of its volume versus temperature reaches a local maximum, to the state in which the same graph reaches a local minimum. Express your answer in joules assuming that $R = 8.31$ J/K is the universal gas constant.



Problem 1.2 (3.0 points)

The figure on the right shows an AC bridge circuit. The resistance $R_1 = 2.5\text{ k}\Omega$, the inductance $L = 1\text{ H}$, the resistance of the inductance $r_L = 1\ \Omega$ are all known. At the frequency of the alternating sinusoidal voltage $\nu = 100\text{ Hz}$, the balance of the bridge occurs at $R_2 = 800\ \Omega$. It turnes out that when the frequency of the alternating current is doubled, the balance of the bridge is not violated. Find the leakage resistance r_C of the capacitor and its capacitance C.



Problem 1.3 (4.0 points)

The Nobel Prize in Physics for 2019 was awarded to the Swiss astronomers M. Major and D. Quelo for the discovery of non-luminous satellite planets (exoplanets) around stars. Consider the planet Jupiter in our solar system as an exoplanet for the Sun. The orbital period of Jupiter is $T_J = 11.9$ year at a mass of $M_J = 1.90 \cdot 10^{27}$ kg, while the mass of the Sun is equal to $M_S = 1.99 \cdot 10^{30}$ kg. Consider also known the speed of the Earth on its orbit $v_E = 29.8$ km/s, and assume that the orbits of all planets are circular. Let an observation be carried out by a spectrometer located far away from outside the solar system so that the observer is in the plane of motion of the Sun–Jupiter system. Find the minimum resolution of the spectrometer $R_{\min}$, which allows one to positively detect the presence of the massive exoplanet Jupiter near the Sun. Consider the speed of light to be $c = 2.99 \cdot 10^8$ m/s.

Note: The resolution of a spectrometer is its ability to distinguish between two closely spaced spectral lines, which is characterized by a dimensionless parameter $R = \lambda / \Delta\lambda$, where $\Delta\lambda$ denotes the smallest difference between the wavelengths of two lines still recorded as separate by the instrument, and λ stands for the average wavelength of those two resolvable lines.

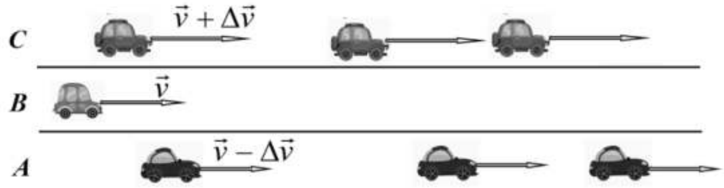
Problem 2. Fermi acceleration (10.0 points)

Cosmic rays contain extremely high energy particles. A possible mechanism for the appearance of such particles is called Fermi acceleration.

Fermi acceleration is a stochastical mechanism for the acceleration that charged particles experience when they are repeatedly reflected, usually by magnetic mirrors. In this problem, we consider the main ideas underlying this paradoxical, at first glance, phenomenon.

Why are there more oncoming cars than overtaking cars?

On a motorway with three lanes ***A, B, C,*** cars move at a constant speed in one direction: along the central lane ***B***, a car moves at a speed of $v = 90$ km/h; cars move along lane ***A*** at speeds $v - \Delta v = 80$ km/h; cars move along lane ***C*** at speeds $v + \Delta v == 100$ km/h. The distance between cars is the same on each of lanes ***A*** and C, the number of cars per unit of lane length for each of them is $n = 5.0\,\text{km}^{-1}$.

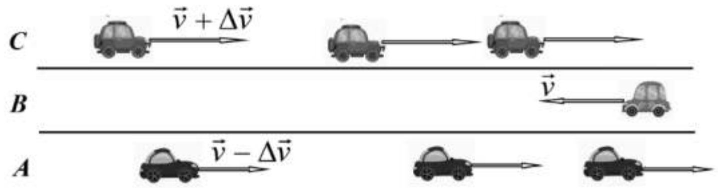


Consider the car moving in lane ***B***.

2.1 Calculate how many cars N_1 in lane ***A*** overtake the car ***B*** within the time period of $t = 1.0$ min, as well as the time τ_1 between two consecutive overtakes.

2.2 Calculate how many cars N_2 in lane ***C*** overtake the car in lane ***B*** within the time period of $t = 1.0$ min, as well as the time τ_2 between two consecutive overtakes.

Now let the car move in lane ***B*** towards the cars moving in lanes ***A*** and ***C***. The speeds of all cars and their density on each lane remain the same.

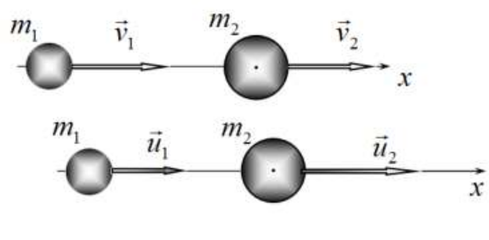


2.3 Calculate the number of cars N_3 in lane ***A*** and N_4 in lane ***C*** that the car ***B*** encounters within the time period of $t = 1.0$ min, as well as the corresponding times τ_3 and τ_4 between two consecutive encounters.

Elastic collision

In this part, we consider the classical problem of elastic collision of two bodies. The main purpose of this consideration is to determine the conditions under which the kinetic energy of one of the selected bodies increases as a result of the collision.

Two elastic balls, whose masses are equal to m_1 and m_2, respectively, move along the axis x. The speed of the first ball before the collision is v_1, whereas the speed of the second is v_2. Let us denote the speeds of the balls after an absolutely elastic central collision as u_1 and u_2, respectively. The speeds of the balls should be understood as the projections of their velocities on the x axis, therefore, they can be both positive and/or negative.



2.4 Express the velocities of the balls u_1 and u_2 after the collision in terms of their velocities before the collision v_1 and v_2, as well as their masses.

Let us denote the ratio of the ball masses as $\mu = \frac{m_2}{m_1}$, the ratio of the speeds of the first ball after and before the collision as $\eta_1 = \frac{u_1}{v_1}$ and the ratio of the velocities of the balls before the collision as $\eta_2 = \frac{v_2}{v_1}$. For definiteness, consider that $v_1 > 0$.

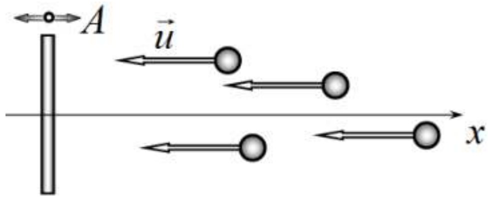
2.5 Draw a set of graphs that represents the dependence of the parameter η_1 on the parameter η_2 for all characteristic values of the ball mass ratios μ.

2.6 Find the relation between the parameters η_2 and μ, at which the first ball increases its energy as a result of the collision.

2.7 Consider the case of a light ball colliding with a heavy one such that $m_2 >> m_1$. In this limiting case, find the speed of the first ball $\tilde{u}_1$ after the collision and determine the range of velocities η_2 of the heavy ball before the collision, at which the energy of the light ball increases as a result of the collision.

The simplest Fermi acceleration model

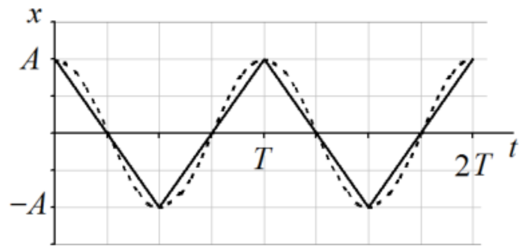
A massive plate, located perpendicular to the x axis, performs harmonic oscillations in the direction of the x axis. The oscillation amplitude is A with the period being T. Light balls move with equal speeds u in the direction of the plate along the x axis such that the times of ball arrivals to the still plate are randomly and uniformly distributed.



2.8 Express the maximum speed of the plate V_0 in terms of the amplitude and period of its oscillations.

2.9 Calculate the fraction φ of incident balls that are to increase their kinetic energy after the collision. Express your answer in terms of u and V_0. For a numerical estimate, consider the following two cases separately: a) $u = 1.5V_0$; b) $u = 0.50V_0$.

Let us approximate the harmonic law of the plate motion by a piecewise linear function, see the figure below, i.e. assume that the modulus of the plate speed remains constant at the same values of the amplitude and period of oscillations.



2.10 Express the value of the plate speed modulus V in terms of the amplitude and period of its oscillations.

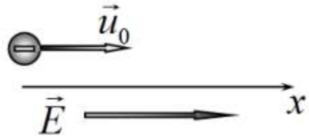
2.11 Calculate how many times $\varepsilon = \frac{E}{E_0}$ the average energy of the incident balls changes, where E_0 is the kinetic energy of the balls before the collision, and E symbolizes the average energy of the balls after the collision with the oscillating plate. For a numerical estimate, consider the following two cases separately: a) $u = 1.5V$; b) $u = 0.50V$.

Problem 3. Magnetron (10.0 points)

Both electric and magnetic fields act on moving charged particles, which in a certain way change the character of their motion. Everywhere below it is assumed that charged particles move in a vacuum, and the radiation of electromagnetic waves can be neglected.

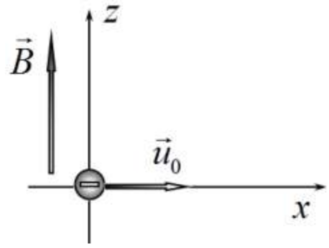
In this problem, the motion of electrons, which are classical point-like particles, is considered. When carrying out numerical calculations, consider, where necessary, that an electron has a negative electric charge, whose modulus is equal to $e = 1.60\cdot10^{-19}$ Cl, and its mass is $m = 9.11\cdot10^{-31}$ kg. The electric and magnetic constants are equal $\varepsilon_0 = 8.85\cdot10^{-12}$ F/m and $\mu_0 = 1.26\cdot10^{-6}$ H/m, respectively, the Boltzmann constant is denoted as $k_B = 1.38\cdot10^{-23}$ J/K.

Electron motion in electric and magnetic fields



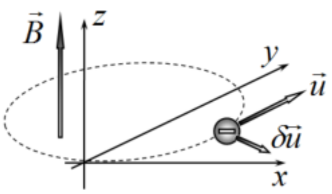
At the initial moment of time $t = 0$ an electron is at rest at the origin of coordinates, and a uniform electric field with the strength E is applied along the positive direction of the x axis. The electron is given an initial velocity u_0, also directed in the positive direction of the x axis.

3.1 Determine the maximum value of the electron coordinate x_{max} for the entire time of its motion.

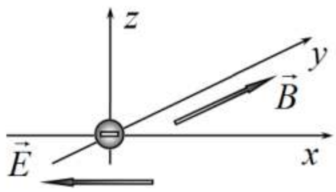


Now, at the initial moment of time $t = 0$, an electron is at rest at the origin of coordinates, and a uniform magnetic field with the induction B is applied along the positive direction of the z axis. The electron is given an initial velocity u_0 directed in the positive direction of the x axis.

3.2 Determine the maximum value of the electron coordinate x_{max} for the entire time of its motion.

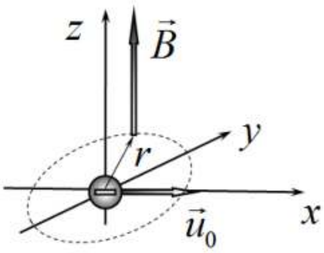


3.3 In the process of moving along the trajectory, at some moment of time, an additional velocity δu is given to the electron in the direction perpendicular to its current velocity u and lying in the plane of its initial trajectory, such that $\delta u \quad u$. Determine the period of the arisen two-dimensional oscillations of the electron with respect to its initial unperturbed trajectory.



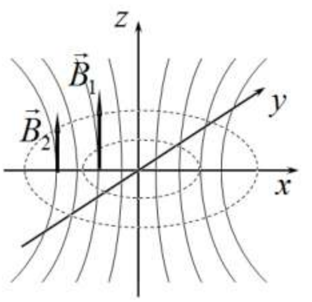
Again, at the initial moment of time $t = 0$ an electron is at rest at the origin of coordinates, and a uniform electric field with the strength E is applied along the negative direction of the x axis. In addition, a uniform magnetic field with induction B is created along the positive direction of the y axis. The electron is released with the zero initial velocity.

3.4 Determine the maximum value of the electron coordinate x_{max} for the entire time of its motion.



At the initial moment of time $t = 0$ an electron is at rest at the origin of coordinates, and a magnetic field is applied along the positive direction of the z axis, such that the inductionшт in the plane xy depends on the distance r to the z axis according to the law $B = \alpha r = \alpha(x^2 + y^2)^{1/2}$. The electron is given an initial velocity directed in the positive direction of the x axis.

3.5 Determine the maximum distance r_{max} from the electron to the z axis for the entire time of its motion.



In some region of space, an axisymmetric magnetic field is created with respect to the z axis. At the moment of time $t = 0$, the magnetic field is absent, and then it begins to slowly increase, such that at all points in the xy plane the field induction vector is directed along the z axis. After a certain period of time, the induction of the magnetic field reaches its final value and ceases to change. The final field distribution in the xy plane has the form

$$B(r)=\begin{cases} B_1, & 0<r\le r_1 \\ B_2, & r_1<r\le r_2 \\ 0, & r>r_2 \end{cases},$$

where r denotes the distance to the z axis, r_1 and r_2 are known quantities.

At a moment in time $t=0$, the electron is at rest at some point lying in the xy plane. When the field is turned on, the electron begins to move along a circle in the xy plane, whose center lies on the z axis.

3.6 Determine under what conditions for the ratio B_1/B_2 the described situation turns possible.

Cylindrical magnetron

Magnetron is an electronic electrovacuum device, in which the amount of flowing current is controlled by electric and magnetic fields. Let us consider the simplest magnetron, consisting of a conducting coaxial long cylindrical cathode and anode with radii $a=3.00\cdot10^{-1}\,\text{mm}$ and $b=6.00\text{ mm}$, respectively, located inside a vacuum tube. The lamp is placed in the center of a cylindrical solenoid, whose axis coincides with the axes of the cathode and anode, and the number of turns is $N=2590$ with the length being equal $L=210\text{ mm}$ and the diameter of $D=105\text{ mm}$. The potential difference between the cathode and the anode is constant and equal to $V_0=75.0\text{ V}$. Consider that the electrons leaving the cathode are formed due to thermionic emission and have a zero initial velocity, neglect the space charge of the electron cloud in the lamp.

3.7 Calculate the potential difference V between the cathode and a point in space located at a distance $r=3.00$ мм from the common axis of the solenoid and the lamp.

3.8 Calculate the smallest solenoid current $I_{\min}$ at which the current in the magnetron between the cathode and anode vanishes completely.

3.9 Find the condition for the cathode temperature T, under which the initial electron velocity can indeed be considered zero.

Mathematical hints for the theoretical problems

The following formulas may be useful:

$\int x^n dx=\dfrac{x^{n+1}}{n+1}+C$, where $n\neq-1$ is a fixed number, C refers to an arbitrary constant;

$\int \dfrac{dx}{x}=\ln|x|+C$, where C stands for an arbitrary constant;

$\int \dfrac{dx}{(x^2+a^2)^{3/2}}=\dfrac{x}{a^2(x^2+a^2)^{3/2}}+C$, where a, C stand for arbitrary constants;

$(1+x)^\gamma\approx1+\gamma x+\dfrac{\gamma(\gamma-1)}{2}x^2$, for $|x|\ll 1$ and arbitrary γ;

$\ln(1+x)\approx x$, for $|x|\ll 1$.

ТЕОРИЯЛЫҚ ТУР

2 ақпан, 2023 жыл

Алдымен мынаны оқып шығыңыз:

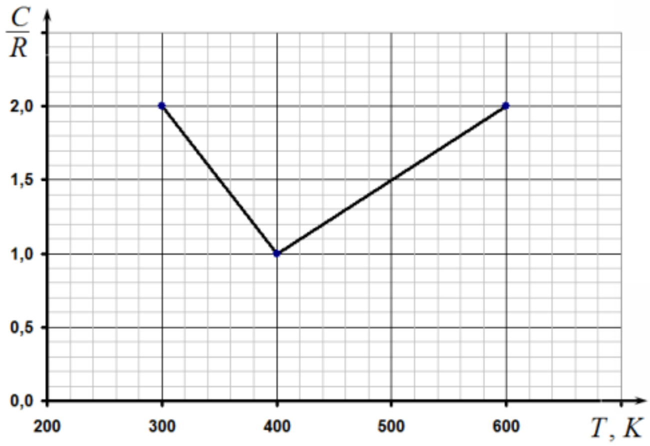
1. Теориялық тур үш тапсырмадан тұрады. Турдың ұзақтығы 4 сағат.
2. Есептеу үшін өзіңіздің калькуляторыңызды пайдалануға болады.
3. Сізге бос қағаз және ***Жазу парақтары*** (***Writing sheets***) беріледі. Бос қағаз парақтарын өз қалауыңызша пайдалануға болады, олар тексерілмейді. Жазу парақтарында жұмысты тексеру кезінде бағаланатын есептердің шешімдері жазылуы тиіс. Шешімдеріңізде ауызша сипаттамаларды мүмкіндігінше аз пайдаланыңыз. Шешіміңізді түсіндіру үшін негізінен өрнектерді, сандарды, әріптерді, суреттерді және графиктерді пайдаланғаныңыз жөн.
4. Жазу парақтарының (***Writing sheets)*** тек алдыңғы бетін пайдаланыңыз, артына жазуға болмайды. Жазу кезінде белгіленген рамкадан шықпаңыз.
5. Әрбір жаңа есептің шешімін жаңа парақтан (***Writing sheets)*** бастау керек.
6. Әрбір пайдаланылған жазу парақтарында, бұл үшін берілген бағандарда сіз еліңізді **(*Country*)**, кодыңызды **(*Student Code*)**, есептің реттік нөмірін **(*Question Number*)**, әрбір парақтың ағымдағы нөмірін **(*Page Number*)** және барлық есептерді шешуде пайдаланылған парақтардың жалпы санын ***(Total Number of pages)*** көрсетуіңіз керек. Жауабыңызға кейбір қолданылған Жазу парақтарын (***Writing sheets)*** қосуды қаламасаңыз, оларды бүкіл парақтың үстінен үлкен крестпен сызып тастаңыз және оларды парақтардың жалпы санына қоспаңыз

1-есеп (10,0 ұпай)

Бұл тапсырма бір-бірімен байланысы жоқ үш бөліктен тұрады.

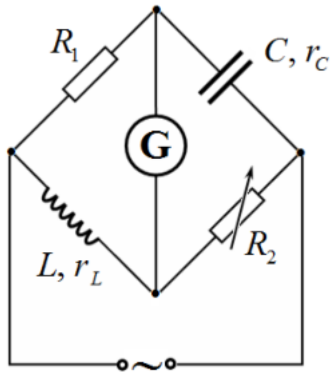
1.1 есеп (3,0 ұпай)

Поршень астындағы цилиндрде орналасқан идеал бір атомдық газдың бір молін T_1=300 K температурасынан T_2=600 K -ге дейін квазистатикалық түрде қыздырады және бұл кезде оның көлемін бұл газдың жылу сыйымдылығының температураға тәуелділігі төмендегі суретте көрсетілгендей етіп өзгертеді. Осы жағдайдағы газ көлемінің температураға тәуелділік графигінің максимумға жеткен күйінен сол графикдің минимумға жеткен күйіне дейінгі аралықта газға жасалған жұмысты табыңыз. Жауабыңызды джоульмен көрсетіңіз, әмбебап газ тұрақтысын R=8.31 Дж/К.



Есеп 1.2 (3.0 ұпай)

Суретте айнымалы ток көпірі тізбегі көрсетілген. Мұндағы кедергі $R_1 = 2 Ом$, индуктивтілік $L = Гн$, индуктив катушкасының кедергісі $r_L = Ом$. Айнымалы синусоидалық кернеудің $\nu = 100$ жиілігінде көпірдегі тепе-теңдік кедергі $R_2 = 800$ болғанда жүзеге асады. Айнымалы токтың жиілігін екі есе арттырғанда көпірдегі тепе теңдік бұзылмайды екен. Конденсатордың *C* сыйымдылығын және конденсатордан r_C ағу кедергісін анықтаңыз.



Есеп 1.3 (4.0 ұпай)

2019 жылғы физика бойынша Нобель сыйлығы швейцариялық астрономдар М.Майджор мен Д.Квелоға жұлдыздардың айналасында жарықырамайтын серіктер (экзопланеталар) бар екенін ашқаны үшін берілді. Осы тұрғыдан күн жүйесіндегі Юпитер планетасын Күннің экзопланетасы ретінде қарастырайық. Юпитердің айналу периоды $T_ю$ = 11.9 жыл, массасы $M_J = 1.90 \cdot 10^{27}$, ал Күннің массасы $M_S = 1.99 \cdot 10^{30}$. Жердің өз орбитасымен қозғалу жылдамдығы $v_E = 29.8$ екенін белгілі деп есептеңіз. Планеталардың орбиталары шеңбер болсын. Бақылаушы Күн-Юпитер жүйесінің қозғалыс жазықтығында болсын және бақылауды күн жүйесінен тыс жерде орналасқан спектрометр арқылы жүзеге асырсын. Күннің жанында массивті экзопланета Юпитердің бар екендігін анықтауға мүмкіндік беретін спектрометрдің минималды ажыратқыштық қабілеті R_{min} -ді табыңыз. Жарық жылдамдығы $c = 2.99 \cdot 10^8$.

Ескерту: Спектрометрдің ажыратқыштық қабілеті деп оның жақын орналасқан екі спектр сызығын ажырата алу мүмкіндігін сипаттайтын өлшем бірліксіз мынадай параметрді айтады

$R = \lambda / \Delta\lambda$, мұндағы $\Delta\lambda$ – спектрометр жеке сызықтар деп тіркей алатын екі спектр сызығының толқын ұзындығының айырымы, λ – ажыратылатын екі сызықтың орташа толқын ұзындығы.

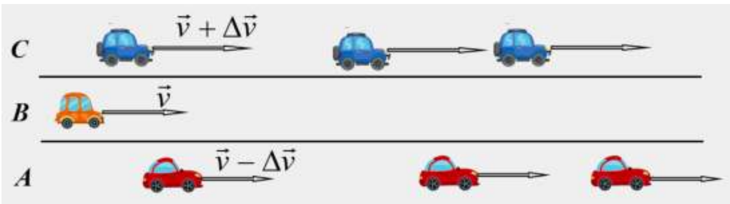
Есеп 2. Ферми үдеуі (10.0 ұпай)

Ғарыштық сәулелердің құрамында өте жоғары энергиялы бөлшектер бар. Мұндай бөлшектердің пайда болуының мүмкін механизмі Ферми үдеуі деп аталады.

Ферми үдеуі – зарядталған бөлшектер әдетте магниттік айнадан қайта-қайта шағылысқан кезде пайда болатын стохастикалық үдеу механизмі. Бұл есепте біз бір қарағанда қарама-қайшылықты болып көрінетін осы құбылыстың негізінде жатқан идеяларды қарастырамыз.

Неліктен бағыттас көліктерге қарағанда қарсы келе жатқан көліктер көп?

A, B, C деп белгіленген үш жолы бар автомагистральмен бір бағытта тұрақты жылдамдықтармен автомобильдер қозғалыпп келеді: ортадағы ***B*** жолымен келе жатқан автомобильдердің жылдамдығы $v = 90$ км/сағ, ***A*** жолымен жүріп келе жатқан автомобильдердің жылдамдығы $v - \Delta v = 90$ км/сағ, ал ***C*** жолымен келе жатқан автомобильдердікі $v + \Delta v = 100$ км/сағ. ***A*** және ***C*** жолдарының әр қайсысымен келе жатқан автомобильдердің өзара ара қашықтары бірдей. Ол жолдардың бірлік ұзындығындағы автомобильдердің саны n = 50 км$^{-1}$

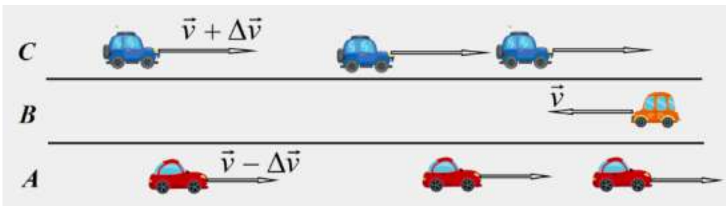


B жолымен жүріп келе жатқан автомобильді қарастыралық.

2.1 $t =$ [illegible] уақытта ***A*** жолымен келе жатқан қанша N_1 автомобиль ***B*** жолымен келе жатқан автомобильді басып озатынын анықтаңыз және қатар екі басып озудың арасындағы τ_1 уақытты анықтаңыз.

2.2 $t =$ [illegible] уақытта ***C*** жолымен келе жатқан қанша N_2 автомобиль ***B*** жолымен келе жатқан автомобильді басып озатынын анықтаңыз және қатар екі басып озудың арасындағы τ_2 уақытты анықтаңыз.

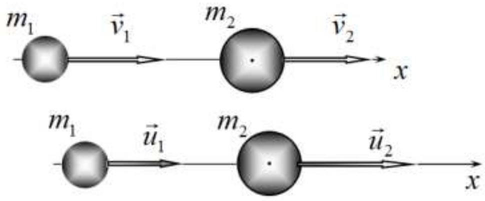
Енді ***B*** жолымен келе жатқан автомобиль ***A*** және ***C*** жолымен келе жатқан автомобильдерге қарсы бағытта қозғалсын. Автомобильдердің жылдамдықтары және жол бойындағы тығыздықтары бұрынғы жағдайдағыдай.



2.3 $t =$ [illegible] уақытта ***A*** жолымен келе жатқан қанша N_3 автомобиль және ***C*** жолымен келе жатқан қанша N_4 автомобиль ***B*** жолымен келе жатқан автомобильді кездестіреді? Қатар екі кездесуге сәйкес келетін τ_3 және τ_4 уақыт мәндерін анықтаңыз.

Серпімді соқтығысу

Есептің бұл бөлімінде екі дененің серпімді соқтығысуын қарастырамыз. Бұл қарастырудың негізгі мақсаты соқтығысудың нәтижесінде таңдалған денелердің біреуінің кинетикалық энергиясының өсу шартын анықтау болып табылады.



Массалары m_1 және m_2 серпімді шарлар x осінің бойымен қозғалады. Соқтығысқанға дейінгі бірінші шардың жылдамдығы v_1, ал екіншісінікі – v_2. Абсолют серпімді орталық соқтығысудан кейінгі сәйкес шариктердің жылдамдықтарын u_1 және u_2 деп белгілейміз. Шариктердің жылдамдығы деп олардың x осіне проекциясын түсіну қажет, сондықтан ол оң да, теріс те болуы мүмкін.

2.4 Шариктердің соқтығысқаннан кейінгі u_1 және u_2 жылдамдықтарын олардың соқтығысқанға дейінгі v_1 және v_2 жылдамдықтары және массалары арқылы өрнектеп жазыңыз.

Шариктердің массаларының қатынасын $\mu = \frac{m_2}{m_1}$ деп белгілейміз, бірінші шариктің соқтығысқанға дейінгі жылдамдығының соқтығысқаннан кейінгі жылдамдығына қатынасы $\eta_1 = \frac{u_1}{v_1}$, ал шариктердің соқтығысқанға дейінгі жылдамдықтарының қатынасы $\eta_2 = \frac{v_2}{v_1}$. Нақтылық үшін $v_1 > 0$ деп есептеңіз.

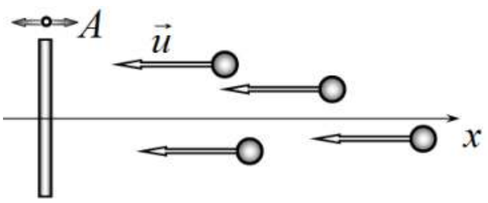
2.5 Шариктер массасының μ қатынысының әртүрлі мүмкін мәндері үшін η_1 параметрінің η_2 параметрінен тәуелділігінің графиктерін тұрғызыңыз.

2.6 η_2 және μ параметрлерінің арасында қандай қатынас болған кезде бірінші шарик соқтығысудың нәтижесінде өзінің энергиясын арттырады?

2.7 Жеңіл шариктің ауыр шарикпен соқтығысуын қарастырыңыз, яғни $m_2 >> m_1$. Осы шектік жағдайдағы бірінші шариктің соқтығысқаннан кейінгі $\tilde{u}_1$ жылдамдығын және соқтығысу нәтижесінде жеңіл шариктің энергиясы артатын η_2 -нің мәндерінің аймағын анықтаңыз.

Ферми үдеуінің қарапайым моделі

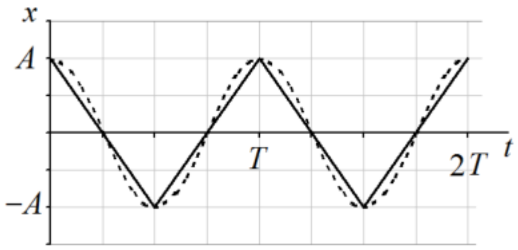
x осьіне перпендикуляр орналасқан ауыр плита x осы бағытында гармоникалық тербелістер жасайды. Тербеліс амплитудасы A, ал периоды T. Плитаға қарата x осы бойымен жеңіл шарлар бірдей u жылдамдықпен қозғалады. Шарлардың плитаға келу уақыты кездейсоқ және біркелкі таралған.



2.8 Плитаның қозғалысының V_0 максимальді жылдамдығын оның тербелісінің амплитудасы және периоды арқылы анықтаңыз.

2.9 Соқтығысқаннан кейін ұшып келе жатқан шариктердің қандай φ бөлігі өзінің энергиясын арттыратынын анықтаңыз. Жауаптарыңызды u және V_0 арқылы өрнектеп жазыңыз. Мынадай екі жағдай үшін: а) $u = 1.5V_0$; б) $u = 0.50V_0$ нақтылы есептеулер жүргізіңіз.

Плита қозғалысының гармоникалық заңын бөлікті-сызықтық функциямен жуықтаймыз, төмендегі суретті қараңыз, яғни амплитудасы мен тербеліс периоды бірдей мәндерінде плитаның қозғалыс жылдамдығының модулі тұрақты болып қалады деп есептейміз.



2.10 Плитаның V жылдамдық модулінің мәнін оның тербелістерінің амплитудасы мен периоды бойынша өрнектеңіз.

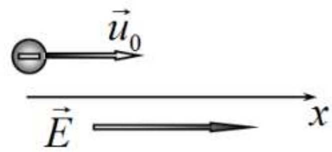
2.11 Түскен шарлардың орташа энергиясы $\varepsilon = \frac{E}{E_0}$ қанша есе өзгеретінін есептеңіз, мұндағы E_0 –шарлардың соқтығысқанға дейінгі кинетикалық энергиясы, E – тербелмелі плиткамен соқтығысқаннан кейінгі шарлардың орташа энергиясы. Мынадай екі жағдайлар үшін: а) $u = 1.5V$; б) $u = 0.50V$ нақтылы есептеулер жүргізіңіздер.

Есеп 3. Магнетрон (10.0 ұпай)

Электр және магнит өрістері оларда қозғалатын зарядталған бөлшектерге әсер етеді, бұл олардың қозғалысының сипатын белгілі бір түрде өзгертеді. Төмендегі барлық қарастыруларда зарядталған бөлшектер вакуумда қозғалады, ал электромагниттік сәуле шығаруды елемеуге болады. деп есептелінеді.

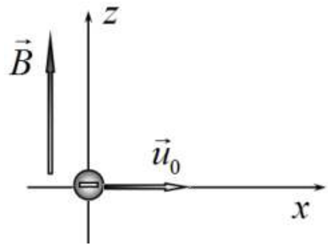
Бұл есепте классикалық нүктелік бөлшек деп есептеуге болатын электрондардың қозғалысы қарастырылады. Сандық есептеулерді жүргізгенде, қажет болған жағдайда электронның теріс $e = 1.60\cdot10^{-19}$ заряды бар екенін ескеріңіз, ал оның массасы $m = 9.11\cdot10^{-31}$. Электр және магнит тұрақтылары $\varepsilon_0 = 8.85\cdot10^{-12}$ и $\mu_0 = 1.26\cdot10^{-6}$, Больцман тұрақтысы $k_B = 1.38\cdot10^{-23}$.

Электр және магнит өрістеріндегі электрондардың қозғалысы



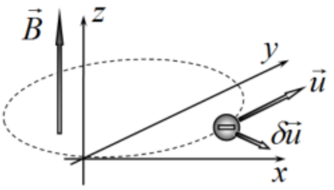
Бастапқы $t = 0$ уақыт мезетінде электрон координаттың бас нүктесінде тыныштықта тұр, ал x осінің оң бағытында кернеулігі E болатын біртекті электр өрісі бар. Электронға x осінің оң бағытында u_0 бастапқы жылдамдық береді.

3.1 Электронның бүкіл қозғалған уақытындағы координаттың ең үлкен $x_{\max}$ мәнін анықтаңыз.

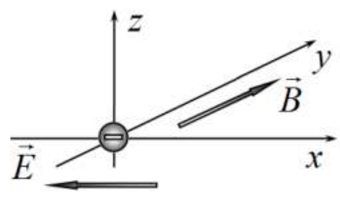


Бастапқы $t = 0$ уақыт мезетінде электрон координаттың бас нүктесінде тыныштықта тұрсын, ал z осінің оң бағытында индукциясы B біртекті магнит өрісі берілген болсын. Электронға x осінің оң бағытымен бағытталған u_0 бастапқы жылдамдық береді.

3.2 Электронның бүкіл қозғалған уақытындағы координаттың ең үлкен $x_{\max}$ мәнін анықтаңыз.

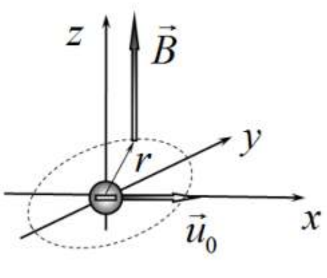


3.3 Өз траекториясымен қозғалып келе жатқан электронға уақыттың бір мезетінде оның сол мезеттегі u жылдамдығына перпендикуляр бағытта және оның алғашқы траекториясы жатқан жазықтықта қосымша δu жылдамдығы беріледі және $\delta u << u$. Электронның бастапқы ұйытқымаған траекториясына қатысты екі өлшемді тербелісінің периодын анықтаңыз.



Бастапқы $t = 0$ уақыт мезетінде электрон координаттың бас нүктесінде тыныштықта тұр, ал x осінің теріс бағытында кернеулігі E болатын біртекті электр өрісі бар. Сонымен қатар, y осінің оң бағыты бойынша индукциясы B біртекті магнит өрісі берілген. Электронды бастапқы жылдамдықсыз босатады.

3.4 Электронның бүкіл қозғалған уақытындағы координаттың ең үлкен $x_{\max}$ мәнін анықтаңыз.

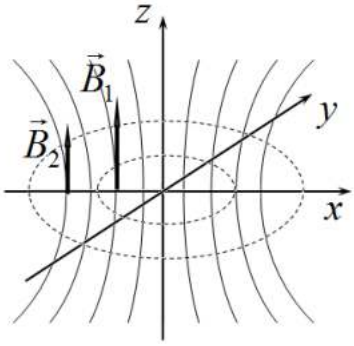


Бастапқы $t = 0$ уақыт мезетінде электрон координаттың бас нүктесінде тыныштықта тұрсын, z осінің оң бағытында магнит индукциясы xy жазықтығында z осінен r ара қашықтықтан мына түрде $B = \alpha r = \alpha(x^2 + y^2)^{1/2}$ тәуелді магнит өрісі берілсін. Электронға x осінің оң бағытында u_0 бастапқы жылдамдығы беріледі.

3.5 Қозғалыс кезіндегі электроннан z осіне дейінгі ең үлкен $r_{\max}$ ара қашықтығын анықтаңыз.

Кеңістіктің бірқатар бөлігінде z осіне қатысты оське симметриялы магниттік өріс туғызылған. Уақыттың $t = 0$ мезетінде магниттік өріс жоқ, ал содан соң жайлап өсе бастайды және өрістің магнит индукциясы xy жазықтығының кез келген нүктесінде z осімен бағытталған. Біраз уақыттан соң магнит индукциясы өзінің шектік мәніне жетеді де, одан әрі өспейді. Және де өрістің xy жазықтығында таралуы мынадай:

$$B \begin{cases} B_1, & 0 < r < r_1 \\ B_2, & r_1 < r < r_2 \\ 0, & r > r_2 \end{cases},$$



мұндағы r – сол z осіне дейінгі ара қашықтық, ал r және r_1 – белгілі шамалар.

Уақыттың $t = 0$ мезетінде электрон xy жазықтығының қандай да бір нүктесінде тыныштықта тұр. Өріс қосылған кезде электрон xy жазықтығында жатқан, центрі z осіндегі шеңбер бойынша қозғала бастайды.

3.6 B_1 / B_2 қатынасының қандай мәнінде жоғарыда айтылған қозғалыс мүмкін болады?

Цилиндрлік магнетрон

Магнетрон – электрондық электровакуумдық қондырғы, ондағы өтетін тоқтың шамасы электр және магнит өрісі арқылы реттеледі. Вакуумдық лампаның ішінде орналасқан, осьтері бір-біріне сәйкес, радиустары $a = 3.00 \cdot 10^{-1}$ және $b = 6.00$ болатын цилиндрлік катод пен анодтан тұратын, қарапайым магнетронды қарастыралық. Лампа осы катод пен анодтың осьтерімен сәйкес келетін цилиндр соленоидтың ортасында орналасқан. Соленоидтың орам саны $N = 2590$, ұзындығы $L = 30.0$ и диаметрі – $D = 10.5$. Анод пен катодтың арасындағы потенциалдар айырымы $V_a = 35.0$. Катодтан шығатын электрондар термоэлектрондық эмиссияның салдарынан пайда болады, бастапқы жылдамдығы нөлге тең деп есептеңіз және электрондық лампадағы кеңістіктік зарядтарды ескермеңіз.

3.7 Лампа мен соленоидтың ортақ осьінен $r = 3.00$ қашықтықта орналасқан нүкте мен катод арасындағы V потенциалдар айырымын есептеңіз.

3.8 Магнетрондағы катод пен анодтың арасындағы ток нөлге тең болатын соленоидтың ең аз $I_{\min}$ тогын есептеңіз.

3.9 Катодтың T температурасы үшін электрондардың жылдамдығы шын мәнінде нөлге тең болатын шартты табыңыз.

Теориялық есептерді шешуге арналған математикалық кеңес

Сіздерге төмендегі интегралдарды білу қажет болуы мүмкін:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$, где $n \neq -1$ және C – произвольная постоянная;

$$\int \frac{dx}{x} = \ln|x| + C$$, где C – бос мүше;

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2 (x^2 + a^2)^{3/2}} + C$$, мұндағы a, C – бос мүшелер;

$$(1+x)^\gamma \approx 1 + \gamma x + \frac{\gamma(\gamma-1)}{2} x^2$$, кез келген γ және |x|<<1 үшін;

$$\ln(1+x) \approx x$$, бұл |x|<<1 үшін.

ТЕОРЕТИЧЕСКИЙ ТУР
2 февраля 2023 года

Сначала, пожалуйста, прочитайте следующее:

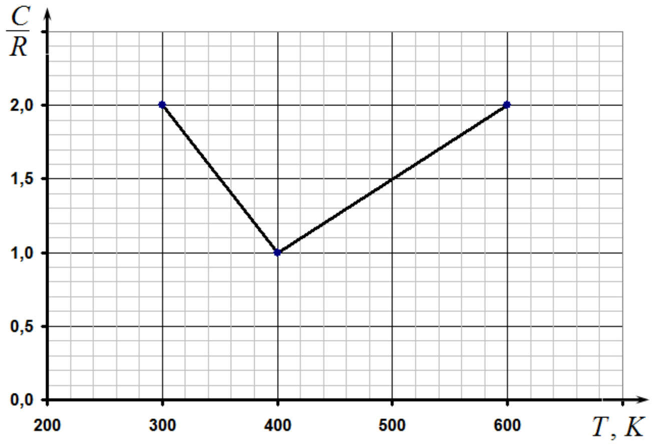
1. Теоретический тур состоит из трех задач. Продолжительность тура 4 часа.
2. Для расчетов Вы можете использовать свой калькулятор.
3. Вам предоставлены чистые листы бумаги и ***Листы для записи*** (***Writing sheets***). Чистые листы бумаги предназначены для черновых записей, их Вы можете использовать по Вашему усмотрению, они не проверяются. На ***Writing sheets*** следует записывать решения задач, которые будут оценены при проверке работы. В решениях как можно меньше используйте словесные описания. В основном Вы должны использовать уравнения, числа, буквенные обозначения, рисунки и графики.
4. Используйте только лицевую сторону ***Writing sheets***. При записи не выходите за пределы отмеченной рамки.
5. Решение каждой задачи следует начинать с новой страницы ***Writing sheets***.
6. На каждом использованном ***Writing sheets,*** в отведенных для этого графах, необходимо указать Вашу страну (***Country***), Ваш код (***Student Code***)**,** порядковый номер задачи (***Question Number***), текущий номер каждого листа (***Page Number***) и полное количество листов, использованных при решении всех задач (***Total Number of Pages)***. Если Вы не хотите, чтобы некоторые использованные ***Writing sheets*** были включены в ответ, тогда перечеркните их большим крестом на весь лист и не включайте в Ваш подсчёт полного количества листов.

