2020 Online Physics Olympiad (OPhO): Open Contest v1.5

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Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

- Use $g = 9.81$ m/s^2 in this contest. See the constants sheet on the following page for other constants.
- This test contains 55 short answer questions. Each problem will have three possible attempts.
- The weight of each question depends on our scoring system found here. Put simply, the later questions are worth more, and the overall amount of points from a certain question decreases with the number of attempts and days that you take to solve a problem as well as the number of teams who solve it. This means that your score decreases with the number of tries and days you take to solve a given problem.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should contain at least **three** significant figures, unless otherwise specified. All answers within the 1% range will be accepted.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used. Technology and computer algebra systems like Wolfram Alpha or the one in the TI nSpire will not be needed. Attempts to use these tools will be classified as cheating.
- You are *allowed* to use Wikipedia or books in this exam. Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions.
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in SI units (meter, second, kilogram, watt) unless otherwise specified. Please answer all questions in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put letters in your answer on the submission portal! If your answer is "x meters" , input only the value x into the submission portal.
- **Do not communicate information to anyone else apart from your team-members before the exam ends on May 29, 2020 at 11:59 PM UTC**.

Sponsors

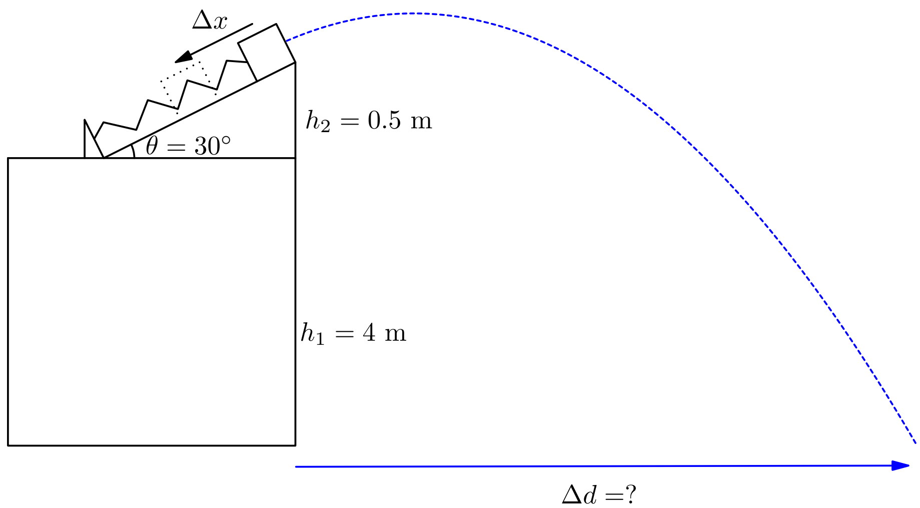
List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^8$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\mathrm{N \cdot m^2})/\mathrm{kg^2}$$
- Acceleration due to gravity, $g = 9.81$ m/s^2
- 1 unified atomic mass unit,
$$1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15}\ \mathrm{eV \cdot s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9\ (\mathrm{N \cdot m^2})/\mathrm{C^2}$$
- Permeability of free space, $\mu_0 = 4\pi \cdot 10^{-7}$ T $\cdot$ m/A
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\mathrm{T \cdot m})/\mathrm{A}$$
- 1 atmospheric pressure,
$$1\ \mathrm{atm} = 1.01 \cdot 10^5\ \mathrm{N/m^2} = 1.01 \cdot 10^5\ \mathrm{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$$

Problems

Pr 1. Angry Birds

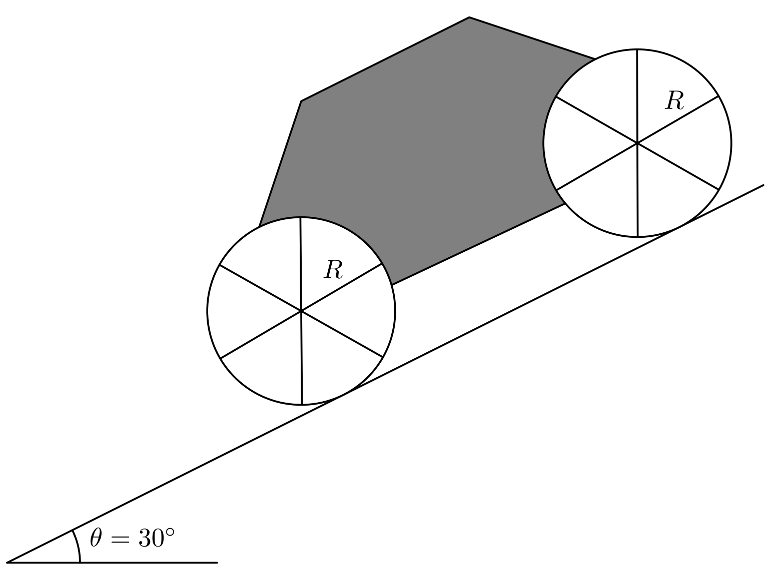
A quarantined physics student decides to perform an experiment to land a small box of mass $m = 60$ g onto the center of a target a distance Δd away. The student puts the box on a top of a frictionless ramp with height $h_2 = 0.5$ m that is angled $\theta = 30°$ to the horizontal on a table that is $h_1 = 4$ m above the floor. If the student pushes the spring with spring constant $k = 6.5$ N/m down by $\Delta x = 0.3$ m compared to its rest length and lands the box exactly on the target, what is Δd? Answer in meters. You may assume friction is negligible.



Pr 2. The Wheels on the Monster Truck go Round and Round

A wooden bus of mass $M = 20,000$ kg (M represents the mass excluding the wheels) is on a ramp with angle $30°$. Each of the four wheels is composed of a ring of mass $\frac{M}{2}$ and radius $R = 1$ m and 6 evenly spaced spokes of mass $\frac{M}{6}$ and length R. All components of the truck have a uniform density. Find the acceleration of the bus down the ramp assuming that it rolls without slipping.

Answer in m/s^2.



Pr 3. Don't make me reach my Tipping Point

THIS QUESTION HAS BEEN REMOVED FROM THE EXAM.

Pr 4. District 12

In an old coal factory, a conveyor belt will move at a constant velocity of 20.3 m/s and can deliver a maximum power of 15 MW. Each wheel in the conveyor belt has a diameter of 2 m. However a changing demand has pushed the coal factory to fill their coal hoppers with a different material with a certain constant specific density. These "coal" hoppers have been modified to deliver a constant 18 $\mathrm{m^3 s^{-1}}$ of the new material to the conveyor belt. Assume that the kinetic and static friction are the same and that there is no slippage. What is the maximum density of the material?

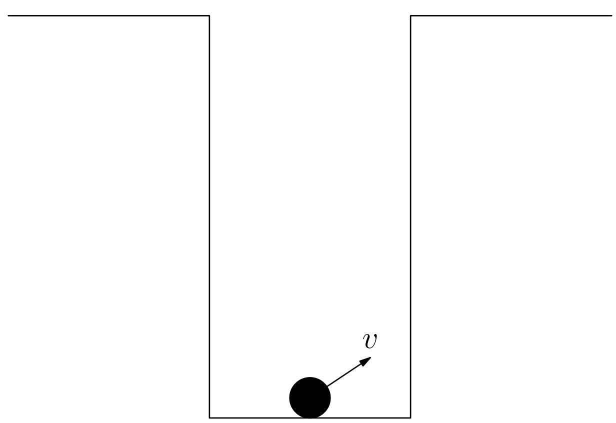
Pr 5. Neutrino Party

Neutrinos are extremely light particles and rarely interact with matter. The Sun emits neutrinos, each with an energy of 8×10^{-14} J and reaches a flux density of 10^{11} neutrinos/(s cm^2) at Earth's surface.

In the movie *2012*, neutrinos have mutated and now are completely absorbed by the Earth's inner core, heating it up. Model the inner core as a sphere of radius 1200 km, density 12.8 g/cm^3, and a specific heat of 0.400J/g K. The time scale, in seconds, that it will take to heat up the inner core by 1°C is $t = 1 \times 10^N$ where N is an integer. What is the value of N?

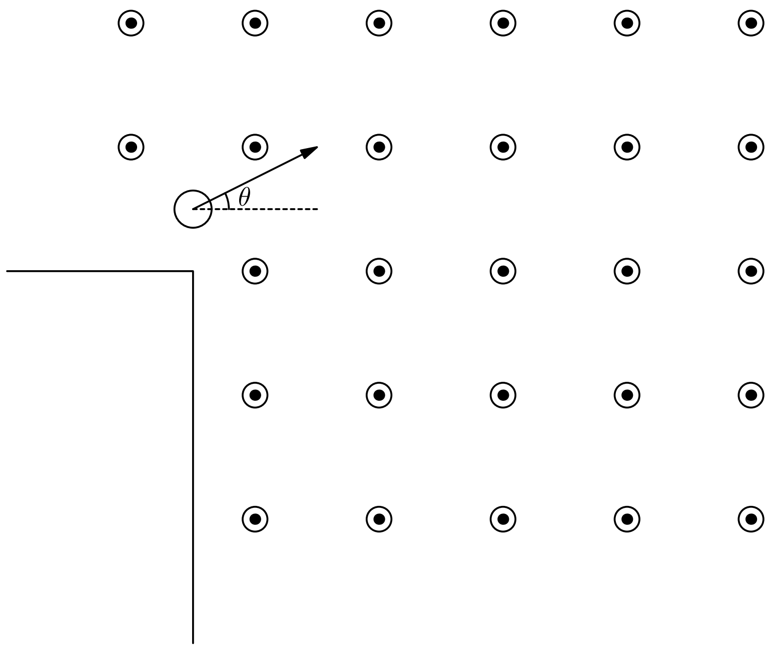
Pr 6. Quarantine Secrets

A ball is situated at the midpoint of the bottom of a rectangular ditch with width 1 m. It is shot at a velocity $v = 5$ m/s at an angle of 30° relative to the horizontal. How many times does the ball collide with the walls of the ditch until it hits the bottom of the ditch again? Assume all collisions to be elastic and that the ball never flies out of the ditch.



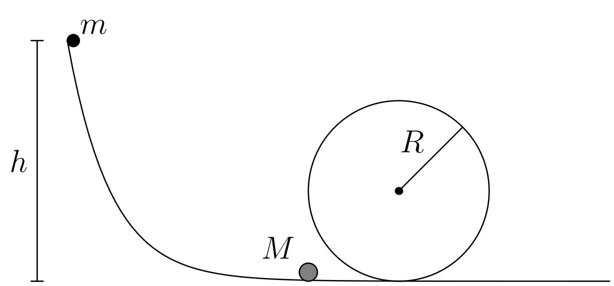
Pr 7. Planetary Proton

Professor Proton has discovered a new planet on one of his planetary expeditions. He wants to measure the magnetic field of the planet he has found. Professor Proton has brought all the necessary equipment required to carry out the following experiment. A proton is launched off a large cliff at a non-relativistic speed v and an angle $\theta = 30°$ with respect to the horizontal plane at the magnetic equator of a distant planet. The magnetic field acting perpendicularly on the particle can be assumed to be perfectly horizontal and coming out of the page, as shown in the diagram. How strong is the magnetic field at the magnetic equator of this planet if the period of oscillation of v_x is 4.94×10^{-4} s? Write your answer in terms of μT (micro-Teslas).



Pr 8. Angel Coaster

A frictionless track contains a loop of radius $R = 0.5$ m. Situated on top of the track lies a small ball of mass $m = 2$ kg at a height h. It is then dropped and collides with another ball (of negligible size) of mass $M = 5$ kg.



Let h be the minimum height that m was dropped such that M would be able to move all the way around the loop. The coefficient of restitution for this collision is given as $e = \frac{1}{2}$.

Now consider a different scenario. Assume that the balls can now collide perfectly inelastically, which means that they stick to each other instantaneously after collision for the rest of the motion. If m was dropped from a height $3R$, find the minimum value of $\frac{m}{M}$ such that the combined mass can fully move all the way around the loop. Let this minimum value be k. Compute $\alpha = \frac{k^2}{h^2}$. (Note that this question is *only* asking for α but you need to find h to find α). Assume the balls are point masses (neglect rotational effects).

Pr 9. Wannabe Twoset

Eddie is experimenting with his sister's violin. Allow the "A" string of his sister's violin have an ultimate tensile strength σ_1. He tunes a string up to its highest possible frequency f_1 before it breaks. He then builds an exact copy of the violin, where all lengths have been increased by a factor of $\sqrt{2}$ and tunes the same string again to its highest possible frequency f_2. What is f_2/f_1? The density of the string does not change.

Note: The ultimate tensile strength is maximum amount of stress an object can endure without breaking. Stress is defined as $\frac{F}{A}$, or force per unit area.

Pr 10. Waterhorse or Flyinghorse

A one horsepower propeller powered by a battery and is used to propel a small boat initially at rest. You have two options:

1. Put the propeller on top of the boat and push on the air with an initial force F_1
2. Put the propeller underwater and push on the water with an initial force F_2.

The density of water is 997 $\mathrm{kg/m^3}$ while the density of air is 1.23 $\mathrm{kg/m^3}$. Assume that the force is both cases is dependent upon only the density of the medium, the surface area of the propeller, and the power delivered by the battery. What is F_2/F_1? You may assume (unrealistically) the efficiency of the propeller does not change. Round to the nearest tenths.

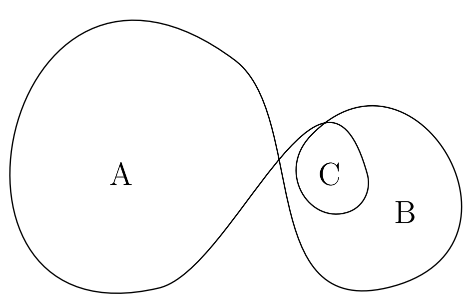
Pr 11. Charlie And The Chocolate Factory

A professional pastry chef is making a sweet which consists of 3 sheets of chocolate. The chef leaves a gap with width $d_1 = 0.1$ m between the top and middle layers and fills it with a chocolate syrup with uniform viscosity $\eta_1 = 10\,\mathrm{Pa \cdot s}$ and a gap with width $d_2 = 0.2$ m between the middle and bottom sheet and fills it with caramel with uniform viscosity $\eta_2 = 15\,\mathrm{Pa \cdot s}$. If the chef pulls the top sheet with a velocity $2\,\mathrm{m/s}$ horizontally, at what speed must he push the bottom sheet horizontally such that the middle sheet remains stationary initially? Ignore the weight of the pastry sheets throughout the problem and the assume the sheets are equally sized.

Note: Shear stress is governed by the equation $\tau = \eta \times$ rate of strain.

Pr 12. Loopy Wire

The following diagram depicts a single wire that is bent into the shape below. The circuit is placed in a magnetic field pointing out of the page, uniformly increasing at the rate $\frac{dB}{dt} = 2.34$ T/s. Calculate the magnitude of induced electromotive force in the wire, in terms of the following labelled areas ($\mathrm{m^2}$). Note that B is non-inclusive of C and that $A = 4.23$, $B = 2.74$, and $C = 0.34$.



The following information applies to the next two problems. A magnetic field is located within a region enclosed by an elliptical island with semi-minor axis of $a = 100$ m and semi-major axis of $b = 200$ m. A car carrying charge $+Q = 1.5$ C drives on the boundary of the island at a constant speed of $v = 5$ m/s and has mass $m = 2000$ kg. Any dimensions of the car can be assumed to be much smaller than the dimensions of the island. Ignore any contributions to the magnetic field from the moving car and assume that the car has enough traction to continue driving in its elliptical path.

Let the center of the island be located at the point $(0, 0)$ while the semi major and semi minor axes lie on the x and y-axes, respectively.

On this island, the magnetic field varies as a function of x and y: $B(x, y) = k_b e^{c_b xy} \hat{z}$ (pointing in the upward direction, perpendicular to the island plane in the positive z-direction). The constant $c_b = 10^{-4}$ m^{-2} and the constant $k_b = 2.1$ μT

Pr 13. Journey 2: The Magnetic Island 1

At what point on the island is the force from the magnetic field a maximum? Write the distance of this point from the x-axis in metres.

Pr 14. Journey 2: The Magnetic Island 2

Assuming no slipping, what is the magnitude of the net force on the car at the point of the maximum magnetic field? (Answer in Newtons.)

Pr 15. Tuning Outside

Inside a laboratory at room temperature, a steel tuning fork in the shape of a U is struck and begins to vibrate at $f = 426$ Hz. The tuning fork is then brought outside where it is $10°C$ hotter and the experiment is performed again. What is the change in frequency, Δf of the tuning fork? (A positive value will indicate an increase in frequency, and a negative value will indicate a decrease.)

Note: The linear thermal coefficient of expansion for steel is $\alpha = 1.5 \times 10^{-5}$ K^{-1} and you may assume the expansion is isotropic and linear. When the steel bends, there is a restoring torque $\tau = -\kappa\theta$ such that $\kappa \equiv GJ$ where $G = 77$ GPa is constant and J depends on the geometry and dimensions of the cross-sectional area.

Pr 16. Too Much Potential

A large metal conducting sphere with radius 10 m at an initial potential of 0 and an infinite supply of smaller conducting spheres of radius 1 m and potential 10 V are placed into contact in such a way: the large metal conducting sphere is contacted with each smaller sphere one at a time. You may also assume the spheres are touched using a thin conducting wire that places the two spheres sufficiently far away from each other such that their own spherical charge symmetry is maintained. What is the least number of smaller spheres required to be touched with the larger sphere such that the potential of the larger sphere reaches 9 V? Assume that the charges distribute slowly and that the point of contact between the rod and the spheres is not a sharp point.

Pr 17. Particle in the Box

During high speed motion in a strong electric field, a charged particle can ionize air molecules it collides with.

A charged particle of mass $m = 0.1$ kg and charge $q = 0.5$ μC is located in the center of a cubical box. Each vertex of the box is fixed in space and has a charge of $Q = -4$ μC. If the side length of the box is $l = 1.5$ m what minimum speed (parallel to an edge) should be given to the particle for it to exit the box (even if it's just momentarily)? Let the energy loss from Corona discharge and other radiation effects be $E = 0.00250$ J.

Pr 18. Room of Mirrors 1

Max finds himself trapped in the center of a mirror walled equilateral triangular room. What minimum beam angle must his flashlight have so that any point of illumination in the room can be traced back to his flashlight with at most 1 bounce? (Answer in degrees.) Since the room is large, assume the person is a point does not block light. Visualize the questions in a 2D setup. The floor/ceiling is irrelevant.

The point of illumination refers to any point in the room that is lit.

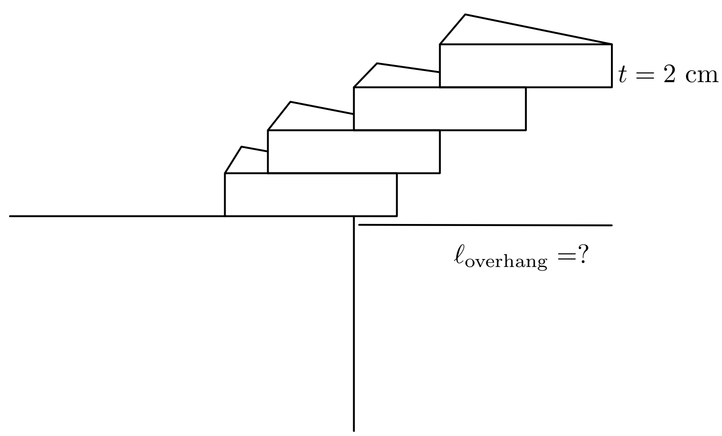
Pr 19. Room of Mirrors 2

Max is again trapped at the center of the same room (from the previous problem) but with a much less powerful flashlight. What minimum beam angle must his flashlight have so that any point of illumination in the room can be traced back to his flashlight with at most 100 bounces? (Answer in degrees.)

Pr 20. Secret Society

For his art project, Weishaupt cut out $N = 20$ wooden equilateral triangular blocks with a side length of $\ell = 10$ cm and a thickness of $t = 2$ cm, each with the same mass and uniform density. He wishes to stack one on top of the other overhanging the edge of his table. In centimeters, what is the maximum overhang? Round to the nearest centimeter. A side view is shown below. Assume that all triangles are parallel to each other.

Note: This diagram is not to scale.



The following information applies to the following three problems. Kushal finds himself trapped in a large room with mirrors as walls. Being scared of the dark, he has a powerful flashlight to light the room. All references to "percent" refer to area. Since the room is large, assume the person is a point does not block light. Visualize the questions in a 2D setup. The floor/ceiling is irrelevant. *The point of illumination refers to any point in the room that is lit.*

Pr 21. Focus On That Not This! 1

What percent of a large circular room can be lit up using a flashlight with a 20 degree beam angle if Kushal stands in the center?

Pr 22. Focus On That Not This! 2

Kushal stands at a focus of an elliptical room with eccentricity 0.5 and semi major axis = 20 m. He points the flashlight along the semi-major axis away from the other focus. Find the ideal position where the torch can be placed to catch fire easily by the beam from the flashlight. What is the distance from this point to Kushal? Note that the torch cannot be at the same location as the flashlight. (Answer in metres.)

Pr 23. Focus On That Not This! 3

Now Kushal stands at a focus of the same elliptical room as in Problem 22. Determine the minimum percent of the elliptical room that can be lit up with a flashlight of beam angle 1 degree.

Pr 24. Two Star Crossed Lovers...

Two identical neutron stars with mass $m = 4 \times 10^{30}$ kg and radius 15 km are orbiting each other a distance $d = 700$ km away from each other (d refers to the initial distance between the cores of the neutron stars). Assume that they orbit as predicted by classical mechanics, except that they generate gravitational waves. The power dissipated through these waves is given by:

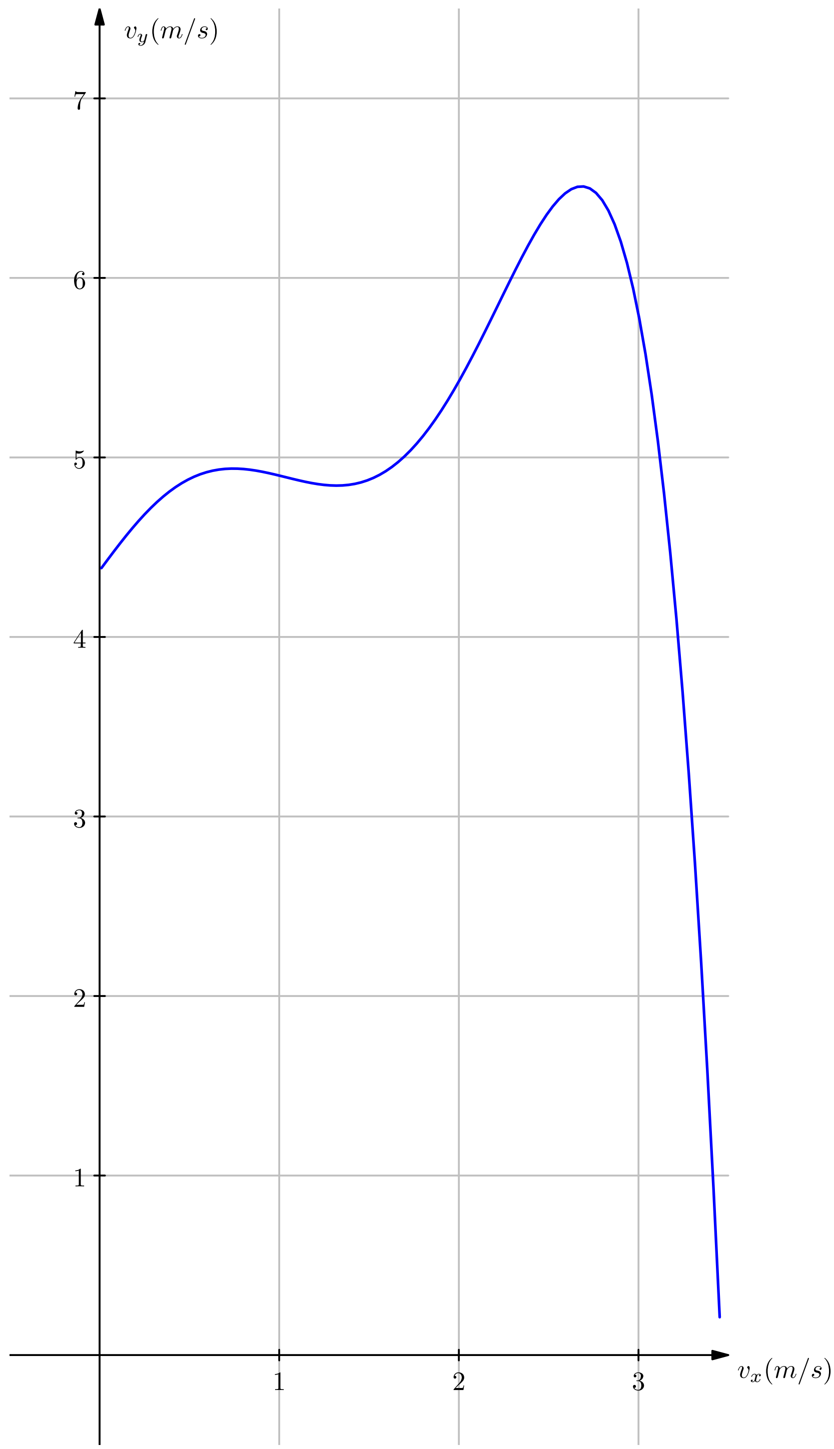
$$P = \frac{32G^4}{5}\left(\frac{m}{dc}\right)^5$$

How long does it take for the two stars to collide? Answer in seconds. *Note:* d is the distance between the *cores* of the stars.

Pr 25. Too Bored

The graph provided plots the y-component of the velocity against the x-component of the velocity of a kiddie roller coaster at an amusement park for a certain duration of time. The ride takes place entirely in a two dimensional plane.

Some students made a remark that at one time, the acceleration was perpendicular to the velocity. Using this graph, what is the minimum x-velocity the ride could be travelling at for this to be true? Round to the nearest integer and answer in meters per second. The diagram is drawn to scale, and you may print this page out and make measurements.



The following information applies to the next two problems: In the cosmic galaxy, the Sun is a main-sequence star, generating its energy mainly by nuclear fusion of hydrogen nuclei into helium. In its core, the Sun fuses hydrogen to produce deuterium (2H) and tritium (3H), then makes about 600 million metric tons of helium (4He) per second. Of course, there are also some relatively smaller portions of fission reactions in the Sun's core, e.g. a nuclear fission reaction with Uranium-235 (235U). The Fusion reaction:

$$^2H + {}^3H \to {}^4He + n + \text{Released Energy}$$

The Fission reaction:

$$^{235}U + n + (\text{Initial Energy}) \to {}^{144}Ba + {}^{90}Kr + 2n + \text{Released Energy}$$

Isotope Mass (at rest)

Isotope Names	Mass (at rest) (u)
Deuterium (^2H)	2.0141
Tritium (^3H)	3.0160
Helium (^4He)	4.0026
Neutron (n)	1.0087
Uranium-235 (^{235}U)	235.1180
Barium-144 (^{144}Ba)	143.8812
Krypton-90 (^{90}Kr)	89.9471

Pr 26. You are my Sunshine 1

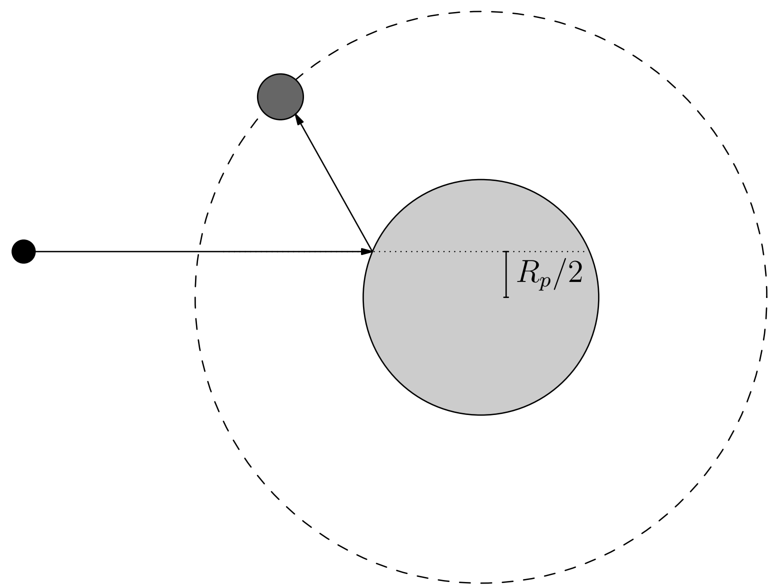
Calculate the kinetic energy (in MeV) released by the products in one fusion reaction.

Pr 27. You are my Sunshine 2

Calculate the energy produced in the core of the Sun per second from helium fusion. Answer in Joules.

Pr 28. Be Reflected It Must

While exploring outer space, Darth Vader comes upon a purely reflective spherical planet with radius $R_p = 40,000$ m and mass $M_p = 8.128 \times 10^{24}$ kg. Around the planet is a strange moon of orbital radius $R_s = 6,400,000$ m ($R_s \gg R_p$) and mass $M_s = 9.346 \times 10^{19}$ kg($M_s \ll M_p$). The moon can be modelled as a blackbody and absorbs light perfectly. Darth Vader is in the same plane that the planet orbits in. Startled, Darth Vader shoots a laser with constant intensity and power $P_0 = 2 \times 10^{32}$ W at the reflective planet and hits the planet a distance of $\frac{R_p}{2}$ away from the line from him to the center of the planet. Upon hitting the reflective planet, the light from the laser is plane polarized. The angle of the planet's polarizer is always the same as the angle of reflection. After reflectance, the laser lands a direct hit on the insulator planet. Darth Vader locks the laser in on the planet until it moves right in front of him, when he turns the laser off. Determine the energy absorbed by the satellite. Assume the reflective planet remains stationary and that the reflective planet is a perfect polarizer of light.

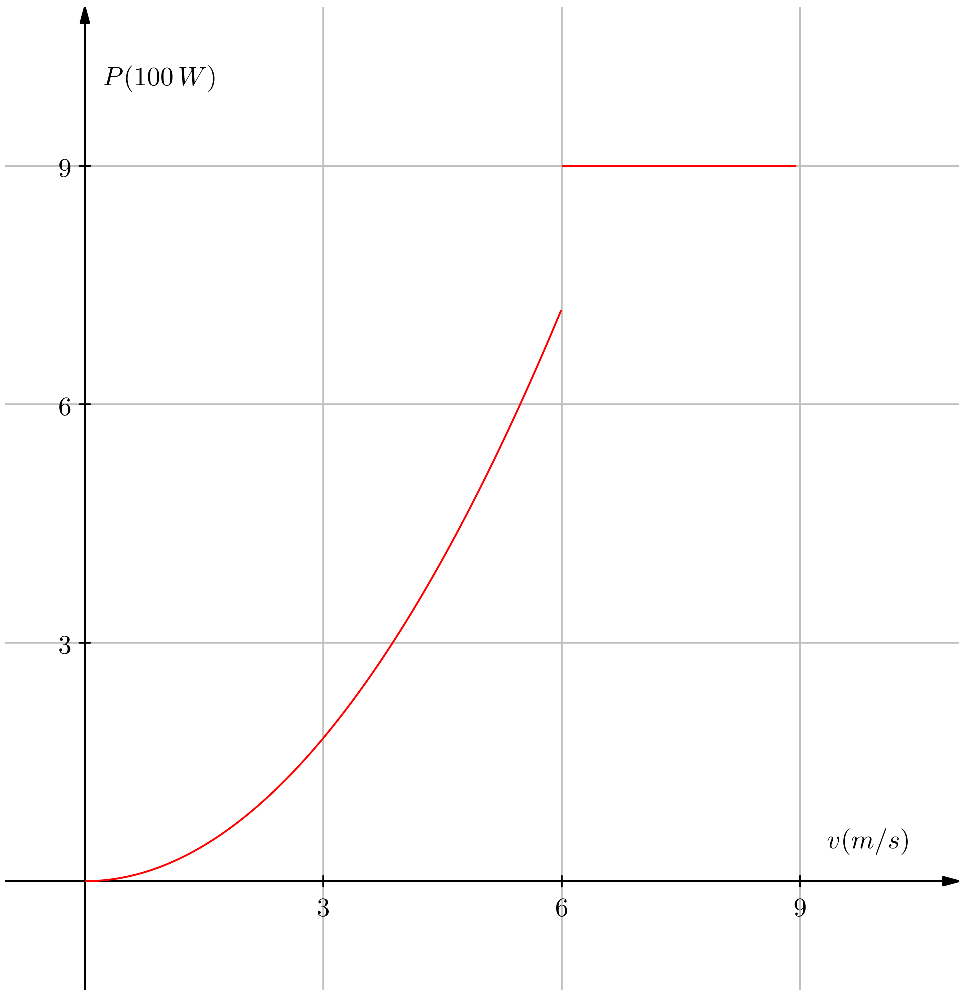


Pr 29. Braking Up

A particle of rest mass m moving at a speed $v = 0.7c$ decomposes into two photons which fly off at a separated angle θ. What is the minimum value of the angle of separation assuming that the two photons have equal wavelength. (Answer in degrees)

Pr 30. With Great Power

Mario is racing with Wario on Moo Moo Meadows when a goomba, ready to avenge all of his friends' deaths, came and hijacked Mario's kart. A graph representing the motion of Mario at any instant is shown below. The velocity acquired by Mario is shown on the x-axis, and the net power of his movement is shown on the y-axis. When Mario's velocity is 6 m/s, he eats a mushroom which gives him a super boost.



You may need to make measurements. Feel free to print this picture out as the diagram is drawn to scale. Find the total distance from Mario runs from when his velocity is 0 m/s to when his velocity just reaches 9 m/s given that Mario's mass is $m = 89$ kg. Answer in meters and round to one significant digit.

Pr 31. I'm a little teacup

At an amusement park, there is a ride with three "teacups" that are circular with identical dimensions. Three friends, Ethan, Rishab, and Kushal, all pick a teacup and sit at the edge. Each teacup rotates about its own axis clockwise at an angular speed $\omega = 1$ rad/s and can also move linearly at the same time.

The teacup Ethan is sitting on (as always) is malfunctional and can only rotate about its own axis. Rishab's teacup is moving linearly at a constant velocity 2 m/s [N] and Kushal's teacup is also moving linearly at a constant velocity of 4 m/s [N 60° E]. All three teacups are rotating as described above. Interestingly, they observe that at some point, all three of them are moving at the same velocity. What is the radius of each teacup?

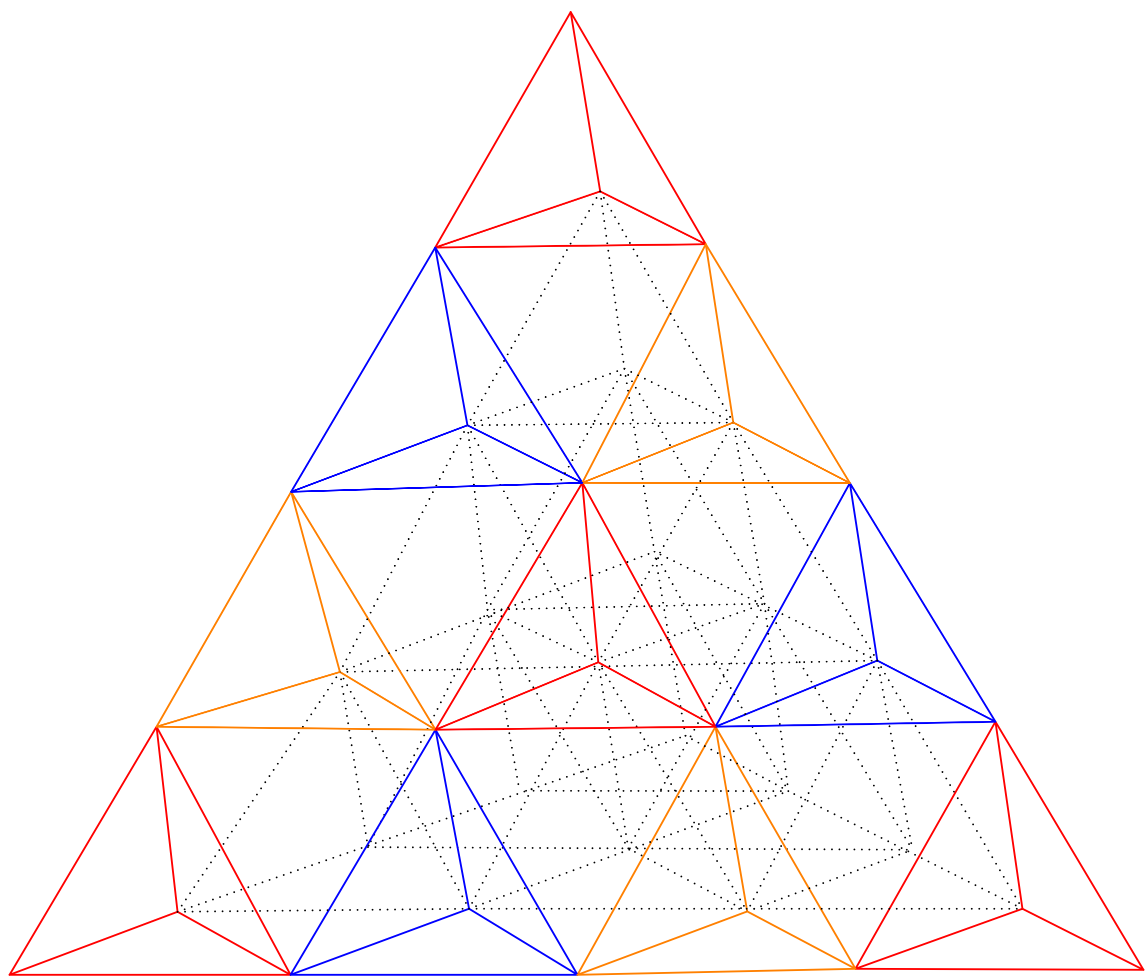
Note: [N 60° E] means 60° clockwise from north e.g. 60° east of north.

Pr 32. Tetrahedron Resistance

An engineer has access to a tetrahedron building block with side length $\ell = 10$ cm. The body is made of a thermal insulator but the edges are wrapped with a thin copper wiring with cross sectional area $S = 2\ \text{cm}^2$. The thermal conductivity of copper is 385.0 W/(m K). He stacks these tetrahedrons (all facing the same direction) to form a large lattice such that the copper wires are all in contact. In the diagram, only the front row of a small section is coloured. Assume that the lattice formed is infinitely large.

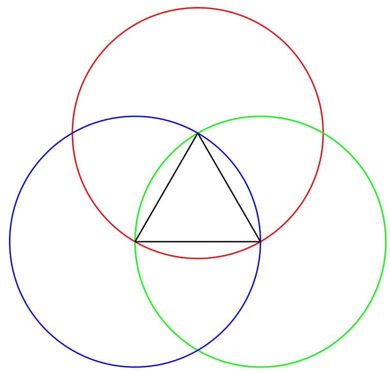
At some location in the tetrahedral building block, the temperature difference between two adjacent points is 1°C. What is the heat flow across these two points? Answer in Watts.

Note: Two adjacent points refer to two adjacent points on the tetrahedron.



Pr 33. AIME

Three unit circles, each with radius 1 meter, lie in the same plane such that the center of each circle is one intersection point between the two other circles, as shown below. Mass is uniformly distributed among all area enclosed by at least one circle. The mass of the region enclosed by the triangle shown above is 1 kg. Let x be the moment of inertia of the area enclosed by all three circles (intersection, *not* union) about the axis perpendicular to the page and through the center of mass of the triangle. Then, x can be expressed as $\frac{a\pi - b\sqrt{c}}{d\sqrt{e}}$ kg m^2, where a, b, c, d, e are integers such that $\gcd(a, b, d) = 1$ and both c and e are squarefree. Compute $a + b + c + d + e$.



Pr 34. Global Warming

Life on Earth would not exist as we know it without the atmosphere. There are many reasons for this, but one of which is temperature. Let's explore how the atmosphere affects the temperature on Earth. Assume that all thermal energy striking the earth uniformly and ideally distributes itself across the Earth's surface.

- Assume that the Earth is a perfect black body with no atmospheric effects. Let the equilibrium temperature of Earth be T_0. (The sun outputs around 3.846×10^{26} W, and is 1.496×10^{8} km away.)
- Now assume the Earth's atmosphere is isothermal. The short wavelengths from the sun are nearly unaffected and pass straight through the atmosphere. However, they mostly convert into heat when they strike the ground. This generates longer wavelengths that do interact with the atmosphere. Assume that the albedo of the ground is 0.3 and e, the emissivity and absorptivity of the atmosphere, is 0.8. Let the equilibrium average temperature of the planet now be T_1.

What is the percentage increase from T_0 to T_1?

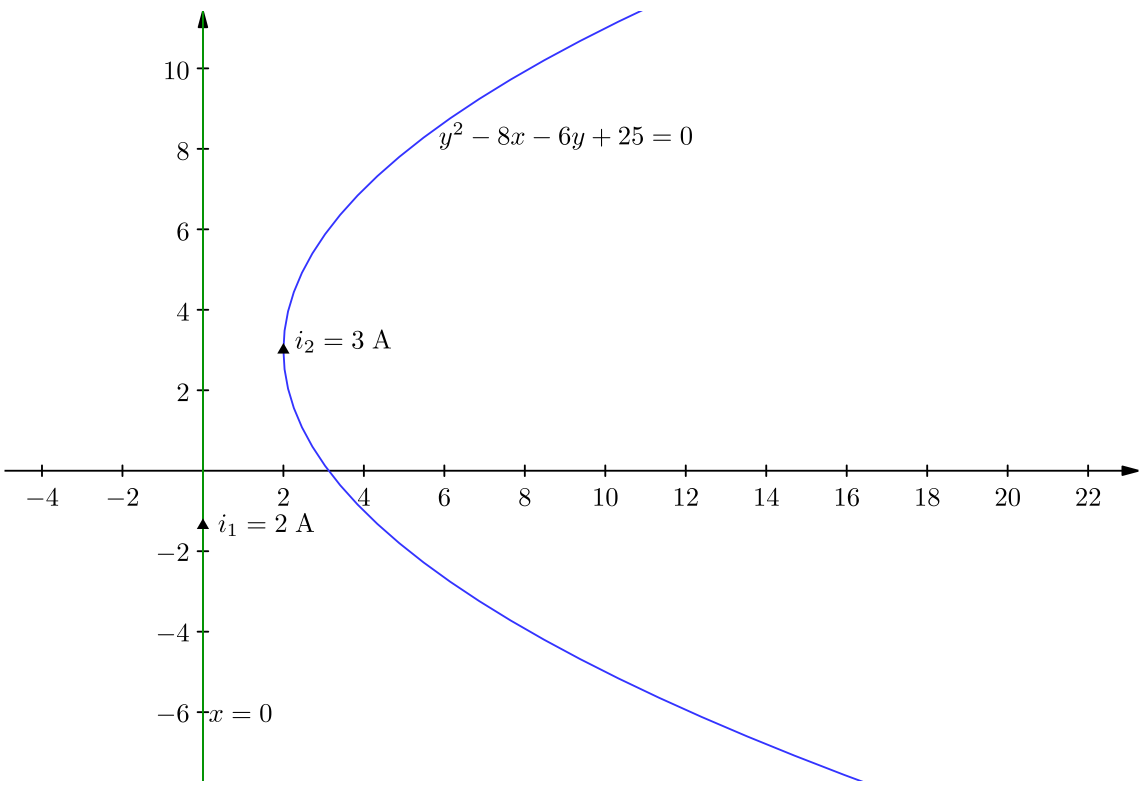
Note: The emissivity is the degree to which an object can emit longer wavelengths (infrared) and the absorptivity is the degree to which an object can absorb energy. Specifically, the emissivity is the ratio between the energy emitted by an object and the energy emitted by a perfect black body at the same temperature. On the other hand, the absorptivity is the ratio of the amount of energy absorbed to the amount of incident energy.

Pr 35. Power Dissipation

THIS QUESTION HAS BEEN REMOVED FROM THE EXAM.

Pr 36. Flattening the Curve

Two infinitely long current carrying wires carry constant current $i_1 = 2$ A and $i_2 = 3$ A as shown in the diagram. The equations of the wire curvatures are $y^2 - 8x - 6y + 25 = 0$ and $x = 0$. Find the magnitude of force (in Newtons) acting on one of the wires due to the other.



Note: The current-carrying wires are rigidly fixed. The units for distances on the graph should be taken in metres.

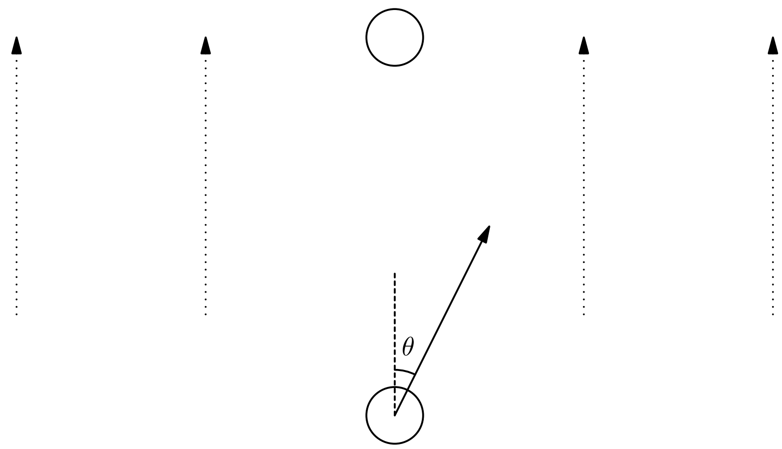
Pr 37. Hiking in Mountains

Mountains have two sides: windward and leeward. The windward side faces the wind and typically receives warm, moist air, often from an ocean. As wind hits a mountain, it is forced upward and begins to move towards the leeward side. During social distancing, Rishab decides to cross a mountain from the windward side to the leeward side of the mountain. What he finds is that the air around him has warmed when he is on the leeward side of the mountain.

Let us investigate this effect. Consider the warm, moist air mass colliding with the mountain and moving upwards on the mountain. Disregard heat exchange with the air mass and the mountain. Let the humidity of the air on the windward side correspond to a partial vapor pressure 0.5 kPa at 100.2 kPa and have a molar mass of $\mu_a = 28$ g/mole. The air predominantly consists of diatomic molecules of oxygen and nitrogen. Assume the mountain to be very high which means that at the very top of the mountain, all of the moisture in the air condenses and falls as precipitation. Let the precipitation have a heat of vaporization $L = 2.4 \cdot 10^6$ J/kg and molar mass $\mu_p = 18.01$ g/mole. Calculate the total change in temperature from the windward side to the leeward side in degrees Celsius.

Pr 38. Me and my Crush

Two electrons are in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$ where $E_0 = 10^{-11}$ N/C. One electron is at the origin, and another is 10 m above the first electron. The electron at the origin is moving at $u = 10$ m/s at an angle of 30° from the line connecting the electrons at $t = 0$, while the other electron is at rest at $t = 0$. Find the minimum distance between the electrons. You may neglect relativistic effects.



Pr 39. Can't or can

Consider a long uniform conducting cylinder. First, we divide the cylinder into thirds and remove the middle third. Then, we perform the same steps on the remaining two cylinders. Again, we perform the same steps on the remaining four cylinders and continuing until there are 2048 cylinders.

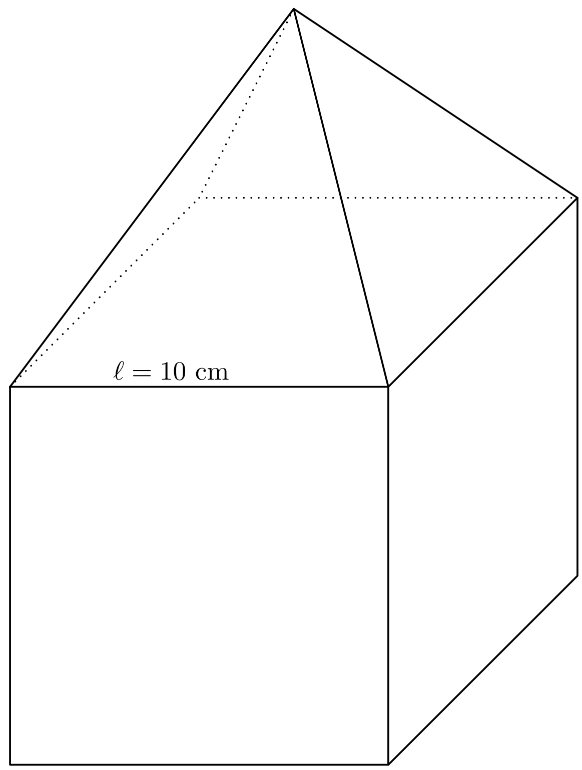
We then connect the terminals of the cylinder to a battery and measure the effective capacitance to be C_1. If we continue to remove cylinders, the capacitance will reach an asymptotic value of C_0. What is C_1/C_0?

You may assume each cylindrical disk to be wide enough to be considered as an infinite plate, such that the radius R of the cylinders is much larger than the d between any successive cylinders.

Note: The diagram is not to scale.

Pr 40. Mom Trust the Physics!

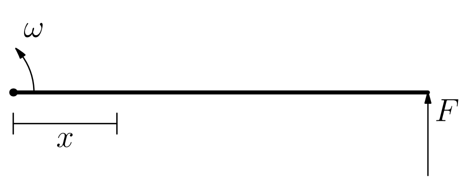
A square based pyramid (that is symmetrical) is standing on top of a cube with side length $\ell = 10$ cm such that their square faces perfectly line up. The cube is initially standing still on flat ground and both objects have the same uniform density. The coefficient of friction between every surface is the same value of $\mu = 0.3$. The cube is then given an initial speed v in some direction parallel to the floor. What is the maximum possible value of v such that the base of the pyramid will always remain parallel to the top of the cube? Answer in meters per second.



Pr 41. FBI Open Up!

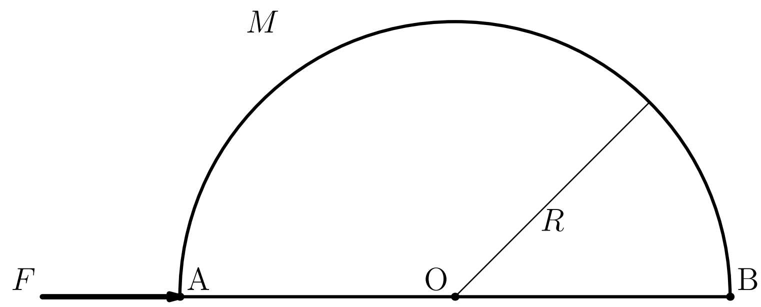
During quarantine, the FBI has been monitoring a young physicists suspicious activities. After compiling weeks worth of evidence, the FBI finally has had enough and searches his room.

The room's door is opened with a high angular velocity about its hinge. Over a very short period of time, its angular velocity increases to $\omega = 8.56$ rad/s due to the force applied at the end opposite from the hinge. For simplicity, treat the door as a uniform thin rod of length $L = 1.00$ m and mass $M = 9.50$ kg. The hinge (pivot) is located at one end of the rod. Ignore gravity. At what distance from the hinge of the door is the door most likely to break? Assume that the door will break at where the bending moment is largest. (Answer in metres.)



Pr 42. Pappu's Half Disk

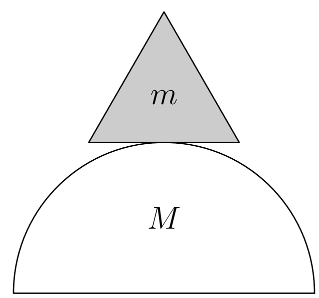
A solid half-disc of mass $m = 1$ kg in the shape of a semi-circle of radius $R = 1$ m is kept at rest on a smooth horizontal table. QiLin starts applying a constant force of magnitude $F = 10$ N at point A as shown, parallel to its straight edge. What is the initial linear acceleration of point B? (Answer in m/s^2)



Note: the diagram above is a *top down view*.

Pr 43. Don't Fall

A regular tetrahedron of mass $m = 1$ g and unknown side length is balancing on top of a hemisphere of mass $M = 100$ kg and radius $R = 100$ m. The hemisphere is placed on a flat surface such that it is at its lowest potential. For a certain value of the length of the regular tetrahedron, the oscillations become unstable. What is this side length of the tetrahedron?



Pr 44. Heartbreak

A planet has a radius of 10 km and a uniform density of 5 g/cm^3. A powerful bomb detonates at the center of the planet, releasing 8.93×10^{17} J of energy, causing the planet to separate into three large sections each with equal masses. You may model each section as a perfect sphere of radius r'. The initial and final distances between the centers of any two given sections is $2r'$. How long does it take for the three sections to collide again?

Pr 45. Hot or Not

THIS QUESTION HAS BEEN REMOVED FROM THE EXAM.

Pr 46. Sandwiched!

A point charge $+q$ is placed a distance a away from an infinitely large conducting plate. The force of the electrostatic interaction is F_0. Then, an identical conducting plate is placed a distance $3a$ from the charge, parallel to the first one such that the charge is "sandwiched in." The new electrostatic force the particle feels is F'. What is F'/F_0? Round to the nearest hundredths.

The following information applies to the next three problems. Jerry spots a truckload of his favourite golden yellow Swiss cheese being transported on a cart moving at a constant velocity $v_0 = 5$ m/s $\hat{\imath}$ along the x-axis, which is initially placed at $(0, 0)$. Jerry, driven by desire immediately starts pursuing the cheese-truck in such a way that his velocity vector always points towards the cheese-truck; however, Jerry is smart and knows that he must maintain a constant distance $\ell = 10$ m from the truck to avoid being caught by anyone, no matter what. Note that Jerry starts at coordinates $(0, \ell)$.

Pr 47. Tom and Jerry 1

Let the magnitude of velocity (in m/s) and acceleration (in m/s^2) of Jerry at the moment when the (acute) angle between the two velocity vectors is $\theta = 60°$ be α and β respectively. Compute $\alpha^2 + \beta^2$.

Pr 48. Tom and Jerry 2

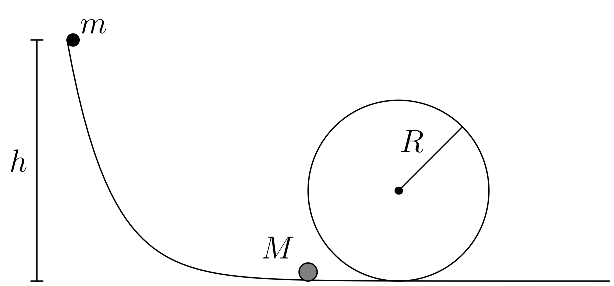
At a certain instant during Jerry's motion, when his distance from the x-axis is 2 m, let his distance from the y-axis be ξ (in metres), and let his speed at $t = 1$ second be ψ m/s. Compute $\xi^2 + \psi^2$.

Pr 49. Tom and Jerry 3

Tom spots Jerry's footprints in the mud after Jerry has already travelled a distance $\ell = 10$ m towards the cheese truck. He starts moving at a constant speed of 5 m/s (except for a very large acceleration at the start, a result of his dislike for Jerry) along Jerry's trail. Alas, as is destined, he will never be able to catch Jerry. After a long period of time, what will be the separation between them? (in meters) Assume that Tom and Jerry have the energy to maintain their velocities for a very long period of time. Tom starts chasing Jerry from the same place Jerry started running towards the cheese truck.

Pr 50. Ghoster Coaster

A frictionless track contains a loop of radius $R = 0.5$ m. Situated on top of the track lies a small ball of mass $m = 2$ kg at a height h. It is then dropped and collides with another ball of mass $M = 5$ kg.



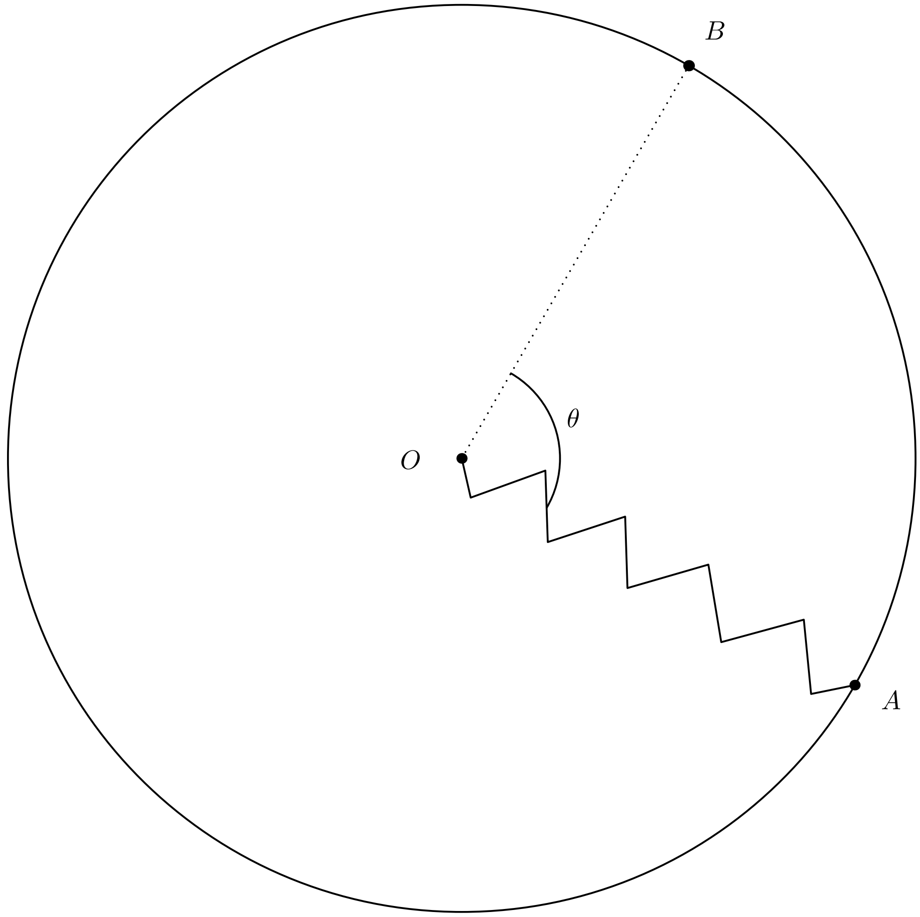
The coefficient of restitution for this collision is given as $e = \frac{1}{2}$. Now consider a different alternative. Now let the circular loop have a uniform coefficient of friction $\mu = 0.6$, while the rest of the path is still frictionless. Assume that the balls can once again collide with a restitution coefficient of $e = \frac{1}{2}$. Considering the balls to be point masses, find the minimum h such that the ball of mass M would be able to move all the way around the loop. Both balls can be considered as point masses.

Pr 51. Galactic Games

Two astronauts, Alice and Bob, are standing inside their cylindrical spaceship, which is rotating at an angular velocity ω clockwise around its axis in order to simulate the gravitational acceleration g on earth. The radius of the spaceship is R. For this problem, we will only consider motion in the plane perpendicular to the axis of the spaceship. Let point O be the center of the spaceship. Initially, an ideal zero-length spring has one end fixed at point O, while the other end is connected to a mass m at the "ground" of the spaceship, where the astronauts are standing (we will call this point A). From the astronauts' point of view, the mass remains motionless.

Next, Alice fixes one end of the spring at point A, and attaches the mass to the other end at point O. Bob starts at point A, and moves an angle θ counterclockwise to point B (such that AOB is an isosceles triangle). At time $t = 0$, the mass at point O is released. Given that the mass comes close enough for Bob to catch it, find the value of θ to the nearest tenth of a degree.

Assume that the only force acting on the mass is the spring's tension, and that the astronauts' heights are much less than R.

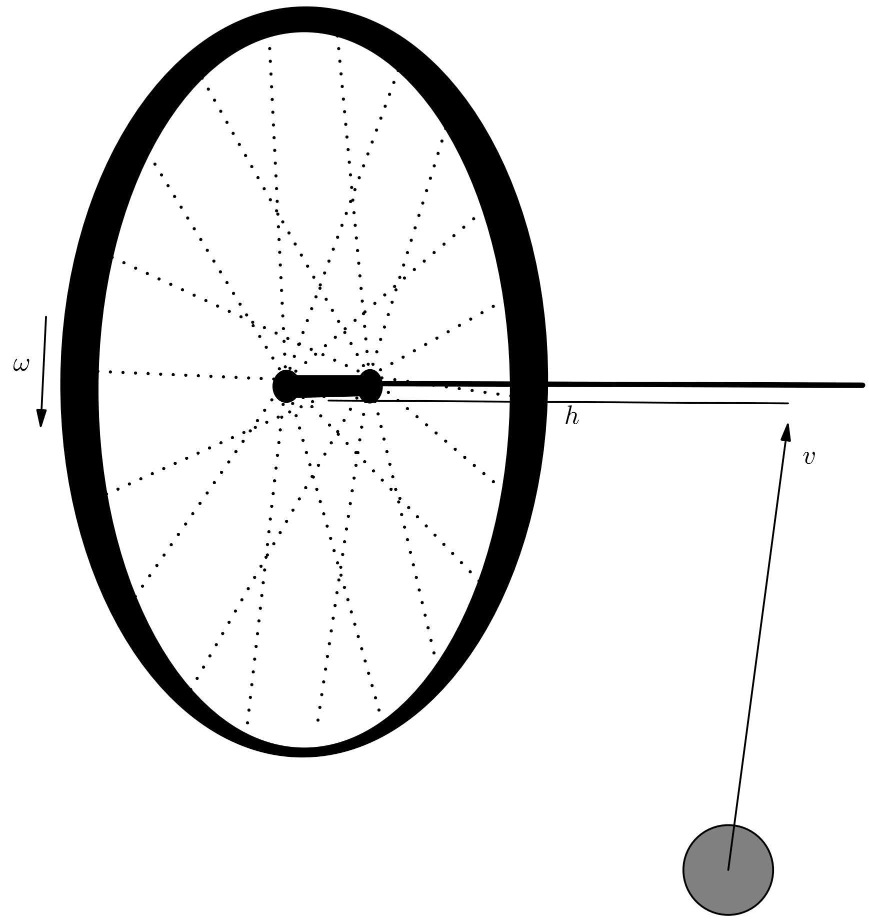


Pr 52. Cramped Up

Consider an LC circuit with one inductor and one capacitor. The amplitude of the charge on the plates of the capacitor is $Q = 10$ C and the two plates are initially at a distance $d = 1$ cm away from each other. The plates are then slowly pushed together to a distance 0.5 cm from each other. Find the resultant amplitude of charge on the parallel plates of the capacitor after this process is completed. Note that the initial current in the circuit is zero and assume that the plates are grounded.

Pr 53. I knew I should've stayed home today

A bicycle wheel of mass $M = 2.8$ kg and radius $R = 0.3$ m is spinning with angular velocity $\omega = 5$ rad/s around its axis in outer space, and its center is motionless. Assume that it has all of its mass uniformly concentrated on the rim. A long, massless axle is attached to its center, extending out along its axis. A ball of mass $m = 1.0$ kg moves at velocity $v = 2$ m/s parallel to the plane of the wheel and hits the axle at a distance $h = 0.5$ m from the center of the wheel. Assume that the collision is elastic and instantaneous, and that the ball's trajectory (before and after the collision) lies on a straight line.



Find the time it takes for the axle to return to its original orientation. Answer in seconds and round to three significant figures.