Задача 1 (10.0 балла)

Эта задача состоит из трех частей, не связанных друг с другом.

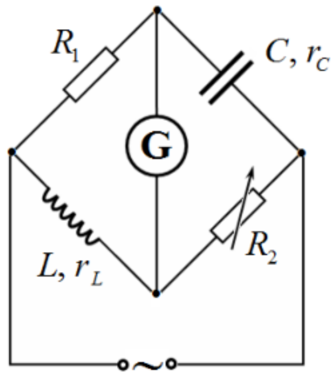
Задача 1.1 (3.0 балла)

Один моль идеального одноатомного газа, находящийся в цилиндре под поршнем, квазистатически нагревают от температуры $T_1 = 300$ К до $T_2 = 600$ К, изменяя объём таким образом, что зависимость теплоёмкости газа C от температуры имеет вид, показанный на рисунке. Найдите работу, совершённую над газом от состояния, в котором график зависимости его объёма от температуры достигал локального максимума, до состояния, в котором этот же график достигал локального минимума. Ответ выразите в джоулях, считая универсальную газовую постоянную равной $R = 8.31$ Дж/К.



Задача 1.2 (3.0 балла)

На рисунке показана схема моста переменного тока. Сопротивление $R_1 = 2.5$ кОм, индуктивность $L = 1$ Гн, сопротивление катушки индуктивности $r_L = 1$ Ом. На частоте переменного синусоидального напряжения $\nu = 100$ Гц балланс моста наступает при $R_2 = 800$ Ом. Оказалось, что при увеличении частоты переменного тока в два раза балланс моста не нарушается. Найдите сопротивление утечки r_C конденсатора и его емкость C.



Задача 1.3 (4.0 балла)

Нобелевская премия по физике за 2019 г. была присуждена швейцарским астрономам М. Майору и Д. Кело за открытие несветящихся спутников (экзопланет) у звёзд. Рассмотрим планету Юпитер в нашей солнечной системе как экзопланету для Солнца. Период обращения Юпитера составляет $T_J = 11.9$ года при массе $M_J = 1.90 \cdot 10^{27}$ кг, а масса Солнца равна $M_S = 1.99 \cdot 10^{30}$ кг. Считайте также известной скорость движения Земли по орбите $v_E = 29.8$ км/с, а орбиты планет предполагайте круговыми. Пусть наблюдение проводится спектрометром далеко из-за пределов солнечной системы так, что наблюдатель находится в плоскости движения системы Солнце–Юпитер. Найдите минимальную разрешающую способность спектрометра $R_{\min}$, которая позволяет определить наличие у Солнца массивной экзопланеты Юпитер. Считайте скорость света равной $c = 2.99 \cdot 10^8$ м/с.

Примечание: разрешающей способностью спектрометра называется его способность различать две близко расположенные спектральные линии, которая характеризуется безразмерным параметром $R = \lambda / \Delta\lambda$, где $\Delta\lambda$ – наименьшая разность длин волн двух линий еще регистрируемых прибором как отдельные, λ – средняя длина волны двух разрешаемых линий.

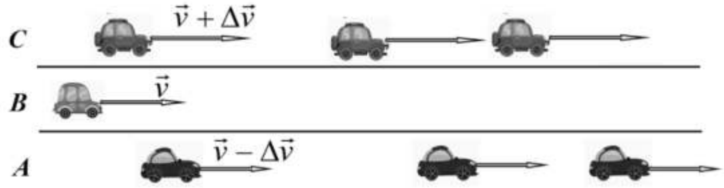
Задача 2. Ускорение Ферми (10.0 балла)

В космических лучах обнаруживаются частицы, обладающие чрезвычайно большой энергией. Возможный механизм появления таких частиц называется ускорением Ферми.

Ускорение Ферми – это стохастический механизм ускорения, которое заряженные частицы испытывают при многократном отражении, обычно от магнитного зеркала. В данной задаче рассматриваются основные идеи, лежащие в основе этого, парадоксального на первый взгляд, явления.

Почему встречных машин больше, чем попутных?

По автомагистрали, имеющей три полосы ***A, B, C***, движутся с постоянной скоростью в одном направлении автомобили: по центральной полосе ***B*** автомобиль движется со скоростью $v = 90$ км/ч; по полосе ***A*** движутся автомобили со скоростями $v - \Delta v = 80$ км/ч; по полосе ***C*** движутся автомобили со скоростями $v + \Delta v == 100$ км/ч. Расстояние между автомобилями одинаковое на каждой из полос ***A*** и ***C***, число автомобилей на единицу длины полосы для каждой из них составляет $n = 5.0\,\text{км}^{-1}$.

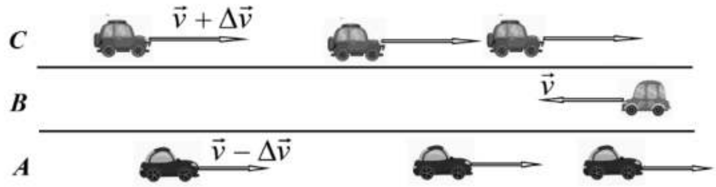


Рассмотрим автомобиль, движущийся по полосе ***B***.

2.1 Рассчитайте, сколько автомобилей N_1, движущихся по полосе ***A***, обгоняет автомобиль ***B*** за время $t = 1.0$ мин, а также время τ_1 между двумя последовательными обгонами.

2.2 Рассчитайте, сколько автомобилей N_2, движущихся по полосе ***C,*** которые обгоняют автомобиль на полосе ***B*** за время $t = 1.0$ мин, а также время τ_2 между двумя последовательными обгонами.

Пусть теперь автомобиль движется по полосе ***B*** навстречу автомобилям, движущимся по полосам ***A*** и ***C***. Скорости автомобилей и их плотность на дороге остаются прежними.

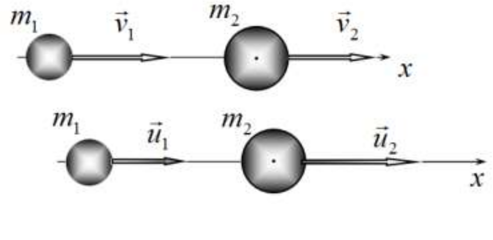


2.3 Рассчитайте числа автомобилей N_3, движущихся по полосе ***A***, и N_4, движущихся по полосе ***C***, которые встречает автомобиль ***B*** за время $t = 1.0$ мин. а также соответствующие времена τ_3 и τ_4 между двумя последовательными встречами.

Упругое столкновение

В данной части рассмотрим классическую задачу об упругом столкновении двух тел. Основной целью данного рассмотрения является определение условий, при которых кинетическая энергия одного из выбранных тел возрастает в результате удара.

Два упругих шарика, массы которых равны m_1 и m_2, движутся вдоль оси x. Скорость первого шарика до столкновения равна v_1, скорость второго – v_2. Обозначим скорости шариков после абсолютно упругого центрального столкновения u_1 и u_2, соответственно. Под скоростями шариков следует понимать проекции скоростей на ось x, поэтому они могут быть как положительными, так и отрицательными.



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2.4 Выразите скорости шариков u_1 и u_2 после удара через их скорости до удара v_1 и v_2, а также массы шариков.

Обозначим отношение масс шариков как $\mu = \frac{m_2}{m_1}$, отношение скорости первого шарика после и до удара как $\eta_1 = \frac{u_1}{v_1}$ и отношение скоростей шариков до удара как $\eta_2 = \frac{v_2}{v_1}$. Для определенности считайте, что $v_1 > 0$.

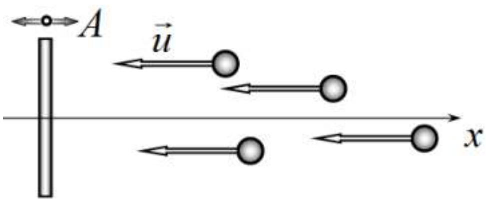
2.5 Постройте семейство графиков зависимостей параметра η_1 от параметра η_2 для всех характерных значений отношений масс шариков μ.

2.6 Найдите соотношение между параметрами η_2 и μ, при котором первый шарик увеличивает свою энергию в результате столкновения.

2.7 Рассмотрите случай столкновения легкого шарика с тяжелым $m_2 >> m_1$. Найдите в этом предельном случае скорость первого шарика после столкновения $\tilde{u}_1$ и определите область значений скоростей тяжелого шарика η_2 до столкновения, при которых энергия легкого шарика возрастает в результате столкновения.

Простейшая модель ускорения Ферми

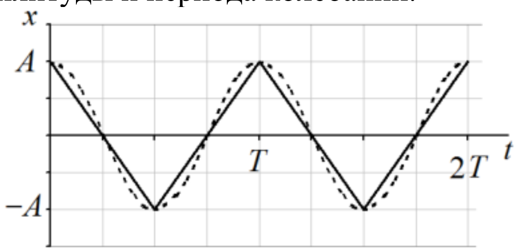
Массивная плита, расположенная перпендикулярно оси x, совершает гармонические колебания в направлении оси x. Амплитуда колебаний равна A, их период – T. В направлении плиты вдоль оси x с одинаковыми скоростями u движутся легкие шарики. Времена подлета шариков к плите являются случайными и равномерно распределенными.



2.8 Выразите максимальную скорость движения плиты V_0 через амплитуду и период ее колебаний.

2.9 Рассчитайте долю φ налетающих шариков, которые после столкновения увеличат свою кинетическую энергию. Ответ выразите через u и V_0. Для численной оценки отдельно рассмотрите два случая: а) $u = 1.5V_0$; б) $u = 0.50V_0$.

Аппроксимируем гармонический закон движения плиты кусочно-линейной функцией, смотрите рисунок ниже, то есть будем считать, что модуль скорости V движения плиты остается постоянным при тех же значениях амплитуды и периода колебаний.



2.10 Выразите значение модуля скорости движения плиты V через амплитуду и период ее колебаний.

2.11 Рассчитайте, во сколько раз изменится средняя энергия налетающих шариков $\varepsilon = \frac{E}{E_0}$, где E_0 – кинетическая энергия шариков до столкновения, E – средняя энергия шариков после столкновения с колеблющейся плитой. Для численной оценки отдельно рассмотрите два случая: а) $u = 1.5V$; б) $u = 0.50V$.

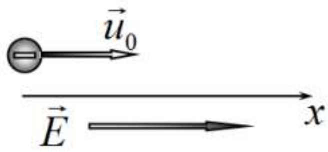
Задача 3. Магнетрон (10.0 баллов)

И электрическое и магнитное поля воздействуют на движущиеся в них заряженные частицы, которые определенным образом меняют характер своего движения. Всюду в дальнейшем предполагается, что заряженные частицы двигаются в вакууме, а излучением электромагнитных волн можно пренебречь.

В данной задаче рассматривается движение электронов, которые представляют собой классические точечные частицы. При проведении численных расчетов считайте, где необходимо, что электрон обладает отрицательным электрическим зарядом, модуль которого равен $e = 1.60 \cdot 10^{-19}$ Кл, а его масса равна $m = 9.11 \cdot 10^{-31}$ кг. Электрическая и магнитная постоянные равны $\varepsilon_0 = 8.85 \cdot 10^{-12}$ Ф/м и $\mu_0 = 1.26 \cdot 10^{-6}$ Гн/м, постоянная Больцмана $k_B = 1.38 \cdot 10^{-23}$ Дж/К.

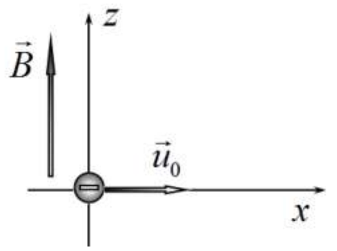
Движение электрона в электрическом и магнитном полях

Пусть в начальный момент времени $t = 0$ электрон покоится в начале координат, а вдоль положительного направления оси x приложено однородное электрическое поле напряженностью E. Электрону сообщают начальную скорость u_0, также направленную в положительном направлении оси x.

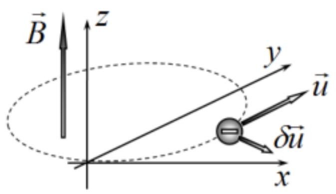


3.1 Определите максимальное значение координаты электрона $x_{\max}$ за все время его движения.

Пусть в начальный момент времени $t = 0$ электрон покоится в начале координат, а вдоль положительного направления оси z приложено однородное магнитное поле с индукцией B. Электрону сообщают начальную скорость u_0, направленную в положительном направлении оси x.

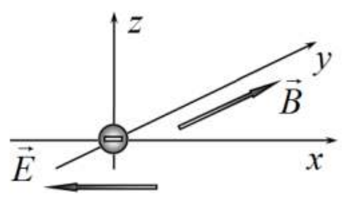


3.2 Определите максимальное значение координаты электрона $x_{\max}$ за все время его движения.

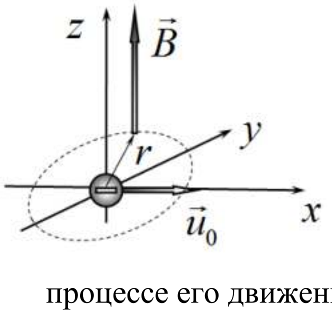


3.3 В процессе движения по траектории электрону в некоторый момент времени сообщили дополнительную скорость δu в направлении, перпендикулярном его текущей скорости u и лежащем в плоскости его начальной траектории, причем $\delta u \quad u$. Определите период возникших двумерных колебаний электрона относительно его первоначальной невозмущенной траектории.

Пусть в начальный момент времени $t = 0$ электрон покоится в начале координат, а вдоль отрицательного направления оси x приложено однородное электрическое поле напряженностью E. Помимо этого, вдоль положительного направления оси y создано однородное магнитное поле с индукцией B. Электрон освобождают с нулевой начальной скоростью.



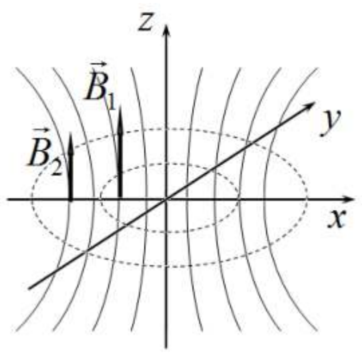
3.4 Определите максимальное значение координаты электрона $x_{\max}$ за все время его движения.



Пусть в начальный момент времени $t = 0$ электрон покоится в начале координат, а вдоль положительного направления оси z приложено магнитное поле, индукция которого в плоскости xy зависит от расстояния r до оси z по закону $B = \alpha r = \alpha (x^2 + y^2)^{1/2}$. Электрону сообщают начальную скорость u_0, направленную в положительном направлении оси x.

3.5 Определите максимальное расстояние $r_{\max}$ от электрона до оси z в процессе его движения.

В некоторой области пространства создается осесиммеричное относительно оси z магнитное поле. В момент времени $t=0$ магнитное поле отсутствует, а затем начинает медленно нарастать, причем во всех точках плоскости xy вектор индукции поля направлен вдоль оси z. По прошествии некоторого промежутка времени индукция магнитного поля достигает предельного значения и перестает изменяться. При этом в плоскости xy распределение поля имеет вид



$$B=\begin{cases} B_1, & 0<r<r_1 \\ B_2, & r_1<r<r_2, \\ 0, & r>r_2 \end{cases}$$

где r – расстояние до оси z, а r_1 и r_2 – известные величины.

В момент времени $t=0$ электрон покоится в некоторой точке, лежащей в плоскости xy. При включении поля электрон начинает двигаться по окружности в плоскости xy, центр которой находится на оси z.

3.6 Определите, при каком отношении B_1/B_2 возможно описанное движении электрона.

Цилиндрический магнетрон

Магнетрон – электронный электровакуумный прибор, величина протекающего тока в котором управляется электрическим и магнитным полем. Рассмотрим простейший магнетрон, состоящий из проводящих соосных длинных цилиндрических катода и анода с радиусами $a=3.00\cdot10^{-1}$мм и $b=6.00$ мм соответственно, расположенных внутри вакуумной лампы. Лампа находится в центре цилиндрического соленоида, ось которого совпадает с осями катода и анода, а число витков составляет $N=2590$, длина равна $L=210$ мм и диаметр – $D=105$ мм. Разность потенциалов между катодом и анодом постоянна и равна $V_0=75.0$ В. Считайте, что покидающие катод электроны образуются вследствие термоэлектронной эмиссии и имеют нулевую начальную скорость, а также пренебрегайте пространственным зарядом электронов в лампе.

3.7 Рассчитайте разность потенциалов V между катодом и точкой в пространстве, расположенной на расстоянии $r=3.00$ мм до общей оси лампы и соленоида.

3.8 Рассчитайте наименьший ток соленоида $I_{\min}$, при котором ток в магнетроне между катодом и анодом обращается в ноль.

3.9 Найдите условие для температуры катода T, при выполнении которого начальную скорость электронов действительно можно считать нулевой.

Математическая подсказка для задач теоретического тура

Вам может понадобиться знание следующих интегралов:

$$\int x^n dx=\frac{x^{n+1}}{n+1}+C, \text{ где } n\neq-1 \text{ и } C \text{ – произвольная постоянная;}$$

$$\int\frac{dx}{x}=\ln|x|+C, \text{ где } C \text{ – произвольная постоянная;}$$

$$\int\frac{dx}{(x^2+a^2)^{3/2}}=\frac{x}{a^2(x^2+a^2)^{3/2}}+C, \text{ где } a, C \text{ – произвольные постоянные;}$$

$$(1+x)^\gamma\approx1+\gamma x+\frac{\gamma(\gamma-1)}{2}x^2, \text{ для } |x|\ll1 \text{ и любых } \gamma;$$

$$\ln(1+x)\approx x, \text{ для } |x|\ll1.$$

ТЕОРИЯЛЫҚ ТУРДЫҢ ЕСЕПТЕРІНІҢ ШЕШІМІ

Назар аударыңыз: ұпайлар бағаға бөлінбейді

Тапсырма 1 (10.0 ұпай)

Тапсырма 1.1 (4.0 ұпай)

Термодинамиканың бірінші бастамасынан белгілі

$$\delta Q = dU + dA, \tag{1}$$

мұндағы δQ - берілген жылу мөлшері, dU – ішкі энергия өзгерісі, dA – газдың жасаған жұмысы.

Идеал газдың бір молі үшін бұл шамаларды белгілі p қысым кезіндегі көлем dV және температура dT өзгерісі түрінде келесі түрде жазуға болады

$$\delta A = pdV, \tag{2}$$

$$dU = C_V dT. \tag{3}$$

Жылу сыйымдылығының анықтамасы бойынша бізде бар

$$C = \frac{\delta Q}{dT}, \tag{4}$$

онда (1)-(4) қатынасынан аламыз

$$p\frac{dV}{dT} = C - C_V, \tag{5}$$

тең тұрақты көлемдегі бір атомды газдың молярлық жылу сыйымдылығында

$$C_V = \frac{3}{2}R. \tag{6}$$

Шартта берілген графиктен көруге болады, мына температурада

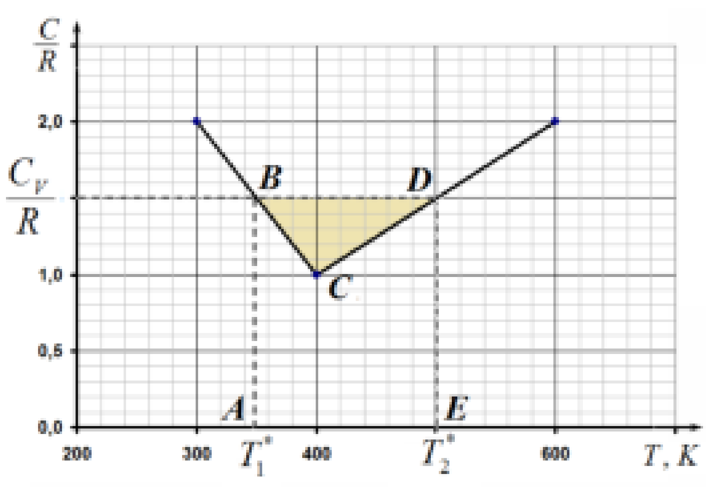
$$T_1^* = 350 \tag{7}$$

жылусыйымдылық $C = C_V$ және, сәйкесінше, $\frac{dV}{dT} = 0$. Осы температура арқылы өткенде туынды $\frac{dV}{dT}$ белгісі плюс минусқа өзгереді. Бұл осы температурада газ көлемі жергілікті максимумға жетеді дегенді білдіреді: $T_{max} = T_1^* = 350$.

Мына температурада

$$T_2^* = 500 \tag{8}$$

туынды $\frac{dV}{dT}$ да нөлге тең және осы нүктеден өткенде оның туынды белгісі минустан плюсқа өзгереді. Бұл T_2^* жергілікті көлемнің минимум нүктесі екенін білдіреді: $T_{min} = T_2^* = 500$.



$T_1^* = 350$ -ден до $T_2^* = 500$ -ге дейінгі бөлікте газ Q жылу мөлшерін алады, $C(T)$ тәуелділігінің ауданына сандық түрде тең, яғни ***ABCDE*** фигураның ауданы болып табылады. Ішкі энергия өзгерісі

$\Delta U = C_V(T_2^* - T_1^*)$ ***ABDE*** тіктөртбұрышының ауданына сандық түрде тең. Термодинамиканың бірінші бастамасынан $A = \Delta U - Q$, сондықтан T_1^* -ден T_2^* -ге дейінгі газда атқарылған жұмыс сандық жағынан ***ABDE*** тіктөртбұрышының аудандары мен ***ABCDE*** фигурасы арасындағы айырмашылыққа тең, яғни. ***BDC*** штрихталған фигураның ауданы:

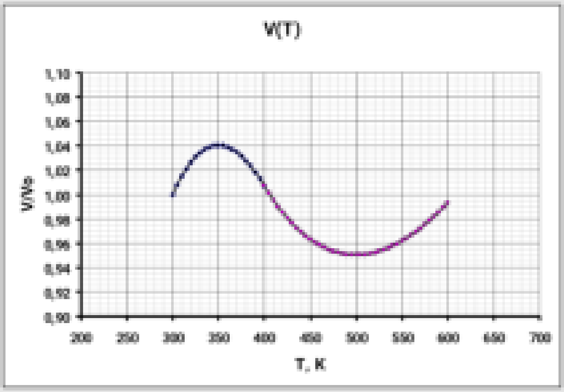
$$A = \frac{1}{4}R(T_{max} - T_{min}) = 312 \quad . \tag{9}$$

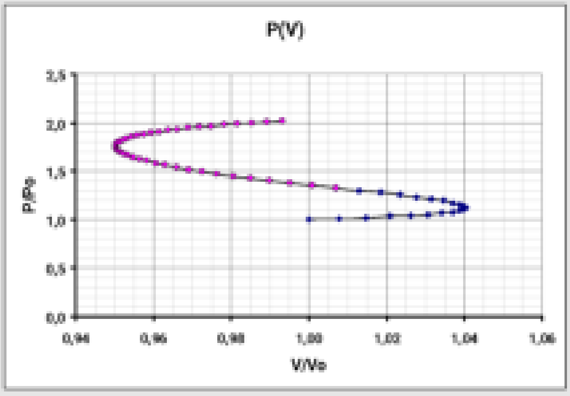
Қосымша: айқын тәуелділік $V(T)$:

$$\frac{V}{V_1} = \left(\frac{T}{T_1}\right)^{7/2} \exp\left(-\frac{T - T_1}{\Delta T_1}\right)$$, егер $T_1 = 300 \le T \le 400$ K және $\Delta T_1 = 100$ K ;

$$\frac{V}{V_0} = \left(\frac{T}{T_0}\right)^{7/2} \exp\left(\frac{T - T_0}{\Delta T_2}\right)$$, егер $T_0 = 400 \le T \le 600$ K және $\Delta T_1 = 200$ K .

Газды қыздыру процесіндегі $V(T)$ және $P(V)$ тәуелділіктері төмендегі суреттерде көрсетілген.



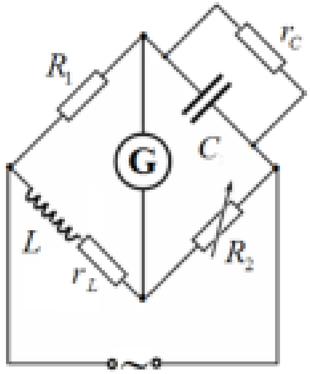


Мазмұны	Ұпайлар
Формула (1): $\delta Q = dU + dA$	0.2
Формула (2): $\delta A = pdV$	0.2
Формула (3): $dU = C_V dT$	0.2
Формула (4): $C = \frac{\delta Q}{dT}$	0.2
Формула (5): $p\frac{dV}{dT} = C - C_V$	0.4
Формула (6): $C_V = \frac{3}{2}R$	0.2
Формула (7): $T_1^* = 350$	0.4
Формула (8): $T_2^* = 500$	0.4
Формула (9): $A = \frac{1}{4}R(T_{max} - T_{min})$	0.2
(9) формуланың сандық мәні: $A = 312$	0.4

Барлығы	3.0

Тапсырма 1.2 (3.0 ұпай)

Көпірдің эквиваленттi тізбегі төмендегі суретте көрсетілген, ол идеалды емес индуктивтілік тізбегі эквивалентті екенін ескереді - бұл идеалды катушкалар L және резистор r_L тізбектей жалғанған; ағып тұрған конденсатордың эквивалентті тізбегі идеалды конденсатор C -ға параллель қосылған r_C резисторы болып табылады.



Шешім 1. Күрделі сандардағы көпір балансының шарты былай жазылады

$$Z_L Z_C = R_1 R_2, \tag{1}$$

мұндағы кедергілер сәйкесінше

$$Z_L = r_L + i\omega L \tag{2}$$

және

$$Z_C = \frac{r_C}{1 + i\omega C r_C}. \tag{3}$$

(1)-(3) өрнектерді түрлендірулерден кейін аламыз:

$$i\omega\left(L - R_1 R_2 C\right) = r_L - \frac{R_1 R_2}{r_C}. \tag{4}$$

Жиілікті өзгерткенде, теңдеудің екі жағы да нөлге тең болса, бұл теңдік бұзылмайды, сондықтан

$$C = \frac{L}{R_1 R_2} = 0{,}1\ \text{мкФ}, \tag{5}$$

$$r_C = \frac{R_1 R_2}{r_L} = 3\ \text{МОм}. \tag{6}$$

Шешім 2. Конденсатордағы кернеу болсын

$$U_C = U_0 \cos\omega t, \tag{1}$$

содан кейін ол арқылы ток өтеді

$$I_C = -C\omega \cos\omega t, \tag{2}$$

және оның ағып кету кедергісі арқылы өтетін ток

$$I_{r_C} = \frac{U_0 \cos\omega t}{r_C}. \tag{3}$$

Конденсатор бар тұтқа арқылы өтетін жалпы ток тең

$$I_1 = I_C + I_{r_C}\ , \qquad (4)$$

және көпір теңдестірілген болғандықтан, сол ток R_1 кедергісі арқылы өтеді, демек,

$$U_{R_1} = I_1 R_1\ . \qquad (5)$$

екінші жағынан, бұл кернеу индуктивтілігі

$$U_L = U_{R_1}\ , \qquad (6)$$

иықтағы кернеуінің төмендеуіне тең, ол үшін кернеудің төмендеуі

$$U_L = \frac{dI_2}{dt} + I_2 r_L\ , \qquad (7)$$

өрнегімен берілген, онда ток күші

$$I_2 = I_{R_1} = \frac{U_C}{R_2}\ . \qquad (8)$$

баланстық теңдеуімен берілген, себебі

$$U_{R_2} = U_C\ . \qquad (9)$$

(1)-(9) теңдеулерді бірге жинап, аламыз

$$\left(-\frac{\omega L}{R_2} + C\omega R_1\right) U_0 \sin \omega t = \left(\frac{R_1}{r_C} - \frac{r_L}{R_2}\right) U_0 \cos \omega t\ .$$

(10)

Бұл теңдіктен теңдеудің екі жағы да нөлге тең болса, жиілікке тәуелсіз тепе-теңдік шарты орындалатынын көруге болады, яғни жауабын аламыз.

$$C = \frac{L}{R_1 R_2} = 0.5\ мкФ\ ,$$

(11)

$$r_C = \frac{R_1 R_2}{r_L} = 3\ МОм\ .$$

(12)

Мазмұны	Ұпайлар
Шешім 1	
Эквивалентті диаграмма: Барлық элементтер дұрыс орналастырылған	0.5
Формула (1): $Z_L Z_C = R_1 R_2$	0.3
Формула (2): $Z_L = r_L + i\omega L$	0.3
Формула (3): $Z_C = \frac{r_C}{1 + i\omega C r_C}$	0.3
Формула (4): $i\omega\left(L - R_1 R_2 C\right) = r_L - \frac{R_1 R_2}{r_C}$	0.4
Формула (5): $C = \frac{L}{R_1 R_2}$	0.4
Сандық мән формула (5)-те: $C = 0.5\ мкФ$	0.2

Формула (6): $r_C = \dfrac{R_1 R_2}{r_L}$	0.4
Сандық мән формула (6)-да: $r_C =$ [illegible]МОм	0.2
Барлығы	**3.0**
Шешім 2	
Эквиваленттi диаграмма: Барлық элементтер дұрыс орналастырылған	0.5
Формула (1): $U_C = U_0 \cos\omega t$	0.1
Формула (2): $I_C = -C\omega \cos\omega t$	0.1
Формула (3): $I_{r_C} = \dfrac{U_0 \cos\omega t}{r_C}$	0.1
Формула (4): $I_1 = I_C + I_{r_C}$	0.1
Формула (5): $U_{R_1} = I_1 R_1$	0.1
Формула (6): $U_L = U_{R_1}$	0.1
Формула (7): $U_L = \dfrac{dI_2}{dt} + I_2 r_L$	0.1
Формула (8): $I_2 = I_{R_2} = \dfrac{U_C}{R_2}$	0.1
Формула (9): $U_{R_2} = U_C$	0.1
Формула (10): $\left(-\dfrac{\omega L}{R_2} + C\omega R_1\right) U_0 \sin\omega t = \left(\dfrac{R_1}{r_C} - \dfrac{r_L}{R_2}\right) U_0 \cos\omega t$	0.4
Формула (11): $C = \dfrac{L}{R_1 R_2}$	0.4
Сандық мән формула (11)-де: $C =$ [illegible]Ф	0.2
Формула (12): $r_C = \dfrac{R_1 R_2}{r_L}$	0.4
Сандық мән формула (12)-де: $r_C =$ [illegible]МОм	0.2
Барлығы	**3.0**

Тапсырма 1.3 (4.0 ұпай)

Массасы m планета Күнді айналмалы орбита радиусы R бойынша v жылдамдықпен қозғалсын, онда планетаның радиалды бағытқа проекциясында қозғалыс теңдеуі былай жазылады

$$\frac{mv^2}{R} = G\frac{mM_S}{R^2}, \qquad (1)$$

одан

$$v = \sqrt{G\frac{M_S}{R}}, \qquad (2)$$

мұндағы G – гравитациялық тұрақты .

J индексі бар Юпитер және *E* индексі бар Жер үшін формуланы (2) жазып, бөлгеннен кейін аламыз

$$\frac{v_J}{v_E} = \sqrt{\frac{R_E}{R_J}}, \qquad (3)$$

ал екінші жағынан, Кеплердің үшінші заңына сәйкес, T_E және T_J айналу периодтарының қатынасы үшін бізде бар

$$\frac{T_E^2}{T_J^2} = \frac{R_E^3}{R_J^3}. \qquad (4)$$

Юпитердің қозғалысын спектрометрмен анықтау мүмкін емес, бірақ ол Күн үшін де жасалуы мүмкін, өйткені ол Күн-Юпитер жүйесінің массалар центрін де айналып өтеді. Күннің жылдамдығын мына өрнектен оңай табуға

$$v_S = v_J\frac{M_J}{M_S}. \qquad (5)$$

Күн жүйенің жалпы масса центрі айналасында қозғалатындықтан және бақылаушы бір жазықтықта орналасқандықтан, Доплер эффектінің формуласы бойынша анықтау кезінде келесі шарт орындалады

$$\frac{\Delta\lambda}{\lambda} = \frac{2v_S}{c}. \qquad (6)$$

(3)-(6) теңдеулерді бірге жинап, соңғы жауапты аламыз

$$R_{\min} = \frac{M_S}{M_J}\left(\frac{T_J}{T_E}\right)^{1/3}\frac{c}{2v_E} = 1.20\cdot 10^7. \qquad (7)$$

Бұл рұқсат әлемнің әртүрлі елдерінде шығарылған көптеген заманауи спектрометрлер үшін қол жетімді.

Мазмұны	Ұпайлар
Формула (1): $\frac{mv^2}{R} = G\frac{mM_S}{R^2}$	0.2
Формула (2): $v = \sqrt{G\frac{M_S}{R}}$	0.2
Формула (3): $\frac{v_J}{v_E} = \sqrt{\frac{R_E}{R_J}}$	0.2
Формула (4): $\frac{T_E^2}{T_J^2} = \frac{R_E^3}{R_J^3}$	0.4
Формула (5): $v_S = v_J\frac{M_J}{M_S}$	1.0

Формула (6): $\frac{\Delta\lambda}{\lambda} = \frac{2v_S}{c}$	1.0
Формула (7): $R_{\min} = \frac{M_S}{M_J}\left(\frac{T_J}{T_E}\right)^{1/3}\frac{c}{2v_E}$	0.5
Сандық мән формула (7)-де: $R_{\min} = 1.20 \cdot 10^7$	0.5
Барлығы	**4.0**

Тапсырма 2. Ферми үдеуі (10.0 ұпай)

Неліктен өтіп бара жатқан көліктерге қарағанда қарсы келе жатқан көліктер көп?

2.1 t уақыт ішінде ***B*** жолағындағы көлік өзінен

$$l = (v - (v - \Delta v))t = \Delta v\, t \quad (1)$$

қашықтықта тұрған көліктерді ғана басып озады. Сондықтан бұл көліктердің саны

$$N_1 = nl = n\Delta vt \approx 0.83 \quad (2)$$

Бірін-бірі озу арасындағы уақыт

$$\tau_1 = \frac{1}{n\Delta v} = 0.02 \quad 72 \text{ с} \quad (3)$$

2.2 Ұқсас пайымдаулар басып озулар саны мен басып озулар арасындағы уақыт өзгеріссіз қалады деген қорытындыға әкеледі, яғни.

$$N_2 = nl = n\Delta vt \approx 0.83, \quad (4)$$

$$\tau_2 = \frac{1}{n\Delta v} = 0.02 \quad 72 \text{ с} \quad (5)$$

2.3 Қарсы келе жатқан көліктердің санына қарай қозғалу кезінде және қатарынан екі кездесу арасындағы уақыт мына формулалар бойынша есептеледі:

$$N_{3,4} = n(v + (v \pm \Delta v))t = n(2v \pm \Delta v)t$$

$$\tau_{3,4} = \frac{1}{n(2v \pm \Delta v)}, \quad (6)$$

және сандық есептеулер келесі мәндерді береді

$$N_3 = 14.2; \quad \tau_3 = 4.2 \text{ с};$$

$$N_3 = 15.8; \quad \tau_3 = 3.8 \text{ с}. \quad (7)$$

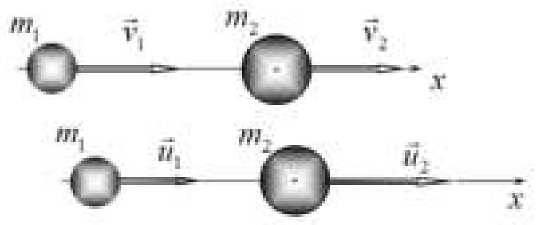
Серпімді соқтығыс

2.4 Импульстің сақталу заңын

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \quad (8)$$

және механикалық энергияның сақталу заңын жазамыз

$$\frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1 u_1^2}{2} + \frac{m_2 u_2^2}{2}. \quad (9)$$



Осы теңдеулерді мына түрде қайттан жазамыз

$$m_1v_1 - m_1u_1 = m_2u_2 - m_2v_2$$
$$m_1v_1^2 - m_1u_1^2 = m_2u_2^2 - m_2v_2^2$$

(10)

ал, екінші теңдеуді біріншіге бөліп, мына нәтижені аламыз

$$v_1 + u_1 = u_2 + v_2 \,.$$

(11)

Осы теңдіктен $u_2 = v_1 + u_1 - v_2$ өрнектеп, импульстің сақталу заңын теңдеуге қоямыз

$$(m_1 + m_2)u_1 = (m_1 - m_2)v_1 + 2m_2v_2 \,,$$

(12)

осыдан шығады

$$u_1 = \frac{m_1 - m_2}{m_1 + m_2}v_1 + \frac{2m_2}{m_1 + m_2}v_2 \,.$$

(13)

Екінші шардың жылдамдығын (13) формуладағы «1» және «2» индекстерін өзгерту арқылы оңай алуға болады.

$$u_2 = \frac{2m_1}{m_1 + m_2}v_1 + \frac{m_2 - m_1}{m_1 + m_2}v_2 \,.$$

(14)

2.5 (13) формуланы пайдаланып, қажетті параметрлер арасындағы тәуелділіктің айқын түрін аламыз

$$\frac{u_1}{v_1} = \frac{m_1 - m_2}{m_1 + m_2} + \frac{2m_2}{m_1 + m_2}\frac{v_2}{v_1} = \frac{1 - \frac{m_2}{m_1}}{1 - \frac{m_2}{m_1}} + \frac{2\frac{m_2}{m_1}}{1 + \frac{m_2}{m_1}}\frac{v_2}{v_1} \Rightarrow$$

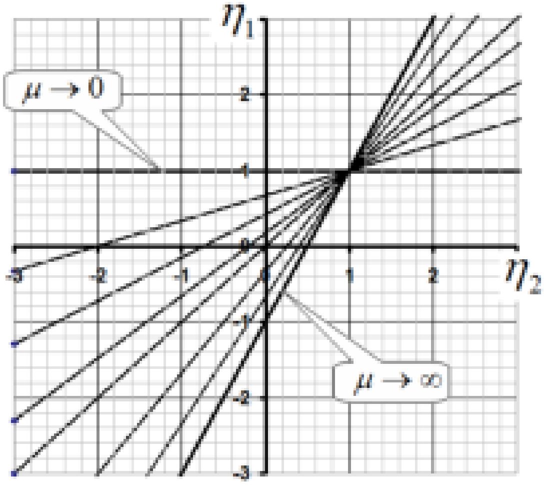
$$\eta_1 = \frac{1-\mu}{1+\mu} + \frac{2\mu}{1+\mu}\eta_2 \,. \qquad (15)$$

Как Алынған өрнектен шығатындай, μ массалық қатынасының кез келген мәндері үшін $\eta_1(\eta_2)$ тәуелділігі сызықты, яғни. оның графигі түзу болады. Бұл түзулердің барлығы $\eta_1 = 1; \eta_2 = 1$ нүктесі арқылы өтетінін байқау да қиын емес. $\mu \to 0$ нүктесінде көлбеу коэффициенті $\eta_1 = 1$ нөлге ұмтылады, яғни тәуелділік графигі көлденең түзуге ұмтылады. $\mu \to \infty$ нүктесінде қалаған тәуелділік ұмтылады:

$$\eta_1 = -1 + 2\eta_2 \,.$$

(16)

Бұл функциялардың графиктік тобы төмендегі суретте көрсетілген.



2.6 Шардың кинетикалық энергиясы, егер соққыдан кейінгі доптың жылдамдығының модулі соққыға дейінгі жылдамдық модулінен үлкен болса, яғни теңсіздіктер орындалса артады

$$|\eta_1| > 1 \quad \Rightarrow \quad \begin{cases} \eta_1 > 1 \\ \eta_1 < -1 \end{cases}. \tag{17}$$

η_1 шамасы үшін орнына (15) өрнегін қойып, теңсіздіктерді аламыз

$$\begin{cases} \dfrac{1-\mu}{1+\mu} + \dfrac{2\mu}{1+\mu}\eta_2 > 1 \\ \dfrac{1-\mu}{1+\mu} + \dfrac{2\mu}{1+\mu}\eta_2 < -1 \end{cases}. \tag{18}$$

Бұл теңсіздіктер келесі қатынастар арқылы шешіледі:

а)

$$\eta_2 > 1, \tag{19}$$

яғни бұл шартты орындау үшін екінші доп біріншісін қуып жетуі керек;

б)

$$\eta_2 < -\frac{1}{\mu}, \tag{20}$$

бұл жағдайда екінші доп қарай жылжу керек және оның жылдамдығының модулі көрсетілген мәннен асып кетуі керек.

2.7 Шектеу жағдайында $m_2 \gg m_1$ соқтығысқаннан кейінгі бірінші доптың жылдамдығы

$$\bar{u}_1 = -v_1 + 2v_2, \tag{21}$$

яғни бірінші доптың жылдамдығы таңбасын өзгертеді (доп шағылысады) және оның модулі екінші ауыр доптың жылдамдығынан екі есеге өзгереді.

Жеңіл шар өзінің жылдамдығын және, демек, кинетикалық энергиясын арттырады, егер:

а) ауыр доп оны қуып жетеді (артынан соғу) $v_2 > 1$;

б) ауыр доп оған қарай жылжиды $v_2 < 0$.

Ферми үдеуінің ең қарапайым моделі

2.8 Плитаның қозғалыс заңын дәстүрлі түрде жазамыз

$$x(t) = A\cos(\omega t),$$

(22)

онда жылдамдықтың уақытқа тәуелділігі функция арқылы сипатталады

$$v(t) = -A\omega\sin(\omega t),$$

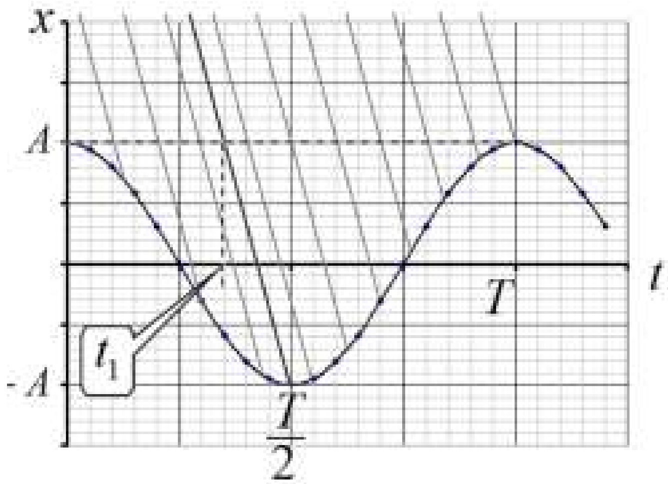
(23)

сондықтан платформаның максималды жылдамдығы

$$V_0 = A\omega = 2\pi\frac{A}{T}.$$

(24)

2.9 Бұл сұраққа жауап беру үшін платформалық тербелістердің бір кезеңін қарастыру жеткілікті. Платформа координатасының уақытқа (22) тәуелділігінің графигін тұрғызайық және оған $x = x_0 - ut$ түзу болып келетін түскен бөлшектердің координаталарының уақытқа тәуелділік графиктерін салайық..



Суретте $u > V_0$ жағдай көрсетілген. Соққы нәтижесінде платформа x осінің оң бағытына қарай жылжыған сәттерде платформамен соқтығысатын бөлшектер жылдамдығын арттырады, бұл жағдайда соқтығыстар $\frac{T}{2}$ және T уақыт аралығында болуы керек. Дегенмен, соқтығыс уақыттары біркелкі бөлінген кездейсоқ шама емес, бірақ пластинаның өзіне жақындау уақыттары біркелкі бөлінген, сондықтан $x = A$ жазықтықты қарастырамыз, жақындау уақыттары бірдей ықтимал. Платформамен соқтығысқан бөлшектің қозғалыс заңын сипаттайтын түзу жүргізейік $t = \frac{T}{2}$ уақытында (суретте – жуан сызық). Бұл бөлшек $x = A$ жазықтығымен кесіп өткен уақыт моментін t_1 деп белгілейік. Осы нүктеден кейін платформамен соқтығысқан бөлшектер жылдамдығы мен энергиясын арттырады. Бірақ бұл бөлшектер t_1 және T уақыт интервалында $x = A$ жазықтығын кесіп өтеді, сондықтан бұл бөлшектердің үлесі келесідей есептеледі

$$\eta = \frac{T - t_1}{T}.$$

(25)

Бұл бөлшектің қозғалыс заңынан t_1 уақыт моментін табу оңай

$$t_1 = \frac{T}{2} - \frac{2A}{u},$$

(26)

онда үдетілген бөлшектердің үлесі тең болады

$$\eta = \frac{T - t_1}{T} = \frac{1}{2} + \frac{2A}{uT} = \frac{1}{2} + \frac{V_0}{\pi u}.$$

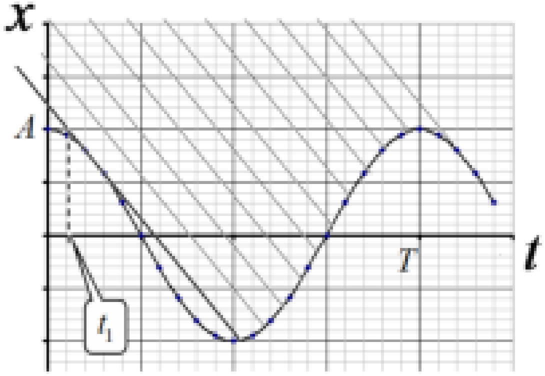
(27)

Мұнда (24) формуладан келетін қатынасты қолданамыз.: $\frac{2A}{T} = \frac{V_0}{\pi}$. Көрсетілген $u = 1.5V_0$ сандық мәнін ауыстырсақ, мынаны аламыз:

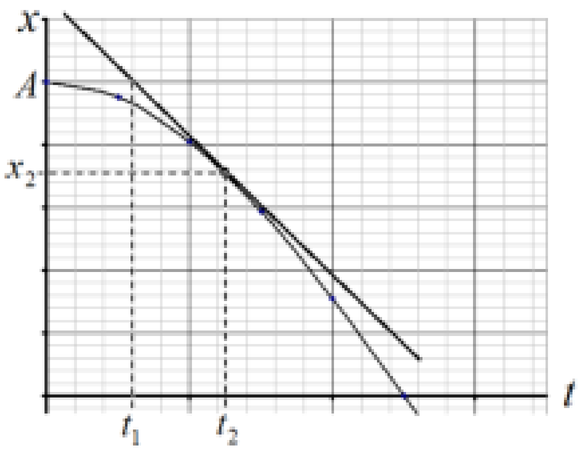
$$\eta = \frac{1}{2} + \frac{1}{1.5\pi} \approx 0.71.$$

(28)

Төмендегі суретте көрсетілген $u < V_0$-де біршама басқа жағдай орын алады.



Бұл жағдайда үдетілген және тежелген бөлшектер арасындағы t_1 «шектері» төмендегі суретте көрсетілгендей платформаның қозғалыс заңының графигіне жанама болатын түзу сызықпен анықталады.



Екі функцияның графиктері t_2 уақытында тиіп тұрғанда, функциялардың өздерінің де, олардың туындыларының да мәндері сәйкес келеді, яғни платформа мен шардың жылдамдықтары, сондықтан

$$-A\omega \sin(\omega t_2) = -u,$$

(29)

одан табамыз

$$t_2 = \frac{1}{\omega} \arcsin \frac{u}{A\omega} = \frac{T}{2\pi} \arcsin \frac{u}{V_0}.$$

(30)

$$x_2 = A\cos\omega t_2 = A\sqrt{1-\sin^2\omega t_2} = A\sqrt{1-\frac{u^2}{V_0^2}}.$$

(31)

Бұл өрнектер $x = A$ жазықтыққа жақындау уақытын анықтауға мүмкіндік береді

$$t_1 = t_2 - \frac{A - x_2}{u} = \frac{T}{2\pi}\left(\arcsin\frac{u}{V_0} - \frac{V_0}{u}\left(1-\sqrt{1-\frac{u^2}{V_0^2}}\right)\right).$$

(32)

Осы уақыттың тербеліс кезеңіне қатынасы платформамен соқтығысатын, оны қуып жететін бөлшектердің үлесін анықтайды, сондықтан олардың энергиясы төмендейді:

$$1-\eta = \frac{1}{2\pi}\left(\arcsin\frac{u}{V_0} - \frac{V_0}{u}\left(1-\sqrt{1-\frac{u^2}{V_0^2}}\right)\right) \approx 0.04,$$

(33)

сондықтан соққыдан кейін энергиясы өсетін бөлшектердің үлесі тең

$$\eta \approx 0.96.$$

(34)

2.10 Тербелістің бір кезеңінде платформа $4A$ жолымен жүреді, сондықтан оның жылдамдығының модулі тең,

$$V = \frac{4A}{T}.$$

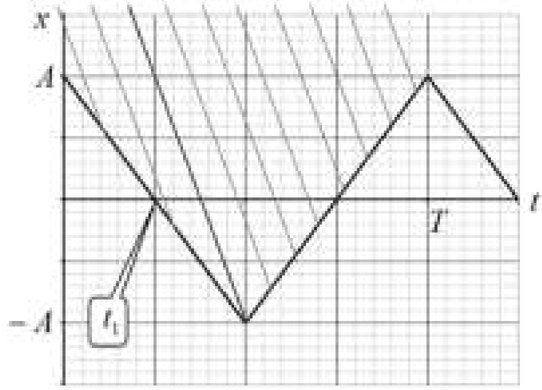
(35)

2.11 Шардың жылдамдығы платформаның жылдамдығынан үлкен болған кезде, соғу нәтижесінде энергиясы артқан шарлардың үлесі (27) формулаға ұқсас формула бойынша есептеледі:

$$\eta = \frac{T - t_1}{T} = \frac{1}{2} + \frac{2A}{uT} = \frac{1}{2} + \frac{V}{2u},$$

(36)

және сәйкес сурет төменде көрсетілген.



Платформа жылдамдығының модулі тұрақты деп қабылданғандықтан, соққыдан кейінгі бөлшектердің жылдамдығының модулі тең болады

$$u_+ = u + 2V.$$

(37)

0-ден t_1-ге дейінгі уақыт аралығында платформамен соқтығысқан бөлшектердің жылдамдықтары тең болады.

$$u_- = u - 2V.$$

(38)

Осылайша, соққыдан кейінгі бөлшектердің орташа энергиясы тең болады

$$E = \eta \frac{mu_+^2}{2} + (1-\eta)\frac{mu_-^2}{2} = \frac{m}{2}\left(\left(\frac{1}{2}+\frac{V}{2u}\right)(u+2V)^2 + \left(\frac{1}{2}-\frac{V}{2u}\right)(u-2V)^2\right) =$$

$$= \frac{mu^2}{4}\left(\left(1+\frac{V}{u}\right)\left(1+2\frac{V}{u}\right)^2 + \left(1-\frac{V}{u}\right)\left(1-2\frac{V}{u}\right)^2\right) = \frac{mu^2}{2}\left(1+8\left(\frac{V}{u}\right)^2\right),$$

(39)

және, демек, орташа энергияның ұлғаюы тең

$$\varepsilon = 1 + 8\left(\frac{V}{u}\right)^2 \approx 4.6 .$$

(40)

Бөлшектердің жылдамдығы платформаның жылдамдығынан аз болса, онда барлық бөлшектер платформамен кері бағытта соқтығысады, сондықтан барлық бөлшектер жылдамдығы мен энергиясын арттырады. Соқтығысқаннан кейін бөлшектердің жылдамдықтары $u_+ = u + 2V$ және олардың энергиясына тең болады

$$E = \frac{m}{2}(u+2V)^2 = \frac{mu^2}{2}\left(1+2\frac{V}{u}\right)^2,$$

(41)

және, демек, соқтығысудан кейінгі және соқтығысқа дейінгі бөлшектердің энергияларының қатынасы теңр

$$\varepsilon = \left(1+2\frac{V}{u}\right)^2 = 25.0 .$$

(42)

	Мазмұны	Ұпайлар	
2.1	Формула (2): $N_1 = n\Delta v t$	0.1	0.4
	Сандық мәні формула (2)-де: $N_1 \approx 0.83$	0.1	
	Формула (3): $\tau_1 = \frac{1}{n\Delta v}$	0.1	
	Сандық мәні формула (3)-те: $\tau_1 =$ [illegible] 72 с	0.1	
2.2	Формула (4): $N_2 = n\Delta v t$	0.1	0.4
	Сандық мәні формула (4)-те: $N_2 \approx 0.83$	0.1	
	Формула (5): $\tau_2 = \frac{1}{n\Delta v}$	0.1	
	Сандық мәні формула (5)-те: $\tau_2 =$ [illegible] 72 с	0.1	

2.3	Формулы (6): $N_{3,4} = n(2v \pm \Delta v)t$ $\tau_{3,4} = \dfrac{1}{n(2v \pm \Delta v)}$	0.4	0.8
	Сандық мәні формула (7)-де: $N_3 = 14.2; \quad \tau_3 = 4.2$ с; $N_3 = 15.8; \quad \tau_3 = 3.8$ с.	0.4	
2.4	Формула (8): $m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$	0.1	0.6
	Формула (9): $\dfrac{m_1v_1^2}{2} + \dfrac{m_2v_2^2}{2} = \dfrac{m_1u_1^2}{2} + \dfrac{m_2u_2^2}{2}$	0.1	
	Формула (13): $u_1 = \dfrac{m_1 - m_2}{m_1 + m_2}v_1 + \dfrac{2m_2}{m_1 + m_2}v_2$	0,2	
	Формула (14): $u_2 = \dfrac{2m_1}{m_1 + m_2}v_1 + \dfrac{m_2 - m_1}{m_1 + m_2}v_2$	0,2	
2.5	Формула (15): $\eta_1 = \dfrac{1-\mu}{1+\mu} + \dfrac{2\mu}{1+\mu}\eta_2$	0.2	1.6
	Диаграммада тек түзу сызықтар бар, әйтпесе диаграмма бағаланбайды	0.2	
	Барлық түзулер $\eta_1 = 1; \eta_2 = 1$ нүкте арқылы өтеді	0.4	
	$\eta_1 = 1$ түзу бар	0.2	
	$\eta_1 = -1 + 2\eta_2$ түзу бар	0.4	
	Барлық сызықтар $\eta_1 = 1$ және $\eta_1 = -1 + 2\eta_2$ арасында орналасқан	0,2	
2.6	Теңсіздік (7): $\lvert\eta_1\rvert > 1 \quad \Rightarrow \quad \begin{cases} \eta_1 > 1 \\ \eta_1 < -1 \end{cases}$	0.2	0.4
	Теңсіздік (19): $\eta_2 > 1$	0.1	
	Теңсіздік (20): $\eta_2 < -\dfrac{1}{\mu}$	0.1	
2.7	Формула (21): $\tilde{u}_1 = -v_1 + 2v_2$	0.1	0.3
	Теңсіздік а): $v_2 > 1$	0.1	
	Теңсіздік б): $v_2 < 0$	0.1	
2.8	Формула (22): $x(t) = A\cos(\omega t)$	0.1	0.4
	Формула (23): $v(t) = -A\omega\sin(\omega t)$	0.1	

	Формула (24): $V_0 = A\omega = 2\pi \dfrac{A}{T}$	0,2	
2.9	Формула (25): $\eta = \dfrac{T - t_1}{T}$	0.3	2.7
	Формула (26): $t_1 = \dfrac{T}{2} - \dfrac{2A}{u}$	0.3	
	Формула (27): $\eta = \dfrac{1}{2} + \dfrac{V_0}{\pi u}$	0.3	
	Сандық мәні формула (28)-де: $\eta \approx 0.71$	0.3	
	Формула (29): $-A\omega \sin(\omega t_2) = -u$	0.2	
	Формула (30): $t_2 = \dfrac{1}{\omega}\arcsin\dfrac{u}{A\omega} = \dfrac{T}{2\pi}\arcsin\dfrac{u}{V_0}$	0.2	
	Формула (31): $x_2 = A\sqrt{1-\dfrac{u^2}{V_0^2}}$	0.3	
	Формула (32): $t_1 = \dfrac{T}{2\pi}\left(\arcsin\dfrac{u}{V_0} - \dfrac{V_0}{u}\left(1-\sqrt{1-\dfrac{u^2}{V_0^2}}\right)\right)$	0.3	
	Формула (33): $1-\eta = \dfrac{1}{2\pi}\left(\arcsin\dfrac{u}{V_0} - \dfrac{V_0}{u}\left(1-\sqrt{1-\dfrac{u^2}{V_0^2}}\right)\right)$	0.3	
	Сандық мәні формула (34)-те: $\eta \approx 0.96$	0.2	
2.10	Формула (35): $V = \dfrac{4A}{T}$	0.2	0.2
2.11	Формула (36): $\eta = \dfrac{1}{2} + \dfrac{V}{2u}$	0.3	2.2
	Формула (37): $u_+ = u + 2V$	0.2	
	Формула (38): $u_- = u - 2V$	0.2	
	Формула (39): $E = \eta\dfrac{mu_+^2}{2} + (1-\eta)\dfrac{mu_-^2}{2}$	0.3	
	Формула (40): $\varepsilon = 1 + 8\left(\dfrac{V}{u}\right)^2$	0.3	
	Сандық мәні формула (40)-та: $\varepsilon \approx 4.6$	0.2	
	Формула (41): $E = \dfrac{m}{2}(u+2V)^2$	0.2	
	Формула (42): $\varepsilon = \left(1 + 2\dfrac{V}{u}\right)^2$	0.3	
	Сандық мәні формула (42)-де: $\varepsilon = 25.0$	0.2	
Барлығы			**10.0**

Тапсырма 3. Магнетрон

Электр және магнит өрістеріндегі электрондардың қозғалысы

3.1 Біртекті электр өрісінің әсерінен электрон тұрақты үдеумен қозғалады

$$a = \frac{eE}{m}, \tag{1}$$

ол x осінің теріс бағытына бағытталған, сондықтан қол жеткізілген координатаның максималды мәні өрнекпен анықталады

$$x_{max} = \frac{u_0^2}{2a} = \frac{mu_0^2}{2eE}. \tag{2}$$

3.2 Біртекті магнит өрісінде қозғалған кезде Лоренц күші электронға әсер етеді, тең

$$F_L = eu_0B. \tag{3}$$

және ол радиусы R Ньютонның екінші заңынан анықталған шеңбер бойымен қозғалады

$$m\frac{u_0^2}{R} = F_L, \tag{4}$$

Одан шығады

$$R = \frac{mu_0}{eB}. \tag{5}$$

Бұл жағдайда координатаның максималды мәні тең болатыны анық

$$x_{max} = R = \frac{mu_0}{eB}. \tag{6}$$

3.3 Есеп зертханалық анықтамалық жүйеде оңай шешіледі, онда электрон төменгі түрінде (5) формуламен анықталған жиілікпен шеңбер бойымен қозғалады.

$$\omega = \frac{u_0}{R} = \frac{eB}{m}. \tag{7}$$

Электронға шамалы қосымша жылдамдық берілгенде, ол бастапқыға жақын және онымен диаметральді қарама-қарсы екі нүктеде қиылысатын шеңбер бойымен қозғалады, бұл периодты жабық екі өлшемді траектория бойынша қозғалыс ретінде қарастырылуы мүмкін.

$$T = \frac{2\pi}{\omega} = \frac{2\pi m}{eB}. \tag{8}$$

3.4 Координат x максималды болған кезде, u бөлшегінің жылдамдығы z осі бойымен бағытталған және энергияның сақталу заңы бойынша тең

$$eEx_{max} = \frac{mu^2}{2}. \tag{9}$$

Қозғалыс теңдеуі z оське проекцияда шекті айырмашылықтар мына түрінде жазылады

$$m\frac{\Delta u_z}{\Delta t} = eBu_x, \tag{10}$$

$\Delta x = u_x \Delta t$ есебімен мына қатынасқа әкеледі

$$m\Delta u_z = eB\Delta x, \tag{11}$$

ол қажетті момент үшін мына түрді қабылдайды

$$mu = eBx_{max}$$.

(12)

(9) және (12) теңдеулерін бірге шешіп, біз соңында аламыз

$$x_{max} = \frac{2mE}{eB^2}$$,

(13)

3.5 Магнит өрісі жұмыс істемейтіндіктен, электронның жылдамдығы абсолютті мәнде тұрақты болып қалады және бастапқыға тең

$$u = u_0 = const$$.

(14)

Толық жылдамдықты $u_r = dr/dt$ радиалды және $u_\varphi = rd\varphi/dt$ азимуттық құраушыларға бөлейік. Электронның бастапқы нүктеге қатысты бұрыштық импульсі анық тең

$$L = mru_\varphi$$,

(15)

және сол нүктеге қатысты Лоренц күшінің моменті

$$M = eBu_r r$$.

(16)

Моменттер теңдеуі бойынша аламыз

$$\frac{dL}{dt} = M$$,

(17)

$u_r = dr/dt$ пайдаланумен мына қатынасқа алып келеді

$$d(mru_\varphi) = e\alpha r^2 dr$$.

(18)

Оське дейінгі қашықтық максимум болған уақытта радиалды жылдамдық жоғалады, ал азимуттық жылдамдық (14) формулаға сәйкес бастапқыға тең, сондықтан (18) қатынасты интегралдау теңдеуге әкеледі.

$$mr_{max}u_0 = e\alpha\frac{r_{max}^3}{3}$$,

(19)

одан шығады

$$r_{max} = \sqrt{\frac{3mu_0}{e\alpha}}$$.

(20)

3.6 Электрон барлық уақытта шеңбер бойымен қозғалатындықтан, (7) теңдеу бойынша оның орбитасындағы магнит өрісі B_0 артқан сайын импульстің туындысы заңға сәйкес өзгереді

$$\frac{dp}{dt} = er\frac{dB_0}{dt}$$.

(21)

Электрон құйынды электр өрісінің әсерінен қозғалады, оның қарқындылығы E қатынасымен анықталады

$$E = \frac{1}{2\pi r}\frac{d\Phi}{dt},$$

(22)

ол Фарадей заңы бойынша электрон орбитасы арқылы магнит индукциясының ағынын қамтиды, тең

$$\Phi = \int_0^r B(r)2\pi r dr.$$

(23)

Орбитадағы электронның үдеуіне арналған Ньютонның екінші заңының теңдеуі мынадай түрге ие

$$\frac{dp}{dt} = eE.$$

(24)

(21)-(24) теңдеулерінің бірлескен шешімі магнит өрісі үшін келесі теңдікке әкеледі, ол астындағы өрнек циклотрон шарты деп аталады

$$\int_0^r B(r)2\pi r dr = 2\pi r^2 B_0.$$

(25)

(25) формуладан оның орындалуы электрон индукциямен $B_0 = B_2$ магнит өрісі аймағында қозғалған жағдайда ғана мүмкін болады деген қорытындыға келеміз, сондықтан шартта берілген магнит индукциясын қашықтыққа байланысты интегралдасақ, қатынасты аламыз

$$B_1\pi r_1^2 + B_2\pi(r^2 - r_1^2) = 2\pi r^2 B_2,$$

(26)

оның шешімі мына түрінде болады

$$\frac{B_1}{B_2} = 1 + \frac{r^2}{r_1^2}.$$

(27)

Электронның шеңбер бойымен қозғалысы индукциясы B_2 тең болатын ауданда ғана мүмкін, яғни $r_1 < r < r_2$ кезінде, бұл қажетті қатынас интервалда жату керек дегенді білдіреді

$$2 < \frac{B_1}{B_2} < 1 + \frac{r_2^2}{r_1^2}.$$

(28)

Цилиндрлік магнетрон

3.7 Цилиндрлік катод пен анодтың ұзындық бірліктерінің заряды λ-ға тең болсын, ал электродтардың жалпы ұзындығы l. Сонда Гаусс теоремасы бойынша катод пен анод арасындағы кеңістіктегі электр өрісінің кернеулігі теңдеу арқылы анықталады.

$$E2\pi r l = \frac{\lambda l}{\varepsilon_0},$$

(29)

одан аламыз

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}.$$

(30)

мұнда r –магнетрон осіне дейінгі қашықтық.

Потенциалдар айырмасының r қашықтыққа тәуелділігі, анықтамасы бойынша интеграл ретінде жазылады

$$V = \int_a^r Edr = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r}{a},$$

(31)

жеке $r = b$ үшін береді

$$V_0 = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{a}.$$

(32)

(31) және (32) теңдеулерін бірге шешіп, аламыз

$$V = V_0\frac{\ln(r/a)}{\ln(b/a)} = 87.6.$$

(33)

3.8 Радиусы R жұқа сақинаны қарастырайық, ол арқылы j ток өтеді және сақина осінің нүктесіндегі магнит индукциясының шамасын есептеңіз, содан кейін ол z қашықтықта оның орталығы болады. Сақинаны $d\vec{l}$ кіші элементтерге бөлеміз, онда магнит индукциясы келесі Био-Саварра заңымен анықталады

$$d\vec{B} = \frac{\mu_0 j}{4\pi}\frac{d\vec{l}\times\vec{r}}{r^3},$$

(34)

онда $\vec{r}$ векторы ток $d\vec{l}$ элементінің орналасқан жерінен магниттік индукция ізделетін *O* нүктесіне дейін сызылады

Бұл геометриялық қатынастардан туындайды

$$d\vec{l}\times\vec{r} = dl\cdot r,$$

(35)

және алынған магнит индукциясы сақина осі бойымен бағытталғандықтан

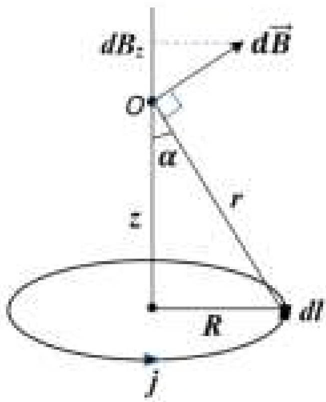
$$dB_z = dB\sin\alpha,$$

(36)

содан кейін $R = r\sin\alpha$ геометриялық қатынасын қолданып, ең соңында аламыз

$$dB_z = \frac{\mu_0 j}{4\pi}\frac{Rdl}{r^3}.$$

(37)



(37) формулаға енгізілген арақашықтықтар тұрақты және

$$r^2 = R^2 + z^2 ,$$

(38)

содан кейін сақинаның барлық элементтерін қосқаннан кейін табамыз

$$B_z = \frac{\mu_0 j}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} .$$

(39)

Енді соленоидтың центріндегі магнит өрісінің индукциясын есептейік, өйткені бұл жерде магнетрон шамы орналасқан. Ол үшін центрден z бастап $z + dz$ -қа дейінгі қашықтықта орналасқан бұрылыстарды қарастырайық, олар арқылы ток өтеді

$$dj = \frac{NI}{L} dz .$$

(40)

Бұл бұрылыстарды магнит индукциясы формула (39) бойынша анықталатын сақина ретінде қарастыруға болады, одан біз аламыз

$$dB = \frac{\mu_0 NI}{2L} \frac{R^2}{(R^2 + z^2)^{3/2}} dz ,$$

(41)

ол интеграциядан кейін соңғы өрнекті береді

$$B = \frac{\mu_0 NIR^2}{2L} \int_{-L/2}^{L/2} \frac{dz}{(R^2 + z^2)^{3/2}} = \frac{\mu_0 NI}{L\sqrt{1 + D^2 / L^2}} ,$$

(42)

мұнда $D = 2R$ диаметр үшін өрнек қолданылады.

Магнетрондағы электрондардың қозғалысы үшін (18) формулаға ұқсас және төменгі түріндегі формула жарамды

$$d(mru_\varphi) = eBrdr ,$$

(43)

тұрақты магниттік индукция жағдайында интегралдау береді

$$mru_\varphi = \frac{1}{2} eBr^2 .$$

(44)

Екінші жағынан, энергияның сақталу заңынан шығатыны

$$\frac{m}{2}(u_r^2 + u_\varphi^2) = eV .$$

(45)

Токтың критикалық мәніне жеткен кезде анодтың жанындағы магниттік индукция электрондардың радиалды жылдамдығы жойылатындай болады, бұл жағдайға әкеледі.

$$u_r = 0, \quad r = b, \quad V = V_0 ,$$

(46)

(44) және (45) өрнектерін пайдаланып, магнит өрісінің критикалық мәнін береді

$$B = \sqrt{\frac{8mV_0}{eb^2}} .$$

(47)

(42) формуланы пайдаланып, соленоидтағы сәйкес токты табамыз

$$I_{\min} = \sqrt{\frac{8mV_0}{e}\left(1 + D^2 / L^2\right)}\,\frac{L}{\mu_0 Nb} = 0,701$$.

(48)

3.9 Катодқа жақын шамдағы электрондардың бастапқы энергиясы катодтың температурасымен анықталады және

$$E_T = k_B T$$.

(49)

Бұл энергия анодтың жанындағы электрондардың энергиясынан әлдеқайда аз болуы керек E_0, яғни

$$E_T \ll E_0$$,

(50)

мұнда $E_0 = eV_0$, одан ізделінген бағаны аламыз

$$T \ll \frac{eE_0}{k_B} = \text{K}70 \cdot 10^5$$,

(51)

бұл шын мәнінде пайдаланылған жуықтауды қолдану мүмкіндігін білдіреді, өйткені катод температурасы әдетте кем дегенде екі рет төмен.

	Мазмұны	Ұпайлар	
3.1	Формула (1): $a = \frac{eE}{m}$	0.1	**0.2**
	Формула (2): $x_{\max} = \frac{mu_0^2}{2eE}$	0.1	
3.2	Формула (3): $F_L = eu_0B$	0.1	**0.4**
	Формула (4): $m\frac{u_0^2}{R} = F_L$	0.1	
	Формула (5): $R = \frac{mu_0}{eB}$	0.1	
	Формула (6): $x_{\max} = R = \frac{mu_0}{eB}$	0.1	
3.3	Формула (7): $\omega = \frac{u_0}{R} = \frac{eB}{m}$	0.2	**0.4**
	Формула (8): $T = \frac{2\pi}{\omega} = \frac{2\pi m}{eB}$	0.2	
3.4	Формула (9): $eEx_{\max} = \frac{mu^2}{2}$	0.2	**1.0**
	Формула (10): $m\frac{\Delta u_z}{\Delta t} = eBu_x$	0.2	
	Формула (11): $m\Delta u_z = eB\Delta x$	0.2	

	Формула (12): $mu = eBx_{\max}$	0.2	
	Формула (13): $x_{\max} = \frac{2mE}{eB^2}$	0.2	
3.5	Формула (14): $u = u_0 = const$	0.2	**1.4**
	Формула (15): $L = mru_\varphi$	0.2	
	Формула (16): $M = eBu_r r$	0.2	
	Формула (17): $\frac{dL}{dt} = M$	0.2	
	Формула (18): $d(mru_\varphi) = e\alpha r^2 dr$	0.2	
	Формула (19): $mr_{\max}u_0 = e\alpha \frac{r_{\max}^3}{3}$	0.2	
	Формула (20): $r_{\max} = \sqrt{\frac{3mu_0}{e\alpha}}$	0.2	
3.6	Формула (21): $\frac{dp}{at} = er\frac{dB_0}{dt}$	0.2	**1.6**
	Формула (22): $E = \frac{1}{2\pi r}\frac{d\Phi}{dt}$	0.2	
	Формула (23): $\Phi = \int_0^r B(r)2\pi r dr$	0.2	
	Формула (24): $\frac{dp}{dt} = eE$	0.2	
	Формула (25): $\int_0^r B(r)2\pi r dr = 2\pi r^2 B_0$	0.2	
	Формула (26): $B_1\pi r_1^2 + B_2\pi(r^2 - r_1^2) = 2\pi r^2 B_2$	0.2	
	Формула (27): $\frac{B_1}{B_2} = 1 + \frac{r^2}{r_1^2}$	0.2	
	Формула (28): $2 < \frac{B_1}{B_2} < 1 + \frac{r_2^2}{r_1^2}$	0.2	
3.7	Формула (29): $E2\pi rl = \frac{\lambda l}{\varepsilon_0}$	0.1	**1.0**
	Формула (30): $E = \frac{\lambda}{2\pi\varepsilon_0 r}$	0.1	
	Формула (31): $V = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r}{a}$	0.2	

	Формула (32): $V_0 = \frac{\lambda}{2\pi\varepsilon_0} \ln\frac{b}{a}$	0.2	
	Формула (33): $V = V_0 \frac{\ln(r/a)}{\ln(b/a)}$	0.2	
	Сандық мән формула (33)-де: $V = 87.6$	0.2	
3.8	Формула (34): $d\vec{B} = \frac{\mu_0 j}{4\pi} \frac{d\vec{l} \times \vec{r}}{r^3}$	0.2	3.2
	Формула (35): $d\vec{l} \times \vec{r} = dl \cdot r$	0.2	
	Формула (36): $dB_z = dB \sin\alpha$	0.2	
	Формула (37): $dB_z = \frac{\mu_0 j}{4\pi} \frac{Rdl}{r^3}$	0.2	
	Формула (38): $r^2 = R^2 + z^2$	0.2	
	Формула (39): $B_z = \frac{\mu_0 j}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}$	0.2	
	Формула (40): $dj = \frac{NI}{L} dz$	0.2	
	Формула (41): $dB = \frac{\mu_0 NI}{2L} \frac{R^2}{(R^2 + z^2)^{3/2}} dz$	0.2	
	Формула (42): $B = \frac{\mu_0 NI}{L\sqrt{1 + D^2 / L^2}}$	0.2	
	Формула (43): $d(mru_\varphi) = eBrdr$	0.2	
	Формула (44): $mru_\varphi = \frac{1}{2} eBr^2$	0.2	
	Формула (45): $\frac{m}{2}(u_r^2 + u_\varphi^2) = eV$	0.2	
	Формула (46): $u_r = 0, \quad r = b, \quad V = V_0$	0.2	
	Формула (47): $B = \sqrt{\frac{8mV_0}{eb^2}}$	0.2	
	Формула (48): $I_{min} = \sqrt{\frac{8mV_0}{e}(1 + D^2 / L^2)} \frac{L}{\mu_0 Nb}$	0.2	
	Сандық мән формула (48): $I_{min} = 0,701$	0.2	
3.9	Формула (49): $E_T = k_B T$	0.2	0.8
	Формула (50): $E_T \gg E_0$	0.2	
	Формула (51): $T \gg \frac{eE_0}{k_B}$	0.2	

	Сандық мән формула (51): T [illegible] $K70 \cdot 10^{5}$	0.2	
Барлығы			**10.0**

SOLUTIONS TO THE PROBLEMS OF THE THEORETICAL COMPETITION

Attention. Points in grading are not divided!

Problem 1 (10.0 points)

Problem 1.1 (3.0 points)

It follows from the first law of thermodynamics that

$$\delta Q = dU + dA, \tag{1}$$

where δQ is the amount of heat supplied, dU is the change in internal energy, dA is the work done by the gas.

For one mole of an ideal gas, these quantities can be written in terms of a change in volume dV and temperature dT at a known pressure p in the following form

$$\delta A = pdV, \tag{2}$$

$$dU = C_V dT. \tag{3}$$

By definition of heat capacity, we have

$$C = \frac{\delta Q}{dT}, \tag{4}$$

then from relations (1)-(4) one obtains

$$p\frac{dV}{dT} = C - C_V, \tag{5}$$

at the molar heat capacity of a monatomic gas at a constant volume equal to

$$C_V = \frac{3}{2}R. \tag{6}$$

From the graph given in the problem statement, it can be seen that at a temperature

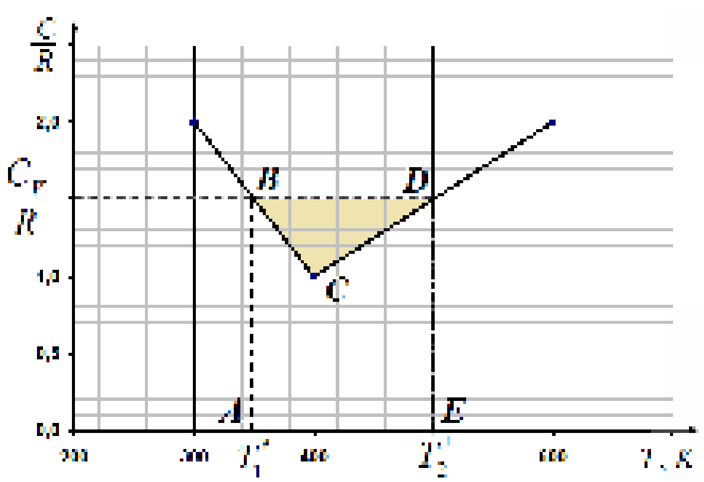
$$T_1^* = 350\ \text{K} \tag{7}$$

the heat capacity is $C = C_V$ and, accordingly, $\frac{dV}{dT} = 0$. When passing through this temperature, the derivative sign changes from plus to minus. This means that at this temperature the gas volume reaches a local maximum: $T_{\max} = T_1^* = 350\ \text{K}$.

At a temperature

$$T_2^* = 500\ \text{K} \tag{8}$$

the derivative $\frac{dV}{dT}$ also equals zero, and when passing through this point, the sign of the derivative changes from minus to plus. This means that T_2^* is the point of the local minimum of the volume: $T_{\min} = T_2^* = 500\ \text{K}$.



In the section from $T_1^* = 350\ \text{K}$ to $T_2^* = 500\ \text{K}$, the gas receives heat Q, numerically equal to the area under the dependence $C(T)$, i.e. the area of the figure ***ABCDE***. The change in internal energy $\Delta U = C_V(T_2^* - T_1^*)$ is numerically equal to the area of the rectangle ***ABDE***. According to the first law of thermodynamics, therefore, the work on the gas from T_1^* to T_2^* is numerically equal to the difference in the areas of the rectangle ***ABDE*** and the figure ***ABCDE***, i.e. area of the shaded figure ***BDC***:

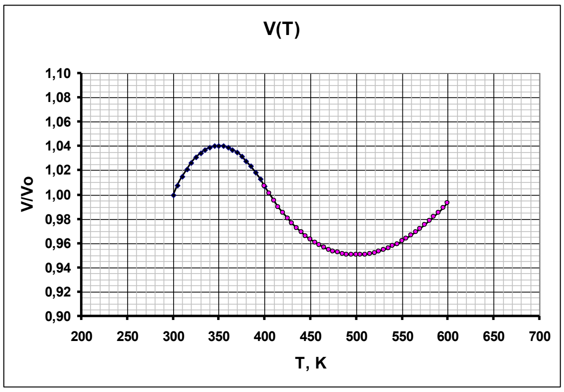
$$A = \frac{1}{4} R(T_{\max} - T_{\min}) = 312 \text{ J} . \qquad (9)$$

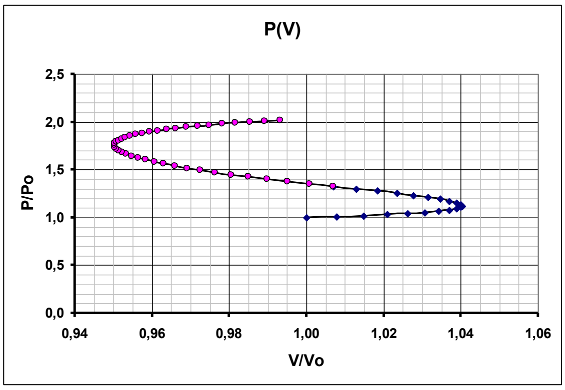
Note: Exact dependence $V(T)$:

$$\frac{V}{V_1} = \left(\frac{T}{T_1}\right)^{7/2} \exp\left(-\frac{T - T_1}{\Delta T_1}\right), \text{ at } T_1 = 300 \text{ K} \le T \le T_0 = 400 \text{ K and } \Delta T_1 = 100 \text{ K};$$

$$\frac{V}{V_0} = \left(\frac{T}{T_0}\right)^{-5/2} \exp\left(\frac{T - T_0}{\Delta T_2}\right), \text{ at } T_0 = 400 \text{ K} \le T \le T_2 = 600 \text{ K and } \Delta T_2 = 200 \text{ K}.$$

Dependences $V(T)$ and $P(V)$ in the process of gas heating are shown in the figures below.

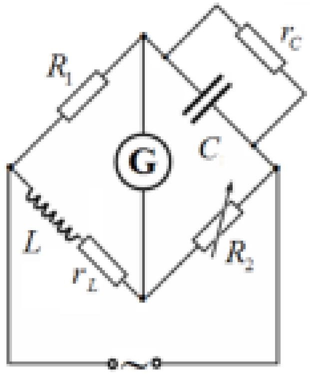




Content	Points
Formula (1): $\delta Q = dU + dA$	0.2
Formula (2): $\delta A = pdV$	0.2
Formula (3): $dU = C_V dT$	0.2
Formula (4): $C = \dfrac{\delta Q}{dT}$	0.2
Formula (5): $p\dfrac{dV}{dT} = C - C_V$	0.4
Formula (6): $C_V = \dfrac{3}{2}R$	0.2
Formula (7): $T_1^* = 350$ K	0.4
Formula (8): $T_2^* = 500$ K	0.4
Formula (9): $A = \dfrac{1}{4}R(T_{\max} - T_{\min})$	0.4
Numerical value in formula (9): $A = 312$ J	0.4
Total	**3.0**

Problem 1.2 (3.0 points)

The equivalent circuit of the bridge is shown in the figure below, which takes into account that the non-ideal inductance circuit is equivalent to an ideal coil L and resistor r_L connected in series, whereas the equivalent circuit of a leaky capacitor is a resistor r_C connected in parallel to an ideal capacitor C.