Pr 54. Fun with a String

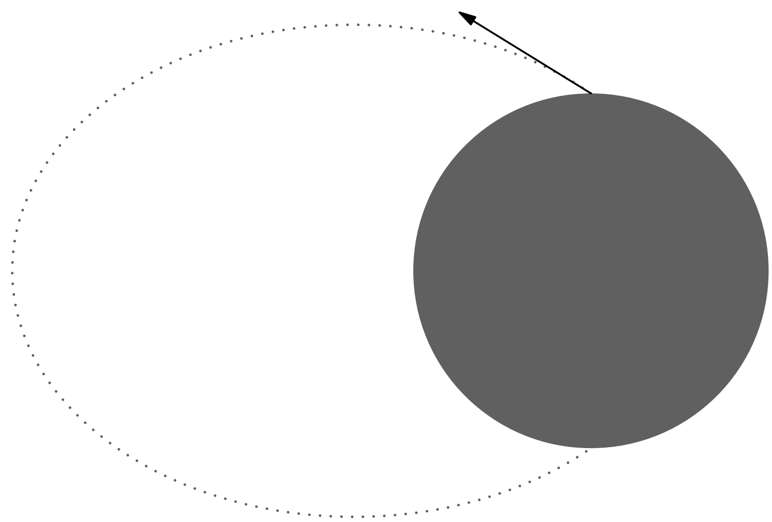
A child attaches a small rock of mass $M = 0.800$ kg to one end of a uniform elastic string of mass $m = 0.100$ kg and natural length $L = 0.650$ m. He grabs the other end and swings the rock in uniform circular motion around his hand, with angular velocity $\omega = 6.30$ rad/s. Assume his hand is stationary, and that the elastic string behaves like a spring with spring constant $k = 40.0$ N/m. After that, at time $t = 0$, a small longitudinal perturbation starts from the child's hand, traveling towards the rock. At time $t = T_0$, the perturbation reaches the rock. How far was the perturbation from the child's hand at time $t = \frac{T_0}{2}$? Ignore gravity.

Pr 55. When Rocket Scientists Play Catch

During the cold war, there was tension between the USSR and the U.S. But now, contrary to popular belief, American and Russian astronauts pass time by hanging out, enjoying the view from the moon, and even playing catch by launching projectiles at each other:

A projectile is launched with a speed $v_0 = 2200$ m/s from the North Pole to the South Pole of a moon with radius $r_0 = 1.7 \times 10^6$ m and $M = 7.4 \times 10^{22}$ kg.

How long does the flight take? Answer in seconds.



2020 Online Physics Olympiad (OPhO): Open Contest

OPhO Lead Organizers and Writers: Adithya Balachandran, Eddie Chen, Ashmit Dutta, Ethan Hu, Viraj Jayam, Rishab Parthasarathy, Kushal Thaman, Max Wang, QiLin Xue

OPhO Website Developer: Ben Oommen

Advisors: Dr. Branislav Kisačanin, Kye Shi, Kevin Zhou

Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

- Use $g = 9.81$ m/s^2 in this contest. See the constants sheet on the following page for other constants.
- This test contains 55 short answer questions. Each problem will have three possible attempts.
- The weight of each question depends on our scoring system found here. Put simply, the later questions are worth more, and the overall amount of points from a certain question decreases with the number of attempts and days that you take to solve a problem as well as the number of teams who solve it. This means that your score decreases with the number of tries and days you take to solve a given problem.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should contain at least **three** significant figures, unless otherwise specified. All answers within the 1% range will be accepted.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used. Technology and computer algebra systems like Wolfram Alpha or the one in the TI nSpire will not be needed. Attempts to use these tools will be classified as cheating.
- You are *allowed* to use Wikipedia or books in this exam. Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions.
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in SI units (meter, second, kilogram, watt) unless otherwise specified. Please answer all questions in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put letters in your answer on the submission portal! If your answer is "x meters" , input only the value x into the submission portal.
- **Do not communicate information to anyone else apart from your team-members before the exam ends on May 29, 2020 at 11:59 PM UTC**.

Sponsors

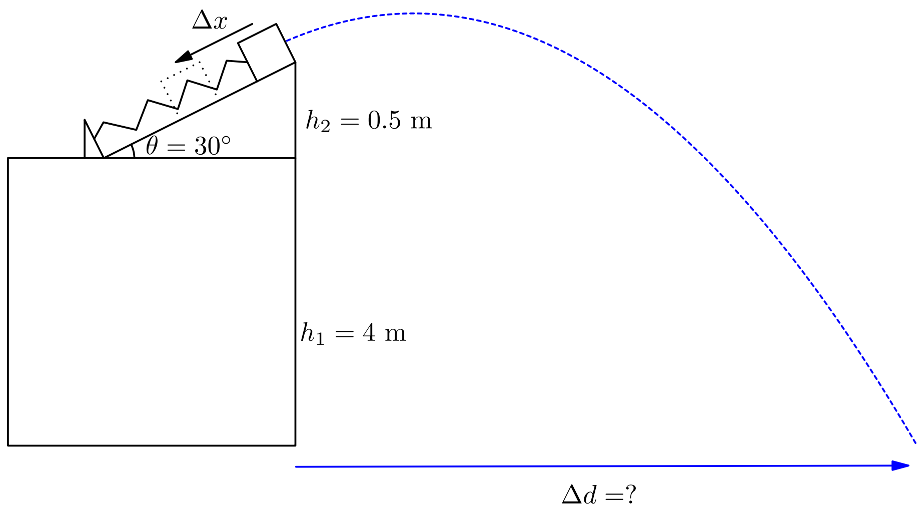
List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^8$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\mathrm{N} \cdot \mathrm{m}^2)/\mathrm{kg}^2$$
- Acceleration due to gravity, $g = 9.81$ m/s^2
- 1 unified atomic mass unit,
$$1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15}\ \mathrm{eV \cdot s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9\ (\mathrm{N \cdot m^2})/\mathrm{C^2}$$
- Permeability of free space, $\mu_0 = 4\pi \cdot 10^{-7}$ T $\cdot$ m/A
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\mathrm{T \cdot m})/\mathrm{A}$$
- 1 atmospheric pressure,
$$1\ \mathrm{atm} = 1.01 \cdot 10^5\ \mathrm{N/m^2} = 1.01 \cdot 10^5\ \mathrm{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$$

Problems

Pr 1. Angry Birds

A quarantined physics student decides to perform an experiment to land a small box of mass $m = 60$ g onto the center of a target a distance Δd away. The student puts the box on a top of a frictionless ramp with height $h_2 = 0.5$ m that is angled $\theta = 30°$ to the horizontal on a table that is $h_1 = 4$ m above the floor. If the student pushes the spring with spring constant $k = 6.5$ N/m down by $\Delta x = 0.3$ m compared to its rest length and lands the box exactly on the target, what is Δd? Answer in meters. You may assume friction is negligible.



Solution: By conservation of energy, we have that

$$\frac{1}{2}kx^2 = mgx\sin\theta + \frac{1}{2}mv^2 \implies v = \sqrt{\frac{k}{m}x^2 - 2gx\sin\theta}.$$

By kinematic formulae for motion with constant acceleration,

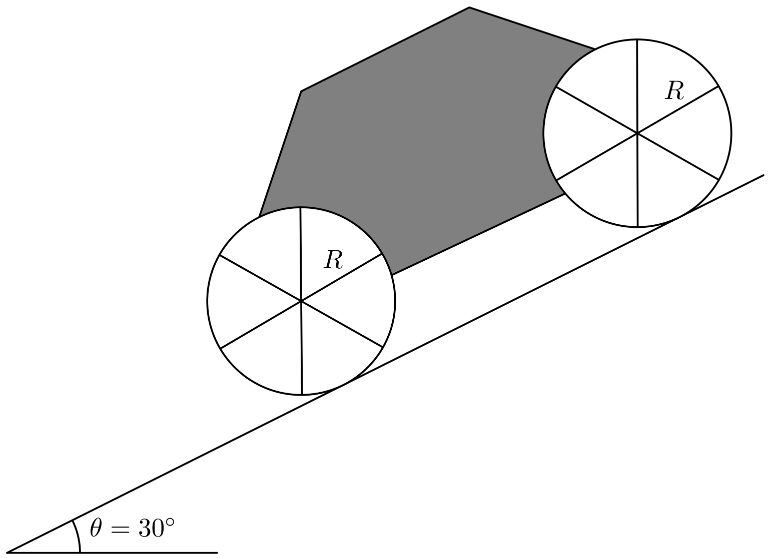
$$-4.5 \text{ m} = v\sin\theta t - \frac{1}{2}gt^2.$$

Solving for t through the quadratic, $t = 1.099$ s and the velocity is $v = 2.60$ m/s. Therefore, the distance, $\Delta d = v\cos\theta t = \boxed{2.47 \text{ m}}$.

Pr 2. The Wheels on the Monster Truck go Round and Round

A wooden bus of mass $M = 20,000$ kg (M represents the mass excluding the wheels) is on a ramp with angle $30°$. Each of the four wheels is composed of a ring of mass $\frac{M}{2}$ and radius $R = 1$ m and 6 evenly spaced spokes of mass $\frac{M}{6}$ and length R. All components of the truck have a uniform density. Find the acceleration of the bus down the ramp assuming that it rolls without slipping.

Answer in m/s^2.



Solution: The moment of inertia of each wheel can be thought of as a superposition of the six spokes and a ring. Therefore, we get:

$$I_{\text{wheel}} = \frac{M}{2}R^2 + 6\left(\frac{1}{3}\frac{M}{6}R^2\right) = \frac{5}{6}MR^2.$$

The moment of inertia of four wheels is:

$$I_{\text{bus}} = 4\left(\frac{5}{6}MR^2\right) = \frac{10}{3}MR^2.$$

The total mass of the bus is

$$m = M + 4\left(\frac{M}{2} + 6 \cdot \frac{M}{6}\right) = 7M.$$

Using Newton's second law down the ramp,

$$7Ma = 7Mg\sin\theta - 4f$$

if f is the friction at each wheel, and the torque balance on each wheel is:

$$\frac{5}{6}MR^2\alpha = fR.$$

Letting $a = \alpha r$ for the no slip condition, we can solve for f to be:

$$f = \frac{5}{6}Ma.$$

so our force balance equation becomes:

$$7Ma = 7Mg\sin\theta - \frac{10}{3}Ma \implies a = \frac{21}{31}g\sin\theta = \boxed{3.32 \text{ m/s}^2}$$

Pr 3. District 12

In an old coal factory, a conveyor belt will move at a constant velocity of 20.3 m/s and can deliver a maximum power of 15 MW. Each wheel in the conveyor belt has a diameter of 2 m. However a changing demand has pushed the coal factory to fill their coal hoppers with a different material with a certain constant specific density. These "coal" hoppers have been modified to deliver a constant 18 m^3s^{-1} of the new material to the conveyor belt. Assume that the kinetic and static friction are the same and that there is no slippage. What is the maximum density of the material?

Solution: The maximal force the convey belt can provide to a particle is:

$$F = \frac{P}{v}$$

The conveyor belt must provide an impulse to the particles to have a momentum of $p = mv$, where m is the mass of the particle and v is the velocity.

$$F = \frac{dp}{dt}$$

where $\frac{dp}{dt}$ is:

$$\rho \dot{V} v$$

Solving for for the maximum density we get:

$$\rho = \frac{P}{\dot{V} v^2}$$

$$\rho = \boxed{2022.2 \ \frac{\text{kg}}{\text{m}^3}}$$

Pr 4. Neutrino Party

Neutrinos are extremely light particles and rarely interact with matter. The Sun emits neutrinos, each with an energy of 8×10^{-14} J and reaches a flux density of 10^{11} neutrinos/(s cm^2) at Earth's surface.

In the movie *2012*, neutrinos have mutated and now are completely absorbed by the Earth's inner core, heating it up. Model the inner core as a sphere of radius 1200 km, density 12.8 g/cm^3, and a specific heat of 0.400J/g K. The time scale, in seconds, that it will take to heat up the inner core by 1°C is $t = 1 \times 10^N$ where N is an integer. What is the value of N?

Solution: The cross sectional area is πr^2, so the incoming power generated by the neutrinos is:

$$P = \pi r^2 E \Phi$$

where E is the energy of each neutrino and Φ is the flux density. We want to cause a change in energy of:

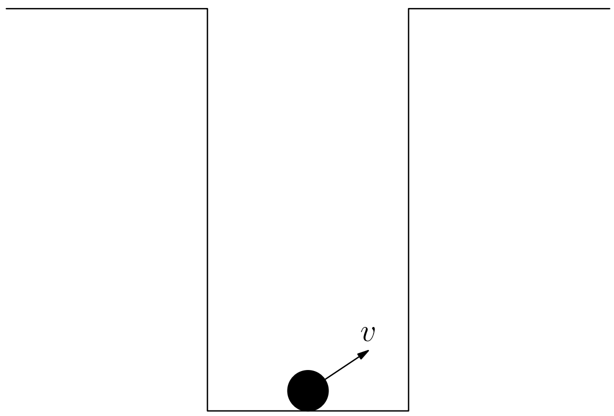
$$\Delta Q = mc\Delta T = \rho \frac{4}{3} \pi r^3 c \Delta T$$

which can be accomplished in a time:

$$Pt = \Delta Q \implies t = \frac{\rho(4\pi r^3)c\Delta T}{3\pi r^2 E\Phi} = \frac{4\rho r c \Delta T}{3E\Phi} = \boxed{1 \times 10^{14} \text{ s}}$$

Pr 5. Quarantine Secrets

A ball is situated at the midpoint of the bottom of a rectangular ditch with width 1 m. It is shot at a velocity $v = 5$ m/s at an angle of 30° relative to the horizontal. How many times does the ball collide with the walls of the ditch until it hits the bottom of the ditch again? Assume all collisions to be elastic and that the ball never flies out of the ditch.



Solution: We use the idea of mirroring the walls of the ditch. We can then draw out the normal path of the projectile and find the number of intersections the projectile makes with the mirror walls. The total time of the projectile to travel a path is given by

$$t = \frac{2v_0 \sin\theta}{g}.$$

The projectile will cover a horizontal distance

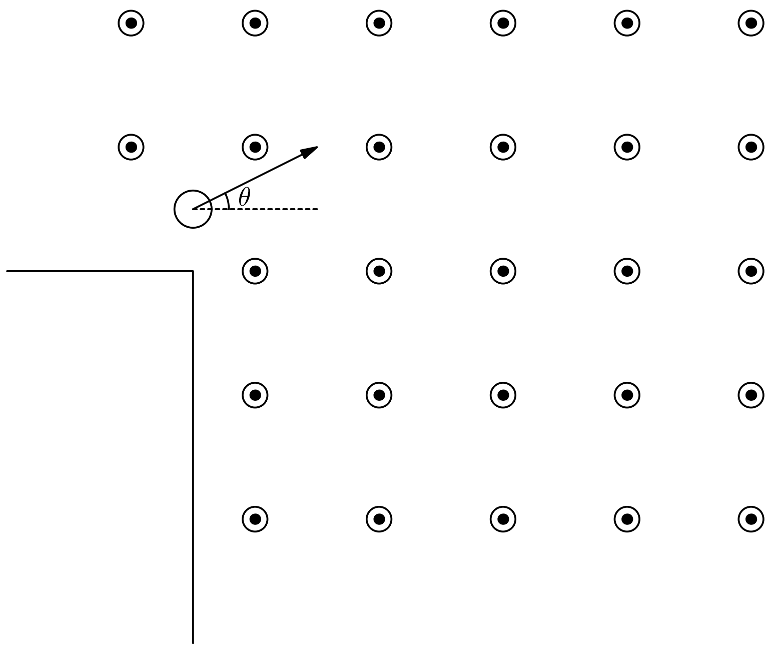
$$\left(N + \frac{1}{2}\right) a = v\cos\theta t \implies N = \frac{v}{a}\cos\theta t - \frac{1}{2}.$$

Taking the ceiling of this gives:

$$N = \left\lceil \frac{R - a/2}{a} \right\rceil = \boxed{2}.$$

Pr 6. Planetary Proton

Professor Proton has discovered a new planet on one of his planetary expeditions. He wants to measure the magnetic field of the planet he has found. Professor Proton has brought all the necessary equipment required to carry out the following experiment. A proton is launched off a large cliff at a non-relativistic speed v and an angle $\theta = 30^\circ$ with respect to the horizontal plane at the magnetic equator of a distant planet. The magnetic field acting perpendicularly on the particle can be assumed to be perfectly horizontal and coming out of the page, as shown in the diagram. How strong is the magnetic field at the magnetic equator of this planet if the period of oscillation of v_x is 4.94×10^{-4} s? Write your answer in terms of μT (micro-Teslas).



Solution: The separate time-dependent equations for the x and y components of velocity are

$$\dot{v_y} = \frac{-qv_xB - mg}{m}$$

$$\dot{v_x} = \frac{qv_yB}{m}$$

Taking the derivative of the second equation:

$$\ddot{v_x} = \frac{q\dot{v_y}B}{m}$$

Plugging,

$$\dot{v_y} = \frac{\ddot{v_x}m}{qB}$$

into

$$\dot{v_y} = \frac{-qv_xB - mg}{m}$$

we get a differential equation that has a sinusoidal solution. The angular frequency is therefore: $\omega = \frac{qB}{m}$. Solving for period B we get that

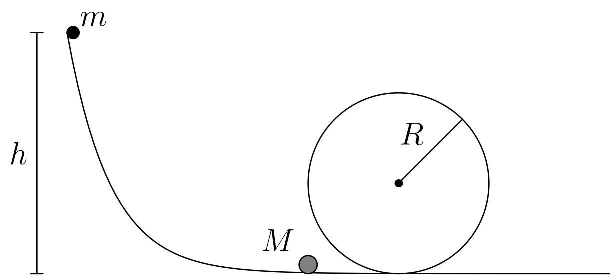
$$B = \frac{\omega m}{q}$$

$$B = \frac{\frac{2\pi}{T}m}{q}$$

$$B = \boxed{132.76\ \mu\text{T}}$$

Pr 7. Angel Coaster

A frictionless track contains a loop of radius $R = 0.5$ m. Situated on top of the track lies a small ball of mass $m = 2$ kg at a height h. It is then dropped and collides with another ball (of negligible size) of mass $M = 5$ kg.



Let h be the minimum height that m was dropped such that M would be able to move all the way around the loop. The coefficient of restitution for this collision is given as $e = \frac{1}{2}$.

Now consider a different scenario. Assume that the balls can now collide perfectly inelastically, which means that they stick to each other instantaneously after collision for the rest of the motion. If m was dropped from a height $3R$, find the minimum value of $\frac{m}{M}$ such that the combined mass can fully move all the way around the loop. Let this minimum value be k. Compute $\alpha = \frac{k^2}{h^2}$. (Note that this question is *only* asking for α but you need to find h to find α). Assume the balls are point masses (neglect rotational effects).

Solution: The velocity of m when it gets to the bottom of the track will be given by

$$v_b = \sqrt{2gh}.$$

Claim: The velocity of M after collision will be given by $v_M = \frac{(1+e)m}{m+M}\sqrt{2gh}$.

Proof: Conservation of momentum before and after the collision is expressed by:

$$mv_b = mv_m + Mv_M$$

By coefficient of restitution,

$$v_M - v_m = ev_b$$

These equations may be solved directly to find v_m, v_M to give

$$v_M = \frac{(1+e)m}{m+M}v_b \quad \square$$

Next, when the objects get to the top of the loop, conservation of energy gives the speed of M when it gets to the top as

$$\frac{1}{2}v_M^2 = \frac{1}{2}v_t^2 + 2gR.$$

At the top of the loop, M must at least have an acceleration g to maintain circular motion, and thus

$$\frac{v_t^2}{R} = g \implies v_t = \sqrt{gR}.$$

Substituting these results into our conservation equation gives us

$$\begin{aligned}\frac{(1+e)^2m^2}{(m+M)^2}(\sqrt{2gh})^2 &= gR+4gR\\ \frac{2(1+e)^2m^2gh}{(m+M)^2} &= 5gR\\ h &= \frac{5(m+M)^2R}{2(1+e)^2m^2}\\ &= 6.805\text{ m}\end{aligned}$$

In the second scenario, conservation of momentum before and after the collision gives:

$$mv_b = (M+m)v_f \implies v_f = \frac{mv_b}{M+m}$$

The same conservation of energy formula as in the first scenario yields

$$\begin{aligned}\frac{m^2}{(M+m)^2}\left(\sqrt{2gh}\right)^2 &= 5gR\\ \frac{2m^2gh}{(M+m)^2} &= 5gR\\ \frac{6m^2gR}{(M+m)^2} &= 5gR\\ \left(\frac{k}{k+1}\right)^2 &= \frac{5}{6}\\ \frac{k}{k+1} &= \sqrt{\frac{5}{6}}\\ k &= \frac{\sqrt{5}}{\sqrt{6}-\sqrt{5}}\\ &= 10.477\end{aligned}$$

Thus,

$$\alpha = \frac{k^2}{h^2} = \boxed{2.37\text{ m}^2}$$

Pr 8. Wannabe Twoset

Eddie is experimenting with his sister's violin. Allow the "A" string of his sister's violin have an ultimate tensile strength σ_1. He tunes a string up to its highest possible frequency f_1 before it breaks. He then builds an exact copy of the violin, where all lengths have been increased by a factor of $\sqrt{2}$ and tunes the same string again to its highest possible frequency f_2. What is f_2/f_1? The density of the string does not change.

Note: The ultimate tensile strength is maximum amount of stress an object can endure without breaking. Stress is defined as $\frac{F}{A}$, or force per unit area.

Solution: We note from a simple dimensional analysis that the angular frequency of the string ω will consist of the tension T, the length of the string L and the mass of the string m.

$$T = [MLT^{-2}]$$
$$L = [L]$$
$$m = [M]$$
$$\omega = [T^{-1}]$$

Therefore, by rearranging, we find that

$$\omega = T^{\alpha} L^{\beta} m^{\gamma}$$
$$[T^{-1}] = [MLT^{-2}]^{\alpha}[L]^{\beta}[M]^{\gamma}$$

Distributing the exponents, and rearranging gives us

$$T^{-1} = M^{\alpha+\gamma} L^{\alpha+\beta} T^{-2\alpha}$$

We now have three equations

$$\alpha + \gamma = 0$$
$$\alpha + \beta = 0$$
$$-2\alpha = -1$$

From here, we find that $\alpha = 1/2$. Substituting this into the first equation gives us

$$1/2 + \gamma = 0 \implies \gamma = -1/2$$

then substituting α into the second equation gives us

$$1/2 + \beta = 0 \implies \beta = -1/2$$

We now find that the angular frequency is given by

$$\omega = A\sqrt{\frac{T}{Lm}}$$

where A is an arbitrary constant. Noting that $\omega = 2\pi f$, we find that

$$f = \frac{A}{2\pi}\sqrt{\frac{T}{Lm}}.$$

From this analysis, we can then see that $f_2/f_1 = \sqrt{2}/2 \approx \boxed{0.707}$.

Pr 9. Waterhorse or Flyinghorse

A one horsepower propeller powered by a battery and is used to propel a small boat initially at rest. You have two options:

1. Put the propeller on top of the boat and push on the air with an initial force F_1
2. Put the propeller underwater and push on the water with an initial force F_2.

The density of water is 997 kg/m^3 while the density of air is 1.23 kg/m^3. Assume that the force is both cases is dependent upon only the density of the medium, the surface area of the propeller, and the power delivered by the battery. What is F_2/F_1? You may assume (unrealistically) the efficiency of the propeller does not change. Round to the nearest tenths.

Solution: The force exerted on the fluid is roughly proportional to the change in momentum with respect to time:

$$F = \frac{dp}{dt} = v\frac{dm}{dt} = v\frac{d}{dt}(\rho A x) = \rho A v^2$$

It is kept at a constant power $P = Fv$, which can allow us to solve for the speed v of the propellers.

$$P = \rho A v^3 \implies v = \left(\frac{P}{\rho A}\right)^{1/3}$$

so the force is given by:

$$F = \rho A\left(\frac{P}{\rho A}\right)^{2/3} \implies F \propto \rho^{1/3}$$

Therefore:

$$F_2/F_1 = (997/1.23)^{1/3} = \boxed{9.26} \text{ times}$$

Pr 10. Charlie And The Chocolate Factory

A professional pastry chef is making a sweet which consists of 3 sheets of chocolate. The chef leaves a gap with width $d_1 = 0.1$ m between the top and middle layers and fills it with a chocolate syrup with uniform viscosity $\eta_1 = 10\,\mathrm{Pa\cdot s}$ and a gap with width $d_2 = 0.2$ m between the middle and bottom sheet and fills it with caramel with uniform viscosity $\eta_2 = 15\,\mathrm{Pa\cdot s}$. If the chef pulls the top sheet with a velocity $2\,\mathrm{m/s}$ horizontally, at what speed must he push the bottom sheet horizontally such that the middle sheet remains stationary initially? Ignore the weight of the pastry sheets throughout the problem and the assume the sheets are equally sized.

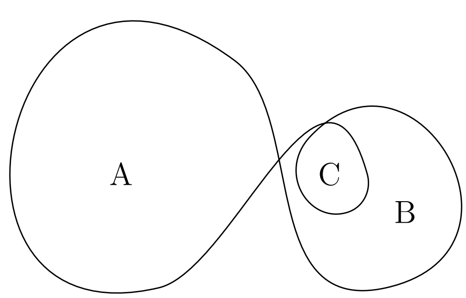
Note: Shear stress is governed by the equation $\tau = \eta \times$ rate of strain.

Solution: The plates are equal sizes so all we have to do is simply balance the shear stresses which act in opposing directions on the middle plate:

$$\begin{aligned}\tau_1 &= \tau_2\\ \eta_1\cdot\frac{v_1}{d_1} &= \eta_2\cdot\frac{v_2}{d_2}\\ 10\cdot\frac{2}{0.1} &= 15\frac{v}{0.2}\\ v &= \boxed{2.667\text{ m/s}}.\end{aligned}$$

Pr 11. Loopy Wire

The following diagram depicts a single wire that is bent into the shape below. The circuit is placed in a magnetic field pointing out of the page, uniformly increasing at the rate $\frac{dB}{dt} = 2.34$ T/s. Calculate the magnitude of induced electromotive force in the wire, in terms of the following labelled areas (m^2). Note that B is non-inclusive of C and that $A = 4.23$, $B = 2.74$, and $C = 0.34$.



Solution: Without loss of generality, let the current around A flow in the counterclockwise direction and let the flux through A be positive. Note that the current will flow in the clockwise direction in C and around B, the area enclosed by the loop is $B + C$. The flux will be negative here. Therefore, the total flux is then proportional to:

$$\Phi \propto A - B - 2C = 0.81$$

and the magnitude of the induced electromotive force is:

$$\varepsilon = (A - B - 2C)\frac{dB}{dt} = \boxed{1.9\ \mathrm{T\,m^2\,s^{-1}}}.$$

The following information applies to the next two problems. A magnetic field is located within a region enclosed by an elliptical island with semi-minor axis of $a = 100$ m and semi-major axis of $b = 200$ m. A car carrying charge $+Q = 1.5$ C drives on the boundary of the island at a constant speed of $v = 5$ m/s and has mass $m = 2000$ kg. Any dimensions of the car can be assumed to be much smaller than the dimensions of the island. Ignore any contributions to the magnetic field from the moving car and assume that the car has enough traction to continue driving in its elliptical path.

Let the center of the island be located at the point $(0, 0)$ while the semi major and semi minor axes lie on the x and y-axes, respectively.

On this island, the magnetic field varies as a function of x and y: $B(x, y) = k_b e^{c_b xy}\hat{z}$ (pointing in the upward direction, perpendicular to the island plane in the positive z-direction). The constant $c_b = 10^{-4}$ m^{-2} and the constant $k_b = 2.1$ μT

Pr 12. Journey 2: The Magnetic Island 1

At what point on the island is the force from the magnetic field a maximum? Write the distance of this point from the x-axis in metres.

Pr 13. Journey 2: The Magnetic Island 2

Assuming no slipping, what is the magnitude of the net force on the car at the point of the maximum magnetic field? (Answer in Newtons.)

Solution:

(12) To find a minimum or maximum, the gradient of the constraint function $f(x, y) = \frac{x^2}{b^2} + \frac{y^2}{a^2} - 1$ and the gradient of the B field function should be scalar multiples of each other.

$$\frac{2x}{b^2}\mu = c_b y e^{xy}$$

$$\frac{2y}{a^2}\mu = c_b x e^{xy}$$

Solving the two equations, we get that a maximum point (x, y) is of the form $(\frac{b}{\sqrt{2}}, \frac{a}{\sqrt{2}})$ or $(-\frac{b}{\sqrt{2}}, -\frac{a}{\sqrt{2}})$. The distance from the y-axis is thus $\frac{a}{\sqrt{2}} = \boxed{70.7 \text{ m}}$.

(13) The net force acting on the car must provide a_c. We have

$$r(x) = x\hat{i} + a\sqrt{1 - \frac{x^2}{b^2}}\hat{j}$$

and we simply have to find the radius of curvature of this function. This is given by

$$R = \frac{[1 + (y'(x))^2]^{\frac{3}{2}}}{|y''(x)|}.$$

We can evaluate this to find

$$R = \frac{(a^2 + b^2)^{3/2}}{2\sqrt{2}ab}.$$

The total force acting on the car is $m\frac{v^2}{R} = \boxed{252.98 \text{ N}}$.

Pr 14. Tuning Outside

Inside a laboratory at room temperature, a steel tuning fork in the shape of a U is struck and begins to vibrate at $f = 426$ Hz. The tuning fork is then brought outside where it is 10°C hotter and the experiment is performed again. What is the change in frequency, Δf of the tuning fork? (A positive value will indicate an increase in frequency, and a negative value will indicate a decrease.)

Note: The linear thermal coefficient of expansion for steel is $\alpha = 1.5 \times 10^{-5}$ K^{-1} and you may assume the expansion is isotropic and linear. When the steel bends, there is a restoring torque $\tau = -\kappa\theta$ such that $\kappa \equiv GJ$ where $G = 77$ GPa is constant and J depends on the geometry and dimensions of the cross-sectional area.

Solution: Note that κ has units of torque so dimensionally, J must be proportional to L^3. Therefore, we have:

$$\beta M L^2 \alpha \propto -L^3\theta \implies f \propto \sqrt{L}$$

Therefore, we have:

$$\frac{\Delta f}{f} = \frac{\Delta L}{2L}$$

Since $\frac{\Delta L}{L} = \alpha \Delta T$, this gives us:

$$\Delta f = \frac{1}{2} f \alpha \Delta T = \boxed{0.0320 \text{ Hz}}.$$

Pr 15. Too Much Potential

A large metal conducting sphere with radius 10 m at an initial potential of 0 and an infinite supply of smaller conducting spheres of radius 1 m and potential 10 V are placed into contact in such a way: the large metal conducting sphere is contacted with each smaller sphere one at a time. You may also assume the spheres are touched using a thin conducting wire that places the two spheres sufficiently far away from each other such that their own spherical charge symmetry is maintained. What is the least number of smaller spheres required to be touched with the larger sphere such that the potential of the larger sphere reaches 9 V? Assume that the charges distribute slowly and that the point of contact between the rod and the spheres is not a sharp point.

Solution: Let each sphere with radius 1 m have charge q. Note that each time the large metal conducting sphere is contacted with each of the smaller spheres, the potential is equalized between the two objects. The potential on a sphere is proportional to $\frac{q}{r}$, so the large conducting sphere must retain $\frac{10}{11}$ of the total charge after it is contacted with a smaller sphere. Furthermore, to reach 9 V, the required end charge on the sphere of radius 10 m is at least $9q$. Thus, we get a recursion for the charge of the large square Q in terms of the number of small spheres touched n.

$$Q(n+1) = (Q(n) + q) \cdot \frac{10}{11}$$

Inductively applying this recursion, we obtain

$$Q(n) = q\left[\left(\frac{10}{11}\right)^n + \cdots + \frac{10}{11}\right].$$

We can now sum this geometric series:

$$Q(n) = q\left(\frac{10}{11} \cdot \frac{\left(\frac{10}{11}\right)^n - 1}{\frac{10}{11} - 1}\right).$$

Thus, using $Q(n) \geq 9q$, we find that $\left(\frac{10}{11}\right)^n \leq 0.1$, which provides $n \geq 24.1588$, or $n = \boxed{25}$ as the answer.

Pr 16. Particle in the Box

During high speed motion in a strong electric field, a charged particle can ionize air molecules it collides with.

A charged particle of mass $m = 0.1$ kg and charge $q = 0.5\ \mu$C is located in the center of a cubical box. Each vertex of the box is fixed in space and has a charge of $Q = -4\ \mu$C. If the side length of the box is $l = 1.5$ m what minimum speed (parallel to an edge) should be given to the particle for it to exit the box (even if it's just momentarily)? Let the energy loss from Corona discharge and other radiation effects be $E = 0.00250$ J.

Solution: Conservation of energy gives:

$$T_i + U_i = T_f + U_f + E.$$

Solving for the initial potential energy gives

$$U_i = -8\frac{kqQ}{l\sqrt{3}/2} = -\frac{16kqQ}{\sqrt{3}l}.$$

And since the final kinetic energy is zero, the final potential energy is

$$U_f = -4\frac{kqQ}{l/\sqrt{2}} - 4\frac{kqQ}{l\sqrt{\frac{3}{2}}} = -\left(4\sqrt{2} + 4\sqrt{\frac{2}{3}}\right)\frac{kqQ}{l}$$

and thus solving for the initial kinetic energy:

$$\frac{1}{2}mv^2 = \left(-4\sqrt{2} - 4\sqrt{\frac{2}{3}} + \frac{16}{\sqrt{3}}\right)\frac{kqQ}{l} + E.$$

The final answer is $v = \boxed{0.354\text{ m/s}}$.

Pr 17. Room of Mirrors 1

Max finds himself trapped in the center of a mirror walled equilateral triangular room. What minimum beam angle must his flashlight have so that any point of illumination in the room can be traced back to his flashlight with at most 1 bounce? (Answer in degrees.) Since the room is large, assume the person is a point does not block light. Visualize the questions in a 2D setup. The floor/ceiling is irrelevant.

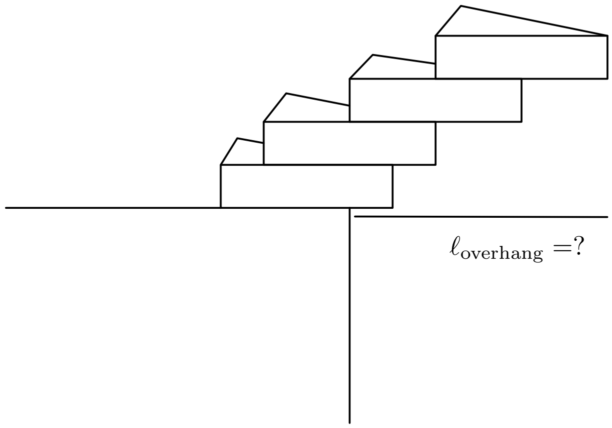
The point of illumination refers to any point in the room that is lit.

Solution: Each time light hits a mirror, we can reflect the entire equilateral triangle about that mirror and continue to trace the straight-line path of the light. For a maximum of 1 bounce, we can reflect our triangle about each of its initial sides. In order for the light to hit every part of the triangle (or an image of that part), by symmetry we require a $\boxed{120^\circ}$ angle. In this case, we would just shine the flashlight so that light directly reaches the entirety of one side, and reflections of light will fully reach the other two sides of the triangle.

Pr 18. Secret Society

For his art project, Weishaupt cut out $N = 20$ wooden equilateral triangular blocks with a side length of $\ell = 10$ cm and a thickness of $t = 2$ cm, each with the same mass and uniform density. He wishes to stack one on top of the other overhanging the edge of his table. In centimeters, what is the maximum overhang? Round to the nearest centimeter. A side view is shown below. Assume that all triangles are parallel to each other.

Note: This diagram is not to scale.



Solution: Let us consider $N = 1$ equilateral triangles. From inspection, we need to place the triangle such that the center of mass lies at the edge of the table. The maximum overhang in this case is $(1-f)h$ where $h = \frac{\ell\sqrt{3}}{2}$ is the height of the triangle and $fh = \frac{h}{3}$ is the location of the center of mass.
If we wish to place a second triangle on top, we want to maximize the center of mass to be as far right as possible without the top block toppling. Placing the second block such that its center of mass is at the tip of the first triangle accomplishes this. However, the center of mass of the two triangles combined is now past the edge. Their center of mass is:

$$x_{\text{cm}} = \frac{fh + h}{2} = \frac{f+1}{2}h$$

Thus the maximum overhang of the first block is now:

$$h - \frac{f+1}{2}h = \frac{1-f}{2}h$$

Now, we will place a third block such that it has the maximum overhang with respect to the top block and then shift the entire setup so that the center of mass of the system lies at the edge of the table. Following the same procedures, we find that the maximum overhang of the first block is:

$$\frac{1-f}{3}h$$

The overhang of the top two blocks are $(1-f)h$ and $\frac{1-f}{2}h$, unchanged from earlier. You can show via induction that the maximum overhang of the n^{th} block (counting from the top downwards) is:

$$\frac{1-f}{n}h$$

so if there are 20 such blocks, then the total overhang (summing over all the blocks) is:

$$\sum_{k=0}^{20} \frac{1-f}{k}h = (1-f)hH_{20} = \frac{2}{3}\frac{\ell\sqrt{3}}{2}H_{20} = \frac{\ell\sqrt{3}}{3}H_{20} = 20.77 \text{ cm} \approx \boxed{21 \text{ cm}}$$

where H_N is the N^{th} harmonic number.

The following information applies to the following three problems. Kushal finds himself trapped in a large room with mirrors as walls. Being scared of the dark, he has a powerful flashlight to light the room. All references to "percent" refer to area. Since the room is large, assume the person is a point does not block light. Visualize the questions in a 2D setup. The floor/ceiling is irrelevant. *The point of illumination refers to any point in the room that is lit.*

Pr 19. Focus On That Not This! 1

What percent of a large circular room can be lit up using a flashlight with a 20 degree beam angle if Kushal stands in the center?

Pr 20. Focus On That Not This! 2

Kushal stands at a focus of an elliptical room with eccentricity 0.5 and semi major axis = 20 m. He points the flashlight along the semi-major axis away from the other focus. Find the ideal position where the torch can be placed to catch fire easily by the beam from the flashlight. What is the distance from this point to Kushal? Note that the torch cannot be at the same location as the flashlight. (Answer in metres.)

Pr 21. Focus On That Not This! 3

Now Kushal stands at a focus of the same elliptical room as in Problem 22. Determine the minimum percent of the elliptical room that can be lit up with a flashlight of beam angle 1 degree.

Solution:

(19) Each ray emitted follows a straight line, even when it is reflected since it originates from the center of a circle. Thus, the light rays in total trace out two circular sectors with an angle of $\theta = 20°$ each. Thus, the total percent of the room illuminated is:

$$f = \frac{2\theta}{360} = \boxed{11.1\%}.$$

(20) By the property of an ellipse, any light that is emitted from one focus and bounces off the sides will arrive at the other focus. The distance between the two foci is:

$$2ae = \boxed{20\text{ m}}$$

(21) Let c be the distance between the foci and the center. The overlap area is: $\frac{1}{2}(2c)\left(\frac{c\theta}{6}\right) = \frac{c^2\theta}{6}$. PIE then gives:

$$A = \frac{1}{2}(a+c)^2\left(\frac{\theta}{3} + \frac{\theta}{9}\right) - \frac{c^2\theta}{6}$$

Letting $c = ae$ gives:

$$A = \frac{2}{9}a^2(1+e)^2\theta - \frac{a^2e^2\theta}{6} = a^2\theta\left(\frac{2}{9}(1+e)^2 - \frac{e^2}{6}\right)$$

the area of the ellipse is $\pi a^2\sqrt{1-e^2}$ so the fraction is:

$$f = \frac{\theta\left(\frac{2}{9}(1+e)^2 - \frac{e^2}{6}\right)}{\pi\sqrt{1-e^2}} \approx \boxed{0.294\%}$$

Pr 22. Two Star Crossed Lovers...

Two identical neutron stars with mass $m = 4 \times 10^{30}$ kg and radius 15 km are orbiting each other a distance $d = 700$ km away from each other (d refers to the initial distance between the cores of the neutron stars). Assume that they orbit as predicted by classical mechanics, except that they generate gravitational waves. The power dissipated through these waves is given by:

$$P = \frac{32G^4}{5}\left(\frac{m}{dc}\right)^5$$

How long does it take for the two stars to collide? Answer in seconds. *Note:* d is the distance between the *cores* of the stars.

Solution: Due to Virial theorem, we have:

$$K = -\frac{1}{2}U$$

so the total energy is:

$$E = U - \frac{1}{2}U = -\frac{Gm^2}{2R}$$

We know that the power dissipated gives the change in energy, or:

$$P = \frac{32G^4}{5}\left(\frac{m}{Rc}\right)^5 = \frac{d}{dt}\frac{Gm^2}{2R}$$

or:

$$\frac{32G^4}{5}\left(\frac{m}{Rc}\right)^5 \, dt = -\frac{Gm^2}{2R^2} \, dR \implies \int_0^t \frac{64G^3}{5}\frac{m^3}{c^5} \, dt = \int_d^{2r} -R^3 \, dR$$

Solving this leads us to:

$$\frac{64G^3m^3}{5c^5}t = \frac{d^4 - r^4}{4} \implies t = \frac{5c^5(d^4 - 16r^4)}{256G^3m^3}$$

Plugging in the numbers gives:

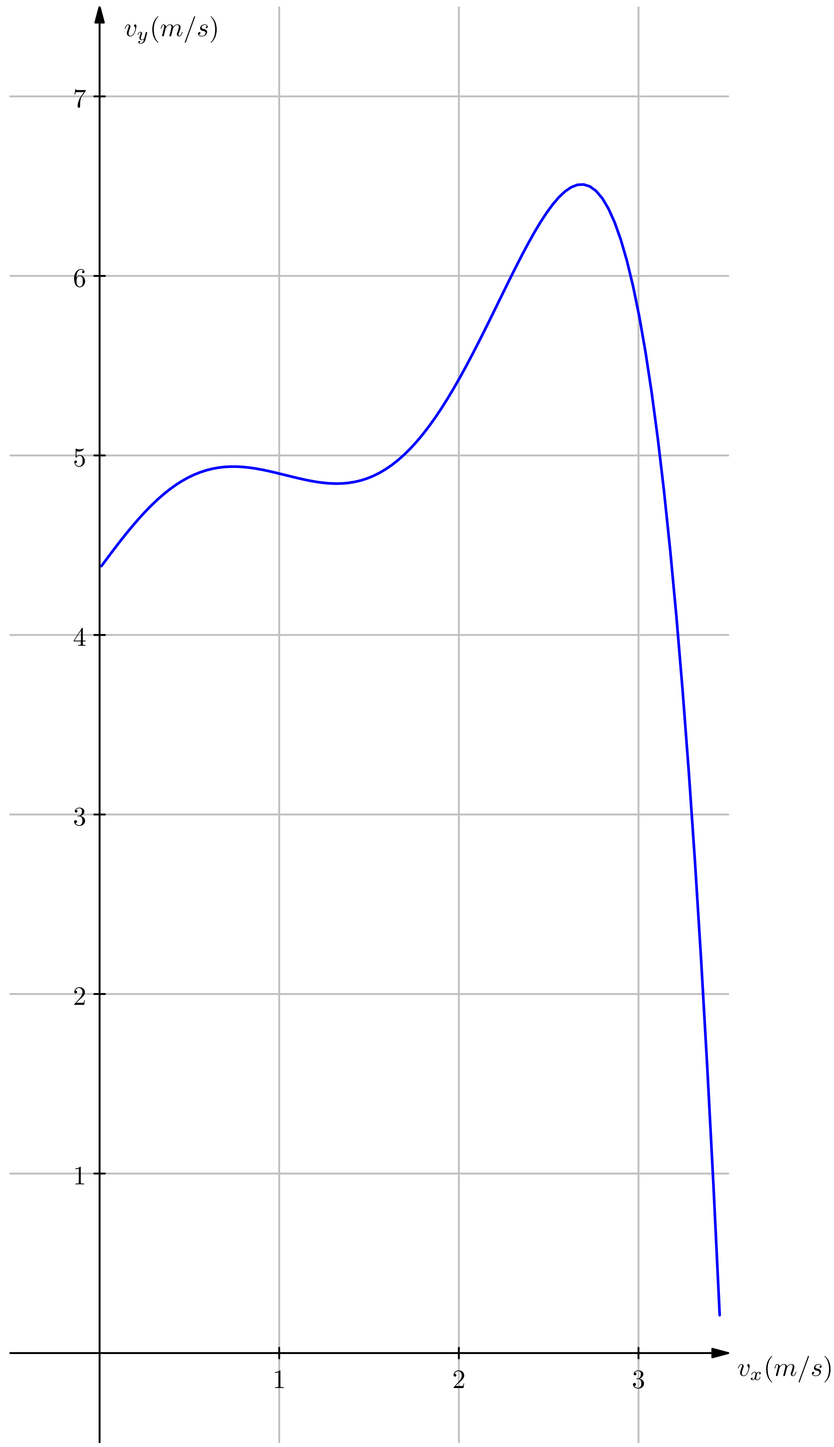
$$\boxed{t = 590 \text{ sec}}$$

Note that we can also assume that $d^4 \gg (2r)^4$ which will simplify calculations, and not introduce any noticeable error.

Pr 23. Too Bored

The graph provided plots the y-component of the velocity against the x-component of the velocity of a kiddie roller coaster at an amusement park for a certain duration of time. The ride takes place entirely in a two dimensional plane.

Some students made a remark that at one time, the acceleration was perpendicular to the velocity. Using this graph, what is the minimum x-velocity the ride could be travelling at for this to be true? Round to the nearest integer and answer in meters per second. The diagram is drawn to scale, and you may print this page out and make measurements.



Solution: The solution revolves around the idea that when the acceleration is perpendicular to the velocity, the work done is 0, and thus, the instantaneous rate of change of the magnitude of velocity $v_x^2 + v_y^2$ is 0. Thus, at such points, when the vertical velocity is plotted against the horizontal velocity, the curve will be tangent to a circle centered at the origin because $v_y^2 + v_x^2$ is nonchanging at that instant.

This is equivalent to stating that the line from the origin to the curve is perpendicular to the curve. Drawing such lines to the curve, the first time this occurs is at $v_x = \boxed{1 \text{ m/s}}$.

The following information applies to the next two problems: In the cosmic galaxy, the Sun is a main-sequence star, generating its energy mainly by nuclear fusion of hydrogen nuclei into helium. In its core, the Sun fuses hydrogen to produce deuterium (2H) and tritium (3H), then makes about 600 million metric tons of helium (4He) per second. Of course, there are also some relatively smaller portions of fission reactions in the Sun's core, e.g. a nuclear fission reaction with Uranium-235 (235U). The Fusion reaction:

$$^2H + ^3H \to ^4He + n + \text{Released Energy}$$

The Fission reaction:

$$^{235}U + n + (\text{Initial Energy}) \to ^{144}Ba + ^{90}Kr + 2n + \text{Released Energy}$$

Isotope Mass (at rest)

Isotope Names	Mass (at rest) (u)
Deuterium (^{2}H)	2.0141
Tritium (^{3}H)	3.0160
Helium (^{4}He)	4.0026
Neutron (n)	1.0087
Uranium-235 (^{235}U)	235.1180
Barium-144 (^{144}Ba)	143.8812
Krypton-90 (^{90}Kr)	89.9471

Pr 24. You are my Sunshine 1

Calculate the kinetic energy (in MeV) released by the products in one fusion reaction.

Pr 25. You are my Sunshine 2

Calculate the energy produced in the core of the Sun per second from helium fusion. Answer in Joules.

Solution:
(24) Let the kinetic energy released be

$$KE_{released} = -\Delta mc^2$$

Let the mass of helium be m_h, deuterium be m_d, tritium m_t, and mass of neutron m_n Therefore.

$$-\Delta m = m_d + m_t - m_h - m_n = 3.0160 + 2.0141 - 4.0026 - 1.0087 = 0.0188 \text{ u}$$

which gives

$$KE_{released} = (0.0188 \text{ u}) \cdot \left(931.494 \, \frac{\text{MeV}}{\text{u}}\right) = \boxed{17.51 \text{ MeV}}$$

(25) We perform the following conversions:

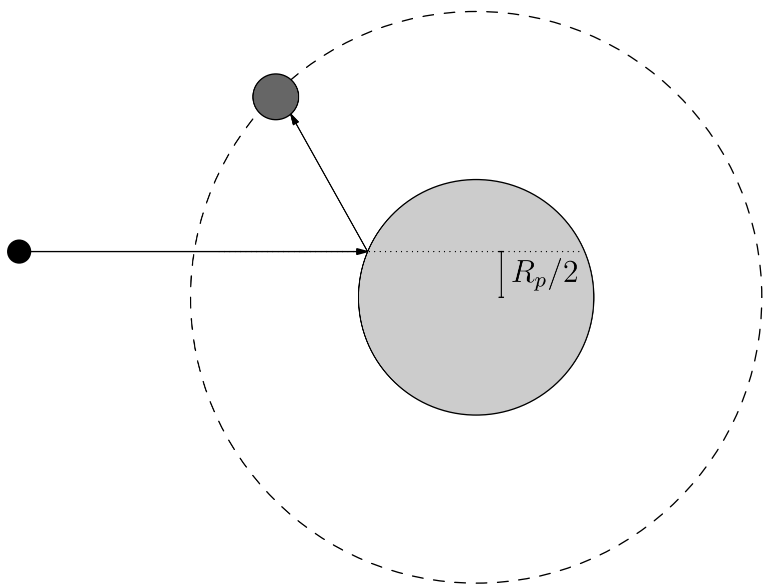
$$600 \cdot 10^6 \text{ tons He} = 6 \cdot 10^{14} \text{ g He}$$

$$\frac{6 \cdot 10^{14} \text{ g He}}{4.0026 \, \frac{\text{g He}}{\text{mol He}}} = 1.499 \cdot 10^{14} \text{ mol He}$$

$$1.499 \cdot 10^{14} \text{ mol He} \cdot \frac{6.02 \cdot 10^{23} \text{ molecules He}}{1 \text{ mol He}} \cdot \frac{17.51 \text{ MeV}}{1 \text{ molecule He}} \cdot \frac{1.6 \cdot 10^{-13} \text{ J}}{1 \text{ MeV}} = \boxed{2.528 \cdot 10^{26} \text{ J}}$$

Pr 26. Be Reflected It Must

While exploring outer space, Darth Vader comes upon a purely reflective spherical planet with radius $R_p = 40,000$ m and mass $M_p = 8.128 \times 10^{24}$ kg. Around the planet is a strange moon of orbital radius $R_s = 6,400,000$ m ($R_s \gg R_p$) and mass $M_s = 9.346 \times 10^{19}$ kg($M_s \ll M_p$). The moon can be modelled as a blackbody and absorbs light perfectly. Darth Vader is in the same plane that the planet orbits in. Startled, Darth Vader shoots a laser with constant intensity and power $P_0 = 2 \times 10^{32}$ W at the reflective planet and hits the planet a distance of $\frac{R_p}{2}$ away from the line from him to the center of the planet. Upon hitting the reflective planet, the light from the laser is plane polarized. The angle of the planet's polarizer is always the same as the angle of reflection. After reflectance, the laser lands a direct hit on the insulator planet. Darth Vader locks the laser in on the planet until it moves right in front of him, when he turns the laser off. Determine the energy absorbed by the satellite. Assume the reflective planet remains stationary and that the reflective planet is a perfect polarizer of light.



Solution: The original angle of reflectance can be found with some optical geometry to be $\theta_0 = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. By Malus' Law, when the laser hits the planet with an angle θ, the final power after reflection is $P_0 \cos^2(\theta)$. Also, note that the satellite has to be at an angle 2θ due to the law of reflection. Therefore, by Kepler's Third law, the time for the satellite to reach $\theta = 0$ is

$$t = 2\theta\sqrt{\frac{r^3}{GM_p}} \implies dt = 2\sqrt{\frac{r^3}{GM_p}}d\theta.$$

Thus, the total power absorbed by the satellite is

$$\int_0^{\theta_0} P_0 \cos^2(\theta)dt = 2P_0\sqrt{\frac{r^3}{GM_p}}\int_0^{\theta_0} \cos^2(\theta)d\theta = P_0\sqrt{\frac{r^3}{GM_p}}(\theta_0 + \sin(\theta_0)\cos(\theta_0)) = \boxed{1.33 \cdot 10^{35}\text{J}}.$$

Pr 27. Braking Up

A particle of rest mass m moving at a speed $v = 0.7c$ decomposes into two photons which fly off at a separated angle θ. What is the minimum value of the angle of separation assuming that the two photons have equal wavelength. (Answer in degrees)

Solution: Conservation of momentum and energy gives:

$$p_m = 2E_\gamma \cos(\theta/2)$$

$$E_m = 2E_\gamma$$

Relativity demands that:

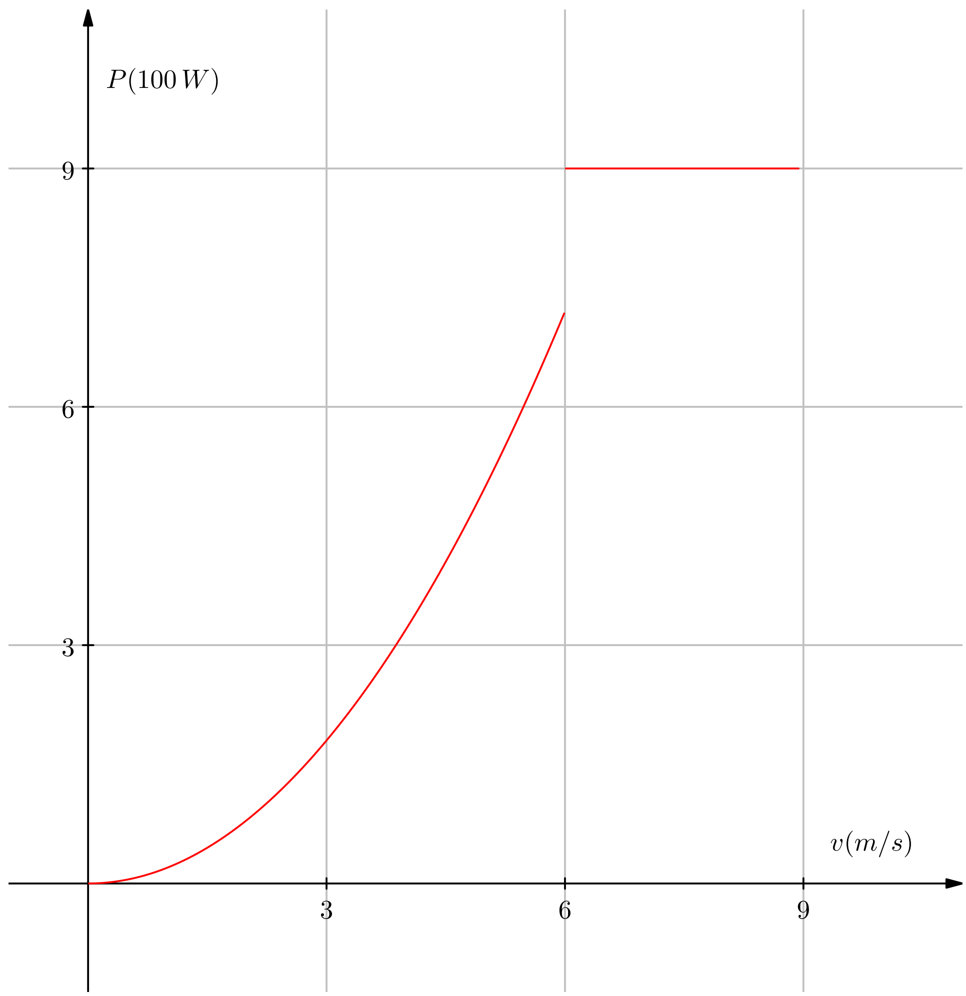
$$E_m^2 = m^2 + p_m^2$$

Solving this system of three equations gives:

$$\begin{aligned}
4E_\gamma^2 &= m^2 + p_m^2 \\
\frac{\gamma^2 m^2 v^2}{\cos^2(\theta/2)} &= m^2 + \gamma^2 m^2 v^2 \\
\gamma^2 v^2 \left(-1 + \frac{1}{\cos^2(\theta/2)}\right) &= 1 \\
\frac{1}{\cos^2(\theta/2)} &= \frac{1}{\gamma^2 v^2} + 1 \\
\cos(\theta/2) &= v \\
\theta &= \boxed{91.1^\circ}
\end{aligned}$$

Pr 28. With Great Power

Mario is racing with Wario on Moo Moo Meadows when a goomba, ready to avenge all of his friends' deaths, came and hijacked Mario's kart. A graph representing the motion of Mario at any instant is shown below. The velocity acquired by Mario is shown on the x-axis, and the net power of his movement is shown on the y-axis. When Mario's velocity is 6 m/s, he eats a mushroom which gives him a super boost.



You may need to make measurements. Feel free to print this picture out as the diagram is drawn to scale. Find the total distance from Mario runs from when his velocity is 0 m/s to when his velocity just reaches 9 m/s given that Mario's mass is $m = 89$ kg. Answer in meters and round to one significant digit.

Solution: Our first goal is to find an expression for the power curve $P(v)$. To do this, let us select a few points on the curve. The easiest point to pick is $(0, 0)$ since it is fixed at the origin. The next two easiest points to pick are those that are on the lines $x = 3$ and $x = 6$ which are given as approximately $(3, 1.8)$ and $(6, 7.2)$. Note that y-axis is in units of 100 W so in reality these two points are given as $(3, 180)$ and $(6, 720)$. This curve is resemblant of a quadratic in the form of $y = k_1x^2$ and upon solving for k_1 we find that the curve is given as $P = 20v^2$. Secondly, the next line remains constant with respect to time as a line $P = 900$. Therefore, we can write a piecewise function for power defined by

$$P(v) = \begin{cases} 20v^2 & \text{if } v \geq 0, \quad \text{and} \quad v < 6 \\ 900 & \text{if } v \geq 6 \end{cases}.$$

We need to find the relationship between power, velocity, and displacement of Mario. Consider dividing the displacement into tiny rectangular pieces with width Δt such that

$$s = \sum_{i \in \mathbb{N}} \Delta s_i = \sum_i v_i(t) \cdot \Delta t.$$

We want to the displacement to be expressed in terms without Δt. This means that we have to find a relationship for Δt. Note that

$$\Delta t = \Delta v \cdot \frac{\Delta t}{\Delta v} = \Delta v \cdot \frac{1}{\Delta v / \Delta t} = \frac{\Delta v}{a}.$$

Therefore, we now know the displacement to be expressed as

$$s = \sum_{i \to 0} v_i(t) \frac{\Delta v}{a} = \int \frac{v}{a(v)}\, dv.$$

To find an expression for $a(v)$ in terms of power, we note that

$$P(v) = F(v) \cdot v \implies a(v) = \frac{P(v)}{vm}$$

which means that upon substituting,

$$s = \int \frac{v^2 m}{P(v)}\, dv \implies s = \int_0^6 \frac{89v^2}{20v^2} dv + \int_6^9 \frac{89v^2}{900} dv = 43.61 \approx \boxed{40 \text{ m}}.$$

Pr 29. I'm a little teacup

At an amusement park, there is a ride with three "teacups" that are circular with identical dimensions. Three friends, Ethan, Rishab, and Kushal, all pick a teacup and sit at the edge. Each teacup rotates about its own axis clockwise at an angular speed $\omega = 1$ rad/s and can also move linearly at the same time.

The teacup Ethan is sitting on (as always) is malfunctional and can only rotate about its own axis. Rishab's teacup is moving linearly at a constant velocity 2 m/s [N] and Kushal's teacup is also moving linearly at a constant velocity of 4 m/s [N 60° E]. All three teacups are rotating as described above. Interestingly, they observe that at some point, all three of them are moving at the same velocity. What is the radius of each teacup?

Note: [N 60° E] means 60° clockwise from north e.g. 60° east of north.

Solution: We can plot the motion on a $v_y - v_z$ graph instead of carrying out calculations. We have three points at locations $(0, 0)$, $(0, 2)$, and $(2\sqrt{3}, 2)$ which represent the velocity of the center of mass of the teacups. The velocity that they are moving at can be traced as a circle with radius $r\omega$, centered at these points.

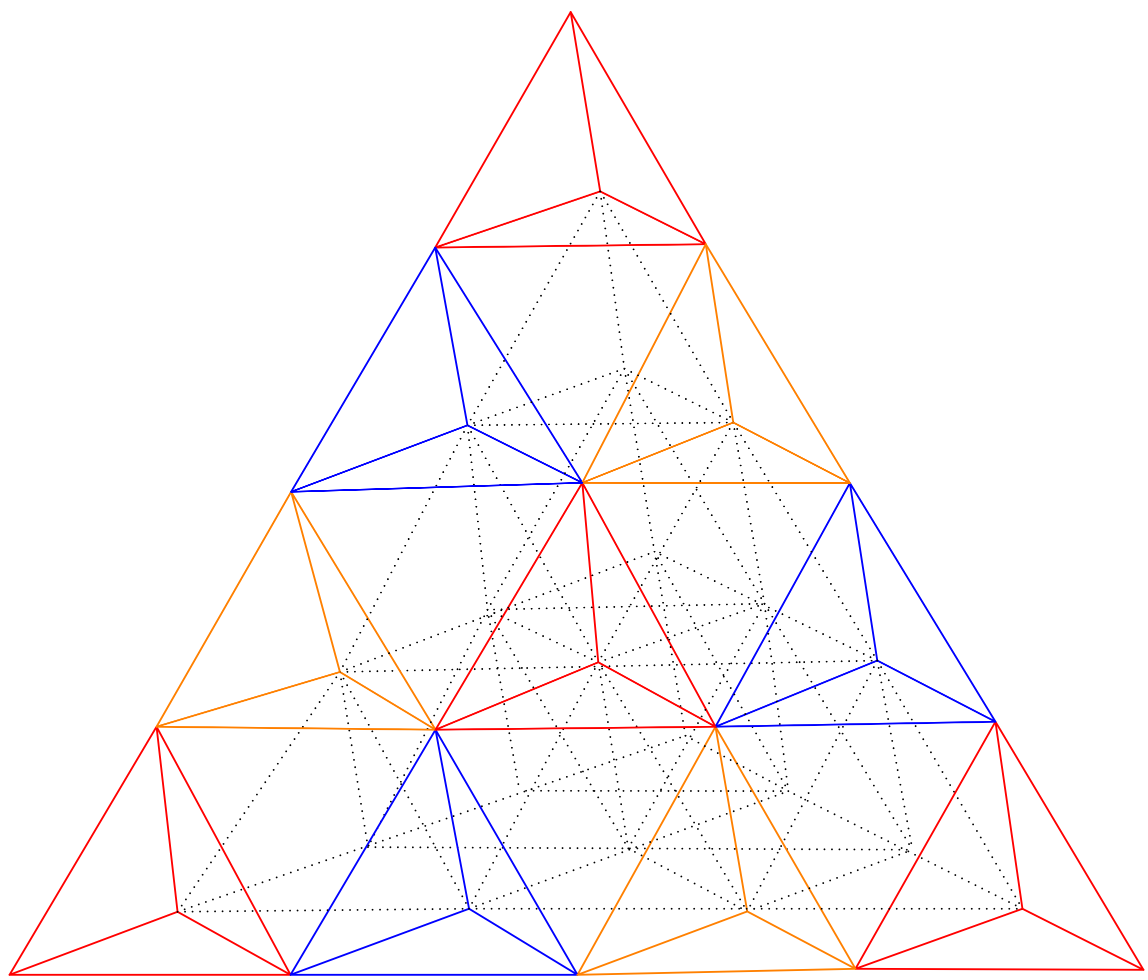
The problem now becomes, at what value r will the three circles intersect. Drawing a diagram, or carrying out trigonometric calculations gives $r = \boxed{2 \text{ m}}$.