Solution 1. The bridge balance condition in complex numbers is written as

$$Z_L Z_C = R_1 R_2 , \tag{1}$$

where the impedances are respectively

$$Z_L = r_L + i\omega L \tag{2}$$

and

$$Z_C = \frac{r_C}{1+i\omega C r_C} . \tag{3}$$

After some transformation we get from expressions (1)-(3):

$$i\omega\left(L - R_1 R_2 C\right) = r_L - \frac{R_1 R_2}{r_C} . \tag{4}$$

While varying the frequency, this equality is not violated if both sides of the equation are equal to zero, therefore

$$C = \frac{L}{R_1 R_2} = 0.5\ \mu\text{F} , \tag{5}$$

$$r_C = \frac{R_1 R_2}{r_L} = 2\ \text{M}\Omega . \tag{6}$$

Solution 2. Let the voltage across the capacitor be

$$U_C = U_0 \cos \omega t , \tag{1}$$

then current through it is found as

$$I_C = -C\omega \sin \omega t , \tag{2}$$

and the current through its leakage resistance is

$$I_{r_C} = \frac{U_0 \cos \omega t}{r_C} . \tag{3}$$

The total current through the upper arm containing the capacitor is

$$I_1 = I_C + I_{r_C} , \tag{4}$$

and since the bridge is balanced, the same current flows through the resistance R_1, therefore

$$U_{R_1} = I_1 R_1 . \tag{5}$$

On the other hand, this voltage is equal to the voltage drop across the arm with the inductance

$$U_L = U_{R_1} , \tag{6}$$

for which the voltage drop is given by

$$U_L = L\frac{dI_2}{dt} + I_2 r_L , \tag{7}$$

in which the current is determined by the balance equation

$$I_2 = I_{R_2} = \frac{U_C}{R_2} . \tag{8}$$

since

$$U_{R_2} = U_C . \tag{9}$$

Collecting equations (1)-(9) together, we obtain

$$\left(-\frac{\omega L}{R_2}+C\omega R_1\right)U_0\sin\omega t=\left(\frac{R_1}{r_C}-\frac{r_L}{R_2}\right)U_0\cos\omega t\ . \tag{10}$$

It can be seen from this equality that the frequency-independent balance condition is satisfied if both sides of the equation are equal to zero, that is, one obtains the final answer

$$C=\frac{L}{R_1R_2}=0.5\ \mu\text{F}\,, \tag{11}$$

$$r_C=\frac{R_1R_2}{r_L}=2\ \text{M}\Omega\,. \tag{12}$$

Content	Points
Solution 1	
Equivalent circuit: All elements are correctly connected	0.5
Formula (1): $Z_LZ_C=R_1R_2$	0.3
Formula (2): $Z_L=r_L+i\omega L$	0.3
Formula (3): $Z_C=\dfrac{r_C}{1+i\omega Cr_C}$	0.3
Formula (4): $i\omega\left(L-R_1R_2C\right)=r_L-\dfrac{R_1R_2}{r_C}$	0.4
Formula (5): $C=\dfrac{L}{R_1R_2}$	0.4
Numerical value in formula (5): $C=0.5\ \mu\text{F}$	0.2
Formula (6): $r_C=\dfrac{R_1R_2}{r_L}$	0.4
Numerical value in formula (6): $r_C=2\ \text{M}\Omega$	0.2
Total	**3.0**
Solution 2	
Equivalent circuit: All elements are correctly connected	0.5
Formula (1): $U_C=U_0\cos\omega t$	0.1
Formula (2): $I_C=-C\omega\sin\omega t$	0.1
Formula (3): $I_{r_C}=\dfrac{U_0\cos\omega t}{r_C}$	0.1
Formula (4): $I_1=I_C+I_{r_C}$	0.1
Formula (5): $U_{R_1}=I_1R_1$	0.1
Formula (6): $U_L=U_{R_1}$	0.1
Formula (7): $U_L=\dfrac{dI_2}{dt}+I_2r_L$	0.1
Formula (8): $I_2=I_{R_2}=\dfrac{U_C}{R_2}$	0.1
Formula (9): $U_{R_2}=U_C$	0.1
Formula (10): $\left(-\dfrac{\omega L}{R_2}+C\omega R_1\right)U_0\sin\omega t=\left(\dfrac{R_1}{r_C}-\dfrac{r_L}{R_2}\right)U_0\cos\omega t$	0.4

Formula (11): $C = \dfrac{L}{R_1 R_2}$	0.4
Numerical value in formula (11): $C = 0.5\ \mu\text{F}$	0.2
Formula (12): $r_C = \dfrac{R_1 R_2}{r_L}$	0.4
Numerical value in formula (12): $r_C = 2\ \text{M}\Omega$	0.2
Total	**3.0**

Problem 1.3 (4.0 points)

Let a planet of mass m move around the Sun in a circular orbit of radius R with a speed v, then the equation of motion of the planet in the projection onto the radial direction is written as

$$\frac{mv^2}{R} = G\frac{mM_S}{R^2}, \tag{1}$$

which results in

$$v = \sqrt{G\frac{M_S}{R}}, \tag{2}$$

with G being the gravitational constant.

Writing formula (2) for Jupiter with the index *J* and Earth with the index *E*, we get after dividing

$$\frac{v_J}{v_E} = \sqrt{\frac{R_E}{R_J}}, \tag{3}$$

and, on the other hand, we have according to Kepler's third law for the ratio of rotation periods

$$\frac{T_E^2}{T_J^2} = \frac{R_E^3}{R_J^3}. \tag{4}$$

The motion of Jupiter cannot be detected with a spectrometer, but it can be done for the Sun, since it also moves around the center of mass of the Sun-Jupiter system. The speed of the Sun is easy to find from the expression

$$v_S = v_J\frac{M_J}{M_S}. \tag{5}$$

Since the Sun moves around the common center of mass of the system, and the observer is located in the same plane, according to the Doppler effect formula, the following condition is satisfied for detection

$$\frac{\Delta\lambda}{\lambda} = \frac{2v_S}{c}. \tag{6}$$

Putting together equations (3)-(6), we get the final answer

$$R_{\min} = \frac{M_S}{M_J}\left(\frac{T_J}{T_E}\right)^{1/3}\frac{c}{2v_E} = 1.20\cdot 10^7. \tag{7}$$

Such resolution is achievable for many modern spectrometers manufactured in different countries of the world.

Content	Points
Formula (1): $\dfrac{mv^2}{R} = G\dfrac{mM_S}{R^2}$	0.2
Formula (2): $v = \sqrt{G\dfrac{M_S}{R}}$	0.2

Formula (3): $\frac{v_J}{v_E}=\sqrt{\frac{R_E}{R_J}}$	0.2
Formula (4): $\frac{T_E^2}{T_J^2}=\frac{R_E^3}{R_J^3}$	0.4
Formula (5): $v_S=v_J\frac{M_J}{M_S}$	1.0
Formula (6): $\frac{\Delta\lambda}{\lambda}=\frac{2v_S}{c}$	1.0
Formula (7): $R_{\min}=\frac{M_S}{M_J}\left(\frac{T_J}{T_E}\right)^{1/3}\frac{c}{2v_E}$	0.5
Numerical value in formula (7): $R_{\min}=1.20\cdot10^7$	0.5
Total	**4.0**

Problem 2. Fermi acceleration (10.0 points)

Why are there more oncoming cars than overtaking cars?

2.1 Within the time period t, a car in lane ***B*** overtakes only those cars that are located at the distance no longer than

$$l=\left(v-\left(v-\Delta v\right)\right)t=\Delta v\,t\,. \tag{1}$$

Therefore, the number of those cars is

$$N_1=nl=n\Delta vt\approx0.83\,. \tag{2}$$

The time between overtakes is found as

$$\tau_1=\frac{1}{n\Delta v}=0.02\text{ h}=72\text{ s}\,. \tag{3}$$

2.2 Similar reasoning leads to the conclusion that the number of overtakes and the time between overtakes remain the same, i.e.

$$N_2=nl=n\Delta vt\approx0.83\,, \tag{4}$$

$$\tau_2=\frac{1}{n\Delta v}=0.02\text{ h}=72\text{ s}\,. \tag{5}$$

2.3 When driving towards oncoming cars, the number of cars and the time between two consecutive meetings are calculated by the formulas

$$\begin{aligned}&N_{3,4}=n\left(v+\left(v\pm\Delta v\right)\right)t=n\left(2v\pm\Delta v\right)t\\&\tau_{3,4}=\frac{1}{n\left(2v\pm\Delta v\right)}\end{aligned}\,, \tag{6}$$

and numerical calculations give the following values

$$\begin{aligned}&N_3=14.2;\quad \tau_3=4.2\text{ s};\\&N_3=15.8;\quad \tau_3=3.8\text{ s}.\end{aligned} \tag{7}$$

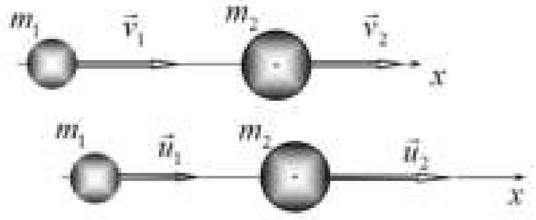
Elastic collision

2.4 Let us write down the momentum conservation law as

$$m_1v_1+m_2v_2=m_1u_1+m_2u_2 \tag{8}$$

together with the conservation of kinetic energy

$$\frac{m_1v_1^2}{2}+\frac{m_2v_2^2}{2}=\frac{m_1u_1^2}{2}+\frac{m_2u_2^2}{2}\,. \tag{9}$$



Rewriiting these equations in the following form

$$\begin{aligned} m_1v_1 - m_1u_1 &= m_2u_2 - m_2v_2 \\ m_1v_1^2 - m_1u_1^2 &= m_2u_2^2 - m_2v_2^2 \end{aligned} \tag{10}$$

and dividing then, yields the relation

$$v_1 + u_1 = u_2 + v_2 . \tag{11}$$

From this equality, we express $u_2 = v_1 + u_1 - v_2$ and substitute it into the equation of conservation of momentum

$$(m_1 + m_2)u_1 = (m_1 - m_2)v_1 + 2m_2v_2 , \tag{12}$$

from which it follows that

$$u_1 = \frac{m_1 - m_2}{m_1 + m_2} v_1 + \frac{2m_2}{m_1 + m_2} v_2 . \tag{13}$$

The speed of the second ball can be easily obtained by changing the indices "1" and "2" in formula (13)

$$u_2 = \frac{2m_1}{m_1 + m_2} v_1 + \frac{m_2 - m_1}{m_1 + m_2} v_2 . \tag{14}$$

2.5 Using formula (13), we obtain an explicit form of the dependence between the required parameters

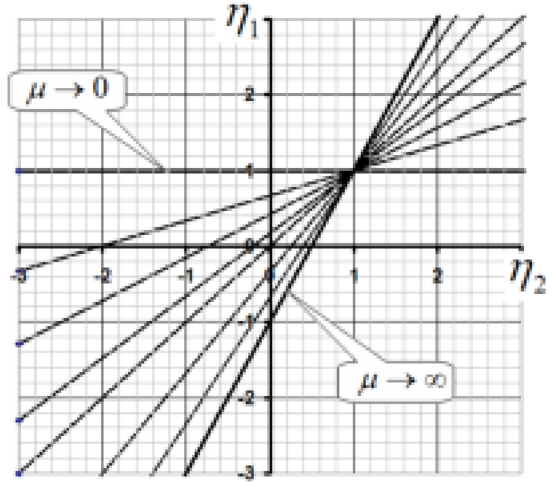
$$\frac{u_1}{v_1} = \frac{m_1 - m_2}{m_1 + m_2} + \frac{2m_2}{m_1 + m_2}\frac{v_2}{v_1} = \frac{1 - \frac{m_2}{m_1}}{1 - \frac{m_2}{m_1}} + \frac{2\frac{m_2}{m_1}}{1 + \frac{m_2}{m_1}}\frac{v_2}{v_1} \quad \Rightarrow \quad . \tag{15}$$

$$\eta_1 = \frac{1-\mu}{1+\mu} + \frac{2\mu}{1+\mu}\eta_2$$

As follows from the resulting expression, for any values of the mass ratio μ, the dependence is linear, i.e. its graph is a straight line. It is also not difficult to see that all these lines pass through the point $\eta_1 = 1; \eta_2 = 1$. When $\mu \to 0$, the slope coefficient tends to zero, that is, the dependence graph tends to a horizontal straight line $\eta_1 = 1$. At $\mu \to \infty$, the desired dependence tends to

$$\eta_1 = -1 + 2\eta_2 . \tag{16}$$

The set of graphs of function (15) is shown in the figure below.



2.6 Кинетическая энергия шарика увеличится, если модуль скорости шарика после удара станет больше модуля скорости до удара, то есть при выполнении неравенств The kinetic energy of the ball increases if the modulus of its velocity after the collision becomes greater than the modulus of its velocity before the collision, that is, if the following inequalities are fulfilled

$$|\eta_1| > 1 \quad \Rightarrow \quad \begin{cases} \eta_1 > 1 \\ \eta_1 < -1 \end{cases}. \tag{17}$$

Substituting expression (15) for the quantity η_1, we obtain the following two inequalities

$$\begin{cases} \dfrac{1-\mu}{1+\mu} + \dfrac{2\mu}{1+\mu}\eta_2 > 1 \\ \dfrac{1-\mu}{1+\mu} + \dfrac{2\mu}{1+\mu}\eta_2 < -1 \end{cases}. \tag{18}$$

The solutions of these inequalities are the following relations:

a)

$$\eta_2 > 1, \tag{19}$$

that is, to fulfill this condition, the second ball must catch up with the first one;;

b)

$$\eta_2 < -\frac{1}{\mu}, \tag{20}$$

in this case, the second ball must move towards the first one and the modulus of its velocity must exceed the above specified value.

2.7 In the limiting case $m_2 \gg m_1$, the speed of the first ball after the collision is

$$\tilde{u}_1 = -v_1 + 2v_2, \tag{21}$$

that is, the speed of the first ball changes sign (the ball is reflected) and its modulus changes to twice the speed of the second, heavy ball.

The light ball increases its speed, and, consequently, its kinetic energy, if:

a) the heavy ball catches up with the light ball (hit from behind) $v_2 > 1$;

б) the heavy ball moves towards the light ball $v_2 < 0$.

The simplest Fermi acceleration model

2.8 We write the law of motion of the plate in the traditional form

$$x(t) = A\cos(\omega t), \tag{22}$$

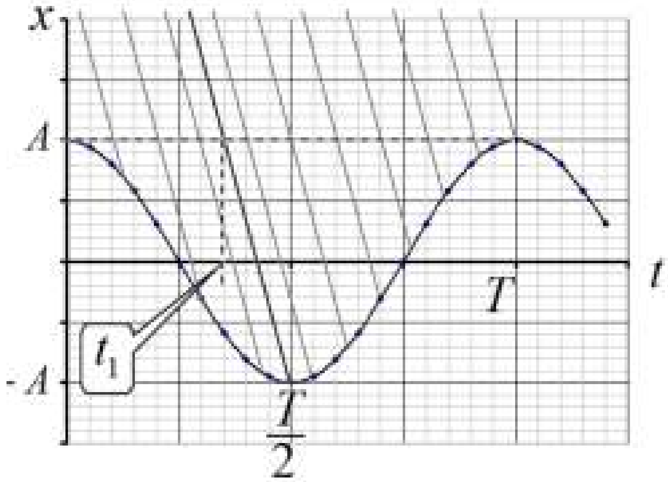
then the dependence of the velocity on time is described by the function

$$v(t) = -A\omega\sin(\omega t), \tag{23}$$

thus, the maximum speed of the platform is

$$V_0 = A\omega = 2\pi\frac{A}{T}. \tag{24}$$

2.9 To answer the question posted, it is enough to consider one period of plate oscillations. Let us plot the dependence of the plate coordinates on time (22) and plot on the same graph the dependences of the incoming particle coordinates on time, which are straight lines $x = x_0 - ut$.



The figure shows the case $u > V_0$. As a result of the collision, balls that collide with the plate increase their speed at those time moments when the plate moves towards the positive direction of the axis , while collisions must occur in the time interval from $\frac{T}{2}$ to T. However, the collision times are not randomly and uniformly distributed, but the times of approach to the plate itself are uniformly distributed, so we consider a plane $x = A$, the times of approach to which are equally probable. Let us draw a straight line that describes the law of motion of a ball colliding with the plate at the moment of time $t = \frac{T}{2}$ (the thick line in the figure). Let us denote t_1 as the moment of time when this ball crosses the plane $x = A$. Balls that collide with the plate after this moment of time increase their speed and energy. But these balls cross the plane in the time interval from t_1 to T, so the fraction of these particles is obtained as

$$\eta = \frac{T - t_1}{T}. \tag{25}$$

The moment of time t_1 is easy to find from the law of the ball motion

$$t_1 = \frac{T}{2} - \frac{2A}{u}, \tag{26}$$

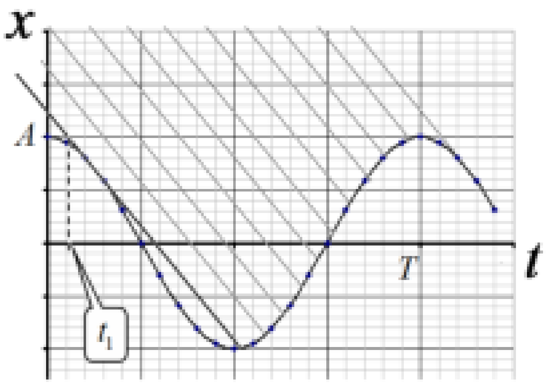
then the fraction of accelerated particles is equal to

$$\eta = \frac{T - t_1}{T} = \frac{1}{2} + \frac{2A}{uT} = \frac{1}{2} + \frac{V_0}{\pi u}. \tag{27}$$

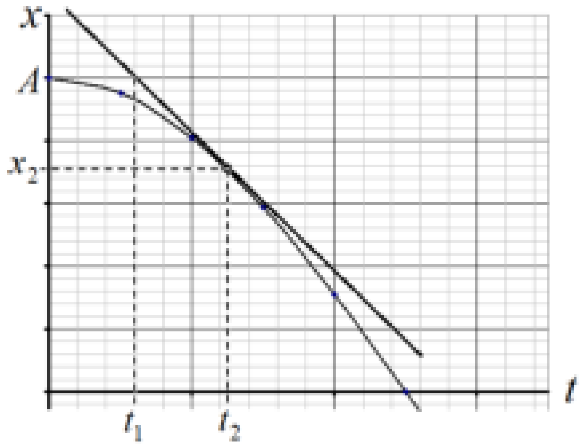
Here we use the relation that follows from formula (24): $\frac{2A}{T} = \frac{V_0}{\pi}$. Substituting the specified numerical value $u = 1.5V_0$, we get:

$$\eta = \frac{1}{2} + \frac{1}{1.5\pi} \approx 0.71. \tag{28}$$

A somewhat different situation is realized at $u < V_0$, which is shown in the figure below.



In this case, the "border time" t_1 between accelerated and decelerated balls is determined by a straight line, which is tangent to the graph of the plate law of motion, as shown in the figure below.



When the graphs of two functions touch at the moment of time t_2, the values of both functions themselves and their derivatives, that is, the speeds of the plate and the ball, coincide, therefore

$$-A\omega\sin(\omega t_2) = -u, \tag{29}$$

which gives rise to

$$t_2 = \frac{1}{\omega}\arcsin\frac{u}{A\omega} = \frac{T}{2\pi}\arcsin\frac{u}{V_0}. \tag{30}$$

$$x_2 = A\cos\omega t_2 = A\sqrt{1-\sin^2\omega t_2} = A\sqrt{1-\frac{u^2}{V_0^2}}. \tag{31}$$

These expressions allow us to determine the time of approach to the plane $x = A$

$$t_1 = t_2 - \frac{A - x_2}{u} = \frac{T}{2\pi}\left(\arcsin\frac{u}{V_0} - \frac{V_0}{u}\left(1-\sqrt{1-\frac{u^2}{V_0^2}}\right)\right). \tag{32}$$

The ratio of this time to the oscillation period determines the fraction of particles that collide with the plate, catching it up, such that their energy decreases:

$$1-\eta = \frac{1}{2\pi}\left(\arcsin\frac{u}{V_0} - \frac{V_0}{u}\left(1-\sqrt{1-\frac{u^2}{V_0^2}}\right)\right) \approx 0.04, \tag{33}$$

therefore, the fraction of balls whose energy increases after the collision is equal to

$$\eta \approx 0.96. \tag{34}$$

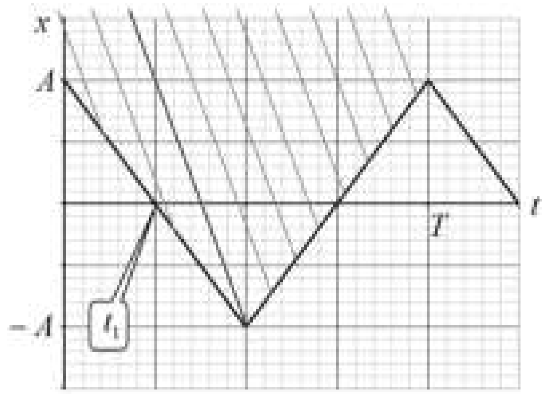
2.10 In one period of oscillation, the plate travels a path $4A$, so the modulus of its speed is equal to

$$V = \frac{4A}{T}. \tag{35}$$

2.11 When the ball speed is greater than the platform speed, the proportion of balls that increase their energy as a result of the collision is calculated by a formula similar to formula (27):

$$\eta = \frac{T - t_1}{T} = \frac{1}{2} + \frac{2A}{uT} = \frac{1}{2} + \frac{V}{2u}, \tag{36}$$

and the corresponding figure is shown below.



Since the modulus of the plate velocity is assumed to be constant, the ball velocity modulus after the impact becomes equal to

$$u_+ = u + 2V\,. \tag{37}$$

The velocities of balls that collide with the plate in the time interval from 0 to t_1, are equal to

$$u_- = u - 2V\,. \tag{38}$$

Thus, the average ball energy after the collision becomes equal to

$$\begin{aligned} E &= \eta\frac{mu_+^2}{2} + (1-\eta)\frac{mu_-^2}{2} = \frac{m}{2}\left(\left(\frac{1}{2}+\frac{V}{2u}\right)(u+2V)^2 + \left(\frac{1}{2}-\frac{V}{2u}\right)(u-2V)^2\right) = \\ &= \frac{mu^2}{4}\left(\left(1+\frac{V}{u}\right)\left(1+2\frac{V}{u}\right)^2 + \left(1-\frac{V}{u}\right)\left(1-2\frac{V}{u}\right)^2\right) = \frac{mu^2}{2}\left(1+8\left(\frac{V}{u}\right)^2\right) \end{aligned}, \tag{39}$$

and, consequently, the increase in the average energy is equal to

$$\varepsilon = 1 + 8\left(\frac{V}{u}\right)^2 \approx 4.6\,. \tag{40}$$

If the speed of the balls is less than the speed of the plate, then all the balls collide with the plate when it moves in the opposite direction, so all the balls increase their speed and energy. After the collision, the particle velocities become equal $u_+ = u + 2V$, and their energy

$$E = \frac{m}{2}(u+2V)^2 = \frac{mu^2}{2}\left(1+2\frac{V}{u}\right)^2, \tag{41}$$

and, consequently, the ratio of the ball energies after and before the collision is equal to

$$\varepsilon = \left(1+2\frac{V}{u}\right)^2 = 25.0\,. \tag{42}$$

	Content	Points	
2.1	Formula (2): $N_1 = n\Delta vt$	0.1	0.4
	Numerical value in formula (2): $N_1 \approx 0.83$	0.1	
	Formula (3): $\tau_1 = \dfrac{1}{n\Delta v}$	0.1	
	Numerical value in formula (3): $\tau_1 = 0.02\text{ h} = 72\text{ s}$	0.1	
2.2	Formula (4): $N_2 = n\Delta vt$	0.1	0.4
	Numerical value in formula (4): $N_2 \approx 0.83$	0.1	
	Formula (5): $\tau_2 = \dfrac{1}{n\Delta v}$	0.1	
	Numerical value in formula (5): $\tau_2 = 0.02\text{ h} = 72\text{ s}$	0.1	

2.3	Formulas (6): $N_{3,4} = n(2v \pm \Delta v)t$; $\tau_{3,4} = \dfrac{1}{n(2v \pm \Delta v)}$	0.4	**0.8**
	Numerical values in formula (7): $N_3 = 14.2;\quad \tau_3 = 4.2$ s; $N_3 = 15.8;\quad \tau_3 = 3.8$ s.	0.4	
2.4	Formula (8): $m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$	0.1	**0.6**
	Formula (9): $\dfrac{m_1v_1^2}{2} + \dfrac{m_2v_2^2}{2} = \dfrac{m_1u_1^2}{2} + \dfrac{m_2u_2^2}{2}$	0.1	
	Formula (13): $u_1 = \dfrac{m_1 - m_2}{m_1 + m_2} v_1 + \dfrac{2m_2}{m_1 + m_2} v_2$	0,2	
	Formula (14): $u_2 = \dfrac{2m_1}{m_1 + m_2} v_1 + \dfrac{m_2 - m_1}{m_1 + m_2} v_2$	0,2	
2.5	Formula (15): $\eta_1 = \dfrac{1-\mu}{1+\mu} + \dfrac{2\mu}{1+\mu}\eta_2$	0.2	**1.6**
	There are only straight lines on the graph, otherwise the graph is not graded	0.2	
	All lines pass through the point $\eta_1 = 1; \eta_2 = 1$	0.4	
	There is a straight line $\eta_1 = 1$	0.2	
	There is a straight line $\eta_1 = -1 + 2\eta_2$	0.4	
	All lines are located in between $\eta_1 = 1$ and $\eta_1 = -1 + 2\eta_2$	0,2	
2.6	Inequalities (7): $\lvert\eta_1\rvert > 1 \quad \Rightarrow \quad \begin{cases} \eta_1 > 1 \\ \eta_1 < -1 \end{cases}$	0.2	**0.4**
	Inequality (19): $\eta_2 > 1$	0.1	
	Inequality (20): $\eta_2 < -\dfrac{1}{\mu}$	0.1	
2.7	Formula (21): $\tilde{u}_1 = -v_1 + 2v_2$	0.1	**0.3**
	Inequality a): $v_2 > 1$	0.1	
	Inequality b): $v_2 < 0$	0.1	
2.8	Formula (22): $x(t) = A\cos(\omega t)$	0.1	**0.4**
	Formula (23): $v(t) = -A\omega\sin(\omega t)$	0.1	
	Formula (24): $V_0 = A\omega = 2\pi\dfrac{A}{T}$	0,2	
2.9	Formula (25): $\eta = \dfrac{T - t_1}{T}$	0.3	**2.7**
	Formula (26): $t_1 = \dfrac{T}{2} - \dfrac{2A}{u}$	0.3	
	Formula (27): $\eta = \dfrac{1}{2} + \dfrac{V_0}{\pi u}$	0.3	
	Numerical value in formula (28): $\eta \approx 0.71$	0.3	
	Formula (29): $-A\omega\sin(\omega t_2) = -u$	0.2	
	Formula (30): $t_2 = \dfrac{T}{2\pi}\arcsin\dfrac{u}{V_0}$	0.2	

	Formula (31): $x_2 = A\sqrt{1-\frac{u^2}{V_0^2}}$	0.3	
	Formula (32): $t_1 = \frac{T}{2\pi}\left(\arcsin\frac{u}{V_0} - \frac{V_0}{u}\left(1-\sqrt{1-\frac{u^2}{V_0^2}}\right)\right)$	0.3	
	Formula (33): $1-\eta = \frac{1}{2\pi}\left(\arcsin\frac{u}{V_0} - \frac{V_0}{u}\left(1-\sqrt{1-\frac{u^2}{V_0^2}}\right)\right)$	0.3	
	Numerical value in formula (34): $\eta \approx 0.96$	0.2	
2.10	Formula (35): $V = \frac{4A}{T}$	0.2	**0.2**
2.11	Formula (36): $\eta = \frac{1}{2} + \frac{V}{2u}$	0.3	**2.2**
	Formula (37): $u_+ = u + 2V$	0.2	
	Formula (38): $u_- = u - 2V$	0.2	
	Formula (39): $E = \eta\frac{mu_+^2}{2} + (1-\eta)\frac{mu_-^2}{2}$	0.3	
	Formula (40): $\varepsilon = 1 + 8\left(\frac{V}{u}\right)^2$	0.3	
	Numerical value in formula (40): $\varepsilon \approx 4.6$	0.2	
	Formula (41): $E = \frac{m}{2}(u+2V)^2$	0.2	
	Formula (42): $\varepsilon = \left(1+2\frac{V}{u}\right)^2$	0.3	
	Numerical value in formula (42): $\varepsilon = 25.0$	0.2	
Total			**10.0**

Problem 3. Magnetron

Electron motion in electric and magnetic fields

3.1 Under the action of a uniform electric field, an electron moves with a constant acceleration

$$a = \frac{eE}{m}, \tag{1}$$

which is directed in the negative direction of the x axis, so the maximum value of the achieved coordinate is determined by the expression

$$x_{\max} = \frac{u_0^2}{2a} = \frac{mu_0^2}{2eE}. \tag{2}$$

3.2 When moving in a uniform magnetic field, the Lorentz force acts on an electron, equal to

$$F_L = eu_0B. \tag{3}$$

and it moves in a circle whose radius R is determined from Newton's second law

$$m\frac{u_0^2}{R} = F_L, \tag{4}$$

which yeilds

$$R = \frac{mu_0}{eB}. \tag{5}$$

It is obvious that the maximum value of the coordinate in this case is equal to

$$x_{max} = R = \frac{mu_0}{eB}. \tag{6}$$

3.3 The problem is most easily solved in the laboratory reference frame, in which the electron moves along the circle with the frequency determined by formula (5) in the form

$$\omega = \frac{u_0}{R} = \frac{eB}{m}. \tag{7}$$

When an electron is given a small additional speed, it begins to move along a circle that is close to the original one and intersects with it at two diametrically opposite points, which can be considered as motion along a closed two-dimensional trajectory with the period

$$T = \frac{2\pi}{\omega} = \frac{2\pi m}{eB}. \tag{8}$$

3.4 В момент, когда координата x максимальна, скорость частицы u направлена вдоль оси z и по закону сохранения энергии равна At the moment when the coordinate x is maximum, the particle velocity u is directed along the z axis and, according to the law of conservation of energy, is equal to

$$eEx_{max} = \frac{mu^2}{2}. \tag{9}$$

In the projection onto the z axis, the equation of motion is written in finite differences in the form

$$m\frac{\Delta u_z}{\Delta t} = eBu_x, \tag{10}$$

which, with account of $\Delta x = u_x \Delta t$, leads to the relation

$$m\Delta u_z = eB\Delta x, \tag{11}$$

which for the time moment sought takes the form

$$mu = eBx_{max}. \tag{12}$$

Solving equations (9) and (12) simultaneously, we finally obtain

$$x_{max} = \frac{2mE}{eB^2}, \tag{13}$$

3.5 Since the magnetic field does not perform any work, the electron velocity remains constant in absolute value and equal to its initial value

$$u = u_0 = const. \tag{14}$$

Let us divide the total velocity into radial $u_r = dr/dt$ and $u_\varphi = rd\varphi/dt$ azimuthal components. The angular momentum of the electron relative to the origin is obviously equal to

$$L = mru_\varphi, \tag{15}$$

and the torque of the Lorentz force about the same point is

$$M = eBu_r r. \tag{16}$$

According to the moment equation, we have

$$\frac{dL}{dt} = M, \tag{17}$$

which together with the use of $u_r = dr/dt$ provides to the relation

$$d(mru_\varphi) = e\alpha r^2 dr. \tag{18}$$

At the moment of time when the distance to the z axis is maximum, the radial velocity vanishes, and the azimuthal velocity is equal to the initial one in accordance with formula (14), so the integration of relation (18) leads to the equation

$$mr_{max}u_0 = e\alpha\frac{r_{max}^3}{3}, \tag{19}$$

which finally gives rise to

$$r_{max} = \sqrt{\frac{3mu_0}{e\alpha}}. \tag{20}$$

3.6 Since the electron moves all the time along a circle, then, according to equation (5), with an increase in the magnetic field B_0 at its orbit, the derivative of the momentum changes according to the law

$$\frac{dp}{dt} = er\frac{dB_0}{dt}. \tag{21}$$

The electron is set in motion due to the vortex electric field, whose strength E is determined by the relation

$$E = \frac{1}{2\pi r}\frac{d\Phi}{dt}, \tag{22}$$

which, according to the Faraday law, includes the flux of magnetic induction through the electron orbit, equal to

$$\Phi = \int_0^r B(r)2\pi r dr. \tag{23}$$

The equation of Newton's second law for the acceleration of an electron in orbit has the form

$$\frac{dp}{dt} = eE. \tag{24}$$

The joint solution of equations (21)-(24) leads to the following equality for the magnetic field, which is called the cyclotron condition

$$\int_0^r B(r)2\pi r dr = 2\pi r^2 B_0. \tag{25}$$

From formula (25) we conclude that its satisfaction is possible only in the case when the electron moves in the region of a magnetic field with induction $B_0 = B_2$, therefore, integrating the magnetic induction given in the formulation as a function of distance, we obtain the relation

$$B_1\pi r_1^2 + B_2\pi(r^2 - r_1^2) = 2\pi r^2 B_2, \tag{26}$$

whose solution has the following form

$$\frac{B_1}{B_2} = 1 + \frac{r^2}{r_1^2}. \tag{27}$$

The motion of an electron in a circle is possible only in the area in which the induction is equal B_2, that is, at $r_1 < r < r_2$, which means that the ratio sought must lie in the interval

$$2 < \frac{B_1}{B_2} < 1 + \frac{r_2^2}{r_1^2}. \tag{28}$$

Cylindrical magnetron

3.7 Let the unit length of the cylindrical cathode and anode have a charge equal to λ, and the total length of the electrodes is l. Then, according to the Gauss theorem, the electric field strength in the space between the cathode and anode is determined by the equation

$$E2\pi rl = \frac{\lambda l}{\varepsilon_0}, \tag{29}$$

which immediately yields

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}. \tag{30}$$

Here r stands for the distance to the magnetron axes.

The dependence of the potential difference on the distance r, by definition, is written as an integral

$$V = \int_a^r Edr = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r}{a}, \tag{31}$$

which in particularly for $r = b$ gives rise to

$$V_0 = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{a}. \tag{32}$$

Solving equations (31) and (32) together, we obtain

$$V = V_0 \frac{\ln(r/a)}{\ln(b/a)} = 57.6 \text{ V}. \tag{33}$$

3.8 Рассмотрим тонкое кольцо радиуса R, по которому протекает ток j, и рассчитаем величину магнитной индукции в точке на оси кольца, отстоящей то его центра на расстоянии z. Разобьем кольцо на малые элементы dl, тогда магнитная индукция определяется следующим законом Био-Саварра Consider a thin ring of radius R, through which the current j flows, and calculate the magnitude of the magnetic induction at a point on the axis of the ring, which is located at a distance z from its center. Let us divide the ring into small elements dl, then the magnetic induction is determined by the following Biot-Savart law

$$d\bar{B} = \frac{\mu_0 j}{4\pi} \frac{d\bar{l} \times \bar{r}}{r^3}, \tag{34}$$

in which the vector $\dot{r}$ is drawn from the location of the current element dl to the point O where the magnetic induction is sought.

It follows from geometric relations that

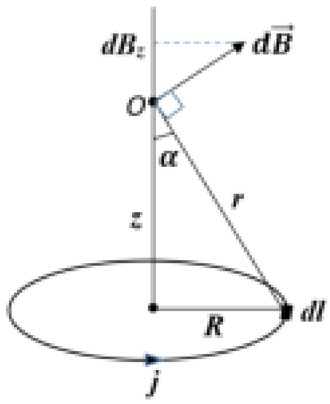
$$dl \times r = dl \cdot r, \tag{35}$$

and since the resulting magnetic induction is directed along the axis of the ring

$$dB_z = dB \sin\alpha, \tag{36}$$

then, using the geometric relation $R = r\sin\alpha$, we finally obtain

$$dB_z = \frac{\mu_0 j}{4\pi} \frac{Rdl}{r^3}. \tag{37}$$



Considering that the distances included in formula (37) are constant and

$$r^2 = R^2 + z^2, \tag{38}$$

then after summing over all elements of the ring one finds

$$B_z = \frac{\mu_0 j}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}. \tag{39}$$

Let us now calculate the magnetic field induction at the center of the solenoid, since this is where the magnetron lamp is located. To do this, consider the turns located at a distance from the center from z to $z + dz$, through which the current flows

$$dj = \frac{NI}{L} dz. \tag{40}$$

These turns can be considered as a ring, whose magnetic induction is determined by formula (39), such that

$$dB = \frac{\mu_0 NI}{2L} \frac{R^2}{(R^2 + z^2)^{3/2}} dz, \tag{41}$$

which after integration gives the final expression

$$B = \frac{\mu_0 NIR^2}{2L} \int_{-L/2}^{L/2} \frac{dz}{(R^2 + z^2)^{3/2}} = \frac{\mu_0 NI}{L\sqrt{1 + D^2/L^2}}, \tag{42}$$

where the expression $D = 2R$ is used for the diameter.

For the motion of electrons in a magnetron, a formula is valid that is similar to formula (18) and has the form

$$d(mru_\varphi) = eBrdr\,, \tag{43}$$

whose integration under the conditions of constant magnetic induction and $a << b$ gives

$$mru_\varphi = \frac{1}{2}eBr^2\,. \tag{44}$$

On the other hand, it follows from the law of conservation of energy that

$$\frac{m}{2}(u_r^2 + u_\varphi^2) = eV\,. \tag{45}$$

At the moment when the critical value of the current is reached, the magnetic induction near the anode becomes such that the radial velocity of the electrons vanishes, which leads to the conditions

$$u_r = 0, \quad r = b, \quad V = V_0\,, \tag{46}$$

which, using expressions (44) and (45), results in the critical value of the magnetic field

$$B = \sqrt{\frac{8mV_0}{eb^2}}\,. \tag{47}$$

Using formula (42), we find the corresponding current in the solenoid

$$I_{\min} = \sqrt{\frac{8mV_0}{e}(1 + D^2/L^2)}\,\frac{L}{\mu_0 Nb} = 0{,}701\ \text{A}\,. \tag{48}$$

3.9 The initial energy of electrons in a lamp near the cathode is determined by the temperature of the cathode itself and is on the order of

$$E_T = k_B T\,. \tag{49}$$

This energy is obviously must be much less than the energy of electrons near the anode, i.e.

$$E_T \ll E_0\,, \tag{50}$$

where $E_0 = eV_0$, whence we obtain the desired estimate

$$T \ll \frac{eE_0}{k_B} = 8.70 \cdot 10^5\ \text{K}\,, \tag{51}$$

which actually means the applicability of the approximation used, since the cathode temperature is usually at least two orders of magnitude lower.

	Content	Points	
3.1	Formula (1): $a = \frac{eE}{m}$	0.1	**0.2**
	Formula (2): $x_{\max} = \frac{mu_0^2}{2eE}$	0.1	
3.2	Formula (3): $F_L = eu_0B$	0.1	**0.4**
	Formula (4): $m\frac{u_0^2}{R} = F_L$	0.1	
	Formula (5): $R = \frac{mu_0}{eB}$	0.1	
	Formula (6): $x_{\max} = R = \frac{mu_0}{eB}$	0.1	
3.3	Formula (7): $\omega = \frac{u_0}{R} = \frac{eB}{m}$	0.2	**0.4**
	Formula (8): $T = \frac{2\pi}{\omega} = \frac{2\pi m}{eB}$	0.2	

3.4	Formula (9): $eEx_{\max} = \frac{mu^2}{2}$	0.2	**1.0**
	Formula (10): $m\frac{\Delta u_z}{\Delta t} = eBu_x$	0.2	
	Formula (11): $m\Delta u_z = eB\Delta x$	0.2	
	Formula (12): $mu = eBx_{\max}$	0.2	
	Formula (13): $x_{\max} = \frac{2mE}{eB^2}$	0.2	
3.5	Formula (14): $u = u_0 = const$	0.2	**1.4**
	Formula (15): $L = mru_\varphi$	0.2	
	Formula (16): $M = eBu_r r$	0.2	
	Formula (17): $\frac{dL}{dt} = M$	0.2	
	Formula (18): $d(mru_\varphi) = e\alpha r^2 dr$	0.2	
	Formula (19): $mr_{\max}u_0 = e\alpha\frac{r_{\max}^3}{3}$	0.2	
	Formula (20): $r_{\max} = \sqrt{\frac{3mu_0}{e\alpha}}$	0.2	
3.6	Formula (21): $\frac{dp}{dt} = er\frac{dB_0}{dt}$	0.2	**1.6**
	Formula (22): $E = \frac{1}{2\pi r}\frac{d\Phi}{dt}$	0.2	
	Formula (23): $\Phi = \int_0^r B(r)2\pi r dr$	0.2	
	Formula (24): $\frac{dp}{dt} = eE$	0.2	
	Formula (25): $\int_0^r B(r)2\pi r dr = 2\pi r^2 B_0$	0.2	
	Formula (26): $B_1\pi r_1^2 + B_2\pi(r^2 - r_1^2) = 2\pi r^2 B_2$	0.2	
	Formula (27): $\frac{B_1}{B_2} = 1 + \frac{r^2}{r_1^2}$	0.2	
	Formula (28): $2 < \frac{B_1}{B_2} < 1 + \frac{r_2^2}{r_1^2}$	0.2	
3.7	Formula (29): $E2\pi rl = \frac{\lambda l}{\varepsilon_0}$	0.1	**1.0**
	Formula (30): $E = \frac{\lambda}{2\pi\varepsilon_0 r}$	0.1	
	Formula (31): $V = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r}{a}$	0.2	
	Formula (32): $V_0 = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{a}$	0.2	

	Formula (33): $V = V_0 \dfrac{\ln(r/a)}{\ln(b/a)}$	0.2	
	Numerical value in formula (33): $V = 57.6$ V	0.2	
3.8	Formula (34): $d\bar{B} = \dfrac{\mu_0 j}{4\pi} \dfrac{d\bar{l} \times \bar{r}}{r^3}$	0.2	**3.2**
	Formula (35): $dl \times r = dl \cdot r$	0.2	
	Formula (36): $dB_z = dB \sin\alpha$	0.2	
	Formula (37): $dB_z = \dfrac{\mu_0 j}{4\pi} \dfrac{Rdl}{r^3}$	0.2	
	Formula (38): $r^2 = R^2 + z^2$	0.2	
	Formula (39): $B_z = \dfrac{\mu_0 j}{2} \dfrac{R^2}{(R^2 + z^2)^{3/2}}$	0.2	
	Formula (40): $dj = \dfrac{NI}{L} dz$	0.2	
	Formula (41): $dB = \dfrac{\mu_0 NI}{2L} \dfrac{R^2}{(R^2 + z^2)^{3/2}} dz$	0.2	
	Formula (42): $B = \dfrac{\mu_0 NI}{L\sqrt{1 + D^2 / L^2}}$	0.2	
	Formula (43): $d(mru_\varphi) = eBrdr$	0.2	
	Formula (44): $mru_\varphi = \dfrac{1}{2} eBr^2$	0.2	
	Formula (45): $\dfrac{m}{2}(u_r^2 + u_\varphi^2) = eV$	0.2	
	Formula (46): $u_r = 0, \quad r = b, \quad V = V_0$	0.2	
	Formula (47): $B = \sqrt{\dfrac{8mV_0}{eb^2}}$	0.2	
	Formula (48): $I_{\min} = \sqrt{\dfrac{8mV_0}{e}(1 + D^2 / L^2)} \dfrac{L}{\mu_0 Nb}$	0.2	
	Numerical value in formula (48): $I_{\min} = 0{,}701$ A	0.2	
3.9	Formula (49): $E_T = k_B T$	0.2	**0.8**
	Formula (50): $E_T \ll E_0$	0.2	
	Formula (51): $T \ll \dfrac{eE_0}{k_B}$	0.2	
	Numerical value in formula (51): $T \ll 8.70 \cdot 10^5$ K	0.2	
Total			**10.0**

РЕШЕНИЕ ЗАДАЧ ТЕОРЕТИЧЕСКОГО ТУРА

Внимание: баллы в оценках не делятся!