Pr 30. Tetrahedron Resistance

An engineer has access to a tetrahedron building block with side length $\ell = 10$ cm. The body is made of a thermal insulator but the edges are wrapped with a thin copper wiring with cross sectional area $S = 2$ cm^2. The thermal conductivity of copper is 385.0 W/(m K). He stacks these tetrahedrons (all facing the same direction) to form a large lattice such that the copper wires are all in contact. In the diagram, only the front row of a small section is coloured. Assume that the lattice formed is infinitely large.

At some location in the tetrahedral building block, the temperature difference between two adjacent points is 1°C. What is the heat flow across these two points? Answer in Watts.

Note: Two adjacent points refer to two adjacent points on the tetrahedron.



Solution: There are many ways to solve this problem. We first identify that this is exactly the same as an infinite lattice resistor problem. To solve these, we can imagine injecting a current at a node and seeing how this current spreads out. However, a faster approach is by applying **Foster's Theorem** on this lattice.

The resistance of a single wire is:

$$R = \frac{\ell}{kS} = 1.299 \text{ W/K}$$

Foster's theorem tells us that

$$ER = V - 1$$

where V is number of vertices and E is edges. Taking the limit as $E, V \to \infty$, we get: $E = 6V$ (since each vertex is connected to 12 edges, but each edge is shared by two vertices). Therefore:

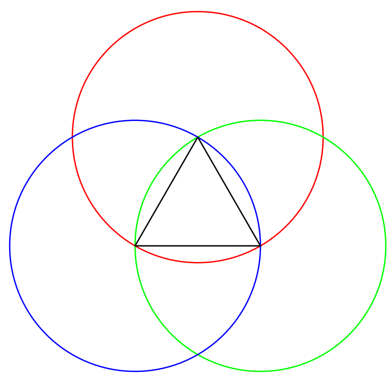
$$R_{\text{eff}} = \frac{1}{6}R = 0.2165 \text{ W/K}$$

From Fourier's Law, we have:

$$\dot{Q} = \frac{\Delta T}{R_{\text{eff}}} = \boxed{4.62 \text{ W}}.$$

Pr 31. AIME

Three unit circles, each with radius 1 meter, lie in the same plane such that the center of each circle is one intersection point between the two other circles, as shown below. Mass is uniformly distributed among all area enclosed by at least one circle. The mass of the region enclosed by the triangle shown above is 1 kg. Let x be the moment of inertia of the area enclosed by all three circles (intersection, *not* union) about the axis perpendicular to the page and through the center of mass of the triangle. Then, x can be expressed as $\frac{a\pi - b\sqrt{c}}{d\sqrt{e}}$ kg m^2, where a, b, c, d, e are integers such that $\gcd(a, b, d) = 1$ and both c and e are squarefree. Compute $a + b + c + d + e$.



Solution: Define point O as the point in the plane that the axis of rotation passes through. Since moments of inertia simply add about a given axis, we can calculate the moments of inertia of the three "sectors" whose union forms the given area and subtract twice the moment of inertia of the triangle, so our answer will be $3I_{s,O} - 2I_{t,O}$.

Claim: The center of mass of a sector is $\frac{2}{\pi}$ away from the vertex of the sector along its axis of symmetry.

Proof: We can divide the sector into arbitrarily small sectors that can be approximated as isosceles triangles. It's well known that the center of mass of one such isosceles triangle is $\frac{2}{3}$ of the way from the central vertex to the base. Therefore, the center of mass of the sector is the center of mass of the arc with central angle $\frac{\pi}{3}$ and same center with radius $\frac{2}{3}$ contained within the sector. Since the center of mass has to lie on the axis of symmetry, we set that as the x axis with the vertex of the sector being $x = 0$. Then, the x-coordinate of a point on the arc whose corresponding radius makes an angle of θ with the axis of symmetry is $\frac{2}{3}\cos(\theta)$. We can integrate this over all possible angles $(-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6})$ and then divide by the range $(\frac{\pi}{3})$ to get the average x-coordinate, or the center of mass.

$$\frac{\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{2}{3}\cos(\theta)\mathrm{d}\theta}{\frac{\pi}{3}}$$

$$\frac{2}{\pi}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos(\theta)\mathrm{d}\theta$$

$$\frac{2}{\pi}\left(\sin\left(\frac{\pi}{6}\right) - \sin\left(-\frac{\pi}{6}\right)\right)$$

$$\frac{2}{\pi}$$

This concludes the proof. $\square$
Now define point X as the vertex of a sector and point M as the center of mass of that sector. According to the parallel axis theorem,

$$I_{s,X} = I_{s,M} + m_s\left(\frac{2}{\pi}\right)^2$$

. It's well known that $I_{s,X} = \frac{1}{2}m_s r^2 = \frac{m_s}{2}$, and so

$$I_{s,M} = \frac{m_s}{2} - \frac{4m_s}{\pi^2} = m_s\left(\frac{\pi^2-8}{2\pi^2}\right)$$

It's also well known that O is on the line of symmetry and a distance of $\frac{1}{\sqrt{3}}$ away from X, and so $MX = \frac{2}{\pi} - \frac{1}{\sqrt{3}}$. Therefore,

$$I_{s,O} = I_{s,M} + m_s\left(\frac{2}{\pi} - \frac{1}{\sqrt{3}}\right)^2 = m_s\left(\frac{5\pi - 8\sqrt{3}}{6\pi}\right)$$

It's well known that, since O is the center of mass of the triangle,

$$I_{t,O} = \frac{1}{12}$$

Now we just need to calculate m_s. Since the mass of the triangle is 1 kg, this is equivalent to finding the ratio of the area of a sector to the area of a triangle. Through geometry, this is found to be $\frac{2\pi}{3\sqrt{3}}$. Finally, we get our answer to be

$$\left(\frac{2\pi}{\sqrt{3}}\right)\left(\frac{5\pi - 8\sqrt{3}}{6\pi}\right) - \frac{1}{6} = \left(\frac{10\pi - 17\sqrt{3}}{6\sqrt{3}}\right)$$

and $a+b+c+d+e = 10+17+3+6+3 = \boxed{039}$

Pr 32. Global Warming

Life on Earth would not exist as we know it without the atmosphere. There are many reasons for this, but one of which is temperature. Let's explore how the atmosphere affects the temperature on Earth. Assume that all thermal energy striking the earth uniformly and ideally distributes itself across the Earth's surface.

- Assume that the Earth is a perfect black body with no atmospheric effects. Let the equilibrium temperature of Earth be T_0. (The sun outputs around $3.846 \times 10^{2^6}$ W, and is 1.496×10^8 km away.)
- Now assume the Earth's atmosphere is isothermal. The short wavelengths from the sun are nearly unaffected and pass straight through the atmosphere. However, they mostly convert into heat when they strike the ground. This generates longer wavelengths that do interact with the atmosphere. Assume that the albedo of the ground is 0.3 and e, the emissivity and absorptivity of the atmosphere, is 0.8. Let the equilibrium average temperature of the planet now be T_1.

What is the percentage increase from T_0 to T_1?

Note: The emissivity is the degree to which an object can emit longer wavelengths (infrared) and the absorptivity is the degree to which an object can absorb energy. Specifically, the emissivity is the ratio between the energy emitted by an object and the energy emitted by a perfect black body at the same temperature. On the other hand, the absorptivity is the ratio of the amount of energy absorbed to the amount of incident energy.

Solution: Let us solve this problem in the case of the Earth being a graybody first and then substitute values for when it is a blackbody. The portion of energy that reaches the Earth is given by the ratio between the cross-sectional area of the satellite and the area of an imaginary sphere centered around the sun with a radius of L. Thus, the incoming radiation is multiplied by a factor of $\gamma = (R/2L)^2$. The energy from the sun that the surface absorbs is $\gamma(1-\alpha)E$, where E is the energy output of the sun. Here $\gamma = 1/4$ as the sphere encompassing will be 4 times the area of its intercept.
We can now write two systems of equations at the atmosphere and the ground of the Earth. At the top of the atmosphere, we require equilibrium meaning that zero net radiation leaves the atmosphere or:

$$-\frac{1}{4}S_0(1-\alpha) + \varepsilon\sigma T_a^4 + (1-\varepsilon)\sigma T_s^4 = 0.$$

Similarly, at the ground, we write another equilibrium equation of:

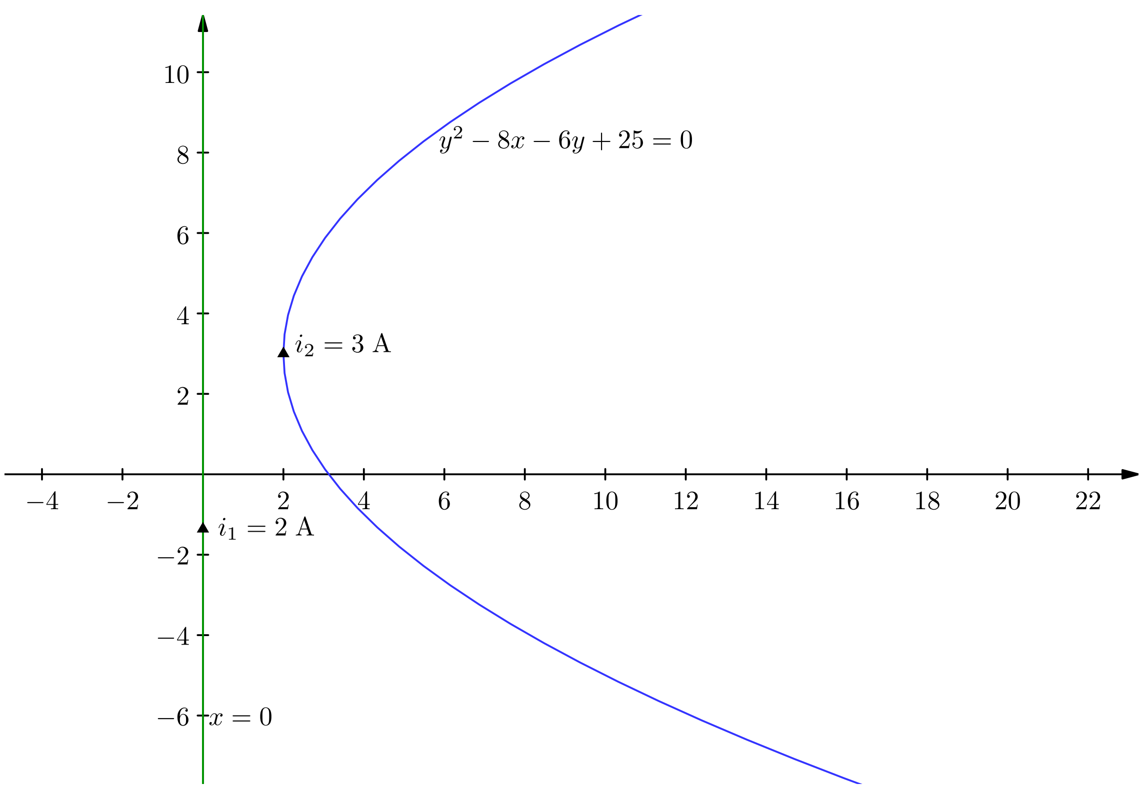
$$\frac{1}{4}S_0(1-\alpha) + \varepsilon\sigma T_a^4 - \sigma T_s^4 = 0.$$

Thus, solving the ground equilibrium equation yields us $T_a = 2^{-1/4}T_s$ and plugging back into the atmosphere equilibrium equation tells us:

$$\frac{1}{4}S_0(1-\alpha) = \left(1-\frac{\varepsilon}{2}\right)\sigma T_s^4 \implies T_s = \boxed{289.601 \text{ K}}.$$

Pr 33. Flattening the Curve

Two infinitely long current carrying wires carry constant current $i_1 = 2$ A and $i_2 = 3$ A as shown in the diagram. The equations of the wire curvatures are $y^2 - 8x - 6y + 25 = 0$ and $x = 0$. Find the magnitude of force (in Newtons) acting on one of the wires due to the other.



Note: The current-carrying wires are rigidly fixed. The units for distances on the graph should be taken in metres.

Solution: The magnetic field from the wire is given by $B = \frac{\mu_0 i_1}{2\pi x}$. Let θ be the direction of a component of force from the vertical. It is then seen that

$$dF = Bi_2 d\ell \implies dF_x = Bi_2 d\ell \sin\theta = Bi_2 dy.$$

We only consider the force in the x-direction which means that

$$F_x = \int_{\infty}^{\infty} dF_x = \frac{\mu_0 i_1 i_2}{2\pi}\int_{-\infty}^{\infty}\frac{dy}{x}.$$

Solving the equation in terms of x and then plugging in gives us

$$F_x = \frac{8\mu_0 i_1 i_2}{2\pi}\int_{-\infty}^{\infty}\frac{dy}{y^2-6y+25} = \frac{8\mu_0 i_1 i_2}{2\pi}\cdot\frac{\pi}{4} = \mu_0 i_1 i_2 = \boxed{7.5398\cdot 10^{-6}\text{ N}}.$$

Pr 34. Hiking in Mountains

Mountains have two sides: windward and leeward. The windward side faces the wind and typically receives warm, moist air, often from an ocean. As wind hits a mountain, it is forced upward and begins to move towards the leeward side. During social distancing, Rishab decides to cross a mountain from the windward side to the leeward side of the mountain. What he finds is that the air around him has warmed when he is on the leeward side of the mountain.

Let us investigate this effect. Consider the warm, moist air mass colliding with the mountain and moving upwards on the mountain. Disregard heat exchange with the air mass and the mountain. Let the humidity of the air on the windward side correspond to a partial vapor pressure 0.5 kPa at 100.2 kPa and have a molar mass of $\mu_a = 28$ g/mole. The air predominantly consists of diatomic molecules of oxygen and nitrogen. Assume the mountain to be very high which means that at the very top of the mountain, all of the moisture in the air condenses and falls as precipitation. Let the precipitation have a heat of vaporization $L = 2.4\cdot 10^6$ J/kg and molar mass $\mu_p = 18.01$ g/mole. Calculate the total change in temperature from the windward side to the leeward side in degrees Celsius.

Solution: We use the first law of thermodynamics to solve this problem. For diatomic molecules, the internal energy per mole is given by $\frac{5}{2}RT$. If the molar mass of the air is μ_a, then we have that the change in internal energy of the air is given by

$$\Delta U = \frac{5}{2}\frac{M}{\mu_a}R\Delta T.$$

We also note that the total work performed by the gas is

$$W = P_2V_2 - P_1V_1$$

since the process is adiabatic, we can use the ideal gas equation $PV = \nu RT = (M/\mu)RT$ to express the total work as

$$W = \frac{M}{\mu_a}R\Delta T.$$

The heat that is taken away during condensation at the top of the mountain is given by $Q = L\Delta m$ where Δm is the total mass of the precipitation. According to the ideal gas law, we have that

$$PV_1 = \frac{\Delta m}{\mu_p}RT_1, \quad P_1V_1 = \frac{M}{\mu_a}RT_1$$

recombining these equations and equating them gives us

$$\frac{\Delta m}{\mu_p P}RT_1 = \frac{M}{\mu_a P_1}RT_1$$

$$\Delta m = M\frac{\mu_p P}{\mu_a P_1}.$$

Therefore,

$$Q = LM\frac{\mu_a P}{\mu_p P_1}.$$

We finally can now use the first law of thermodynamics

$$Q = \Delta U + W \implies LM\frac{\mu_p P}{\mu_a P_1} = \frac{5}{2}\frac{M}{\mu_a}R\Delta T + \frac{M}{\mu_a}R\Delta T.$$

We then simplify this equation to get

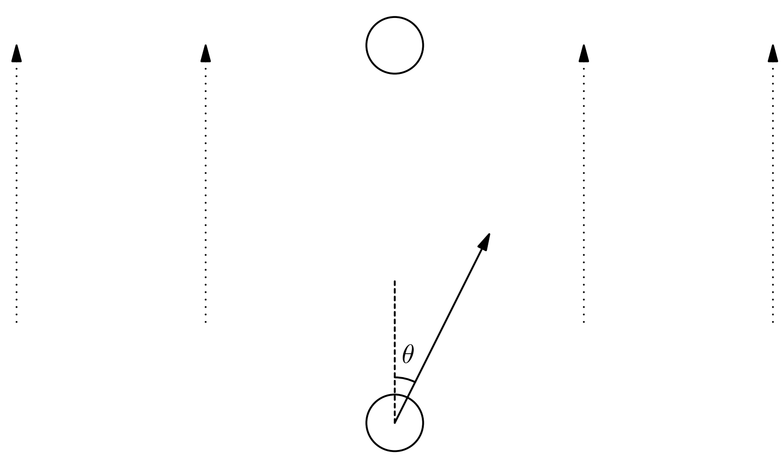
$$LM\frac{\mu_p P}{\mu_a P_1} = \frac{7}{2}\frac{M}{\mu_a}R\Delta T$$
$$\Delta T = \frac{2}{7}\frac{L\mu_p P}{RP_1} = \boxed{7.41\text{ K}}$$

A simpler approach could be to assume that the number of moles of water vapour in the atmosphere is equal to number of moles of water condensed. Then, the mass of precipitated water is $\mu_p n\frac{P}{P_1}$, where n is the number of moles of air. Thus,

$$\mu_p n\frac{P}{P_1}L = \frac{7}{2}nR\Delta T.$$

Pr 35. Me and my Crush

Two electrons are in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$ where $E_0 = 10^{-11}$ N/C. One electron is at the origin, and another is 10 m above the first electron. The electron at the origin is moving at $u = 10$ m/s at an angle of 30° from the line connecting the electrons at $t = 0$, while the other electron is at rest at $t = 0$. Find the minimum distance between the electrons. You may neglect relativistic effects.



Solution: Let $\ell = 10$ m. First, switch into the reference frame accelerating at $-\frac{Eq}{m}\hat{z}$. In this frame, the electrons are not affected by the electric field. Now, switch into the center of mass reference frame from here. In this frame, we have both conservation of angular momentum and conservation of energy. Both electrons in this frame are moving at $\frac{u}{2}$ initially at an angle of $\theta = 30°$. At the smallest distance, both electrons will be moving perpendicular to the line connecting them. Suppose that they both move with speed v and are a distance r from the center of mass. By conservation of angular momentum,

$$2m \cdot \frac{u}{2} \cdot \frac{\ell}{2}\sin\theta = 2mvr$$

$$vr = \frac{u\ell}{4}\sin\theta.$$

Now, by conservation of energy,

$$mv^2 + \frac{ke^2}{2r} = \frac{1}{4}mu^2 + \frac{ke^2}{\ell}.$$

Now, we just solve this system of equations to determine the value of r. Substituting $v = \frac{ul}{4r}\sin\theta$ into the conservation of energy equation, we can solve the ensuing quadratic to find:

$$r = \frac{\frac{ke^2}{2} + \sqrt{\left(\frac{ke^2}{2}\right)^2 + \left(mu^2 + \frac{4ke^2}{\ell}\right)\left(\frac{mu^2\ell^2}{16}\sin^2(\theta)\right)}}{\frac{1}{2}mu^2 + \frac{2ke^2}{\ell}}.$$

Finally, remembering that the distance between the electrons is actually $2r$, we obtain $2r = \boxed{6.84 \text{ m}}$ as the final answer.

Pr 36. Can't or can

Consider a long uniform conducting cylinder. First, we divide the cylinder into thirds and remove the middle third. Then, we perform the same steps on the remaining two cylinders. Again, we perform the same steps on the remaining four cylinders and continuing until there are 2048 cylinders.

We then connect the terminals of the cylinder to a battery and measure the effective capacitance to be C_1. If we continue to remove cylinders, the capacitance will reach an asymptotic value of C_0. What is C_1/C_0?

You may assume each cylindrical disk to be wide enough to be considered as an infinite plate, such that the radius R of the cylinders is much larger than the d between any successive cylinders.

Note: The diagram is not to scale.

Solution: The capacitance is proportional to $C \propto \frac{1}{d}$, where d is the distance between successive parallel plates. When we add capacitor plates in series, their effective capacitance will be:

$$C \propto \left(\frac{1}{1/d_1} + \frac{1}{1/d_2} + \cdots\right)^{-1} = \frac{1}{d_1 + d_2 + \cdots} \implies C \propto \frac{1}{d_{\text{total}}}$$

Therefore, this essentially becomes a math problem: What is the total length of the spacing in between? Between successive 'cuts', the length of each cylinder is cut down by 1/3, but the number of gaps double. Therefore, the spacing grows by a factor of 2/3 each time. For $n = 2^1$, the spacing starts off as 1/3. For $n = 2^{10}$, the spacing becomes:

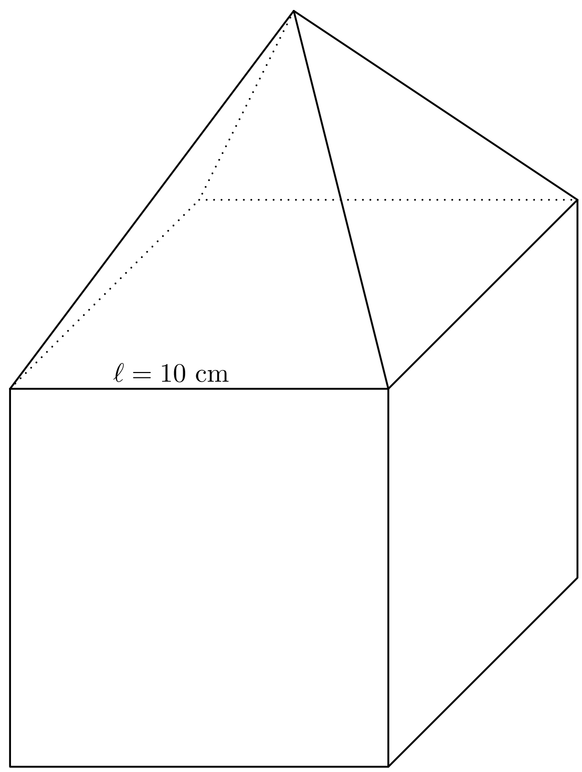
$$\frac{1}{C_{\text{eff}}} \propto d = \frac{1}{3}\left(\frac{1-(2/3)^{10}}{1-2/3}\right)L = 0.983L$$

for $n \to \infty$, it is clear the total spacing will converge to L. Therefore:

$$C_1/C_0 = \boxed{1.017}$$

Pr 37. Mom Trust the Physics!

A square based pyramid (that is symmetrical) is standing on top of a cube with side length $\ell = 10$ cm such that their square faces perfectly line up. The cube is initially standing still on flat ground and both objects have the same uniform density. The coefficient of friction between every surface is the same value of $\mu = 0.3$. The cube is then given an initial speed v in some direction parallel to the floor. What is the maximum possible value of v such that the base of the pyramid will always remain parallel to the top of the cube? Answer in meters per second.



Solution: Let x be the relative displacement of the two objects. Then:

$$v_i^2 = 2ax$$

where the relative acceleration is:

$$a = \frac{2h}{3\ell} g\mu$$

work The acceleration of pyramid is $g\mu$ and the acceleration of cube is:

$$\rho\ell^3 a = \rho\left(\ell^3 + \frac{\ell^2 h}{3}\right) g\mu + \rho\frac{\ell^2 h}{3} g\mu \implies a = \frac{3\ell + 2h}{3\ell} g\mu$$

Therefore, the relative acceleration is:

$$a = \frac{6\ell + 2h}{3\ell} g\mu$$

We want to direct the motion of the cube diagonally (such that horizontal sides of the cube form a 45 degree angle with the displacement). Initially, we may think that we need to let $x = \frac{\sqrt{2}}{2}\ell$ but it can start tipping before that. Moving into a non-inertial reference frame for the pyramid, we see that the effective gravity needs to point towards the back corner of the cube, so it needs to satisfy the criteria

$$\frac{h_{\text{cm}}}{\frac{\sqrt{2}}{2}\ell - x} = \frac{mg}{mg\mu} \implies x = \frac{\sqrt{2}}{2}\ell - \mu\frac{1}{4}h$$

Here I used the fact that the center of mass was 1/4 of the way up. Substituting, we get:

$$v = \sqrt{2\left(\frac{6\ell + 2h}{3\ell} g\mu\right)\left(\frac{\sqrt{2}}{2}l - \frac{\mu}{4}h\right)} = \sqrt{2g\mu\left(2 + \frac{2h}{3\ell}\right)\left(\frac{\sqrt{2}}{2}\ell - \frac{\mu}{4}h\right)}$$

The maximum v occurs at

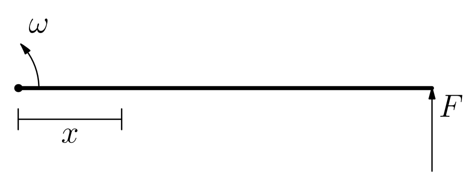
$$h/\ell = \frac{-3 + \frac{2\sqrt{2}}{\mu}}{2}$$

giving a maximum v of $v_{\text{max}} = \boxed{1.07 \text{ m/s}}$.

Pr 38. FBI Open Up!

During quarantine, the FBI has been monitoring a young physicists suspicious activities. After compiling weeks worth of evidence, the FBI finally has had enough and searches his room.

The room's door is opened with a high angular velocity about its hinge. Over a very short period of time, its angular velocity increases to $\omega = 8.56$ rad/s due to the force applied at the end opposite from the hinge. For simplicity, treat the door as a uniform thin rod of length $L = 1.00$ m and mass $M = 9.50$ kg. The hinge (pivot) is located at one end of the rod. Ignore gravity. At what distance from the hinge of the door is the door most likely to break? Assume that the door will break at where the bending moment is largest. (Answer in metres.)



Solution: Let N be the force from the pivot and F be the applied force at the end. Let α be the angular acceleration. Writing the torque equation and Newton's 2nd law for the whole door, we get:

$$F \cdot L = \frac{1}{3}ML^2\alpha$$

$$N + F = \frac{1}{2}ML\alpha$$

Solving, we get $F = \frac{1}{3}ML\alpha$ and $N = \frac{1}{6}ML\alpha$. Now, we consider the part of the door with length x attached to the pivot. The rest of the door applies a torque τ and shear force f on our system. (There is also tension force). Let $\lambda = \frac{M}{L}$. We can write the torque equation and Newton's 2nd law for our system:

$$\tau + fx = \frac{1}{3}\lambda x^3\alpha$$

$$N + f = \lambda x \cdot \frac{x}{2}\alpha$$

Solving, we get

$$\tau = \frac{1}{6}\lambda x\alpha(L^2 - x^2)$$

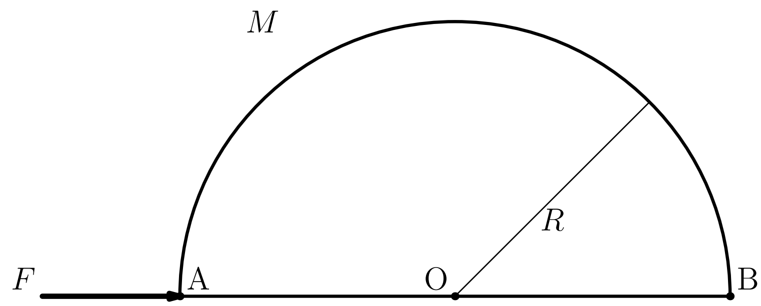
and

$$f = \frac{1}{6}\lambda\alpha(3x^2 - L^2)$$

We maximize τ (which is equivalent to maximizing bending moment) to get $x = \frac{L}{\sqrt{3}} = \boxed{0.577 \text{ m}}$

Pr 39. Pappu's Half Disk

A solid half-disc of mass $m = 1$ kg in the shape of a semi-circle of radius $R = 1$ m is kept at rest on a smooth horizontal table. QiLin starts applying a constant force of magnitude $F = 10$ N at point A as shown, parallel to its straight edge. What is the initial linear acceleration of point B? (Answer in m/s^2)



Note: the diagram above is a *top down view.*

Solution: Let C denote the location of the centre of mass of the disc. It is well known that $OC = \frac{4R}{3\pi}$. Note that the initial angular velocity of the disc about its centre of mass is 0, and the linear acceleration is simply

$$a_{\text{CM}} = \frac{F}{m}\hat{i}$$

Now we use the $\tau = \vec{r} \times \vec{F} = I_{\text{CM}}\alpha_{\text{CM}}$ about the centre of mass of the disc

$$\left(\frac{4R}{3\pi}\right) F = I_{\text{CM}}\alpha_{\text{CM}}$$

To compute the moment of inertia of the disc about its centre of mass, we use Steiner's theorem:

$$I_{\text{CM}} = \frac{MR^2}{2} - M\left(\frac{4R}{3\pi}\right)^2$$

so the $\tau = I\alpha$ equation becomes

$$\left(\frac{4R}{3\pi}\right) F = \left[\frac{MR^2}{2} - M\left(\frac{4R}{3\pi}\right)^2\right]\alpha_{\text{CM}} \Rightarrow \alpha_{\text{CM}} = \frac{\frac{4FR}{3\pi}}{MR^2\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)}$$

Using kinematics equation for rotational motion, we have

$$\vec{a}_B = \vec{a}_{\text{CM}} + \vec{\alpha}_{\text{CM}} \times \vec{\text{CB}}$$

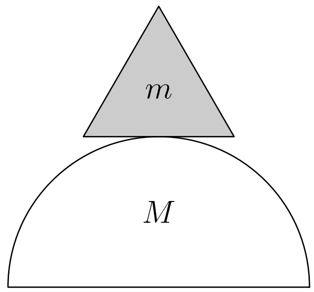
Substituting the values of $\vec{\text{CB}} = R\hat{i} - \frac{4R}{3\pi}\hat{j}$ and $\alpha_{\vec{\text{CM}}} = \frac{\frac{4FR}{3\pi}}{MR^2\left(\frac{1}{2}-\frac{16}{9\pi^2}\right)}\hat{k}$ we get

$$a_B = \frac{F}{m}\left[\left(1 + \frac{1}{-1 + \frac{9\pi^2}{32}}\right)\hat{i} + \frac{2}{3\pi\left(\frac{1}{2} + \frac{16}{9\pi^2}\right)}\hat{j}\right] = \boxed{15.9395 \text{ m/s}^2}$$

and we are done. $\square$

Pr 40. Don't Fall

A regular tetrahedron of mass $m = 1$ g and unknown side length is balancing on top of a hemisphere of mass $M = 100$ kg and radius $R = 100$ m. The hemisphere is placed on a flat surface such that it is at its lowest potential. For a certain value of the length of the regular tetrahedron, the oscillations become unstable. What is this side length of the tetrahedron?



Solution: For stable equilibrium the height resultant from a slight displacement must be greater than the original height of the center of mass of the cube. A small displacement from the original position can be modeled as an x displacement of

$$Rd\theta,$$

which raises the height by

$$Rd\theta \sin(d\theta)$$

added to the height

$$s\cos(d\theta)$$

This must be greater than s, where s is the distance from the center of mass to the point of contact to the sphere. Approximating to the second degree of θ using Taylor series, we solve the inequality and get that $s = R$. The altitude of the tetrahedron is therefore $4s = 400$

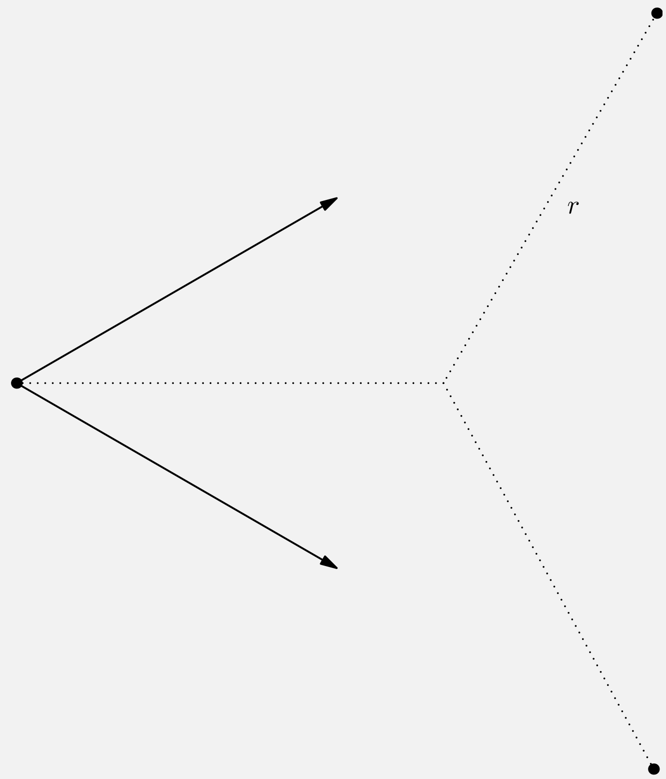
$$l^2 - (l\sqrt{3}/3)^2 = 400^2$$

Therefore, $l = \sqrt{400^2 \cdot 3/2} = \boxed{490 \text{ m}}$

Pr 41. Heartbreak

A planet has a radius of 10 km and a uniform density of 5 g/cm^3. A powerful bomb detonates at the center of the planet, releasing 8.93×10^{17} J of energy, causing the planet to separate into three large sections each with equal masses. You may model each section as a perfect sphere of radius r'. The initial and final distances between the centers of any two given sections is $2r'$. How long does it take for the three sections to collide again?

Solution: Due to conservation of momentum, the three masses must form an equilateral triangle at all times. Let us determine the force as a function of r, the distance between each mass and the center.



The vector sum of the net force on any individual mass is

$$F = \frac{Gm^2}{d^2}\sqrt{2-2\cos 120^\circ} = \frac{\sqrt{3}Gm^2}{d^2}$$

where d is the distance between the mass and the center.

$$d^2 = 3r^2$$

The net force is thus

$$F = \frac{Gm^2}{\sqrt{3}r^2}$$

The system behaves as if there was a stationary mass $m' = m/\sqrt{3}$ at the center, simplifying the problem greatly into a restricted two body system. Next, we need to figure out the height of the apoapsis. This can be done via conservation of energy.

$$E_{\text{binding,initial}} + E = 3E_{\text{binding,final}} - \frac{Gm^2}{\sqrt{3}\ell} \implies \ell = -\frac{Gm^2}{\sqrt{3}\left(-\frac{3GM^2}{5R} + 8.93\cdot 10^{17} - \left(-3\frac{3Gm^2}{5r_f}\right)\right)} = 101,000 \text{ m}$$

If you have a stationary mass M at the center. The time it takes for an object to fall into it is:

$$T = \pi\sqrt{\frac{\ell^3}{8GM}}$$

our time will be double this, and the mass in the center will be $M = m/\sqrt{3}$. So plugging in numbers gives:

$$t = 2\pi\sqrt{\frac{l^3}{8G\left(\frac{m}{\sqrt{3}}\right)}} = 138,000$$

If we take into account a nonzero radius so final separation is $10/\cos(30°)$ km, then the answer should be:

$$-2\int_{l}^{\frac{10000}{\cos\left(\frac{\pi}{6}\right)}}\left(\sqrt{\frac{xl}{2G\left(\frac{m}{\sqrt{3}}\right)(l-x)}}\right)dx = \boxed{136,000 \text{ s}}$$

Note if you set the upper bound to zero, you get the same answer as before. Both these answers will be accepted.

Pr 42. Sandwiched!

A point charge $+q$ is placed a distance a away from an infinitely large conducting plate. The force of the electrostatic interaction is F_0. Then, an identical conducting plate is placed a distance $3a$ from the charge, parallel to the first one such that the charge is "sandwiched in." The new electrostatic force the particle feels is F'. What is F'/F_0? Round to the nearest hundredths.

Solution: We solve this via the method of image charges. Let us first reflect the charge $+q$ across the closest wall. The force of this interaction will be:

$$F_0 = \frac{q^2}{4\pi\epsilon_0 a^2}\frac{1}{2^2}$$

and this will be the only image charge we need to place if there were only one conducting plane. Since there is another conducting plane, another charge will be reflected to a distance $a+4a$ past the other conducting plane, and thus will be $a+4a+3a=8a$ away from the original charge. All these reflections cause a force that points in the same direction, which we will label as the positive $+$ direction. Therefore:

$$F_+ = \frac{q^2}{4\pi\epsilon_0 a^2}\left(\frac{1}{2^2}+\frac{1}{8^2}+\frac{1}{10^2}+\frac{1}{16^2}\frac{1}{18^2}+\frac{1}{24^2}+\cdots\right)$$

Now let us look at what happens if we originally reflect the charge $+q$ across the other wall. Repeating the steps above, we see that through subsequent reflections, each force will point in the negative $-$ direction. Therefore:

$$F_- = \frac{-q^2}{4\pi\epsilon_0 a^2}\left(\frac{1}{6^2}+\frac{1}{8^2}+\frac{1}{14^2}+\frac{1}{16^2}+\frac{1}{22^2}+\frac{1}{24^2}+\cdots\right)$$

The net force is a result of the superposition of these two forces, giving us:

$$\begin{aligned}F' = \frac{q^2}{4\pi\epsilon_0 a^2}\Big(&\frac{1}{2^2}+\frac{1}{8^2}+\frac{1}{10^2}+\frac{1}{16^2}\frac{1}{18^2}+\frac{1}{24^2}+\cdots \\ &-\frac{1}{6^2}-\frac{1}{8^2}-\frac{1}{14^2}-\frac{1}{16^2}-\frac{1}{22^2}-\frac{1}{24^2}-\cdots\Big)\end{aligned}$$

Even terms can be cancelled out to give:

$$\begin{aligned}F' &= \frac{q^2}{4\pi\epsilon_0 a^2}\left(\frac{1}{2^2}-\frac{1}{6^2}+\frac{1}{10^2}-\frac{1}{14^2}+\frac{1}{18^2}-\frac{1}{22^2}+\cdots\right) \\ &= \frac{q^2}{4\pi\epsilon_0 a^2}\frac{1}{4}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\cdots\right)\end{aligned}$$

You may recognize the infinite series inside the parentheses to be Catalan's constant $G \approx 0.916$. Alternatively, you can use a calculator and evaluate the first seven terms to get a rough answer (but will still be correct since we asked for it to be rounded). Therefore:

$$F'/F = \frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\cdots = G \approx \boxed{0.916}$$

The following information applies to the next three problems. Jerry spots a truckload of his favourite golden yellow Swiss cheese being transported on a cart moving at a constant velocity $v_0 = 5$ m/s $\hat{\imath}$ along the x-axis, which is initially placed at $(0, 0)$. Jerry, driven by desire immediately starts pursuing the cheese-truck in such a way that his velocity vector always points towards the cheese-truck; however, Jerry is smart and knows that he must maintain a constant distance $\ell = 10$ m from the truck to avoid being caught by anyone, no matter what. Note that Jerry starts at coordinates $(0, \ell)$.

Pr 43. Tom and Jerry 1

Let the magnitude of velocity (in m/s) and acceleration (in m/s^2) of Jerry at the moment when the (acute) angle between the two velocity vectors is $\theta = 60°$ be α and β respectively. Compute $\alpha^2 + \beta^2$.

Pr 44. Tom and Jerry 2

At a certain instant during Jerry's motion, when his distance from the x-axis is 2 m, let his distance from the y-axis be ξ (in metres), and let his speed at $t = 1$ second be ψ m/s. Compute $\xi^2 + \psi^2$.

Pr 45. Tom and Jerry 3

Tom spots Jerry's footprints in the mud after Jerry has already travelled a distance $\ell = 10$ m towards the cheese truck. He starts moving at a constant speed of 5 m/s (except for a very large acceleration at the start, a result of his dislike for Jerry) along Jerry's trail. Alas, as is destined, he will never be able to catch Jerry. After a long period of time, what will be the separation between them? (in meters) Assume that Tom and Jerry have the energy to maintain their velocities for a very long period of time. Tom starts chasing Jerry from the same place Jerry started running towards the cheese truck.

Solution:

(43) If the distance between Jerry and the cheese truck is constant, then Jerry moves in circle of radius ℓ in the reference frame of the cheese truck. There is no radial component of Jerry's velocity in this reference frame, so we must have $\alpha = v_0 \cos\theta = \frac{5}{2}$. In this case, the tangential velocity is $v_0 \sin\theta$. Furthermore, the radial acceleration in this frame is given by the centripetal acceleration which is $\frac{(v_0 \sin\theta)^2}{\ell} = \frac{v_0^2 \sin^2\theta}{\ell}$. The tangential acceleration is

$$\frac{d}{dt}(v_0 \sin\theta) = v_0 \cos\theta \cdot \frac{d\theta}{dt} = v_0 \cos\theta \cdot \frac{-v_0 \sin\theta}{\ell} = -\frac{v_0^2 \sin\theta \cos\theta}{\ell}.$$

The vector sum of these accelerations has magnitude

$$\beta = \sqrt{\left(\frac{v_0^2 \sin^2\theta}{\ell}\right)^2 + \left(\frac{v_0^2 \sin\theta \cos\theta}{\ell}\right)^2} = \frac{v_0^2 \sin\theta}{\ell} = \frac{5\sqrt{3}}{4}.$$

The final answer is $\alpha^2 + \beta^2 = 10.9375$.

(44) As in the previous part, we can work in the reference frame of the cheese truck. As in the previous problem, we know that

$$\frac{d\theta}{dt} = -\frac{v_0 \sin\theta}{\ell}.$$

We can solve this differential equation more explicitly:

$$\frac{d\theta}{\sin\theta} = -\frac{v_0}{\ell} dt$$

$$\int_{\frac{\pi}{2}}^{\theta} \frac{d\theta'}{\sin\theta'} = -\frac{v_0}{\ell} \int_0^t dt'.$$

$$\ln|\cot\theta + \csc\theta| = \frac{v_0}{\ell} t$$

From the value of t, we can solve for θ and find the speed from $\psi = v_0 \cos\theta = 2.3106$ (Note a quick way to find θ is to use $\cot\frac{\theta}{2} = \cot\theta + \csc\theta$). Now, the distance from the x-axis is given by $\ell\sin\theta$, so we can easily find θ and substitute in to our equation to find the time. At this time, the cheese truck has moved a distance $v_0 t$, but Jerry is a horizontal distance $\ell\cos\theta$ behind the truck, so the distance to the y-axis is $\xi = v_0 t - \ell\cos\theta = 13.1264$. Finally, $\xi^2 + \psi^2 = 177.6$.

(45) First, we will calculate the change in the horizontal distance between Tom and the cheese truck from the time Tom starts moving. When Jerry was moving along this path, in a small time dt, the angle of Jerry's motion changes by $d\theta = -\frac{\ell}{v_0 \sin\theta} d\theta$ from the results in the previous problem. For the Tom's motion, the speed is faster by a factor of $\frac{1}{\cos\theta}$, so in time dt for the Tom, we have $dt = -\frac{\ell}{v_0}\cot\theta d\theta$. Now, the cheese truck continues to the right at speed v_0, while Tom has a horizontal velocity $v_0\cos\theta$. Thus, the total change in horizontal distance between Tom and the cheese truck is

$$\int_0^\infty (v_0 - v_0\cos\theta)dt = \int_{\frac{\pi}{2}}^0 (v_0 - v_0\cos\theta)\left(-\frac{\ell}{v_0}\cot\theta d\theta\right) = \ell\int_0^{\frac{\pi}{2}}(\cot\theta - \cos\theta\cot\theta)d\theta = \ell(1-\ln 2).$$

The initial horizontal distance between Tom and the cheese truck can be found with $v_0 t$ where t is the time at which Jerry has traveled a distance ℓ. The arc length of Jerry's path is

$$\int (v_0\cos\theta)dt = \int_{\frac{\pi}{2}}^0 (-\ell\cot\theta)d\theta = -\ell\ln|\sin\theta| = \ell.$$

Thus, we find $\sin\theta = \frac{1}{e}$ and $\cos\theta = \frac{\sqrt{e^2-1}}{e}$. From our equation above, distance the cheese truck travels in this time is

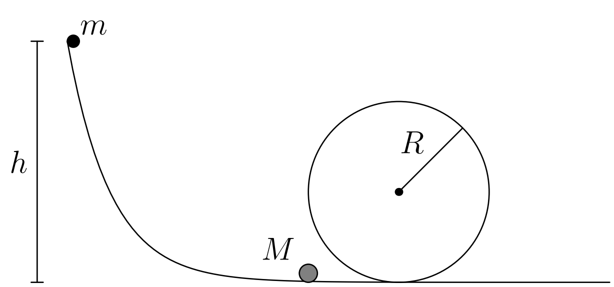
$$v_0 t = \ell\ln|\cot\theta + \csc\theta| = \ell\ln(e + \sqrt{e^2-1}).$$

The distance between Tom and the cheese truck then approaches $\ell(\ln(e+\sqrt{e^2-1}) + 1 - \ln 2)$ after a long time. Since Jerry lags the truck by a distance ℓ, the distance between Tom and Jerry approaches

$$\ell(\ln(e+\sqrt{e^2-1}) - \ln 2) = \boxed{9.64\text{ m}}.$$

Pr 46. Ghoster Coaster

A frictionless track contains a loop of radius $R = 0.5$ m. Situated on top of the track lies a small ball of mass $m = 2$ kg at a height h. It is then dropped and collides with another ball of mass $M = 5$ kg.



The coefficient of restitution for this collision is given as $e = \frac{1}{2}$. Now consider a different alternative. Now let the circular loop have a uniform coefficient of friction $\mu = 0.6$, while the rest of the path is still frictionless. Assume that the balls can once again collide with a restitution coefficient of $e = \frac{1}{2}$. Considering the balls to be point masses, find the minimum h such that the ball of mass M would be able to move all the way around the loop. Both balls can be considered as point masses.

Solution: Let the angle formed by M at any moment of time be angle θ with the negative y-axis. The normal force experienced by M is just

$$N = Mg\cos\theta + M\frac{v(\theta)^2}{R}$$

by balancing the radial forces at this moment. Now, applying the work energy theorem, we have

$$\int -\mu\left[Mg\cos\theta + M\frac{v(\theta)^2}{R}\right]R\mathrm{d}\theta = \frac{1}{2}Mv(\theta)^2 - \frac{1}{2}M{v_0}^2 + MgR(1-\cos\theta)$$

$$\Rightarrow -\mu\left[Mg\cos\theta + M\frac{v(\theta)^2}{R}\right]R = \frac{M}{2}\frac{\mathrm{d}(v(\theta)^2)}{\mathrm{d}\theta} + MgR\sin\theta$$

Rearranging, we have

$$\frac{\mathrm{d}(v(\theta)^2))}{\mathrm{d}\theta} + 2\mu v(\theta)^2 = -2gR(\sin\theta + \mu\cos\theta)$$

Let $v^2(\theta) = y$. Thus we have a first order linear ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} + P(\theta)y = Q(\theta)$$

This is easily solvable using the integrating factor $e^{\int P(\theta)\mathrm{d}\theta}$. Here the integrating factor is

$$e^{\int 2\mu \mathrm{d}\theta} = e^{2\mu\theta}$$

So multiplying by the integrating factor, we get

$$\int \mathrm{d}(e^{2\mu\theta}y) = \int -2gR(\sin\theta + \mu\cos\theta)e^{2\mu\theta}\mathrm{d}\theta$$

$$\Rightarrow y = \frac{\int -2gR(\sin\theta + \mu\cos\theta)e^{2\mu\theta}\mathrm{d}\theta}{e^{2\mu\theta}}$$

Now we use the well known integrals

$$\int e^{ax}\sin x \ \mathrm{d}x = \frac{e^{ax}}{1+a^2}(a\sin x - \cos x)$$

$$\int e^{ax}\cos x \ \mathrm{d}x = \frac{e^{ax}}{1+a^2}(a\cos x + \sin x)$$

(These integrals can be computed using integration by parts.) Thus, plugging and chugging these integration formulas into our expression for y and integrating from $\theta = 0$ to $\theta = \phi$, we have upon solving

$$v^2(\phi) - {v_0}^2 = \frac{-2gR}{1+4\mu^2}\left[(3\mu\sin\phi + (2\mu^2-1)\cos\phi - (2\mu^2-1)e^{-2\mu\phi}\right]$$

where v_0 is the velocity at $\phi = 0$. Solving gives us the velocity as a function of angle covered

$$v(\phi) = \sqrt{{v_0}^2 - \frac{2gR}{1+4\mu^2}\left[(3\mu\sin\phi + (2\mu^2-1)\cos\phi - (2\mu^2-1)e^{-2\mu\phi}\right]}$$

But to cover a complete circle, at the top most point

$$N = mg - \frac{mv^2(\pi)}{R} \geq 0 \Rightarrow v(\pi) \leq \sqrt{gR}$$

Thus

$$v_0 \leq \sqrt{gR\left[1 + \frac{2(1-2\mu^2)}{1+4\mu^2}(1+e^{-2\mu\pi})\right]}$$

From the previous expression,

$$v_0 = \frac{m(1+e)\sqrt{2gh}}{M+m} \geq \sqrt{gR\left[1 + \frac{2(1-2\mu^2)}{1+4\mu^2}(1+e^{-2\mu\pi})\right]}$$

Hence

$$\boxed{h \geq \frac{R(M+m)^2}{2m^2(1+e)^2}\left[1 + \frac{2(1-2\mu^2)}{1+4\mu^2}(1+e^{-2\mu\pi})\right]}$$

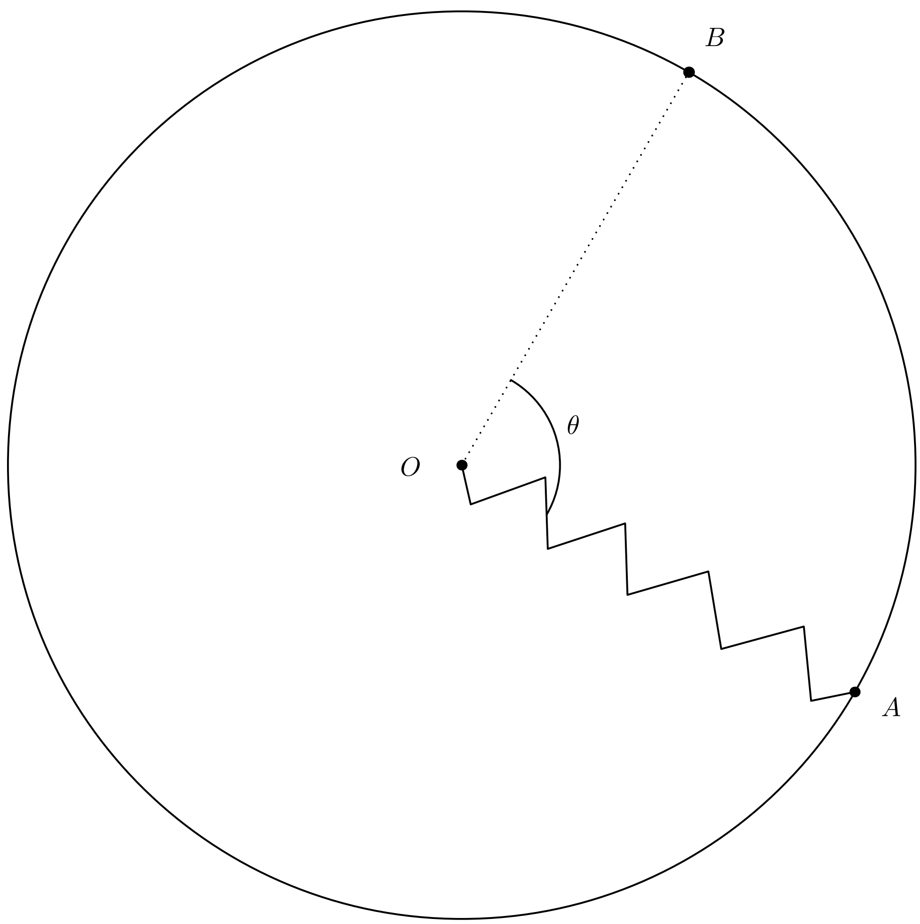
We get $h \geq \boxed{72.902 \text{ m}}$ and we are done.

Pr 47. Galactic Games

Two astronauts, Alice and Bob, are standing inside their cylindrical spaceship, which is rotating at an angular velocity ω clockwise around its axis in order to simulate the gravitational acceleration g on earth. The radius of the spaceship is R. For this problem, we will only consider motion in the plane perpendicular to the axis of the spaceship. Let point O be the center of the spaceship. Initially, an ideal zero-length spring has one end fixed at point O, while the other end is connected to a mass m at the "ground" of the spaceship, where the astronauts are standing (we will call this point A). From the astronauts' point of view, the mass remains motionless.

Next, Alice fixes one end of the spring at point A, and attaches the mass to the other end at point O. Bob starts at point A, and moves an angle θ counterclockwise to point B (such that AOB is an isosceles triangle). At time $t = 0$, the mass at point O is released. Given that the mass comes close enough for Bob to catch it, find the value of θ to the nearest tenth of a degree.

Assume that the only force acting on the mass is the spring's tension, and that the astronauts' heights are much less than R.



Solution: First, we must have $\omega^2 R = g$ and $k = m\omega^2$. Now, we step into the frame of point A, rotating around with the spaceship. We will thus have three fictitious forces: translational, centrifugal, and coriolis. Note that because centrifugal is $m\omega^2 r$, pointing away from A, it cancels with the spring force. Thus, the only forces left to consider are translational and coriolis.

The translational force points "down" with a constant magnitude of mg, like gravity. The coriolis force

points perpendicular to the velocity with magnitude $2m\omega v$. We recognize this setup is analogous to that of a charged particle moving in an E field and B field. It is well known that the mass will follow a cycloid shape. Writing the equation of the cycloid, and finding where the cycloid hits the circle (spaceship), we can find θ. Note that we have to use numerical methods.
Specifically, the cycloid can be parametrized as $\frac{R}{4}(\alpha - \sin\alpha, 1 - \cos\alpha)$, and we need to find where this intersects the circle $x^2 + y^2 = R^2$, so $(\alpha - \sin\alpha)^2 + (1 - \cos\alpha)^2 = 16$. Solving gives $\alpha = 3.307$, and since $\cot\theta = \frac{1-\cos\alpha}{\alpha-\sin\alpha}$, we have $\theta = \boxed{60.2^\circ}$.

Pr 48. Cramped Up

Consider an LC circuit with one inductor and one capacitor. The amplitude of the charge on the plates of the capacitor is $Q = 10$ C and the two plates are initially at a distance $d = 1$ cm away from each other. The plates are then slowly pushed together to a distance 0.5 cm from each other. Find the resultant amplitude of charge on the parallel plates of the capacitor after this process is completed. Note that the initial current in the circuit is zero and assume that the plates are grounded.

Solution: In slow steady periodic processes (when the time for the change in parameters τ is much less than the total systems frequency f), a quantity called the adiabatic invariant I is conserved[a]. The adiabatic invariant corresponds to the area of a closed contour in phase space (a graph with momentum p and position x as its axes). Note the we can electrostatically map this problem to a mechanics one as the charge corresponds to position, while the momentum would correspond to LI where I is the current and L is the inductance. Thus, in phase space, we have an elliptical contour corresponding to the equation: $\frac{Q^2}{2C} + \frac{(LI)^2}{2L} = C$ where C is a constant in the system. As the area under the curve is conserved, then it can be written that $\pi Q_0 LI_0 = \pi Q_f LI_f$. It is also easy to conserve energy such that $LI^2 = \frac{Q^2}{C}$ which tells us $I = \frac{Q}{\sqrt{LC}}$. As $C \propto 1/x$, we then can write the adiabatic invariant as xq^4 which tells us $\boxed{Q_f = \sqrt[4]{2}Q}$.

We can also solve this regularly by looking at the changes analytically. From Gauss's law, the electric field between the plates of the capacitators initially can be estimated as

$$E = \frac{Q}{2\varepsilon_0 A}$$

where A is the area of the plate. The plates of the capacitator is attracted to the other one with a force of

$$F = QE = \frac{Q^2}{2\varepsilon_0 A}.$$

The charges of the plates as a function of time can be approximated as

$$Q_c = \pm Q\sin(\omega t + \phi).$$

where $\omega = \frac{1}{\sqrt{LC}}$. Using this equation, we estimate the average force $\langle F\rangle$ applied on the plate after a period of oscillations to be

$$\langle F\rangle = \frac{\langle Q^2\rangle}{2\varepsilon_0 A} = \frac{Q^2}{2\varepsilon_0 A}\left\langle \sin^2(\omega t + \phi)\right\rangle = \frac{Q^2}{2\varepsilon_0 A}\cdot\left(\frac{1}{2\pi}\int_0^{2\pi}\sin^2(x)dx\right) = \frac{Q^2}{4\varepsilon_0 A}$$

this means that after one period, the amount of work done to push the plates closer together is given by

$$W_F = \langle F\rangle\, dx = \frac{Q^2}{4\varepsilon_0 A}dx.$$

In this cycle, the amount of incremental work done by the LC circuit will be given by

$$dW_{\mathrm{LC}} = \Delta(Fx) = \Delta\left(\frac{Q^2 x}{2\varepsilon_0 A}\right) = \frac{Qx}{\varepsilon_0 A}dQ + \frac{Q^2}{2\varepsilon_0 A}dx.$$

From conservation of energy, $W_F = W_{LC}$. Or in other words,

$$\frac{Q^2}{4\varepsilon_0 A}dx = \frac{Qx}{\varepsilon_0 A}dQ + \frac{Q^2}{2\varepsilon_0 A}dx$$

simplifying gives us

$$\begin{aligned}\frac{Qx}{\varepsilon_0 A}dQ &= -\frac{Q^2}{4\varepsilon_0 A}dx\\ \frac{1}{4}\int \frac{dx}{x} &= -\int \frac{dQ}{Q}\\ \frac{1}{4}\ln x + \ln Q &= \text{const.}\end{aligned}$$

We now find our adiabatic invariant to be

$$xQ^4 = \text{const.}$$

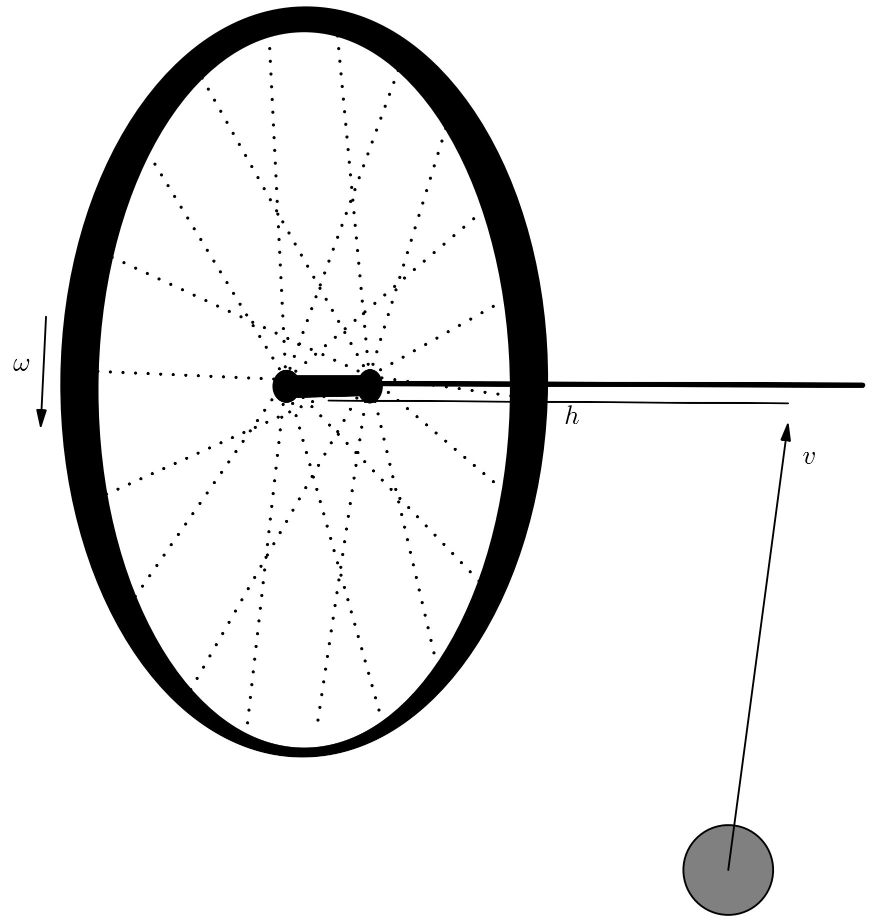
Substituting values into our equation, we find that

$$dQ_i^4 = \frac{d}{2}Q_f^4 \implies Q_f = \sqrt[4]{2}Q = \boxed{11.892 \text{ C}}$$

[a] This will not be proved in this solution, but the proof can be found in any good mechanics book.

Pr 49. I knew I should've stayed home today

A bicycle wheel of mass $M = 2.8$ kg and radius $R = 0.3$ m is spinning with angular velocity $\omega = 5$ rad/s around its axis in outer space, and its center is motionless. Assume that it has all of its mass uniformly concentrated on the rim. A long, massless axle is attached to its center, extending out along its axis. A ball of mass $m = 1.0$ kg moves at velocity $v = 2$ m/s parallel to the plane of the wheel and hits the axle at a distance $h = 0.5$ m from the center of the wheel. Assume that the collision is elastic and instantaneous, and that the ball's trajectory (before and after the collision) lies on a straight line.



Find the time it takes for the axle to return to its original orientation. Answer in seconds and round to three significant figures.

Solution: After the collision, let the wheel have speed v_1 and the ball have speed v_2. Conserving momentum, energy, and angular momentum gives:

$$mv = Mv_1 + mv_2$$

$$\frac{1}{2}mv^2 + \frac{1}{2}MR^2\omega^2 = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}MR^2\omega^2 + \frac{1}{2}\cdot\frac{1}{2}MR^2\omega_1^2$$

$$m(v - v_2)h = \frac{1}{2}MR^2\omega_1$$

where ω_1 is the angular velocity (after collision) of the wheel in the direction perp. to the axis and the velocity of the ball.

Solving for ω_1, we get

$$\omega_1 = \frac{4hmv}{m(R^2+2h^2)+MR^2}.$$

Now, we realize that the angular momentum of the wheel is given by $I_x\omega\hat{x} + I_y\omega_1\hat{y}$ where the wheel's axis is the x-axis and the y-axis is in the direction of ω_1. Since angular momentum is conserved, the wheel must precess about its angular momentum vector. Let $\hat{L}$ represent the direction of the angular momentum vector. To find the rate of precession, we can decompose the angular velocity vector $\omega\hat{x}+\omega_1\hat{y}$ into a $\hat{L}$ component and a $\hat{x}$ component. Since $I_x = 2I_y$, the $\hat{L}$ component is $\sqrt{(2\omega)^2+\omega_1^2}$, resulting in a precession period of

$$T = \frac{\pi}{\sqrt{\omega^2+\frac{\omega_1^2}{4}}} = \boxed{0.568s}$$

Pr 50. Fun with a String

A child attaches a small rock of mass $M = 0.800$ kg to one end of a uniform elastic string of mass $m = 0.100$ kg and natural length $L = 0.650$ m. He grabs the other end and swings the rock in uniform circular motion around his hand, with angular velocity $\omega = 6.30$ rad/s. Assume his hand is stationary, and that the elastic string behaves like a spring with spring constant $k = 40.0$ N/m. After that, at time $t = 0$, a small longitudinal perturbation starts from the child's hand, traveling towards the rock. At time $t = T_0$, the perturbation reaches the rock. How far was the perturbation from the child's hand at time $t = \frac{T_0}{2}$? Ignore gravity.

Solution: Let x be the distance from a point on the unstretched elastic string to the center of rotation (child's hand). Note that x varies from 0 to L. However, the string stretches, so let r be the distance from a point on the stretched string (in steady state) to the center of rotation. Let T be the tension in the string as a function of position. Let $\lambda = \frac{m}{L}$. Consider a portion of the string dx. We know that the portion as spring constant $k\frac{L}{dx}$ and it is stretched by $dr - dx$, so by Hooke's Law, we have $T = k\frac{L}{dx}(dr-dx) = kL(\frac{dr}{dx}-1)$. Also, by applying Newton's Second Law on the portion, we get $dT = -\lambda dx \cdot \omega^2 r$, which implies $\frac{dT}{dx} = -\lambda\omega^2 r$. Combining the two equations, we obtain

$$T = -kL\left(\frac{1}{\lambda\omega^2}T'' + 1\right).$$

We know that

$$T'(x=0) = 0,$$

since $r = 0$ when $x = 0$. The general solution is

$$T = A\cos\left(\frac{\omega}{L}\sqrt{\frac{m}{k}}x\right) - kL,$$

for some constant A. Thus, we have

$$r = -\frac{1}{\lambda\omega^2}T' = \frac{A}{\omega\sqrt{km}}\sin\left(\frac{\omega}{L}\sqrt{\frac{m}{k}}x\right).$$

Also, we have that

$$T(x=L) = M\omega^2\int_0^L rxdx = M\omega^2\int_0^L\left(\frac{T}{kL}+1\right)x.$$

Plugging in our general solution, we can get $A\cos\left(\omega\sqrt{\frac{m}{k}}\right) - kL = M\omega^2\cdot\frac{A}{kL}\frac{L}{\omega}\sqrt{\frac{k}{m}}\sin\left(\omega\sqrt{\frac{m}{k}}\right)$. Solving for A, we obtain

$$A = \frac{kL}{\cos\left(\omega\sqrt{\frac{m}{k}}\right) - \frac{M\omega}{\sqrt{km}}\sin\left(\omega\sqrt{\frac{m}{k}}\right)}.$$

We now introduce a claim:

Claim. The speed of a longitudinal wave on a spring with spring constant k, length L, and mass m is given by $v = L\sqrt{\frac{k}{m}}$

Proof. Let a spring with spring constant k and mass m be stretched to length L. The spring constant of a small portion dx of the spring is $k\frac{L}{dx}$, and the excess tension is $\delta T = k\frac{L}{dx}ds = kL\frac{ds}{dx}$, where s is the displacement from equilibrium. By Newton's second law on the portion, we get $dT = \frac{m}{L}dx \cdot \frac{d^2s}{dt^2}$, or $\frac{dT}{dx} = \frac{m}{L}\frac{d^2s}{dt^2}$. Thus, $L^2\frac{k}{m}\frac{d^2s}{dx^2} = \frac{d^2s}{dt^2}$, which we recognize as the wave equation with speed $v = L\sqrt{\frac{k}{m}}$ and the time it takes to traverse the spring is $\sqrt{\frac{m}{k}}$. □

Thus, we have

$$t = \int dt = \int_0^x \sqrt{\frac{\lambda dx}{k \cdot \frac{L}{dx}}} = \int_0^x \sqrt{\frac{\lambda}{kL}}dx = \sqrt{\frac{m}{k}}\frac{x}{L}.$$

Since we know $x = L$ when $t = T_0$, we have $x = \frac{L}{2}$ when $t = \frac{T_0}{2}$. Therefore, our answer is

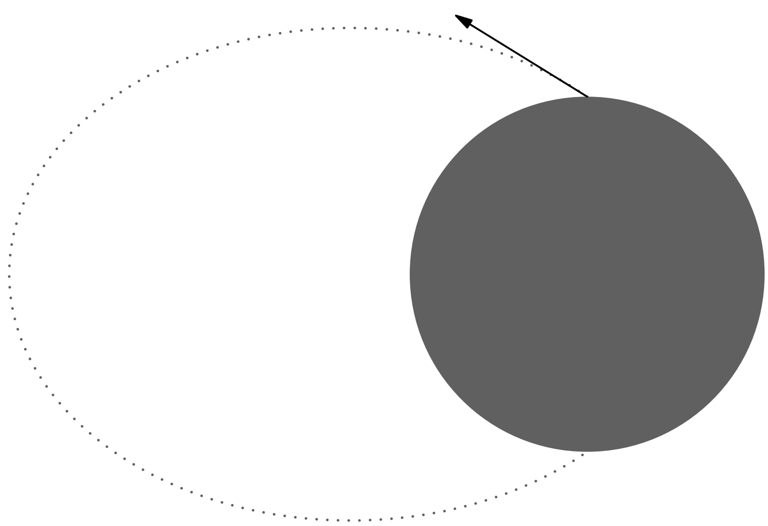
$$r = \frac{A}{\omega\sqrt{km}}\sin\left(\frac{\omega}{L}\sqrt{\frac{m}{k}}x\right) = \frac{1}{\omega}\sqrt{\frac{k}{m}}\frac{L\sin\left(\frac{1}{2}\omega\sqrt{\frac{m}{k}}\right)}{\cos\left(\omega\sqrt{\frac{m}{k}}\right) - \frac{M\omega}{\sqrt{km}}\sin\left(\omega\sqrt{\frac{m}{k}}\right)} = \boxed{1.903 \text{ m}}$$

Pr 51. When Rocket Scientists Play Catch

During the cold war, there was tension between the USSR and the U.S. But now, contrary to popular belief, American and Russian astronauts pass time by hanging out, enjoying the view from the moon, and even playing catch by launching projectiles at each other:

A projectile is launched with a speed $v_0 = 2200$ m/s from the North Pole to the South Pole of a moon with radius $r_0 = 1.7 \times 10^6$ m and $M = 7.4 \times 10^{22}$ kg.

How long does the flight take? Answer in seconds.



Solution: The projectile will follow an elliptical path. It is easiest to represent this path in terms of an ellipse:

$$r = \frac{a(1-e^2)}{1 - e\cos\theta}$$

At $\theta = 90°$, the numerator becomes $a(1-e^2) = r_0$. We will now show that the angular momentum is given by:

$$L = m\sqrt{GMr_0}.$$

If the apoapsis is r_a and the speed of the projectile at apoapsis is v_a, then conservation of angular momentum and energy gives:

$$L = mv_f r_f$$

$$\frac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \frac{1}{2}mv_f^2 - \frac{GMm}{r_f}$$

Solving these equations, we get:

$$r_a = GM \cdot \frac{-1 + \sqrt{\frac{v^2 r}{GM} - 1}}{2E}$$

where E is the energy per unit mass of the projectile. By setting $\theta = 180°$, we can write the location of the apoapsis to be

$$r_a = a(1+e) = 9274582m$$

Therefore, to determine the apoapsis a, we just need to determine e. The eccentricity is given by:

$$e = \sqrt{1 + \frac{2Er_0}{GM}} = 0.8167.$$

Plugging in numbers, we find the semi-major axis to be $a = 461,670$ m and the orbital period to be

$$T = 2\pi\sqrt{\frac{a^3}{GM}} = 32622 \text{ s}$$

However, we are only interested in the section of the orbit that occurs above the surface. If we are able to determine the area the center of the moon subtends with the curve inside the moon, then we can apply Kepler's second law to determine the orbital period.

This area can be determined via polar integration to be:

$$\frac{1}{2}\int_{\frac{5\pi}{2}}^{\frac{7\pi}{2}} \left(\frac{a\left(1-e_c^2\right)}{1-e_c\cos\theta}\right)^2 d\theta = 2.1633 \times 10^{12} \text{ m}^2$$

The fraction of area covered by the projectile while outside the moon will be:

$$f = \frac{\pi ab - 2.1633 \times 10^{12}}{\pi ab} = \frac{\pi a^2\sqrt{1-e^2} - 2.1633 \times 10^{12}}{\pi a^2\sqrt{1-e^2}} = 0.9542$$

where $b = c^2 - a^2$ and $c = e/a$ were used to simplify the expression. Applying Kepler's second law, the time of flight is then:

$$t = fT = \boxed{31128.8 \text{ s}}$$

2020 Online Physics Olympiad (OPhO): Invitational Contest v1.4

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Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

- This test contains 10 long answer questions.
- The total **base** score for the exam is 300 points; a factor incorporating the number of teams who solved the question will be added in the marking scheme. The final scores of all the teams will be available a few days after the contest ends.
- The team leader should submit their final solution document in this here. We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it.
- You can write on paper, type it up online, or a mix of both. If you wish to use a pre-made LaTeX template, we made one which you can choose to use here.
- Since this is a long answer response, you will be judged on the quality of your work. To receive full points, you need to show your work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in this formula sheet) can be cited without proof. Remember to state any approximations made, which system of equations were solved after every step. Explicitly showing every step of algebra is not necessary.
- You may leave all final answers in symbolic form (in terms of variables) unless otherwise specified.2
- Be sure to state all assumptions.

Sponsors

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^8$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\mathrm{N \cdot m^2})/\mathrm{kg}^2$$
- Acceleration due to gravity, $g = 9.81$ m/s^2
- 1 unified atomic mass unit,
$$1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15}\ \mathrm{eV \cdot s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9\ (\mathrm{N \cdot m^2})/\mathrm{C^2}$$
- Permeability of free space, $\mu_0 = 4\pi \cdot 10^{-7}$ T $\cdot$ m/A
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\mathrm{T \cdot m})/\mathrm{A}$$
- 1 atmospheric pressure,
$$1\ \mathrm{atm} = 1.01 \cdot 10^5\ \mathrm{N/m^2} = 1.01 \cdot 10^5\ \mathrm{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$$

Problem 1: A Party Trick (20 pts)

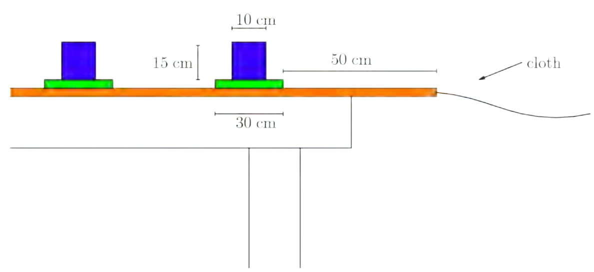
It is a well known party trick that by pulling the tablecloth very quickly and suddenly, the plates on top of the table can stay nearly in place.

(a) **(2 pt)** Start with a circular table $D = 1$ m in diameter with a very flat and small (and dimensionless for now) plate in the centre. How fast must you pull the tablecloth so the plate remains on the table? Assume the static friction coefficient is $\mu_s = 0.5$ and kinetic friction is $\mu_k = 0.3$ (for any contacts with the tablecloth) and that the tablecloth accelerates instantaneously. The tablecloth does not overhang the table.

(b) **(4 pts)** Mythbusters famously attempted to replicate the same trick with a giant tablecloth and a motorbike. We will simplify the experiment by assuming that the table was only set with plates placed $d = 0.5$ m apart from each other and from both ends. Assuming a $\ell = 7$ m long table, how fast must the bike be travelling to successfully carry out this experiment? The mass of the cloth and plates are negligible compared to the mass of the motorbike. Assume small plates and that the tablecloth does not overhang the table.

(c) **(6 pts)** A young boy places his toy car in the centre on the table from part (b). He believes that since it has wheels, it will stay on the table easier. Confirm or deny this statement and find the significant speeds that the tablecloth is pulled at that would cause the car to stay on the table. Assume the car (including wheels) has mass $2m$, and that each wheel is a uniform disk of mass $m/4$. The car is small in comparison to the table.

(d) **(8 pts)** We run the experiment one last time with the glass, with same mass as a plate, placed on top of each plate on the tablecloth, which in turn, is on a frictionless table. The static and kinetic coefficient between the glass and the plate are $\mu_s' = 0.30$ and $\mu_k' = 0.15$ respectively. How fast must the tablecloth be pulled so that the glasses stay completely on the plate and the plates stay completely on the table? Do not assume that the either plate or glass is dimensionless this time. A diagram is provided below:



Problem 2: Solar Sails (26 pts)

Much research is being done on the possibility of using solar sails to reach far away reaches in out galaxy. This is a method of propulsion that use light from the sun to exert a pressure on what is usually a large mirror. The sails themselves are often made of a thin reflective film. Compared to traditional spacecraft, while these sails have very limited payloads, they offer long operational lifetimes and are relatively low cost. The most significant advantage is speed: since solar sails do not depend on onboard propellant, they can travel much faster than a standard rocket, with the possibility of reaching a significant percentage of the speed of light. For all parts, assume the solar sails are only under the influence of gravity from the sun only. In addition, you may necessary variables such as the mass and power output of the sun.

(a) **(2 pt)** Assume the solar sails are not revolving around the sun. Assume they are very reflective thin discs and that all of the mass is in the sail. What is the maximum area density so that the sail does not fall towards the sun? Does the distance matter (find the general density to distance equation if it does matter)?

(b) **(3 pts)** Assume the solar sails are a thin spherical shell and made of a perfectly absorbent material instead. The area density is exactly half the maximum area density so that it does not fall towards the sun. What is its final speed if it starts at a stationary position 1 Au from the sun?

(c) **(5 pts)** Assume the solar sails are a thin spherical shell and made of material having reflectance r. With the same area density as in part (b), what is the final speed (it can reach this speed either when crashing into the sun, escaping the solar system, or remain in orbit)? This time, assume it starts in circular orbit 1 Au away from the sun as the starting condition. What is the orbital shape?

(d) **(6 pts)** Simply not falling into the sun is insufficient for solar sails. Most are planned to reach relativistic speeds. A way to achieve this is to fire large Earthbound lasers at the sail. How powerful must the lasers be to accelerate the sail up to $v = 0.2c$ in 50 days? Since this far exceeds the value obtained in part 3, neglect the effects of the sun.

(e) **(10 pts)** Suppose a perfectly reflecting solar sail in the shape of a thin disk (with mass m and radius r) orbiting around the star in a circular orbit of radius $d \gg r$. It is also spinning around itself, such that the spin angular velocity is in the same direction as the orbital angular velocity, and its axis of symmetry always remains parallel to the plane of the orbit. If its initial spin velocity is ω_0 find its spin velocity after it revolves an angle θ around the star.

Problem 3: Electron Escape (28 pts)

An infinite wire with current I has a radius a. The wire is made out of a material with resistivity ρ and heat conductivity κ. The temperature outside the wire is a constant T_0.

(a) **(4 pts)** After a long time, determine the temperature $T(r)$ at a distance r from the center of the wire. Assume that the current in the wire is uniformly distributed.

(b) **(4 pts)** Now, the outer surface of the wire is maintained at a potential of $-V$, where V is positive. The wire is surrounded by an infinite cylindrical shell with radius $b > a$ that is grounded. Somehow, an electron is able to escape from the wire. Assume that it is at rest just as it escapes. You can neglect radiation from the electron.

Draw a qualitative graph of the physical path that the electron takes, along with a diagram of the wire.

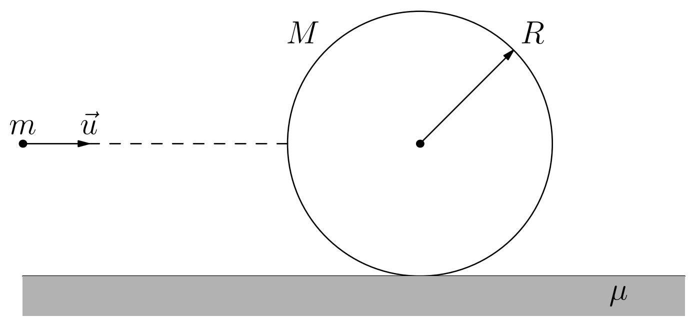
(c) **(6 pts)** Find the maximum distance $r_{\max}$ of the electron from the center of the wire in the subsequent motion as a function of V, and also in terms of I, a, and b. Ignore all relativistic effects in this part only.

(d) **(12 pts)** Redo the calculation in the previous part with relativistic effects.

(e) **(2 pts)** Graph the maximum radius $r_{\max}$ according to part (d) as a function of V.

Problem 4: Bullet in Cylinder (20 pts)

A hollow cylinder of mass M and radius R rests on a rough horizontal surface. A projectile of mass $m < M$ having a velocity u directed horizontally exactly towards the middle of the cylinder as shown in the figure. The shell gets stuck in the cylinder wall, after which the shell starts to move, slipping on the surface. The coefficient of static and kinetic friction between the cylinder and the horizontal surface are the same, and are equal to $\mu < 2$.



(a) **(4 pt)** In which direction does the cylinder rotate? State your answers for different values of μ.

(b) **(6 pts)** Find its angular acceleration about the centre of the cylinder just after the impact.

(c) **(10 pts)** It is known that some time after the impact, the horizontal projection of the velocity of the center of mass of the system is equal to v, and the angular velocity of the cylinder is Ω. Till this point, the cylinder has rotated through an angle ϕ.

How much heat was released in the system till this point, if the cylinder all the time after the impact moved with slipping, rotating in one direction? Assume that all energy losses are dissipated in the form of heat.

Problem 5: Mathematical Physics (18 pts)

The study of mathematics has almost always paved the way for the development of new ideas in physics. Newtonian mechanics could not be possible without first inventing calculus, and general relativity could not have existed without heavy development in tensors. However, there are numerous cases where physical insight have paved the way for mathematics.

Perhaps the most notable would be the Brachistochrone problem, which asks for the path that leads to the fastest descent influenced by gravity between two given points. While it is solvable through the calculus of variations, Newton proposed an easier solution by modelling the path of light through a medium with a variable index of refraction. You may read about this problem and the fascinating history behind it here.

We will not be dealing with this specific problem, but rather multiple short mathematics problems that can be can be represented with a physical analog. To receive points, you must use the suggested physical set-up.

(a) **(4 pts)** Show that for small values of x, we have

$$\cos(x) = 1 - \frac{x^2}{2}.$$

Physical Setup: Consider a small object moving in a circle.

(b) **(8 pts)** A ladder of length ℓ with a thickness of 0.3 m is transported around a right angled corner where the two hallways leading up to it have a width of 3 m and 5 m. What is the maximum length of the ladder such that it can be successfully transported across?

Physical Setup: Consider the ladder as a compressed spring that can freely expand in only its longitudinal direction. Do not explicitly take derivatives. Instead, consider a force/torque balance. You may or may not need to solve an equation numerically.

(c) **(6 pts)** Prove the AM-QM inequality:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}$$

Physical Setup: Design circuit(s) and compare their measurable quantities with each other qualitatively (or otherwise).

Problem 6: Flat Earth (22 pts)

In this problem, we will explore the true gravitational model of the earth, not the one that is claimed in most textbooks. Contrary to popular belief, the Earth is a flat circle of radius R and has a uniform mass per unit area σ. The Earth rotates with angular velocity ω.

(a) **(5 pts)** A pendulum of length ℓ that is constrained to only move in one plane is placed on the ground at the center of the Earth. The pendulum has more than one angular frequency of small oscillations. Find the value of each angular frequency of small oscillations $\Omega(0), \Omega_1(0), ...$ in terms of σ, ω, ℓ, and physical constants and the equilibrium angle $\theta, \theta_1, ...$ that the frequency occurs at. Assume for all parts that $\ell \ll R$.

An equilibrium angle corresponds to the angle with respect to the vertical where there is an equilibrium point.

(b) **(2 pt)** Investigate the stability of each equilibrium position with varying angular velocity of the Earth.

(c) **(12 pts)** The entire pendulum is moved a horizontal distance $r \ll R$ away from the center of the Earth. It is oriented so that it is constrained to only move in the radial direction. Now, find the new angular frequency $\Omega(r)$ of small oscillations about the lowest equilibrium point in terms of the given parameters, assuming that $\omega^2 r$ is much less than the local gravitational acceleration.

(d) **(3 pts)** The angular frequencies $\Omega(0)$ and $\Omega(r)$ are both measured and the difference is found to be $\Delta\Omega$. Assuming that $\Delta\Omega \ll \Omega(0)$ and $\omega^2 \ll \frac{g}{\ell}$, determine σ in terms of $\omega, r, \Omega(0), \Delta\Omega$, and physical constants.

Problem 7: Boltzmann Statistics (24 pts)

In this problem, we will explore Boltzmann Statistics and using it to build similar models for quantum particles such as bosons and fermions.

(a) **(8 pts)** Consider a energy of a gaseous molecule in space is given by 5

$$E = E_0(|x|^r + |y|^r + |z|^r)$$

where the coordinates of the molecule are represented by (x, y, z), E_0 is a constant with appropriate units, and r is a non-negative real number. The system is in thermal equilibrium with a reservoir of temperature T. Calculate explicitly using appropriate statistical methods, the average energy of a thermodynamic system consisting of such gaseous molecules, considering the Maxwell-Boltzmann distribution. Analyse your result and provide a qualitative argument to support it.

Hint: If you are not familiar with how to solve this part, try part (b) first.

(b) **(6 pts)** Consider a specific case in which

$$E = E_0(x^2 + y^2 + z^2)$$

except where $|x|, |y|, |z| < 2$ and particles can only exist at integer values of x, y, z. If the system is still in thermal equilibrium at a temperature T, calculate the average energy.

(c) **(6 pts)** In this system, there are two particles. What is the probability that at least one of these particles will be in the ground state (energy is zero) if:

(i) The two particles are distinct.

(ii) The two particles are identical bosons.

(iii) The two particles are identical fermions (fermions follow Pauli exclusion principle).

For each of the parts, assume that there are no other interactions (e.g. electromagnetism) and focus mainly on a statistical argument.

(d) **(4 pts)** Rank the probabilities in part (c) from highest to lowest when the temperature is

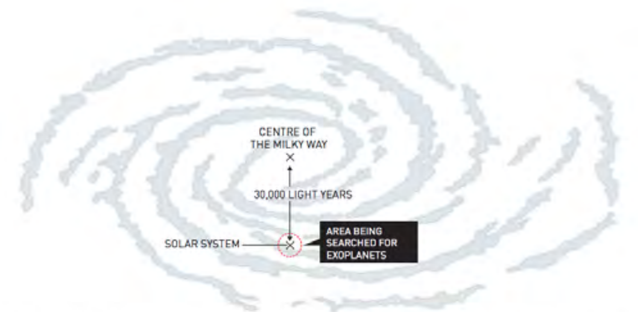
(i) high

(ii) low

For each, explain qualitatively why this must be true.

Problem 8: Radiation (40 pts)

The Nobel Prize in Physics 2019 was awarded for providing a new understanding of the universe's structure and history, and the first discovery of a planet orbiting a solar-type star outside our solar system.



The Sun is one of several hundred billion stars in our home galaxy, the Milky Way, and there should be planets orbiting most of those stars. So far, astronomers have discovered over 4,000 planets around other stars and they are continuing their search in the area of space closest to us.

Since ancient times, humans have speculated whether there are worlds like our own, with points of views at the extremes expressed thousands of years ago. In modern times, the possibility of observing planets orbiting stars other than the Sun was proposed more than 50 years ago, and has grown into a vast and ever-expanding theory to make the evolution of the universe more clear to us than ever before. In 1995, the very first discovery of a planet outside our solar system, an exoplanet, orbiting a solar-type star was made. This discovery challenged our ideas about these strange worlds and led to a revolution in astronomy. The more than 4,000 known exoplanets are surprising in their richness of forms, as most of these planetary systems look nothing like our own, with the Sun and its planets. These discoveries have led researchers to develop new theories about the physical processes responsible for the birth of planets.

(Taken from the Nobel Prize in Physics 2019 summary, and the Laureates' popular science and scientific views.)

In this problem, we analyse and create a model for a system of two fictitious celestial bodies: an exoplanet and a solar-type star. Unless specified otherwise, consider the two bodies to be solely in each other's gravitational influence and rotate about their barycentre. In the three parts that follow, we will model the physics of a star, of the star-planet model, and the planet respectively.

Part A

The star, with mass $M_s = 2M_\odot$ (twice the mass of our Sun) and radius $R_\odot$ uses nuclear fusion reactions to provide pressure against gravity and electron degeneracy pressure, so as to maintain hydrostatic equilibrium in the star. As long as the hydrostatic equilibrium is preserved, the star is said to be in "main sequence". However, once the energy from the reactions taking place in its core start running out, the star's outer layers swell out to form a red giant. The core of the star (having a radius R_c) starts to shrink, becoming hot and dense; the temperature of the core rises to over a 100 billion degrees, and the pressure from the proton-proton interactions in the core exceeds that of gravity, causing the core to recoil out from the heart of the star in an explosive shock wave. In one of the most spectacular events in the Universe, the shock propels the material away from the star in a tremendous explosion called a supernova. The material spews off into interstellar space.

Being solar-type, this star has the same proton-proton nuclear fusion chain reaction as our Sun: essentially, this is conversion of four protons (mass of a proton is m_P) into 1 He nucleus having mass m_{He}. The star is said to have a "stable lifetime" as long as it is in its "main sequence". The energy emitted by the star passing a sphere of radius r per unit time is $P(r)$, constant over time and the surface of the imaginary sphere of radius r. The density of the exoplanet having radius r_{E} is a constant, ρ, and it orbits around the star in a circular orbit of radius r_{SE}. Neglect any convection effects in the star.

(a) **(4 pts)** Treating the solar-type star as a perfect black-body, estimate the temperature of the surface of the star $T_\odot$ (assumed in thermal equilibrium) by integrating over all frequencies using Planck's distribution for the energy density (defined as the energy per unit volume for a given frequency interval (ν, $\nu + \mathrm{d}\nu$):

$$u(\nu)\mathrm{d}\nu = \frac{8\pi h\nu^3}{c^3}\frac{\mathrm{d}\nu}{(e^{\frac{h\nu}{kT_\odot}} - 1)}$$

where the constants h and k have their usual meanings. For this part only, note that the energy flux from the star onto the exoplanet is J_0. You can use $T_\odot$ as the surface temperature in the later parts.

(b) **(8 pts)** Estimate an expression for, during the main sequence of the star:

(i) the number of protons being fused together per second.

(ii) the stable lifetime of the star, assuming $\eta =$ 1% mass of the star can undergo nuclear fusion. The change in the temperature or size of the star is insignificant. Assume the only fusion is between protons.

(c) **(8 pts)** Find the temperature gradient $\mathrm{d}T(r)/\mathrm{d}r$ of the star as a function of the radial distance r from the center, such that $R_c \leq r \leq R_\odot$ if the star is in its main sequence, or in a hydrostatic equilibrium. Neglect any quantum-mechanical pressure effects such as electron degeneracy pressure, and assume that pressure from electromagnetic radiation is much larger than any other pressure. State all assumptions.

(d) **(1 pts)** What is the temperature at $r = R_c$, the outermost layer of the core?

(e) **(2 pt)** From experiments, it was found that the temperature gradient of the star is actually

$$\frac{\mathrm{d}T(r)}{\mathrm{d}r} = -\frac{9kGM_\odot cP(r)}{128\pi^2\sigma r^2 T^3}$$

Here the modulus of k is one, and has appropriate dimensions. For what value of $P(R_\odot)$ (P_r evaluated at the surface of the star) will the star's main sequence end, leading to the formation of a supernova?

Part B

In this part, we will analyse the radiation effects from the star onto the exoplanet. Assume only black body radiation from the star on the exoplanet. No light is absorbed in the region between the star's and the exoplanet's surface.

(f) **(5 pts)** The distance between the star and the exoplanet is r_{SE}. For this part, assume the surface of the exoplanet has a constant and uniform reflectance γ. What is the force exerted by the radiation from the star on the exoplanet? For the exoplanet's gravitational force to completely balance out the radiation force, how large must the radius of the exoplanet r_E be? Comment on your results and their feasibility.

Part C

(g) **(2 pt)** Find the temperature T_E of the outermost surface of the planet, assumed constant over the whole surface from (a). Assume the planet's surface to be a perfect black body.

(h) **(10 pts)** Model the exoplanet to be made up of N concentric shells equally spaced across the volume of the planet. Between the shells is a peculiar kind of thick type of tectonic rocks which allow no emission, reflection or absorption of energy. However, absorption or emission of radiation energy may take place. The emissivity of all the shells are the same, and are equal to ε, constant and uniform over a surface. Reflection, emission and absorption of any energy due to radiation from the shells, however, may take place. Assume all conduction and convection effects to also be negligible. The temperature of the exoplanet as a function of r is represented by

$$T(r) = T_0\left(1 - \frac{n}{10N}\right)$$

where n is the n^{th} shell from the centre of the planet and T_0 is an appropriate constant as calculated from the previous part (which is unknown, meaning that you need to answer in any variables calculated before). Calculate the total thermal energy due to radiation falling on the outermost shell per unit time. The planet is maintained in a state of thermal equilibrium; this is done by an atmospheric material that allows a fraction (β, which is unknown) of energy from the star falling on the exoplanet. This material only absorbs a fraction of energy it receives from the star. Do NOT assume any such effects for any of the other parts, since they are meant to be crude estimates of the actual calculation. Also compute β.

Problem 9: Piston Gun (40 pts)

In this problem, we examine a model for a certain type of gun that works by using the expansion of a gas to propel a bullet. We can model the bullet as a piston. Since we are assuming atmospheric pressure is negligible, we can assume that the whole setup is in a vacuum. Also, the gun is insulated.

An ideal monatomic gas of initial temperature T_0 is inside a long cylindrical container of cross-section area A. One side of the container is a wall, while the other side is a piston of mass M that can slide freely along the container without friction. The total mass of the gas is m, and it is made up of N particles. Initially, the piston is at rest and a distance L_0 away from the opposite wall. Then, the piston is released. After a time t, the piston moves at a speed v. Assume that throughout the process, the particles on average move very fast.

(a) **(5 pts)** Assume that m is negligible. Find v.

(b) **(6 pts)** From now on, do not assume that m is negligible.

Find the time at which the pressure at the wall opposite the piston changes. Also, does it increase or decrease? State all assumptions.

(c) **(14 pts)** From now on, assume t is much smaller than the mean free time of the particles of the gas, and L_0 is much smaller than the mean free path. (During this time interval t, assume that all the particles still collide many, many times with the walls, but they don't collide with each other.) Find v.

(d) **(6 pts)** Find the recoil impulse of the gun over the time t.

(e) **(9 pts)** Let $r > 1$ be a dimensionless parameter. Suppose at time t, the piston is a distance rL_0 away from the wall; then the piston is stopped, and the gas is allowed to come to equilibrium (after a time much greater than the mean free time). Find the total entropy change (throughout the whole process) of the gas in terms of r, and verify the Second Law of Thermodynamics.

Problem 10: Magnetostatics (62 pts)

Part A

In 3-D space, a permeable medium covers the region $x > 0$, while the rest of the space is vacuum. The medium's relative magnetic permeability is $\mu_r > 1$. A magnetic dipole with dipole moment m is placed a distance d away from the permeable medium, at position $(-d, 0, 0)$. The dipole is pointed towards the +x direction. Treat the dipole as ideal (point-sized).

(a) (10 pts) Find the force required to keep the dipole in place.

(b) (3 pts) How much work does it take to slowly pull the dipole from its original position to infinity (at $x = -\infty$)?

(c) (5 pts) How much work does it take to slowly rotate the dipole from its original orientation to one that makes an angle θ with the +x-axis?

After the dipole is rotated an angle θ, a superconducting ring with radius R and self-inductance L is brought in from infinity (with initially no current). It is placed so that the dipole is located at its center and its axis is the x-axis. Assume that $R \gg d$.

(d) (16 pts) Find the current I in the ring.

(e) (8 pts) Find the force required to hold the dipole in place (not the torque).

Part B

From now on, there is no permeable medium. Ignore any radiation loss for all parts.

(f) (7 pts) The dipole (mass M), starts at a distance h from the centre of the ring (kept fixed) and pointed towards the centre of the ring (along its axis), and is projected with a small velocity v_0 towards the centre. Find its speed v as a function of h. Ignore gravity.

(g) (7 pts) Consider another scenario, in which the dipole is placed on the axis of a thin infinite magnetic tube with surface conductivity σ (defined as the ratio of surface current density and the electric field) and radius R, placed at an arbitrary location inside it. (You may neglect the self inductance of the solenoid for the sake of this part.) We find that the motion of the dipole in this case is damped. Find the damping parameter of this motion. (Damping parameter is defined as the ratio of the resistive force to the speed.) Ignore gravity.

(h) (6 pts) Determine the terminal velocity of the magnet, assuming that it now falls under gravity. The tube may be considered infinitely long for all calculation purposes in this part.

2020 Online Physics Olympiad (OPhO): Invitational Contest Partial Solutions

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Problem 1: A Party Trick (20 pts)

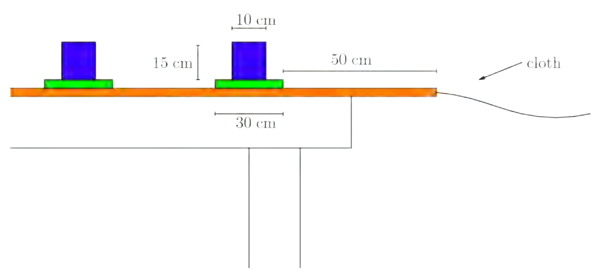
It is a well known party trick that by pulling the tablecloth very quickly and suddenly, the plates on top of the table can stay nearly in place.

(a) **(2 pt)** Start with a circular table $D = 1$ m in diameter with a very flat and small (and dimensionless for now) plate in the centre. How fast must you pull the tablecloth so the plate remains on the table? Assume the static friction coefficient is $\mu_s = 0.5$ and kinetic friction is $\mu_k = 0.3$ (for any contacts with the tablecloth) and that the tablecloth accelerates instantaneously. The tablecloth does not overhang the table.

(b) **(4 pts)** Mythbusters famously attempted to replicate the same trick with a giant tablecloth and a motorbike. We will simplify the experiment by assuming that the table was only set with plates placed $d = 0.5$ m apart from each other and from both ends. Assuming a $\ell = 7$ m long table, how fast must the bike be travelling to successfully carry out this experiment? The mass of the cloth and plates are negligible compared to the mass of the motorbike. Assume small plates and that the tablecloth does not overhang the table.

(c) **(6 pts)** A young boy places his toy car in the centre on the table from part (b). He believes that since it has wheels, it will stay on the table easier. Confirm or deny this statement and find the significant speeds that the tablecloth is pulled at that would cause the car to stay on the table. Assume the car (including wheels) has mass $2m$, and that each wheel is a uniform disk of mass $m/4$. The car is small in comparison to the table.

(d) **(8 pts)** We run the experiment one last time with the glass, with same mass as a plate, placed on top of each plate on the tablecloth, which in turn, is on a frictionless table. The static and kinetic coefficient between the glass and the plate are $\mu_s' = 0.30$ and $\mu_k' = 0.15$ respectively. How fast must the tablecloth be pulled so that the glasses stay completely on the plate and the plates stay completely on the table? Do not assume that the either plate or glass is dimensionless this time. A diagram is provided below:



Solution 1: TBD

Problem 2: Solar Sails (26 pts)

Much research is being done on the possibility of using solar sails to reach far away reaches in out galaxy. This is a method of propulsion that use light from the sun to exert a pressure on what is usually a large mirror. The sails themselves are often made of a thin reflective film. Compared to traditional spacecraft, while these sails have very limited payloads, they offer long operational lifetimes and are relatively low cost. The most significant advantage is speed: since solar sails do not depend on onboard propellant, they can travel much faster than a standard rocket, with the possibility of reaching a significant percentage of the speed of light. For all parts, assume the solar sails are only under the influence of gravity from the sun only. In addition, you may define necessary variables such as the mass and power output of the sun.

(a) **(2 pt)** Assume the solar sails are not revolving around the sun. Assume they are very reflective thin discs and that all of the mass is in the sail. What is the maximum area density so that the sail does not fall towards the sun? Does the distance matter (find the general density to distance equation if it does matter)?

(b) **(3 pts)** Assume the solar sails are a thin spherical shell and made of a perfectly absorbent material instead. The area density is exactly half the maximum area density so that it does not fall towards the sun. What is its final speed if it starts at a stationary position 1 Au from the sun?

(c) **(5 pts)** Assume the solar sails are a thin spherical shell and made of material having reflectance r. With the same area density as in part (b), what is the final speed (it can reach this speed either when crashing into the sun, escaping the solar system, or remain in orbit)? This time, assume it starts in circular orbit 1 Au away from the sun as the starting condition. What is the orbital shape?

(d) **(6 pts)** Simply not falling into the sun is insufficient for solar sails. Most are planned to reach relativistic speeds. A way to achieve this is to fire large Earthbound lasers at the sail. How powerful must the lasers be to accelerate the sail up to $v = 0.2c$ in 50 days? Since this far exceeds the value obtained in part 3, neglect the effects of the sun.

(e) **(10 pts)** Suppose a perfectly reflecting solar sail in the shape of a thin disk (with mass m and radius r) orbiting around the star in a circular orbit of radius $d \gg r$. It is also spinning around itself, such that the spin angular velocity is in the same direction as the orbital angular velocity, and its axis of symmetry always remains parallel to the plane of the orbit. If its initial spin velocity is ω_0 find its spin velocity after it revolves an angle θ around the star.

Solution 2:

(a) Let the sail have an area of A and area density σ and the sun have mass M. The gravitational force acting on this sail is:

$$F_G = \frac{GM(A\sigma)}{R^2}$$

The photons emitted from the sun have a momentum of $\frac{E}{c}$. On the reflective sail, the change in momentum is:

$$\frac{2E}{c}$$

The force imparted from this change is:

$$F = \frac{dp}{dt} = \frac{2}{c}\frac{dE}{dt}$$

Now taking the projection area of the disk which is A in this case, we can find the fraction of the sun's power (denoted by P) reaching the sail:

$$\frac{PA}{4\pi R^2}$$

So the force from the photons emitted by the sun is

$$F_P = \frac{PA}{2\pi c R^2}$$

Now force balancing and solving algebraically we get,

$$\sigma = \frac{P}{2\pi G M c}$$

(b) The change in momentum for the photons is reduced by a factor of two because of the new perfectly absorbing sails:

$$F = \frac{1}{c}\frac{dE}{dt}$$

If we let the spherical shell have a surface area of A, the projection area of the spherical shell is $\frac{1}{4}A$. Using the same analysis in part a we find that

$$\sigma_{\text{max}} = \frac{PA}{16\pi c R^2}$$

and

$$F_P = \frac{PA}{16\pi c R^2}$$

Now using $\frac{1}{2}\sigma_{max}$ for our force balance equation we get,

$$(A\sigma)\ddot{r} = F_P - F_G$$

$$(A\sigma)\ddot{r} = -\frac{PA}{32\pi c R^2}$$

We find that the acceleration is $\frac{GM}{R^2}$ which means the magnitude of the gravitational potential field is $\frac{GM}{R}$. Now using conservation of energy (the initial kinetic energy and the final potential energy are zero):

$$\frac{1}{2}mv_f^2 = \frac{GMm}{R}$$

$$v_f = \sqrt{\frac{2GM}{R}}$$

(c) The absorbance is $1 - r$, so the force contribution from absorbed photons is:

$$F_{P1} = (1 - r)\frac{PA}{4\pi c R^2}$$

where A is the area of projection of the sphere. For reflected photons, we only consider the change of momentum in the same direction of the original travelling direction since any impulse in the perpendicular directions cancel due to spherical symmetry. Incoming photons that are reflected at an incident angle θ have a change in momentum of

$$\frac{E}{c}(1 + \cos(2\theta))$$

in the direction of travel. This is better illustrated in the following diagram:

It is important to note that at different angles there are varying frequencies of photons reflecting off at that angle due to a varying area of projection. It is easy to show that the area of projection at an angle θ is

$$2\pi r^2 \sin(\theta)\cos(\theta)\mathrm{d}\theta \ = A\sin(2\theta)\mathrm{d}\theta.$$

Taking the sum over all contributions from rings at all angles, we get the expression

$$F_{P2} = r\frac{PA}{4\pi cR^2}\int_0^{\pi/2}(1+2\cos(2\theta))(\sin(2\theta))\mathrm{d}\theta$$

After evaluating, we get that

$$F_{P2} = r\frac{PA}{4\pi cR^2}$$

We can conclude that the force is independent of the reflectance value r because the force from absorbance and reflectance contributions sum to $\frac{PA}{4\pi cR^2}$.

Now to solve for v, we can use conservation of energy (similar to part b, except this time we have an initial kinetic energy as well) to get:

$$\frac{1}{2}mv_f^2 = \frac{GMm}{R} + \frac{1}{2}mv_o^2$$

$$v_o^2 = \frac{GM}{R}$$

$$v_f = \sqrt{\frac{3GM}{R}}$$

It is well known that the eccentricity of an orbit in an inverse square relationship with two masses is:

$$e = \sqrt{1+\frac{2L^2E}{m_r k^2}}$$

With E being the total energy, L the angular momentum, m_r the reduced mass, and k a constant. Because the energy is positive, the eccentricity is greater than one. Therefore it is a hyperbolic orbit. (Note: points for describing the orbit were given to teams that went through the complete derivation of the eccentricity or at least provided some sort of mathematical analysis)

(d) We need to take into account both relativistic effects as well as the doppler effect. The force of radiation is given by

$$F_{\text{rad}} = \frac{2P}{c}\sqrt{\frac{1-\beta}{1+\beta}}$$

which gives rise to the net force

$$\gamma^3 mc\frac{d\beta}{dt} = \frac{2P}{c}\sqrt{\frac{1-\beta}{1+\beta}}.$$

Solving this differential equation gives us

$$P = \frac{0.113mc^2}{t}.$$

(e) Assume that the intensity of the solar radiation at the solar sail is I. Let the solar sail have mass m uniformly distributed across a disc of radius r, and be perfectly reflecting. Suppose it is rotating with angular speed ω, about an axis on its plane (so by perpendicular axis theorem, it has moment of inertia $\frac{1}{4}mr^2$ about that axis). We break the solar sail into many tiny pieces of area dA and analyze the momentum transferred to each piece by the solar radiation.

Suppose a piece of area dA has its normal oriented an angle θ from the oncoming radiation, and it is moving a speed v directly along its normal. Consider what happens after a small time dt. Assume that the total momentum of the photons it intercepts is dp. The total momentum imparted to the area by those photons can be calculated by doppler shift, resulting in $2(\cos\theta + \frac{v}{c})dp$. Next, drawing a picture will show that the total momentum of the photons it intercepts is $(1+\frac{v}{c\cos\theta})\frac{I}{c}\cos\theta dAdt$.

Thus, we know $2(\cos\theta + \frac{v}{c})(1 + \frac{v}{c\cos\theta})\frac{I}{c}\cos\theta dAdt$ gives the total momentum imparted to the piece of area by the radiation.

Now, we integrate all those small pieces of area of find the total angular impulse in time dt. Let α be the angular position of the small piece of area with respect to the axis of rotation of the solar sail. Then we get

$$\tau = \int_0^\pi 2(\cos\theta + \frac{\omega r\sin\alpha}{c})(1 + \frac{\omega r\sin\alpha}{c\cos\theta})\frac{I}{c}\cos\theta * 2(r\cos\alpha)^2 r\sin\alpha d\alpha -$$

$$\int_0^\pi 2(\cos\theta - \frac{\omega r\sin\alpha}{c})(1 - \frac{\omega r\sin\alpha}{c\cos\theta})\frac{I}{c}\cos\theta * 2(r\cos\alpha)^2 r\sin\alpha d\alpha$$

where τ is the angular impulse delivered per unit time. Note that the first integral represents the side of the disc that moves toward the sun, while the second integral represents the side of the disc that moves away from the sun. Also, notice we replaced v with $\omega r\sin\alpha$ in the first integral and v with $-\omega r\sin\alpha$ in the second integral. We also included the total area of the small pieces that were a distance $r\sin\alpha$ from the axis of rotation, which is $2(r\cos\alpha)^2 d\alpha$, and the lever arm is $r\sin\alpha$. After simplification, we get

$$\tau = 16\frac{I}{c^2}\omega r^4\cos\theta\int_0^\pi \cos^2\alpha\sin^2\alpha d\alpha$$

Finally, we get the average value of $|\tau|$ over θ is $\tau = \frac{4}{c^2}I\omega r^4 = -\frac{1}{4}mr^2\dot{\omega}$, since the torque is against the direction of rotation. Solving the differential equation gives $\omega(t) = \omega_0 e^{-\frac{16Ir^2}{mc^2}t}$.

Problem 3: Electron Escape (28 pts)

An infinite wire with current I has a radius a. The wire is made out of a material with resistivity ρ and heat conductivity κ. The temperature outside the wire is a constant T_0.

(a) (4 pts) After a long time, determine the temperature $T(r)$ at a distance r from the center of the wire. Assume that the current in the wire is uniformly distributed.

(b) (4 pts) Now, the outer surface of the wire is maintained at a potential of $-V$, where V is positive. The wire is surrounded by an infinite cylindrical shell with radius $b > a$ that is grounded. Somehow, an electron is able to escape from the wire. Assume that it is at rest just as it escapes. You can neglect radiation from the electron.

Draw a qualitative graph of the physical path that the electron takes, along with a diagram of the wire.

(c) (6 pts) Find the maximum distance $r_{\max}$ of the electron from the center of the wire in the subsequent motion as a function of V, and also in terms of I, a, and b. Ignore all relativistic effects in this part only.

(d) (12 pts) Redo the calculation in the previous part with relativistic effects.

(e) (2 pts) Graph the maximum radius $r_{\max}$ according to part (d) as a function of V.

Solution 3:

(a) As the current density is uniform, the current through a concentric circle with radius r will have total current $\frac{r^2}{a^2}I$. The resistance of the portion of the wire of length ℓ up to a radius r is given by

$$R = \frac{\rho\ell}{A} = \frac{\rho\ell}{\pi r^2}.$$

Therefore, the power emitted out of a cylindrical shell with radius r and length ℓ is

$$P = I^2R = \frac{r^4}{a^4}I^2\frac{\rho\ell}{\pi r^2} = \frac{\rho I^2\ell r^2}{\pi a^4}.$$

By Fourier's Law,

$$\frac{\rho I^2\ell r^2}{\pi a^4} = -\kappa A\frac{dT}{dr} = -\kappa(2\pi r\ell)\frac{dT}{dr}.$$

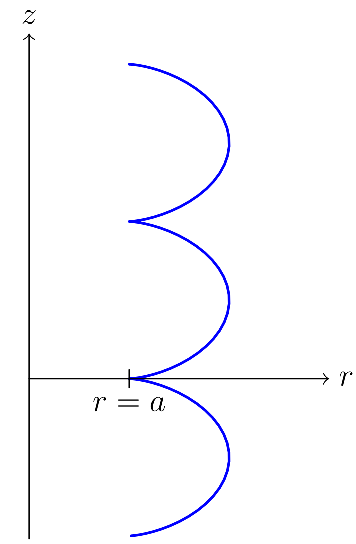
Therefore,

$$\frac{dT}{dr} = -\frac{\rho I^2}{2\pi^2a^4}r \implies T(r) = -\frac{\rho I^2}{4\pi^2a^4}r^2 + C.$$

Since $T(a) = T_0$, so we see $C = \frac{\rho I^2}{4\pi^2a^4}\cdot a^2 + T_0$. Thus,

$$\boxed{T(r) = T_0 + \frac{\rho I^2}{4\pi^2a^4}(a^2 - r^2)}.$$

(b) The electric force is constantly outward, while the magnetic force is perpendicular to the motion in the plane determined by the z axis and the radial direction. Note that the motion of the electron is always in this plane. Therefore, the electron will move outward and continuously change direction until it reaches a maximum radius. Then, it will move back towards the wire in a motion that is completely symmetric. When it reaches the wire again, the motion will repeat in a somewhat "cycloidal" motion. See the graph below.



(c) Clearly, there is only movement in the radial and z directions. The magnetic field at a radius r is given by $\frac{\mu_0 I}{2\pi r}$, and it is tangential. From Newton's second law in the z direction,

$$m\frac{dv_z}{dt} = e\dot{r}\left(\frac{\mu_0 I}{2\pi r}\right) \implies v_z = \frac{\mu_0 e I}{2\pi m}\ln\left(\frac{r}{a}\right).$$

By energy conservation, the energy of the electron at a distance r from the z-axis is given by

$$E = \frac{eV}{\ln\left(\frac{b}{a}\right)}\ln\left(\frac{r}{a}\right) = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_z^2.$$

At the maximum radius, $v_r = 0$, so we get

$$\frac{eV}{\ln\left(\frac{b}{a}\right)}\ln\left(\frac{r_{\max}}{a}\right) = \frac{1}{2}mv_z^2 = \frac{\mu_0^2 e^2 I^2}{8\pi^2 m}\left(\ln\frac{r_{\max}}{a}\right)^2.$$

Solving for $r_{\max}$,

$$\ln\left(\frac{r_{\max}}{a}\right) = \frac{8\pi^2 mV}{\mu_0^2 e I^2 \ln\left(\frac{b}{a}\right)} \implies \boxed{r_{\max} = a\exp\left(\frac{8\pi^2 mV}{\mu_0^2 e I^2\ln\left(\frac{b}{a}\right)}\right)}.$$

Noe that when this expression is greater than b, the electron stops at a radius b.

(d) Clearly, there is only movement in the radial and z directions. From Newton's second law in the z direction,

$$\frac{dp_z}{dt} = e\dot{r}\left(\frac{\mu_0 I}{2\pi r}\right) \implies p_z = \frac{\mu_0 e I}{2\pi}\ln\left(\frac{r}{a}\right).$$

Also, note by energy conservation that the energy of the electron at a distance r from the z-axis is

$$E = mc^2 + \frac{eV}{\ln\left(\frac{b}{a}\right)}\ln\left(\frac{r}{a}\right).$$

Therefore,we have,

$$E^2 = (mc^2)^2 + (pc)^2 = (mc^2)^2 + (p_r^2 + p_z^2)c^2 = \left(mc^2 + \frac{eV}{\ln\left(\frac{b}{a}\right)}\ln\left(\frac{r}{a}\right)\right)^2,$$

$$p_r^2 + p_z^2 = \frac{e^2V^2 \ln\left(\frac{r}{a}\right)^2}{c^2\left(\ln\left(\frac{b}{a}\right)\right)^2} + \frac{2meV\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)},$$

$$p_r^2 = \frac{e^2V^2 \ln\left(\frac{r}{a}\right)^2}{c^2\left(\ln\left(\frac{b}{a}\right)\right)^2} + \frac{2meV\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)} - \frac{\mu_0^2e^2I^2}{4\pi^2}\left(\ln\left(\frac{r}{a}\right)\right)^2$$

At the maximum radius, r_{max}, we have $p_r = 0$. Therefore,

$$\frac{e^2V^2 \ln\left(\frac{r_{\text{max}}}{a}\right)^2}{c^2\left(\ln\left(\frac{b}{a}\right)\right)^2} + \frac{2meV\ln\left(\frac{r_{\text{max}}}{a}\right)}{\ln\left(\frac{b}{a}\right)} - \frac{\mu_0^2e^2I^2}{4\pi^2}\left(\ln\left(\frac{r_{\text{max}}}{a}\right)\right)^2 = 0.$$

We can solve for r_{max} to get

$$r_{\text{max}} = a\exp\left(\frac{2meV}{\ln\left(\frac{b}{a}\right)}\left(\frac{\mu_0^2e^2I^2}{4\pi^2} - \frac{e^2V^2}{c^2\left(\ln\left(\frac{b}{a}\right)\right)^2}\right)^{-1}\right).$$

However, we also have to note that $r_{\text{max}} \leq b$. We will now find the condition on V such that the electron's maximum radius is b. The critical voltage V_{crit} is when $p_r = 0$ at $r = b$. We have

$$\frac{e^2V_{\text{crit}}^2 \ln\left(\frac{b}{a}\right)^2}{c^2\left(\ln\left(\frac{b}{a}\right)\right)^2} + \frac{2meV_{\text{crit}}\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)} - \frac{\mu_0^2e^2I^2}{4\pi^2}\left(\ln\left(\frac{b}{a}\right)\right)^2 = 0,$$

$$e^2V_{\text{crit}}^2 + 2mc^2eV_{\text{crit}} - \frac{\mu_0^2e^2I^2c^2}{4\pi^2}\left(\ln\left(\frac{b}{a}\right)\right)^2 = 0.$$

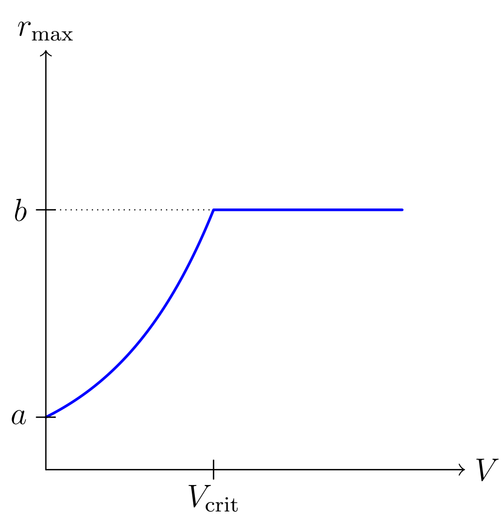
By the quadratic formula,

$$\boxed{V_{\text{crit}} = \frac{\sqrt{4m^2c^4e^2 + \frac{\mu_0^2e^4I^2c^2}{\pi^2}\left(\ln\frac{b}{a}\right)^2} - 2mc^2e}{2e^2}}.$$

Therefore, we have

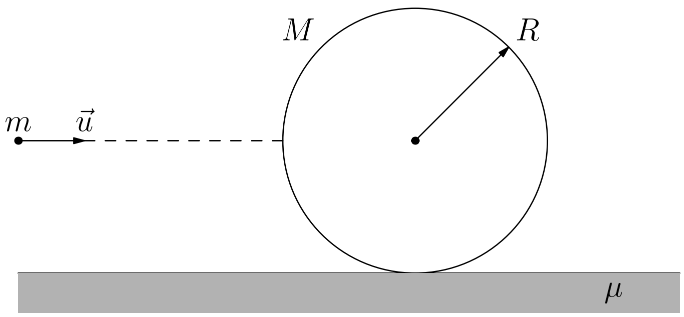
$$\boxed{r_{\text{max}} = \begin{cases} a\exp\left(\frac{2meV}{\ln\left(\frac{b}{a}\right)}\left(\frac{\mu_0^2e^2I^2}{4\pi^2} - \frac{e^2V^2}{c^2\left(\ln\left(\frac{b}{a}\right)\right)^2}\right)^{-1}\right) & 0 \leq V < V_{\text{crit}} \\ b & V \geq V_{\text{crit}} \end{cases}}.$$

(e) The graph is roughly exponential in V until $r_{\text{max}} = b$, at which point it is flat.



Problem 4: Bullet in Cylinder (20 pts)

A hollow cylinder of mass M and radius R rests on a rough horizontal surface. A projectile of mass $m < M$ having a velocity u directed horizontally exactly towards the middle of the cylinder as shown in the figure. The shell gets stuck in the cylinder wall, after which the shell starts to move, slipping on the surface. The coefficient of static and kinetic friction between the cylinder and the horizontal surface are the same, and are equal to $\mu < 2$.



(a) **(4 pt)** In which direction does the cylinder rotate? State your answers for different values of μ.

(b) **(6 pts)** Find its angular acceleration about the centre of the cylinder just after the impact.

(c) **(10 pts)** It is known that some time after the impact, the horizontal projection of the velocity of the center of mass of the system is equal to v, and the angular velocity of the cylinder is Ω. Till this point, the cylinder has rotated through an angle ϕ.

How much heat was released in the system till this point, if the cylinder all the time after the impact moved with slipping, rotating in one direction? Assume that all energy losses are dissipated in the form of heat.

Solution 4:

(a) Combining both force equations in the vertical direction and our 'torque' equation and that $F_0 = \mu N$, we get

$$\alpha = \frac{g}{R} \cdot \frac{(\mu - 1)m + \mu M}{(1 - \mu)m + M}$$

$$N = \frac{(2m + M)Mg}{M + (1 - \mu)m}$$

Note that α and N tend to infinity as $\mu \to \mu_0 = 1 + \frac{M}{m}$. This means that the normal force can even increase infinitely if a large impulse is provided by the bullet. For $m = M$, $\mu_0 = 2$, and since the problem says that $\mu < 2$, we know that both N and α are finite and always remain true.

Now, for the direction of rotation, let us observe when α changes parity. Notice that for $\mu < \mu_1 = \frac{m}{M+m}$, we have $\alpha < 0$, meaning that the cylinder will rotate counter-clockwise. Let us try to understand what could've caused such a situation? This could've been caused by a large impulse by the bullet, when it exceeded the impulse due to friction. Similarly, for $\mu > \mu_1$, we have that the cylinder rotates clockwise. Note that since $m < M$, $0 < \mu_1 < \frac{1}{2}$. So are answers for the first part are

$$\text{Direction of rotation} = \begin{cases} \text{Clockwise} & \text{if } \mu > \frac{m}{M+m} \\ \text{Counter-clockwise} & \text{if } \mu < \frac{m}{M+m} \end{cases}$$

(b) Now we try to find the heat released in the system at any time. For the sake of a definite solution, assume that the cylinder has rotated an angle ϕ clockwise. At this moment, we have the velocity v of the cylinder along the horizontal, angular velocity Ω, and the linear velocity of rotation of the bullet v_1. (clearly $v_1 = \Omega R$) Now let us write the energy of the bullet in the reference frame of the axis of the cylinder:

$$E = mgR\sin\phi + \frac{1}{2}m{v_1}^2$$

Transcending back to the original lab frame, since we need to replace the velocity of the bullet accordingly, and using $v_1 = \Omega R$, we get

$$E = mgR\sin\phi + \frac{1}{2}m{v_0}^2 + \frac{1}{2}m\Omega^2 R^2 + m\Omega R v_0 \sin\phi$$

and we have the energy of the cylinder in the lab frame

$$E_c = \frac{1}{2}M{v_0}^2 + \frac{1}{2}M\Omega^2 R^2$$

Also note that from transitioning from the cylinder axis' frame to the lab frame, we convert the centre of mass velocity as

$$v = v_0 + \Omega r \sin\phi$$

where r is the distance of the cylinder's centre to the new centre of mass. The value of r can be found easily:

$$Mr = m(R-r) \Rightarrow r = \frac{m}{M+m}R$$

So the work-energy theorem becomes

$$\Delta H = E + E_c - \frac{1}{2}m{v_0}^2$$

Substituting the values of E and E_c, we finally get

$$\Delta H = \begin{cases} \frac{1}{2}mu^2 + \frac{m^2\Omega^2R^2\sin^2\phi}{2(m+M)} - \frac{m+M}{2}(v^2 + \Omega^2R^2) + mgR\sin\phi & \text{if counter-clockwise} \\ \frac{1}{2}mu^2 + \frac{m^2\Omega^2R^2\sin^2\phi}{2(m+M)} - \frac{m+M}{2}(v^2 + \Omega^2R^2) - mgR\sin\phi & \text{if clockwise} \end{cases}$$

Comment (not for grading purposes):

In the process, we assumed that the combined system of bullet and cylinder does not start its motion by "jumping up". This directly affects our approach to the problem and is not entirely obvious.
In the perfectly inelastic collision, the angular momentum of the bullet-cylinder system is conserved. As above, we assume the velocity of the centre of mass v to be directed along the horizontal.

Problem 5: Mathematical Physics (18 pts)

The study of mathematics has almost always paved the way for the development of new ideas in physics. Newtonian mechanics could not be possible without first inventing calculus, and general relativity could not have existed without heavy development in tensors. However, there are numerous cases where physical insight have paved the way for mathematics.

Perhaps the most notable would be the Brachistochrone problem, which asks for the path that leads to the fastest descent influenced by gravity between two given points. While it is solvable through the calculus of variations, Newton proposed an easier solution by modelling the path of light through a medium with a variable index of refraction. You may read about this problem and the fascinating history behind it here.

We will not be dealing with this specific problem, but rather multiple short mathematics problems that can be can be represented with a physical analog. To receive points, you must use the suggested physical set-up.

(a) **(4 pts)** Show that for small values of x, we have

$$\cos(x) = 1 - \frac{x^2}{2}.$$

Physical Setup: Consider a small object moving in a circle.

(b) **(8 pts)** A ladder of length ℓ with a thickness of 0.3 m is transported around a right angled corner where the two hallways leading up to it have a width of 3 m and 5 m. What is the maximum length of the ladder such that it can be successfully transported across?

Physical Setup: Consider the ladder as a compressed spring that can freely expand in only its longitudinal direction. Do not explicitly take derivatives. Instead, consider a force/torque balance. You may or may not need to solve an equation numerically.

(c) **(6 pts)** Prove the AM-QM inequality:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}$$

Physical Setup: Design circuit(s) and compare their measurable quantities with each other qualitatively (or otherwise).

Solution 5:

(a) Consider a small particle starting from rest on a circular path. It quickly and uniformly accelerates to a speed v in a very small time interval Δt such that $\frac{v\Delta t}{R} \ll 1$ where R is the radius of the circle.

Since this time interval is so small, the acceleration is directed inwards. Without loss of generality, let us define the inwards direction as the positive $+y$ direction. The displacement Δy is given by $R\,(1 - \cos\theta)$ which gives:

$$R\,(1 - \cos\theta) = \frac{1}{2}at^2$$

At time t, since the particle hasn't moved very far, the inwards acceleration is still pointed in the $+y$ direction and using equations for circular motion, it has a magnitude of:

$$a = \frac{v^2}{R}$$

Making the substituting, we can simplify:

$$1 - \cos\theta = \frac{1}{2}\left(\frac{vt}{R}\right)^2$$

Since $vt = R\theta$ is the distance traveled, this simplifies to:

$$1 - \cos\theta = \frac{1}{2}\theta^2 \implies \boxed{\cos\theta = 1 - \frac{1}{2}\theta^2}$$

(b) Treating the ladder as a spring, we see that its length is directly related to its potential energy. Determining the point at which the ladder's length is at a minimum is equivalent to determining the point at which the rigid spring's potential energy is at a minimum. Using the fact that:

$$F = -\frac{dU}{dx}$$

implies that when the potential energy is at a minimum, the net force must be equal to zero. Therefore, we can set up a force and torque balance in order to solve for the shortest length. There are three normal forces to take into account. Balancing forces gives two equations, and a torque balance gives the third. Solving a system of three equations will allow us to solve the problem.

Perhaps a slicker way is to realize that these three forces must be concurrent. At least two of the normal force vectors will intersect, and performing a torque balance where the pivot is selected to be that intersection point, these two normal force vectors will not contribute to a torque. In order to satisfy $\sum \tau = 0$, the third normal force vector must also intersect at this same place. This turns it into a geometry problem.

Solving the problem geometrically or analytically, you end up solving the equation:

$$\frac{a\cos\theta - \ell}{\sin^2\theta} = \frac{b\sin\theta - \ell}{\cos^2\theta}$$

which is possible to be solved numerically (or solved through an approximation). This wasn't specified on the exam, so most teams who arrived at this equation would have received the marks.

Solving, we get $L \approx 10.6$ m.

(c) We connect several batteries with an electromotive force ε_i and an identical internal resistance r in parallel with one another, and the entire system is in parallel with a wire with zero resistance. Using Kirchoff's Loop rule, the voltage drops across each internal resistor is going to be the same. Therefore, the power dissipation is:

$$P_1 = \frac{\varepsilon_1^2 + \varepsilon_2^2 + \cdots + \varepsilon_N^2}{r}$$

if there are N such resistors. We then create a second circuit where all of these resistors are in series. The current is now the same and the total power dissipation is:

$$P_2 = \frac{(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)^2}{Nr}$$

If we make the claim that $P_1 \geq P_2$, the AM-QM inequality will naturally follow. To do this, we can look at the amp-hour capacity of batteries in parallel vs in series. In series, the amp-hour capacity of each resistor must be the same (and be equal to the capacity of the weakest battery). If this was not the case, then some batteries will be depleted before others. This is not the case for parallel and will only be the same if $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_N$, in which case we have $P_1 = P_2$. Due to this, the power dissipation in the first circuit must be equal or greater than the one in the second circuit, proving the AM-QM inequality.

Problem 6: Flat Earth (22 pts)

In this problem, we will explore the true gravitational model of the earth, not the one that is claimed in most textbooks. Contrary to popular belief, the Earth is a flat circle of radius R and has a uniform mass per unit area σ. The Earth rotates with angular velocity ω.