Задача 1 (10.0 балла)

Задача 1.1 (4.0 балла)

Из первого начала термодинамики следует, что

$$\delta Q = dU + dA, \qquad (1)$$

где δQ представляет собой количество подводимого тепла, dU – изменение внутренней энергии, dA – совершенная газом работа.

Для одного моля идеального газа можно записать эти величины через изменение объема dV и температуры dT при известном давлении p в следующем виде

$$\delta A = pdV, \qquad (2)$$

$$dU = C_V dT. \qquad (3)$$

По определению теплоемкости имеем

$$C = \frac{\delta Q}{dT}, \qquad (4)$$

тогда из соотношений (1)-(4) получаем

$$p\frac{dV}{dT} = C - C_V, \qquad (5)$$

при молярной теплоемкости одноатомного газа при постоянном объеме равной

$$C_V = \frac{3}{2}R. \qquad (6)$$

Из приведенного в условии графика видно, что при температуре

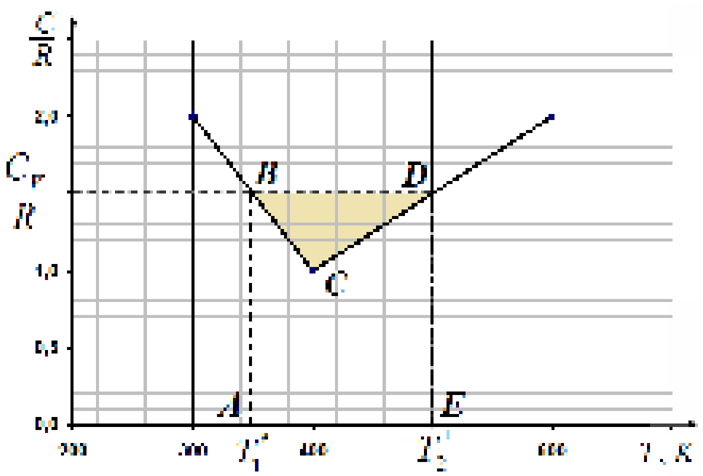
$$T_1^* = 350 \text{ К} \qquad (7)$$

теплоемкость $C = C_V$ и, соответственно, $\frac{dV}{dT} = 0$. При переходе через эту температуру знак производной $\frac{dV}{dT}$ изменяется с плюса на минус. Значит, при этой температуре объём газа достигает локального максимума: $T_{\max} = T_1^* = 350$ К.

При температуре

$$T_2^* = 500 \text{ К} \qquad (8)$$

производная $\frac{dV}{dT}$ также равняется нулю, а при переходе через эту точку знак ее производной изменяется с минуса на плюс. Это означает, что T_2^* является точкой локального минимума объёма: $T_{\min} = T_2^* = 500$ К.



На участке от $T_1^* = 350$ К до $T_2^* = 500$ К газ получает теплоту Q, численно равную площади под зависимостью $C(T)$, то есть площади фигуры ***ABCDE***. Изменение внутренней энергии $\Delta U = C_V(T_2^* - T_1^*)$ численно равно площади прямоугольника ***ABDE***. По первому началу термодинамики $A = \Delta U - Q$, поэтому работа над газом от T_1^* до T_2^* численно равна разности площадей прямоугольника ***ABDE*** и фигуры ***ABCDE***, т.е. площади заштрихованной фигуры ***BDC***:

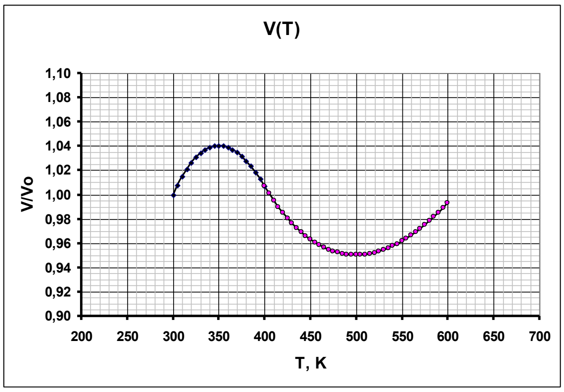
$$A = \frac{1}{4} R(T_2^* - T_1^*) = 312 \text{ Дж}. \qquad (9)$$

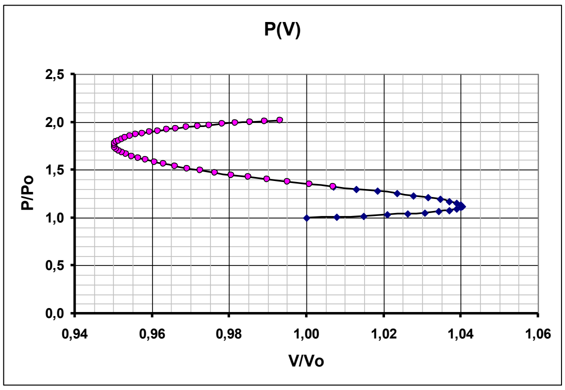
Дополнение: Явная зависимость $V(T)$:

$$\frac{V}{V_1} = \left(\frac{T}{T_1}\right)^{7/2} \exp\left(-\frac{T - T_1}{\Delta T_1}\right), \text{ при } T_1 = 300 \text{ К} \le T \le T_0 = 400 \text{ К и } \Delta T_1 = 100 \text{ К};$$

$$\frac{V}{V_0} = \left(\frac{T}{T_0}\right)^{-5/2} \exp\left(\frac{T - T_0}{\Delta T_2}\right), \text{ при } T_0 = 400 \text{ К} \le T \le T_2 = 600 \text{ К и } \Delta T_2 = 200 \text{ К}.$$

Зависимости $V(T)$ и $P(V)$ в процессе нагрева газа показаны на рисунках ниже.

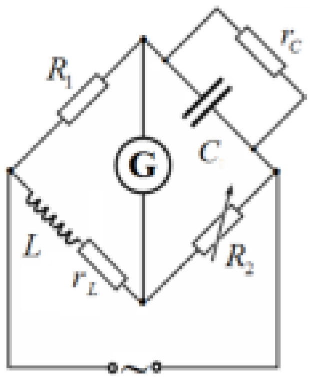




Содержание	Баллы
Формула (1): $\delta Q = dU + dA$	0.2
Формула (2): $\delta A = pdV$	0.2
Формула (3): $dU = C_V dT$	0.2
Формула (4): $C = \frac{\delta Q}{dT}$	0.2
Формула (5): $p\frac{dV}{dT} = C - C_V$	0.4
Формула (6): $C_V = \frac{3}{2}R$	0.2
Формула (7): $T_1^* = 350$ К	0.4
Формула (8): $T_2^* = 500$ К	0.4
Формула (9): $A = \frac{1}{4}R(T_2^* - T_1^*)$	0.4
Численное значение в формуле (9): $A = 312$ Дж	0.4
Итого	**3.0**

Задача 1.2 (3.0 балла)

Эквивалентная схема моста показана на рисунке ниже, на котором учтено, что эквивалентная схема неидеальной индуктивности – это последовательно соединенные идеальная катушка L и резистор r_L; эквивалентная схема конденсатора с утечкой – это резистор r_C параллельно подсоединённый к идеальному конденсатору C.



Решение 1. Условие баланса моста в комплексных числах записывается в виде

$$Z_L Z_C = R_1 R_2\,, \tag{1}$$

где импедансы равны соответственно

$$Z_L = r_L + i\omega L \tag{2}$$

и

$$Z_C = \frac{r_C}{1 + i\omega C r_C}\,. \tag{3}$$

После преобразований из выражений (1)-(3) получаем:

$$i\omega\left(L - R_1 R_2 C\right) = r_L - \frac{R_1 R_2}{r_C}\,. \tag{4}$$

При изменении частоты это равенство не нарушается, если обе части уравнения равны нулю, поэтому

$$C = \frac{L}{R_1 R_2} = 0.5\ \text{мкФ}\,, \tag{5}$$

$$r_C = \frac{R_1 R_2}{r_L} = 2\ \text{МОм}\,. \tag{6}$$

Решение 2. Пусть напряжение на конденсаторе равно

$$U_C = U_0 \cos\omega t\,, \tag{1}$$

тогда через него протекает ток

$$I_C = -CU_0\omega \sin\omega t\,, \tag{2}$$

а ток через его сопротивление утечки составляет

$$I_{r_C} = \frac{U_0 \cos\omega t}{r_C}\,. \tag{3}$$

Полный ток через плечо, содержащее конденсатор, равен

$$I_1 = I_C + I_{r_C}\,, \tag{4}$$

а так как мост сбалансирован, то такой же ток идёт через сопротивление R_1, поэтому

$$U_{R_1} = I_1 R_1\,. \tag{5}$$

С другой стороны, это напряжение равно падению напряжения на плече с индуктивностью

$$U_L = U_{R_1}\,, \tag{6}$$

для которой падение напряжения определяется выражением

$$U_L = L\frac{dI_2}{dt} + I_2 r_L\,, \tag{7}$$

в котором ток определяется уравнением балланса

$$I_2 = I_{R_2} = \frac{U_C}{R_2}\,. \tag{8}$$

так как

$$U_{R_2} = U_C\,. \tag{9}$$

Собирая совместно уравнения (1)-(9), получаем

$$\left(-\frac{\omega L}{R_2}+C\omega R_1\right)U_0\sin\omega t=\left(\frac{R_1}{r_C}-\frac{r_L}{R_2}\right)U_0\cos\omega t\ . \qquad (10)$$

Из этого равенства видно, что условие баланса, независящего от частоты, выполняется, если обе части уравнения равны нулю, то есть получаем ответ

$$C=\frac{L}{R_1R_2}=0.5\ \text{мкФ}\ , \qquad (11)$$

$$r_C=\frac{R_1R_2}{r_L}=2\ \text{МОм}\ . \qquad (12)$$

Содержание	**Баллы**
Решение 1	
Эквивалентная схема: все элементы расположены правильно	0.5
Формула (1): $Z_LZ_C=R_1R_2$	0.3
Формула (2): $Z_L=r_L+i\omega L$	0.3
Формула (3): $Z_C=\dfrac{r_C}{1+i\omega Cr_C}$	0.3
Формула (4): $i\omega\left(L-R_1R_2C\right)=r_L-\dfrac{R_1R_2}{r_C}$	0.4
Формула (5): $C=\dfrac{L}{R_1R_2}$	0.4
Численное значение в формуле (5): $C=0.5$ мкФ	0.2
Формула (6): $r_C=\dfrac{R_1R_2}{r_L}$	0.4
Численное значение в формуле (6): $r_C=2$ МОм	0.2
Итого	**3.0**
Решение 2	
Эквивалентная схема: все элементы расположены правильно	0.5
Формула (1): $U_C=U_0\cos\omega t$	0.1
Формула (2): $I_C=-CU_0\omega\sin\omega t$	0.1
Формула (3): $I_{r_C}=\dfrac{U_0\cos\omega t}{r_C}$	0.1
Формула (4): $I_1=I_C+I_{r_C}$	0.1
Формула (5): $U_{R_1}=I_1R_1$	0.1
Формула (6): $U_L=U_{R_1}$	0.1
Формула (7): $U_L=\dfrac{dI_2}{dt}+I_2r_L$	0.1
Формула (8): $I_2=I_{R_2}=\dfrac{U_C}{R_2}$	0.1
Формула (9): $U_{R_2}=U_C$	0.1
Формула (10): $\left(-\dfrac{\omega L}{R_2}+C\omega R_1\right)U_0\sin\omega t=\left(\dfrac{R_1}{r_C}-\dfrac{r_L}{R_2}\right)U_0\cos\omega t$	0.4

Формула (11): $C=\frac{L}{R_1R_2}$	0.4
Численное значение в формуле (11): $C=0.5$ мкФ	0.2
Формула (12): $r_C=\frac{R_1R_2}{r_L}$	0.4
Численное значение в формуле (12): $r_C=2$ МОм	0.2
Итого	**3.0**

Задача 1.3 (4.0 балла)

Пусть планета массы m движется вокруг Солнца по круговой орбите радиуса R со скоростью v, тогда уравнение движения планеты в проекции на радиальное направление записывается в виде

$$\frac{mv^2}{R}=G\frac{mM_S}{R^2}\,, \tag{1}$$

откуда

$$v=\sqrt{G\frac{M_S}{R}}\,, \tag{2}$$

где G – гравитационная постоянная.

Записывая формулу (2) для Юпитера с индексом J и Земли с индексом E, после деления получаем

$$\frac{v_J}{v_E}=\sqrt{\frac{R_E}{R_J}}\,, \tag{3}$$

а с другой стороны по третьему закону Кеплера для отношения периодов вращения T_E и T_J имеем

$$\frac{T_E^2}{T_J^2}=\frac{R_E^3}{R_J^3}\,. \tag{4}$$

Движение Юпитера нельзя обнаружить с помощью спектрометра, зато это можно сделать для Солнца, так как оно тоже двигается вокруг центр масс системы Солнце-Юпитер. Скорость Солнца легко найти из выражения

$$v_S=v_J\frac{M_J}{M_S}\,. \tag{5}$$

Так как Солнце двигается вокруг общего центра масс системы, а наблюдатель расположен в этой же плоскости, то согласно формуле эффекта Доплера при обнаружении выполняется следующее условие

$$\frac{\Delta\lambda}{\lambda}=\frac{2v_S}{c}\,. \tag{6}$$

Собирая вместе уравнения (3)-(6), получаем окончательный ответ

$$R_{\min}=\frac{M_S}{M_J}\left(\frac{T_J}{T_E}\right)^{1/3}\frac{c}{2v_E}=1.20\cdot10^7\,. \tag{7}$$

Такая разрешающая способность доступна многим современным спектрометрам, производимым в разных странах мира.

Содержание	Баллы
Формула (1): $\frac{mv^2}{R}=G\frac{mM_S}{R^2}$	0.2
Формула (2): $v=\sqrt{G\frac{M_S}{R}}$	0.2

Формула (3): $\frac{v_J}{v_E}=\sqrt{\frac{R_E}{R_J}}$	0.2
Формула (4): $\frac{T_E^2}{T_J^2}=\frac{R_E^3}{R_J^3}$	0.4
Формула (5): $v_S=v_J\frac{M_J}{M_S}$	1.0
Формула (6): $\frac{\Delta\lambda}{\lambda}=\frac{2v_S}{c}$	1.0
Формула (7): $R_{\min}=\frac{M_S}{M_J}\left(\frac{T_J}{T_E}\right)^{1/3}\frac{c}{2v_E}$	0.5
Численное значение в формуле (7): $R_{\min}=1.20\cdot10^7$	0.5
Итого	**4.0**

Задача 2. Ускорение Ферми (10.0 балла)

Почему встречных машин больше, чем попутных?

2.1 За время t автомобиль в полосе ***B*** обгонит только те автомобили, которые находятся от него на расстоянии

$$l=\left(v-\left(v-\Delta v\right)\right)t=\Delta v\,t\,. \tag{1}$$

Следовательно, число этих автомобилей равно

$$N_1=nl=n\Delta vt\approx 0.83\,. \tag{2}$$

Время между обгонами составляет

$$\tau_1=\frac{1}{n\Delta v}=0.02\text{ час}=72\text{ с}\,. \tag{3}$$

2.2 Аналогичные рассуждения приводят к выводу, что число обгонов и время между обгонами остаются прежними, то есть

$$N_2=nl=n\Delta vt\approx 0.83\,, \tag{4}$$

$$\tau_2=\frac{1}{n\Delta v}=0.02\text{ час}=72\text{ с}\,. \tag{5}$$

2.3 При движении навстречу число встречных автомобилей и время между двумя последовательными встречами рассчитываются по формулам

$$\begin{gathered}N_{3,4}=n\left(v+\left(v\pm\Delta v\right)\right)t=n\left(2v\pm\Delta v\right)t\\ \tau_{3,4}=\frac{1}{n\left(2v\pm\Delta v\right)}\end{gathered}\,, \tag{6}$$

а численные расчеты дают следующие значения

$$\begin{gathered}N_3=14.2;\quad \tau_3=4.2\text{ с};\\ N_3=15.8;\quad \tau_3=3.8\text{ с}.\end{gathered} \tag{7}$$

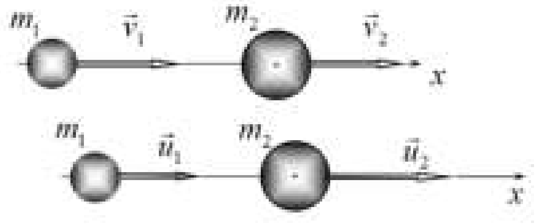
Упругое столкновение

2.4 Запишем закон сохранения импульса

$$m_1v_1+m_2v_2=m_1u_1+m_2u_2 \tag{8}$$

и закон сохранения механической энергии

$$\frac{m_1v_1^2}{2}+\frac{m_2v_2^2}{2}=\frac{m_1u_1^2}{2}+\frac{m_2u_2^2}{2}\,. \tag{9}$$



Перепишем эти уравнения в виде

$$\begin{aligned} m_1v_1 - m_1u_1 &= m_2u_2 - m_2v_2 \\ m_1v_1^2 - m_1u_1^2 &= m_2u_2^2 - m_2v_2^2 \end{aligned} \qquad (10)$$

и, разделив второе уравнение на первое, в результате получим

$$v_1 + u_1 = u_2 + v_2 . \qquad (11)$$

Из этого равенства выразим $u_2 = v_1 + u_1 - v_2$ и подставим в уравнение закона сохранения импульса

$$(m_1 + m_2)u_1 = (m_1 - m_2)v_1 + 2m_2v_2 , \qquad (12)$$

из которого следует

$$u_1 = \frac{m_1 - m_2}{m_1 + m_2}v_1 + \frac{2m_2}{m_1 + m_2}v_2 . \qquad (13)$$

Скорость второго шарика легко получить, если поменять индексы «1» и «2» в формуле (13)

$$u_2 = \frac{2m_1}{m_1 + m_2}v_1 + \frac{m_2 - m_1}{m_1 + m_2}v_2 . \qquad (14)$$

2.5 Используя формулу (13), получим явный вид зависимости между требуемыми параметрами

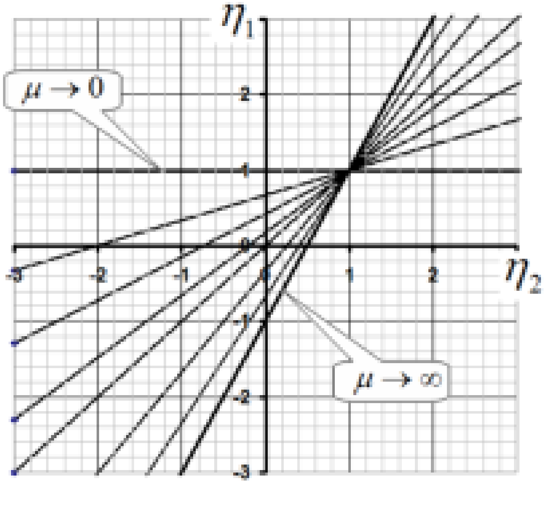
$$\frac{u_1}{v_1} = \frac{m_1 - m_2}{m_1 + m_2} + \frac{2m_2}{m_1 + m_2}\frac{v_2}{v_1} = \frac{1 - \frac{m_2}{m_1}}{1 - \frac{m_2}{m_1}} + \frac{2\frac{m_2}{m_1}}{1 + \frac{m_2}{m_1}}\frac{v_2}{v_1} \quad \Rightarrow \qquad (15)$$

$$\eta_1 = \frac{1-\mu}{1+\mu} + \frac{2\mu}{1+\mu}\eta_2$$

Как следует из полученного выражения, при любых значениях отношения масс μ зависимость $\eta_1(\eta_2)$ является линейной, т.е. ее графиком является прямая линия. Также не сложно заметить, что все эти прямые проходят через точку $\eta_1 = 1; \eta_2 = 1$. При $\mu \to 0$ коэффициент наклона стремится к нулю $\eta_1 = 1$, то есть график зависимости стремится к горизонтальной прямой. При $\mu \to \infty$ искомая зависимость стремится к

$$\eta_1 = -1 + 2\eta_2 . \qquad (16)$$

Семейство графиков этих функций показано на рисунке ниже.



2.6 Кинетическая энергия шарика увеличится, если модуль скорости шарика после удара станет больше модуля скорости до удара, то есть при выполнении неравенств

$$|\eta_1| > 1 \quad \Rightarrow \quad \begin{cases} \eta_1 > 1 \\ \eta_1 < -1 \end{cases}. \tag{17}$$

Подставляя выражение (15) для величины η_1, получим неравенства

$$\begin{cases} \dfrac{1-\mu}{1+\mu} + \dfrac{2\mu}{1+\mu}\eta_2 > 1 \\ \dfrac{1-\mu}{1+\mu} + \dfrac{2\mu}{1+\mu}\eta_2 < -1 \end{cases}. \tag{18}$$

Решением этих неравенств являются следующие соотношения:

а)

$$\eta_2 > 1, \tag{19}$$

то есть для выполнения этого условия второй шарик должен догонять первый;

б)

$$\eta_2 < -\frac{1}{\mu}, \tag{20}$$

в этом случае второй шарик должен двигаться навстречу и модуль его скорости должен превышать указанное значение.

2.7 В предельном случае $m_2 \ggg m_1$ скорость первого шарика после столкновения равна

$$\tilde{u}_1 = -v_1 + 2v_2, \tag{21}$$

то есть скорость первого шарика изменяет знак (шарик отражается) и его модуль изменяется на удвоенную скорость второго, тяжелого шарика.

Легкий шарик увеличит свою скорость, а, следовательно, и кинетическую энергию, если:

а) тяжелый шарик его догоняет (удар сзади) $v_2 > 1$;

б) тяжелый шарик движется ему навстречу $v_2 < 0$.

Простейшая модель ускорения Ферми

2.8 Запишем закон движения плиты в традиционном виде

$$x(t) = A\cos(\omega t), \tag{22}$$

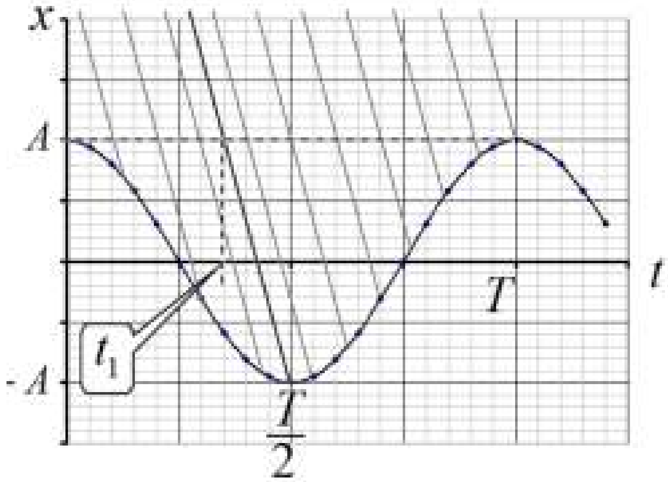
тогда зависимость скорости от времени описывается функцией

$$v(t) = -A\omega\sin(\omega t), \tag{23}$$

поэтому максимальная скорость движения платформы равна

$$V_0 = A\omega = 2\pi\frac{A}{T}. \tag{24}$$

2.9 Чтобы ответить на поставленный вопрос, достаточно рассмотреть один период колебаний платформы. Построим график зависимости координаты платформы от времени (22) и нанесем на него графики зависимостей координат налетающих частиц от времени, представляющих собой прямые линии $x = x_0 - ut$.



На рисунке изображен случай $u > V_0$. Свою скорость в результате удара увеличат частицы, которые столкнутся с платформой в те моменты времени, когда платформа движется навстречу в положительном направлении оси x, при этом столкновения должны произойти в интервале времени от $\frac{T}{2}$ до T. Однако времена столкновений не являются равномерно распределенной случайной величиной, а равномерно распределены времена подлета к самой пластине, поэтому рассмотрим плоскость $x = A$, времена подлета к которой равновероятны. Проведем прямую, описывающую закон движения частицы, сталкивающуюся с платформой в момент времени $t = \frac{T}{2}$ (на рисунке – жирная линия). Обозначим t_1 как момент времени, когда эта частица пересекает плоскость $x = A$. Частицы, которые столкнулись с платформой после этого момента, увеличат свою скорость и энергию. Но эти частицы пересекут плоскость $x = A$ в интервале времен от t_1 до T, поэтому доля этих частиц рассчитывается как

$$\eta = \frac{T - t_1}{T}. \tag{25}$$

Момент времени t_1 легко найти из закона движения этой частицы

$$t_1 = \frac{T}{2} - \frac{2A}{u}, \tag{26}$$

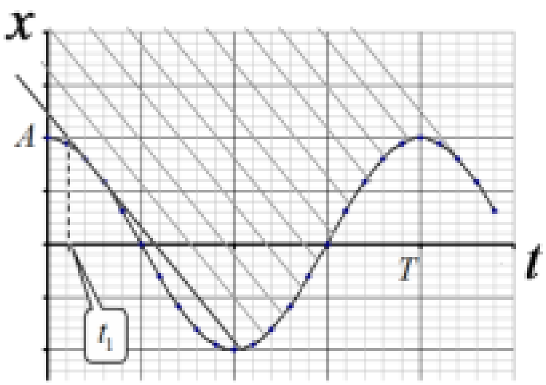
тогда доля ускорившихся частиц равна

$$\eta = \frac{T - t_1}{T} = \frac{1}{2} + \frac{2A}{uT} = \frac{1}{2} + \frac{V_0}{\pi u}. \tag{27}$$

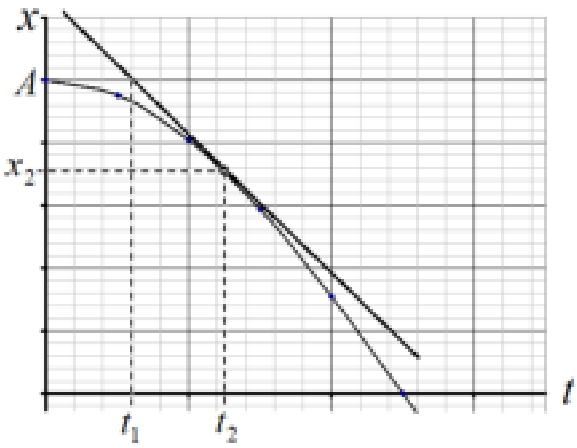
Здесь использовано соотношение, следующее из формулы (24): $\frac{2A}{T} = \frac{V_0}{\pi}$. Подставим указанное численное значение $u = 1.5V_0$, получим:

$$\eta = \frac{1}{2} + \frac{1}{1.5\pi} \approx 0.71. \tag{28}$$

Несколько иная ситуация реализуется при $u < V_0$, которая показана на рисунке ниже.



В этом случае «граница» t_1 между частицами ускорившимися и затормозившимися определяется прямой, которая является касательной к графику закона движения платформы, как показано на рисунке ниже.



При касании графиков двух функций в момент времени t_2 совпадают значения как самих функций, так и их производных, то есть скоростей движения платформы и шарика, поэтому

$$-A\omega\sin(\omega t_2) = -u, \tag{29}$$

откуда находим

$$t_2 = \frac{1}{\omega}\arcsin\frac{u}{A\omega} = \frac{T}{2\pi}\arcsin\frac{u}{V_0}. \tag{30}$$

$$x_2 = A\cos\omega t_2 = A\sqrt{1-\sin^2\omega t_2} = A\sqrt{1-\frac{u^2}{V_0^2}}. \tag{31}$$

Эти выражения позволяют определить время подлета к плоскости $x = A$

$$t_1 = t_2 - \frac{A - x_2}{u} = \frac{T}{2\pi}\left(\arcsin\frac{u}{V_0} - \frac{V_0}{u}\left(1-\sqrt{1-\frac{u^2}{V_0^2}}\right)\right). \tag{32}$$

Отношение этого времени к периоду колебаний определяет долю частиц, которые столкнутся с платформой, догоняя ее, поэтому их энергия уменьшится:

$$1-\eta = \frac{1}{2\pi}\left(\arcsin\frac{u}{V_0} - \frac{V_0}{u}\left(1-\sqrt{1-\frac{u^2}{V_0^2}}\right)\right) \approx 0.04, \tag{33}$$

следовательно, доля частиц, энергия которых увеличиться после удара, равна

$$\eta \approx 0.96. \tag{34}$$

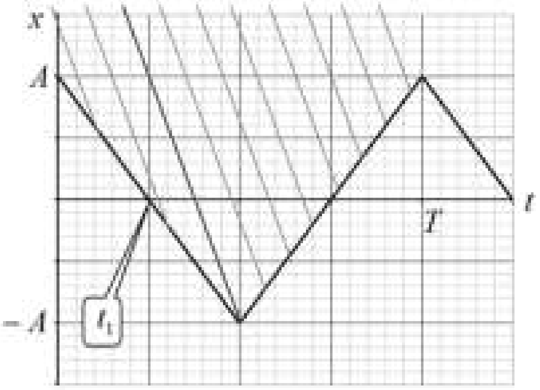
2.10 За один период колебаний платформа проходит путь $4A$, поэтому модуль ее скорости равен

$$V = \frac{4A}{T}. \tag{35}$$

2.11 При скорости шарика больше скорости платформы доля шариков, увеличивших свою энергию в следствие удара, рассчитывается по формуле, аналогичной формуле (27):

$$\eta = \frac{T - t_1}{T} = \frac{1}{2} + \frac{2A}{uT} = \frac{1}{2} + \frac{V}{2u}, \tag{36}$$

а соответствующий рисунок показан ниже.



Так как модуль скорости платформы принимается постоянным, то модуль скорости частиц после удара станет равным

$$u_+ = u + 2V. \tag{37}$$

Скорости частиц, которые столкнуться с платформой в интервале времени от 0 до t_1, будут равны

$$u_- = u - 2V. \tag{38}$$

Таким образом, средняя энергия частиц после удара станет равной

$$\begin{aligned} E &= \eta \frac{mu_+^2}{2} + (1-\eta)\frac{mu_-^2}{2} = \frac{m}{2}\left(\left(\frac{1}{2} + \frac{V}{2u}\right)(u+2V)^2 + \left(\frac{1}{2} - \frac{V}{2u}\right)(u-2V)^2\right) = \\ &= \frac{mu^2}{4}\left(\left(1+\frac{V}{u}\right)\left(1+2\frac{V}{u}\right)^2 + \left(1-\frac{V}{u}\right)\left(1-2\frac{V}{u}\right)^2\right) = \frac{mu^2}{2}\left(1+8\left(\frac{V}{u}\right)^2\right) \end{aligned}, \tag{39}$$

а, следовательно, увеличение средней энергии равно

$$\varepsilon = 1 + 8\left(\frac{V}{u}\right)^2 \approx 4.6. \tag{40}$$

Если, скорость частиц меньше скорости платформы, то все частицы столкнутся с платформой на встречном движении, поэтому все частицы увеличат свою скорость и энергию. После столкновения скорости частиц станут равными $u_+ = u + 2V$, а их энергия

$$E = \frac{m}{2}(u+2V)^2 = \frac{mu^2}{2}\left(1 + 2\frac{V}{u}\right)^2, \tag{41}$$

а, следовательно, отношение энергий частиц после и до столкновения равно

$$\varepsilon = \left(1 + 2\frac{V}{u}\right)^2 = 25.0. \tag{42}$$

<table>
<tr><th></th><th>Содержание</th><th colspan="2">Баллы</th></tr>
<tr><td rowspan="4">2.1</td><td>Формула (2): $N_1 = n\Delta v t$</td><td>0.1</td><td rowspan="4">0.4</td></tr>
<tr><td>Численное значение в формуле (2): $N_1 \approx 0.83$</td><td>0.1</td></tr>
<tr><td>Формула (3): $\tau_1 = \frac{1}{n\Delta v}$</td><td>0.1</td></tr>
<tr><td>Численное значение в формуле (3): $\tau_1 = 0.02$ час $= 72$ с</td><td>0.1</td></tr>
<tr><td rowspan="2">2.2</td><td>Формула (4): $N_2 = n\Delta v t$</td><td>0.1</td><td rowspan="2">0.4</td></tr>
<tr><td>Численное значение в формуле (4): $N_2 \approx 0.83$</td><td>0.1</td></tr>
</table>

	Формула (5): $\tau_2 = \dfrac{1}{n\Delta v}$	0.1	
	Численное значение в формуле (5): $\tau_2 = 0.02$ час $= 72$ с	0.1	
2.3	Формулы (6): $N_{3,4} = n(2v \pm \Delta v)t$; $\tau_{3,4} = \dfrac{1}{n(2v \pm \Delta v)}$	0.4	**0.8**
	Численное значение в формуле (7): $N_3 = 14.2; \quad \tau_3 = 4.2$ с; $N_3 = 15.8; \quad \tau_3 = 3.8$ с.	0.4	
2.4	Формула (8): $m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$	0.1	**0.6**
	Формула (9): $\dfrac{m_1v_1^2}{2} + \dfrac{m_2v_2^2}{2} = \dfrac{m_1u_1^2}{2} + \dfrac{m_2u_2^2}{2}$	0.1	
	Формула (13): $u_1 = \dfrac{m_1 - m_2}{m_1 + m_2}v_1 + \dfrac{2m_2}{m_1 + m_2}v_2$	0,2	
	Формула (14): $u_2 = \dfrac{2m_1}{m_1 + m_2}v_1 + \dfrac{m_2 - m_1}{m_1 + m_2}v_2$	0,2	
2.5	Формула (15): $\eta_1 = \dfrac{1-\mu}{1+\mu} + \dfrac{2\mu}{1+\mu}\eta_2$	0.2	**1.6**
	На графике только прямые линии, иначе график не оценивается	0.2	
	Все прямые проходят через точку $\eta_1 = 1; \eta_2 = 1$	0.4	
	Имеется прямая $\eta_1 = 1$	0.2	
	Имеется прямая $\eta_1 = -1 + 2\eta_2$	0.4	
	Все прямые расположены между $\eta_1 = 1$ и $\eta_1 = -1 + 2\eta_2$	0,2	
2.6	Неравенства (7): $\lvert\eta_1\rvert > 1 \quad \Rightarrow \quad \begin{cases}\eta_1 > 1 \\ \eta_1 < -1\end{cases}$	0.2	**0.4**
	Неравенство (19): $\eta_2 > 1$	0.1	
	Неравенство (20): $\eta_2 < -\dfrac{1}{\mu}$	0.1	
2.7	Формула (21): $\tilde{u}_1 = -v_1 + 2v_2$	0.1	**0.3**
	Неравенство а): $v_2 > 1$	0.1	
	Неравенство б): $v_2 < 0$	0.1	
2.8	Формула (22): $x(t) = A\cos(\omega t)$	0.1	**0.4**
	Формула (23): $v(t) = -A\omega\sin(\omega t)$	0.1	
	Формула (24): $V_0 = A\omega = 2\pi\dfrac{A}{T}$	0,2	
2.9	Формула (25): $\eta = \dfrac{T - t_1}{T}$	0.3	**2.7**
	Формула (26): $t_1 = \dfrac{T}{2} - \dfrac{2A}{u}$	0.3	
	Формула (27): $\eta = \dfrac{1}{2} + \dfrac{V_0}{\pi u}$	0.3	
	Численное значение (28): $\eta \approx 0.71$	0.3	
	Формула (29): $-A\omega\sin(\omega t_2) = -u$	0.2	

	Формула (30): $t_2 = \frac{1}{\omega}\arcsin\frac{u}{A\omega} = \frac{T}{2\pi}\arcsin\frac{u}{V_0}$	0.2	
	Формула (31): $x_2 = A\sqrt{1-\frac{u^2}{V_0^2}}$	0.3	
	Формула (32): $t_1 = \frac{T}{2\pi}\left(\arcsin\frac{u}{V_0} - \frac{V_0}{u}\left(1-\sqrt{1-\frac{u^2}{V_0^2}}\right)\right)$	0.3	
	Формула (33): $1-\eta = \frac{1}{2\pi}\left(\arcsin\frac{u}{V_0} - \frac{V_0}{u}\left(1-\sqrt{1-\frac{u^2}{V_0^2}}\right)\right)$	0.3	
	Численное значение (34): $\eta \approx 0.96$	0.2	
2.10	Формула (35): $V = \frac{4A}{T}$	0.2	**0.2**
2.11	Формула (36): $\eta = \frac{1}{2} + \frac{V}{2u}$	0.3	**2.2**
	Формула (37): $u_+ = u + 2V$	0.2	
	Формула (38): $u_- = u - 2V$	0.2	
	Формула (39): $E = \eta\frac{mu_+^2}{2} + (1-\eta)\frac{mu_-^2}{2}$	0.3	
	Формула (40): $\varepsilon = 1 + 8\left(\frac{V}{u}\right)^2$	0.3	
	Численное значение в формуле (40): $\varepsilon \approx 4.6$	0.2	
	Формула (41): $E = \frac{m}{2}(u+2V)^2$	0.2	
	Формула (42): $\varepsilon = \left(1 + 2\frac{V}{u}\right)^2$	0.3	
	Численное значение в формуле (42): $\varepsilon = 25.0$	0.2	
Итого			**10.0**

Задача 3. Магнетрон

Движение электрона в электрическом и магнитном полях

3.1 Под действием однородного электрического поля электрон движется с постоянным ускорением

$$a = \frac{eE}{m}, \tag{1}$$

которое направлено в отрицательном направлении оси x, поэтому максимальное значение достигаемой координаты определяется выражением

$$x_{\max} = \frac{u_0^2}{2a} = \frac{mu_0^2}{2eE}. \tag{2}$$

3.2 При движении в однородном магнитном поле на электрон действует сила Лоренца, равная

$$F_L = eu_0B. \tag{3}$$

и он совершает движение по окружности, радиус R которой определяется из второго закона Ньютона

$$m\frac{u_0^2}{R} = F_L, \tag{4}$$

откуда получаем

$$R=\frac{mu_0}{eB}. \tag{5}$$

Очевидно, что максимальное значение координаты при этом равно

$$x_{\max}=R=\frac{mu_0}{eB}. \tag{6}$$

3.3 Задача проще всего решается в лабораторной системе отсчета, в которой электрон двигается по окружности с частотой, определяемой формулой (5) в виде

$$\omega=\frac{u_0}{R}=\frac{eB}{m}. \tag{7}$$

При сообщении электрону малой дополнительной скорости он перейдет на движение по окружности, близкой к первоначальной и пересекающейся с ней в двух диаметрально противоположных точках, что можно рассматривать как движение по замкнутой двумерной траектории с периодом

$$T=\frac{2\pi}{\omega}=\frac{2\pi m}{eB}. \tag{8}$$

3.4 В момент, когда координата x максимальна, скорость частицы u направлена вдоль оси z и по закону сохранения энергии равна

$$eEx_{\max}=\frac{mu^2}{2}. \tag{9}$$

В проекции на ось z уравнение движения записывается в конечных разностях виде

$$m\frac{\Delta u_z}{\Delta t}=eBu_x, \tag{10}$$

что с учетом $\Delta x=u_x\Delta t$ приводит к соотношению

$$m\Delta u_z=eB\Delta x, \tag{11}$$

которое для искомого момента принимает вид

$$mu=eBx_{\max}. \tag{12}$$

Решая совместно уравнения (9) и (12), окончательно получаем

$$x_{\max}=\frac{2mE}{eB^2}, \tag{13}$$

3.5 Так как магнитное поле работы не совершает, то скорость электрона остается постоянной по модулю и равной начальной

$$u=u_0=const. \tag{14}$$

Разобьем полную скорость на радиальную $u_r=dr/dt$ и азимутальную $u_\varphi=rd\varphi/dt$ составляющие. Момент импульса электрона относительно начала координат очевидно равен

$$L=mru_\varphi, \tag{15}$$

а момент силы Лоренца относительно той же точки составляет

$$M=eBu_r r. \tag{16}$$

Согласно уравнению моментов имеем

$$\frac{dL}{dt}=M, \tag{17}$$

что с использованием $u_r=dr/dt$ приводит к соотношению

$$d(mru_\varphi)=e\alpha r^2dr. \tag{18}$$

В момент времени, когда расстояние до оси z максимально, радиальная скорость обращается в ноль, а азимутальная скорость равна начальной в соответствии с формулой (14), поэтому интегрирование соотношения (18) приводит к уравнению

$$mr_{\max}u_0=e\alpha\frac{r_{\max}^3}{3}, \tag{19}$$

из которого следует, что

$$r_{\max}=\sqrt{\frac{3mu_0}{e\alpha}}\,. \qquad (20)$$

3.6 Так как электрон все время движется по окружности, то, согласно уравнению (5), при нарастании магнитного поля на его орбите B_0 производная импульса меняется по закону

$$\frac{dp}{dt}=er\frac{dB_0}{dt}\,. \qquad (21)$$

Электрон приходит в движение за счет вихревого электрического поля, напряженность E которого определяется соотношением

$$E=\frac{1}{2\pi r}\frac{d\Phi}{dt}\,, \qquad (22)$$

в которое по закону Фарадея входит поток магнитной индукции через орбиту электрона, равный

$$\Phi=\int_0^r B(r)2\pi r dr\,. \qquad (23)$$

Уравнение второго закона Ньютона для ускорения электрона по орбите имеет вид

$$\frac{dp}{dt}=eE\,. \qquad (24)$$

Совместное решение уравнений (21)-(24) приводит к следующему равенству для магнитного поля, которое называется циклотронным условием

$$\int_0^r B(r)2\pi r dr=2\pi r^2 B_0\,. \qquad (25)$$

Из формулы (25) заключаем, что его выполнение возможно только в случае, когда электрон движется в области магнитного поля с индукцией $B_0=B_2$, поэтому интегрируя заданную в условии магнитную индукцию как функцию расстояния, получаем соотношение

$$B_1\pi r_1^2+B_2\pi(r^2-r_1^2)=2\pi r^2 B_2\,, \qquad (26)$$

решение которого имеет вид

$$\frac{B_1}{B_2}=1+\frac{r^2}{r_1^2}\,. \qquad (27)$$

Движение электрона по окружности возможно только в той области, в которой индукция равна B_2, то есть при $r_1<r<r_2$, а значит искомое отношение должно лежать в интервале

$$2<\frac{B_1}{B_2}<1+\frac{r_2^2}{r_1^2}\,. \qquad (28)$$

Цилиндрический магнетрон

3.7 Пусть на единицы длины цилиндрического катода и анода приходится заряд, равный λ, а полная длина электродов составляет l. Тогда по теореме Гаусса, напряженность электрического поля в пространстве между катодом и анодом определяется уравнением

$$E2\pi r l=\frac{\lambda l}{\varepsilon_0}\,, \qquad (29)$$

откуда получаем

$$E=\frac{\lambda}{2\pi\varepsilon_0 r}\,. \qquad (30)$$

Здесь r – расстояние до оси магнетрона.