(a) **(5 pts)** A pendulum of length ℓ that is constrained to only move in one plane is placed on the ground at the center of the Earth. The pendulum has more than one angular frequency of small oscillations. Find the value of each angular frequency of small oscillations $\Omega(0), \Omega_1(0), ...$ in terms of σ, ω, ℓ, and physical constants and the equilibrium angle $\theta, \theta_1, ...$ that the frequency occurs at. Assume for all parts that $\ell \ll R$.

An equilibrium angle corresponds to the angle with respect to the vertical where there is an equilibrium point.

(b) **(2 pt)** Investigate the stability of each equilibrium position with varying angular velocity of the Earth.

(c) **(12 pts)** The entire pendulum is moved a horizontal distance $r \ll R$ away from the center of the Earth. It is oriented so that it is constrained to only move in the radial direction. Now, find the new angular frequency $\Omega(r)$ of small oscillations about the lowest equilibrium point in terms of the given parameters, assuming that $\omega^2 r$ is much less than the local gravitational acceleration.

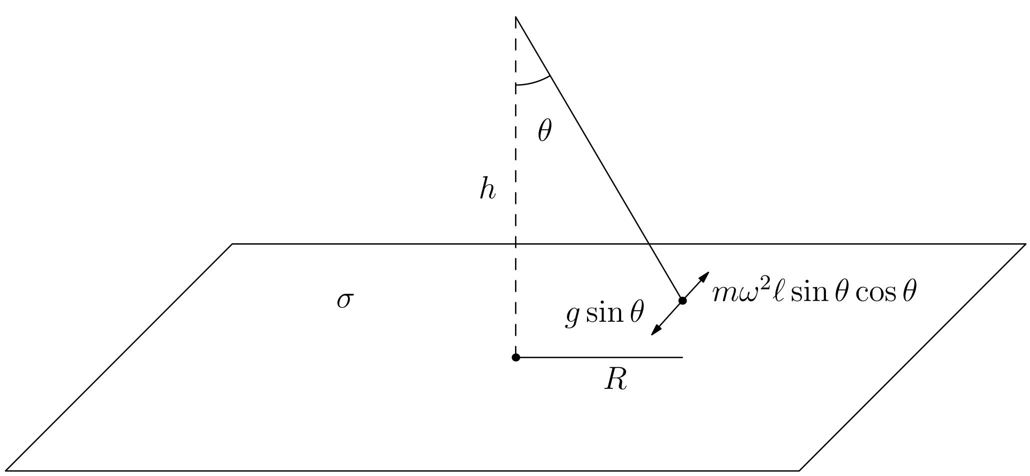
(d) **(3 pts)** The angular frequencies $\Omega(0)$ and $\Omega(r)$ are both measured and the difference is found to be $\Delta\Omega$. Assuming that $\Delta\Omega \ll \Omega(0)$ and $\omega^2 \ll \frac{g}{\ell}$, determine σ in terms of $\omega, r, \Omega(0), \Delta\Omega$, and physical constants.

Solution 6:

(a) The first thing that we note is that the acceleration due to the gravitational force is different as compared to the one on Earth. Consider a cylindrical Gaussian surface that has it's circular faces parallel to the Earth. From Gauss's Law of gravitation, we can calculate the acceleration due to gravity on a flat plane to be

$$4\pi G m = 2 \times \pi r^2 g \implies g = 2\pi G\sigma$$

by substituting $m = \sigma\pi r^2$. Let us now consider a small displacement of the mass m from the origin. We see that there are two forces involved: the centrifugal force, and the gravitational force. We now draw a free-body diagram as shown below:



Note that the gravitational force will be restoring and is given by $mg\sin\theta \approx mg\theta$. The centrifugal force is directed outwards and is given by $m\omega^2\ell\sin\theta\cos\theta \approx m\omega^2\ell\theta$. Writing Newton's Laws on the pendulum now gives us

$$m\ell\ddot{\theta} = m\omega^2\ell\theta - mg\theta \implies \ddot{\theta} = -\left(\frac{g}{\ell} - \omega^2\right)\theta.$$

An equilibrium position occurs when $\ddot{\theta} = 0$. This tells us that the two equilibrium positions are defined by

$$\theta_1 = 0, \qquad \theta_2 = \arccos\left(\frac{g}{\omega^2\ell}\right).$$

For oscillations near $\theta_1 = 0$, we can use small-angle approximations gives us

$$\ddot{\theta} = -\left(\frac{g}{\ell} - \omega^2\right)\theta.$$

Therefore, we see $\Omega(0) = \sqrt{\frac{g}{\ell} - \omega^2} = \boxed{\sqrt{\frac{2\pi G\sigma}{\ell} - \omega^2}}$.

For oscillations near $\theta_2 = \arccos\left(\frac{g}{\omega^2\ell}\right)$, we let $\theta = \theta_1 + \varphi$ where $\varphi \ll \theta_1$. Using a first order approximation tells us that

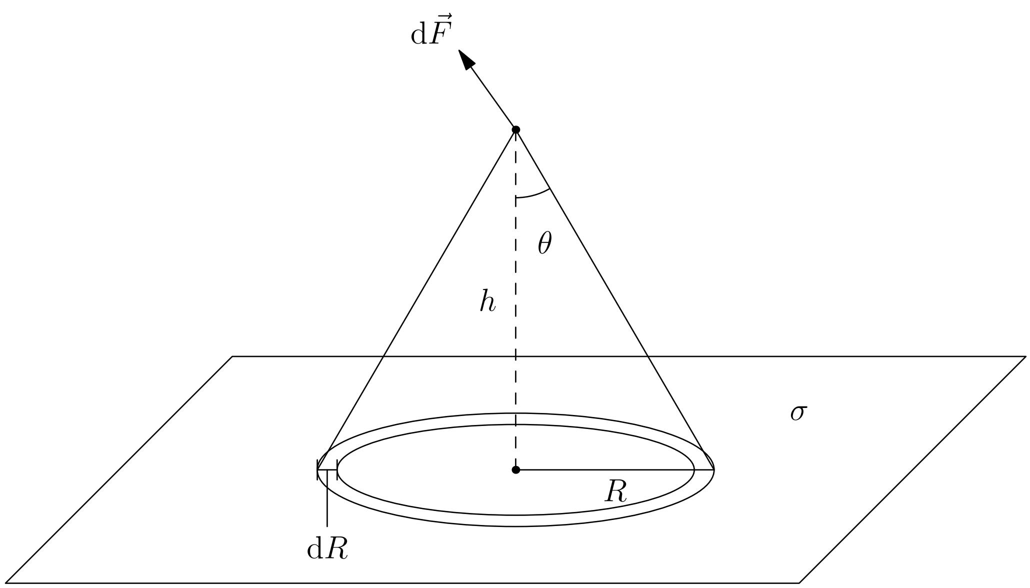
$$\ddot{\varphi} + \left(\frac{g}{\ell} + \omega^2\varphi\sin\theta_1 - \omega^2\cos\theta_1\right)(\sin\theta_1 + \varphi\cos\theta_1) = 0.$$

Simplifying gives us

$$\ddot{\varphi} = -(\omega^2\sin^2\theta_1)\varphi.$$

This tells us that $\Omega_1(0) = \omega\sin\theta = \omega\sqrt{1 - \frac{g^2}{\omega^4\ell^2}} = \boxed{\frac{1}{\omega\ell}\sqrt{\omega^4\ell^2 - 4\pi^2G^2\sigma^2}}$.

Remark 1: We can find the gravitational acceleration on the planet by in fact another way of analyzing the force per each infinitesimal ring. We can split the flat Earth into many tiny rings as shown below



Each ring provides a force of $d\vec{F}$. If each ring has a thickness dR, our force will be given by

$$dF = \frac{Gm\sigma}{h^2 + R^2}\cos\theta \cdot (\pi(R + dR)^2 - \pi R^2).$$

Using the fact that $\cos\theta = \frac{h}{\sqrt{h^2+R^2}}$, we then find that

$$dF = \frac{Gm\sigma}{h^2+R^2}\frac{h}{\sqrt{h^2+R^2}} \cdot (\pi(R+dR)^2 - \pi R^2).$$

Using a first order approximation gives us

$$(\pi(R+dR)^2 - \pi R^2) \approx 2RdR.$$

We then find by substituting that,

$$dF = \frac{2GmR\sigma h}{(h^2+R^2)^{3/2}}dR.$$

Now, integrating this force dF from 0 to ∞ gives us

$$F = \int_0^\infty dF = 2Gm\sigma h \int_0^\infty \frac{R}{(h^2+R^2)^{3/2}}dR = 2\pi G\sigma m.$$

The gravitational acceleration is then given by

$$g \equiv \frac{2\pi G\sigma m}{m} = 2\pi G\sigma.$$

■

Remark 2: We can find the equation of motion with the Euler-Lagrange equations. Let us consider a rotating frame at the center of the Earth. The pendulum's coordinates are then given by

$$(x, y, z) = (\ell\sin\theta, 0, -\ell\cos\theta)$$

which implies that the velocity of the pendulum is given by

$$(\dot{x}, \dot{y}, \dot{z}) = (\ell\dot{\theta}\cos\theta, 0, \ell\dot{\theta}\sin\theta).$$

In the fixed frame, the pendulum has an additional velocity from the centrifugal force which is given by

$$\vec{\omega} \times \vec{r} = (0, 0, \omega) \times (\ell\sin\theta, 0, -\ell\cos\theta) = (0, \omega\ell\sin\theta, 0).$$

We now write the lagrangian of the system as

$$\begin{aligned}
\mathcal{L} &\equiv T - V \\
&= \frac{1}{2}m\sqrt{(\ell\dot{\theta}\cos\theta)^2 + 0^2 + (\ell\dot{\theta}\sin\theta)^2} + \frac{1}{2}m\ell^2\omega^2\sin^2\theta + mg\ell\cos\theta \\
&= \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}m\ell^2\omega^2\sin^2\theta + mg\ell\cos\theta.
\end{aligned}$$

Now, using the Euler-Lagrange equations gives us

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathcal{L}}{\partial\theta} \implies \ddot{\theta} = -\left(\frac{g}{\ell} - \omega^2\cos\theta\right)\sin\theta.$$

■

(b) We have two cases:

Case 1: (oscillations for $\Omega(0)$) If $\omega < \sqrt{\frac{g}{\ell}} = \sqrt{\frac{2\pi G\sigma}{\ell}}$, we see that the oscillations are stable. The oscillations would follow the equation

$$\theta(t) = A_1\cos(\Omega(0) + \phi_1)$$

where A_1 and ϕ_1 are constants to be determined from initial conditions.

Case 2: (oscillations for $\Omega_1(0)$) If $\omega > \sqrt{\frac{2\pi G\sigma}{\ell}}$ we see that the oscillations are stable. The oscillations would follow the equation

$$\varphi(t) = A_2 \cos(\Omega_1(0) + \phi_2)$$

where A_2 and ϕ_2 are constants to be determined from initial conditions.

If $\omega = \sqrt{\frac{2\pi G\sigma}{\ell}}$ the oscillations are neutrally stable but do not display stable small oscillations.

(c) We use the fact that, for small oscillations of an object subject to a potential $U(x)$, the frequency of small oscillations will be defined as

$$\omega = \sqrt{\frac{U''(x)}{m_{\text{eff}}}}.$$

The potential energy is due to the gravitational potential and the centrifugal potential:

$$U = -mg\ell\cos\theta - \frac{1}{2}m\omega^2(r + \ell\sin\theta)^2$$

We see that

$$\frac{d^2U}{d\theta^2} = mg\ell\cos\theta + m\omega^2 r\ell\sin\theta + m\omega^2\ell^2(\sin^2\theta - \cos^2\theta)$$

When evaluated at $\theta = \theta_0$, the equilibrium position,

$$U''(\theta_0) = mg\ell\cos\theta_0 + m\omega^2 r\ell\sin\theta_0 + m\omega^2\ell^2(\sin^2\theta_0 - \cos^2\theta_0) = k_{\text{eff}}.$$

We now need to find the effective mass. Note that the kinetic energy is given by

$$\frac{1}{2}m\ell^2\dot{\theta}^2 = \frac{1}{2}m_{\text{eff}}\dot{\theta}^2 \implies m_{\text{eff}} = m\ell^2.$$

Thus, the frequency is given by

$$\Omega(r) = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{g}{\ell}\cos\theta_0 + \frac{\omega^2 r}{\ell}\sin\theta_0 + \omega^2(\sin^2\theta_0 - \cos^2\theta_0)}$$

Now, we have $\cos\theta_0 = \frac{g}{\sqrt{\omega^4 r^2 + g^2}} \approx 1 - \frac{\omega^4 r^2}{2g^2}$ and $\sin\theta_0 \approx \frac{\omega^2 r}{g}$. Substituting,

$$\Omega(r) = \sqrt{\frac{g}{\ell} - \frac{\omega^4 r^2}{2g\ell} + \frac{\omega^4 r^2}{g\ell} + \frac{\omega^6 r^2}{g^2} - \omega^2 + \frac{\omega^6 r^2}{2g^2}}$$

$$\Omega(r) = \sqrt{\Omega(0)^2 + \frac{\omega^4 r^2}{2g\ell} + \frac{3\omega^6 r^2}{2g^2}} = \boxed{\sqrt{\Omega(0)^2 + \frac{\omega^4 r^2}{4\pi G\sigma\ell} + \frac{3\omega^6 r^2}{8\pi^2 G^2\sigma^2}}}$$

(d) Note that we can write

$$\Omega(r) = \sqrt{\Omega(0)^2 + \frac{\omega^4 r^2}{2g^2}\left(\frac{g}{\ell} + 3\omega^2\right)} = \Omega(0)\sqrt{1 + \frac{\omega^4 r^2}{2g^2}\left(\frac{\frac{g}{\ell} + 3\omega^2}{\frac{g}{\ell} - \omega^2}\right)}$$

Therefore, we have

$$\Omega(r) \approx \Omega(0)\sqrt{1 + \frac{\omega^4 r^2}{2g^2}} \approx \Omega(0)\left(1 + \frac{\omega^4 r^2}{4g^2}\right)$$

Therefore, $\Delta\Omega = \frac{\omega^4 r^2}{4g^2}\Omega(0)$. This implies that

$$g^2 = 4\pi^2 G^2\sigma^2 = \frac{\omega^4 r^2 \Omega(0)}{4\Delta\Omega}$$

We can solve for σ to get $\sigma = \boxed{\frac{\omega^2 r}{4\pi G}\sqrt{\frac{\Omega(0)}{\Delta\Omega}}}$.

Problem 7: Boltzmann Statistics (24 pts)

In this problem, we will explore Boltzmann Statistics and using it to build similar models for quantum particles such as bosons and fermions.

(a) **(8 pts)** Consider a energy of a gaseous molecule in space is given by 5

$$E = E_0(|x|^r + |y|^r + |z|^r)$$

where the coordinates of the molecule are represented by (x, y, z), E_0 is a constant with appropriate units, and r is a non-negative real number. The system is in thermal equilibrium with a reservoir of temperature T. Calculate explicitly using appropriate statistical methods, the average energy of a thermodynamic system consisting of such gaseous molecules, considering the Maxwell-Boltzmann distribution. Analyse your result and provide a qualitative argument to support it.

Hint: If you are not familiar with how to solve this part, try part (b) first.

(b) **(6 pts)** Consider a specific case in which

$$E = E_0(x^2 + y^2 + z^2)$$

except where $|x|, |y|, |z| < 2$ and particles can only exist at integer values of x, y, z. If the system is still in thermal equilibrium at a temperature T, calculate the average energy.

(c) **(6 pts)** In this system, there are two particles. What is the probability that at least one of these particles will be in the ground state (energy is zero) if:

(i) The two particles are distinct.

(ii) The two particles are identical bosons.

(iii) The two particles are identical fermions (fermions follow Pauli exclusion principle). For simplicity, ignore spin.

For each of the parts, assume that there are no other interactions (e.g. electromagnetism) and focus mainly on a statistical argument.

(d) **(4 pts)** Rank the probabilities in part (c) from highest to lowest when the temperature is

(i) high

(ii) low

For each, explain qualitatively why this must be true.

Solution 7:

(a) In Maxwell-Boltzmann's distribution, the probability that an isolated particle occupies the point $A(x, y, z)$ is given by

$$\mathrm{d}\mathbb{P}(E) = \mathcal{P}(E)\mathrm{d}V = -\frac{e^{-\frac{E_0}{k_B T}}}{Z}$$

where Z denotes the partition function. Note that although we write the probability such that an isolated particle occupied the space, but for a system of N particles, there would be an extra factor of $N!$ in the partition function, but since the particles are considered independent, this factor would cancel out in further calculation anyway. Now for the calculation of Z, we normalize the probability $\mathrm{d}\mathbb{P}(E)$ over all possible states; meaning that we convert the probability function to the probability density function by summing total probability as 1. This gives us

$$Z = \iiint_{\mathbb{R}^3} e^{-\beta \cdot E_0 \cdot (|x|^r + |y|^r + |z|^r)} \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

(More accurately, the partition function of a set of molecules would be found by summing, and not integrating over the space, but since it is a continuous function, the sum reduces to an integral.) From the statistical definition of average energy per unit particle, we have

$$\langle E \rangle = \iiint_{\mathbb{R}^3} E \cdot \mathrm{d}\mathbb{P}(E) = \iiint_{\mathbb{R}^3} E_0(|x|^r + |y|^r + |z|^r)\mathrm{d}\mathbb{P}(E)$$

Substituting the partition function from above, we obtain

$$\langle E\rangle = \frac{\iiint_{\mathbb{R}^3} E_0(|x|^r+|y|^r+|z|^r)e^{\left(-\frac{E_0}{k_BT}(|x|^r+|y|^r+|z|^r)\right)}\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\iiint_{\mathbb{R}^3} e^{\left(-\frac{E_0}{k_BT}(|x|^r+|y|^r+|z|^r)\right)}\mathrm{d}x\mathrm{d}y\mathrm{d}z}$$

$$\langle E\rangle = \frac{\sum_{x_i\in(x,y,z)}\iiint_{\mathbb{R}^3} e^{-\frac{E_0 {x_i}^r}{kT}}E_0 {x_i}^r\mathrm{d}x_i}{\sum_{x_i\in(x,y,z)}\iiint_{\mathbb{R}^3} e^{-\frac{E_0 {x_i}^r}{k_BT}}\mathrm{d}x_i}$$

Now let us evaluate some integrals to obtain a closed form for $\langle E\rangle$ above. Applying by-parts, we have

$$\int_0^\infty e^{-\frac{E_0 {x_i}^r}{k_BT}}\mathrm{d}x_i = \int_0^\infty e^{-\left(\left(\frac{E_0}{k_BT}\right)^{\frac{1}{r}}x_i\right)^r}\mathrm{d}x_i$$

$$= x_i\left(\frac{E_0}{K_BT}\right)^{1/r}e^{-\frac{E_0x_i^r}{r_BT}}\Bigg|_0^\infty + r\int_0^\infty \frac{E_0x_i^r}{K_BT}e^{-\frac{E_0x_i^r}{k_BT}}\mathrm{d}\left(x_i\left(\frac{E_0}{k_BT}\right)^{\frac{1}{r}}\right)$$

Note that after applying the bounds, $x_i\left(\frac{E_0}{K_BT}\right)^{1/r}e^{-\frac{E_0x_i^r}{r_BT}}\Big|_0^\infty$ reduces to 0. This gives

$$\int_0^\infty e^{-\left(\left(\frac{E_0}{k_BT}\right)^{\frac{1}{r}}x_i\right)^r}\mathrm{d}x_i = r\int_0^\infty \frac{E_0x_i^r}{K_BT}e^{-\frac{E_0x_i^r}{k_BT}}\mathrm{d}\left(x_i\left(\frac{E_0}{k_BT}\right)^{\frac{1}{r}}\right)$$

The same integral ratio appeared in our original average energy per unit particle expression. However, the above integral was performed across one-dimension. In three dimensions (the integral is non-degenerate across all three dimensions), our average energy per particle expression becomes

$$\langle E\rangle = \frac{\iiint_{\mathbb{R}^3} e^{-\frac{E_0 {x_i}^r}{kT}}E_0 {x_i}^r\mathrm{d}x_i}{\sum_{x_i\in(x,y,z)}\iiint_{\mathbb{R}^3} e^{-\frac{E_0 {x_i}^r}{k_BT}}\mathrm{d}x_i} = E_0\times\frac{3k_BT}{rE_0} = \frac{3k_BT}{r}$$

Hence, the average energy of a thermodynamic system of N such particles is

$$\langle E_s\rangle = \boxed{\frac{3Nk_BT}{r}}$$

As a sanity check, note that the average energy is independent of E_0, which was expected. This is because for any quadratic function in space (specifically, here this is $r = 2$) we have the "Law of Equipartition", which states that every degree of freedom has an average energy of $1/2k_BT$, which is confirmed by the obtained expression.

Aliter: Consider the partition function:

$$Z = \sum e^{-\beta E} = \sum e^{-\beta E_0(x^r+y^r+z^r)}$$

We can turn this into the integral form:

$$Z = \frac{1}{\delta x\delta y\delta z}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-\beta E_0(x^r+y^r+z^r)}\,dx\,dy\,dz$$

Letting $a = (\beta E_0)^{1/r}x$, $b = (\beta E_0)^{1/r}y$, and $c = (\beta E_0)^{1/r}z$, we can rewrite this as:

$$Z = \frac{1}{\delta x\delta y\delta z(\beta E_0)^{3/r}}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-(a^r+b^r+c^r)}\,da\,db\,dc = \frac{C}{\delta x\delta y\delta z(\beta E_0)^{3/r}}$$

where C is a constant, equal to the integral. The average energy is given by:

$$\begin{aligned}
E_{\text{avg}} &= \frac{-1}{Z}\frac{\partial Z}{\partial \beta} \\
&= \frac{-\delta x \delta y \delta z (\beta E_0)^{3/r}}{C}\frac{C}{\delta x \delta y \delta z}\left(\frac{-3}{r}E_0(\beta E_0)^{\frac{-3-r}{r}}\right) \\
&= \frac{3}{r\beta} \\
&= \frac{3}{r}k_B T
\end{aligned}$$

(b) The energy of any given particle can be $0, 1, 2, 3$ (working in units where $E_0 = 1$). We need to calculate the Boltzmann factor for each energy as well as how many ways each energy can be achieved.

- Energy 0: $e^{0\beta} = 1 \to 1$ way.
- Energy 1: $e^{1\beta} \to 6$ ways.
- Energy 2: $e^{2\beta} \to 12$ ways.
- Energy 3: $e^{3\beta} \to 8$ ways.

In total, the partition function is:

$$Z = 1 + 6e^{\beta} + 12e^{2\beta} + 8e^{3\beta}$$

The average energy is thus:

$$\begin{aligned}
E_{\text{avg}} &= -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \\
&= -\frac{6e^{-\beta} + 24e^{-2\beta} + 24e^{-3\beta}}{1 + 6e^{-\beta} + 12e^{-2\beta} + 8e^{-3\beta}}
\end{aligned}$$

Note that this can be factored further.

(c) The probability of at least one particle being in the ground state is equal to one minus the probability of no particles being in the ground state. The probability of having some energy E_i is:

$$P_i = \frac{(\text{number of ways this can happen})e^{-\beta E_i}}{Z}$$

Summing up all the probabilities for nonzero energies gives us:

$$P_{\text{at least 1 in ground state}} = 1 - \frac{Z_{\text{restrictions}}}{Z_{\text{no restrictions}}}$$

(i) If two particles are distinct, they can take on energies $1, 2, 3, 4, 5, 6$. Similar to part (b), we list out all the possible ways to achieve these energies if there are no restrictions:

- Energy 0: $1^2 = 1$ way.
- Energy 1: $2(1 \cdot 6) = 12$ ways.
- Energy 2: $2(1 \cdot 12) + 6^2 = 60$ ways.
- Energy 3: $2(1 \cdot 8) + 2(6 \cdot 12) = 160$ ways.
- Energy 4: $2(6 \cdot 8) + 12^2 = 240$ ways.
- Energy 5: $2(12 \cdot 8) = 192$ ways.
- Energy 6: $8^2 = 64$ ways.

If no particles are in the ground state, then:

- Energy 2: $6^2 = 36$ ways.
- Energy 3: $2(6 \cdot 12) = 144$ ways.
- Energy 4: $2(6 \cdot 8) + 12^2 = 240$ ways.
- Energy 5: $2(12 \cdot 8) = 192$ ways.
- Energy 6: $8^2 = 64$ ways.

The probability is thus:

$$P_1 = 1 - \frac{36e^{-2\beta} + 144e^{-3\beta} + 240e^{-4\beta} + 192e^{-5\beta} + 64e^{-6\beta}}{1 + 12e^{-\beta} + 60e^{-2\beta} + 160e^{-3\beta} + 240e^{-4\beta} + 192e^{-5\beta} + 64e^{-6\beta}}$$

Perhaps an easier way to do this problem is to think of the partition functions as generating functions. The partition function for placing two particles with no restrictions is the square of placing a single particle:

$$Z_{2,\text{ no restriction}} = Z^2_{1,\text{ no restriction}} = \left(1 + 6e^{-\beta} + 12e^{-2\beta} + 8e^{-3\beta}\right)\left(1 + 6e^{-\beta} + 12e^{-2\beta} + 8e^{-3\beta}\right)$$

Any combination of two terms will yield a different distinct possibility. Therefore, it is possible to express the answer in terms of

$$Z \equiv \left(1 + 6e^{-\beta} + 12e^{-2\beta} + 8e^{-3\beta}\right)$$

but we will not be doing that here, time to bash away!

(ii) If two particles are identical bosons, they can take on the same energy, except we have to adjust for overcounting. With no restrictions:

- Energy 0: 1 way.
- Energy 1: $1 \cdot 6 = 6$ way.
- Energy 2: $1 \cdot 12 + \binom{6}{2} + 6 = 33$ ways.
- Energy 3: $1 \cdot 8 + 6 \cdot 12 = 80$ ways.
- Energy 4: $6 \cdot 8 + \left(\binom{12}{2} + 12\right) = 126$ ways.
- Energy 5: $12 \cdot 8 = 96$ ways.
- Energy 6: $\binom{8}{2} + 8 = 36$ ways.

With restrictions:

- Energy 2: $\binom{6}{2} + 6 = 21$ ways.
- Energy 3: $6 \cdot 12 = 72$ ways.
- Energy 4: $6 \cdot 8 + \left(\binom{12}{2} + 12\right) = 126$ ways.
- Energy 5: $12 \cdot 8 = 96$ ways.
- Energy 6: $\binom{8}{2} + 8 = 36$ ways.

So:

$$P_2 = 1 - \frac{21e^{-2\beta} + 72e^{-3\beta} + 126e^{-4\beta} + 96e^{-5\beta} + 36e^{-6\beta}}{1 + 6e^{-\beta} + 33e^{-2\beta} + 80e^{-3\beta} + 126e^{-4\beta} + 96e^{-5\beta} + 36e^{-6\beta}}$$

(iii) If two particles are identical fermions, they can take on the same energy (except ground state), except they can't be in the same state.

- Energy 1: $1 \cdot 6 = 6$ ways.
- Energy 2: $1 \cdot 12 + 1 \cdot \binom{6}{2} = 27$ ways.
- Energy 3: $1 \cdot 8 + 6 \cdot 12 = 80$ ways.
- Energy 4: $6 \cdot 8 + \left(\binom{12}{2}\right) = 114$ ways.
- Energy 5: $12 \cdot 8 = 96$ ways.

- Energy 6: $\binom{8}{2} = 28$ ways.

With restrictions:

- Energy 2: $\cdot\binom{6}{2} = 15$ ways.
- Energy 3: $6 \cdot 12 = 72$ ways.
- Energy 4: $6 \cdot 8 + \left(\binom{12}{2}\right) = 114$ ways.
- Energy 5: $12 \cdot 8 = 96$ ways.
- Energy 6: $\binom{8}{2} = 28$ ways.

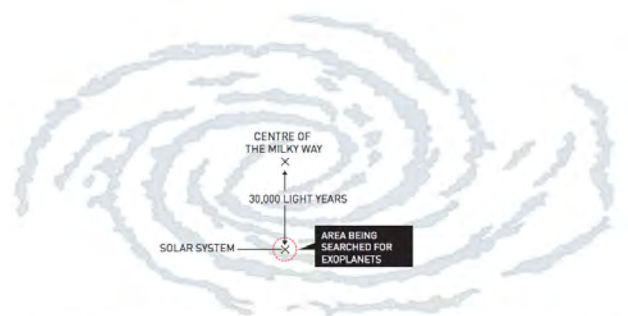
$$P_3 = 1 - \frac{15e^{-2\beta} + 72e^{-3\beta} + 114e^{-4\beta} + 96e^{-5\beta} + 28e^{-6\beta}}{6e^{-\beta} + 27e^{-2\beta} + 80e^{-3\beta} + 114e^{-4\beta} + 96e^{-5\beta} + 28e^{-6\beta}}$$

(d) For high temperature: fermion ¿ distinct ¿ boson. As well as including the math, justification needs to be on the lines of every state having an equal probability, and thus we want the ratio between desired and total states.

For low temperatures, it's reversed. Particles will tend to settle in the lowest energy state (which is what we desire), and distinct particles have more ways to do it than identical bosons. Fermions cannot have two particles in the same state, further decreasing its chances.

Problem 8: Radiation (40 pts)

The Nobel Prize in Physics 2019 was awarded for providing a new understanding of the universe's structure and history, and the first discovery of a planet orbiting a solar-type star outside our solar system.



The Sun is one of several hundred billion stars in our home galaxy, the Milky Way, and there should be planets orbiting most of those stars. So far, astronomers have discovered over 4,000 planets around other stars and they are continuing their search in the area of space closest to us.

Since ancient times, humans have speculated whether there are worlds like our own, with points of views at the extremes expressed thousands of years ago. In modern times, the possibility of observing planets orbiting stars other than the Sun was proposed more than 50 years ago, and has grown into a vast and ever-expanding theory to make the evolution of the universe more clear to us than ever before. In 1995, the very first discovery of a planet outside our solar system, an exoplanet, orbiting a solar-type star was made. This discovery challenged our ideas about these strange worlds and led to a revolution in astronomy. The more than 4,000 known exoplanets are surprising in their richness of forms, as most of these planetary systems look nothing like our own, with the Sun and its planets. These discoveries have led researchers to develop new theories about the physical processes responsible for the birth of planets.

(Taken from the Nobel Prize in Physics 2019 summary, and the Laureates' popular science and scientific views.)

In this problem, we analyse and create a model for a system of two fictitious celestial bodies: an exoplanet and a solar-type star. Unless specified otherwise, consider the two bodies to be solely in each other's gravitational influence and rotate about their barycentre. In the three parts that follow, we will model the physics of a star, of the star-planet model, and the planet respectively.

Part A

The star, with mass $M_s = 2M_\odot$ (twice the mass of our Sun) and radius $R_\odot$ uses nuclear fusion reactions to provide pressure against gravity and electron degeneracy pressure, so as to maintain hydrostatic equilibrium in the star. As long as the hydrostatic equilibrium is preserved, the star is said to be in "main sequence". However, once the energy from the reactions taking place in its core start running out, the star's outer layers swell out to form a red giant. The core of the star (having a radius R_c) starts to shrink, becoming hot and dense; the temperature of the core rises to over a 100 billion degrees, and the pressure from the proton-proton interactions in the core exceeds that of gravity, causing the core to recoil out from the heart of the star in an explosive shock wave. In one of the most spectacular events in the Universe, the shock propels the material away from the star in a tremendous explosion called a supernova. The material spews off into interstellar space.

Being solar-type, this star has the same proton-proton nuclear fusion chain reaction as our Sun: essentially, this is conversion of four protons (mass of a proton is m_P) into 1 He nucleus having mass m_{He}. The star is said to have a "stable lifetime" as long as it is in its "main sequence". The energy emitted by the star passing a sphere of radius r per unit time is $P(r)$, constant over time and the surface of the imaginary sphere of radius r. The density of the exoplanet having radius r_{E} is a constant, ρ, and it orbits around the star in a circular orbit of radius r_{SE}. Neglect any convection effects in the star.

(a) **(4 pts)** Treating the solar-type star as a perfect black-body, estimate the temperature of the surface of the star $T_\odot$ (assumed in thermal equilibrium) by integrating over all frequencies using Planck's distribution for the energy density (defined as the energy per unit volume for a given frequency interval (ν, $\nu + \mathrm{d}\nu$):

$$u(\nu)\mathrm{d}\nu = \frac{8\pi h \nu^3}{c^3} \frac{\mathrm{d}\nu}{(e^{\frac{h\nu}{kT_\odot}} - 1)}$$

where the constants h and k have their usual meanings. For this part only, note that the energy flux from the star onto the exoplanet is J_0. You can use $T_\odot$ as the surface temperature in the later parts.

(b) **(8 pts)** Estimate an expression for, during the main sequence of the star:

(i) the number of protons being fused together per second.

(ii) the stable lifetime of the star, assuming $\eta =$ 1% mass of the star can undergo nuclear fusion. The change in the temperature or size of the star is insignificant. Assume the only fusion is between protons.

(c) **(8 pts)** Find the temperature gradient $\mathrm{d}T(r)/\mathrm{d}r$ of the star as a function of the radial distance r from the center, such that $R_c \le r \le R_\odot$ if the star is in its main sequence, or in a hydrostatic equilibrium. Neglect any quantum-mechanical pressure effects such as electron degeneracy pressure, and assume that pressure from electromagnetic radiation is much larger than any other pressure. State all assumptions.

(d) **(1 pts)** What is the temperature at $r = R_c$, the outermost layer of the core?

(e) **(2 pt)** From experiments, it was found that the temperature gradient of the star is actually

$$\frac{\mathrm{d}T(r)}{\mathrm{d}r} = -\frac{9kGM_\odot cP(r)}{128\pi^2\sigma r^2 T^3}$$

Here the modulus of k is one, and has appropriate dimensions. For what value of $P(R_\odot)$ (P_r evaluated at the surface of the star) will the star's main sequence end, leading to the formation of a supernova?

Part B

In this part, we will analyse the radiation effects from the star onto the exoplanet. Assume only black body radiation from the star on the exoplanet. No light is absorbed in the region between the star's and the exoplanet's surface.

(f) **(5 pts)** The distance between the star and the exoplanet is r_{SE}. For this part, assume the surface of the exoplanet has a constant and uniform reflectance γ. What is the force exerted by the radiation from the star on the exoplanet? For the exoplanet's gravitational force to completely balance out the radiation force, how large must the radius of the exoplanet r_E be? Comment on your results and their feasibility.

Part C

(g) **(2 pt)** Find the temperature T_E of the outermost surface of the planet, assumed constant over the whole surface from (a). Assume the planet's surface to be a perfect black body.

(h) **(10 pts)** Model the exoplanet to be made up of N concentric shells equally spaced across the volume of the planet. Between the shells is a peculiar kind of thick type of tectonic rocks which allow no emission, reflection or absorption of energy. However, absorption or emission of radiation energy may take place. The emissivity of all the shells are the same, and are equal to ε, constant and uniform over a surface. Reflection, emission and absorption of any energy due to radiation from the shells, however, may take place. Assume all conduction and convection effects to also be negligible. The temperature of the exoplanet as a function of r is represented by

$$T(r) = T_0\left(1 - \frac{n}{10N}\right)$$

where n is the n^{th} shell from the centre of the planet and T_0 is an appropriate constant as calculated from the previous part (which is unknown, meaning that you need to answer in any variables calculated before). Calculate the total thermal energy due to radiation falling on the outermost shell per unit time. The planet is maintained in a state of thermal equilibrium; this is done by an atmospheric material that allows a fraction (β, which is unknown) of energy from the star falling on the exoplanet. This material only absorbs a fraction of energy it receives from the star. Do NOT assume any such effects for any of the other parts, since they are meant to be crude estimates of the actual calculation. Also compute β.

Solution 8:

(a) Using Planck's distribution,

$$u(\nu) = \int_0^\infty \frac{8\pi h\nu^3}{c^3}\frac{\mathrm{d}\nu}{(e^{\frac{h\nu}{kT_\odot}} - 1)}$$

This integration is well known and equal to $\zeta(4) = \pi^4/15$.

Returning to our problem,

$$u(\nu) = \frac{8\pi^5 k {T_\odot}^4}{15h^3c^3}$$

The energy flux onto the exoplanet is

$$J_0 = cu\left(\frac{\Omega}{4\pi}\right)$$

where $\Omega = \frac{\pi {R_\odot}^2}{r_{\mathrm{SE}}^2}$. So

$$J_0 = c \times \frac{8\pi^5 k {T_\odot}^4}{15h^3c^3} \times \frac{\pi {R_\odot}^2}{4\pi r_{\mathrm{SE}}^2} \Rightarrow \boxed{T_\odot = \left(\frac{60 {J_0}^3 c^2 {r_{\mathrm{SE}}}^2}{8\pi^5 k {R_\odot}^2}\right)^{\frac{1}{4}}}$$

We get the same result by applying Stefan-Boltzmann's radiation law for a black body, since this is how the law is derived.

(b) (i) In a single nuclear fusion reaction, four protons fuse into one helium nucleus. Let $\mathcal{N}$ be the rate of such fusion reactions (per second), and $P(R_\odot) = 4\pi\sigma {R_\odot}^2 {T_\odot}^4$ be the rate of total energy emitted by the star at its surface. By Einstein's law of mass energy conservation, we have

$$P(R_\odot) = \mathcal{N}(4m_P - m_{\mathrm{He}})c^2 = 4\pi\sigma {R_\odot}^2 {T_\odot}^4$$

which gives

$$\boxed{\mathcal{N} = \frac{4\pi\sigma {R_\odot}^2 {T_\odot}^4}{(4m_P - m_{\mathrm{He}})c^2}}$$

(ii) Let the stable lifetime of the star be T_0. The total number of fusion reactions in the lifetime is then (roughly) $\mathcal{N} \times T_0$, and we have the following expression by substituting $\mathcal{N}$ from (i):

$$\mathcal{N} \times T_0 = \frac{2M_\odot \eta}{4m_p} \Rightarrow \boxed{T_0 = \frac{2M_\odot \eta c^2}{4\pi\sigma {R_\odot}^2 {T_\odot}^4}\left[1 - \frac{m_{\mathrm{He}}}{4m_p}\right]}$$

(c) Consider a circular strip between radial distances r and $r + \mathrm{d}r$ from the centre of the star. For hydrostatic equilibrium to be established, the net force on this strip must be zero, or the pressure forces and gravitational forces on the layer must exactly balance. Let $m(r)$ and $\rho(r)$ be the mass and density of a sphere of radius r respectively. A pressure force of $P(r)$ on this sphere. We have from force balance:

$$-\frac{Gm(r)\rho(r)4\pi r^2 \mathrm{d}r}{r^2} = [P(r + \mathrm{d}r) - P(r)]4\pi r^2$$

$$\Rightarrow \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}$$

Note that radiation pressure can be written as $P_r = \frac{4\sigma T^4}{3c}$ by Stefan Boltzmann's law. Using this with the pressure gradient equation, we can write the temperature gradient of the star as:

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{3c}{16\sigma T^3}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{3Gm(r)\rho(r)c}{16\sigma T^3 r^2} = \boxed{-\frac{9G{M_\odot}^2 rc}{16\pi\sigma {R_\odot}^6 T^3}}$$

(d) To find the temperature at any point on the star, we solve the gradient equation obtained in (c). Applying the boundary condition $T(R_\odot) = T_\odot$ and separating variables, we have

$$T(r) = \left[T_\odot^4 + \frac{9GM_\odot^2 c}{16\pi\sigma R_\odot^6}(R_\odot^2 - r^2)\right]^{\frac{1}{4}}$$

At $r = R_c$, we have

$$\boxed{T(R_c) = \left[T_\odot^4 + \frac{9GM_\odot^2 c}{16\pi\sigma R_\odot^6}(R_\odot^2 - R_c^2)\right]^{\frac{1}{4}}}$$

(e) From the problem statement, we know that the experiments suggest the following:

$$\frac{\mathrm{d}T(r)}{\mathrm{d}r} = -\frac{9kGM_\odot cP(r)}{128\pi^2\sigma r^2 T^3}$$

The main sequence of the star lasts as long as pressure from the gravitational force is greater than the net radiation pressure.

Comment (not for grading purposes):

Throughout the problem, density of the star is assumed constant and equal to $\rho(r) = \frac{6M_\odot}{4\pi {R_\odot}^3}$.

Part (a): The integration was required to be carried out in the examination. A well known method to do this is to split the integral using by parts and express it as the limit of a sum.

Part (c): It is assumed that the pressure balancing the gravitational pressure arises from radiation. There is pressure due to the gas/fuel in the star and a few quantum mechanical effects, but they can be neglected for the sake of this problem.

Problem 9: Piston Gun (40 pts)

In this problem, we examine a model for a certain type of gun that works by using the expansion of a gas to propel a bullet. We can model the bullet as a piston. Since we are assuming atmospheric pressure is negligible, we can assume that the whole setup is in a vacuum. Also, the gun is insulated.

An ideal monatomic gas of initial temperature T_0 is inside a long cylindrical container of cross-section area A. One side of the container is a wall, while the other side is a piston of mass M that can slide freely along the container without friction. The total mass of the gas is m, and it is made up of N particles. Initially, the piston is at rest and a distance L_0 away from the opposite wall. Then, the piston is released. After a time t, the piston moves at a speed v. Assume that throughout the process, the particles on average move very fast.

(a) **(5 pts)** Assume that m is negligible. Find v.

(b) **(6 pts)** From now on, do not assume that m is negligible.

Find the time at which the pressure at the wall opposite the piston changes. Also, does it increase or decrease? State all assumptions.

(c) **(14 pts)** From now on, assume t is much smaller than the mean free time of the particles of the gas, and L_0 is much smaller than the mean free path. (During this time interval t, assume that all the particles still collide many, many times with the walls, but they don't collide with each other.) Find v.

(d) **(6 pts)** Find the recoil impulse of the gun over the time t.

(e) **(9 pts)** Let $r > 1$ be a dimensionless parameter. Suppose at time t, the piston is a distance rL_0 away from the wall; then the piston is stopped, and the gas is allowed to come to equilibrium (after a time much greater than the mean free time). Find the total entropy change (throughout the whole process) of the gas in terms of r, and verify the Second Law of Thermodynamics.

Solution 9:

(a) Since $m \ll M$, the expansion of the gas can be treated as a reversible process, because pressure, density, etc. will be uniformly distributed. The process is also adiabatic because the container is insulated. If x is the position of the piston, we have px^γ is constant, where p is the pressure and $\gamma = \frac{C_p}{C_v} = \frac{5}{3}$ for monatomic gas. Also, Newton's 2nd Law gives $pA = M\ddot{x}$. We have $\ddot{x} = \frac{p_0 A L_0^\gamma}{M x^\gamma}$, where $p_0 = \frac{NkT_0}{AL_0}$. To find v, we need to solve this differential equation and plug in the boundary conditions of $x(0) = L_0$ and $\dot{x}(0) = 0$. Since the final answer is very complicated, we have decided to award (almost) full credit for those who end up with correct differential equation and boundary conditions (and have accurate, thorough reasoning).

(b) Now, we cannot neglect m anymore, so the pressure will not be uniform throughout the gas. In fact, a pressure wave will start at the piston and travel to the opposite wall at speed $c = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{5NkT_0}{3m}}$.

The time it takes is $\frac{L_0}{c} = \boxed{L_0\sqrt{\frac{3m}{5NkT_0}}}$.

(c) Let $L(t)$ be the position of the piston as a function of time, where $L(0) = L_0$. Let $v = \dot{L}$. Let the x direction be along the cylinder's axis. Consider a particle in the gas with speed $v_x(t)$ in the x direction. We will show that Lv_x is a conserved quantity. Since the mean free time is large, we can assume the particles rarely collide with each other. Since the gas is monatomic, the particle makes elastic collisions with the walls. When it collides with a stationary wall, the particle's speed stays the same. However, if it moves at speed v_x and collides with the piston moving at speed v, it rebounds with a new speed $v_x - 2v$, which can be seen by shifting to the piston's frame and noting that the particle's mass is much smaller than the piston's. Thus, its decrease in speed is $2v$. Since it is moving very fast, on average, it loses speed $2v$ every time it takes to go back and forth, which is $\frac{2L}{v_x}$. We can write $\frac{dv_x}{dt} = -\frac{2v}{\frac{2L}{v_x}}$. This simplifies to $L\dot{v_x} + vv_x = 0$. Integrating, we get $Lv_x = c$ where c is an integration constant. Plugging in initial conditions, we have $Lv_x = L_0 v_{x0}$, where v_{x0} is the initial x-velocity of the particle.

Next, we write the conservation of energy equation after time t. For simplicity, we redefine m as the mass of each particle.

$$\int_{-\infty}^{\infty} \frac{1}{2} Nm(v_{x0}^2 - v_x^2) f(v_{x0})\, \mathrm{d}v_{x0} = \frac{1}{2} M v^2$$

where $f(v_{x0}) = \sqrt{\frac{m}{2\pi k T_0}} e^{-\frac{mv_{x0}^2}{2kT_0}}$ is the Maxwell-Boltzmann distribution in the x direction. Using the fact that $Lv_x = L_0 v_{x0}$ and substituting for f, we get

$$\int_{-\infty}^{\infty} Nmv_{x0}^2 (1 - \frac{L_0^2}{L^2}) \sqrt{\frac{m}{2\pi k T_0}} e^{-\frac{mv_{x0}^2}{2kT_0}}\, \mathrm{d}v_{x0} = Mv^2$$

Integrating and simplifying, we get

$$\dot{L} = \sqrt{\frac{NkT_0}{M}} \sqrt{1 - \frac{L_0^2}{L^2}}$$

$$\frac{L}{\sqrt{L^2 - L_0^2}} \mathrm{d}L = \sqrt{\frac{NkT_0}{M}} \mathrm{d}t$$

Integrating and plugging in initial conditions gives

$$\sqrt{L^2 - L_0^2} = \sqrt{\frac{NkT_0}{M}} t$$

$$L = \sqrt{L_0^2 + \frac{NkT_0}{M} t^2}$$

Finally, we have $\boxed{v = \frac{NkT_0 t}{\sqrt{M^2 L_0^2 + NkT_0 M t^2}}}$

(d) The recoil impulse is simply the momentum of the gas + piston. The center of mass of the gas moves at half the speed of the piston, so we have the total momentum $Mv + \frac{mv}{2} = \boxed{\left(M + \frac{m}{2}\right) \frac{NkT_0 t}{\sqrt{M^2 L_0^2 + NkT_0 M t^2}}}$

(e) After a long time, the pressure will again become uniform, and we can use the formula for entropy of an ideal gas: $Nk \ln PV_0^{\gamma}$. To do this, we need to first realize that when the piston is stopped, $\frac{1}{2}Mv^2$ of energy is lost, and combining this fact with the ideal gas law gives us enough information to solve for the final pressure of the gas. After calculation, we get $\Delta S = \boxed{Nk \left(\frac{3}{2} \ln \left(\frac{2}{3} + \frac{1}{3r^2}\right) + \ln r\right)}$. Note that there are no surroundings (vacuum). Also, it is easy to see ΔS is positive if one finds that the derivative with respect to r is positive for $r > 1$. Thus, the Second Law of Thermodynamics is verified.

Problem 10: Magnetostatics (62 pts)

Part A

In 3-D space, a permeable medium covers the region $x > 0$, while the rest of the space is vacuum. The medium's relative magnetic permeability is $\mu_r > 1$. A magnetic dipole with dipole moment m is placed a distance d away from the permeable medium, at position $(-d, 0, 0)$. The dipole is pointed towards the +x direction. Treat the dipole as ideal (point-sized).

(a) **(10 pts)** Find the force required to keep the dipole in place.

(b) **(3 pts)** How much work does it take to slowly pull the dipole from its original position to infinity (at $x = -\infty$)?

(c) **(5 pts)** How much work does it take to slowly rotate the dipole from its original orientation to one that makes an angle θ with the +x-axis?

After the dipole is rotated an angle θ, a superconducting ring with radius R and self-inductance L is brought in from infinity (with initially no current). It is placed so that the dipole is located at its center and its axis is the x-axis. Assume that $R \gg d$.

(d) **(16 pts)** Find the current I in the ring.

(e) **(8 pts)** Find the force required to hold the dipole in place (not the torque).

Part B

From now on, there is no permeable medium. Ignore any radiation loss for all parts.

(f) **(7 pts)** The dipole (mass M), starts at a distance h from the centre of the ring (kept fixed) and pointed towards the centre of the ring (along its axis), and is projected with a small velocity v_0 towards the centre. Find its speed v as a function of h. Ignore gravity.

(g) **(7 pts)** Consider another scenario, in which the dipole is placed on the axis of a thin infinite magnetic tube with surface conductivity σ (defined as the ratio of surface current density and the electric field) and radius R, placed at an arbitrary location inside it. (You may neglect the self inductance of the solenoid for the sake of this part.) We find that the motion of the dipole in this case is damped. Find the damping parameter of this motion. (Damping parameter is defined as the ratio of the resistive force to the speed.) Ignore gravity.

(h) **(6 pts)** Determine the terminal velocity of the magnet, assuming that it now falls under gravity. The tube may be considered infinitely long for all calculation purposes in this part.

Solution 10: For permeable media, two of Maxwell's equations become

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{free}}$$

where $\mathbf{H} = \mathbf{B}/\mu$ and $\mathbf{J}_{free} = 0$ because there is no free current. Note that $\mu = \mu_r \mu_0$. Thus, the equations become $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{H} = 0$. Applying these to the boundary, we get the following boundary conditions:

$$B_{1,\perp} = B_{2,\perp} \tag{1}$$
$$H_{1,\parallel} = H_{2,\parallel} \tag{2}$$

where region 1 is $x < 0$ and region 2 is $x > 0$. We claim that if a magnetic monopole of magnetic charge q is placed a distance d away from the permeable medium instead of a dipole,
(1) in the region $x < 0$, the magnetic field is equivalent to that of the original monopole and a magnetic monopole placed at $(d, 0, 0)$ with magnetic charge $q_1 = -\frac{\mu_r - 1}{\mu_r + 1} q_0$ (2) in the region $x > 0$, the magnetic field is equivalent to that of the original monopole at its current position but with charge $q_2 = \frac{2\mu_r}{\mu_r + 1} q_0$.
To show this, note that as long as we show that the boundary conditions are satisfied, we are done by the uniqueness theorem because the field in the rest of the space satisfies Laplace's equation. For the region

$x < 0$, the boundary at the real magnetic monopole is already satisfied. Now, the only boundary we need to consider is the interface $x = 0$. Consider a point P on the plane $x = 0$ such that the line from $(-d, 0, 0)$ to P forms an angle θ with the x-axis. Define B_i to be the magnitude of the field at P from q_i. According to the claim, the magnetic field near P in region 1 has a magnetic field component perpendicular to the interface of $B_{1,\perp} = B_0 \cos\theta - B_1 \cos\theta$. This must be equal to the perpendicular component in region 2, so we have

$$B_0 \cos\theta - B_1 \cos\theta = B_2 \cos\theta \tag{3}$$

Now, the parallel component of $\mathbf{H}$ in region 1 is $\frac{B_0}{\mu_0}\sin\theta + \frac{B_1}{\mu_0}\sin\theta$, which must be equal to the parallel component of $\mathbf{H}$ in region 2, so

$$\frac{B_0}{\mu_0}\sin\theta + \frac{B_1}{\mu_0}\sin\theta = \frac{B_2}{\mu}\sin\theta \tag{4}$$

Solving the two equations gives

$$B_1 = -\frac{\mu_r - 1}{\mu_r + 1} B_0 \tag{5}$$

$$B_2 = \frac{2\mu_r}{\mu_r + 1} B_0 \tag{6}$$

Noting that $B_i \propto q_i$, we see that the claim is true.
We apply the claim to the problem via superposition. We treat the magnetic dipole as two magnetic monopoles of magnetic charges q and $-q$ that are a small distance s away from each other. Note that $m = qs$.

(a) If the dipole is at $(-d, 0, 0)$ and pointed towards the +x direction, the field in region 1 is equivalent to that of the same dipole superposed with another dipole at $(d, 0, 0)$ pointing in the +x direction with moment $m_1 = q_1 s = \frac{\mu_r - 1}{\mu_r + 1} m$. The field produced by the latter magnetic dipole at position of the original dipole is given by $\frac{\mu_0 m_1}{2\pi r^3}$, where $r = 2d$. The magnetic force on the original dipole is given by

$$F = m\frac{\partial B_x}{\partial x} = \frac{3\mu_0 m m_1}{2\pi r^4} = \frac{3\mu_0 m m_1}{32\pi d^4} = \boxed{\frac{3\mu_0(\mu_r - 1)m^2}{32\pi(\mu_r + 1)d^4}} \tag{7}$$

Note that the force is attractive.

(b) We integrate the force from (a) to get the work done:

$$W = \int_d^\infty \frac{3\mu_0(\mu_r - 1)m^2}{32\pi(\mu_r + 1)x^4}\,\mathrm{d}x = \boxed{\frac{\mu_0(\mu_r - 1)m^2}{32\pi(\mu_r + 1)d^3}} \tag{8}$$

(c) Instead of rotating the dipole from the original orientation, we first bring the dipole to infinity, rotate it, and then bring it back to $(-d, 0, 0)$. From part (b), we have the work it takes to bring the dipole to infinity. Rotating it at infinity requires no work. Finally, to calculate the amount of work required to bring it back, we find the potential energy of the final system. From our claim, we can apply superposition to see that the setup is equivalent to one in which the permeable medium acts like a magnetic dipole at $(d, 0, 0)$ with moment $m_1 = \frac{\mu_r - 1}{\mu_r + 1}$ and pointed in a direction that makes an angle $-\theta$ with the +x-axis. The general formula for dipole-dipole interaction is given by:

$$U = \frac{\mu_0}{4\pi r^3}(\mathbf{m}_1 \cdot \mathbf{m}_2 - 3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}})) \tag{9}$$

which equals

$$U = \frac{\mu_0}{4\pi(2d)^3}(m m_1 \cos 2\theta - 3 m m_1 \cos^2\theta) = -\frac{\mu_0 m m_1}{32\pi d^3}(\cos^2\theta + 1) \tag{10}$$

However, this is not the work we do to bring it back from infinity; rather, it is twice our work. To see why, imagine an external agent bringing in the imaginary dipole from $+\infty$ as we are bringing in the real dipole from $-\infty$. By Newton's third law, the force they apply is equal and opposite to ours, and so they do the same amount of work as we do. Thus, the work it takes to bring in the dipole from $-\infty$ is

$$W = -\frac{\mu_0 m m_1}{64\pi d^3}(\cos^2\theta + 1) \tag{11}$$

In total, the amount of work done is

$$W = \boxed{\frac{\mu_0(\mu_r - 1)m^2\sin^2\theta}{64\pi(\mu_r + 1)d^3}} \tag{12}$$

(d) By similar reasoning from the previous parts, we can see that the image of the ring with current I will be a ring centered at $x = d$ with current $\frac{\mu_r - 1}{\mu_r + 1}I$ in the same direction.

Claim 1: The mutual inductance between the real ring and the image dipole is given by $\frac{\mu_0 R^2\cos\theta}{2(R^2+4d^2)^{\frac{3}{2}}} * \frac{\mu_r - 1}{\mu_r + 1}A$, where A is the area of the dipole.

Proof: By the mutual inductance reciprocity theorem ($M_{12} = M_{21}$), we just need to calculate the flux through the dipole over a current I in the ring. Since the dipole is ideal, we can find the field at the dipole and do $\Phi = BA\cos\theta$. After plugging in well-known expression $B = \frac{\mu_0 R^2}{2(R^2+x^2)^{\frac{3}{2}}}$, where $x = 2d$, we get the desired answer.

A natural corollary of this claim is that the mutual inductance between the real ring and the real dipole is $\frac{\mu_0\cos\theta}{2R}m$ **Claim 2**: The mutual inductance between the real ring and the image ring is given by $\mu_0 R\ln\frac{R}{d}$. **Proof**: Consider the field lines from the ring. Since $d << R$, the field lines near the edge of the ring are circular, so the flux through the image ring from the real ring is the same as the flux through a circle coplanar with the real ring with radius $R - d$. This is because of the fact that magnetic field has zero divergence. To find the flux through the circle of radius $R - d$, we use the fact

$$\iint_S \mathbf{B}\cdot d\mathbf{S} = \oint_C \mathbf{A}\cdot d\mathbf{l}$$

where $\mathbf{A}$ is the vector potential. By rotational symmetry, we only need to calculate the tangential component of $\mathbf{A}$ at a point $R - d$ from the center, and then multiply by $2\pi(R - d)$. We have

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{Id\mathbf{l}}{r} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{R\cos\theta d\theta}{\sqrt{R^2 + (R-d)^2 - 2R(R-d)\cos\theta}}\hat{\theta}$$

We simplify to get

$$A_\theta = \frac{\mu_0 IR}{2\pi\sqrt{R(R-d)}}\int_0^\pi\frac{\cos\theta}{\sqrt{2 + \frac{d^2}{R^2} - 2\cos\theta}}d\theta \approx \frac{\mu_0 I}{2\pi}\int_0^\pi\frac{\cos\theta}{\sqrt{2 + \frac{d^2}{R^2} - 2\cos\theta}}d\theta$$

Now, we realize that the integral diverges if $d = 0$. Thus, since $d << R$, we see that the integral is dominated by the region where θ is close to zero. We can use the small angle approximation $\cos\theta \approx 1 - \frac{\theta^2}{2}$. We simplify to get

$$A \approx \frac{\mu_0 I}{2\pi}\int_0^\pi\frac{1}{\sqrt{\theta^2 + \frac{d^2}{R^2}}}d\theta = \frac{\mu_0 I}{2\pi}\ln\left(\sqrt{(\theta * \frac{R}{d})^2 + 1} + \theta * \frac{R}{d}\right)\Big|_0^\pi \approx \frac{\mu_0 I}{2\pi}\ln\frac{R}{d}$$

Finally, we multiply by $2\pi R$ and divide by I to get the desired mutual inductance. Note: the result can also be achieved using Grover's formula.

The flux through the superconducting ring must remain zero. The flux contributed by the ring itself is simply LI. The flux contributed by each dipole is the mutual inductance over the area times the dipole moment. The flux contributed by the image ring is the mutual inductance times the current in the image ring. Combining all the terms and setting it to zero gives: $LI - \frac{\mu_0 \cos\theta}{2R}m - \frac{\mu_0 R^2 \cos\theta}{2(R^2+4d^2)^{\frac{3}{2}}} * \frac{\mu_r-1}{\mu_r+1}m + \mu_0 R \ln\frac{R}{d} * \frac{\mu_r-1}{\mu_r+1}I = 0$ Solving, we get $\boxed{I = \frac{\mu_0 m \cos\theta}{2} \cdot \frac{\frac{1}{R} - \frac{\mu_r-1}{\mu_r+1} * \frac{R^2}{(R^2+4d^2)^{\frac{3}{2}}}}{L + \mu_0 R\frac{\mu_r-1}{\mu_r+1}\ln\frac{R}{d}}}$

(e) The force on a dipole is given by $\mathbf{F} = \nabla(\mathbf{p}\cdot\mathbf{B})$. We can break this force up into components. The x-component is given by $F_x = p_x\frac{\partial B_x}{\partial x} + p_y\frac{\partial B_x}{\partial y}$, where the dipole is in the x-y plane. Note that the ring contributes no gradient of $\mathbf{B}$ field at the dipole, so we only need to consider the field from the image ring and image dipole. First we find the force on the dipole due to the image ring. The field from the image ring is given by $B = \frac{\mu_0 I_1 R^2}{2(R^2+(2d)^2)^{\frac{3}{2}}}$, so the gradient in the x direction is $\frac{3\mu_0 I_1 R^2 d}{(R^2+(2d)^2)^{\frac{5}{2}}}$. The x-component of the field from the image ring doesn't change significantly in the y-direction, by symmetry, so we get $\frac{\partial B_x}{\partial y} = 0$.

Now we consider the y-component $F_y = p_x\frac{\partial B_y}{\partial x} + p_y\frac{\partial B_y}{\partial y}$. The y-component of the field from the image ring along its axis is 0, so $\frac{\partial B_y}{\partial x} = 0$. To find $\frac{\partial B_y}{\partial y}$, we use the fact that $\nabla\cdot\mathbf{B} = 0$, so we get $\frac{\partial B_x}{\partial x} + 2\frac{\partial B_y}{\partial y} = 0$, where here we are only considering the field from the image ring. Thus, $\frac{\partial B_y}{\partial y} = -\frac{\partial B_x}{2\partial x} = -\frac{3\mu_0 I_1 R^2 d}{2(R^2+(2d)^2)^{\frac{5}{2}}}$.

It remains to find the force between the image dipole and the real dipole. The dipole-dipole interaction is given by equation (9). To find the x-component of the force F_x, we can calculate the derivative of the potential energy in the x direction. Moving the dipole in the x direction only changes r, so we have $F_x = \frac{dU}{dr} = -\frac{3\mu_0}{4\pi r^4}(mm_1\cos(2\theta) - 3mm_1\cos^2\theta) = \frac{3\mu_0}{4\pi(2d)^4}mm_1(\cos^2\theta + 1) = \frac{3\mu_0}{64\pi d^4}mm_1(\cos^2\theta+1)$.

To find the y-component of the force F_y, we can calculate the derivative of the potential energy in the y direction. Moving the dipole in the y direction does not change r to first order, but does change $\hat{\mathbf{r}}$. Thus, $F_y = -\frac{dU}{dy} = -\frac{3\mu_0}{4\pi r^4}(2mm_1\cos\theta\sin\theta) = -\frac{3\mu_0}{32\pi d^4}mm_1\cos\theta\sin\theta$.

Finally, we add up all the forces to get

$$\boxed{\mathbf{F} = \mu_0\frac{\mu_r-1}{\mu_r+1}m((-\frac{3IR^2 d\cos\theta}{(R^2+4d^2)^{\frac{5}{2}}} + \frac{3}{64\pi d^4}m(\cos^2\theta+1))\hat{\mathbf{x}} + (\frac{3IR^2 d\sin\theta}{2(R^2+4d^2)^{\frac{5}{2}}} - \frac{3}{32\pi d^4}m\cos\theta\sin\theta)\hat{\mathbf{y}})}$$

, where I is given in part (d).

2021 Online Physics Olympiad: Open Contest (Version 1.1)

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Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

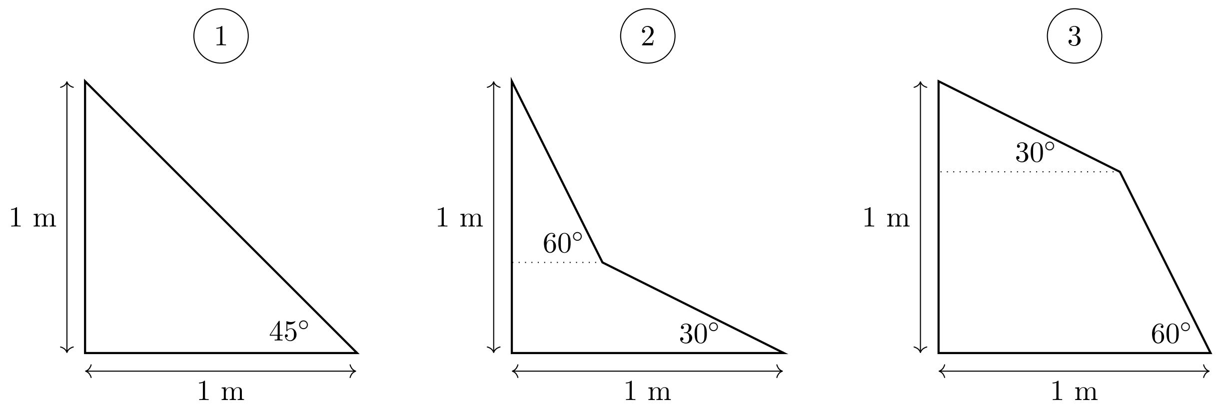
- Use $g = 9.81\ \mathrm{m/s^2}$ in this contest. See the constants sheet on the following page for other constants.
- This test contains 35 short answer questions. Each problem will have three possible attempts.
- The weight of each question depends on our scoring system found here. Put simply, the later questions are worth more, and the overall amount of points from a certain question decreases with the number of attempts that you take to solve a problem as well as the number of teams who solve it. **Note:** Unlike last year, the time it takes will no longer be a factor.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should contain **three** significant figures, unless otherwise specified. All answers within the 1% range will be accepted.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used. Technology and computer algebra systems like Wolfram Alpha or the one in the TI nSpire will not be needed or allowed. Attempts to use these tools will be classified as cheating.
- You are *allowed* to use Wikipedia or books in this exam. Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions.
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in SI units (meter, second, kilogram, watt, etc.) unless otherwise specified. Please input all angles in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put units in your answer on the submission portal! If your answer is "x meters", input only the value x into the submission portal.
- **Do not communicate information to anyone else apart from your team-members before the exam ends on June 6, 2021 at 11:59 PM UTC**.