Зависимость разности потенциалов от расстояния r по определению записывается в виде интеграла

$$V=\int_a^r Edr=\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r}{a}\,, \qquad (31)$$

который в частности для $r = b$ дает

$$V_0 = \frac{\lambda}{2\pi\varepsilon_0} \ln\frac{b}{a}. \qquad (32)$$

Решая совместно уравнения (31) и (32), получаем

$$V = V_0 \frac{\ln(r/a)}{\ln(b/a)} = 57.6 \text{ В}. \qquad (33)$$

3.8 Рассмотрим тонкое кольцо радиуса R, по которому протекает ток j, и рассчитаем величину магнитной индукции в точке на оси кольца, отстоящей то его центра на расстоянии z. Разобьем кольцо на малые элементы dl, тогда магнитная индукция определяется следующим законом Био-Саварра

$$d\bar{B} = \frac{\mu_0 j}{4\pi} \frac{d\bar{l} \times \bar{r}}{r^3}, \qquad (34)$$

в котором вектор $\dot{r}$ проведен из точки расположения элемента тока dl в точку O, в которой ищется магнитная индукция.

Из геометрических соотношений следует, что

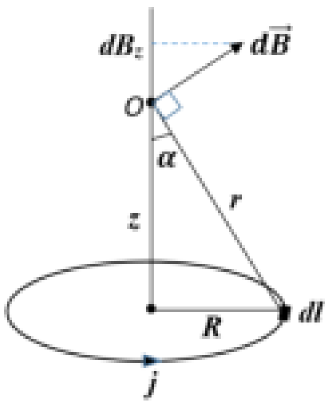
$$dl \times r = dl \cdot r, \qquad (35)$$

а так как результирующая магнитная индукция направлена вдоль оси кольца

$$dB_z = dB \sin\alpha, \qquad (36)$$

то, используя геометрическое соотношение $R = r\sin\alpha$, окончательно получаем

$$dB_z = \frac{\mu_0 j}{4\pi} \frac{Rdl}{r^3}. \qquad (37)$$



Учитывая, что расстояния, входящие в формулу (37), являются постоянными и

$$r^2 = R^2 + z^2, \qquad (38)$$

то после суммирования по всем элементам кольца находим

$$B_z = \frac{\mu_0 j}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}. \qquad (39)$$

Рассчитаем теперь индукцию магнитного поля в центре соленоида, так как именно там расположена лампа магнетрона. Для этого рассмотрим витки, расположенные на расстоянии от центра от z до $z + dz$, по которым протекает ток

$$dj = \frac{NI}{L} dz. \qquad (40)$$

Эти витки можно рассматривать как кольцо, магнитная индукция которого определяется формулой (39), из которой получаем

$$dB = \frac{\mu_0 NI}{2L} \frac{R^2}{(R^2 + z^2)^{3/2}} dz, \qquad (41)$$

что после интегрирования дает окончательное выражение

$$B = \frac{\mu_0 NIR^2}{2L} \int_{-L/2}^{L/2} \frac{dz}{(R^2 + z^2)^{3/2}} = \frac{\mu_0 NI}{L\sqrt{1 + D^2/L^2}}, \qquad (42)$$

где использовано выражение для диаметра $D = 2R$.

Для движения электронов в магнетроне справедлива формула, аналогичная формуле (18) и имеющая вид

$$d(mru_\varphi) = eBrdr\,, \tag{43}$$

интегрирование которой при условии постоянства магнитной индукции и $a << b$ дает

$$mru_\varphi = \frac{1}{2}eBr^2\,. \tag{44}$$

С другой стороны, из закона сохранения энергии следует, что

$$\frac{m}{2}(u_r^2 + u_\varphi^2) = eV\,. \tag{45}$$

В момент достижения критической величины тока магнитная индукция вблизи анода становится такой, что радиальная скорость электронов обращается в ноль, что приводит к условиям

$$u_r = 0,\quad r = b,\quad V = V_0\,, \tag{46}$$

что с использованием выражений (44) и (45) дает критическую величину магнитного поля

$$B = \sqrt{\frac{8mV_0}{eb^2}}\,. \tag{47}$$

Используя формулу (42), находим соответствующую величину тока в соленоиде

$$I_{\min} = \sqrt{\frac{8mV_0}{e}(1 + D^2/L^2)}\,\frac{L}{\mu_0 Nb} = 0{,}701\text{ A}\,. \tag{48}$$

3.9 Начальная энергия электронов в лампе вблизи катода определяется температурой самого катода и составляет порядка

$$E_T = k_B T\,. \tag{49}$$

Эта энергия, очевидно, должна быть много меньше энергии электронов вблизи анода E_0, то есть

$$E_T \lll E_0\,, \tag{50}$$

где $E_0 = eV_0$, откуда получаем искомую оценку

$$T \lll \frac{eE_0}{k_B} = 8.70\cdot 10^5\text{ K}\,, \tag{51}$$

что фактически означает применимость использованного приближения, так как температура катода обычно минимум на два порядка меньше.

	Содержание	**Баллы**	
3.1	Формула (1): $a = \frac{eE}{m}$	0.1	**0.2**
	Формула (2): $x_{\max} = \frac{mu_0^2}{2eE}$	0.1	
3.2	Формула (3): $F_L = eu_0B$	0.1	**0.4**
	Формула (4): $m\frac{u_0^2}{R} = F_L$	0.1	
	Формула (5): $R = \frac{mu_0}{eB}$	0.1	
	Формула (6): $x_{\max} = R = \frac{mu_0}{eB}$	0.1	
3.3	Формула (7): $\omega = \frac{u_0}{R} = \frac{eB}{m}$	0.2	**0.4**
	Формула (8): $T = \frac{2\pi}{\omega} = \frac{2\pi m}{eB}$	0.2	

3.4	Формула (9): $eEx_{\max}=\dfrac{mu^2}{2}$	0.2	1.0
	Формула (10): $m\dfrac{\Delta u_z}{\Delta t}=eBu_x$	0.2	
	Формула (11): $m\Delta u_z=eB\Delta x$	0.2	
	Формула (12): $mu=eBx_{\max}$	0.2	
	Формула (13): $x_{\max}=\dfrac{2mE}{eB^2}$	0.2	
3.5	Формула (14): $u=u_0=const$	0.2	1.4
	Формула (15): $L=mru_\varphi$	0.2	
	Формула (16): $M=eBu_r r$	0.2	
	Формула (17): $\dfrac{dL}{dt}=M$	0.2	
	Формула (18): $d(mru_\varphi)=e\alpha r^2dr$	0.2	
	Формула (19): $mr_{\max}u_0=e\alpha\dfrac{r_{\max}^3}{3}$	0.2	
	Формула (20): $r_{\max}=\sqrt{\dfrac{3mu_0}{e\alpha}}$	0.2	
3.6	Формула (21): $\dfrac{dp}{at}=er\dfrac{dB_0}{dt}$	0.2	1.6
	Формула (22): $E=\dfrac{1}{2\pi r}\dfrac{d\Phi}{dt}$	0.2	
	Формула (23): $\Phi=\int\limits_0^r B(r)2\pi rdr$	0.2	
	Формула (24): $\dfrac{dp}{dt}=eE$	0.2	
	Формула (25): $\int\limits_0^r B(r)2\pi rdr=2\pi r^2B_0$	0.2	
	Формула (26): $B_1\pi r_1^2+B_2\pi(r^2-r_1^2)=2\pi r^2B_2$	0.2	
	Формула (27): $\dfrac{B_1}{B_2}=1+\dfrac{r^2}{r_1^2}$	0.2	
	Формула (28): $2<\dfrac{B_1}{B_2}<1+\dfrac{r_2^2}{r_1^2}$	0.2	
3.7	Формула (29): $E2\pi rl=\dfrac{\lambda l}{\varepsilon_0}$	0.1	1.0
	Формула (30): $E=\dfrac{\lambda}{2\pi\varepsilon_0 r}$	0.1	
	Формула (31): $V=\dfrac{\lambda}{2\pi\varepsilon_0}\ln\dfrac{r}{a}$	0.2	
	Формула (32): $V_0=\dfrac{\lambda}{2\pi\varepsilon_0}\ln\dfrac{b}{a}$	0.2	

	Формула (33): $V = V_0 \dfrac{\ln(r/a)}{\ln(b/a)}$	0.2	
	Численное значение в формуле (33): $V = 57.6$ В	0.2	
3.8	Формула (34): $d\bar{B} = \dfrac{\mu_0 j}{4\pi} \dfrac{d\bar{l} \times \bar{r}}{r^3}$	0.2	3.2
	Формула (35): $dl \times r = dl \cdot r$	0.2	
	Формула (36): $dB_z = dB \sin\alpha$	0.2	
	Формула (37): $dB_z = \dfrac{\mu_0 j}{4\pi} \dfrac{Rdl}{r^3}$	0.2	
	Формула (38): $r^2 = R^2 + z^2$	0.2	
	Формула (39): $B_z = \dfrac{\mu_0 j}{2} \dfrac{R^2}{(R^2+z^2)^{3/2}}$	0.2	
	Формула (40): $dj = \dfrac{NI}{L} dz$	0.2	
	Формула (41): $dB = \dfrac{\mu_0 NI}{2L} \dfrac{R^2}{(R^2+z^2)^{3/2}} dz$	0.2	
	Формула (42): $B = \dfrac{\mu_0 NI}{L\sqrt{1+D^2/L^2}}$	0.2	
	Формула (43): $d(mru_\varphi) = eBrdr$	0.2	
	Формула (44): $mru_\varphi = \dfrac{1}{2} eBr^2$	0.2	
	Формула (45): $\dfrac{m}{2}(u_r^2 + u_\varphi^2) = eV$	0.2	
	Формула (46): $u_r = 0, \quad r = b, \quad V = V_0$	0.2	
	Формула (47): $B = \sqrt{\dfrac{8mV_0}{eb^2}}$	0.2	
	Формула (48): $I_{\min} = \sqrt{\dfrac{8mV_0}{e}(1+D^2/L^2)} \dfrac{L}{\mu_0 Nb}$	0.2	
	Численное значение в формуле (48): $I_{\min} = 0,701$ А	0.2	
3.9	Формула (49): $E_T = k_B T$	0.2	0.8
	Формула (50): $E_T \ll E_0$	0.2	
	Формула (51): $T \ll \dfrac{eE_0}{k_B}$	0.2	
	Численное значение в формуле (51): $T \ll 8.70 \cdot 10^5$ К	0.2	
Итого			**10.0**

3 февраля 2023 года

Сначала, пожалуйста, прочитайте следующее:

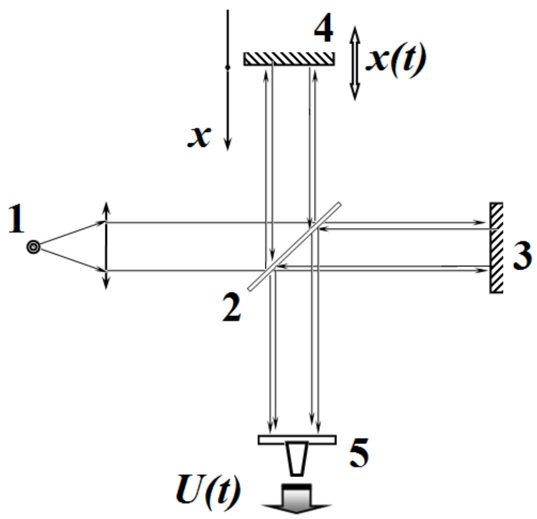
1. Экспериментальный тур состоит из одной задачи. Продолжительность тура 4 часа.
2. Для расчетов Вы можете использовать свой калькулятор.
3. Вам предоставлены чистые листы бумаги, которые предназначены для черновых записей, их Вы можете использовать по Вашему усмотрению, они не проверяются. На ***Writing sheets*** следует записывать решение задачи, которое будет оценено при проверке работы. Можете записывать очень короткие пояснения, используя уравнения, числа, буквенные обозначения.
4. Используйте только лицевую сторону ***Writing sheets***. При записи не выходите за пределы отмеченной рамки.
5. Графики необходимо строить на ***Writing sheets*** в области с миллиметровой бумагой. Все построения производите ручкой, а не карандашом! Если вам нужны дополнительные ***Writing sheets*** с миллиметровой бумагой, то вы можете найти их в конце вашего комплекта заданий.
6. На каждом использованном ***Writing sheets,*** в отведенных для этого графах, необходимо указать Вашу страну **(*Country*)**, Ваш код **(*Student Code*),** порядковый номер задачи **(*Question Number*)**, текущий номер каждого листа **(*Page Number*)** и полное количество листов, использованных при решении всех задач **(*Total Number of Pages)***. Если Вы не хотите, чтобы некоторые использованные ***Writing sheets*** были включены в ответ, тогда перечеркните их большим крестом на весь лист и не включайте в Ваш подсчет полного количества листов.

Фурье-спектрометр[1]
Введение

Основной элемент Фурье спектрометра – интерферометр Майкельсона.

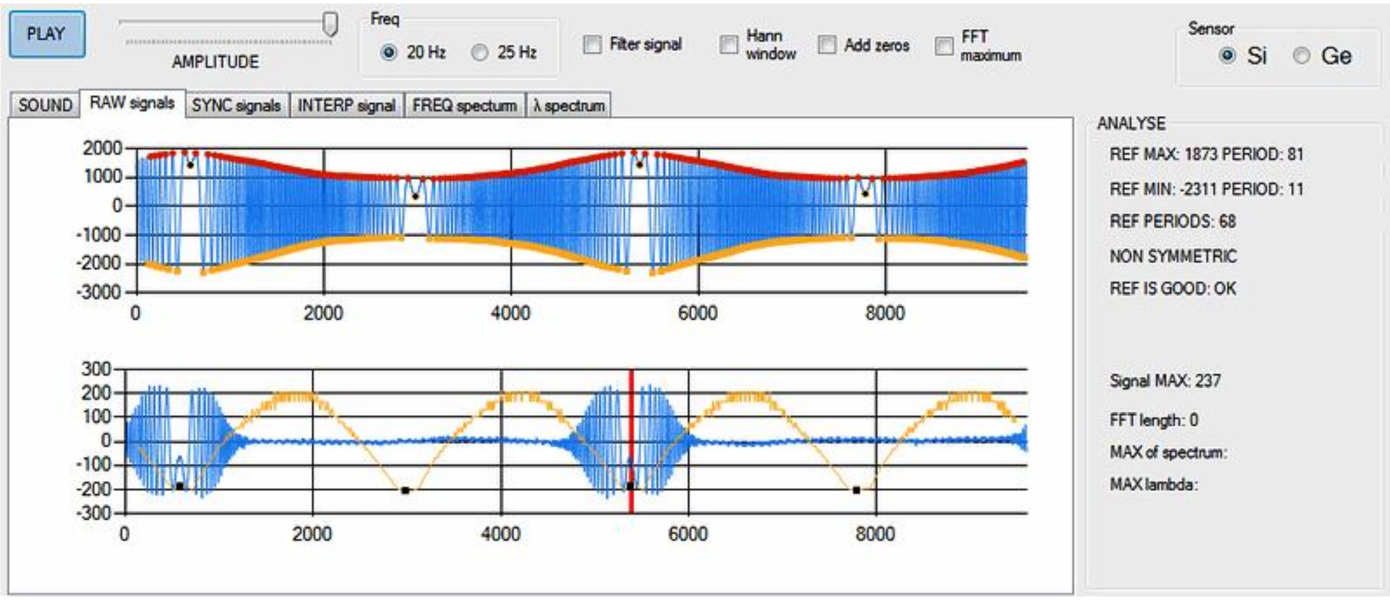
Допустим, у нас имеется когерентный источник излучения с определенной длиной волны. Когда разность хода двух лучей, пришедших в приемник, кратна половине длине волны, то есть лучи пришли в противофазе, интенсивность регистрируемого излучения близка к нулю. При перемещении одного из зеркал интерферометра разность хода лучей изменяется, поэтому изменяется и интенсивность света, регистрируемая приемником. Известно, что интенсивность будет максимальная, когда разность хода кратна длине волны.

Если одно из зеркал интерферометра движется с постоянной скоростью, тогда на выходе приемника будет наблюдаться синусоидальный сигнал. Амплитуда синусоиды пропорциональна интенсивности света, а ее период зависит от длины волны.



На рисунке показана схема интерферометра Майкельсона. Световой поток от источника 1 с помощью линз формируется в параллельный пучок лучей и направляется на светоделительную пластинку 2. Часть света (половина по интенсивности) проходит через пластинку, попадает на неподвижное зеркало 3, полностью отражается от него и снова попадает на пластинку 2, отражается от нее и попадает на экран с фотоприемником 5. Вторая половина светового потока от источника отражается от светоделительной пластинки 2 и попадает на подвижное зеркало 4, отражается от него, проходит через пластинку 2 и также попадает на экран с фотоприемником 5. Таким образом, на экран попадают две когерентные волны, отраженные от зеркал 3 и 4. Эти волны интерферируют между собой, а приемник регистрирует результирующую интенсивность света на экране, как функцию времени. Эта функция записывается в память компьютера для последующей обработки.

Для иллюстрации приводим фотографию зарегистрированного сигнала из указанной статьи.



Теперь представим, что зеркало 4 движется неравномерно и/или источник света не является монохроматическим, то есть содержит в себе несколько длин волн. Таким образом. на выходе будем иметь более сложный чем синусоидальный сигнал. При соответствующей математической обработке

[1] Данное задание разработано на основе материалов статьи **«Самодельный Фурье-спектрометр»** (https://habr.com/ru/post/253947/).

этого сигнала можно получить закон движения зеркала 4 или спектр излучения источника света, то есть интенсивность излучения на различных длинах волн.

При выполнении заданий используйте упрощающие положения и обозначения:

1) регистрируемое фотоприемником напряжение $U(t)$ пропорционально интенсивности света, чувствительность фотоприемника не зависит от длины волны света; с помощью электронной схемы постоянная составляющая сигнала отрезается, поэтому на графиках отражается только переменная составляющая;
2) подвижное зеркало колеблется по гармоническому закону $x(t)$ с частотой 20 Гц и постоянной амплитудой; можно считать, что при $x = 0$ разность хода интерферирующих волн равна нулю;
3) интенсивности интерферирующих на приемнике волн равны;
4) начало регистрации сигнала согласовано с движением подвижного зеркала и всегда начинается при одном и том же положении зеркала;
5) регистрация сигнала проводится в равноотстоящие моменты времени и записывается в ячейки памяти, которые в дальнейшем нумеруются целыми значениями t. Фактически t есть время регистрации в относительных единицах.
6) на всех рисунках приводятся графики зависимостей регистрируемого напряжения $U(t)$ от номера ячейки памяти t. Для упрощения работы к каждому графику прилагается таблица, в которой указаны положения экстремумов (максимумов и минимумов) зарегистрированного сигнала t_m, эти экстремумы нумеруются буквой m.

Внимание! На отдельных листах ***Writing sheets*** приведены зарегистрированные сигналы зависимости напряжения на фотоприемнике от времени, которые вам предстоит обрабатывать. Отметим, что приведены только часть всех сигналов. При выполнении данного задания Вам нет необходимости использовать все приведенные численные данные. Используйте только те, которые считаете необходимы для расчета требуемых величин. При построении графиков используйте разумное количество данных (10-15 точек), однако помните, что точность расчетов повышается при увеличении числа используемых данных. Обязательно указывайте в решении, какие данные Вы используете, также обязательно приводите формулы, по которым проводятся расчеты. Для проведения расчетов **Вы должны** использовать подготовленные таблицы в ***Writing sheets***. Для построения графиков используйте бланки, приведенные в тех же ***Writing sheets*. Обратите внимание, что только *Writing sheets* будут оцениваться**. Для черновых записей вы можете использовать белые листы бумаги, но они оцениваться не будут!

Задания

1. Теоретическая часть

Интерферометр освещается монохроматическим излучением с длиной волны λ.

1.1 Обозначим интенсивность каждой из интерферирующих волн I_0, сдвиг фаз между волнами $\Delta\varphi$. Запишите формулу для интенсивности I результирующей волны.

1.2 Запишите формулу для интенсивности волны на приемнике $I(x)$ в зависимости от положения подвижного зеркала x.

1.3 Запишите формулы, указывающие, при каких значениях координаты зеркала x_m интенсивность света на экране будет максимальна, а при каких минимальна.

1.4 Запишите общую формулу, определяющую координату зеркала x_m, при которой интенсивность света на экране экстремальна.

2. Монохроматическое излучение известной длины волны – градуировка прибора

На рисунке 1 показана зависимость интенсивности света от времени при освещении интерферометра монохроматическим излучением с длиной волны $\lambda_0 = 0.640$ мкм. В таблице 1 приведены значения времен t_m, при которых интенсивность света экстремальна (максимумы и минимумы), а также значения сигнала U_m в эти моменты времени.

2.1 Определите цену деления $\Delta t = 1$ использованной временной шкалы данного устройства в миллисекундах.

В дальнейшем все расчеты проводите в условных единицах шкалы прибора.

2.2 На основе приведенных экспериментальных данных покажите, что движение зеркала может быть описано функцией

$$x(t) = A\sin\left(\frac{2\pi}{T}(t - t_0)\right). \qquad (1)$$

Определите значения параметров этой функции: период T в единицах цены деления шкалы; амплитуду колебаний A в микрометрах; момент времени t_0, при котором $x = 0$. Постройте линеаризованный график зависимости (1), доказывающий применимость этой формулы для описания колебаний зеркала. Оцените погрешность определения амплитуды колебаний ΔA.

Функцию (1) с найденными значениями параметров следует использовать при выполнении следующих частей задания.

3. Монохроматическое излучение с неизвестной длиной волны

Интерферометр освещается монохроматическим излучением с неизвестной длиной волны, которую вам необходимо определить.

На рисунке 2 показана зависимость интенсивности света от времени в этом случае. В таблице 2 приведены значения координат экстремумов этой функции.

3.1 Постройте график зависимости координаты зеркала x_m, при которых интенсивность экстремальная, от номера экстремума m.

3.2 Используя построенный график, определите с максимальной точностью значение длины волны λ света источника. Оцените погрешность $\Delta\lambda$ найденного значения.

4. Две монохроматические волны

Интерферометр освещается излучением, содержащим две монохроматические волны. Длина волны одной из них равна $\lambda_1 = 0.640$ мкм, а длина волны λ_2 второй неизвестна.

На рисунке 3 приведена зависимость интенсивности света на экране от времени. В таблице 3 приведены значения экстремумов приведенной функции.

4.1 Используя приведенные данные, определите с максимальной точностью длину волны λ_2 второй спектральной компоненты. Оцените погрешность найденного значения длины волны $\Delta\lambda_2$.

4.2 Определите отношение интенсивностей второй и первой волны I_2 / I_1.

February 3, 2023

Please read this first:

1. The duration of the experimental competition is 4 hours. There is one problem.
2. You can use your own calculator for numerical calculations.
3. You are provided with white sheets of paper for drafts of your solutions, but these sheets will not be graded. Your final solutions must only be written on the ***Writing sheets***. You can write very short explanations using equations, numbers, letters.
4. Use only the front side of ***Writing sheets*****.** Write only inside the boxed areas.
5. Graphs must be drawn on ***Writing sheets*** within the graph paper area. All drawings must be done with a pen, not a pencil! If you need additional ***Writing sheets*** with the graph paper, you can find them at the end of your worksheet.
6. Fill in the boxes at the top of each ***Writing sheet*** with your country (**Country),** your student code (**Student Code**), problem number (**Question Number)**, the progressive number of each ***Writing sheet*** **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** that you do not wish to be graded, put a large X across the entire sheet and do not include it in your numbering.

Fourier spectrometer [1]

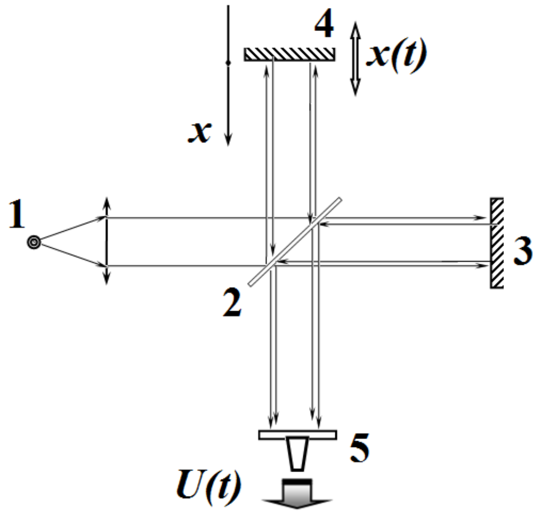
Introduction

The main element of the Fourier spectrometer is the Michelson interferometer.

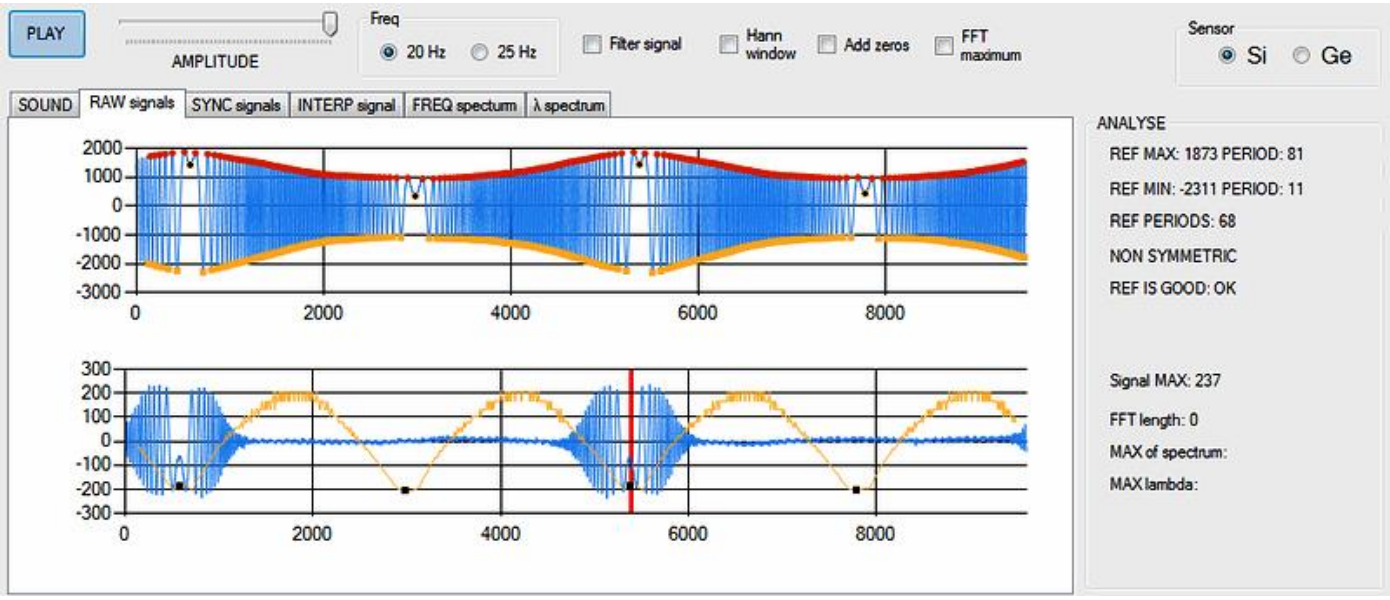
Assume that we have a coherent light source with a certain wavelength. When the difference in the path of the two beams that come to a receiver is a multiple of half the wavelength, that is, the beams come in antiphase, the intensity of the detected intensity is close to zero. When one of the mirrors of the interferometer is moved, the difference in the path of the light rays changes, so the light intensity recorded by the receiver also varies. It is known that the intensity is to be maximum when the path difference is a multiple of the wavelength.

If one of the mirrors of the interferometer moves at a constant speed, then a sinusoidal signal is to be observed at the output of the receiver. The amplitude of the sinusoid is proportional to the light intensity, and its period depends on the wavelength.

The figure shows a setup of the Michelson interferometer. The light flux from source 1 is formed by lenses into a parallel beam of rays and directed to beam splitter 2. Part of the light (half in intensity) passes through the plate to hit the fixed mirror 3 and is completely reflected from it, and then hits the plate 2 again to be reflected from it and hit the screen with the photodetector 5. The second half of the light flux from the source is reflected from the beam-splitting plate 2 to hit the movable mirror 4, it is then reflected to pass through the plate 2 and hit the screen with the photodetector 5. Thus, two coherent waves reflected from the mirrors 3 and 4 fall on the screen. These waves interfere with each other, and the receiver registers the resulting light intensity on the screen as a function of time. This function is written to the computer's memory for further processing.



For illustration, we present a photograph of the registered signal from the specified article.



Let us now imagine that the mirror 4 moves non-uniformly and/or the light source is not monochromatic, i.e. it contains several wavelengths. Thus. at the output a more complex than a sinusoidal signal is to be detected. With appropriate mathematical signal processing, it is possible to obtain the motion law of the mirror 4 or the light spectrum of the source, that is, the light intensity at different wavelengths.

When solving problems, use the following simplifying assumptions and notation:

[1] This problem is developed on the basis of the article « **Homemade Fourier spectrometer**» (https://habr.com/ru/post/253947/).

1) the voltage recorded by the photodetector is proportional to the light intensity, the sensitivity of the photodetector does not depend on the light wavelength; using an electronic circuit, the constant component of the signal is cut off, so only the variable component is reflected on the graphs;

2) the movable mirror oscillates according to a harmonic law $x(t)$ with a frequency of 20 Hz and a constant amplitude; it is assumed that at $x = 0$, the path difference of the interfering waves is equal to zero;

3) the intensities of the interfering waves at the receiver are equal;

4) the beginning of the signal registration is coordinated with the motion of the movable mirror and always starts at the same position of the mirror;

5) signal registration is carried out at equidistant moments of time and recorded in memory cells, which are further numbered with integer values t. In fact, the registration time t is recorded in relative units.

6) all the figures in the problem show graphs of the dependences of the recorded voltage on the number of the memory cell. To simplify the solution, each graph is accompanied by a table, which indicates the positions of the extrema (maxima and minima) of the registered signal t_m, the extrema themselves are numbered with the letter m.

Attention! On separate ***Writing sheets***, the registered signals are given in the form of the dependence of the photodetector voltage on the time, which you are assumed to process. Note that only a part of all signals are given. When solving this problem, you do not need to use all the given numerical data. Use only those that you consider necessary to calculate the required values. When plotting, use a reasonable amount of data (10-15 points), but remember that the accuracy of the calculations increases when the number of data used grows. Be sure to indicate in the solution, which data you use, and be sure to include the formulas used for calculations. For calculations, **you must use** the prepared tables in the ***Writing sheets***. To draw graphs, use the forms given in the same ***Writing sheets***. **Please note that only *Writing sheets* are to be graded**. For draft notes, you can use white sheets of paper, but they will not be graded!

Problems

1. Theoretical part

The interferometer is illuminated by the monochromatic radiation with the wavelength λ.

1. Let us denote I_0 as the intensity of each of the interfering waves, and the phase shift between the waves as $\Delta\varphi$. Write down the formula for the intensity I of the resulting wave.

1.2 Write down the formula for the intensity $I(x)$ of the wave at the receiver as a function of the position of the movable mirror x.

1.3 Write down formulas indicating at what values x_m of the mirror position the light intensity on the screen is to be maximum or, correspondingly, minimum.

1.4 Write down the general formula that determines the mirror coordinate x_m, at which the light intensity on the screen is extreme.

2. Monochromatic radiation of a known wavelength as an instrument calibration

Figure 1 shows the dependence of the light intensity on the time when the interferometer is illuminated with monochromatic radiation with the wavelength $\lambda_0 = 0.640\ \mu$m. Table 1 shows the times t_m at which the light intensity is extreme (maxima and minima), as well as the signal values U_m at corresponding time moments.

2.1 Determine the division value $\Delta t = 1$ of the device in milliseconds.

In the following, all calculations must only be carried out in the units of the instrument scale.

2.2 On the basis of the given experimental data, show that the motion of the mirror can be described by the function

$$x(t) = A\sin\left(\frac{2\pi}{T}(t - t_0)\right). \tag{1}$$

Find the values of the parameters of this function: period T in units of the instrument scale; oscillation amplitude A in micrometers; point in time t_0 at which $x = 0$. Plot a linearized graph of dependence (1), proving the applicability of the above formula to describe the oscillations of the mirror. Estimate the error ΔA in determining the amplitude of oscillations.

Function (1) with the found values of the parameters should be used when performing the following parts of the problem.

3. Monochromatic radiation with unknown wavelength

The interferometer is illuminated by a monochromatic radiation with an unknown wavelength that you have to determine.

Figure 2 shows the dependence of the light intensity on the time for this case. Table 2 shows the values of the extrema coordinates for this particular case.

3.1 Plot the dependence of the mirror coordinates x_m, at which the intensity is extreme, as a function of the number of the extremum .

3.2 Using the plotted graph, determine with maximum accuracy the value of the wavelength λ of the light source. Estimate the error $\Delta\lambda$ of the obtained value.

4. Two monochromatic waves

The interferometer is illuminated by radiation containing two monochromatic waves. The wavelength of one of them is $\lambda_1 = 0.640\ \mu\text{m}$, and the wavelength λ_2 of the second is unknown.

Figure 3 shows the dependence of the light intensity on the screen as a function of the time. Table 3 shows the values of the extrema for this particular case.

4.1 Using the given data, determine with maximum accuracy the wavelength λ_2 of the second spectral component. Estimate the error $\Delta\lambda_2$ of the found value of the wavelength.

4.2 Determine the ratio of the intensities of the second and first waves I_2 / I_1.

3 ақпан 2023 жыл

Алдымен мынаны оқып шығыңыз:

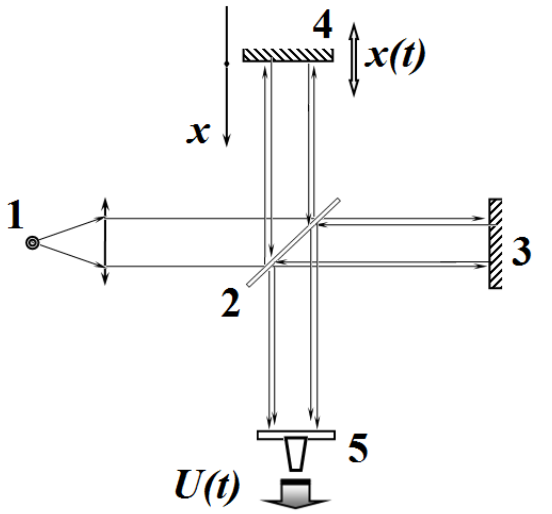
1. Эксперименттік тур бір тапсырмадан тұрады. Тур ұзақтығы 4 сағат.
2. Есептеу үшін калькуляторды пайдалануға болады.
3. Жазба жобаларына арналған бос қағаз парақтары ұсынылады, оларды өз қалауыңыз бойынша пайдалануға болады, олар тексерілмейді. ***Жазу парақтарында*** тапсырманың шешімін жазу керек, ол жұмысты тексеру кезінде бағаланады. Теңдеулерді, сандарды, әріптерді пайдаланып өте қысқа түсініктемелер жазуға болады.
4. ***Жазу парақтарының*** тек алдыңғы жағын пайдаланыңыз. Жазу кезінде белгіленген кадр шеңберінен шықпаңыз.
5. Графиктер графикалық қағазы бар аумақта ***Жазу парақтарында*** салынуы керек. Барлық конструкциялар қарындашпен емес, қаламмен жасалады! Егер сізге графикалық қағазы бар қосымша ***Жазу парақтары*** қажет болса, оларды жұмыс парағының соңында табуға болады.
6. Әрбір пайдаланылған ***Жазу парақтарында***, бұл үшін берілген бағандарда сіз еліңізді **(*Country*)**, кодыңызды (***Student Code***), тапсырманың реттік нөмірін (***Question Number***), әрбір парақтың ағымдағы нөмірін (***Page Number***) көрсетуіңіз керек және барлық есептерді шешуде қолданылатын парақтардың жалпы саны (***Total Number of Pages***). Жауабыңызға кейбір қолданылған Жазу парақтарын қосуды қаламасаңыз, оларды бүкіл парақтың үстінен үлкен крестпен сызып тастаңыз және оларды парақтардың жалпы санына қоспаңыз.

Фурье спектрометрі[1]

Кіріспе

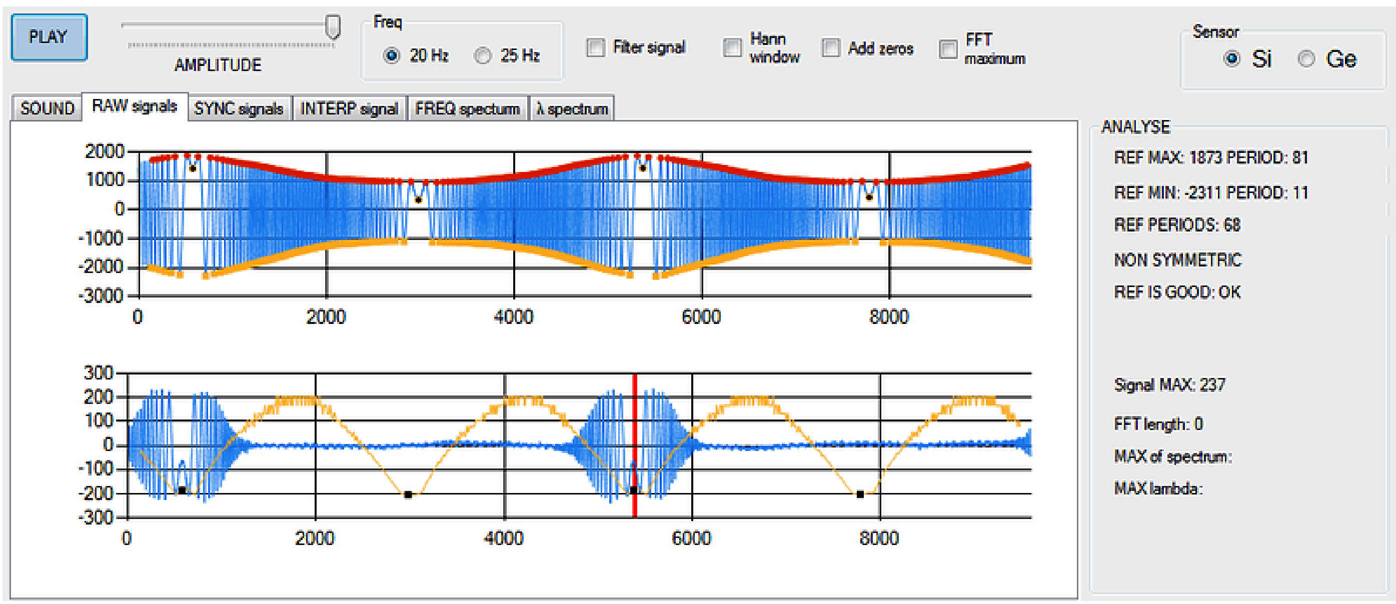
Фурье спектрометрінің негізгі элементі Майкельсон интерферометрі болып табылады.

Бізде белгілі бір толқын ұзындығы бар когерентті сәулелену көзі бар делік. Қабылдағышқа келген екі сәуленің жолындағы айырмашылық толқын ұзындығының жартысына еселі болғанда, яғни сәулелер антифазада келген кезде, анықталған сәулеленудің қарқындылығы нөлге жақын болады. Интерферометрдің айналарының бірін жылжытқанда сәулелер жолындағы айырмашылық өзгереді, сондықтан қабылдағыш жазып алған жарық қарқындылығы да өзгереді. Жол айырымы толқын ұзындығына еселік болғанда қарқындылық максималды болатыны белгілі.



Егер интерферометрдің айналарының бірі тұрақты жылдамдықпен қозғалса, онда қабылдағыштың шығысында синусоидалы сигнал байқалады. Синусоидтың амплитудасы жарықтың қарқындылығына пропорционал, ал оның периоды толқын ұзындығына байланысты. Суретте Майкельсон интерферометрінің диаграммасы көрсетілген. 1 жарық көзінен жарық ағыны линзалар арқылы параллель сәулелер шоғына айналады және сәуле бөлгішке 2 бағытталады. Жарықтың бір бөлігі (интенсивтілігі бойынша жартысы) пластинка арқылы өтіп, қозғалмайтын айнаға 3 түсіп, одан толығымен шағылысып, қайтадан пластинкаға 2 түседі, одан шағылысып, фотоқабылдағыш 5 арқылы экранға түседі. Көзден шыққан жарық ағынының екінші жартысы сәулені бөлетін пластинадан 2 шағылысады және жылжымалы айнаға 4 түседі, одан шағылысады, пластина 2 арқылы өтеді, сонымен қатар фотоқабылдағышпен 5 экранға түседі. Осылайша, 3 және 4 айналардан шағылған екі когерентті толқын экранға түседі.Бұл толқындар бір-біріне кедергі жасайды, ал қабылдағыш пайда болған жарық интенсивтілігін уақыт функциясы ретінде экранға тіркейді. Бұл функция одан әрі өңдеу үшін компьютердің жадына жазылады.

Көрнекілік үшін біз көрсетілген мақаладан тіркелген сигналдың фотосуретін ұсынамыз.



Енді айна 4 біркелкі емес қозғалады және/немесе жарық көзі монохроматикалық емес, яғни оның бірнеше толқын ұзындығы бар екенін елестетіп көрейік. Осылайша, нәтижесінде біз синусоидалы сигналға қарағанда күрделірек сигналға ие боламыз. Бұл сигналды сәйкес

[1] Берілген тапсырма мына мақаланың негізінде құрастырылған **«Өзіміз жасаған Фурье-спектрометр»** (https://habr.com/ru/post/253947/).

математикалық өңдеу арқылы айна 4 қозғалыс заңын немесе жарық көзінің сәулелену спектрін, яғни әртүрлі толқын ұзындығындағы сәулеленудің қарқындылығын алуға болады.

Тапсырмаларды орындау кезінде жеңілдететін ережелер мен белгілерді пайдаланыңыз:

1) фотоқабылдағыш тіркеп алған кернеу $U(t)$ жарық интенсивтілігіне пропорционал, фотоқабылдағыштың сезімталдығы жарықтың толқын ұзындығына тәуелді емес; электрондық схеманы қолдану арқылы сигналдың тұрақты құрамдас бөлігі кесіледі, сондықтан графиктерде тек айнымалы компонент көрсетіледі;
2) жылжымалы айна гармоникалық заң $x(t)$ бойынша жиілігі 20 Гц және тұрақты амплитудасы бойынша тербеледі; $x = 0$ нүктесінде кедергі жасайтын толқындардың жол айырымы нөлге тең деп санауға болады;
3) қабылдағышқа кедергі жасайтын толқындардың интенсивтілігі тең;
4) сигналды тіркеудің басталуы жылжымалы айнаның қозғалысымен келісіледі және әрқашан айнаның сол қалпында басталады;
5) сигналды тіркеу уақыттың тең қашықтықтағы моментінде жүзеге асырылады және жад ұяшықтарына жазылады, олар әрі қарай бүтін t мәндерімен нөмірленеді. Шын мәнінде, t - салыстырмалы бірліктерде тіркеу уақыты.
6) барлық суреттерде жазылған $U(t)$ кернеуінің t жады ұяшығының нөміріне тәуелділік графиктері көрсетілген. Жұмысты жеңілдету үшін әрбір диаграммаға тіркелген t_m сигналының экстремумдарының (максимум және минимум) орындары көрсетілген кесте қоса беріледі, бұл экстремумдар m әрпімен нөмірленеді.

Назар аударыңыз! ***Жазу парақтарының*** бөлек парақтарында кернеудің фотодетекторға уақыт бойынша тәуелділігінің тіркелген сигналдары берілген, оларды өңдеу керек. Барлық сигналдардың бір бөлігі ғана берілгенін ескеріңіз. Бұл тапсырманы орындау кезінде барлық берілген сандық мәліметтерді пайдаланудың қажеті жоқ. Қажетті мәндерді есептеу үшін қажет деп санайтындарды ғана пайдаланыңыз. Графикті құру кезінде деректердің ақылға қонымды көлемін пайдаланыңыз (10-15 ұпай), бірақ есептеулердің дәлдігі пайдаланылған деректер санының ұлғаюымен өсетінін есте сақтаңыз. Шешімде қандай деректерді пайдаланатыныңызды көрсетіңіз және есептеулер үшін қолданылатын формулаларды қосыңыз. Есептеулер үшін ***Жазу парақтарындағы*** дайындалған кестелерді пайдалану керек. Графиктерді құру үшін сол ***Жазу парақтарында*** берілген пішіндерді пайдаланыңыз. Тек жазу парақтары бағаланатынын ескеріңіз. Жазба жобалары үшін ақ парақтарды пайдалануға болады, бірақ олар бағаланбайды!

Тапсырмалар

1. Теориялық бөлім

Интерферометр толқын ұзындығы λ монохроматикалық сәулеленумен жарықтандырылады

1.1 Интерференциялық толқындардың әрқайсысының I_0 интенсивтілігін, $\Delta\varphi$ толқындар арасындағы фазалық ығысуын белгілейік. Пайда болған толқынның I интенсивтілігінің формуласын жазыңыз.

1.2 Жылжымалы x айнасының орнына байланысты $I(x)$ қабылдағыштағы толқынның қарқындылығының формуласын жазыңыз.

1.3 Айна координатасының x_m қандай мәндерінде экрандағы жарық интенсивтілігі максималды болатынын және қай кезде минималды болатынын көрсететін формулаларды жазыңыз.

1.4 Экрандағы жарық интенсивтілігі шектен жоғары болатын x_m айнасының координатасын анықтайтын жалпы формуланы жазыңыз.

2. Белгілі толқын ұзындығының монохроматикалық сәулеленуі – аспапты калибрлеу

1-суретте интерферометр толқын ұзындығы $\lambda_0 = 0.640$ мкм монохроматикалық сәулеленумен жарықтандырылғанда жарық қарқындылығының уақытқа тәуелділігі көрсетілген. 1-кестеде жарық қарқындылығы шектен тыс (максимум және минимум) болатын t_m уақыттары, сондай-ақ осы уақыттағы U_m сигналының мәндері көрсетілген.

2.1 Осы құрылғының пайдаланылған уақыт шкаласы $\Delta t = 1$ бөлімінің мәнін миллисекундпен анықтаңыз.

Болашақта барлық есептеулер аспаптық шкаланың еркті бірліктерімен жүргізіледі.

2.2 Берілген эксперименттік мәліметтер негізінде айна қозғалысын функция арқылы сипаттауға болатынын көрсетіңіз

$$x(t) = A\sin\left(\frac{2\pi}{T}(t - t_0)\right). \qquad (1)$$

Осы функцияның параметрлерінің мәндерін анықтаңыз: шкаланың бөліну бағасының бірлігіндегі T периоды; тербеліс амплитудасы A микрометрде; $x = 0$ нүктесіндегі t_0 уақыты. Айна тербелістерін сипаттау үшін осы формуланың қолданылуын дәлелдей отырып, (1) тәуелділіктің сызықтық графигін құрыңыз. ΔA тербеліс амплитудасын анықтау қателігін бағалаңыз.

Параметрлердің табылған мәндері бар функция (1) тапсырманың келесі бөліктерін орындау кезінде қолданылуы керек.

3. Толқын ұзындығы белгісіз монохроматикалық сәулелену

Интерферометр белгісіз толқын ұзындығы бар монохроматикалық сәулеленумен жарықтандырылады, оны анықтау қажет.

Бұл жағдайда жарық қарқындылығының уақытқа тәуелділігі 2-суретте көрсетілген. 2-кестеде осы функцияның экстремумдарының координаталарының мәндері көрсетілген.

3.1 Интенсивтілігі шектен тыс болатын айна x_m координаталарының m экстремумының санына тәуелділігінің графигін тұрғызыңыз.

3.2 Құрылған графикті пайдаланып, бастапқы жарықтың λ толқын ұзындығының мәнін максималды дәлдікпен анықтаңыз. Табылған мәннің $\Delta\lambda$ қатесін бағалаңыз.

4. Екі монохроматикалық толқындар

Интерферометр екі монохроматикалық толқыны бар сәулеленумен жарықтандырылады. Олардың біреуінің толқын ұзындығы $\lambda_1 = 0.640$ мкм, ал екіншісінің толқын ұзындығы λ_2 белгісіз.

3-суретте экрандағы жарық қарқындылығының уақытқа тәуелділігі көрсетілген. 3-кестеде қысқартылған функцияның экстремумдарының мәндері көрсетілген.

4.1 Берілген мәліметтерді пайдалана отырып, екінші спектрлік компоненттің λ_2 толқын ұзындығын максималды дәлдікпен анықтаңыз. $\Delta\lambda_2$ толқын ұзындығының табылған мәнінің қателігін бағалаңыз.

4.2 Екінші және бірінші I_2 / I_1 толқынының қарқындылықтарының қатынасын анықтаңыз.

SOLUTIONS TO THE PROBLEMS OF THE EXPERIMENTAL COMPETITION

1. Theoretical part

1.1 With the interference of two waves of the same intensity, the resulting intensity is determined by the formula

$$I = 2I_0(1+\cos\Delta\varphi) = 4I_0\cos^2\frac{\Delta\varphi}{2}. \quad (1)$$

1.2 When the mirror is displaced from the initial position by a value x, the path difference changes by $2x$. In this case, a phase difference arisen between the two waves is equal to

$$\Delta\varphi = \frac{4\pi}{\lambda}x, \quad (2)$$

therefore, the dependence of the intensity on the coordinate has the form

$$I = 2I_0\cos^2\frac{2\pi}{\lambda}x. \quad (3)$$

1.3 The intensity maximum is observed if the path difference is equal to an integer number of wavelengths, i.e.

$$2x_m = m\lambda \quad \Rightarrow \quad x_m = m\frac{\lambda}{2}, \quad (4)$$

and intensity minima arise under the following condition

$$2x_m = \left(m+\frac{1}{2}\right)\lambda \quad \Rightarrow \quad x_m = \left(m+\frac{1}{2}\right)\frac{\lambda}{2}. \quad (5)$$

1.4 The intensity changes from maximum to minimum (and vice versa) when the mirror is shifted by a quarter wavelength. Therefore, the sought coordinates of the mirror are described by the formula

$$x_m = m\frac{\lambda}{4}. \quad (6)$$

2. Monochromatic radiation of a known wavelength as an instrument calibration

2.1 It follows from the given figure that the extreme positions of the mirror correspond to the values of the times

$$t_{\min} = 67; \quad t_{\max} = 901. \quad (7)$$

This shift occurs in half the period of the mirror oscillation, so

$$T = 2(t_{\max} - t_{\min}) = 1668. \quad (8)$$

On the other hand, the oscillation period can be expressed in terms of a given mirror oscillation frequency $\nu = 20\ \text{Hz}$:

$$T = \frac{1}{\nu} = 5.0\cdot 10^{-2}\ \text{s} = 50\ \text{ms}. \quad (9)$$

Equating expressions (8) and (9), we find that the division value of the time scale is equal to

$$\Delta t = \frac{50}{2(901-67)} = 0.030\ \text{ms}. \quad (10)$$

2.2 Приведенный график зависимости интенсивности от времени симметричен относительно «центрального» максимума, номер которого равен The given figure of the intensity dependence on time is symmetrical with respect to the "central" maximum, whose number is equal to

$$m_0 = \frac{0+54}{2} = 27 \quad (11)$$

and this maximum corresponds to the time

$$t_0 = 486. \quad (12)$$

For further calculations, we choose 13 extrema (to round it off), approximately symmetrical with respect to the central maximum, see Table 1.

Table 1.

m	t_m	$m-m_0$	$t-t_0$	$\sin\left(\frac{2\pi}{T}(t-t_0)\right)$	x_m, мкм
1	118	-26	-368	-0,983	-4,16
5	224	-22	-262	-0,834	-3,52
10	302	-17	-184	-0,639	-2,72
15	361	-12	-125	-0,454	-1,92
20	414	-7	-72	-0,268	-1,12
25	465	-2	-21	-0,079	-0,32
27	486	0	0	0,000	0,00
30	515	3	29	0,109	0,48
35	567	8	81	0,300	1,28
40	623	13	137	0,493	2,08
45	684	18	198	0,679	2,88
50	766	23	280	0,870	3,68
53	855	26	369	0,984	4,16

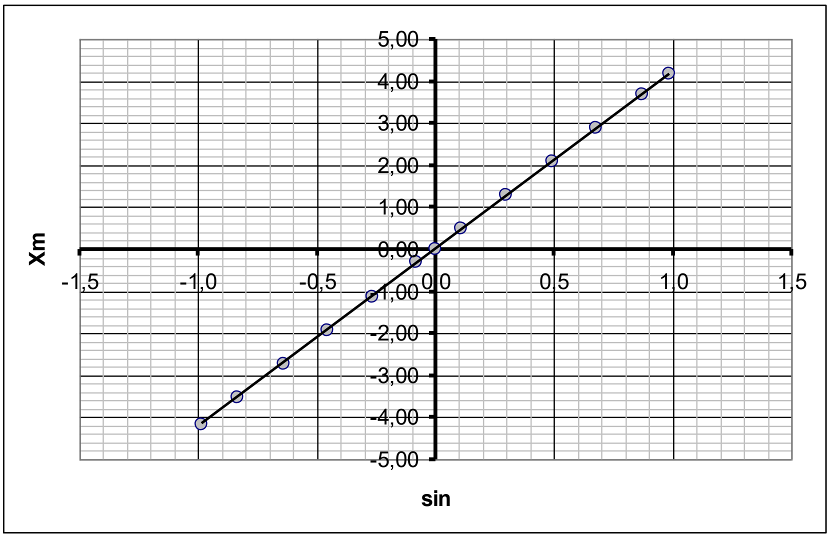
Let us carry out the following calculations:
Extremum number relative to the center $m' = m - m_0$;
Temporal shift from the center $t' = t - t_0$;
The mirror coordinates at intensity extrema $x_m = m'\frac{\lambda_0}{4}$.

To test the applicability of the law of motion

$$x(t) = A\sin\left(\frac{2\pi}{T}(t-t_0)\right) \tag{13}$$

we plot the dependence of coordinates x_m on $S = \sin\left(\frac{2\pi}{T}(t-t_0)\right)$. It should be noted that when choosing points for plotting, you should:
- select points with the maximum range of coordinates change;
- do not include utmost extremes, since they may not satisfy condition (6).

Below is a graph of this relation.



The linearity of this graph confirms the applicability of formula (13) for describing the law of mirror motion. The parameters of this dependence calculated by the least squares method have the following numerical values:

$A = (4.23 \pm 0.01)\ \mu\text{m}$

$b = (0.04 \pm 0.06)\ \mu\text{m}$

(14)

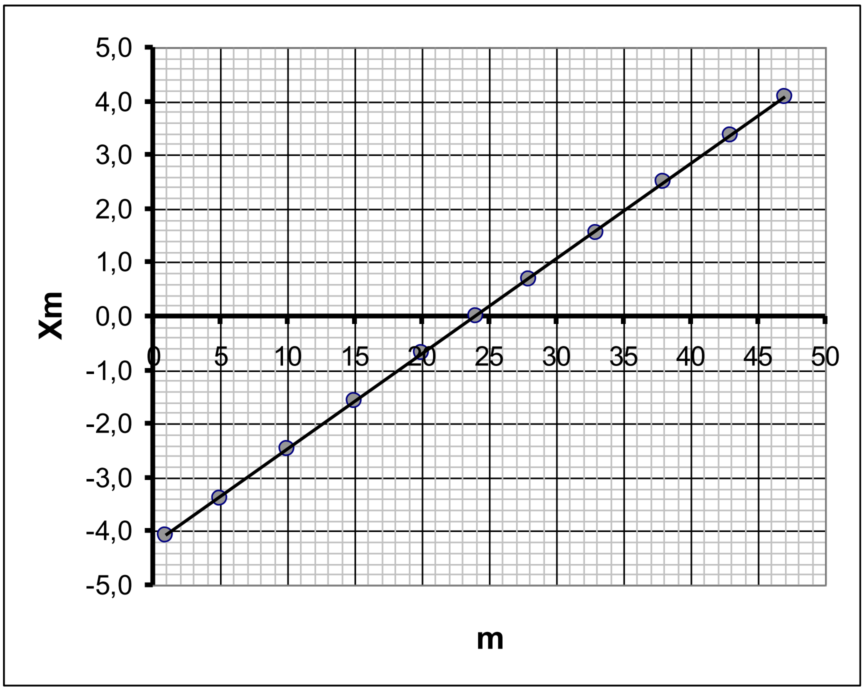
The slope coefficient A is the amplitude of the mirror oscillation. When this numerical value of the shift parameter b is less than its error, therefore, it can be assumed that $b = 0$, and the analyzed dependence is directly proportional.

3. Monochromatic radiation with unknown wavelength

3.1 From the table of extrema, we select symmetrical points (according to the formulated criteria)

Table 2.

m	t_m	x_m
1	138	-4,088
5	240	-3,382
10	320	-2,476
15	384	-1,586
20	442	-0,698
24	486	0,000
28	529	0,682
33	586	1,556
38	653	2,489
43	730	3,363
47	830	4,071



3.2 For each extremum, using formula (13), we calculate the value of the mirror coordinate x_m, after which we plot the dependence of the mirror coordinate on the extremum number m. This dependence is described by the formula

$$x_m = m\frac{\lambda}{4}. \tag{15}$$

The resulting graph confirms this dependence (shift along the number axis m in this case does not play a role and is due to a different numbering of extrema). The coefficient of the slope of the graph calculated by the least squares is equal to

$$a = (0{,}1771 \pm 0{,}0008)\ \mu\text{m}.$$

It follows from the form of function (15) that the radiation wavelength is equal to

$$\lambda = 4a = (0{,}709 \pm 0{,}003)\ \mu\text{m}. \tag{16}$$

4. Two monochromatic waves

4.1 Waves with different wavelengths do not interfere, in this case the recorded signal is the sum of the intensities of these waves. Using formula (3), we write an explicit expression for the dependence of the total intensity on the mirror coordinate and transform it (using the trigonometric formula for the sum of cosines):

$$\begin{aligned} U(x) &= 2I_1\cos\frac{4\pi}{\lambda_1}x + 2I_2\cos\frac{4\pi}{\lambda_2}x = \\ &= 2I_0\left(\cos\frac{4\pi}{\lambda_1}x + \cos\frac{4\pi}{\lambda_2}x\right) + 2(I_2 - I_1)\cos\frac{4\pi}{\lambda_2}x = \\ &= 4I_1\cos\left(2\pi\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)x\right)\cos\left(2\pi\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)x\right) + 2(I_2 - I_1)\cos\frac{4\pi}{\lambda_2}x \end{aligned} \tag{17}$$

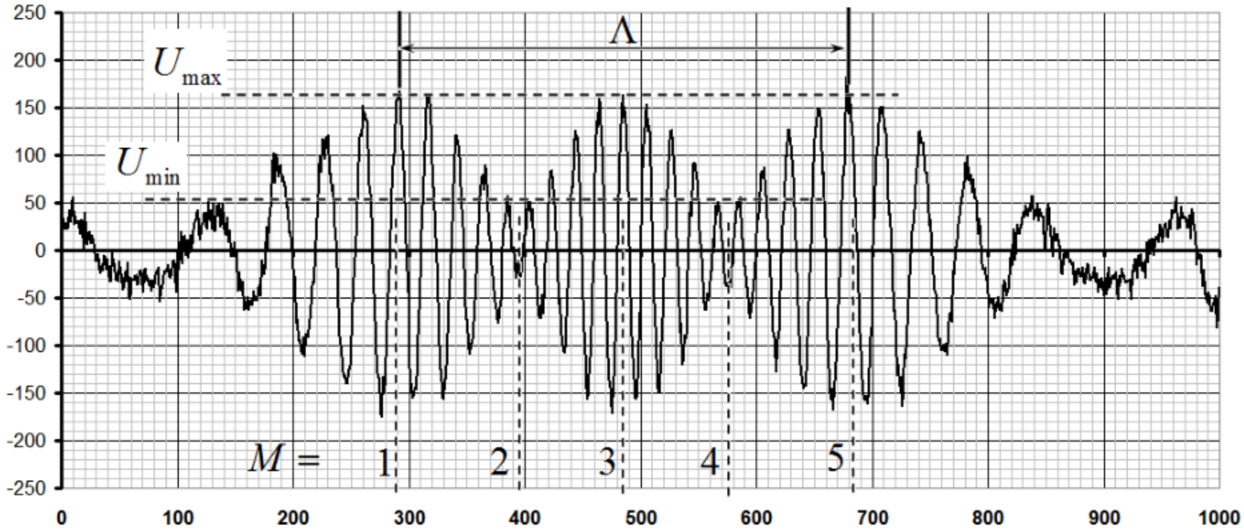
The resulting function describes the modulated signal obtained experimentally. The formula is too complicated to get explicit expressions for the extrema of this function. Therefore, the only way to obtain the required characteristics is to analyze the envelope of the fast-changing signal. Up to a constant term, this envelope is described by the function

$$\bar{U}(x) = U_1 \cos\left(2\pi\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)x\right) = U_1 \cos\left(\frac{2\pi}{\Lambda}x\right), \qquad (18)$$

where we denote

$$\left|\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right| = \frac{1}{\Lambda}. \qquad (19)$$

Here Λ – spatial period of signal modulation (see figure below).



The envelope function (18) is extremal under the condition

$$\frac{2\pi}{\Lambda}x_M = \frac{\pi}{2}M, \qquad (20)$$

where $M = 0, \pm 1, \pm 2...$ – extremum number of the envelope.

In the above signal, 5 such extrema can be distinguished, they are also shown in the figure. It follows from formula (20) that the corresponding coordinate of the mirror is determined by the formula

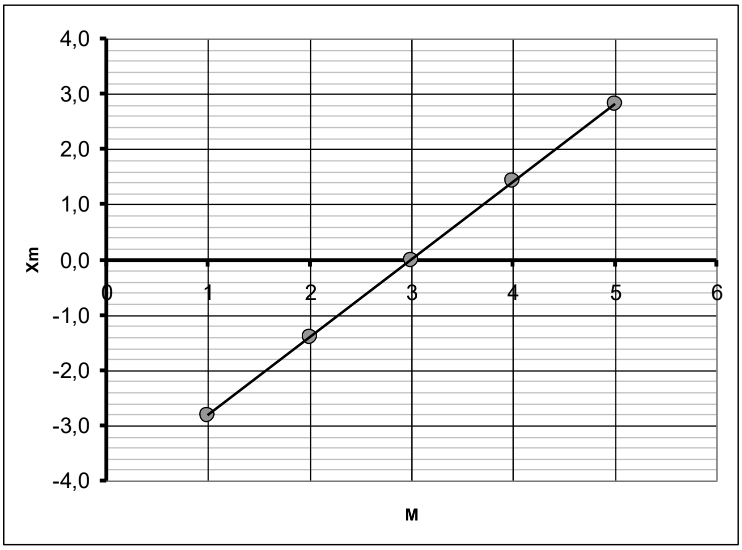
$$x_M = \frac{\Delta}{4}(M - M_0). \qquad (21)$$

Here M_0 is the "initial" number, which is insignificant for further analysis and determines the start of counting the numbers.

Using the table of extrema, we determine the times t_M at which extrema are observed, then, using formula (14), we calculate the values of the mirror coordinates x_M and plot the dependence $x_M(M)$. These values are shown in Table 3 and on the graph.

Table 3.

M	t_M	x_M, мкм
1	292	-2,823
2	397	-1,392
3	486	0,000
4	577	1,422
5	679	2,811



The slope coefficient of this graph, calculated by the least squares, is equal to

$$a = (1,41 \pm 0,01)\ \mu\text{m},$$

and as follows from formula (21), the spatial period is equal to

$$\Lambda = (5,64 \pm 0,04)\ \mu\text{m}. \qquad (22)$$

Finally, from formula (19) we calculate two possible values of the wavelength : λ_2 :

$$\lambda_{21} = \left(\frac{1}{\lambda_1} - \frac{1}{\Lambda}\right)^{-1} = 0,722\ \mu\text{m}$$
$$\Delta\lambda_{21} = \left(\frac{1}{\lambda_1} - \frac{1}{\Lambda}\right)^{-2} \frac{\Delta\Lambda}{\Lambda^2} = \left(\frac{\lambda_1}{\Lambda}\right)^2 \Delta\Lambda = 0,001\ \mu\text{m} \qquad (23)$$

$$\lambda_{22} = \left(\frac{1}{\lambda_1} + \frac{1}{\Lambda}\right)^{-1} = 0,575\ \mu\text{m}$$
$$\Delta\lambda_{22} = \left(\frac{\lambda_2}{\Lambda}\right)^2 \Delta\Lambda = 0,001\ \mu\text{m} \qquad (24)$$

4.2 To estimate the ration of intensities of the two waves, you can use the maximum and minimum values of the modulating function. From formula (17) it follows that

$$U_{\max} \approx 2(I_0 + I_1)$$
$$U_{\max} \approx 2|I_0 - I_1| \qquad (25)$$

The values $U_{\max}, U_{\min}$ can be taken approximately from the graph (or from the table) $U_{\max} \approx 170$, $U_{\min} \approx 40$. Their ratio is $\gamma = \frac{U_{\max}}{U_{\min}} \approx 4,25$. On the other hand, formulas (25) one gets

$$\gamma = \frac{I_1 + I_2}{|I_1 - I_2|} \quad \Rightarrow \quad \frac{I_1}{I_0} = \frac{1-\gamma}{1+\gamma} \approx 0,62. \qquad (26)$$

That is $I_{21} / I_1 \approx 0,62$.

The second option is also possible.: $\frac{I_{22}}{I_1} \approx 1,62$.

№	Content	For the part	points
1. Theoretical part *(incorrect coefficients - the formula is not graded)*		**1,0**	
1.1	formula (1) $I = 2I_0(1+\cos\Delta\varphi) = 4I_0\cos^2\frac{\Delta\varphi}{2}$		0,2
1.2	formula for the phase shift (2) $\Delta\varphi = \frac{4\pi}{\lambda}x$		0,2
	formula for the intensity (3) $I = 4I_0\cos^2\frac{2\pi}{\lambda}x$		0,2
1.3	formula (4) $x_m = m\frac{\lambda}{2}$		0,1
	formula (5) $x_m = \left(m+\frac{1}{2}\right)\frac{\lambda}{2}$		0,1
1.4	formula (6) $x_m = m\frac{\lambda}{4}$		0,2
2. Monochromatic radiation of a known wavelength as an		**8,0**	

instrument calibration			
2.1	Determination of the division value: - *maxima 0 and 54 – utmost positions of the mirror – 0,4;* - *the motion time – half the oscillation period – 0,2;* - *period calculation in relative units* $T = 1668$ *- 0,1;* - *calculation of the period in seconds* $T = 50$ ms *- 0,1;* - *calculation of the division value* $\Delta t = 0,030$ ms *- 0,2*		1,0
2.2	Determination of the law of motion		
	Using the found oscillation period;		0,5
	determination of the center point: - *the number of the maximum* $m_0 = \frac{0+54}{2} = 27$ *- 0,3;* - *time when passing the center point* $t_0 = 486$ *- 0,2;*		0,5
	Choice of points: - *10 or more points are used 0,3 (5 or more -0,1);* - *outmost points not included - 0,2;* - *maximum range used - 0,3;* - *points are roughly symmetrical - 0,2;*		1,0
	coordinate determination method: - *distance between adjacent extrema -* $\frac{\lambda}{4}$ *- 0,5;* - *перенумерация максимумов от среднего – 0,2;* - *formula for calculating coordinates* $x_m = m'\frac{\lambda_0}{4}$ *- 0,3;*		1,0
	coordinate calculation (the correct calculation are carried out for all selected points; the allowable calculation error is 10%)		1,0
	dependence linearization: - *dependence* x *on* $\sin\left(\frac{2\pi}{T}(t-t_0)\right)$ *- 0,4;* - *sines are calculated 0,6;*		1,0
	graph plotting: *(graded if calculations are graded);* - *axes signed and ticked – 0,1;* - *all points are plotted in accordance with the table – 0,2;* - *linear dependence is obtained -0.2;* - *smoothing straight line drawn – 0,2;*		0,7
	calculation of the amplitude of the mirror oscillations: *LSM used – 0,3 (graphically, or averaging over all points – 0,2; using 2 points 0,1);* *(calculation carried out according to the number of extrema – 0,1);* *numerical value obtained in the range 4,2 – 4,3 μm – 0,5 (in the range of 4,0 – 4,5 μm – 0,3; out of the range - 0);*		0,8
	amplitude error calculation: - *LSM used -0,3 (other methos -0,2);* - *numerical value of the order* 10^{-2} μm *- 0,3;*		0,5
3. Monochromatic radiation with unknown wavelength		**5,0**	
3.1	Choice of points: - *10 or more points used 0,3 (5 or more -0,1);* - *utmost points excluded - 0,2;*		1,0

	- maximum range used - 0,3; *- points are roughly symmetrical - 0,2;*		
	extrema coordinates calculation: *- using the correct formula for coordinates – 0,5;* *- the coordinates of extrema calculated with an error of no more than 10% - 1,0 ;*		1,5
	graph plotting: *(graded if calculations are graded);* *- axes signed and ticked – 0,1;* *- all points are plotted in accordance with the table 0,2;* *- linear dependence is obtained -0.2;* *- smoothing straight line drawn – 0,2;*		0,7
3.2	wavelength calculation: *- LSM used – 0,5 (averaging over all point – 0,3 ; 1-2 points used – 0,2);* *- numerical value in the range 0,70 – 0,72 μm – 0,8 (in the range of 0,68 -0,74 μm - 0,4, out of range – 0);*		1,3
	wavelength error calculation: *- LSM used – 0,2 (other reasonable method – 0,1);* *- value of the order* 10^{-2} *μm - 0,3;*		0,5
4. Two monochromatic waves		**6,0**	
4.1	Formula for the resulting intensity (17)		0,5
	Envelope analysis (*calculations based on the position of extrema are not graded);*		0,5
	choice of extremum points of the envelope: *- 0,2 for each extremum;*		1,0
	wavelength calculation formula $\left\lvert \frac{1}{\lambda_0} - \frac{1}{\lambda_1} \right\rvert = \frac{1}{\Lambda}$		0,3
	two solutions for the wavelength: *LSM used (averaging over all points) – 0,4 (by using 2 points – 0,2);* *Numerical values in the ranges* *0,70 – 0,74 μm; 0,56 -0,59 μm – 2x0,6;* *In the ranges* *(0, 67-0,77 μm; 0,53 – 0,62 μm – 2x0,3;)* *Out of range – 0;*		1,6
	error estimation: *formula for error of indirect measurements* $\Delta\lambda_1 = \left(\frac{\lambda_1}{\Lambda}\right)^2 \Delta\Lambda$ *- 0,3;* *Numerical values – 2x0,1;*		0,5
4.2	Formula for calculating ratio of intensities: $\frac{I_2}{I_1} = \frac{1-\gamma}{1+\gamma}$		0,3
	two solutions for intensities ration (reference to two values);		0,3
	calculating ratio of intensities, numerical values: *in the ranges 0,5 – 0,7; 1,5-1,7 – 2x0,5;* *(in the ranges 0,4 – 0,8; 1,4 – 1,8 - 2x0,2)* *out of ranges - 0*		1,0
	TOTAL	**20,0**	

РЕШЕНИЕ ЗАДАЧ ЭКСПЕРИМЕНТАЛЬНОГО ТУРА

Фурье-спектрометр

1. Теоретическая часть

1.1 При интерференции двух волн одинаковой интенсивности, результирующая интенсивность определяется формулой

$$I = 2I_0\left(1+\cos\Delta\varphi\right) = 4I_0\cos^2\frac{\Delta\varphi}{2}. \tag{1}$$

1.2 При смещении зеркала из начального положения на величину x разность хода изменяется на величину $2x$. При этом между двумя волнами разность возникает фаз

$$\Delta\varphi = \frac{4\pi}{\lambda}x, \tag{2}$$

поэтому зависимость интенсивности от координаты имеет вид

$$I = 2I_0\cos^2\frac{2\pi}{\lambda}x. \tag{3}$$

1.3 Максимум интенсивности наблюдается, если разность хода равна целому числу длин волн, т.е.

$$2x_m = m\lambda \quad\Rightarrow\quad x_m = m\frac{\lambda}{2}, \tag{4}$$

а минимумы интенсивности возникают при выполнении условия

$$2x_m = \left(m+\frac{1}{2}\right)\lambda \quad\Rightarrow\quad x_m = \left(m+\frac{1}{2}\right)\frac{\lambda}{2}. \tag{5}$$

1.4 Интенсивность изменяется от максимума до минимума (и наоборот) при смещении зеркала на четверть длины волны. Поэтому искомые координаты зеркала описываются формулой

$$x_m = m\frac{\lambda}{4}. \tag{6}$$

2. Монохроматическое излучение известной длины волны – градуировка прибора

2.1 Из приведенного графика следует, что крайним положениям зеркала соответствуют значения времен

$$t_{\min} = 67; \quad t_{\max} = 901. \tag{7}$$

Это смещение происходит за половину периода колебаний зеркала, поэтому

$$T = 2\left(t_{\max} - t_{\min}\right) = 1668. \tag{8}$$

С другой стороны, период колебаний можно выразить через заданную частоту колебаний зеркала $\nu = 20$ *Гц*:

$$T = \frac{1}{\nu} = 5.0\cdot10^{-2}\ \text{с} = 50\ \text{мс}. \tag{9}$$

Приравнивая выражения (8) и (9), находим, что цена деления временной шкалы равна

$$\Delta t = \frac{50}{2(901-67)} = 0.030\ \text{мс}. \tag{10}$$

2.2 Приведенный график зависимости интенсивности от времени симметричен относительно «центрального» максимума, номер которого равен

$$m_0 = \frac{0+54}{2} = 27 \tag{11}$$

и этому максимуму соответствует момент времени

$$t_0 = 486. \tag{12}$$

Для дальнейших расчетов выберем 13 экстремумов (для ровного счета), примерно симметричных относительно центрального максимума. Таблица 1.

Таблица 1.

m	t_m	$m - m_0$	$t - t_0$	$\sin\left(\frac{2\pi}{T}\left(t-t_0\right)\right)$	x_m, мкм

1	118	-26	-368	-0,983	-4,16
5	224	-22	-262	-0,834	-3,52
10	302	-17	-184	-0,639	-2,72
15	361	-12	-125	-0,454	-1,92
20	414	-7	-72	-0,268	-1,12
25	465	-2	-21	-0,079	-0,32
27	486	0	0	0,000	0,00
30	515	3	29	0,109	0,48
35	567	8	81	0,300	1,28
40	623	13	137	0,493	2,08
45	684	18	198	0,679	2,88
50	766	23	280	0,870	3,68
53	855	26	369	0,984	4,16

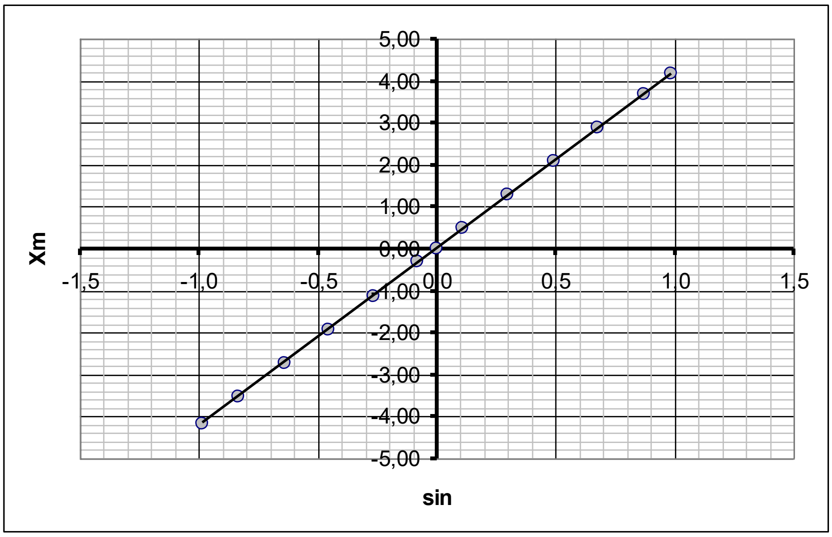
Проведем следующие расчеты:
Номер экстремума относительно центра $m' = m - m_0$;
Временной сдвиг относительно центра $t' = t - t_0$;
Координаты зеркала при экстремумах интенсивности $x_m = m' \dfrac{\lambda_0}{4}$.

Для проверки применимости закона движения

$$x(t) = A\sin\left(\frac{2\pi}{T}(t - t_0)\right) \qquad (13)$$

построим график зависимости координат x_m от $S = \sin\left(\dfrac{2\pi}{T}(t - t_0)\right)$. Следует отметить, что при выборе точек для построения графика следует:
- выбирать точки с максимальным диапазоном изменения координат;
- не включать крайние экстремумы, так как они могут не удовлетворять условию (6).

Ниже приведен график этой зависимости.



Линейность этого графика подтверждает применимость формулы (13) для описания закона движения зеркала. Рассчитанные по методу наименьших квадратов параметры этой зависимости имеют следующие численные значения:

$$A = (4.23 \pm 0.01) \text{ мкм}$$
$$b = (0.04 \pm 0.06) \text{ мкм} \qquad (14)$$

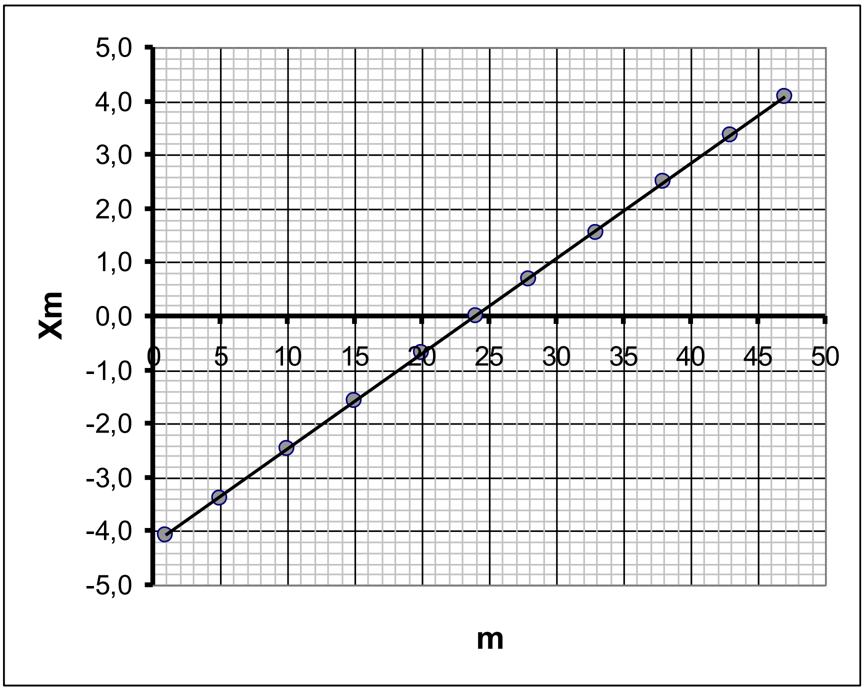
Коэффициент наклона A является амплитудой колебаний зеркала. При это численное значение параметра сдвига меньше его погрешности, поэтому можно принять, что $b = 0$, а анализируемая зависимость является прямо пропорциональной.