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^8$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\mathrm{N} \cdot \mathrm{m}^2)/\mathrm{kg}^2$$
- Acceleration due to gravity, $g = 9.81$ m/s^2
- 1 unified atomic mass unit,
$$1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15}\ \mathrm{eV \cdot s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9\ (\mathrm{N \cdot m^2})/\mathrm{C^2}$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T \cdot m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\mathrm{T \cdot m})/\mathrm{A}$$
- 1 atmospheric pressure,
$$1\ \mathrm{atm} = 1.01 \cdot 10^5\ \mathrm{N/m^2} = 1.01 \cdot 10^5\ \mathrm{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$$

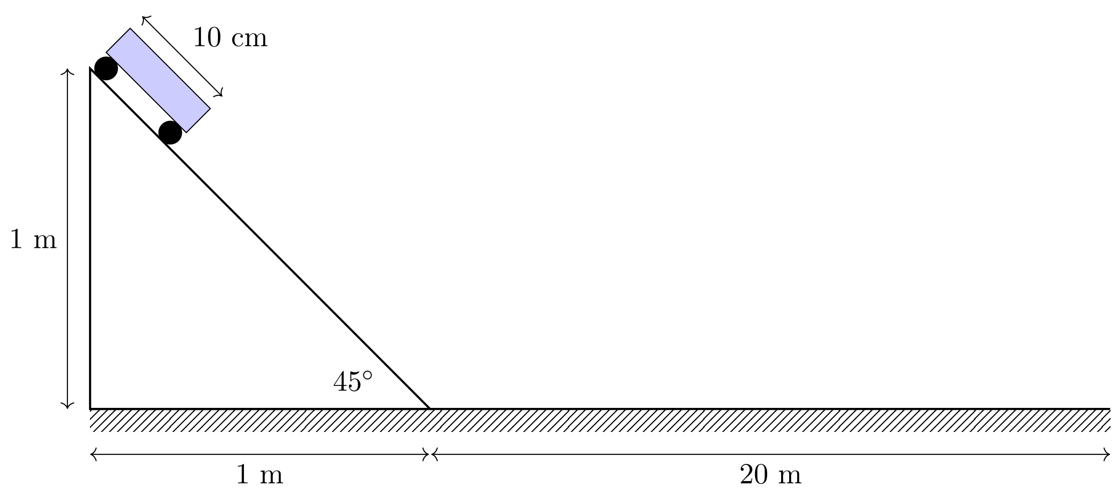
Problems

1. Fastest Path A small toy car rolls down three ramps with the same height and horizontal length, but different shapes, starting from rest. The car stays in contact with the ramp at all times and no energy is lost. Order the ramps from the fastest to slowest time it takes for the toy car to drop the full 1 m. For example, if ramp 1 is the fastest and ramp 3 is the slowest, then enter 123 as your answer choice.



2. Resistor Puzzle What is the smallest number of 1Ω resistors needed such that when arranged in a certain arrangement involving only series and parallel connections, that the equivalent resistance is $\frac{7}{6}\Omega$?

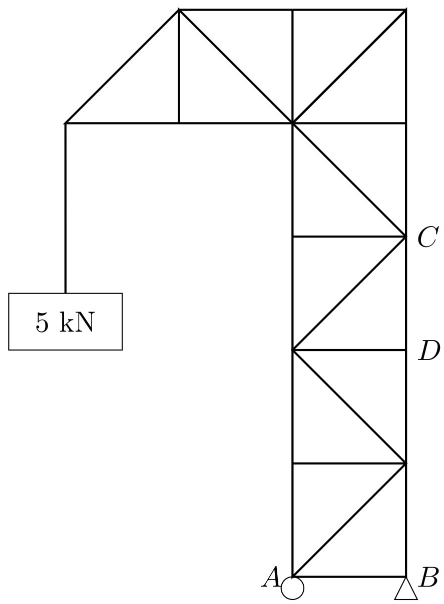
3. Derby Race In a typical derby race, cars start at the top of a ramp, accelerate downwards, and race on a flat track, and are always set-up in the configuration shown below.



A common technique is to change the location of the center of mass of the car to gain an advantage. Alice ensures the center of mass of her car is at the rear and Bob puts the center of mass of his car at the very front. Otherwise, their cars are exactly the same. Each car's time is defined as the time from when the car is placed on the top of the ramp to when the front of the car reaches the end of the flat track. At the competition, Alice's car beat Bob's. What is the ratio of Bob's car's time and Alice's car's time?

Assume that the wheels are small and light compared to the car body, neglect air resistance, and the height of the cars are small compared to the height of the ramp. In addition, neglect all energy losses during the race and the time it takes to turn onto the horizontal surface from the ramp. Express your answer as a decimal greater than 1.

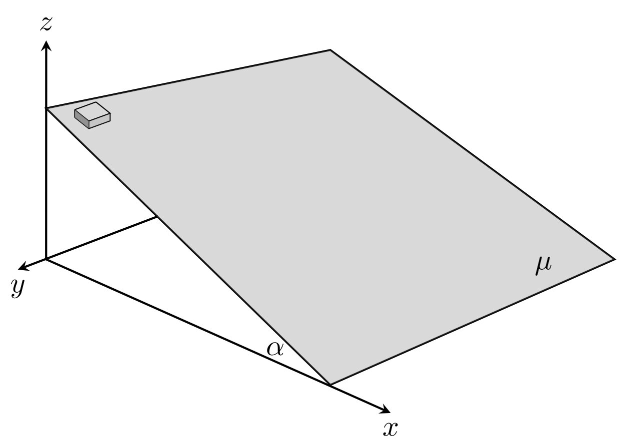
4. Crane A simple crane is shown in the below diagram, consisted of light rods with length 1 m and $\sqrt{2}$ m. The end of the crane is supporting a 5 kN object. Point B is known as a "pin." It is attached to the main body and can exert both a vertical and horizontal force. Point A is known as a "roller" and can only exert vertical forces. Rods can only be in pure compression or pure tension.



In kN, what is the force experienced by the rod CD? Express a positive number if the member is in tension and a negative number if it is in compression.

5. Coaxial Cable A coaxial cable is cylindrically symmetric and consists of a solid inner cylinder of radius $a = 2$ cm and an outer cylindrical shell of inner radius $b = 5$ cm and outer radius $c = 7$ cm. A uniformly distributed current of total magnitude $I = 5$ A is flowing in the inner cylinder and a uniformly distributed current of the same magnitude but opposite direction flows in the outer shell. Find the magnitude $B(r)$ of the magnetic field B as a function of distance r from the axis of the cable. As the final result, submit $\int_0^\infty B(r)\mathrm{d}r$. In case this is infinite, submit 42.

6. Magnetic Block A small block of mass m and charge Q is placed at rest on an inclined plane with a slope $\alpha = 40°$. The coefficient of friction between them is $\mu = 0.3$. A homogenous magnetic field of magnitude B_0 is applied perpendicular to the slope. The speed of the block after a very long time is given by $v = \beta \frac{mg}{QB_0}$. Determine β. Do not neglect the effects of gravity.



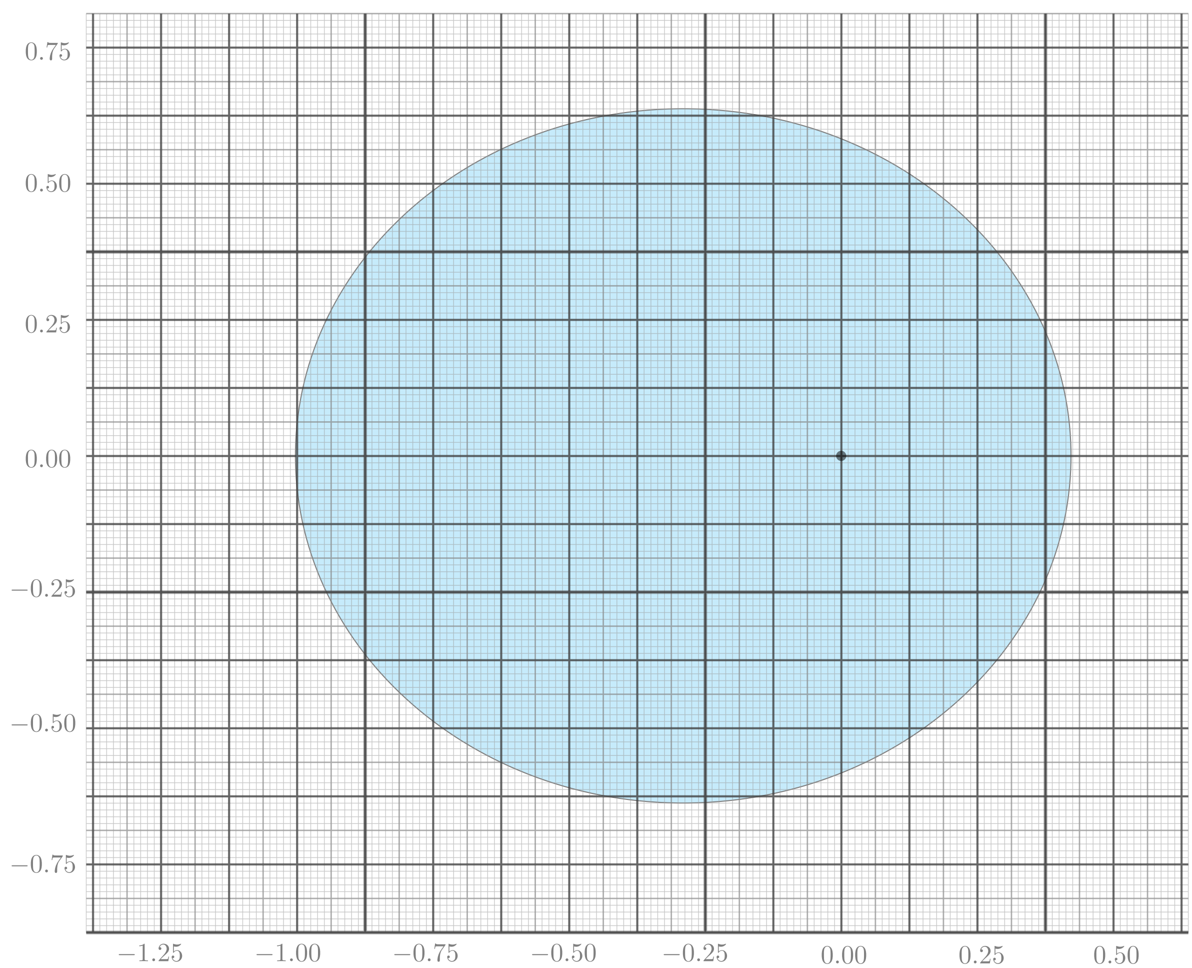
7. Thermal Train A train of length 100 m and mass 10^5 kg is travelling at 20 m/s along a straight track. The driver engages the brakes and the train starts deccelerating at a constant rate, coming to a stop after travelling a distance $d = 2000$ m. As the train decelerates, energy released as heat from the brakes goes into the tracks, which have a linear heat capacity of $5000 \mathrm{J\,m^{-1}\,K^{-1}}$. Assume the rate of heat generation and transfer is uniform across the length of the train at any given moment.

If the tracks start at an ambient temperature of 20°C, there is a function $T(x)$ that describes the temperature (in Celsius) of the tracks at each point x, where the rear of where the train starts is at $x = 0$. Assume (unrealistically) that 100% of the original kinetic energy of the train is transferred to the tracks (the train does not absorb any energy), that there is no conduction of heat along the tracks, and that heat transfer between the tracks and the surroundings is negligible.

Compute $T(20) + T(500) + T(2021)$ in degrees celsius.

8. Fountain A sprinkler fountain is in the shape of a semi-sphere that spews out water from all angles at a uniform speed v such that without the presence of wind, the wetted region around the fountain forms a circle in the XY plane with the fountain centered on it.

Now suppose there is a constant wind blowing in a direction parallel to the ground such that the force acting on each water molecule is proportional to their weight. The wetted region forms the shape below where the fountain is placed at $(0, 0)$. Determine the exit speed of water v in meters per second. Round to two significant digits. All dimensions are in meters.

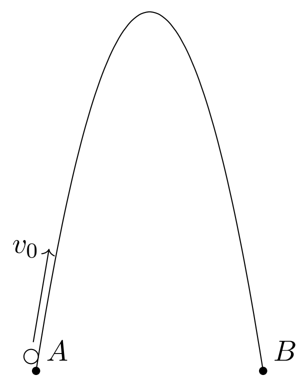


9. Escaping Nieons Consider a gas of mysterious particles called nieons that all travel at the same speed, v. They are enclosed in a cubical box, and there are ρ nieons per unit volume. A very small hole of area A is punched in the side of the box. The number of nieons that escape the box per unit time is given by

$$\alpha v^{\beta} A^{\gamma} \rho^{\delta} \tag{1}$$

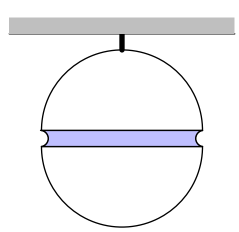
where α, β, γ, and δ are all dimensionless constants. Calculate $\alpha + \beta + \gamma + \delta$.

10. Pico-Pico 1 Poncho is a very good player of the legendary carnival game known as Pico-Pico. Its setup consists of a steel ball, represented by a point mass, of negligible radius and a frictionless vertical track. The goal of Pico-Pico is to flick the ball from the beginning of the track (point A) such that it is able to traverse through the track while never leaving the track, successfully reaching the end (point B). The most famous track design is one of parabolic shape; specifically, the giant track is of the shape $h(x) = 5 - 2x^2$ in meters. The starting and ending points of the tracks are where the two points where the track intersects $y = 0$. If $(v_a, v_b]$ is the range of the ball's initial velocity v_0 that satisfies the winning condition of Pico-Pico, help Poncho find $v_b - v_a$. This part is depicted below:



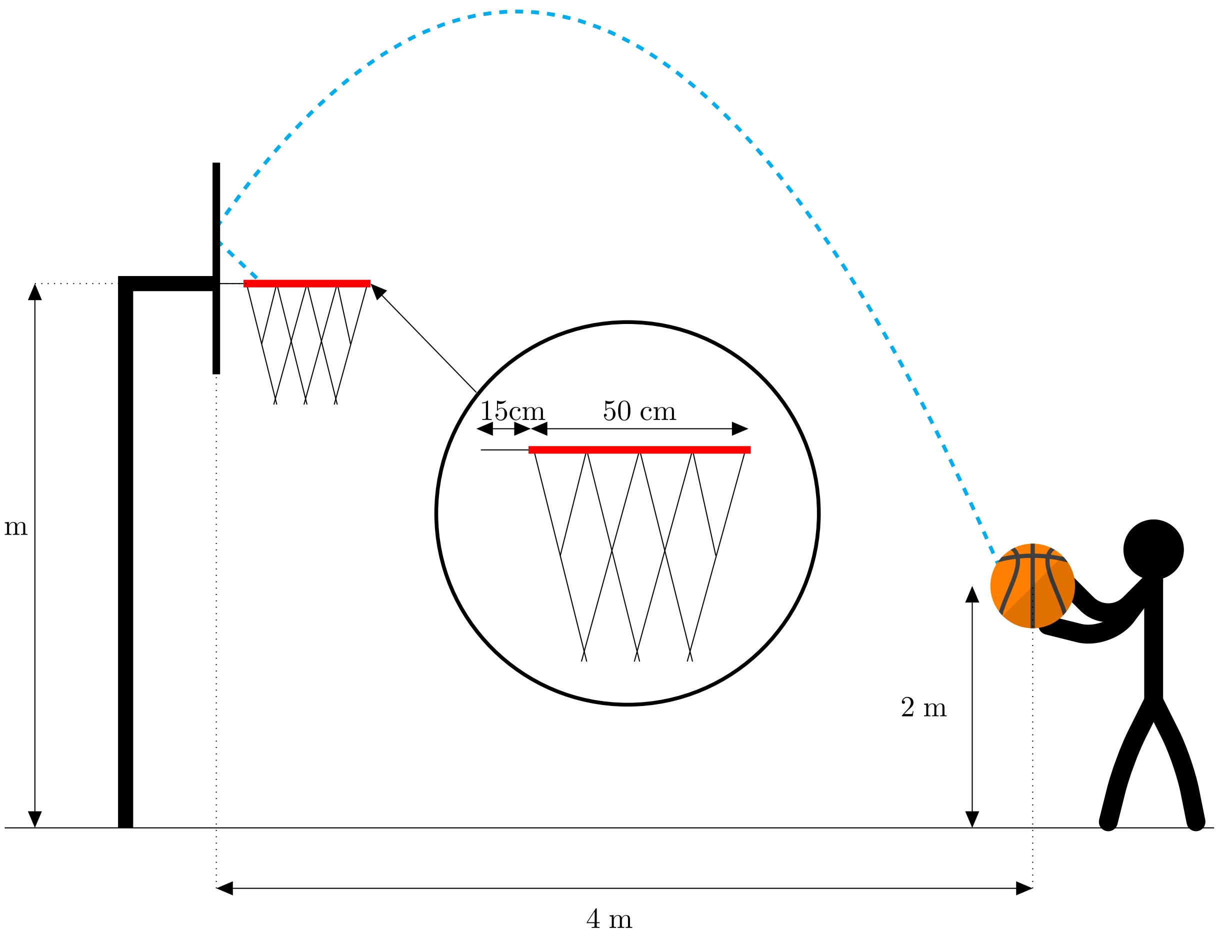
11. Pico-Pico 2 Now, Poncho has encountered a different Pico-Pico game that uses the same shaped frictionless track, but lays it horizontally on a table with friction and coefficient of friction $\mu = 0.8$. In addition, the ball, which can once again be considered a point mass, is placed on the other side of the track as the ball in part 1. Finally, a buzzer on the other side of the track requires the mass to hit with at least velocity $v_f = 2$ m/s in order to trigger the buzzer and win the game. Find the minimum velocity v_0 required for the ball to reach the end of the track with a velocity of at least v_f. The initial velocity must be directed along the track.

12. Golden Apple Anyone who's had an apple may know that pieces of an apple stick together, when picking up one piece a second piece may also come with the first piece. The same idea is tried on a *golden apple*. Consider two uniform hemispheres with radius $r = 4$ cm made of gold of density $\rho_g = 19300 \text{kg m}^{-3}$. The top half is nailed to a support and the space between is filled with water.



Given that the surface tension of water is $\gamma = 0.072 \text{N m}^{-1}$ and that the contact angle between gold and water is $\theta = 10°$, what is the maximum distance between the two hemispheres so that the bottom half doesn't fall? Answer in millimeters.

The following information applies to the next 2 problems. In the following two problems we will look at shooting a basketball. Model the basketball as an elastic hollow sphere with radius 0.1 meters. Model the net and basket as shown below, dimensions marked. Neglect friction between the backboard and basketball, and assume all collisions are perfectly elastic.



13. Free Throw For this problem, you launch the basketball from the point that is 2 meters above the ground and 4 meters from the backboard as shown. You attempt to make a shot by hitting the basketball off the backboard as depicted above. What is the minimum initial speed required for the ball to make this shot?

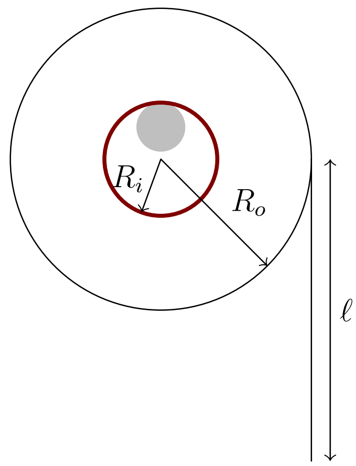
Note: For this problem, you may assume that the size of the ball is negligible.

14. Layup You now wish to practice closer shots. You walk up until you're 1 m away from the backboard (the 4 m changes to a 1 m). You jump 1 m in the air. What is the minimum initial speed of the ball that allows you to score off of the backboard if you release the ball at the top of your jump? Note that scoring off the backboard means that the ball bounces off the backboard and into the net. Do not consider cases where the ball bounces off of the rim or the protrusion. That's just luck and you want a consistent strategy.

Hint: Neglecting the size of the ball may no longer be possible.

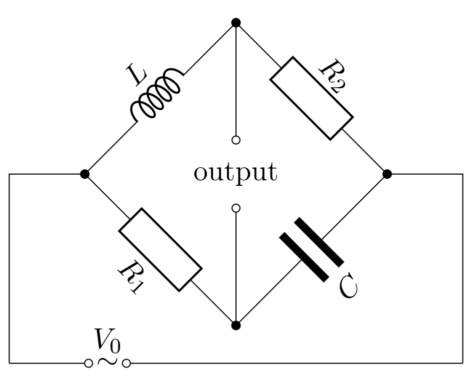
15. Right Triangle Potentials Let ABC be a solid right triangle ($AB = 5s$, $AC = 12s$, and $BC = 13s$) with uniform charge density σ. Let D be the midpoint of BC. We denote the electric potential of a point P by $\phi(P)$. The electric potential at infinity is 0. If $\phi(B) + \phi(C) + \phi(D) = \frac{k\sigma s}{\epsilon_0}$ where k is a dimensionless constant, determine k.

16. Toilet Paper Roll Consider a toilet paper roll with some length of it hanging off as shown. The toilet paper roll rests on a cylindrical pole of radius $r = 1$ cm and the coefficient of static friction between the role and the pole is $\mu = 0.3$.

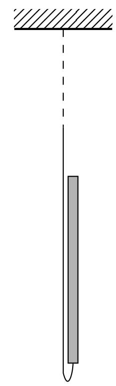


The length of the paper hanging off has length $\ell = 30$ cm and the inner radius of the roll is $R_i = 2$ cm. The toilet paper has thickness $s = 0.1$ mm and mass per unit length $\lambda = 5$ g/m. What is the minimum outer radius R_o such that the toilet paper roll remains static? Answer in centimeters.

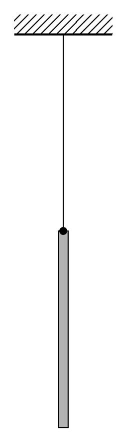
17. Maximum Voltage In the circuit shown below, a capacitor $C = 4$F, inductor $L = 5$H, and resistors $R_1 = 3\Omega$ and $R_2 = 2\Omega$ are placed in a diamond shape and are then fed an alternating current with peak voltage $V_0 = 1$V of unknown frequency. Determine the magnitude of the maximum instantaneous output voltage shown in the diagram.



18. Suspended Rod - 1 A uniform bar of length l and mass m is connected to a very long thread of negligible mass suspended from a ceiling. It is then rotated such that it is vertically upside down and then released. Initially, the rod is in unstable equilibrium. As it falls down, the minimum tension acting on the thread over the rod's entire motion is given by αmg. Determine α.



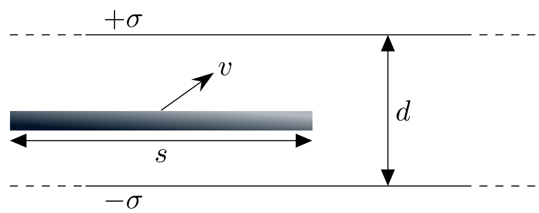
19. Suspended Rod - 2 A uniform bar of length l and mass m is connected to a thread of length $2l$ of negligible mass and is suspended from the ceiling at equilibrium. The rod is then slightly nudged at a point on its body. The largest stable frequency of oscillations of the system is given by $\beta\sqrt{\frac{g}{l}}$. Determine β.



20. One Ladder A straight ladder AB of mass $m = 1$ kg is positioned almost vertically such that point B is in contact with the ground with a coefficient of friction $\mu = 0.15$. It is given an infinitesimal kick at the point A so that the ladder begins rotating about point B. Find the value ϕ_m of angle ϕ of the ladder with the vertical at which the lower end B starts slipping on the ground.

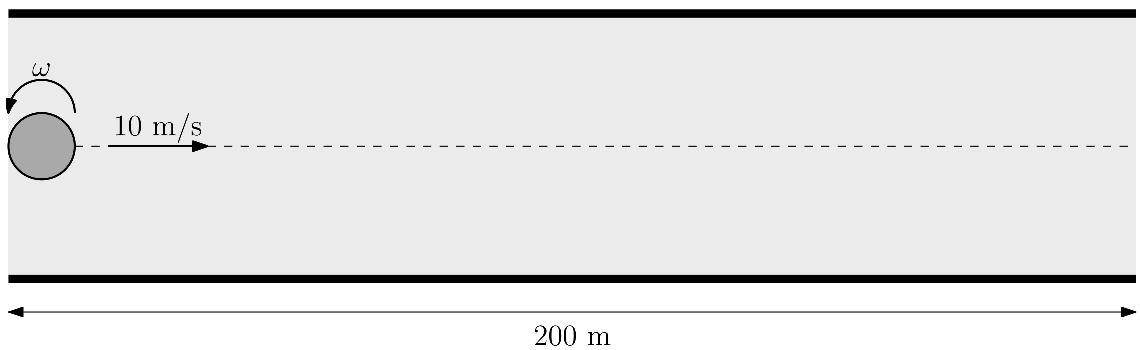
21. Two Ladders Two straight ladders AB and CD, each with length 1 m, are symmetrically placed on smooth ground, leaning on each other, such that they are touching with their ends B and C, ends A and D are touching the floor. The friction at any two surfaces is negligible. Initially both ladders are almost parallel and vertical. Find the distance AD when the points B and C lose contact.

22. Colliding Conducting Slab A thin conducting square slab with side length $s = 5\,\text{cm}$, initial charge $q = 0.1\,\mu\text{C}$, and mass $m = 100\,\text{g}$ is given a kick and sent bouncing between two infinite conducting plates separated by a distance $d = 0.5\,\text{cm} \ll s$ and with surface charge density $\pm\sigma = \pm 50\,\mu\text{C/m}^2$. After a long time it is observed exactly in the middle of the two plates to be traveling with velocity of magnitude $v = 3\,\text{m/s}$ and direction $\theta = 30°$ with respect to the horizontal line parallel to the plates. How many collisions occur after it has traveled a distance $L = 15\,\text{m}$ horizontally from when it was last observed? Assume that all collisions are elastic, and neglect induced charges. Note that the setup is horizontal so gravity does not need to be accounted for.



23. Evil Gamma Photon An evil gamma photon of energy $E_{\gamma 1} = 200$ keV is heading towards a spaceship. The commander's only choice is shooting another photon in the direction of the gamma photon such that they 'collide' head on and produce an electron-positron pair (both have mass m_e). Find the lower bound on the energy $E_{\gamma 2}$ of the photon as imposed by the principles of special relativity such that this occurs. Answer in keV.

24. Spinning Cylinder Adithya has a solid cylinder of mass $M = 10\,\text{kg}$, radius $R = 0.08\,\text{m}$, and height $H = 0.20\,\text{m}$. He is running a test in a chamber on Earth over a distance of $d = 200\,\text{m}$ as shown below. Assume that the physical length of the chamber is much greater than d (i.e. the chamber extends far to the left and right of the testing area). The chamber is filled with an ideal fluid with uniform density $\rho = 700\,\text{kg/m}^3$. Adithya's cylinder is launched with linear velocity $v = 10\,\text{m/s}$ and spins counterclockwise with angular velocity ω. Adithya notices that the cylinder continues on a **horizontal path** until the end of the chamber. Find the angular velocity ω. Do not neglect forces due to fluid pressure differences. Note that the diagram presents a side view of the chamber (i.e. gravity is oriented downwards with respect to the diagram).



Assume the following about the setup and the ideal fluid:

- fluid flow is steady in the frame of the center of mass of the cylinder
- the ideal fluid is incompressible, irrotational, and has zero viscosity
- the angular velocity of the cylinder is approximately constant during its subsequent motion

Hint: For a uniform **cylinder** of radius R rotating counterclockwise at angular velocity ω situated in an ideal fluid with flow velocity u to the **right** far away from the cylinder, the velocity potential Φ is given by

$$\Phi(r, \theta) = ur\cos\theta + u\frac{R^2}{r}\cos\theta + \frac{\Gamma\theta}{2\pi}$$

where (r, θ) is the polar coordinate system with origin at the center of the cylinder. Γ is the circulation and is equal to $2\pi R^2\omega$. The fluid velocity is given by

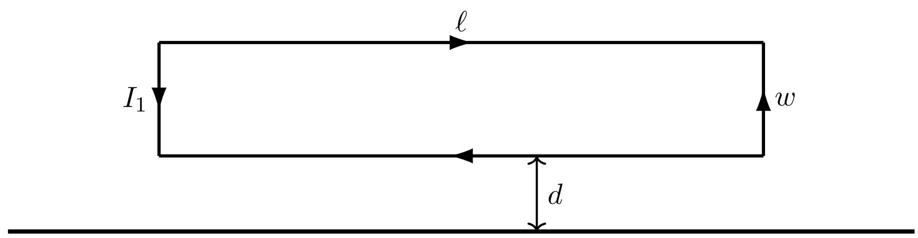
$$\mathbf{v} = \nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}.$$

25. Optimal Launch Adithya is launching a package from New York City (40°43′ N and 73°56′ W) to Guam (13°27′ N and 144°48′ E). Find the minimal launch velocity v_0 from New York City to Guam. Ignore the rotation of the earth, effects due to the atmosphere, and the gravitational force from the sun. Additionally, assume the Earth is a perfect sphere with radius $R_\oplus = 6.37 \times 10^6$ m and mass $M_\oplus = 5.97 \times 10^{24}$ kg.

26. Drag on the Plate Consider a container filled with argon, with molar mass $39.9\text{g}\,\text{mol}^{-1}$ whose pressure is much smaller than that of atmospheric pressure. Suppose there is a plate of area $A = 10\text{mm}^2$ moving with a speed v perpendicular to its plane. If the gas has density $\rho = 4.8 \times 10^{-7}\text{g}\,\text{cm}^{-3}$, and temperature $T = 100\text{K}$, find an approximate value for the drag force acting on the plate. Suppose that the speed of the plate is $v = 100\text{m}\,\text{s}^{-1}$.

27. Superconducting Loop Consider a rectangular loop made of superconducting material with length $\ell = 200$ cm and width $w = 2$ cm. The radius of this particular wire is $r = 0.5$ mm. This superconducting rectangular loop initially has a current $I_1 = 5$A in the counterclockwise direction as shown in the figure below. This rectangular loop is situated a distance $d = 1$ cm above an infinitely long wire that initially contains no current. Suppose that the current in the infinitely long wire is increased to some current I_2 such that there is an attractive force F between the rectangular loop and the long wire. Find the maximum possible value of F. Write your answer in newtons.

Hint: You may neglect the magnetic field produced by the vertical segments in the rectangular loop.



28. Cantor Interference Consider a 1 cm long slit with negligible height. First, we divide the slit into thirds and cover the middle third. Then, we perform the same steps on the two shorter slits. Again, we perform the same steps on the four even shorter slits and continue for a very long time.

Then, we shine a monochromatic, coherent light source of wavelength 500 nm on our slits, which creates an interference pattern on a wall 10 meters away. On the wall, what is the distance between the central maximum and the first side maximum? Assume the distance to the wall is much greater than the width of the slit. Answer in millimeters.

The following information applies to the next 2 problems. A certain planet with radius $R = 3 \times 10^4$ km is made of a liquid with constant density $\rho = 1.5$ g/cm^3 with the exception of a homogeneous solid core of radius $r = 10$ km and mass $m = 2.4 \times 10^{16}$ kg. Normally, the core is situated at the geometric center of the planet. However, a small disturbance has moved the center of the core $x = 1$ km away from the geometric center of the planet. The core is released from rest, and the fluid is inviscid and incompressible.

29. Solid Core - 1 Calculate the magnitude of the force due to gravity that now acts on the core. Work under the assumption that $R \gg r$.

30. Solid Core - 2 Calculate the magnitude of the force due to the pressure from the liquid that now acts on the core.

31. Solenoids A scientist is doing an experiment with a setup consisting of 2 ideal solenoids that share the same axis. The lengths of the solenoids are both ℓ, the radii of the solenoids are r and $2r$, and the smaller solenoid is completely inside the larger one. Suppose that the solenoids share the same (constant) current I, but the inner solenoid has $4N$ loops while the outer one has N, and they have opposite polarities (meaning the current is clockwise in one solenoid but counterclockwise in the other).

Model the Earth's magnetic field as one produced by a magnetic dipole centered in the Earth's core. Let F be the magnitude of the total magnetic force the whole setup feels due to Earth's magnetic field. Now the scientist replaces the setup with a similar one: the only differences are that the the radii of the solenoids are $2r$ (inner) and $3r$ (outer), the length of each solenoid is 7ℓ, and the number of loops each solenoid is $27N$ (inner) and $12N$ (outer). The scientist now drives a constant current $2I$ through the setup (the solenoids still have opposite polarities), and the whole setup feels a total force of magnitude F' due to the Earth's magnetic field. Assuming the new setup was in the same location on Earth and had the same orientation as the old one, find F'/F.

Assume the dimensions of the solenoids are much smaller than the radius of the Earth.

The following information applies to the next 2 problems. Adithya is in a rocket with proper acceleration $a_0 = 3.00 \times 10^8$ m/s^2 to the right, and Eddie is in a rocket with proper acceleration $\frac{a_0}{2}$ to the left. Let the frame of Adithya's rocket be S_1, and the frame of Eddie's rocket be S_2. Initially, both rockets are at rest with respect to each other, and Adithya's clock and Eddie's clock are both set to 0.

32. Accelerating Rockets - 1 At the moment Adithya's clock reaches 0.75 s in S_2, what is the **velocity** of Adithya's rocket in S_2?

33. Accelerating Rockets - 2 At the moment Adithya's clock reaches 0.75 s in S_2, what is the **acceleration** of Adithya's rocket in S_2?

The following information applies to the next 2 problems. Suppose a ping pong ball of radius R, thickness t, made out of a material with density ρ_b, and Young's modulus Y, is hit so that it resonates in mid-air with small amplitude oscillations. Assume $t \ll R$. The density of air around (and inside) the ball is ρ_a, and the air pressure is p, where $\rho_a \ll \rho_b \frac{t}{R}$ and $p \ll Y \frac{t^3}{R^3}$.

34. Ping Pong - 1 An estimate for the resonance frequency is $\omega \sim R^a t^b \rho_b^c Y^d$. Find the value of $4a^2 + 3b^2 + 2c^2 + d^2$.

Hint: The surface of the ball will oscillate by "bending" instead of "stretching", since the former takes much less energy than the latter.

35. Ping Pong - 2 Assuming that the ball loses mechanical energy only through the surrounding air, find an estimate of the characteristic time τ it takes for the ball to stop resonating (or to lose half its mechanical energy), that is $\tau \sim R^\alpha t^\beta \rho_b^\kappa Y^\delta \rho_a^\zeta p^\gamma$. Find the value of $6\alpha^2 + 5\beta^2 + 4\kappa^2 + 3\delta^2 + 2\zeta^2 + \gamma^2$. (Note that in reality, the ball also loses mechanical energy to heat, but we will neglect that for simplicity.)

2021 Online Physics Olympiad: Open Contest Solutions

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Website Developer: Ben Oommen

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Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

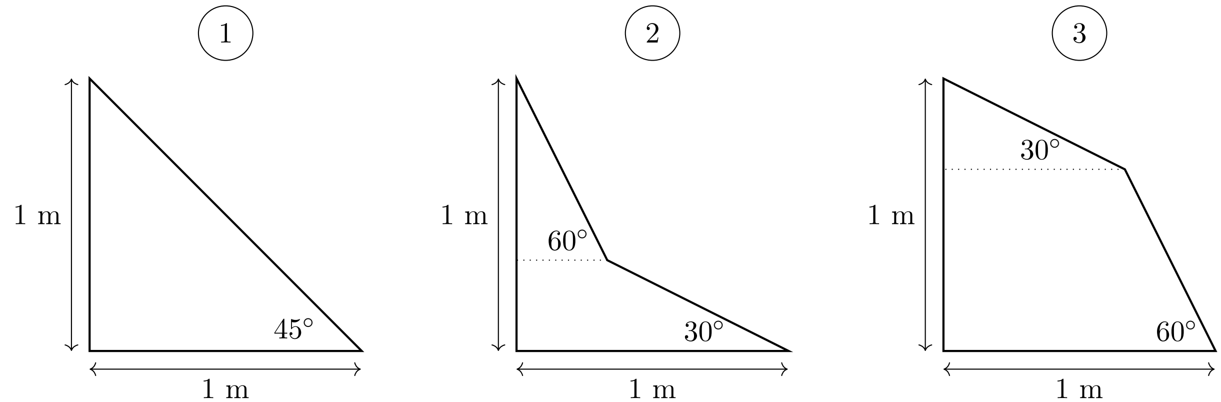
- Use $g = 9.81$ m/s^2 in this contest. See the constants sheet on the following page for other constants.
- This test contains 35 short answer questions. Each problem will have three possible attempts.
- The weight of each question depends on our scoring system found here. Put simply, the later questions are worth more, and the overall amount of points from a certain question decreases with the number of attempts that you take to solve a problem as well as the number of teams who solve it. **Note:** Unlike last year, the time it takes will no longer be a factor.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should contain **three** significant figures, unless otherwise specified. All answers within the 1% range will be accepted.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used. Technology and computer algebra systems like Wolfram Alpha or the one in the TI nSpire will not be needed or allowed. Attempts to use these tools will be classified as cheating.
- You are *allowed* to use Wikipedia or books in this exam. Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions.
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in SI units (meter, second, kilogram, watt, etc.) unless otherwise specified. Please input all angles in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put units in your answer on the submission portal! If your answer is "x meters", input only the value x into the submission portal.
- **Do not communicate information to anyone else apart from your team-members before the exam ends on June 6, 2021 at 11:59 PM UTC**.

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^8$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\mathrm{N} \cdot \mathrm{m}^2)/\mathrm{kg}^2$$
- Acceleration due to gravity, $g = 9.81$ m/s^2
- 1 unified atomic mass unit,
$$1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15}\ \mathrm{eV \cdot s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9\ (\mathrm{N \cdot m^2})/\mathrm{C^2}$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T \cdot m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\mathrm{T \cdot m})/\mathrm{A}$$
- 1 atmospheric pressure,
$$1\ \mathrm{atm} = 1.01 \cdot 10^5\ \mathrm{N/m^2} = 1.01 \cdot 10^5\ \mathrm{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$$

Problems

1. Fastest Path A small toy car rolls down three ramps with the same height and horizontal length, but different shapes, starting from rest. The car stays in contact with the ramp at all times and no energy is lost. Order the ramps from the fastest to slowest time it takes for the toy car to drop the full 1 m. For example, if ramp 1 is the fastest and ramp 3 is the slowest, then enter 123 as your answer choice.



Solution: One can solve this by finding the time it takes for each ramp. For ramp 1:

$$\sqrt{2} = \frac{1}{2} g \sin(45^\circ) t^2$$
$$\implies t = 0.639 \text{ s}$$

For ramps 2, let the length of the dashed region be x. Then:

$$x + x/\tan(30^\circ) = 1 \implies x = 0.366 \text{ m} \tag{1}$$

Due to symmetry, both the steep and shallow regions of both ramps 2 and 3 have a length of $x/\cos(60^\circ) = 0.732$ m. This results in a time for ramp 2 as:

$$0.732 = \frac{1}{2} g \sin(60^\circ) t_1^2 \implies t_1 = 0.415 \text{ s}$$
$$0.732 = \sqrt{2g(1-x)} t_2 + \frac{1}{2} g \sin(30^\circ) t_2^2 \implies t_2 = 0.184 \text{ s}$$

for a total time of $t_1 + t_2 = 0.599$ s. For ramp 3,

$$0.732 = \frac{1}{2} g \sin(30^\circ) t_1^2 \implies t_1 = 0.546 \text{ s}$$
$$0.732 = \sqrt{2g(1-x)} t_2 + \frac{1}{2} g \sin(60^\circ) t_2^2 \implies t_2 = 0.206 \text{ s}$$

which gives a total time of $t_1 + t_2 = 0.752$ s. From fastest to slowest, the answer becomes 213. Note that this answer is easily guessable via intuition.

$\boxed{213}$

2. Resistor Puzzle What is the smallest number of 1Ω resistors needed such that when arranged in a certain arrangement involving only series and parallel connections, that the equivalent resistance is $\frac{7}{6}\Omega$?

Solution: We can write:

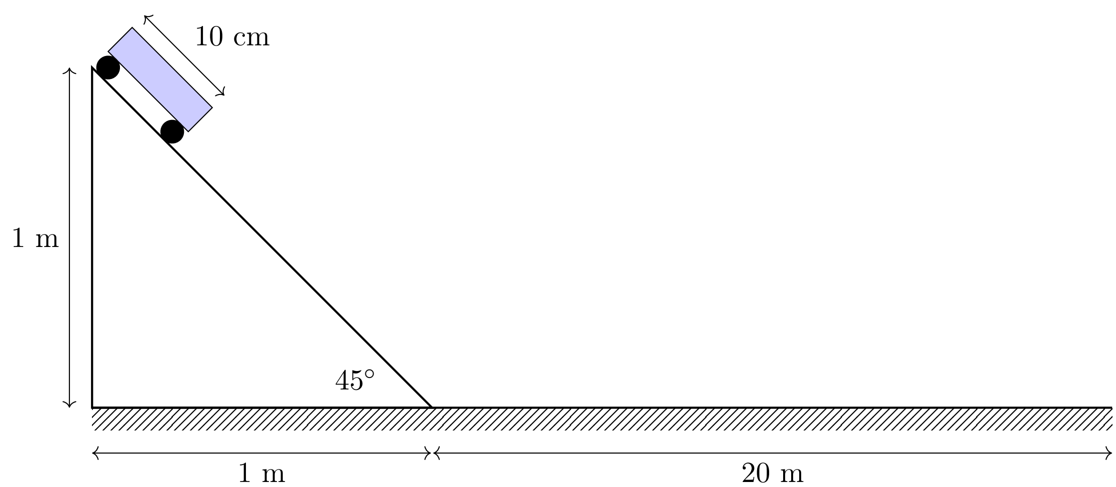
$$\frac{7}{6} = \frac{1}{2} + \frac{2}{3} \tag{2}$$

It takes two resistors (connected in parallel) to create a $\frac{1}{2}\Omega$ resistor. If we write $\frac{2}{3} = \frac{2\cdot 1}{2+1}$, then it takes three resistors to create a $\frac{2}{3}\Omega$ (this is accomplished by connecting a 2 resistor in parallel with a 1 resistor.

Combining the $\frac{1}{2}\Omega$ element in series with the $\frac{2}{3}\Omega$ element gives us our desired amount. To prove this is the minimum, we can easily check all possible combinations using 4 or fewer resistors.

$\boxed{5}$

3. Derby Race In a typical derby race, cars start at the top of a ramp, accelerate downwards, and race on a flat track, and are always set-up in the configuration shown below.



A common technique is to change the location of the center of mass of the car to gain an advantage. Alice ensures the center of mass of her car is at the rear and Bob puts the center of mass of his car at the very front. Otherwise, their cars are exactly the same. Each car's time is defined as the time from when the car is placed on the top of the ramp to when the front of the car reaches the end of the flat track. At the competition, Alice's car beat Bob's. What is the ratio of Bob's car's time and Alice's car's time?

Assume that the wheels are small and light compared to the car body, neglect air resistance, and the height of the cars are small compared to the height of the ramp. In addition, neglect all energy losses during the race and the time it takes to turn onto the horizontal surface from the ramp. Express your answer as a decimal greater than 1.

Solution: The speed at which Alice's car hits the ground is given by $v_A = \sqrt{2g(1)} = 4.43$ m/s from energy conservation. The speed at which Bob's car hits the ground is given by $v_B = \sqrt{2g(1 - 0.1\sin(45°)} = 4.27$ m/s since its center of mass is at the very front of the car.

The time in which Alice and Bob's cars are going down the ramp (and wheels are both on the ramp) is given by:

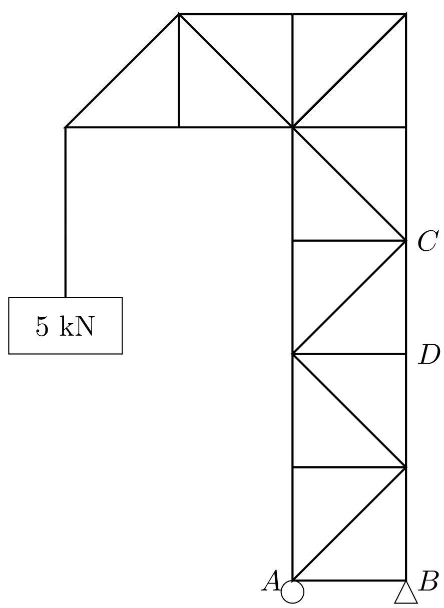
$$\sqrt{2} - 0.1 = \frac{1}{2} g \sin(45°) t^2 \implies t = 0.616 \text{ s} \tag{3}$$

And the time that Alice's car takes on the flat path (and wheels are both on the floor) is given by $t_A = \frac{20}{v_A} = 4.51$, and for Bob, $t_B = \frac{20}{v_B} = 4.68$.
We ignore the time it takes for the car to transition. The justification behind this is that this time would be on the order of magnitude of $\frac{0.1}{v_A} = 0.02 \text{ s} \ll t_A$. In fact, it is smaller than the 1% margin allowed (so teams who did not read the clarification would still have gotten the correct answer).
The ratio asked for was therefore:

$$\frac{0.616 + t_B}{0.616 + t_A} \tag{4}$$

$\boxed{1.03}$

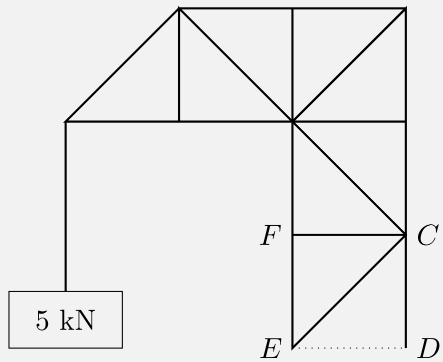
4. Crane A simple crane is shown in the below diagram, consisted of light rods with length 1 m and $\sqrt{2}$ m. The end of the crane is supporting a 5 kN object. Point B is known as a "pin." It is attached to the main body and can exert both a vertical and horizontal force. Point A is known as a "roller" and can only exert vertical forces. Rods can only be in pure compression or pure tension.



In kN, what is the force experienced by the rod CD? Express a positive number if the member is in tension and a negative number if it is in compression.

Solution: One naive method (though perfectly valid) is to solve for each member individually, starting from the two rods that connect to the 5kN weight. At each joint, we can write out force equilibrium equations in the vertical and horizontal directions, and solve a system of linaer equations to get the force in CD.

Instead, we can solve for this force in one line. Consider a horizontal slice right above point D.



Since the net force of this sub-element is still zero, we can do a force balance. The only *external* forces acting on this system is EF, EC, CD, and the 5kN weight. If we do a torque balance about E, we get:

$$5(2L) = CD(L) \tag{5}$$

where L is the length of the rod. This immediately gives $CD = 10\text{kN}$.

$\boxed{10\text{kN}}$

5. Coaxial Cable A coaxial cable is cylindrically symmetric and consists of a solid inner cylinder of radius $a = 2$ cm and an outer cylindrical shell of inner radius $b = 5$ cm and outer radius $c = 7$ cm. A uniformly distributed current of total magnitude $I = 5$ A is flowing in the inner cylinder and a uniformly distributed current of the same magnitude but opposite direction flows in the outer shell. Find the magnitude $B(r)$ of the magnetic field B as a function of distance r from the axis of the cable. As the final result, submit $\int_0^\infty B(r)\mathrm{d}r$. In case this is infinite, submit 42.

Solution: Ampere's law $\int B \cdot \mathrm{d}l = \mu_0 I$ is all we need. For every point on the wire, we can write the magnetic field as a function of the distance from its center r. Thus,

$$B(r) = \begin{cases} \frac{5\mu_0 r}{8\pi} & r \leq 2 \\ \frac{5\mu_0}{2\pi r} & 2 < r < 5 \\ \frac{5\mu_0(-r^2+49)}{48\pi r} & 5 \leq r \leq 7 \\ 0 & r > 7 \end{cases}$$

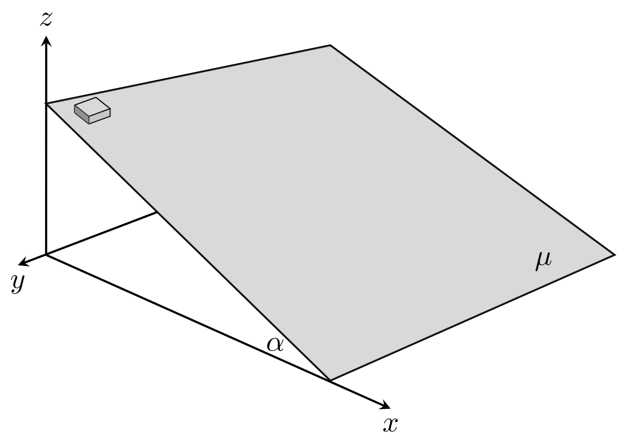
Now we just sum each integral from each interval, or in othe words

$$\int_0^\infty B(r)\mathrm{d}r = \int_0^2 B(r)\mathrm{d}r + \int_2^5 B(r)\mathrm{d}r + \int_5^7 B(r)\mathrm{d}r.$$

This is now straightforward integration.

$\boxed{1.6 \times 10^{-8}}$

6. Magnetic Block A small block of mass m and charge Q is placed at rest on an inclined plane with a slope $\alpha = 40°$. The coefficient of friction between them is $\mu = 0.3$. A homogenous magnetic field of magnitude B_0 is applied perpendicular to the slope. The speed of the block after a very long time is given by $v = \beta \frac{mg}{QB_0}$. Determine β. Do not neglect the effects of gravity.



Solution: Create a free body diagram. The direction of the magnetic field (into or out of the page) does not matter as we only need to know the magnitude of the terminal velocity. In dynamic equilibrium, we have three forces along the plane: the component of gravity along the plane, friction, and the magnetic force. The component of gravity along the plane is $mg\sin\alpha$. Friction has the magnitude $f = \mu mg\cos\alpha$. The magnetic force has magnitude $F_B = QvB_0$. At the terminal velocity, these forces are balanced. To finish, note that friction and the magnetic force are perpendicular so the magnitude of their vector sum is equal to the component of gravity along the plane. By the Pythagorean Theorem,

$$F_B^2 + f^2 = (mg\sin\alpha)^2 \implies v = \frac{mg}{QB_0}\sqrt{\sin^2\alpha - \mu^2\cos^2\alpha}.$$

$\boxed{0.6}$

7. Thermal Train A train of length 100 m and mass 10^5 kg is travelling at 20 m/s along a straight track. The driver engages the brakes and the train starts deccelerating at a constant rate, coming to a stop after travelling a distance $d = 2000$ m. As the train decelerates, energy released as heat from the brakes goes into the tracks, which have a linear heat capacity of $5000\,\mathrm{J\,m^{-1}\,K^{-1}}$. Assume the rate of heat generation and transfer is uniform across the length of the train at any given moment.

If the tracks start at an ambient temperature of 20°C, there is a function $T(x)$ that describes the temperature (in Celsius) of the tracks at each point x, where the rear of where the train starts is at $x = 0$. Assume (unrealistically) that 100% of the original kinetic energy of the train is transferred to the tracks (the train does not absorb any energy), that there is no conduction of heat along the tracks, and that heat transfer between the tracks and the surroundings is negligible.

Compute $T(20) + T(500) + T(2021)$ in degrees celsius.

Solution: Consider a small element of the tracks at position x with width $\mathrm{d}x$. Since the rate of heat generation is uniform along the length of the train L, we have that the rate of heat given to the track element is $mav\frac{\mathrm{d}x}{L}$, where m is the train's mass, a is the train's deceleration, and v is the trains speed. Integrating over time gives the total heat given to the track element: $\mathrm{d}Q = ma\frac{\mathrm{d}x}{L}\Delta x$, where Δx is the total distance the train slips on the track element. Combining with $\mathrm{d}Q = c\mathrm{d}x \cdot \Delta T$, we get $T(x) = T_0 + \frac{ma}{cL}\Delta x$, where c is the linear heat capacity. Now we split into 3 cases:

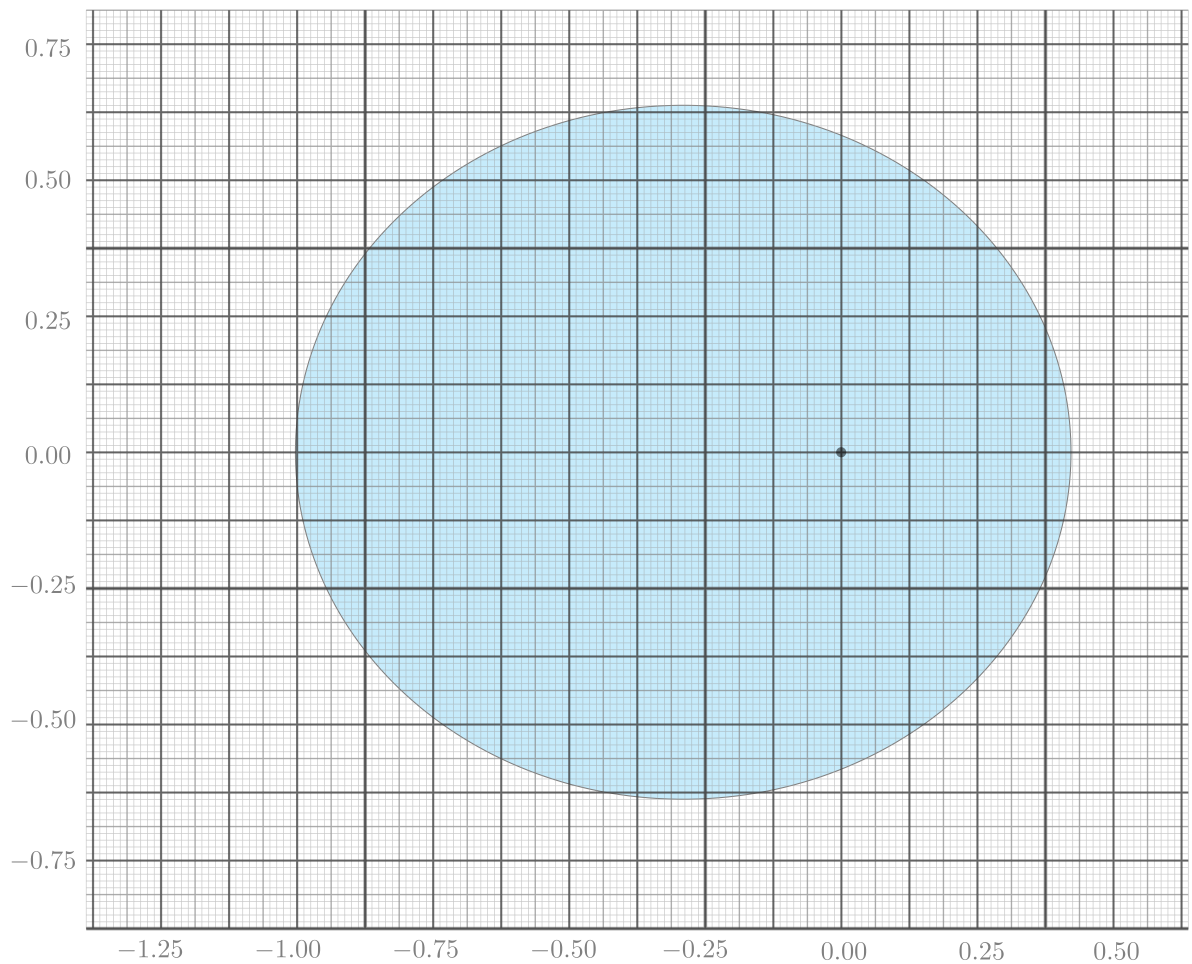
- $0 < x < L$,
- $L < x < d$,
- $d < x < d + L$,

where d is the total distance. If $0 < x < L$, the train slips a distance $\Delta x = x$ on the track element, so we have $T(x) = T_0 + \frac{ma}{cL}x$ (linear). If $L < x < d$, the train slips a distance $\Delta x = L$ on the track element, so we have $T(x) = T_0 + \frac{ma}{c}$ (constant). If $d < x < d + L$, the train slips a distance $\Delta x = d + L - x$, so we have $T(x) = T0 + \frac{ma}{cL}(d + L - x)$ (linear). Note that $a = \frac{v_0^2}{2d}$ where v_0 is the train's initial velocity.

$\boxed{63.98^\circ \text{ C}}$

8. Fountain A sprinkler fountain is in the shape of a semi-sphere that spews out water from all angles at a uniform speed v such that without the presence of wind, the wetted region around the fountain forms a circle in the XY plane with the fountain centered on it.

Now suppose there is a constant wind blowing in a direction parallel to the ground such that the force acting on each water molecule is proportional to their weight. The wetted region forms the shape below where the fountain is placed at $(0, 0)$. Determine the exit speed of water v in meters per second. Round to two significant digits. All dimensions are in meters.



Solution:

Method One: Let us look at only the major axis x. The wind provides a constant acceleration along x, which makes this problem equivalent to throwing a ball up and down a ramp tilted at an angle of $\beta = \tan \frac{a}{g}$. The furthest point along the x axis represents the maximum distance you can throw an object up and down this ramp. This distance can be derived to be:

$$d_{\text{optimal}} = \frac{v^2/g}{1 + \sin \beta} \tag{6}$$

where β is negative if it's tilting downwards. We get the systems of equations:

$$\frac{v^2/g_{\text{eff}}}{1+\sin\beta} = 0.25 + 0.25 \cdot \frac{13}{20} \tag{7}$$

$$\frac{v^2/g_{\text{eff}}}{1-\sin\beta} = 1 \tag{8}$$

where:

$$g_{\text{eff}} = g/\cos\beta \tag{9}$$

and solving gives $v = 2.51\,\mathrm{m\,s^{-1}}$.
Method Two: Looking at the minor axis is much easier. The wind doesn't contribute in this direction, so we know the furthest point must have a vertical displacement of $d_{\max} = \frac{v^2}{g}$. We measure $d_{\max} = 0.50 + 0.25 \cdot \frac{11}{20}$ and solving for v gives $d_{\max} = 2.50$ m.
Note that the answers aren't exactly the same due to measurement inaccuracies, so we asked for two significant digits. An earlier version of the question had the same diagram, but the labels were accidentally shifted. This did not affect the final answer.
$\boxed{2.5 \text{ m/s}}$

9. Escaping Nieons Consider a gas of mysterious particles called nieons that all travel at the same speed, v. They are enclosed in a cubical box, and there are ρ nieons per unit volume. A very small hole of area A is punched in the side of the box. The number of nieons that escape the box per unit time is given by

$$\alpha v^{\beta} A^{\gamma} \rho^{\delta} \tag{10}$$

where α, β, γ, and δ are all dimensionless constants. Calculate $\alpha + \beta + \gamma + \delta$.

Solution: The main idea is this: if a nieon is to escape in time Δt, then it must be traveling towards the hole and be within a hemisphere of radius $v\Delta t$, centered on the hole.

(Note that we will ignore all collisions between nieons. As with many calculations in these kinds of problems, collisions will not affect our answer. Even if they did, we can assume that the hole is so small that we can take $v\Delta t \to 0$, making the radius of the hemisphere much much smaller than the mean free path.)

First, we will break up this hemisphere into volume elements. We will calculate how many nieons are inside each volume element. Then we will find out how many of these nieons are traveling in the direction of the hole.

We define a spherical coordinate system (r, θ, ϕ). The volume of each differential element is $r^2 \sin\theta dr d\theta d\phi$, which means that there are $\rho r^2 \sin\theta dr d\theta d\phi$ nieons inside this volume element.

Within this volume element, how many of these nieons are going towards the hole? We must think about what each nieon "sees." Visualize a sphere of radius r around the volume element. The surface of the sphere will pass through the small hole in the wall. Each nieon is equally likely to go in any direction and is equally likely to end up going through any patch of this spherical surface. Thus, each nieon within this volume element has a probability $A'/4\pi r^2$ of going through the hole, where A' is the *perceived* area of the hole from where the nieon is situated.

A' can be calculated as follows. $\hat{\mathbf{j}}$ is the unit vector pointing perpendicular to the hole. Let $\hat{\mathbf{r}}$ be the unit vector pointing from the hole to the volume element. Then it can be verified that $A' = A\cos\psi$, where ψ is the angle between $\hat{\mathbf{j}}$ and $\hat{\mathbf{r}}$.

Expressing $\hat{\mathbf{r}}$ in terms of the other unit vectors, it can be shown that $A' = A\sin\theta\sin\phi$. Hence, the fraction of nieons inside the volume element that are going towards the hole is

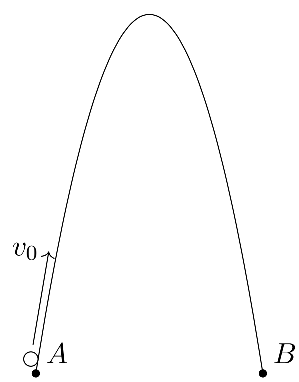
$$\frac{A\sin\theta\sin\phi}{4\pi r^2}.$$

Integrating over the entire hemisphere, the number of nieons escaping through the hole in a time Δt is

$$\int_0^{v\Delta t} dr \int_0^{\pi} d\phi \int_0^{\pi} d\theta \left(\frac{A}{4\pi r^2}\sin\theta\sin\phi\right)(\rho r^2\sin\theta) = \frac{1}{4}vA\rho\cdot\Delta t.$$

$\boxed{3.25}$

10. Pico-Pico 1 Poncho is a very good player of the legendary carnival game known as Pico-Pico. Its setup consists of a steel ball, represented by a point mass, of negligible radius and a frictionless vertical track. The goal of Pico-Pico is to flick the ball from the beginning of the track (point A) such that it is able to traverse through the track while never leaving the track, successfully reaching the end (point B). The most famous track design is one of parabolic shape; specifically, the giant track is of the shape $h(x) = 5 - 2x^2$ in meters. The starting and ending points of the tracks are where the two points where the track intersects $y = 0$. If $(v_a, v_b]$ is the range of the ball's initial velocity v_0 that satisfies the winning condition of Pico-Pico, help Poncho find $v_b - v_a$. This part is depicted below:



Solution: Using conservation of energy, the minimum initial velocity of the ball needed to pass the top of the track is $v_a = \sqrt{2gh} = 9.9045\frac{m}{s}$. To find v_b, the centripetal force at all points on the track must be determined given the initial velocity.

$$F_c = \frac{mv^2}{R} \tag{11}$$

$$= \frac{m(v_b^2 - 2gh)}{\frac{|1+(\frac{d}{dx}h(x))^2|}{\frac{d^2}{dx^2}h(x)}} \tag{12}$$

$$= \frac{m(v_b^2 - 2gh)}{\frac{|1+16x^2|^{\frac{3}{2}}}{4}} \tag{13}$$

For the boundary condition, the ball leaves if the normal force from the track on the ball $N =$

$mg\cos\theta - F_c$ becomes 0.

$$mg\cos\theta - F_c = 0 \tag{14}$$

$$mg\cos\arctan(-4x) = \frac{4m(v_b^2 - 2gh)}{|1+16x^2|^{\frac{3}{2}}} \tag{15}$$

$$\frac{g}{|1+16x^2|^{\frac{1}{2}}} = \frac{4(v_b^2 - 2gh)}{|1+16x^2|^{\frac{3}{2}}} \tag{16}$$

$$g = \frac{4(v_b^2 - 2gh)}{1+16x^2} \tag{17}$$

$$v_{b\mathbf{max}} = \sqrt{\frac{g+16gx^2}{4} + 2gh} \tag{18}$$

From the derivation, $v_{b\mathbf{max}}$ is the lowest at $x = 0$. Thus,

$$v_{b\mathbf{max}} = \sqrt{\frac{g}{4} + 2gh} \tag{19}$$

$$= 10.0276\frac{m}{s} \tag{20}$$

which is our desired v_b. The final answer, $v_b - v_a$, can be calculated.