3. Монохроматическое излучение с неизвестной длиной волны

3.1 Из таблицы максимумов выберем симметричные точки (по сформулированным критериям)

Таблица 2.

m	t_m	x_m
1	138	-4,088
5	240	-3,382
10	320	-2,476
15	384	-1,586
20	442	-0,698
24	486	0,000
28	529	0,682
33	586	1,556
38	653	2,489
43	730	3,363
47	830	4,071



3.2 Для каждого экстремума по формуле (13) рассчитаем значение координаты зеркала x_m, после чего построим график зависимости координаты зеркала от номера экстремума m. Эта зависимость описывается функцией

$$x_m = m\frac{\lambda}{4}. \tag{15}$$

Полученный график подтверждает эту зависимость (сдвиг по оси номеров m в данном случае роли не играет и обусловлен другой нумерацией экстремумов). Рассчитанный по МНК коэффициент наклона графика равен

$$a = (0{,}1771 \pm 0{,}0008) \text{ мкм}.$$

Из вида функции (15) следует, что длина волны излучения равна

$$\lambda = 4a = (0{,}709 \pm 0{,}003) \text{ мкм}. \tag{16}$$

4. Две монохроматические волны

4.1 Волны с разными длинами не интерферируют, в данном случае зарегистрированный сигнал является суммой интенсивностей этих волн. Запишем с помощью формулы (3) явное выражение зависимости суммарной интенсивности от координаты зеркала и преобразуем его (с помощью тригонометрической формулы для суммы косинусов):

$$\begin{aligned} U(x) &= 2I_1\cos\frac{4\pi}{\lambda_1}x + 2I_2\cos\frac{4\pi}{\lambda_2}x = \\ &= 2I_0\left(\cos\frac{4\pi}{\lambda_1}x + \cos\frac{4\pi}{\lambda_2}x\right) + 2(I_2 - I_1)\cos\frac{4\pi}{\lambda_2}x = \\ &= 4I_1\cos\left(2\pi\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)x\right)\cos\left(2\pi\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)x\right) + 2(I_2 - I_1)\cos\frac{4\pi}{\lambda_2}x \end{aligned}. \tag{17}$$

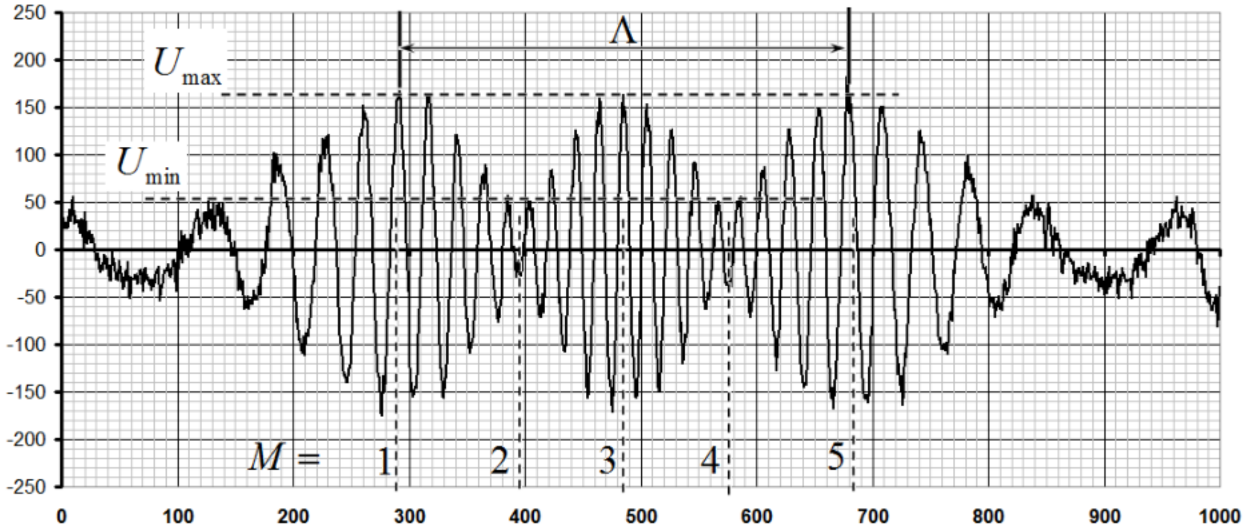
Полученная функция описывает модулированный сигнал, полученный экспериментально. Формула слишком сложна, чтобы получить явные выражения для экстремумов данной функции. Поэтому единственной возможностью для получения требуемых характеристик является анализ огибающей быстропеременного сигнала. С точностью до постоянного слагаемого эта огибающая описывается функцией

$$\bar{U}(x) = U_1 \cos\left(2\pi\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)x\right) = U_1 \cos\left(\frac{2\pi}{\Lambda}x\right), \tag{18}$$

где обозначено

$$\left|\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right| = \frac{1}{\Lambda}. \tag{19}$$

Здесь Λ – пространственный период модуляции сигнала (см. рис. ниже).



Огибающая функция (18) экстремальна при выполнении условия

$$\frac{2\pi}{\Lambda}x_M = \frac{\pi}{2}M\,, \tag{20}$$

где $M = 0, \pm 1, \pm 2...$ – номер экстремума огибающей.

В приведенном сигнале можно выделить 5 таких экстремумов, они также показаны на рисунке. Из формулы (20) следует, что соответствующая координата зеркала определяется по формуле

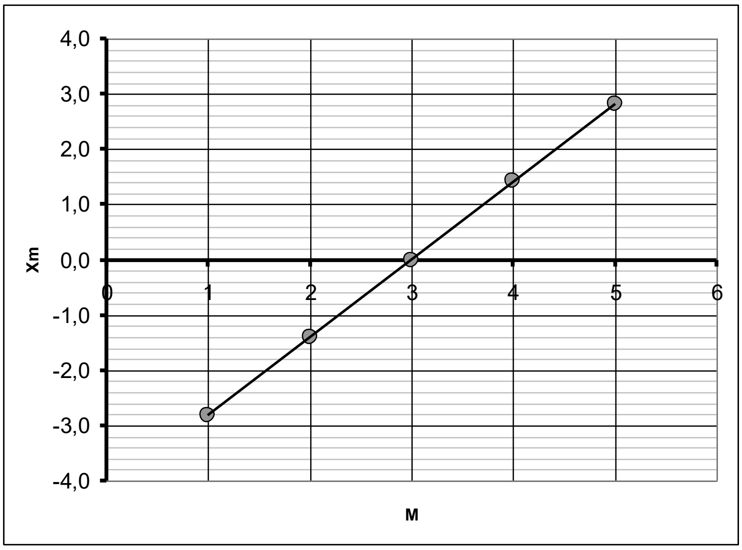
$$x_M = \frac{\Delta}{4}(M - M_0). \tag{21}$$

Здесь M_0 – несущественный для дальнейшего анализа «начальный» номер, определяющий начало отсчета номеров.

По таблице экстремумов определим значения времен t_M, при которых наблюдаются экстремумы, затем по формуле (14) рассчитаем значения координат зеркала x_M и построим график зависимости $x_M(M)$. Эти значения приведены в таблице 3 и на графике.

Таблица 3.

M	t_M	x_M, мкм
1	292	-2,823
2	397	-1,392
3	486	0,000
4	577	1,422
5	679	2,811



Коэффициент наклона этого графика, рассчитанный по МНК, равен

$$a = (1{,}41 \pm 0{,}01)\ \text{мкм},$$

а как следует из формулы (21) , пространственный период равен

$$\Lambda = (5{,}64 \pm 0{,}04)\ \text{мкм}. \tag{22}$$

Наконец, из формулы (19) рассчитаем два возможных значения длины волны λ_2 :

$$\begin{aligned} \lambda_{21} &= \left(\frac{1}{\lambda_1} - \frac{1}{\Lambda}\right)^{-1} = 0{,}722\ \text{мкм} \\ \Delta\lambda_{21} &= \left(\frac{1}{\lambda_1} - \frac{1}{\Lambda}\right)^{-2} \frac{\Delta\Lambda}{\Lambda^2} = \left(\frac{\lambda_1}{\Lambda}\right)^2 \Delta\Lambda = 0{,}001\ \text{мкм} \end{aligned} . \tag{23}$$

$$\begin{aligned} \lambda_{22} &= \left(\frac{1}{\lambda_1} + \frac{1}{\Lambda}\right)^{-1} = 0{,}575\ \text{мкм} \\ \Delta\lambda_{22} &= \left(\frac{\lambda_2}{\Lambda}\right)^2 \Delta\Lambda = 0{,}001\ \text{мкм} \end{aligned} \tag{24}$$

4.2 Для оценки интенсивности второй волны можно воспользоваться максимальным и минимальным значениями модулирующей функции. Из формулы (17) следует, что

$$\begin{aligned} U_{\max} &\approx 2(I_0 + I_1) \\ U_{\max} &\approx 2|I_0 - I_1| \end{aligned} . \tag{25}$$

Значения $U_{\max}, U_{\min}$ можно приближенно снять с графика (или из таблицы) $U_{\max} \approx 170$, $U_{\min} \approx 40$. Их отношение $\gamma = \dfrac{U_{\max}}{U_{\min}} \approx 4{,}25$. С другой стороны, из формул (25) следует

$$\gamma = \frac{I_1 + I_2}{|I_1 - I_2|} \quad \Rightarrow \quad \frac{I_1}{I_0} = \frac{1-\gamma}{1+\gamma} \approx 0{,}62 . \tag{26}$$

То есть $I_{21} / I_1 \approx 0{,}62$.

Возможен и второй вариант: $\dfrac{I_{22}}{I_1} \approx 1{,}62$.

№	Содержание	За часть	баллы
1. Теоретическая часть *(не верные коэффициенты – формула не оценивается)*		**1,0**	
1.1	формула (1) $I = 2I_0(1+\cos\Delta\varphi) = 4I_0\cos^2\dfrac{\Delta\varphi}{2}$		0,2
1.2	формула для сдвига фаз (2) $\Delta\varphi = \dfrac{4\pi}{\lambda}x$		0,2
	формула для интенсивности (3) $I = 4I_0\cos^2\dfrac{2\pi}{\lambda}x$		0,2
1.3	формула (4) $x_m = m\dfrac{\lambda}{2}$		0,1
	формула (5) $x_m = \left(m + \dfrac{1}{2}\right)\dfrac{\lambda}{2}$		0,1
1.4	формула (6) $x_m = m\dfrac{\lambda}{4}$		0,2
2. Монохроматическое излучение известной длины волны – градуировка прибора		**8,0**	

2.1	Определение цены деления: *- максимумы 0 и 54 – крайние положения зеркала – 0,4;* *- время движения – половина периода колебаний – 0,2;* *- расчет периода в отн. единицах* $T = 1668$ *- 0,1;* *- расчет периода в секундах* $T = 50 мс$ *- 0,1;* *- расчет цены деления* $\Delta t = 0{,}030 мс$ *- 0,2*		1,0
2.2	Определение закона движения		
	Использование найденного периода колебаний;		0,5
	определение центральной точки: *- номер максимума* $m_0 = \dfrac{0+54}{2} = 27$ *- 0,3;* *- время прохождения центральной точки* $t_0 = 486$ *- 0,2;*		0,5
	выбор точек: *- использовано 10 и более точек 0,3 (5 и более -0,1);* *- не включены крайние - 0,2;* *- использован максимальный диапазон - 0,3;* *- точки примерно симметричны - 0,2;*		1,0
	метод определения координат: *- расстояние между соседними экстремумами -* $\dfrac{\lambda}{4}$ *- 0,5;* *- перенумерация максимумов от среднего – 0,2;* *- формула для расчета координаты* $x_m = m'\dfrac{\lambda_0}{4}$ *- 0,3;*		1,0
	расчет координат *(проведен правильный расчет по всем выбранным точкам допустимая погрешность расчета 10%)*		1,0
	линеаризация зависимости: *- зависимость* x *от* $\sin\left(\dfrac{2\pi}{T}(t-t_0)\right)$ *- 0,4;* *- проведен расчет синусов 0,6;*		1,0
	построение графика: *(оценивается, если оценены расчеты);* *- оси подписаны и оцифрованы – 0,1;* *- нанесены все точки в соответствии с таблицей 0,2;* *- получена линейная зависимость -0.2;* *- проведена сглаживающая прямая линия – 0,2;*		0,7
	расчет амплитуды колебаний зеркала: *использован МНК – 0,3 (графически, или усреднение по всем точкам – 0,2; по 2 точкам 0,1);* *(расчет проведен по числу экстремумов – 0,1);* *получено численное значение в диапазоне 4,2 – 4,3 мкм – 0,5 (в диапазоне 4,0 – 4,5 мкм – 0,3; вне диапазона - 0);*		0,8
	расчет погрешности амплитуды: *- проведен по МНК -0,3 (иным разумным способом -0,2);* *- численное значение порядка* 10^{-2} *мкм - 0,3;*		0,5
3. Монохроматическое излучение с неизвестной длиной волны		**5,0**	
3.1	Выбор точек: *- использовано 10 и более точек 0,3 (5 и более -0,1);* *- не включены крайние - 0,2;* *- использован максимальный диапазон - 0,3;* *- точки примерно симметричны - 0,2;*		1,0
	расчет координат экстремумов:		1,5

	- использование правильной формулы для координат – 0,5; *- проведен расчет координат экстремумов с погрешностью не более 10% - 1,0 ;*		
	построение графика: *(оценивается, если оценены расчеты);* *- оси подписаны и оцифрованы – 0,1;* *- нанесены все точки в соответствии с таблицей 0,2;* *- получена линейная зависимость -0.2;* *- проведена сглаживающая прямая линия – 0,2;*		0,7
3.2	расчет длины волны: *- использован МНК – 0,5 (усреднение по всем точкам – 0,3 ; по 1-2 точкам – 0,2);* *- получено численное значение в диапазоне 0,70 – 0,72 мкм – 0,8 (в диапазоне 0,68 -0,74 мкм - 0,4, вне диапазона – 0);*		1,3
	расчет погрешности длины волны: *- использован МНК – 0,2 (иной разумный способ – 0,1);* *- получено значение порядка* 10^{-2} *мкм - 0,3;*		0,5
4. Две монохроматические волны		**6,0**	
4.1	формула для суммарной интенсивности (17)		0,5
	Анализ огибающей *(расчеты по положению экстремумов не оцениваются);*		0,5
	выбор точек экстремумов огибающей: *- по 0,2 за каждый экстремум;*		1,0
	расчет длины волны формула $\left\|\frac{1}{\lambda_0}-\frac{1}{\lambda_1}\right\|=\frac{1}{\Lambda}$		0,3
	два решения для длины волны: *Метод расчета по МНК (усреднение по всем точкам) – 0,4 (по двум точкам – 0,2);* *Численные значения в диапазонах* *0,70 – 0,74 мкм; 0,56 -0,59 мкм – 2х0,6;* *В диапазонах* *(0, 67-0,77 мкм; 0,53 – 0,62 мкм – 2х0,3;)* *Вне диапазонов – 0;*		1,6
	оценка погрешностей: *формула для погрешности косвенных измерений* $\Delta\lambda_1=\left(\frac{\lambda_1}{\Lambda}\right)^2\Delta\Lambda$ *- 0,3;* *Численные значения – 2х0,1;*		0,5
4.2	Формула для расчета отношения интенсивностей: $\frac{I_2}{I_1}=\frac{1-\gamma}{1+\gamma}$		0,3
	два решения для отношения интенсивностей (есть указание на два значения);		0,3
	расчет отношения интенсивностей, численные значения: *в диапазонах 0,5 – 0,7; 1,5-1,7 – 2х0,5;* *(в диапазонах 0,4 – 0,8; 1,4 – 1,8 - 2х0,2)* *Вне диапазонов - 0*		1,0
	ВСЕГО	**20,0**	

ЭКСПЕРИМЕНТТІК ТУРДЫҢ ТАПСЫРМАЛАРЫНЫҢ ШЕШІМІ

Фурье-спектрометр

1. Теориялық бөлім

1.1 Қарқындылығы бірдей екі толқын интерференияланғанда қортқы қарқындылық мына өрнекпен анықталады

$$I = 2I_0\left(1 - \cos\Delta\varphi\right) = 4I_0\cos^2\frac{\Delta\varphi}{2}. \tag{1}$$

1.2 Айналарды бастапқы орнынан x өлшемге ауыстырған кезде қашықтық айырмашылығы $2x$ өлшемге өзгереді. Сонымен қатар екі толқын арасында фазалық айырмашылық

$$\Delta\varphi = \frac{4\pi}{\lambda}x, \tag{2}$$

сондықтан интенсивтіліктің координатқа тәуелділігі мына түрде болады

$$I = 2I_0\cos^2\frac{2\pi}{\lambda}x. \tag{3}$$

1.3 Қарқындылық максимумы егер жол айырмасы толқын ұзындығының бүтін санына тең болса байқалады,яғни.

$$2x_{\kappa} = m\lambda \quad \Rightarrow \quad x_{\kappa} = m\frac{\lambda}{2}, \tag{4}$$

ал қарқындылықтың минимумдары мына шартта пайда болады

$$2x_{\kappa} = \left(m + \frac{1}{2}\right)\lambda \quad \Rightarrow \quad x_{\kappa} = \left(m + \frac{1}{2}\right)\frac{\lambda}{2}. \tag{5}$$

1.4 Айна толқын ұзындығының төрттен біріне ауысқанда қарқындылық максимумнан минимумға (және керісінше) өзгереді. Сондықтан айнаның қажетті координаталары мына формуламен сипатталады

$$x_m = m\frac{\lambda}{4}. \tag{6}$$

2. Белгілі толқын ұзындығының монохроматикалық сәуleленуі – аспапты калибрлеу

2.1 Жоғарыдағы графиктен айнаның шеткі жағдайларына мына уақыт мәндеріне сәйкес келетіні шығады

$$t_{min} = 67; \quad t_{max} = 901. \tag{7}$$

Бұл ығысу айнаның тербеліс периодының жартысында орын алады, сондықтан

$$T = 2\left(t_{max} - t_{min}\right) = 1668. \tag{8}$$

Екінші жағынан, тербеліс периодын берілген айна тербеліс жиілігі $\nu = 20$ Гц арқылы көрсетуге болады

$$T = \frac{1}{\nu} = 5{,}0\cdot 10^{-2}\ c = 50. \tag{9}$$

(8) және (9) өрнектерін теңестіре отырып, біз уақыт шкаласының бағасының мынаған тең екенін табамыз.

$$\Delta t = \frac{50}{2(901-67)} = 0{,}030.$$

(10)

2.2 Қарқындылықтың уақытқа тәуелділігінің жоғарыдағы графигі «орталық» максимумға қатысты симметриялы, оның номері мынаған тең

$$m_0 = \frac{0+54}{2} = 27$$

(11)

және бұл максимум мынадай уақытқа сәйкес келеді

$$t_0 = 486 .$$

(12)

Әрі қарай есептеулер үшін орталық максимумға қатысты шамамен симметриялы 13 экстремумды (жақсы өлшем үшін) таңдаймыз. 1-кесте.

1-кесте.

m	t_m	$m - m_0$	$t - t_0$	$\sin\left(\frac{2\pi}{T}(t-t_0)\right)$	x_m, мкм
1	118	-26	-368	-0,983	-4,16
5	224	-22	-262	-0,834	-3,52
10	302	-17	-184	-0,639	-2,72
15	361	-12	-125	-0,454	-1,92
20	414	-7	-72	-0,268	-1,12
25	465	-2	-21	-0,079	-0,32
27	486	0	0	0,000	0,00
30	515	3	29	0,109	0,48
35	567	8	81	0,300	1,28
40	623	13	137	0,493	2,08
45	684	18	198	0,679	2,88
50	766	23	280	0,870	3,68
53	855	26	369	0,984	4,16

Келесі есептеулерді орындайық:

Орталыққа қатысты экстремум саны $m' = m - m_0$;

Орталықтан уақыт бойынша ауытқу $t' = t - t_0$;

Қарқындылықтың экстремумындағы айна координаттары $x_m = m' \frac{\lambda_0}{4}$.

Қозғалыс заңының

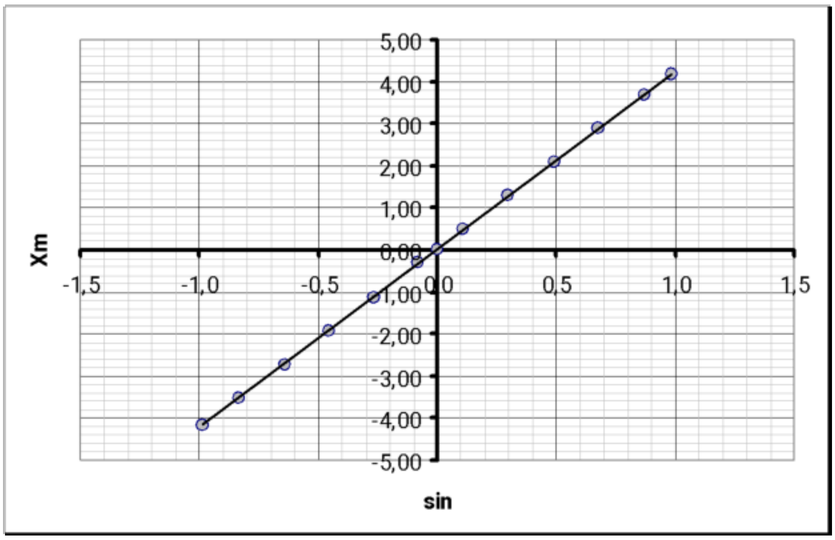
$$x(t) = A\sin\left(\frac{2\pi}{T}(t-t_0)\right)$$

(13)

қолданылуын тексеру үшін координаттардың тәуелділігін сызу x_m от $S = \sin\left(\frac{2\pi}{T}(t-t_0)\right)$. Айта кету керек, сызу үшін нүктелерді таңдаған кезде мыналар қажет:

- координаталарының максималды диапазоны бар нүктелерді таңдаңыз;
- шеткі экстремумдарды қоспаңыз, себебі олар (6) шартты қанағаттандырмауы мүмкін.

Төменде осы қатынастың графигі берілген.



Бұл графиктің сызықтылығы айна қозғалысы заңын сипаттау үшін (13) формуланың қолданылуын растайды. Ең кіші квадраттар әдісімен есептелетін бұл тәуелділіктің параметрлері келесі сандық мәндерге ие:

$$A = (4.23 \pm 0.01)$$

$$b = (0.04 \pm 0.06)$$

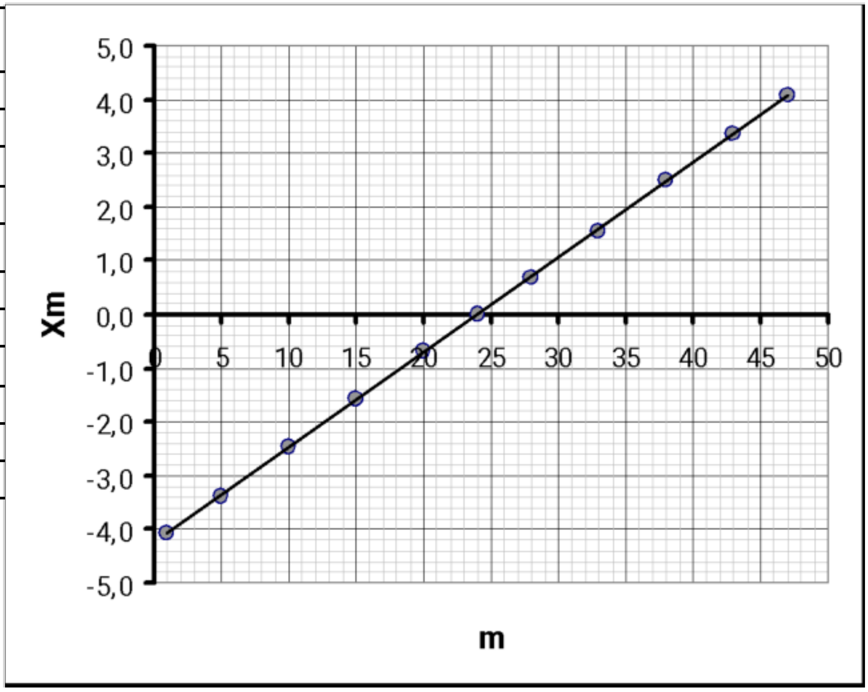
(14)

Көлбеу коэффициенті A– айна тербеліс амплитудасы. Ығысу параметрінің сандық мәні оның қателігінен аз болған кезде, $b = 0$ сондықтан талданатын тәуелділік тура пропорционалды деп болжауға болады.

3. Толқын ұзындығы белгісіз монохроматикалық сәулелену

3.1 Максимумдар кестесінен біз симметриялы нүктелерді таңдаймыз (тұжырымдалған критерийлерге сәйкес). 2-кесте.

m	t_m
1	138
5	240
10	320
15	384
20	442
24	486
28	529
33	586
38	653
43	730
47	830



3.2 Әрбір экстремум үшін (13) формуланы қолданып, айна координатының мәнін есептейміз x_m, содан кейін айна координатының m экстремум санына тәуелділігін сызамыз. Бұл тәуелділік мына функция арқылы сипатталады

$$x_m = m\frac{\lambda}{4}.$$

(15)

Алынған график бұл тәуелділікті растайды (бұл жағдайда сандар осі бойынша жылжу рөл атқармайды және ол экстремумдардың басқа нөмірленуіне байланысты). Ең кіші квадраттармен есептелген графиктің көлбеу коэффициенті мынаған тең

$$a = (0,1771 \pm 0,0008)\ \text{мкм}.$$

Функцияның (15) түрінен сәулелену толқын ұзындығының мынаған тең болатыны шығады

$$\lambda = 4a = (0,709 \pm 0,003)\ \text{мкм}.$$

(16)

4. Екі монохроматикалық толқындар

4.1 Әртүрлі ұзындықтағы толқындар кедергі жасамайды, бұл жағдайда жазылған сигнал осы толқындардың қарқындылығының қосындысы болып табылады. (3) формуланы пайдаланып, толық қарқындылықтың айна координатасына тәуелділігі үшін айқын өрнек жазамыз және оны түрлендіреміз (косинустардың қосындысының тригонометриялық формуласын пайдалана отырып):

$$U(x) = 2I_1\cos\frac{4\pi}{\lambda_1}x + 2I_2\cos\frac{4\pi}{\lambda_2}x =$$
$$= 2I_0\left(\cos\frac{4\pi}{\lambda_1}x + \cos\frac{4\pi}{\lambda_2}x\right) + 2(I_2 - I_1)\cos\frac{4\pi}{\lambda_2}x =$$
$$= 4I_1\cos\left(2\pi\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)x\right)\cos\left(2\pi\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)x\right) + 2(I_2 - I_1)\cos\frac{4\pi}{\lambda_2}x.$$

(17)

Алынған функция тәжірибелік жолмен алынған модуляцияланған сигналды сипаттайды. Бұл функцияның экстремумдары үшін айқын өрнектерді алу үшін формула тым күрделі. Сондықтан қажетті сипаттамаларды алудың бірден-бір жолы - тез өзгеретін сигналды жуықтап сыза отырып талдау. Тұрақты мүшеге дейін бұл жуықтау мына функция арқылы сипатталады

$$\tilde{U}(x) = U_1\cos\left(2\pi\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)x\right) = U_1\cos\left(\frac{2\pi}{\Lambda}x\right),$$

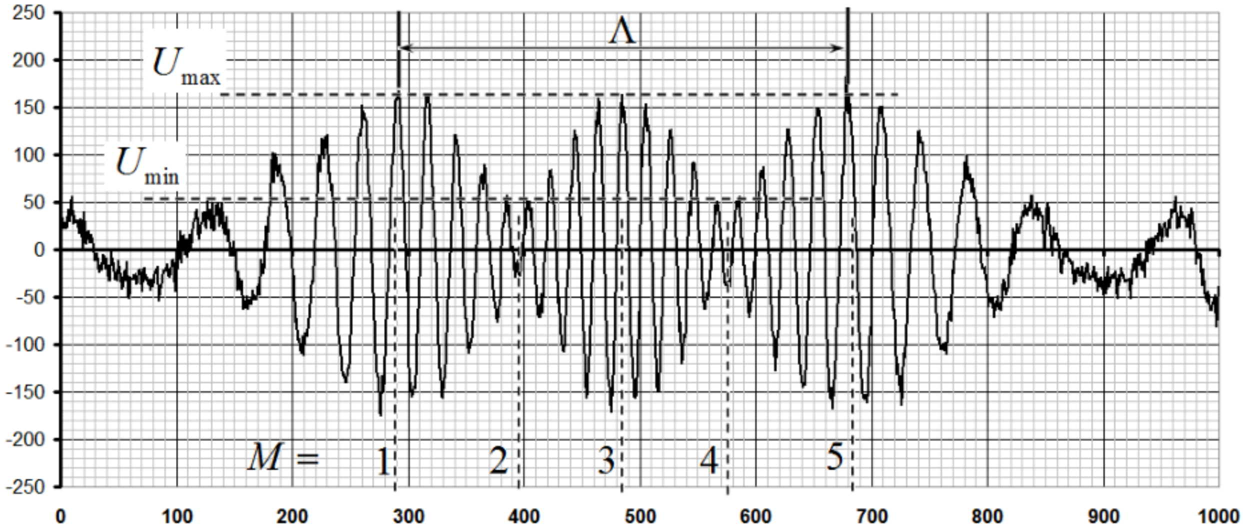
(18)

мұндағы

$$\left|\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right| = \frac{1}{\Lambda}.$$

(19)

Ал Λ – сигнал модуляциясының кеңістіктік периоды (төмендегі суретті қараңыз).



Жуық функция (18) мына шарт орындалғанда экстремальді

$$\frac{2\pi}{\Lambda}x_M = \frac{\pi}{2}M,$$

(20)

мұндағы $M = 0, \pm 1, \pm 2...$ –экстремума номері.

Жоғарыдағы сигналда осындай 5 экстремалды ажыратуға болады, олар да суретте көрсетілген. (20) формуладан айнаның сәйкес координатасы формула бойынша анықталатыны шығады

$$x_M = \frac{\Delta}{4}\left(M - M_0\right).$$

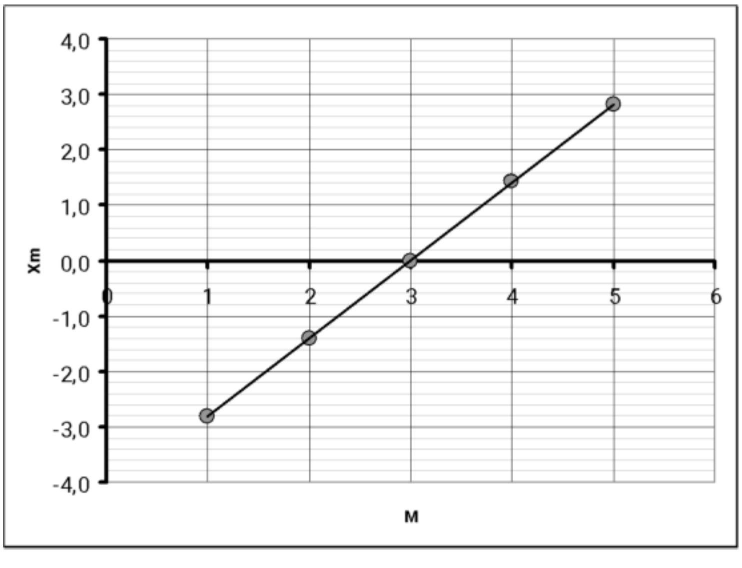
(21)

Мұндағы M_0 – «бастапқы» сан, әрі қарай талдау үшін елеусіз, ол сандарды санаудың басталуын анықтайды.

Экстремумдар кестесін пайдалана отырып, экстремумдардың байқалатын уақыттарын анықтаймыз, содан кейін (14) формуланы пайдаланып, айна координаттарының мәндерін есептеп, тәуелділікті сызамыз. Бұл мәндер 3-кестеде және графикте көрсетілген.

3-кесте.

M	t_M	x_M, мкм
1	292	-2,823
2	397	-1,392
3	486	0,000
4	577	1,422
5	679	2,811



Бұл графиктің ең кіші квадраттармен есептелген көлбеу коэффициенті тең

$$a = (1,[illegible] \pm 0,01)$$,

және (21) формуладан келесідей, кеңістіктік периодын аламыз

$$\Lambda = (5,64 \pm 0,04)$$.

(22)

Соңында (19) формуладан λ_2 толқын ұзындығының екі мүмкін мәнін есептейміз:

$$\lambda_{21} = \left(\frac{1}{\lambda_1} - \frac{1}{\Lambda}\right)^{-1} = 0,722$$

$$\Delta\lambda_{21} = \left(\frac{1}{\lambda_1} - \frac{1}{\Lambda}\right)^{-2} \frac{\Delta\Lambda}{\Lambda^2} = \left(\frac{\lambda_1}{\Lambda}\right)^2 \Delta\Lambda = 0,001$$.

(23)

$$\lambda_{22} = \left(\frac{1}{\lambda_1} - \frac{1}{\Lambda}\right)^{-1} = 0,575$$

$$\Delta\lambda_{22} = \left(\frac{\lambda_2}{\Lambda}\right)^2 \Delta\Lambda = 0,001$$

(24)

4.2 Екінші толқынның қарқындылығын бағалау үшін модуляциялау функциясының максималды және минималды мәндерін пайдалануға болады. (17) формуладан былай шығады

$$U_{max} \approx 2(I_0 + I_1)$$

$$U_{max} \approx 2\, I_0 - I_1$$.

(25)

U_{max}, U_{min} мәндерін жуықтап графиктен (болмаса кестеден) анықтауға болады $U_{max} \approx 170$, $U_{min} \approx 40$. Олардың қатынасы $\gamma = \dfrac{U_{max}}{U_{min}} \approx 4,25$. Екінші жағынан (25) өрнектен мынау шығады

$$\gamma = \frac{I_1 + I_2}{|I_1 - I_2|} \quad \Rightarrow \quad \frac{I_1}{I_0} = \frac{1-\gamma}{1+\gamma} \approx 0,62$$.

(26)

Яғни $I_{21} / I_1 \approx 0,62$.

Мынадай нұсқа да мүмкін: $\dfrac{I_{22}}{I} \approx 1,62$.

№	Содержание	За часть	баллы
1. Теориялық бөлім *(коэффициенттер дұрыс болмаса өрнектер бағаланбайды)*		**1,0**	
1.1	Өрнек (1) $I = 2I_0(1 - \cos\Delta\varphi) = 4I_0 \cos^2 \dfrac{\Delta\varphi}{2}$		0,2
1.2	Фазалық ығысу өрнегі (2) $\Delta\varphi = \dfrac{4\pi}{\lambda} x$		0,2
	Қарқындылық өрнегі (3) $I = 4I_0 \cos^2 \dfrac{2\pi}{\lambda} x$		0,2

1.3	Өрнек (4) $x_m = m\dfrac{\lambda}{2}$		0,1
	Өрнек (5) $x_m = \left(m + \dfrac{1}{2}\right)\dfrac{\lambda}{2}$		0,1
1.4	Өрнек (6) $x_m = m\dfrac{\lambda}{4}$		0,2
2. Белгілі толқын ұзындығының монохроматикалық сәулеленуі – аспапты калибрлеу		**8,0**	
2.1	Бөлік құнын анықтау: - *максимумдар 0 және 54 – айнаның шеткі күйі – 0,4;* - *қозғалу уақыты – жарты период – 0,2;* - *периодты салыстр бірлікте өлшеу* $T = 1668$ *- 0,1;* - *периодты секундпен өлшеу* $T = 50мс$ *- 0,1;* - *бөлік құнын есептеу* $\Delta t \quad 0,030мс$ *- 0,2*		1,0
2.2	Қозғалыс заңын анықтау		
	Табылған тербеліс периодын пайдалану;		0,5
	Орталық нүктені анықтау: - *максимум номері* $m_0 = \dfrac{0+54}{2} = 27$ *- 0,3;* - *орталық нүктені өту уақыты* $t_0 = 486$ *- 0,2;*		0,5
	Нүктелерді таңдау: - *10 және одан көп нүкте пайдаланылған 0,3 (5 и более -0,1);* - *шеткілері ескерілмеген - 0,2;* - *максималь диапазон пайдаланылған - 0,3;* - *нүктелер шамамен симметриялы - 0,2;*		1,0
	Координатты анықтау әдісі: - *көрші экстремумдардың ара қашықтығы -* $\dfrac{\lambda}{4}$ *- 0,5;* - *максимумдарды орталықтан қайта белгілеу – 0,2;* - *координатты есептейтін өрнек* $x_m = m'\dfrac{\lambda_0}{4}$ *- 0,3;*		1,0
	Координатты есептеу *(барлық таңдалған нүктелер үшін дұрыс есептеу жүргізілді, рұқсат етілген есептеу қатесі 10%)*		1,0
	Тәуелділікті линеаризациялау: - x *тің* $\sin\left(\dfrac{2\pi}{T}(t - t_0)\right)$ *-тан тәуелділігі - 0,4;* - *синустар есептелген 0,6;*		1,0
	График тұрғызу: *(есептеулер бағаланса, бағаланады);* - *осьтерге белгіленген және цифрланған – 0,1;* - *барлық нүктелер кестеге сәйкес сызылады 0,2;* - *сызықтық тәуелділік алынады -0.2;* - *түзу сызықты тегістеу – 0,2;*		0,7
	айна тербелістерінің амплитудасын есептеу:		0,8

	Ең кіші квадраттар әдісі пайдаланылған – 0,3 (графикалық түрде немесе барлық нүктелер бойынша орташалау – 0,2; 2 нүктелермен 0,1); (есептеу экстремумдар санына сәйкес жүргізілді – 0,1); диапазондағы сандық мәнді алды 4,2 – 4,3 мкм – 0,5 (диапазонда4,0 – 4,5 мкм – 0,3; диапазоннан тыс - 0);		
	амплитуда қатесін есептеу: *- МНК бойынша-0,3 (басқа әдіспен -0,2);* *- сандық мәні шамамен* 10^{-2} *мкм* *- 0,3;*		0,5
3. Толқын ұзындығы белгісіз монохроматикалық сәулелену		**5,0**	
3.1	Нүктелерді таңдау: *- 10 және одан көп нүкте пайдаланылған 0,3 (5 және көп -0,1);* *- шеткілері ескерілмеген - 0,2;* *- максималь диапазон пайдаланылған - 0,3;* *- нүктелер шамамен симметриялы - 0,2;*		1,0
	Экстремумдар координатын анықтау: *- координат үшін дұрыс формула – 0,5;* *- экстремумдар координатының қателігі 10% тан үлкен емес- 1,0 ;*		1,5
	График тұрғызу: *(есептеулер бағаланса, бағаланады);* *- осьтерге белгіленген және цифрланған – 0,1;* *- барлық нүктелер кестеге сәйкес сызылады 0,2;* *- сызықтық тәуелділік алынады -0.2;* *- түзу сызықты тегістеу – 0,2;*		0,7
3.2	Толқын ұзындығын есептеу: *- МНК пайдаланылған – 0,5 (барлық нүктелер бойынша – 0,3 ; 1-2 нүкелер – 0,2);* *- сандық мән мына диапозонда алынған 0,70 – 0,72 мкм – 0,8 (мына диапозонда 0,68 -0,74 мкм - 0,4, диапазоннан тыс – 0);*		1,3
	Толқын ұзындығының қателігін есептеу: *- МНК пайдаланылған – 0,2 (басқа әдіс – 0,1);* *- алынған мән шамамен* 10^{-2} *мкм* *- 0,3;*		0,5
4. Екі монохромат толқындар		**6,0**	
4.1	Қортынды қарқындылықтың өрнегі (17)		0,5
	Жуықтауды талдау *(экстремумдардың орнын анықтау бағаланбайды);*		0,5
	Жуықтаудың экстремум нүктелерін таңдау: *- әрбір экстремум үшін 0,2 ;*		1,0
	Толқын ұзындығының өрнегі $\left\lvert \frac{1}{\lambda_0} - \frac{1}{\lambda_1} \right\rvert = \frac{1}{\Lambda}$		0,3
	Толқын ұзындығы үшін екі шешім: *МНК бойынша есептеу (барлық нүктелер ескерілген) – 0,4 (екі нүкте – 0,2);* *Сандық мәні мына диапозонда* *0,70 – 0,74 мкм; 0,56 -0,59 мкм – 2х0,6;* *Мына диапозонда* *(0, 67-0,77 мкм; 0,53 – 0,62 мкм – 2х0,3;)* *Диапозоннан тыс – 0;*		1,6

	Қателерді есептеу: *Жанама өлшемдердің қателіктері* $\Delta\lambda = \left(\frac{\lambda_1}{\Lambda}\right)^2 \Delta\Lambda$ *- 0,3;* *Сандық мән – 2х0,1;*		0,5
4.2	Қарқындылықтың қатынасын есептейтін өрнек: $\frac{I_2}{I_1} = \frac{1-\gamma}{1+\gamma}$		0,3
	Қарқындылық қатынасы үшін екі шешім (екі шешімге нұсқау бар);		0,3
	Қарқындылықтың қатынасының сандық мәні: *Мына диапозонда 0,5 – 0,7; 1,5-1,7 – 2х0,5;* *(мына диапозонда 0,4 – 0,8; 1,4 – 1,8 - 2х0,2)* *Диапозоннан тыс - 0*		1,0
	БАРЛЫҒЫ	**20,0**	