$\boxed{0.1231}$

11. Pico-Pico 2 Now, Poncho has encountered a different Pico-Pico game that uses the same shaped frictionless track, but lays it horizontally on a table with friction and coefficient of friction $\mu = 0.8$. In addition, the ball, which can once again be considered a point mass, is placed on the other side of the track as the ball in part 1. Finally, a buzzer on the other side of the track requires the mass to hit with at least velocity $v_f = 2$ m/s in order to trigger the buzzer and win the game. Find the minimum velocity v_0 required for the ball to reach the end of the track with a velocity of at least v_f. The initial velocity must be directed along the track.

Solution: We simply need to find the work done by friction and we can finish with conservation of energy. To get the work done by friction, since the force is constant, we just need to find the arc length of the track. That is

$$\ell = \int_{-\sqrt{5/2}}^{\sqrt{5/2}} \sqrt{1+h'(x)^2}\,\mathrm{d}x = \int_{-\sqrt{5/2}}^{\sqrt{5/2}} \sqrt{1+16x^2}\,\mathrm{d}x = 10.76 \text{ m}.$$

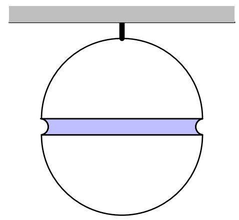
Then by conservation of energy,

$$\frac{1}{2}v_0^2 - \mu g\ell = \frac{1}{2}v_f^2 \implies v_0 = \sqrt{v_f^2 + 2\mu g\ell}.$$

We simply plug in the numbers to finish.

$\boxed{13.1 \text{ m/s}}$

12. Golden Apple Anyone who's had an apple may know that pieces of an apple stick together, when picking up one piece a second piece may also come with the first piece. The same idea is tried on a *golden apple*. Consider two uniform hemispheres with radius $r = 4$ cm made of gold of density $\rho_g = 19300\text{kg}\,\text{m}^{-3}$. The top half is nailed to a support and the space between is filled with water.



Given that the surface tension of water is $\gamma = 0.072\text{N}\,\text{m}^{-1}$ and that the contact angle between gold and water is $\theta = 10°$, what is the maximum distance between the two hemispheres so that the bottom half doesn't fall? Answer in millimeters.

Solution: Let h be the difference in height. There are 3 forces on the bottom hemisphere. The force from gravity, which has magnitude $\frac{2}{3}\pi\rho_g g r^3$, the force from the surface tension, and the force from the pressure difference at the top and bottom. The pressure difference is given by the young-laplace equation,

$$\Delta P = \gamma\left(-\frac{1}{r} + \frac{2\cos\theta}{h}\right) \approx \frac{2\gamma\cos\theta}{h}.$$

The radii are found by some simple geometry. It is likely that r will be much larger than the height, so we can neglect the $1/r$ term. Now the force from surface tension is $2\pi r\gamma\sin\theta$, since we take the vertical component. So we can now set the net force to 0,

$$\frac{2}{3}\pi\rho_g g r^3 = \pi r^2\Delta P + 2\pi r\gamma\sin\theta \implies \frac{2}{3}\rho_g g r^2 = (2r\gamma\cos\theta)\frac{1}{h} + 2\gamma\sin\theta.$$

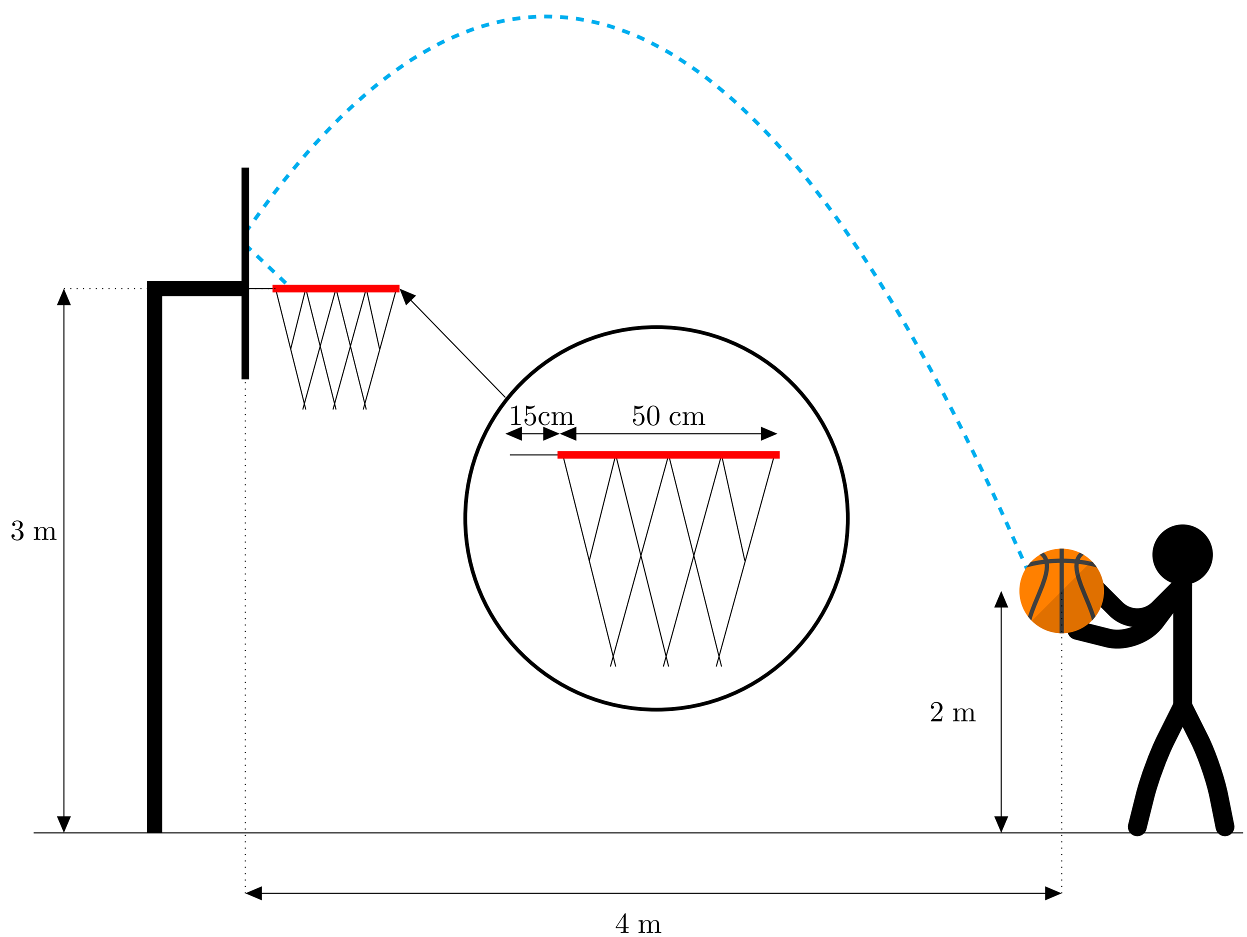
This is simple to solve,

$$h = \frac{2r\gamma\cos\theta}{2\rho_g g r^2/3 - 2\gamma\sin\theta} = 2.81\times10^{-5}\text{ m} = 0.0281\text{ mm}.$$

This is very small so our approximation from earlier is justified.

$\boxed{0.0281\text{ mm}}$

The following information applies to the next 2 problems. In the following two problems we will look at shooting a basketball. Model the basketball as an elastic hollow sphere with radius 0.1 meters. Model the net and basket as shown below, dimensions marked. Neglect friction between the backboard and basketball, and assume all collisions are perfectly elastic.



13. Free Throw For this problem, you launch the basketball from the point that is 2 meters above the ground and 4 meters from the backboard as shown. You attempt to make a shot by hitting the basketball off the backboard as depicted above. What is the minimum initial speed required for the ball to make this shot?

Note: For this problem, you may assume that the size of the ball is negligible.

Solution: For this part, we regard the basketball as a point mass. Now to account for the backboard bounce, we reflect the hoop over the backboard. The problem is now equivalent to making it into the hoop that is 4.15 meters away now. So now we need to find the smallest velocity to reach this point. One way to do it is to use the safety parabola, which makes it so minimal calculations are needed, so we present that method here. However, this problem is also readily solvable with only basic kinematics.

The point that the ball must hit (4.15 meters in front and 1 meter above) lies on the safety parabola when it is thrown at minimum speed. Since the focus of the safety parabola is the point of launch, the directrix is

$$\sqrt{1^2 + 4.15^2} = 4.27 \text{ m}$$

away from the top of the hoop. The distance from the focus to the vertex of the parabola is then $(4.27 + 1)/2 = 2.63$ m. So the minimum velocity to go this height is given by

$$\frac{v^2}{2g} = 2.63\,\text{m},$$

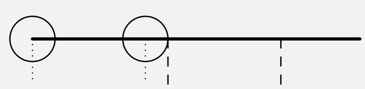
and that gives $v = 7.19\,\text{m/s}$. There is one last bit we need to check to complete this problem. The basketball's trajectory has to actually be possible (i.e. it doesn't go underneath the rim). It is clearly less optimal intuitively. If it passes through the underside, it means that its apex of the trajectory is very near its target.

$\boxed{7.19 \text{ m/s}}$

14. Layup You now wish to practice closer shots. You walk up until you're 1 m away from the backboard (the 4 m changes to a 1 m). You jump 1 m in the air. What is the minimum initial speed of the ball that allows you to score off of the backboard if you release the ball at the top of your jump? Note that scoring off the backboard means that the ball bounces off the backboard and into the net. Do not consider cases where the ball bounces off of the rim or the protrusion. That's just luck and you want a consistent strategy.

Hint: Neglecting the size of the ball may no longer be possible.

Solution: For this part, first notice that the shot is now from a much closer location. This means we must take into account the radius of the sphere, which we'll call r. We first reflect over the entire setup over a vertical line r away from the backboard. The reason it doesn't reflect over the backboard is because the center of the ball bounces before that. Now the basketball must also clear the rim since it's not a point mass. So we can draw a circle of radius r centered at the end of the rim. So the problem is equivalent to clearing the circles shown. Note that we cannot simply set the range equal to 1.05 m because the ball would intersect the rim earlier when it's at an angle (many teams sent challenges with solutions assuming that incorrect fact).



Now consider the minimum case where it clears it. Then the parabola is going to be tangent to the closer circle. Furthermore, the safety parabola will also be tangent because at the boundaries the trajectory parabolas are tangent to the safety parabola. This is the key insight. Now we use geometry.

Consider the center of the right circle. Then suppose the parabola is tangent at some angle θ on the circle from the vertical counterclockwise. Then, the line from the focus of the safety parabola must also make an angle θ with the parabola because that's how a parabola reflects lines from the directrix and focus. Thus, with a little trig, we see that

$$(r\cos\theta)\tan 2\theta = d + r\sin\theta$$

where d is the distance from the launch point to the center of the circle. This distance is in fact just $1 - 0.05 = 0.95$ meters. Now we jsut do a bit of algebra to finish:

$$r\frac{2\sin\theta}{1-\tan^2\theta} = d + r\sin\theta.$$

Rearranging,

$$r\sin\theta\frac{1+\tan^2\theta}{1-\tan^2\theta} = d.$$

Notice that $1+\tan^2\theta = \sec^2\theta$, and so

$$r\frac{\sin\theta}{\cos^2\theta - \sin^2\theta} = d.$$

This is a quadratic in sin,

$$2\sin^2\theta + \frac{r}{d}\sin\theta - 1 = 0.$$

Using the values for r and d and the quadratic formula, we arrive at $\sin\theta \approx 0.681$. Thus, $\theta = 42.9°$. At this point we can now find the distance from the focus to the vertex. The distance from the focus to the point of tangency is

$$\sqrt{(d + r\sin\theta)^2 + (r\cos\theta)^2} \approx 1.02\,\mathrm{m}.$$

So the distance to the directrix is that plus $r\sin\theta$, so the directrix is at $1.09\,\mathrm{m}$ above the axis containing the top of the baskets and the launch point. So the velocity is the same as the velocity required to go up $1.09/2$ meters from the launch point, so we set

$$\frac{v^2}{2g} = 1.09/2 \implies v_{min} = 3.27 \text{ m/s}.$$

There are some things we need to check though. We need to make sure the ball doesn't curve back in quickly enough to hit the rim, but this is pretty unlikely to occur considering the curvatures, the circle has a much smaller radius of curvature than the parabola there, so it won't go back and hit it. A quick drawing also confirms that the safety parabola doesn't hit the other rim, so we can see that the trajectory won't hit the other rim either.

$\boxed{3.27 \text{ m/s}}$

15. Right Triangle Potentials Let ABC be a solid right triangle ($AB = 5s$, $AC = 12s$, and $BC = 13s$) with uniform charge density σ. Let D be the midpoint of BC. We denote the electric potential of a point P by $\phi(P)$. The electric potential at infinity is 0. If $\phi(B) + \phi(C) + \phi(D) = \frac{k\sigma s}{\epsilon_0}$ where k is a dimensionless constant, determine k.

Solution: If we put two of these right triangles together, we can form a rectangle with side lengths $5s$ and $12s$. Let V be the potential at the center of this rectangle. By superposition, $\phi(D) = \frac{V}{2}$. Consider the potential at the corner. It can be decomposed into the potential from each of the right triangles, which is precisely $\phi(B) + \phi(C)$. Now, also note that if we put 4 of these rectangles side by side, to make a larger rectangle with side lengths $10s$ and $24s$, the potential is scaled by a factor of 2 due to dimensional analysis arguments. Thus, the potential at the center of this larger rectangle is $2V$, but this can be decomposed into the sum of the potentials at the

corners of the 4 smaller rectangles. Thus, the potential in the corner of the smaller rectangle is $\frac{V}{2} = \phi(B) + \phi(C)$. Thus, we obtain $\phi(B) + \phi(C) + \phi(D) = V$.
Now, we will find the potential at the center of the rectangle. Note that it suffices to find the potential at the vertex of an isosceles triangle because we can connect the center of the rectangle to the 4 corners to create 4 isosceles triangles. Suppose we have an isosceles triangle with base $2x$ and height y. The potential at the vertex is

$$\int\int \frac{\sigma}{4\pi\epsilon_0 r}(rdrd\theta) = \frac{\sigma}{4\pi\epsilon_0}\int_{-\tan^{-1}(\frac{x}{y})}^{\tan^{-1}(\frac{x}{y})}\int_0^{\frac{y}{\cos\theta}} drd\theta = \frac{\sigma y}{2\pi\epsilon_0}\log\left(\frac{x+\sqrt{x^2+y^2}}{y}\right).$$

If the sides of the rectangle are a and b, we then obtain the potential at the center is

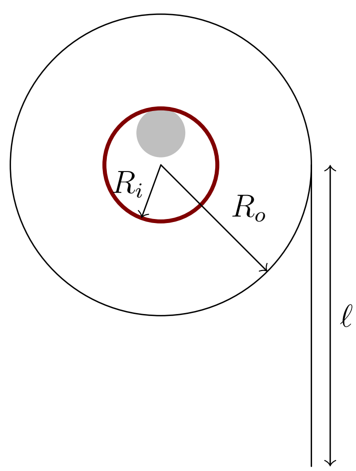
$$V = \frac{\sigma}{2\pi\epsilon_0}\left(a\log\left(\frac{b+\sqrt{a^2+b^2}}{a}\right) + b\log\left(\frac{a+\sqrt{a^2+b^2}}{b}\right)\right).$$

In this case,

$$k = \frac{5}{2\pi}\ln 5 + \frac{6}{\pi}\ln\frac{3}{2}.$$

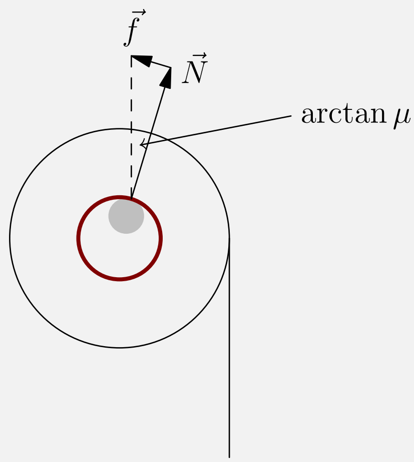
$\boxed{2.055}$

16. Toilet Paper Roll Consider a toilet paper roll with some length of it hanging off as shown. The toilet paper roll rests on a cylindrical pole of radius $r = 1$ cm and the coefficient of static friction between the role and the pole is $\mu = 0.3$.



The length of the paper hanging off has length $\ell = 30$ cm and the inner radius of the roll is $R_i = 2$ cm. The toilet paper has thickness $s = 0.1\,\text{mm}$ and mass per unit length $\lambda = 5$ g/m. What is the minimum outer radius R_o such that the toilet paper roll remains static? Answer in centimeters.

Solution: Due to the length of toilet paper hanging off, the toilet paper will be slightly tilted, in order for torques to balance. The tilt isn't shown in the diagram, since it is meant to be found. So the tilted normal force has to be compensated by the frictional force, and just before slippage, it will look something like:



Let $\theta = \arctan\mu$. So now balancing torques about the contact point, if m is the mass of the toilet paper roll,

$$mgR_i \sin\theta = (\ell\lambda)g(R_o - R_i \sin\theta).$$

And now we need m. The mass per unit area is $\sigma = \lambda/s$, so $m = \pi(R_o^2 - R_i^2)\lambda/s$. So substituting this in,

$$\frac{\pi(R_o^2 - R_i^2)\lambda}{s} gR_i \sin\theta = \ell\lambda g(R_o - R_i \sin\theta).$$

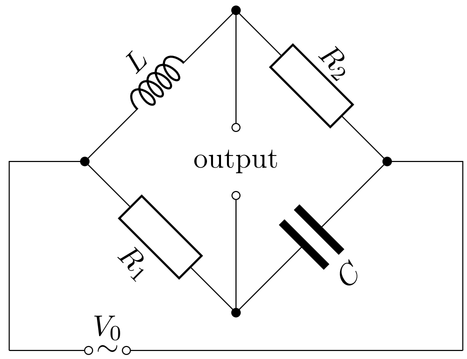
Simplifying a bit,

$$\pi(R_o^2 - R_i^2)R_i \sin\theta = s\ell\left(R_o - R_i \sin\theta\right).$$

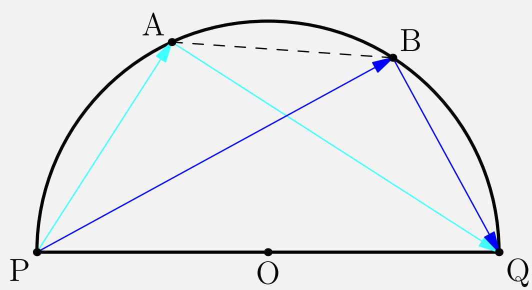
This is a quadratic and we can just plug in the numbers and use the quadratic formula or use a graphing calculator to finish, yielding $R_o = 2.061$ cm.

$\boxed{2.061 \text{ cm}}$

17. Maximum Voltage In the circuit shown below, a capacitor $C = 4\text{F}$, inductor $L = 5\text{H}$, and resistors $R_1 = 3\Omega$ and $R_2 = 2\Omega$ are placed in a diamond shape and are then fed an alternating current with peak voltage $V_0 = 1\text{V}$ of unknown frequency. Determine the magnitude of the maximum instantaneous output voltage shown in the diagram.



Solution: We use the method of phasors. Consider the following phasor diagram:



We define angles as $\angle APB = \angle AQB = \gamma$, $\angle AOB = 2\gamma$, $\angle BPQ = \alpha$, $\angle AQP = \beta$. Note that $\overrightarrow{PA} = I_2\omega L$, $\overrightarrow{AQ} = I_2 R$, $\overrightarrow{PB} = I_1 R_1$, $\overrightarrow{BQ} = 1/I_1\omega C$. We seek to maximize the length of AB. We can write via law of cosines that

$$\text{AB} = \sqrt{r^2 + r^2 - 2r\cos(2\gamma)} = 2r\sin(\gamma).$$

where $r = V_0$ is the radius of the circle of which phasors are inscribed in. Since $\gamma = \alpha + \beta$, we can then rewrite the length of AB to be

$$\text{AB} = r(\cos\alpha\cos\beta - \sin\alpha\sin\beta) = V_0\left(\frac{R_1R_2}{V_0^2} - \frac{L}{CV_0^2}\right)I_1I_2 = V_0\left(R_1R_2 - \frac{L}{C}\right)\left(\frac{1}{|Z_1||Z_2|}\right)$$

where the product of both complex exponentials is simply

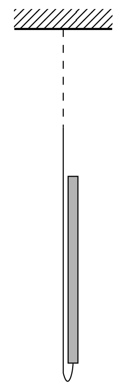
$$|Z_1||Z_2| = \sqrt{\omega^2L^2 + R_2^2}\sqrt{\frac{1}{\omega^2C^2} + R_1^2}$$

which implies the best frequency of the circuit is $\omega = \sqrt{\frac{R_2^2}{L^2R_1C}}$. This means with algebra, the lowest value of $|Z_1||Z_2| = \frac{L}{C} - R_1R_2$. Hence, the maximum value of AB is simply

$$\text{AB} = V_0\left(\frac{R_1R_2 - \frac{L}{C}}{R_1R_2 + \frac{L}{C}}\right).$$

$\boxed{0.65\text{ V}}$

18. Suspended Rod - 1 A uniform bar of length l and mass m is connected to a very long thread of negligible mass suspended from a ceiling. It is then rotated such that it is vertically upside down and then released. Initially, the rod is in unstable equilibrium. As it falls down, the minimum tension acting on the thread over the rod's entire motion is given by αmg. Determine α.



Solution: First note that the thread is given to be very long. Therefore, only vertical tension and gravitational forces act on the rod allowing for its center of mass to move in a straight vertical line. Using this fact, we can now write the acceleration of the rod in terms of angular velocity ω and angular acceleration ε. Let the angle of the rod at any moment to the vertical be φ. For simplicity, we define the length of the rod to be $2l$. Then, by defining the coordinate y to be the change in vertical length of the rod where $y = l\cos\varphi$, one can write for varying $\varphi \in [\varphi, \varphi + \mathrm{d}\varphi)$ that $l\cos(\varphi + \mathrm{d}\varphi) = y + \mathrm{d}y \implies \mathrm{d}y = l\sin\varphi \mathrm{d}\varphi$. Thus, simple differentiation proves that

$$v = l\sin\varphi\omega \implies a = l\varepsilon\sin\varphi + l\omega^2\cos\varphi.$$

We can now also write conservation of energy to get another relationship between velocity and angular velocity. At any given moment, it can be written that $mgl = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 - mgl\cos\varphi$. Since $I = \frac{1}{2}m(2l)^2 = \frac{1}{3}ml^2$, then

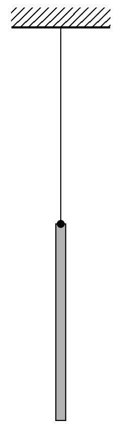
$$mgl(1+\cos\varphi) = ml^2\omega^2\left(\frac{1}{2}\sin^2\varphi + \frac{1}{6}\right).$$

Our third equation comes from the fact that the tension on the rod at any moment can be written as $T = mg - ma \implies ma = mg - T$. We can finally get a fourth equation by equating torques such that $\frac{1}{3}ml^2\varepsilon = Tl\sin\varphi$. With these four equations, simple algebra yields the tension at any point is

$$T(\varphi) = \frac{1+(3\cos\varphi - 1)^2}{(1+3\sin^2\varphi)^2}mg \implies T_{\min} \approx 0.165mg.$$

$\boxed{0.165}$

19. Suspended Rod - 2 A uniform bar of length l and mass m is connected to a thread of length $2l$ of negligible mass and is suspended from the ceiling at equilibrium. The rod is then slightly nudged at a point on its body. The largest stable frequency of oscillations of the system is given by $\beta\sqrt{\frac{g}{l}}$. Determine β.



Solution: Let the angle the rod makes with the vertical be θ, and let the angle the string makes with the vertical be ϕ. Note that the angles are in opposite directions.Throughout the motion, the tension in the string remains at approximately mg. Furthermore, we assume the motion is in simple harmonic oscillation, so $\ddot{\theta} = -\omega^2\theta$ and $\ddot{\phi} = -\omega^2\phi$. Writing the force and torque equations of the rod gives

$$mg\phi = m\left(2l\omega^2\phi - \frac{l}{2}\omega^2\theta\right)$$
$$mg(\theta+\phi)\frac{l}{2} = \frac{1}{12}ml^2\omega^2\theta$$

Solving this gives us a quartic equation $l^2\omega^4 - 8gl\omega^2 + 3g^2 = 0$ which implies that $\omega^2 = (4\pm\sqrt{13})\frac{g}{l}$. Thus, the largest mode of frequency is $f_{\text{max}} = \frac{1}{2\pi}\sqrt{4+\sqrt{13}\frac{g}{l}}$. However, note that a nudge could be interpreted as giving the rod a slight torque or a slight impulse. Which, in the later case would imply, with some calculations, that the largest oscillation frequency is not physically significant as the impulse would be imparted farther away than the rods length. Thus, we accepted either root for this problem.

$$\boxed{\frac{1}{2\pi}\sqrt{4\pm\sqrt{13}}}$$

20. One Ladder A straight ladder AB of mass $m = 1$ kg is positioned almost vertically such that point B is in contact with the ground with a coefficient of friction $\mu = 0.15$. It is given an infinitesimal kick at the point A so that the ladder begins rotating about point B. Find the value ϕ_m of angle ϕ of the ladder with the vertical at which the lower end B starts slipping on the ground.

Solution: By conservation of energy, we have $\frac{1}{2}I\omega_m^2 = mg\frac{L}{2}(1-\cos\phi_m)$ where $I = \frac{1}{3}mL^2$. Thus,

$$\omega_m = \sqrt{\frac{3g(1-\cos\phi_m)}{L}}.$$

Also, by torque analysis about B, we have $\tau = mg\frac{L}{2}\sin\phi_m = I\alpha_m$ which means

$$\alpha_m = \frac{3g}{2L}\sin\phi_m.$$

Thus, the centripetal and tangential accelerations of the ladder are $a_c = \omega_m^2\frac{L}{2} = \frac{3}{2}g(1-\cos\phi_m)$ and $a_t = \alpha_m\frac{L}{2} = \frac{3}{4}g\sin\phi_m$ respectively. The normal force is thus $N = mg - ma_c\cos\phi - ma_t\sin\phi$, so

$$\frac{N}{mg} = 1 - \frac{3}{2}\cos\phi_m(1-\cos\phi_m) - \frac{3}{4}\sin^2\phi_m.$$

The frictional force is thus $f = ma_t\cos\phi_m - ma_c\sin\phi_m$ so

$$\frac{f}{mg} = \frac{3}{4}\sin\phi_m\cos\phi_m - \frac{3}{2}\sin\phi_m(1-\cos\phi_m).$$

Setting $\frac{f}{N} = \mu$, we have $6\sin\phi_m(1-\cos\phi_m) - 3\sin\phi_m\cos\phi_m = -4\mu + 6\mu\cos\phi_m(1-\cos\phi_m) + 3\mu\sin^2\phi_m$. Simplifying

$$6\sin\phi_m - 9\sin\phi_m\cos\phi_m + 9\mu\cos^2\phi_m - 6\mu\cos\phi_m = -\mu.$$

We then can solve for ϕ_m numerically.

$\boxed{11.5^\circ}$

21. Two Ladders Two straight ladders AB and CD, each with length 1 m, are symmetrically placed on smooth ground, leaning on each other, such that they are touching with their ends B and C, ends A and D are touching the floor. The friction at any two surfaces is negligible. Initially both ladders are almost parallel and vertical. Find the distance AD when the points B and C lose contact.

Solution: The center of mass of both of the ladders moves in a circle, centered at the point on the ground directly beneath B/C. So we find when the required normal force between the two ladders is 0. That is, when the total net force on one of the ladders is when the two ladders lose contact. Let $2r = \ell$. Now by conservation of energy,

$$\frac{1}{2}mv^2 + \frac{1}{2}\frac{mr^2}{3}\frac{v^2}{r^2} = mgr(1-\cos\theta),$$

where θ is defined as the angle the ladder makes witht he vertical. So we have

$$v^2\left(1+\frac{1}{3}\right) = 2gr(1-\cos\theta) \implies v^2 = \frac{3}{2}gr(1-\cos\theta).$$

So the centripetal acceleration is

$$a_c = \frac{v^2}{r} = \frac{3}{2}g(1-\cos\theta).$$

And the tangential acceleration is

$$\frac{dv}{dt} = \sqrt{\frac{3gr}{2}} \frac{\sin\theta}{2\sqrt{1-\cos\theta}} \frac{d\theta}{dt}.$$

And $\frac{d\theta}{dt} = \frac{v}{r}$, so

$$a_\theta = \frac{dv}{dt} = \frac{3g}{2}\frac{\sin\theta}{2}.$$

Now for the total acceleration to be vertical, we need

$$a_c \tan\theta = a_\theta,$$

so

$$1 - \cos\theta = \frac{\cos\theta}{2}.$$

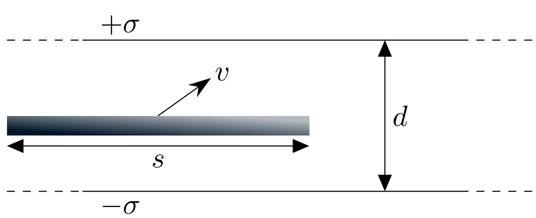
This simplifies to

$$2 = 3\cos\theta.$$

So the distance between the two ends is $4r\sin\theta = 4r\sqrt{1-\frac{4}{9}} = \frac{4r\sqrt{5}}{3}$.

$\boxed{1.491 \text{ m}}$

22. Colliding Conducting Slab A thin conducting square slab with side length $s = 5\,\text{cm}$, initial charge $q = 0.1\,\mu\text{C}$, and mass $m = 100\,\text{g}$ is given a kick and sent bouncing between two infinite conducting plates separated by a distance $d = 0.5\,\text{cm} \ll s$ and with surface charge density $\pm\sigma = \pm 50\,\mu\text{C/m}^2$. After a long time it is observed exactly in the middle of the two plates to be traveling with velocity of magnitude $v = 3\,\text{m/s}$ and direction $\theta = 30^\circ$ with respect to the horizontal line parallel to the plates. How many collisions occur after it has traveled a distance $L = 15\,\text{m}$ horizontally from when it was last observed? Assume that all collisions are elastic, and neglect induced charges. Note that the setup is horizontal so gravity does not need to be accounted for.



Solution: After the first collision, the charge approaches a constant magnitude. Let's look at a collision with the plate at charge q hitting the $-\sigma$ plate. Since the slab is thin, to keep the conducting surface an equipotential, the charge on the conduting slab has to become $-\sigma s^2$. The initial charge doesn't matter because the plates are infinite. So the charge is $-\sigma s^2$ after leaving the negative plate and $+\sigma s^2$ after leaving the positive plate.

Now the acceleration always has a constant magnitude, which is directed toward the plate it is traveling to, which we can find using $F = ma$,

$$(\sigma s^2)\frac{\sigma}{\epsilon_0} = ma \implies a = \frac{\sigma^2 s^2}{\epsilon_0 m} = 7.06\,\text{m/s}^2.$$

Then here we can use the trick of reflecting over the plates, and now it's a basic kinematics problem. The time to travel L is $t = \frac{L}{v\cos\theta}$, and then the distance it travels vertically is

$$y = v\sin\theta\frac{L}{v\cos\theta} + \frac{1}{2}a\left(\frac{L}{v\cos\theta}\right)^2 = 126.36\,\mathrm{m}.$$

And since $25272d + d/2 \approx 87.129$, the number of collisions is 25273.

$\boxed{25273}$

23. Evil Gamma Photon An evil gamma photon of energy $E_{\gamma 1} = 200$ keV is heading towards a spaceship. The commander's only choice is shooting another photon in the direction of the gamma photon such that they 'collide' head on and produce an electron-positron pair (both have mass m_e). Find the lower bound on the energy $E_{\gamma 2}$ of the photon as imposed by the principles of special relativity such that this occurs. Answer in keV.

Solution: The key claim is that energy is minimized when both particles are moving at the same velocity after the collision. This can be proved by transforming into the frame where the total momentum is 0.

This idea is sufficient because it implies both the electron and positron have the same momentum and energy after the collision. Let them both have momentum p. We then have

$$2p = \frac{E_{\gamma_1} - E_{\gamma_2}}{c} \implies pc = \frac{E_{\gamma_1} - E_{\gamma_2}}{2}. \tag{21}$$

By energy conservation, both the electron and positron have energy $\frac{E_{\gamma_1}+E_{\gamma_2}}{2}$. Using the result $E = (pc)^2 + (mc^2)^2$, we obtain,

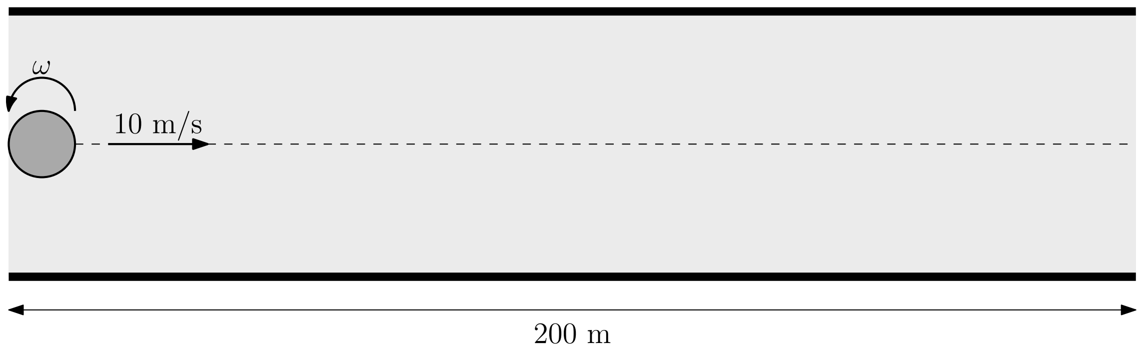
$$\left(\frac{E_{\gamma_1} + E_{\gamma_2}}{2}\right)^2 = (pc)^2 + (m_e c^2)^2. \tag{22}$$

Combining equation 21 and equation 22, we get

$$\left(\frac{E_{\gamma_1} + E_{\gamma_2}}{2}\right)^2 = \left(\frac{E_{\gamma_1} - E_{\gamma_2}}{2}\right)^2 + (m_e c^2)^2 \implies E_{\gamma_2} = \frac{m_e^2 c^4}{E_{\gamma_1}}.$$

$\boxed{1306 \text{ keV}}$

24. Spinning Cylinder Adithya has a solid cylinder of mass $M = 10\,\mathrm{kg}$, radius $R = 0.08\,\mathrm{m}$, and height $H = 0.20\,\mathrm{m}$. He is running a test in a chamber on Earth over a distance of $d = 200\,\mathrm{m}$ as shown below. Assume that the physical length of the chamber is much greater than d (i.e. the chamber extends far to the left and right of the testing area). The chamber is filled with an ideal fluid with uniform density $\rho = 700\,\mathrm{kg/m^3}$. Adithya's cylinder is launched with linear velocity $v = 10\,\mathrm{m/s}$ and spins counterclockwise with angular velocity ω. Adithya notices that the cylinder continues on a **horizontal path** until the end of the chamber. Find the angular velocity ω. Do not neglect forces due to fluid pressure differences. Note that the diagram presents a side view of the chamber (i.e. gravity is oriented downwards with respect to the diagram).



Assume the following about the setup and the ideal fluid:

- fluid flow is steady in the frame of the center of mass of the cylinder
- the ideal fluid is incompressible, irrotational, and has zero viscosity
- the angular velocity of the cylinder is approximately constant during its subsequent motion

Hint: For a uniform **cylinder** of radius R rotating counterclockwise at angular velocity ω situated in an ideal fluid with flow velocity u to the **right** far away from the cylinder, the velocity potential Φ is given by

$$\Phi(r,\theta) = ur\cos\theta + u\frac{R^2}{r}\cos\theta + \frac{\Gamma\theta}{2\pi}$$

where (r,θ) is the polar coordinate system with origin at the center of the cylinder. Γ is the circulation and is equal to $2\pi R^2\omega$. The fluid velocity is given by

$$\mathbf{v} = \nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}.$$

Solution: We will work in the reference frame of the center of mass of the cylinder because the fluid flow is steady in this reference frame. The key intuition here is that the magnitude of the fluid velocity above the cylinder will be higher on the top because the tangential velocity of the cylinder is in the same direction as the velocity of the fluid on the top. By Bernoulli's principle, this means that the pressure on the top is lower than the pressure on the bottom, which will create a lift force on the cylinder.
With the given theory, we can model this quantitatively. In our chosen reference frame, the water

moves with velocity v to the left. The velocity potential around a cylinder with radius R is

$$\Phi(r,\theta) = -vr\cos\theta - v\frac{R^2}{r}\cos\theta + R^2\omega\theta.$$

Therefore, we find

$$\mathbf{v} = \nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} = -v\left(1-\frac{R^2}{r^2}\right)\cos\theta\hat{\mathbf{r}} + \left(v\left(1+\frac{R^2}{r^2}\right)\sin\theta + R\omega\right)\hat{\theta}.$$

As expected from boundary conditions, the radial velocity vanishes when $r = R$. Furthermore, on the surface of the cylinder, we have the tangential velocity of the fluid is $2v\sin\theta + R\omega$ in the counterclockwise direction. Consider points on the cylinder at angles θ and $-\theta$. By Bernoulli's principle (ignoring the height difference which will be accounted with the buoyant force),

$$p_{-\theta} - p_\theta = \frac{1}{2}\rho((2v\sin\theta + R\omega)^2 - (-2v\sin\theta + R\omega)^2) = 4\rho vR\omega\sin\theta.$$

If we integrate this result along the surface of the cylinder, we can find the lift force per unit length. Note that only the vertical components of the pressure will matter as the horizontal components cancel due to symmetry. The vertical component of the pressure difference is then $4\rho vr_0\omega\sin^2\theta$. Thus, the lift force per unit length is

$$\frac{F_{\text{lift}}}{H} = \int_0^\pi 4\rho vR\omega\sin^2(\theta)(Rd\theta) = 2\pi\rho\omega vR^2.$$

The total left force is

$$F_{\text{lift}} = 2\pi\rho R^2 H\omega v.$$

The gravitational force is Mg, and the buoyant force is $\pi R^2 H\rho g$. Therefore, we must have

$$\pi R^2 H\rho g + 2\pi\rho R^2 H\omega v = Mg.$$

Solving for ω, we obtain

$$\omega = \frac{Mg}{2\pi R^2 H\rho v} - \frac{g}{2v}.$$

$\boxed{1.25\,\mathrm{s}^{-1}}$

25. Optimal Launch Adithya is launching a package from New York City ($40°43'$ N and $73°56'$ W) to Guam ($13°27'$ N and $144°48'$ E). Find the minimal launch velocity v_0 from New York City to Guam. Ignore the rotation of the earth, effects due to the atmosphere, and the gravitational force from the sun. Additionally, assume the Earth is a perfect sphere with radius $R_\oplus = 6.37\times10^6$ m and mass $M_\oplus = 5.97\times10^{24}$ kg.

Solution: We first want to find the angular distance between New York City and Guam. Let this be θ. Let New York City be point A and Guam be point B. Consider the north pole P and the spherical triangle PAB. By the spherical law of cosines,

$$\cos\theta = \cos(90° - \phi_A)\cos(90° - \phi_B) + \sin(90° - \phi_A)\sin(90° - \phi_B)\cos(\ell_B - \ell_A). \tag{23}$$

Equation 23 simplifies to

$$\cos\theta = \sin\phi_A \sin\phi_B + \cos\phi_A \cos\phi_B \cos(\ell_B - \ell_A) \tag{24}$$

from which we find $\theta = 115.05°$.
Now that we have determined the angular distance, we will proceed with the orbital mechanics problem. By the vis-viva equation, the speed at the launch point is

$$v_0 = \sqrt{GM_\oplus\left(\frac{2}{R_\oplus} - \frac{1}{a}\right)} \tag{25}$$

where a is the semimajor axis for the orbit. It is clear that in order to minimize v_0, we must minimize a. The orbit is an ellipse with focii F_1 and F_2, where F_1 is the center of the earth. By the definition of an ellipse,

$$AF_1 + AF_2 = 2a. \tag{26}$$

Since $AF_1 = R_\oplus$, is suffices to minimize AF_2. By symmetry, line F_1F_2 is the perpendicular bisector of AB. Since F_2 is on a fixed line, to minimize F_2, we place it at the foot of the perpendicular from A to this line. Then, we obtain $AF_2 = R_\oplus \sin\left(\frac{\theta}{2}\right)$. Thus,

$$a = \frac{1 + \sin\left(\frac{\theta}{2}\right)}{2} R_\oplus,$$

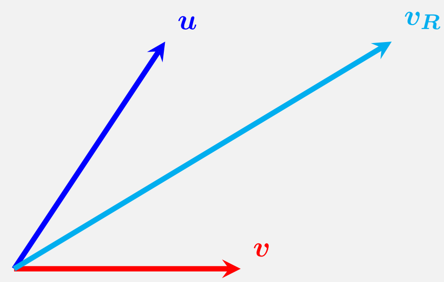
and from Equation 25 we find

$$v_0 = \sqrt{\frac{GM_\oplus}{R_\oplus}\frac{2\sin\frac{\theta}{2}}{1+\sin\frac{\theta}{2}}}.$$

$\boxed{7564\,\mathrm{m/s}}$

26. Drag on the Plate Consider a container filled with argon, with molar mass $39.9\mathrm{g\,mol^{-1}}$ whose pressure is much smaller than that of atmospheric pressure. Suppose there is a plate of area $A = 10\mathrm{mm}^2$ moving with a speed v perpendicular to its plane. If the gas has density $\rho = 4.8 \times 10^{-7}\mathrm{g\,cm^{-3}}$, and temperature $T = 100\mathrm{K}$, find an approximate value for the drag force acting on the plate. Suppose that the speed of the plate is $v = 100\mathrm{m\,s^{-1}}$.

Solution: Let N be the number of particles colliding with the plate in a unit time, and let all the molecules arrive at the plate with the same speed u. The force on the plate will be the momentum imparted over a unit time, i.e. $F = 2Nm(v-u)/t$ where the coefficient off two reflects on the fact that the average speed of the molecules hitting the wall is the same as the molecules departing. Note that Maxwells distribution dictates that the average speed of particles in all direction $\langle u\rangle = 0$ which means that the average force acting on the plate is simply $\langle F\rangle = 2Nmv/t$. During a time period t, these molecules arrive a wall that is of thickness $v_R t$ where v_R is the relative velocity between the molecules and the plate (which is not negligible as the mean free path and velocity of the plate are comparable.) This means the number of collisions found in this layer will be $N = \frac{1}{2}nV \approx Av_R t$ where the factor of 1/2 reflects on how half the molecules go to the plate and the other half go the other direction. Thus the force acting on the plate will become $\langle F_D\rangle = 2mvN/t = 2nmv \cdot v_R A$. To find the relative velocity, consider the vector diagram:



By law of cosines, the magnitude of v_R will simply be

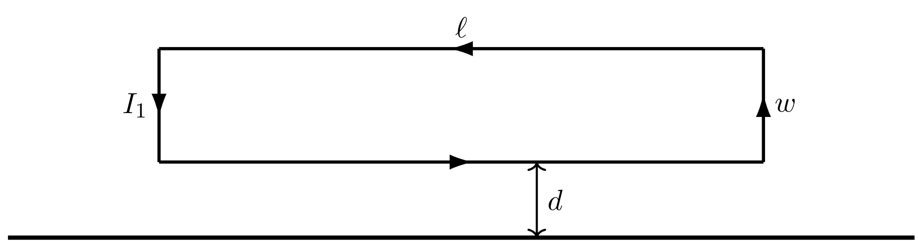
$$v_R^2 = v^2 + u^2 - 2\boldsymbol{v}\cdot\boldsymbol{u}.$$

The direction of $\boldsymbol{u}$ points homogeneously in all directions as the orientation of molecules changes with each individual one. Therefore, $2\boldsymbol{v}\cdot\boldsymbol{u}$ will point in all directions averaging to 0. Thus, the magnitude of v_R will simply be $\langle v_R\rangle = \sqrt{v^2 + \langle u\rangle^2}$ where $\langle u\rangle^2$ is the average thermal velocity of molecules. In terms of density, we can then express the drag force to be $2\rho A v\sqrt{v^2+\langle u\rangle^2}$.

$\boxed{2.41\times 10^{-4}\ \mathrm{N}}$.

27. Superconducting Loop Consider a rectangular loop made of superconducting material with length $\ell = 200$ cm and width $w = 2$ cm. The radius of this particular wire is $r = 0.5$ mm. This superconducting rectangular loop initially has a current $I_1 = 5$A in the counterclockwise direction as shown in the figure below. This rectangular loop is situated a distance $d = 1$ cm above an infinitely long wire that initially contains no current. Suppose that the current in the infinitely long wire is increased to some current I_2 such that there is an attractive force F between the rectangular loop and the long wire. Find the maximum possible value of F. Write your answer in newtons.

Hint: You may neglect the magnetic field produced by the vertical segments in the rectangular loop.



Solution: The key idea is that the superconducting loop must have constant flux. If it did not, by Faraday's Law, an emf

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

would be generated in the loop. Since superconducting materials have no resistance, this would imply an infinite current, hence a contradiction.
We will first compute the flux through the rectangular loop when there is a current I_1. Since $w \ll \ell$, we can assume that the vertical segments produce negligible amounts of magnetic field. We can furthermore approximate the field produced by one of the horizontal wires a distance r away as

$\frac{\mu_0 I}{2\pi r}$ (this is valid for an infinitely long wire, and therefore is also valid in the regime where $w \ll \ell$). Thus, the total flux through the rectangular loop when there is a current I_1 is

$$\Phi_1 = \int_r^w B(\ell dr') = \int_r^{w-r} \left(\frac{\mu_0 I_1}{2\pi r'} + \frac{\mu_0 I_1}{2\pi(w-r')}\right) \ell dr' = \frac{\mu_0 I_1 \ell}{\pi} \ln\left(\frac{w}{r}\right).$$

Note that the self inductance of the loop is $L = \frac{\Phi}{I_1} = \frac{\mu_0 \ell}{\pi} \ln\left(\frac{w}{r}\right)$.
Now, we will determine the flux through the rectangular loop due to the long current-carrying wire. This is

$$\Phi_2 = \int_d^{d+w} \frac{\mu_0 I_2}{2\pi r}(\ell dr) = \frac{\mu_0 I_2 \ell}{2\pi} \ln\left(\frac{d+w}{d}\right).$$

The mutual inductance is $M = \frac{\Phi_2}{I_2} = \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{d+w}{d}\right)$. In to maintain the same flux in the loop, the current will change to I_3 where

$$LI_1 = MI_2 + LI_3,$$

or

$$I_3 = I_1 - \frac{M}{L} I_2.$$

Now, we compute the force between the rectangular loop and the long, current-carrying wire. The forces on the vertical sides cancel out because the current in the loop is in opposite directions on these sides. From the horizontal sides, we have the force is

$$\begin{aligned} F = \sum (I_3 \vec{\ell} \times \vec{B}) &= I_3 \ell \left(\frac{\mu_0 I_2}{2\pi d} - \frac{\mu_0 I_2}{2\pi(d+w)}\right) \\ &= \frac{\mu_0 \ell w}{2\pi d(d+w)} \left[I_2 \left(I_1 - \frac{M}{L} I_2\right)\right]. \end{aligned}$$

This quadratic in I_2 is maximized when $I_2 = \frac{L}{2M} I_1$ in which case the force becomes

$$F = \frac{\mu_0 \ell w}{2\pi d(d+w)} \frac{LI_1^2}{4M} = \frac{\mu_0 \ell w I_1^2}{4\pi d(d+w)} \frac{\ln\left(\frac{w}{r}\right)}{\ln\left(\frac{d+w}{d}\right)}.$$

$\boxed{1.12 \times 10^{-3}\ \text{N}}$

Note: If the size of the wires is considered when computing flux, a slightly different answer is obtained. In the contest, all answers between 1.11×10^{-3} and 1.18×10^{-3} were accepted.

28. Cantor Interference Consider a 1 cm long slit with negligible height. First, we divide the slit into thirds and cover the middle third. Then, we perform the same steps on the two shorter slits. Again, we perform the same steps on the four even shorter slits and continue for a very long time.

Then, we shine a monochromatic, coherent light source of wavelength 500 nm on our slits, which creates an interference pattern on a wall 10 meters away. On the wall, what is the distance between the central maximum and the first side maximum? Assume the distance to the wall is much greater than the width of the slit. Answer in millimeters.

SOLUTION: This problem is essentially an infinite convolution. In detail, consider two separate amplitude functions $f(x)$ and $g(x)$ that correspond to two different interference patterns. One can think of a convolution of f and g as taking f and placing the same function in every position that exists in g. To be more specific, the amplitude of one interference pattern can be thought of as the product of the amplitude patterns of two *separate* amplitude functions. For example, the diffraction pattern of two slits is the same as the product of the amplitudes for one slit and two light sources (this will be given as an exercise to prove). To make more sense of this, let us consider that very example and let f pertain to the amplitude function of a single slit and let g pertain to the amplitude function of two light sources. One can then write in phase space with a generalized coordinate x' that $\{f * g\}(x) = \int_{-\infty}^{\infty} f(x-x')g(x')\mathrm{d}x'$ according to the convolution theorem. The integral then runs through all values and places a copy of $f(x-x')$ at the peaks of $g(x')$.
With this in hand, we can use the convolution theorem to our advantage. Let us designate the amplitude function of the entire Cantor slit to be $F(\theta)$. Since θ is small, we can designate the phase angle as $\phi = kx\theta$ where x is a variable moving through all of the slits. If we consider a single slit that is cut into a third, the angle produced will be a third of its original as well, or the new amplitude function will simply be a function of $\theta/3$ or $F(\theta/3)$. The distance between the midpoints of the two slits will simply be $d = 2/3$ cm which then becomes $2/9$, then $2/27$ and so on. In terms of the original function, we can decompose it into the function of $F(\theta/3)$ and the amplitude function of a single light source. In other words, $F(\theta) = F(\theta/3)\cos(kd\theta/2)$ where $k = \frac{2\pi}{\lambda}$. To the limit of infinity, this can be otherwise written as

$$F(\theta) = \prod_{i=0}^{\infty} \cos\left(\frac{kd\theta}{2\cdot 3^i}\right) = \cos\left(\frac{kd\theta}{2}\right)\cos\left(\frac{kd\theta}{6}\right)\cos\left(\frac{kd\theta}{18}\right)\dots$$

Finding an exact mathematical solution would be difficult, thus it is simple enough to just multiply the first 4 or 5 terms to achieve the final answer.

$\boxed{0.647 \text{ mm}}$

The following information applies to the next 2 problems. A certain planet with radius $R = 3\times 10^4$ km is made of a liquid with constant density $\rho = 1.5$ g/cm^3 with the exception of a homogeneous solid core of radius $r = 10$ km and mass $m = 2.4\times 10^{16}$ kg. Normally, the core is situated at the geometric center of the planet. However, a small disturbance has moved the center of the core $x = 1$ km away from the geometric center of the planet. The core is released from rest, and the fluid is inviscid and incompressible.

29. SOLID CORE - 1 Calculate the magnitude of the force due to gravity that now acts on the core. Work under the assumption that $R \gg r$.

SOLUTION: We solve by simplifying the configuration into progressively simpler but mathematically equivalent formulations of the same problem.

Because the core is spherical and homogeneous, all the gravitational forces on the core are equivalent to the gravitational forces on a point mass located at the center of the core. Thus, the problem is now the force of gravity on a point mass of mass m located x distance away from the center of planet, with a sphere of radius r evacuated around the point mass.

But if we filled this evacuated sphere, would the force on the point mass be any different? No! If we filled up the sphere, all the added liquid (of the same density as the rest of the planet) would add no additional force on the point mass because it is symmetrically distributed around the point mass. Thus, the problem is now the force of gravity on a point mass of mass m located x distance away from the center of the planet.

By shell theorem, we can ignore all the fluid that is more than a distance x away from the center of the planet. Thus, the problem is now the force of gravity on a point mass m situated on the surface of a liquid planet of radius x. This force is not difficult to calculate at all:

$$\begin{aligned} F &= \frac{GMm}{x^2} \\ &= \frac{Gm}{x^2}\left(\frac{4}{3}\pi x^3\rho\right) \\ &= \frac{4}{3}\pi Gm\rho x \end{aligned}$$

$\boxed{1.0058 \times 10^{13}\ \text{N}}$

30. Solid Core - 2 Calculate the magnitude of the force due to the pressure from the liquid that now acts on the core.

Solution:
Let ρ_1 be the density of the planet and ρ_2 be the density of the core. Let the center of the planet be A, and the center of the core be B. Let the gravitational potential energy of system be $U(x)$. Initially, the core and fluid are stationary, but due to the gravitational and pressure forces on the core, the core will accelerate towards A, with some acceleration a. Note that since the mass of the planet is much greater than that of the core, the planet's CM and shape will be affected negligibly. After a small time t, conservation of energy gives $\frac{1}{2}\mathcal{M}(at)^2 = \Delta U = U'(x) \cdot \frac{1}{2}at^2$, where $\mathcal{M}$ takes into account the mass of the core and also the motion of the fluid around the core. Once we determine $\mathcal{M}$ and $U(x)$, we can solve for the acceleration of the core, which will give the total force on the core. Then we subtract off the gravitational force calculated in Part 1 to get the pressure force.

The effective mass $\mathcal{M}$ is given by $\rho_2 \cdot \frac{4}{3}\pi r^3 + \rho_1 \cdot \frac{2}{3}\pi r^3$, where the first term is the actual mass and the second term is the added mass, see here. Thus, $\mathcal{M} = (2\rho_2 + \rho_1) \cdot \frac{2}{3}\pi r^3$.
To find the gravitational potential energy $U(x)$, we consider how much work it takes to move the particles infinitely away. We do the following procedure:

- Step 1: Compress a sphere of density $\rho_2 - \rho_1$ and radius r centered at B so that it becomes nearly a point mass at B.
- Step 2: Move the "point mass" infinitely far away from the planet.
- Step 3: Expand the "point mass" until it becomes a sphere of radius r again.
- Step 4: Expand both the planet and the sphere infinitely, OR simply calculate the gravitational potential energy of this simple arrangement.

Before we do any steps yet, first note that to solve the problem we only need to analyze Steps 1 and 2 because the other steps do not involve x and can be treated as constants (since we are interested in $U'(x)$). Here we include the other steps for completeness.

We first tackle Step 1. Note that the gravitational field inside a uniform solid sphere of density ρ is $\mathbf{g} = -\frac{4\pi G}{3}\rho\mathbf{r}$, where $\mathbf{r}$ is the position vector from the center. Let $\mathbf{v}_1 = \overrightarrow{\mathrm{AB}}$ and $\mathbf{v}_2 = \mathbf{r} - \mathbf{v}_1$, so $\mathbf{g} = -\frac{4\pi G}{3}\rho_1(\mathbf{v}_1 + \mathbf{v}_2)$. Suppose Agent 1 is responsible for countering the force due to $\mathbf{v}_1$, and Agent 2 for $\mathbf{v}_2$. Say Agent 3 is responsible for countering the force due to the sphere's own self gravity. Then, Agent 1 does no work, because the CM of the sphere does not change throughout Step 1. The work Agent 2 does W_2 can be computed by noting that the potential associated with the force due to $\mathbf{v}_2$ is $\frac{2\pi G}{3}\rho r^2$. Thus,

$$W_2 = -\int_0^r (\rho_2 - \rho_1) \cdot 4\pi r^2 \cdot \frac{2\pi G}{3}\rho_1 r^2 dr = -\frac{8\pi^2 G}{15}\rho_1(\rho_2 - \rho_1)r^5.$$

For now, we defined the work done by Agent 3 to be W_3. The total work done in Step 1 is thus $W_2 + W_3 = -\frac{8\pi^2 G}{15}\rho_1(\rho_2 - \rho_1)r^5 + W_3$

Now, we do Step 2. The work it takes to move the point mass to the edge of the planet is $\frac{2\pi G}{3}\rho_1(R^2 - x^2) \cdot \frac{4}{3}\pi r^3(\rho_2 - \rho_1)$. The work it takes to further move the point mass to infinity is $G \cdot \rho_1 \cdot \frac{4}{3}\pi R^2 \cdot \frac{4}{3}\pi r^3(\rho_2 - \rho_1)$. Thus, the total work in Step 2 is $\frac{8\pi^2 G}{9}\rho_1(\rho_2 - \rho_1)r^3(3R^2 - x^2)$.

Next, we do Step 3. The work it takes to expand the point mass back to a sphere of radius r is simply $-W_3$.

Finally, we do Step 4. The potential energy of a uniform sphere of density ρ and radius r is $-\frac{16}{15}\pi^2 G\rho^2 R^5$. Thus, the total potential energy of the planet and the sphere is $-\frac{16}{15}\pi^2 G\rho_1^2 R^5 - \frac{16}{15}\pi^2 G(\rho_2 - \rho_1)^2 r^5$.

Combining all steps gives $U(x) = \frac{8\pi^2 G}{9}\rho_1(\rho_2 - \rho_1)r^3 x^2 + C$, where C is some constant independent of x. We have

$$a = \frac{U'(x)}{\mathcal{M}} = \frac{16\pi^2 G\rho_1(\rho_2 - \rho_1)r^3 x}{9 \cdot (2\rho_2 + \rho_1) \cdot \frac{2}{3}\pi r^3} = \frac{8\pi^2 G\rho_1(\rho_2 - \rho_1)x}{3(2\rho_2 + \rho_1)}.$$

The gravitational force from part 1 was $F_g = \frac{16}{9}\pi^2 G\rho_1\rho_2 r^3 x$. Thus, the pressure force is

$$\begin{aligned} F_p = F_g - Ma &= \frac{16}{9}\pi^2 G\rho_1\rho_2 r^3 x - \frac{4}{3}\pi r^3\rho_2 \cdot \frac{8\pi^2 G\rho_1(\rho_2 - \rho_1)x}{3(2\rho_2 + \rho_1)} \\ &= \frac{16}{9}\pi^2 G\rho_1\rho_2 r^3 x(1 - 2 \cdot \frac{\rho_2 - \rho_1}{2\rho_2 + \rho_1}) = \frac{16}{3}\pi^2 G r^3 x \cdot \frac{\rho_1^2\rho_2}{2\rho_2 + \rho_1}. \end{aligned}$$

$\boxed{3.4926 \times 10^{12}\ \mathrm{N}}$

31. Solenoids A scientist is doing an experiment with a setup consisting of 2 ideal solenoids that share the same axis. The lengths of the solenoids are both ℓ, the radii of the solenoids are r and $2r$, and the smaller solenoid is completely inside the larger one. Suppose that the solenoids share the same (constant) current I, but the inner solenoid has $4N$ loops while the outer one has N, and they have opposite polarities (meaning the current is clockwise in one solenoid but counterclockwise in the other).

Model the Earth's magnetic field as one produced by a magnetic dipole centered in the Earth's core.

Let F be the magnitude of the total magnetic force the whole setup feels due to Earth's magnetic field. Now the scientist replaces the setup with a similar one: the only differences are that the the radii of the solenoids are $2r$ (inner) and $3r$ (outer), the length of each solenoid is 7ℓ, and the number of loops each solenoid is $27N$ (inner) and $12N$ (outer). The scientist now drives a constant current $2I$ through the setup (the solenoids still have opposite polarities), and the whole setup feels a total force of magnitude F' due to the Earth's magnetic field. Assuming the new setup was in the same location on Earth and had the same orientation as the old one, find F'/F.

Assume the dimensions of the solenoids are much smaller than the radius of the Earth.

Solution: We can solve the problem by assuming that the location of the setup is at the North Pole and that the solenoids are oriented so that their axis intersects the Earth's core. Note that if we had some other location or orientation, then both F and F' would be multiplied by the same factor, so their ratio remains the same.

Suppose the radii of the solenoids are r and αr, where the number of inner and outer loops are N and $\frac{N}{\alpha^2}$, respectively. To find the force the Earth's dipole exerts on the solenoids, we can calculate the force the solenoids exert on the dipole. To do this, we need to find the gradient of the magnetic field produced by the solenoids at the dipole's location. Let the radius of the Earth be R.

Consider the field produced by 2 concentric, coaxial, ring currents, the inner ring with current I radius r and the outer one with current $\frac{I}{\alpha^2}$ and radius αr. The currents are in opposite directions. At a distance R away from the center of the rings, along their axis, the magnetic field is given by

$$\begin{aligned} B &= \frac{\mu_0 I r^2}{2(R^2+r^2)^{\frac{3}{2}}} - \frac{\mu_0 I r^2}{2(R^2+(\alpha r)^2)^{\frac{3}{2}}} \\ &= \frac{\mu_0 I r^2}{2R^3}\left(\left(1+\frac{r^2}{R^2}\right)^{-\frac{3}{2}} - \left(1+\frac{\alpha^2 r^2}{R^2}\right)^{-\frac{3}{2}}\right) \\ &\approx \frac{\mu_0 I r^2}{2R^3}\left(\frac{3}{2}(\alpha^2-1)\frac{r^2}{R^2}\right) \\ &= \frac{3\mu_0 I r^4}{4R^5}(\alpha^2-1) \end{aligned}$$

Thus, the gradient of the magnetic field is proportional to $Ir^4(\alpha^2-1)$. Now we consider the actual setup. The new setup multiplies the effective current $\frac{27}{4}\cdot\frac{2}{1}=\frac{27}{2}$ times, while multiplying r by 2. The factor α^2-1 changed from 3 to $\frac{5}{4}$. Combining, we get $\frac{F'}{F}=\frac{27}{2}\cdot 2^4\cdot\frac{5}{12}=90$.

$\boxed{90}$

The following information applies to the next 2 problems. Adithya is in a rocket with proper acceleration $a_0 = 3.00\times10^8\ \mathrm{m/s^2}$ to the right, and Eddie is in a rocket with proper acceleration $\frac{a_0}{2}$ to the left. Let the frame of Adithya's rocket be S_1, and the frame of Eddie's rocket be S_2. Initially, both rockets are at rest with respect to each other, they are at the same location, and Adithya's clock and Eddie's clock are both set to 0.

32. Accelerating Rockets - 1 At the moment Adithya's clock reaches 0.75 s in S_2, what is the **velocity** of Adithya's rocket in S_2?

Solution: Throughout this solution, we will let $c = a_0 = 1$, for simplicity. We will work in the inertial frame that is initially at rest with both rockets at $t_1 = t_2 = 0$. First, we determine the velocity of Adithya's rocket in this frame as a function of the proper time that has elapsed t_1. In frame. S_1, in a time dt_1, the rocket's velocity increases by dt_1, so by velocity addition, the new velocity in the inertial frame is

$$\frac{v_1 + dt_1}{1 + v_1 dt_1} \approx v_1 + (1 - v_1^2)dt_1.$$

Therefore, we have

$$\frac{dv_1}{dt_1} = 1 - v_1^2.$$

Upon separating and integrating, we find $v_1 = \tanh(t_1)$. Similarly, the velocity of Eddie's rocket in the inertial frame is $v_2 = \tanh(t_2/2)$. Now, in this inertial frame, let the time between events A and B be t, and let the distance between rockets A and B be x. By a Lorentz transformation, the time between the events in frame S_2 is

$$t' = \gamma(t - v_2 x) = 0$$

since the events are simultaneous in S_2. Therefore, we must have $t = v_2 x$. Note that $t = t_A - t_B$ where t_A and t_B denote the times of events A and B in the inertial frame, respectively. Also, note that $x = x_A + x_B$ where x_A and x_B are the respective displacements of the two rockets in the inertial frame. By the effects of time dilation, we have

$$t_A = \int \gamma dt_1 = \int \frac{1}{\sqrt{1 - v_1^2}} dt_1 = \int_0^{t_1} \cosh\left(t_1'\right) dt_1' = \sinh(t_1).$$

Similarly, $t_B = 2\sinh(t_2/2)$, and we obtain $t = 2\sinh(t_2/2) - \sinh(t_1)$. Additionally, from the above result,

$$x_A = \int v_1 dt = \int v_1 \cosh(t_1) dt_1 = \int_0^{t_1} \sinh(t_1) dt_1' = \cosh(t_1) - 1.$$

Similarly, $x_B = 2\cosh(t_2/2) - 2$, and $x = \cosh(t_1) + 2\cosh(t_2/2) - 3$. Thus, from $t = v_2 x$,

$$2\sinh(t_2/2) - \sinh(t_1) = \tanh(t_2/2)(\cosh(t_1) + 2\cosh(t_2/2) - 3).$$

$$-\sinh(t_1) = \tanh(t_2/2)(\cosh(t_1) - 3).$$

$$\tanh(t_2/2) = \frac{\sinh(t_1)}{3 - \cosh(t_1)}.$$

Now, this is the velocity of Eddie's rocket as measured from the inertial frame, so by velocity addition, the velocity of Adithya's rocket as seen by Eddie is

$$v = \frac{\tanh(t_1) + \frac{\sinh(t_1)}{3 - \cosh(t_1)}}{1 + \frac{\sinh(t_1)\tanh(t_1)}{3 - \cosh(t_1)}} = \frac{3\tanh(t_1)}{3 - \cosh(t_1) + \sinh(t_1)\tanh(t_1)}.$$

$\boxed{0.855377c = 2.564 \times 10^8 \text{ m/s}}$.

33. Accelerating Rockets - 2 At the moment Adithya's clock reaches 0.75 s in S_2, what is the **acceleration** of Adithya's rocket in S_2?

Solution: Observe that Eddie measures the following acceleration:

$$a = \frac{dv}{dt_2} = \frac{dv}{dt_1} \cdot \frac{dt_1}{dt_2}.$$

Both of these results can be determined from the equations above. Note that from quotient rule:

$$\frac{dv}{dt_1} = \frac{3\operatorname{sech}^2(t_1)(3-\cosh(t_1))}{(3-\cosh(t_1)+\sinh(t_1)\tanh(t_1))^2} = 0.6151.$$

Additionally, we obtain

$$\frac{dt_1}{dt_2} = \frac{\operatorname{sech}^2(t_2/2)(3-\cosh(t_1))^2}{2(3\cosh(t_1)-1)} = 0.3869.$$

$\boxed{7.14 \cdot 10^7 \text{ m/s}^2}$.

The following information applies to the next 2 problems. Suppose a ping pong ball of radius R, thickness t, made out of a material with density ρ_b, and Young's modulus Y, is hit so that it resonates in mid-air with small amplitude oscillations. Assume $t \ll R$. The density of air around (and inside) the ball is ρ_a, and the air pressure is p, where $\rho_a \ll \rho_b \frac{t}{R}$ and $p \ll Y\frac{t^3}{R^3}$.

34. Ping Pong - 1 An estimate for the resonance frequency is $\omega \sim R^a t^b \rho_b^c Y^d$. Find the value of $4a^2 + 3b^2 + 2c^2 + d^2$.

Hint: The surface of the ball will oscillate by "bending" instead of "stretching", since the former takes much less energy than the latter.

Solution: Throughout the problem, we will work to order of magnitude and ignore prefactors.

The hint says the surface of the ball will bend instead of stretch, so we need to develop a theory of bending. First, we consider the simplified scenario of a long beam with thickness t, width w, length L, made out of a material with Young's modulus Y and density ρ. When the beam bends, the top part of the beam will be in tension and the bottom part will be in compression. Thus, this is how the potential energy is stored in the beam. Suppose the beam is bent with curvature κ. Then the top part of the beam will stretch by $Lt\kappa$, and the bottom part will compress by the same amount. Using Hooke's law, we can approximate the total potential energy stored in the beam as $U \sim \frac{Ytw}{L}(Lt\kappa)^2 \sim Yt^3wL\kappa^2$. Note that if the relaxed state of the beam was already curved, we simply replace κ with the change in curvature.

To find the oscillation frequency of the beam, we need to find the kinetic energy in terms of $\dot{\kappa}$. Since curvature is on the order of second derivative of displacement, we can multiply κ by L^2 to get an estimate for displacement. Then $\dot{\kappa}L^2$ gives an estimate for speed, so the kinetic energy is $K \sim \rho twL(\dot{\kappa}L^2)^2 \sim \rho twL^5\dot{\kappa}^2$. Thus, the frequency of oscillations is $\omega \sim \sqrt{\frac{Yt^2}{\rho L^4}}$. Again, if the beam was already curved, we can replace κ everywhere with the change in curvature.

We can model the ping pong ball as a "beam" of length order R, width order R, and thickness t. This is a very crude approximation, but will give a dimensionally correct answer (since we are ignoring prefactors). The angular frequency is thus $\omega \sim \frac{t}{R^2}\sqrt{\frac{Y}{\rho_b}}$.

$\boxed{19.75}$

35. Ping Pong - 2 Assuming that the ball loses mechanical energy only through the surrounding air, find an estimate of the characteristic time τ it takes for the ball to stop resonating (or to lose half its mechanical energy), that is $\tau \sim R^\alpha t^\beta \rho_b^\kappa Y^\delta \rho_a^\zeta p^\gamma$. Find the value of $6\alpha^2 + 5\beta^2 + 4\kappa^2 + 3\delta^2 + 2\zeta^2 + \gamma^2$. (Note that in reality, the ball also loses mechanical energy to heat, but we will neglect that for simplicity.)

Solution:
The ping pong ball loses energy through the surrounding sound waves. Let the surface of the ball oscillate with amplitude on the order of A. Then the energy stored in the ball is on the order of kinetic energy, which is on the order of $E \sim \rho_b R^2 t A^2 \omega^2$. The sound waves have intensity $I \sim \rho_a \omega^2 A^2 c_s$, where the speed of sound $c_s \sim \sqrt{\frac{p}{\rho_a}}$. Thus, the total energy lost per unit time is $P \sim IR^2 \sim \sqrt{\rho_a p} A^2 \omega^2 R^2$. Finally, the characteristic time $\tau \sim \frac{E}{P} \sim \frac{\rho_b t}{\sqrt{\rho_a p}}$.

$\boxed{9.75}$

2021 Online Physics Olympiad: Invitational Contest

v1.1 Theoretical Exam

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This competition could not be possible without the help of our sponsors, who are all doing great things in physics, math, and education.

Instructions for Theoretical Exam

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- The team leader should submit their final solution document in this google form. We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it.
- If you wish to request a clarification, please use this form. To see all clarifications, view this document.
- Each question in this examination are equally worth 25 points. Be sure to spend your time wisely.
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- Since each question is a long answer response, participants will be judged on the quality of your work. To receive full points, participants need to show their work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in the IPhO formula sheet) can be cited without proof.
- Remember to state any approximations made and which system of equations were solved after every step. Explicitly showing every step of algebra is not necessary. Participants may leave all final answers in symbolic form (in terms of variables) unless otherwise specified. Be sure to state all assumptions.

Problems

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Permittivity of free space	$\epsilon_0 = 8.85 \cdot 10^{-12}$ C^2/(N $\cdot$ m^2)
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Optical Tweezers

An optical tweezer (OT) is a device which uses tightly focused laser beams to trap an object in all three spacial dimensions ($x-y-z$). Consider using an OT to trap a dielectric polystyrene nanosphere with mass m, radius R, and relative dielectric constant ε_r. A laser beam is directed to the vertical z-direction (the laser beam is similar to a monochromatic light wave propagating a sparse medium.) The laser beam has a wavelength λ. The time averaged intensity in the z-direction due to light can be considered to follow a Gaussian distribution function:

$$I(\rho, z) = I_0 \left(\frac{W_0}{W(z)}\right)^2 \exp\left(\frac{-2\rho^2}{W(z)^2}\right)$$

where ρ is the distance from the center of the beam and W_0 is known as the waist size, or the measure of the beam size at the point of its focus. Here the waist length, in general, follows $W(z) = W_0\sqrt{1 + z^2/z_R^2}$ where $z_R = \pi W_0^2/\lambda$ denotes the Rayleigh length.

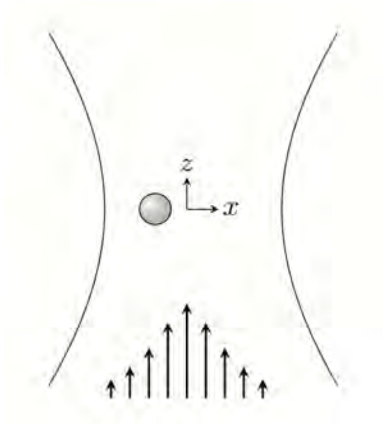

Figure 1: A nanosphere placed off center in a Gaussian beam.

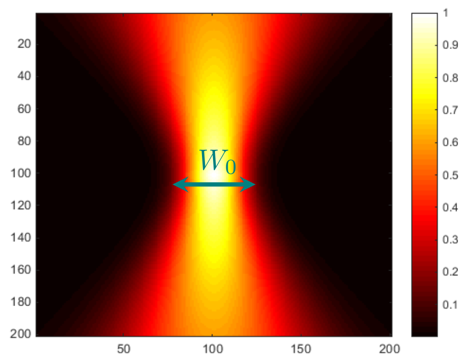

Figure 2: A graphic of the intensity distribution in a Gaussian beam.

The OT traps particles via three different forces: **scattering forces** created by the change in momentum of light scattered or absorbed by a particle; **gradient forces** due to the polarization of the particle created by the strong electric fields of the laser beam; and, **radiation forces** produced by an accelerating charge. The total power radiated by an oscillating electric dipole with dipole moment p_0 at frequency ω will be $P_R = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$, where c is the speed of light.

1. **(6 pts.)** In the Rayleigh spectrum, the nanosphere's size is such that $\lambda \gg R$. Find the oscillation frequency Ω and equilibrium position of the nanosphere when slightly displaced a distance $d \ll W_0$ in the x-direction. Neglect the scattering forces in this part.

In the Mie spectrum, the particle size is no longer negligible such as in part A, and follows $R \gtrsim \lambda$[1]. As a result, a non-homogenous electric field is incident upon the sphere and scattering forces are no longer neglectable. Parts B and C will investigate the nanosphere in the Mie spectrum.

[1] The order of magnitude of R is greater than λ.

2. **(5 pts.)** Determine the scattering force and torque on the nanosphere as a function of a distance $x \ll W_0$ from its origin. We can consider $x \sim R$. Assume that 100% of light is transmitted for simplicity.[2] The index of refraction of the nanosphere is n while the index of refraction of the medium is m. You may express your answer as an integral if needed.

3. **(2 pts.)** In a simplified model, the torque acting on the nanosphere can be represented as $\boldsymbol{\tau} = \kappa \boldsymbol{\omega}$ where κ is a numerical constant and $\boldsymbol{\omega}$ is the angular velocity of the nanosphere. At $t = 0$, the nanosphere is stationary with an angular velocity ω_0 and the OT is turned on. After a time T, the OT is turned off with the nanosphere left floating in the medium again. Determine the angular velocity of the nanosphere after a time t passes where t is the time of the entire process. It can be expressed as a piecewise function.

Acoustic Levitation

Objects can also be trapped via sound waves. Consider the simplest way to model sound waves; that is, in one dimension. A cylindrical tube of length L_0 with ambient temperature and pressure T_0 and P_0 is fixed with a piston at one end[3] of cross-sectional area S that moves periodically as $x(t) = A\cos(2\pi f t)$ where $A \ll L_0$ is the amplitude of the piston and f is the frequency of the process. The tube contains n monatomic particles of mass m per unit volume. As the piston moves back and forth, the air in the tube compresses or expands (rarefies) in the tube as a sound wave. The molecules within the tube move back or forth parallel to their equilibrium position as the air travels within the piston. Neglect any viscous or turbulent friction within the pipe.

4. **(6 pts.)** What is the average power required to move the piston? Consider the limits of $f \gg c_s/A$ and $f \ll c_s/A$ where c_s is the speed of the sound wave.

In sound waves, the density perturbations are very small, so it can be assumed that $\Delta\rho \ll \rho_0$[4]. where ρ_0 is the original density of the pipe. Furthermore, the wavelength of the sound waves are much larger than the mean free path of the gas molecules. The sound wave created by the oscillating piston moves at a speed c_s and dissipates the power given by the piston.

5. **(4 pts.)** As a result of compression, the air within the sound wave has a larger temperature $\Delta T \ll T_0$. Find the change in temperature ΔT. If you were unable to solve problem 4, express the average power to move the piston as P.

6. **(2 pts.)** Consider a small cylindrical object of radius $R < \sqrt{S}$ and width $h \ll R$ in the pipe where variations of pressure on the cylinder's surface are negligible. Determine the force F acting on the cylinder when the sound wave passes through it. If the pipe is placed on a vertical plane where gravity is present, qualitatively describe what location(s) the cylinder would levitate.

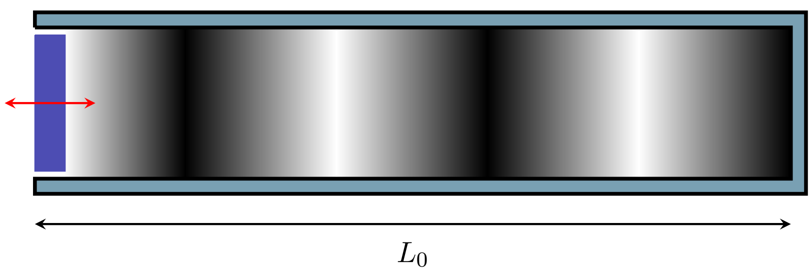

Figure 3: A visualization of how acoustic waves within the pipe are created via the oscillation of the piston.

[2] In reality, some light will be reflected due to Fresnel's equations.
[3] In some models, the piston can represent the moving cone of a loud speaker.
[4] You cannot use $\Delta\rho$ as a variable in this problem, but you can use ρ_0.

T2: Thomas Precession

Sucessive Transformations

In this section, we examine what happens when two successive Lorentz transformations are applied in non-parallel directions.

1. **(6 pts.)** Consider the three reference frames S_1, S_2, and S_3. Events as seen from frame S_i will be labeled with the space-time coordinates (x_i, y_i, z_i, t_i), for $i = 1, 2, 3$. All three frames coincide at $(0, 0, 0, 0)$. Suppose frame S_2 travels with velocity βc in the x_1-direction of frame S_1 and frame S_3 travels with velocity $\beta'_x c\hat{\mathbf{i}}_2 + \beta'_y c\hat{\mathbf{j}}_2$ with respect to frame S_2.

   Perform two successive Lorentz transformations: one expressing the S_3 coordinates in terms of the S_2 coordinates and another expressing the S_2 coordinates in terms of the S_1 coordinates. Then, as the final answer, express the S_3 coordinates in terms of the S_1 coordinates.

   Assume that $\beta' = (\beta'^2_x + \beta'^2_y)^{1/2} \ll \beta$ and work to first order.

2. **(5 pts.)** Now, find the velocity of S_3 in S_1 with the appropriate velocity addition. Perform a single Lorentz transformation to express the S_3 coordinates in terms of the S_1 coordinates. Once again, work to first order. The answer will not be the same as part 1.

3. **(3 pts.)** Show that your answer in Problem 1 differs from your answer in Problem 2 by a spatial rotation. In other words, two successive Lorentz transformations in non-parallel directions cannot be combined as one Lorentz transformation. Rather, they are the combination of one Lorentz transformation and one spatial rotation. Determine the magnitude and direction of this spatial rotation in the current setup.

   Then argue and explain why the answer in Problem 1 and not Problem 2 is the correct transformation.

Precession Frequency

In this section, we examine the precession of the electron's spin magnetic moment within the hydrogen atom.

Electrons possess an attribute known as *spin*. One can imagine the electron as being a small, spherical charged particle that is spinning on some axis. Although this mental picture is not physically correct, it is enough for us. From this mental picture, we can gather that the electron will possess a dipole moment and angular momentum *intrinsic* to itself and not induced by any orbital motion. Respectively, these are known as the *spin magnetic moment* and the *spin angular momentum* that behave, in our model, just as their classical counterparts do. These two vector quantities, $\boldsymbol{\mu}$ and $\mathbf{L}$ respectively, are related by

$$\boldsymbol{\mu} \simeq -\frac{e}{m_e}\mathbf{L},$$

which is determined by experiment.

In this section, we will use the Bohr model of the hydrogen atom, where the electron circles the proton at some radius r, pulled in by the Coulomb force.

4. **(3 pts.)** In the laboratory frame, the electron orbits the proton with some velocity v in the x-y plane. Now switch to the instantaneous rest frame of the electron, where the proton moves with speed v relative to the stationary electron. The moving proton will then induce some magnetic field at the electron's location.

   Suppose the spin angular momentum of the electron points in some direction other than the direction of the magnetic field. Because the electron also possesses a spin magnetic moment, it will precess as a result of the torque done on it. Find the angular frequency of the precession of the electron's spin in terms of r and whatever fundamental constants.

   Ignore any relativistic effects.

5. **(6 pts.)** This problem will use the answer from Part 1. Once again, assume that the electron is orbiting in the x-y plane.

   Consider the instantaneous rest frame of the electron at some time t, S_2. Also consider the instantaneous rest frame of the electron at some time $t + \mathrm{d}t$, S_3. Relative to the laboratory frame, S_1, S_2 will have velocity $\mathbf{v}$. S_3 will have a velocity $\mathrm{d}\mathbf{v}$ relative to S_2, but $\mathbf{v}$ and $\mathrm{d}\mathbf{v}$ won't be parallel.

   The result from Part 1 tells us that S_3 will experience an infinitesimal rotation in this time $\mathrm{d}t$ with respect to the laboratory frame. Since the electron is continuously accelerating, its rest frame, S_3, must then rotate continuously relative to the lab frame. Assume that the spin of the electron always points in the same direction in its rest frame. If the spin is not pointing in the z direction, find the angular frequency of the spin's precession in terms of r and whatever fundamental constants. Ignore the effects of the previous problem.

   Note: although relativistic effects cannot be ignored, you can assume that the electron's velocity v is not comparable to the speed of light in the calculation.

6. **(2 pts.)** Combine your answers from parts 4 and 5 and find the relativistically correct angular frequency of the precession of the electron's spin.

T3: Moving Media

Interesting phenomena can arise in situations where there are 2 media that are moving with respect to each other. In particular, objects can move at much faster speeds than the relative speeds of the media, without using any energy. In this problem, we explore two such examples of this effect.

Moving Cylinders

Suppose we have three cylinders, two small cylinders and a large cylinder, of radii r and R. The frictionless pivots (centers) of the cylinders are attached to a massless triangular frame, such that the large cylinder is in contact with the two small cylinders but the two small cylinders are not touching each other. The small cylinders each have a thin groove along their circumferences (which does not affect the moment of inertia significantly), so that the large cylinder makes contact with the small cylinder at a point with radial distance αr from the center of the small cylinder. The axes of all cylinders are perpendicular to the plane of the triangular frame. The system is placed on a level ground and a long flat horizontal board is put on top of the large cylinder, with the two small cylinders touching the ground (making contact at their outer edge with radial distance r, not αr). Assume that the friction due to contact between all surfaces is large enough to prevent any slipping.

1. **(4 pts.)** The board is moved with speed v in a direction perpendicular to the axes of the cylinders. Find the speed of the cylinder system.

2. **(5 pts.)** The mass of the small and large cylinders are m and M, respectively. The mass of the board is m'. If at a moment in time the board is pushed with speed v and acceleration a, find the power P required to push the board. Assume the cylinders have uniform mass distribution.

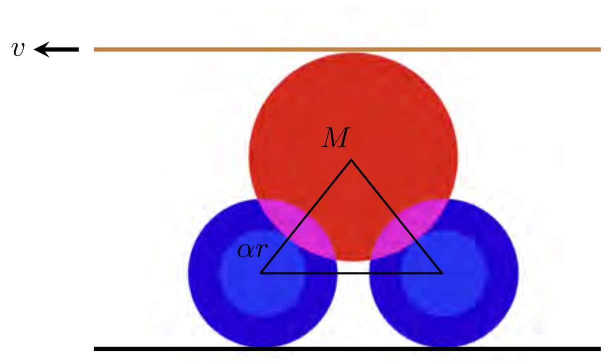

Figure 4: A visual of the three cylinder setup.

Windsurfing

In windsurfing, it is possible to sail faster than the wind without using any energy. Suppose we have a sailboat moving on a large, motionless body of water. The air of density ρ is moving at a speed v uniformly in one direction. If the sailboat is pointed in a certain direction and moves in that direction with velocity $\mathbf{u}$, the drag force from the water $\mathbf{F}$ satisfies $\mathbf{F} \cdot \mathbf{u} = -\gamma u^2$, where γ can be assumed to be a constant drag coefficient.

3. **(10 pts.)** If the wind is moving in the $\hat{x}$ direction, what is the maximum possible sustainable x-component of velocity for the sailboat? Assume that the sailboat can neither generate nor store energy in its interaction with the air. Also, the effective cross-sectional area of the sail is A (this is the component of cross-sectional area that is perpendicular to the wind in the reference frame of the sailboat).

4. **(3 pts.)** What is the power dissipated due to the interaction with the water?

5. **(3 pts.)** It seems that the law of conservation of energy is being violated, as the speed of the sailboat isn't changing despite heat generation in the water. Explain why energy is still conserved.

T4: Missing Energy

Part A

1. **(3 pts.)** Consider a simple circuit with two parallel-plate capacitors of capacitance C_1 and C_2 connected to each other using purely conducting wires and a switch. One of the capacitors is initially charged to a voltage V_0, while the other one is completely uncharged. The circuit is kept in a square shaped figure of side length ℓ throughout the problem, while the diameter of the conducting wires is D. Find the initial total energy of the circuit when the switch is open, given by E_0, and a sufficiently long time after the switch is closed, given by E_∞. Calculate the remaining energy $E_\Delta = E_0 - E_\infty$. What is E_Δ for the case $C_2 \to \infty$?

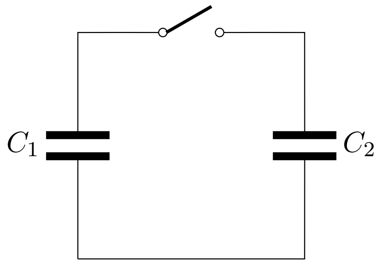

Figure 5: The two parallel plate capacitor-switch circuit.

It seems odd for there to be a difference in energy as the circuit is a closed system. Three young scientists Fermi, Jackson, and Feynman have created different theories to find and verify the correct source of this missing energy.

Thermal Losses: Fermi

To investigate the cause of this missing energy, Fermi assumes that there must be an ohmic resistive load r and a self-inductance L in the circuit responsible for E_Δ.

2. **(3 pts.)** Find the current in the circuit $I(t)$ as a function of time and E_Δ for the circuit.

3. **(1 pt.)** For small values of r, find the oscillation frequency Ω of $I(t)$.

4. **(1 pt.)** For $L = 0$, can Fermi's reasoning be correct for any value of r? If yes, what is this value of r?

Dipole Radiation Losses: Jackson

Jackson believes that the missing energy is dissipated in the form of dipole radiation losses due to the charges accelerating. He assumes that the electric dipole moment of the system remains constant during the process, but the magnetic moment is allowed to vary. Thus, he seeks to determine the maximal possible radiation losses. For this he uses Larmor's formula, which states that for small velocities relative to the speed of light c, total power radiated which the radiation power is defined as:

$$P_r = \frac{\ddot{m}^2}{4\pi\epsilon_0 c^3}$$

where m is the magnetic moment of the circuit as a function of time. Ignore all relativistic effects and the possible charge accumulation in the wires compared to that on the capacitor plates. Moreover, note that he does not assume any resistance or self-inductance in the circuit in his model.

5. **(4 pts.)** Find the total energy dissipation E_r due to this radiation.

6. **(1 pt.)** For what value of time interval $\Delta\tau$ taken by the charges to move from one capacitor to another, can Jackson's theory be reasoned true?

Kinetic Energy: Feynman

Feynman has the following hypothesis:

> The missing energy goes into the kinetic energy of the charge carriers going from C_1 to C_2. Assume that the mean free path of collisions of the carriers is $\lambda > 2\ell$.

7. **(4 pts.)** Find the total kinetic energy ΔK gained by the carriers during a total charge transfer from C_1 to C_2.

8. **(1 pt.)** Could this be a valid hypothesis to explain the cause of the missing energy? When the charges get completely deposited on the plates of C_2, what happens to this kinetic energy?

Part B

Instead of charging one capacitor using the other, we take an ideal parallel-plate capacitor such that the surface charge density $\pm\sigma$ on its plates is uniform throughout both plates, and that the charges are 'fixed' to the surface as the plate expands. The dimensions of the plates are a, b and the plate separation distance is d. The plate is now stretched quasi-statically by a factor of κ in one of the dimensions such that the dimensions of the capacitor plates are now $\kappa a, b$ but the plate separation remains d.

1. **(6 pts.)** Calculate the work $\mathrm{d}W$ done during stretching the plates of this capacitor. Also write a simplified form of this expression for $d \ll \kappa a, b$.

2. **(1 pt.)** In this case, there are no resistive loads, and since the process of plate expansion is quasi-static, there is no gain in kinetic energy of the charge carriers. Where does the energy disappear in this case?

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Theory Solutions

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Problems

- **T1: Levitation** by Ashmit Dutta
- **T2: Thomas Precession** by Jacob Nie
- **T3: Moving Media** by Eddie Chen
- **T4: Missing Energy** by Kushal Thaman

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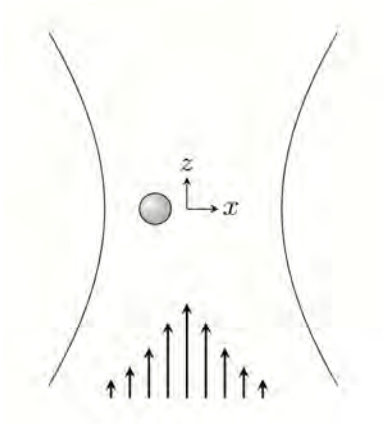

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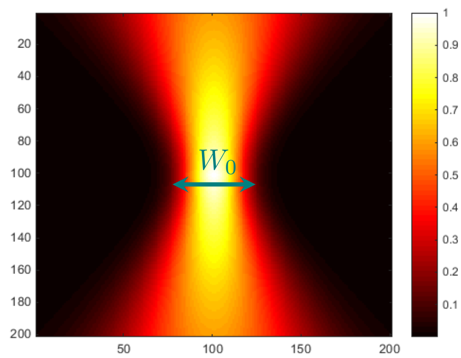

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[1] When the exam was administered, it was said that to neglect the scattering forces in this part. In reality, the radiation forces are a type of scattering force as it takes light and re-emits it. For ambiguity, the equilibrium position was not considered in grading.

[2] The order of magnitude of R is greater than λ.

2. **(5 pts.)** Determine the scattering force and torque on the nanosphere as a function of a distance $x \ll W_0$ from its origin. We can consider $x \sim R$. Assume that 100% of light is transmitted for simplicity.[3] The index of refraction of the nanosphere is n while the index of refraction of the medium is m. You may express your answer as an integral if needed.

3. **(2 pts.)** In a simplified model, the torque acting on the nanosphere can be represented as $\boldsymbol{\tau} = \kappa \boldsymbol{\omega}$ where κ is a numerical constant and $\boldsymbol{\omega}$ is the angular velocity of the nanosphere. At $t = 0$, the nanosphere is stationary with an angular velocity ω_0 and the OT is turned on. After a time T, the OT is turned off with the nanosphere left floating in the medium again. Determine the angular velocity of the nanosphere after a time t passes where t is the time of the entire process. It can be expressed as a piecewise function.

Acoustic Levitation

Objects can also be trapped via sound waves. Consider the simplest way to model sound waves; that is, in one dimension. A cylindrical tube of length L_0 with ambient temperature and pressure T_0 and P_0 is fixed with a piston at one end[4] of cross-sectional area S that moves periodically as $x(t) = A\cos(2\pi f t)$ where $A \ll L_0$ is the amplitude of the piston and f is the frequency of the process. The tube contains n monatomic particles of mass m per unit volume. As the piston moves back and forth, the air in the tube compresses or expands (rarefies) in the tube as a sound wave. The molecules within the tube move back or forth parallel to their equilibrium position as the air travels within the piston. Neglect any viscous or turbulent friction within the pipe.

4. **(6 pts.)** What is the average power required to move the piston? Consider the limits of $f \gg c_s/A$ and $f \ll c_s/A$ where c_s is the speed of the sound wave.

In sound waves, the density perturbations are very small, so it can be assumed that $\Delta\rho \ll \rho_0$[5]. where ρ_0 is the original density of the pipe. Furthermore, the wavelength of the sound waves are much larger than the mean free path of the gas molecules. The sound wave created by the oscillating piston moves at a speed c_s and dissipates the power given by the piston.

5. **(4 pts.)** As a result of compression, the air within the sound wave has a larger temperature $\Delta T \ll T_0$. Find the change in temperature ΔT. If you were unable to solve problem 4, express the average power to move the piston as P.

6. **(2 pts.)** Consider a small cylindrical object of radius $R < \sqrt{S}$ and width $h \ll R$ in the pipe where variations of pressure on the cylinder's surface are negligible. Determine the force F acting on the cylinder when the sound wave passes through it. If the pipe is placed on a vertical plane where gravity is present, qualitatively describe what location(s) the cylinder would levitate.

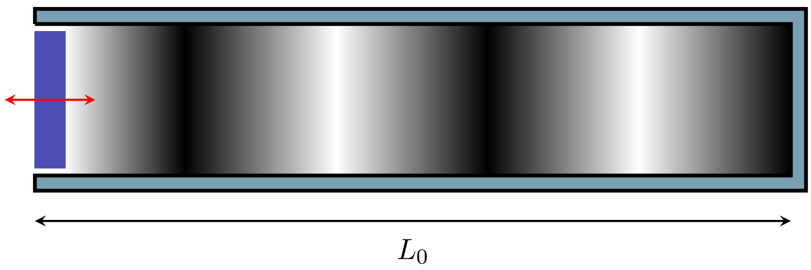


Figure 3: A visualization of how acoustic waves within the pipe are created via the oscillation of the piston.

[3]In reality, some light will be reflected due to Fresnel's equations.
[4]In some models, the piston can represent the moving cone of a loud speaker.
[5]You cannot use $\Delta\rho$ as a variable in this problem, but you can use ρ_0.

Solution

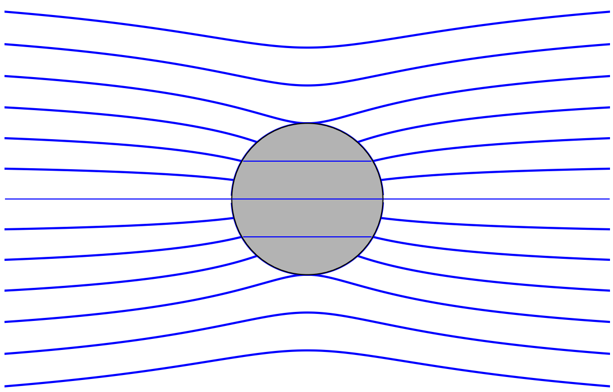
(a) The Rayleigh approximation tells us that $\lambda \gg R$. Thus, the electric field around the microsphere is homogenous; hence, the polarization of the microsphere will be as well. Furthermore, since the nanoball is displaced a small distance $d \ll W_0$, the intensity distribution can be approximated by Taylor expanding $e^x \approx 1 + x$. In other words:

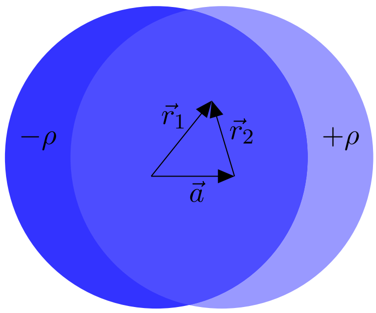
$$I(x) \approx I_0 \left(1 - \frac{2x^2}{W_0^2}\right).$$

Intensity and electric field can be related via the time averaged Poynting vector as as

$$I = |\langle \vec{S} \rangle| = \frac{1}{\mu_0} |\langle \vec{E} \times \vec{B} \rangle| = \frac{1}{c\mu_0} \frac{E_{\mathrm{m}}^2}{2}$$

where E_{m} is the maximum electric field of the Gaussian beam. As the nanoball becomes polarized, it obtains a dipole moment of $\vec{p} = -\alpha\vec{E}$. It is required to find the value of α of a dielectric ball. The polarized sphere can be divided into two different polarized rods of charge densities $+\rho$ and $-\rho$ separated a distance a, replacing them by patches of charge (circles) that are superimposed. Positive charges started accumulating on the upper half of the sphere while negative charges start accumulating on the lower half of the sphere as the electric field redistributes the charges.





Thus, the field outside the small sphere is the sum of E and the field due to an electric dipole that is at the center of the nanoball. The dipole moment expressed in terms of its polarization is

$$\vec{p} = \sum_i q\vec{r}_i = \vec{P} \cdot V = \frac{4}{3}\pi R^3 \vec{P}.$$

The total field at some point outside of the sphere is no longer uniform in the neighborhood of the sphere and is now the sum of the field E and the field generated through the polarized matter, i.e. $\vec{E} = \vec{E}_0 + \vec{E}_{\text{dipole}}$. Thus, the field depends on the polarization vector of the dielectric which can be expressed in terms of its electric susceptibility as

$$\vec{P} = \chi_{e0}\vec{E}_{\text{in}} = ({}_r - 1)_0 \vec{E}_{\text{in}}.$$

Note that the potential at all the points on the spherical boundary is simply the dipole potential or

$$= \frac{p\cos\theta}{4\pi_0 R^2} = \frac{PR\cos\theta}{3_0}.$$

Substituting $R\cos\theta = z$ implies $= Pz/3_0$. By uniqueness theorem, this must be the only solution which means that $E_z = \frac{\partial}{\partial z} = -P/3_0$. Hence, the internal field of the sphere follows $\vec{E}_{\text{in}} = -\vec{P}/3_0$. Superposition yields

$$\vec{E}_{\text{in}} = \vec{E}_0 - \frac{{}_r - 1}{3}\vec{E}_{\text{in}} \implies \vec{E}_{\text{in}} = \frac{3}{2+{}_r}\vec{E}.$$

Inserting this field into the dipole momentum tells us that

$$\vec{p} = \frac{4}{3}\pi R^3 ({}_r - 1)_0 \frac{3}{2 +_r} \vec{E} \implies \alpha = \frac{4\pi R^3 ({}_r - 1)_0}{2 +_r}.$$

The gradient force on the ball depends on the gradients of the various components of its fields. In other words, $\vec{F}(t) = (\vec{p} \cdot \nabla)\vec{E}(t)$. Averaging this quantity over time then allows to write

$$\langle \vec{F}(t) \rangle = \langle (\vec{p} \cdot \nabla)\vec{E}(t) \rangle = \langle (\alpha \vec{E}(t) \cdot \nabla)\vec{E}(t) \rangle = \frac{\alpha}{2} \nabla \langle E(t)^2 \rangle = \frac{\alpha}{2} \nabla \frac{E_{\mathrm{m}}^2}{2} = \frac{\alpha c \mu_0}{2} \partial_x I(x).$$

The standard differential equation for SHO can be retrieved from this as

$$m\ddot{x} = -\frac{2\alpha c \mu_0 I_0}{W_0^2} x = -\kappa_x x \implies \boxed{\Omega = \sqrt{\frac{2\alpha c \mu_0 I_0}{W_0^2}}}.$$

Here, κ_x is collectively known as the trap stiffness of the OT in the x-direction. To find the force due to radiation in the z-direction (as photons are directed on this axis), one can use the formula given in the problem for the power of radiation due to a rotating dipole[6]. As $P = \vec{F} \cdot \vec{v}$, one can write that $F = \frac{P}{c}$. Thus,

$$F = \frac{\mu_0 \alpha^2 E^2 \omega^4}{12\pi c^2} = \frac{\mu_0^2 \alpha^2 \left(\frac{2\pi c}{\lambda}\right)^4}{6\pi c} I(z) = \frac{8\pi^3 c^3 \mu_0^2 \alpha^2}{3\lambda^4} I_0 \left(1 - \frac{2z^2}{W_0^2}\right).$$

This must be equal to the gradient force under equilibrium, thus creating a quadratic equation on x:

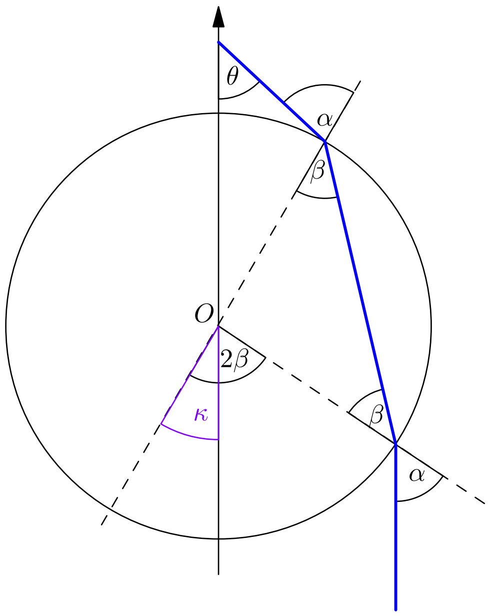
$$F_{\mathrm{rad}} = F_{\mathrm{grad}} \implies \frac{8\pi^3 c^3 \mu_0^2 \alpha^2}{3\lambda^4} I_0 \left(1 - \frac{2z^2}{W_0^2}\right) = -\frac{2\alpha c \mu_0 I_0}{W_0^2} z.$$

Substituting $\zeta \equiv \frac{4\pi^3 c^2 \mu_0 \alpha W_0^2}{3\lambda^4}$ yields a quadratic of

$$\frac{2\zeta}{W_0^2} z^2 - z - \zeta = 0 \implies \boxed{z = \frac{W_0^2}{4\zeta}\left(1 - \sqrt{1 - \frac{8\zeta^2}{W_0^2}}\right)}.$$

[6]This is well derived from the Poynting vector.

(b) In the Mie spectrum, $\lambda \gg R$ which implies that the intensity and electric field distribution over the sphere is no longer homogenous. Create a ray diagram:



The deviation in angle can be described by the angle κ or

$$\kappa = 2(\alpha - \beta) = 2\left(\alpha - \arcsin\left(\frac{m}{n}\sin\alpha\right)\right).$$

The change in momentum of photons that move towards the sphere can then be written as

$$\Delta p = \frac{E}{c}(1-\cos\kappa) \implies \mathrm{d}F = \frac{\Delta p}{\Delta t} = 2I\sin^2\left(\alpha - \arcsin\left(\frac{m}{n}\sin\alpha\right)\right)\mathrm{d}A.$$

We can use the law of cosines for a point away from the origin to find ρ:

$$\rho^2 = x^2 + (R\sin\theta)^2 - 2xR\sin\theta\cos.$$

Thus, putting everything together:

$$F_z = \frac{2I_0R^2}{c}\int_0^{\pi/2}\int_0^{2\pi}\sin^2\left(\alpha - \arcsin\left(\frac{m}{n}\sin\alpha\right)\right)\exp\left(\frac{-2\rho^2}{W_0^2}\right)\sin\theta\cos\theta\mathrm{d}\theta\mathrm{d}.$$

Similarly, you can write the forces in the $\hat{x}$ direction which will be similar to F_z:

$$F_x = \frac{I_0R^2}{c}\int_0^{\pi/2}\int_0^{2\pi}\sin\left(\alpha - \arcsin\left(\frac{m}{n}\sin\alpha\right)\right)\exp\left(\frac{-2\rho^2}{W_0^2}\right)\sin\theta\cos\theta\cos\mathrm{d}\theta\mathrm{d}.$$

The net force is thus $F = \sqrt{F_x^2 + F_z^2}$. The torque acting on the sphere can be written as a volume integral over the cross product of distance and force or:

$$\tau = \int_V \vec{r}\times\vec{F}.$$

Here $\vec{r}$ will have different lengths along each direction, or $\vec{r} = (R\sin\alpha\cos, R\sin\alpha\sin, R\cos\alpha)$.

(c) The nanosphere has an angular momentum $L = I\omega$ which implies the torque acting on it to be

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t} = I\dot{\omega} = \kappa\omega.$$

We now have a simple first order differential equation whose solution is

$$\omega' = \omega_0\exp\left(\frac{5\kappa}{2mR^2}t\right).$$

As the nanosphere is an induced dipole, will it now lose power due to radiative losses? The answer turns out to be no, as the nanosphere rotates on an axis that is perpendicular to the direction of its dipole moment. With no dipole moment directed along the axis of rotation, no radiative power will be lost.

Grading Scheme

A1	It's stated (or taken under consideration) that the entire field on the microsphere is uniform and explanation how oscillations occur is presented	0.50
A3	Intensity and electric field are related under Poynting's vector	1.00
A4	Coefficient α of dipole moment p is calculated	1.00
A5	Gradient force averaged over time is derived	1.00
A6	Answer for Ω	0.75
A7	Radiation force F_{rad} is found and expressed with λ	1.00
A8	Answer for equilibrium position x	0.75
B1	Correct ray diagram and proper deviation angle is derived	2.00
B2	Finds a proper expression for infintessimal force due to momentum change	1.00
B3	Uses law of cosines to obtain a general expression for ρ	0.50
B4	Integral expression for force in x, y, z directions is correct	0.75
B5	Integral expressions for torque is correct	0.75
C1	Differential equation is created and solved	1.0
C2	Correct answer is expressed (can be in a piecewise function)	0.50
C3	Accounts for forces due to radiation and is able to show it does not affect rotation	0.50

Acoustic Levitation

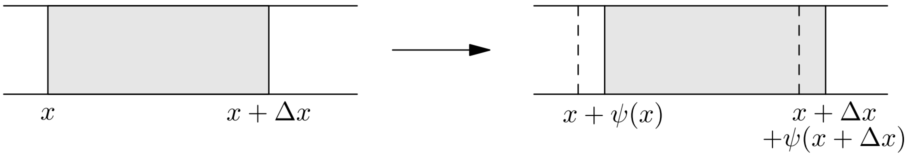
(d) For either case, there are different processes happening which require different analysis.

When f is low

By the condition $f \ll c_s/A$, the acoustic wave produced by the piston travels much faster than the piston as $c_s \gg fA$. For a general traveling wave, we define:

$$\psi(x,t) = A\sin(kx - 2\pi ft)$$

where ψ refers to the longitudinal displacement of the wave and k is the wavenumber. Now consider a part of the gas that is in equilibrium followed by the gas in a later state.



We can approximate $\psi(x+\Delta x)$ to be $\psi(x)+\Delta\psi$. Note that the volume of this slice of gas can be written as $S\Delta x$ but the change in volume of the gas between both successive pictures can be written as $\Delta V = S\Delta\psi$. Furthermore, the ideal gas equation holds for small amplitudes which means that $pV^\gamma = \text{const}$, and by logarithmic differentiation,

$$\frac{\Delta p}{p_0} + \gamma\frac{\Delta V}{V_0} = 0.$$

Thus, we can rewrite the volume terms as:

$$\frac{\Delta V}{V_0} = \frac{S\Delta\psi}{S\Delta x} \to \frac{\partial\psi}{\partial x}.$$

The change in pressure of a segment of the gas at $x=0$ can be characterized as

$$\Delta p(0,t) = -\gamma p_0 \left.\frac{\partial\psi}{\partial x}\right|_{x=0} = \gamma p_0 Ak\cos(2\pi ft).$$

Now the force acting on the piston due to this change in pressure can be accurately described as $\Delta F = S\Delta p(0,t)$. Power can be written as $P = \vec{F}\cdot\vec{v}$ and we know that since $\psi = A\cos(2\pi ft)$, then $v = -2\pi Af\sin(2\pi ft)$. The wavenumber is written as $k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c_s}$. The total and average power acting on the piston for each quarter cycle is then,

$$P = 4\pi^2\gamma p_0 A^2 f^2 c_s\cos^2(2\pi ft) \implies \langle P\rangle = 2\pi^2\gamma f^2 p_0 A^2 c_s.$$

When f is very high

In the limit of $f \gg c_s/A$, the piston will start to move much faster than the actual molecules in the air. Therefore, at some point in time, the piston will actually not have "contact" with the molecular layer. This is because after pushing the gas molecules, there will be an empty void which will eventually be replaced at a distance $d \sim c_s/f$. The piston will only exert a force (and hence power) upon having contact with the layer of gas. Suppose the piston moves a distance $A-d$ in a time τ. We can then find that

$$A\cos(2\pi f\tau) = A - \frac{c_s}{f} \implies 2A\sin^2(\pi f\tau) = \frac{c_s}{f}.$$

Assuming τ is small, we can use approximations such that

$$2\pi^2 Af^2\tau^2 = \frac{c_s}{f} \implies \tau = \frac{1}{\pi}\sqrt{\frac{c_s}{2Af^3}}.$$

The entire cycle takes a time $t = 1/f$ which means we must consider the interval $(\tau, 1/f)$. We can use a similar process as in the analysis for $f \ll c_s/A$ now. Except this time, to find the power used, we will find the energy per cycle and divide by $1/f$. The potential energy per unit length can be written as

$$\mathrm{d}U = \frac{1}{2}S\gamma p_0 \left(\frac{\partial \psi}{\partial x}\right)^2 \implies \Delta U = \frac{1}{2}S\gamma p_0 \int_\tau^{1/f} \left(\frac{\partial \psi}{\partial t}\right)^2 \mathrm{d}t.$$

Thus, we must now evaluate:

$$\Delta U = 8S\gamma p_0 \pi^4 A^2 f^4 \cos^2\left(2\pi f\left(\frac{1}{f} - \tau\right)\right).$$

Having all constants be defined under ξ, we can now write

$$\Delta U \approx \xi\left(1 - \left(2\pi f\left(\frac{1}{f} - \frac{1}{\pi}\sqrt{\frac{c_s}{2Af^3}}\right)\right)^2\right) = \xi\left(1 - 4\pi^2 f^2\left(\frac{1}{f^2} - \frac{2}{\pi}\sqrt{\frac{c_s}{2Af^5}} + \frac{1}{\pi^2}\frac{c_s}{2Af^3}\right)\right).$$

The average power used is thus

$$\langle P\rangle = \frac{\Delta U}{1/f} = 8S\gamma p_0 \pi^3 A^2 f^5\left(\frac{1}{f^2} - \frac{2}{\pi}\sqrt{\frac{c_s}{2Af^5}} + \frac{1}{\pi^2}\frac{c_s}{2Af^3}\right).$$

(e) The internal energy of n moles of the gas is given by $U = nC_V\Delta T$. The power dissipated by the shockwave is then

$$P = \frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}n}{\mathrm{d}t}C_V\Delta T.$$

Note that $\Delta n = \frac{\rho S\Delta x}{m}$ which means that $\mathrm{d}n/\mathrm{d}t = \rho Sc_s/m$ where c_s is the speed of the sound wave. Hence, the power dissipated can be written as $P = 3R\rho Sc_s\Delta T/2m$. Thus, the change in temperature is when both power generation and dissipation is equated. Note that the process is adiabatic as the wavelength of the sound waves are much larger than the mean free path of the gas molecules implying slow motion.

$$\frac{3R\rho Sc_s\Delta T}{2m} = P_{\text{adb}} \implies \Delta T = \frac{2P_{\text{adb}}m}{3R\rho Sc_s}.$$

(f) The force on the cylinder will be

$$F = -\pi R^2(p(y+h) - p(y)) = -\pi R^2 h\frac{\mathrm{d}P}{\mathrm{d}y}.$$

From part D, we know that

$$\Delta p(y,t) = \gamma p_0 Ak\cos(ky - 2\pi ft) \implies \frac{\mathrm{d}p}{\mathrm{d}y} = \gamma p_0 Ak\sin(ky - 2\pi ft).$$

From this, we can see that the cylinder will levitate near the pressure nodes, or where $\frac{\mathrm{d}P}{\mathrm{d}y} = 0$. These will typically be compressions as they imply a stable equilibrium. As the waves are travelling, the puck will float up with the sound wave and then fall repeating the same process over again implying oscillations. For true levitate, we would require the waves to be standing.

Grading Scheme

D1	Correct reasoning for what happens to the piston in the limit of $f \ll c_s/A$	0.25
D2	Shows ideal gas equation holds and relates Δp and ΔV	1.00
D3	Rewrites $\Delta V/V_0$ and derives power $P(t)$	1.25
D4	Correct expression for average power $\langle P\rangle$	0.75
D5	Correct reasoning for what happens to the piston in the limit of $f \gg c_s/A$	0.50
D6	Finds characteristic time τ for when piston hits the "molecular wall"	0.50
D7	Finds potential energy per unit length per cycle	1.25
D8	Correct expression for average power	0.75
E1	Relates internal energy and power dissipation (alternate: uses equipartition theorem)	3.00
E2	Correct answer for ΔT	1.00
F1	Relates force F and pressure gradient p to show sinusoidal nature	1.00
F2	Correct reasoning for how the cylinder levitates and where it does	1.00

T2: Thomas Precession

Sucessive Transformations

In this section, we examine what happens when two successive Lorentz transformations are applied in non-parallel directions.

1. **(6 pts.)** Consider the three reference frames S_1, S_2, and S_3. Events as seen from frame S_i will be labeled with the space-time coordinates (x_i, y_i, z_i, t_i), for $i = 1, 2, 3$. All three frames coincide at $(0, 0, 0, 0)$. Suppose frame S_2 travels with velocity βc in the x_1-direction of frame S_1 and frame S_3 travels with velocity $\beta'_x c\hat{\mathbf{i}}_2 + \beta'_y c\hat{\mathbf{j}}_2$ with respect to frame S_2.

   Perform two successive Lorentz transformations: one expressing the S_3 coordinates in terms of the S_2 coordinates and another expressing the S_2 coordinates in terms of the S_1 coordinates. Then, as the final answer, express the S_3 coordinates in terms of the S_1 coordinates.

   Assume that $\beta' = (\beta'^2_x + \beta'^2_y)^{1/2} \ll \beta$ and work to first order.

2. **(5 pts.)** Now, find the velocity of S_3 in S_1 with the appropriate velocity addition. Perform a single Lorentz transformation to express the S_3 coordinates in terms of the S_1 coordinates. Once again, work to first order. The answer will not be the same as part 1.

3. **(3 pts.)** Show that your answer in Problem 1 differs from your answer in Problem 2 by a spatial rotation. In other words, two successive Lorentz transformations in non-parallel directions cannot be combined as one Lorentz transformation. Rather, they are the combination of one Lorentz transformation and one spatial rotation. Determine the magnitude and direction of this spatial rotation in the current setup.

   Then argue and explain why the answer in Problem 1 and not Problem 2 is the correct transformation.

Precession Frequency

In this section, we examine the precession of the electron's spin magnetic moment within the hydrogen atom.

Electrons possess an attribute known as *spin*. One can imagine the electron as being a small, spherical charged particle that is spinning on some axis. Although this mental picture is not physically correct, it is enough for us. From this mental picture, we can gather that the electron will possess a dipole moment and angular momentum *intrinsic* to itself and not induced by any orbital motion. Respectively, these are known as the *spin magnetic moment* and the *spin angular momentum* that behave, in our model, just as their classical counterparts do. These two vector quantities, $\boldsymbol{\mu}$ and $\mathbf{L}$ respectively, are related by

$$\boldsymbol{\mu} \simeq -\frac{e}{m_e}\mathbf{L},$$

which is determined by experiment.

In this section, we will use the Bohr model of the hydrogen atom, where the electron circles the proton at some radius r, pulled in by the Coulomb force.

4. **(3 pts.)** In the laboratory frame, the electron orbits the proton with some velocity v in the x-y plane. Now switch to the instantaneous rest frame of the electron, where the proton moves with speed v relative to the stationary electron. The moving proton will then induce some magnetic field at the electron's location.

   Suppose the spin angular momentum of the electron points in some direction other than the direction of the magnetic field. Because the electron also possesses a spin magnetic moment, it will precess as a result of the torque done on it. Find the angular frequency of the precession of the electron's spin in terms of r and whatever fundamental constants.

   Ignore any relativistic effects.

5. **(6 pts.)** This problem will use the answer from Part 1. Once again, assume that the electron is orbiting in the x-y plane.

   Consider the instantaneous rest frame of the electron at some time t, S_2. Also consider the instantaneous rest frame of the electron at some time $t + \mathrm{d}t$, S_3. Relative to the laboratory frame, S_1, S_2 will have velocity $\mathbf{v}$. S_3 will have a velocity $\mathrm{d}\mathbf{v}$ relative to S_2, but $\mathbf{v}$ and $\mathrm{d}\mathbf{v}$ won't be parallel.

   The result from Part 1 tells us that S_3 will experience an infinitesimal rotation in this time $\mathrm{d}t$ with respect to the laboratory frame. Since the electron is continuously accelerating, its rest frame, S_3, must then rotate continuously relative to the lab frame. Assume that the spin of the electron always points in the same direction in its rest frame. If the spin is not pointing in the z direction, find the angular frequency of the spin's precession in terms of r and whatever fundamental constants. Ignore the effects of the previous problem.

   Note: although relativistic effects cannot be ignored, you can assume that the electron's velocity v is not comparable to the speed of light in the calculation.

6. **(2 pts.)** Combine your answers from parts 4 and 5 and find the relativistically correct angular frequency of the precession of the electron's spin.

This problem deals with a phenomenon known as *Thomas precession.* It is named after Llewellyn Thomas. Thomas did not discover this phenomenon, but he was the first to discover that its application was important for the electron in the hydrogen atom. He did so in one-page letter, which was printed in *Nature* in 1926. Thomas's contribution was crucial for the calculation of the spin-orbit interaction in the hydrogen atom, which successfully explained the fine structure of the hydrogen spectral lines.

This entire problem is essentially an explanation of his contribution and can all be found in his one-page letter to the editor.

Part 1

Problem 1

(x_2, y_2, z_2, t_2) is related to (x_1, y_1, z_1, t_1) by a simple Lorentz transformation:

$$ct_2 = \gamma(ct_1 - \beta x_1) \tag{1a}$$
$$x_2 = \gamma(x_1 - \beta ct_1) \tag{1b}$$
$$y_2 = y_1 \tag{1c}$$
$$z_2 = z_1, \tag{1d}$$

where γ is the Lorentz factor correlated with β. To go from S_2 to S_3, we must Lorentz transform in a direction that's not the x axis. Our strategy will be to rotate S_2 to S_2' such that S_3 travels along the x_2' axis. Then we can Lorentz transform to find S_3' and rotate back to S_3.

First,

$$ct_2' = ct_2 \tag{2a}$$
$$x_2' = \cos\theta x_2 + \sin\theta y_2 = \frac{\beta_x'}{\beta'}x_2 + \frac{\beta_y'}{\beta'}y_2 \tag{2b}$$
$$y_2' = -\sin\theta x_2 + \cos\theta y_2 = -\frac{\beta_y'}{\beta'}x_2 + \frac{\beta_x'}{\beta'}y_2 \tag{2c}$$
$$z_2' = z_2. \tag{2d}$$

If the Lorentz factor for β' is γ', we can write

$$ct_3' = \gamma'(ct_2' - \beta' x_2') \tag{3a}$$
$$x_3' = \gamma'(x_2' - \beta' ct_2') \tag{3b}$$
$$y_3' = y_2' \tag{3c}$$
$$z_3' = z_2'. \tag{3d}$$

Then we can rotate back and find

$$ct_3 = ct_3' \tag{4a}$$
$$x_3 = \frac{\beta_x'}{\beta'}x_3' - \frac{\beta_y'}{\beta'}y_3' \tag{4b}$$
$$y_3 = \frac{\beta_y'}{\beta'}x_3' + \frac{\beta_x'}{\beta'}y_3' \tag{4c}$$
$$z_3 = z_3'. \tag{4d}$$

Combining all of (1), (2), (3), and (4), we can write

$$
\begin{aligned}
ct_3 &= \gamma\gamma' ct_1 - \gamma\gamma'\beta x_1 - \gamma\gamma'\beta'_x x_1 + \gamma\gamma'\beta'_x\beta ct_1 - \gamma'\beta'_y y_1 \\
&\simeq \gamma(1+\beta'_x\beta)ct_1 - \gamma(\beta+\beta'_x)x_1 - \beta'_y y_1 &\text{(5a)}\\
x_3 &= \gamma\gamma'\left(\beta\beta'_x + \frac{\beta'^2_x}{\beta'^2}\right)x_1 - \gamma\gamma'\left(\beta'_x + \frac{\beta\beta'^2_x}{\beta'^2}\right)ct_1 + \frac{\gamma'\beta'_x\beta'_y}{\beta'^2}y_1 \\
&\quad + \frac{\gamma\beta'^2_y}{\beta'^2}x_1 - \frac{\gamma\beta\beta'^2_y}{\beta'^2}ct_1 - \frac{\beta'_x\beta'_y}{\beta'^2}y_1 \\
&\simeq -\gamma(\beta+\beta'_x)ct_1 + \gamma(1+\beta\beta'_x)x_1 &\text{(5b)}\\
y_3 &= \gamma\gamma'\left(\beta\beta'_y + \frac{\beta'_x\beta'_y}{\beta'^2}\right)x_1 - \gamma\gamma'\left(\beta'_y + \frac{\beta'_x\beta'_y\beta}{\beta'^2}\right)ct_1 + \frac{\gamma'\beta'^2_y}{\beta'^2}y_1 \\
&\quad - \frac{\gamma\beta'_x\beta'_y}{\beta'^2}x_1 + \frac{\gamma\beta\beta'_x\beta'_y}{\beta'^2}ct_1 + \frac{\beta'^2_x}{\beta'^2}y_1 \\
&\simeq -\gamma\beta'_y ct_1 + \gamma\beta\beta'_y x_1 + y_1 &\text{(5c)}\\
z_3 &= z_1. &\text{(5d)}
\end{aligned}
$$

Here, everything has been done to first order, which includes writing $\gamma' \simeq 1$.

Problem 2

To transform from S_1 directly to S_3, we need the proper velocity addition. Let the velocity of S_3 in S_1 be defined by the components β''_x and β''_y. Then

$$\beta''_x = \frac{\beta+\beta'_x}{1+\beta\beta'_x} \text{ and } \beta''_y = \frac{\beta'_y}{\gamma(1+\beta\beta'_x)}. \tag{6}$$

We further define $\beta'' = (\beta''^2_x + \beta''^2_y)^{1/2}$ and γ'' as the Lorentz factor corresponding to β''.

Note that the following approximations are true to first order:

$$
\begin{aligned}
\beta''_x &\simeq \beta + \frac{\beta'_x}{\gamma^2} &\text{(7a)}\\
\gamma'' &\simeq \gamma(1+\beta\beta'_x) &\text{(7b)}\\
\beta'' &\simeq \beta''_x. &\text{(7c)}
\end{aligned}
$$

Once again, the Lorentz transformation needs to be made in a direction other than the x direction. We employ a similar scheme. The details of the calculation are not shown here, since they are similar to what has been done previously. The final result is

$$
\begin{aligned}
ct'_3 &= \gamma'' ct_1 - \gamma''\beta''_x x_1 - \gamma''\beta''_y y_1 \\
&\simeq \gamma(1+\beta\beta'_x)ct_1 - \gamma(\beta+\beta'_x)x_1 - \gamma\beta''_y y_1 &\text{(8a)}\\
x'_3 &= \frac{\gamma''\beta''^2_x}{\beta''^2}x_1 + \frac{\gamma''\beta''_x\beta''_y}{\beta''^2}y_1 - \gamma''\beta'' ct_1 + \frac{\beta''^2_y}{\beta''^2}x_1 - \frac{\beta''_x\beta''_y}{\beta''^2}y_1 &\text{(8b)}\\
&\simeq -\gamma(\beta+\beta'_x)ct_1 + \gamma(1+\beta\beta'_x)x_1 + (\gamma-1)\frac{\beta''_y}{\beta}y_1 &\text{(8c)}\\
y'_3 &= \frac{\gamma''\beta''_x\beta''_y}{\beta''^2}x_1 + \frac{\gamma''\beta''^2_y}{\beta''^2}y_1 - \gamma''\beta''_y ct_1 - \frac{\beta''_x\beta''_y}{\beta''^2}x_1 + \frac{\beta''^2_x}{\beta''^2}y_1 \\
&\simeq -\gamma\beta''_y ct_1 + (\gamma-1)\frac{\beta''_y}{\beta}x_1 + y_1 &\text{(8d)}\\
z'_3 &= z_1. &\text{(8e)}
\end{aligned}
$$

Note that all terms β''^2_y/β''^2 are second order.

Problem 3

Let us write S_3 in terms of S_3'. Using (6), we can write (5) in terms of β_y'' instead of β_y'. As we will see in Problem 5, β_y'', the laboratory-observed difference in velocity between S_2 and S_3, is more physically relevant. This is

$$ct_3 \simeq \gamma(1+\beta_x'\beta)ct_1 - \gamma(\beta+\beta_x')x_1 - \gamma\beta_y'' y_1 \tag{9a}$$
$$x_3 \simeq -\gamma(\beta+\beta_x')ct_1 + \gamma(1+\beta\beta_x')x_1 \tag{9b}$$
$$y_3 \simeq -\gamma^2\beta_y'' ct_1 + \gamma^2\beta\beta_y'' x_1 + y_1 \tag{9c}$$
$$z_3 = z_1 \tag{9d}$$

We can see that this is different from (8). Let us invert (8) such that we have (x_1, y_1, z_1, t_1) in terms of (x_3', y_3', z_3', t_3'). This is just the same Lorentz transformation in Problem 2, except with the velocities reversed. So

$$ct_1 \simeq \gamma(1+\beta\beta_x')ct_3' + \gamma(\beta+\beta_x')x_3' + \gamma\beta_y'' y_3' \tag{10a}$$
$$x_1 \simeq \gamma(\beta+\beta_x')ct_3' + \gamma(1+\beta\beta_x')x_3' + (\gamma-1)\frac{\beta_y''}{\beta}y_3' \tag{10b}$$
$$y_1 \simeq \gamma\beta_y'' ct_3' + (\gamma-1)\frac{\beta_y''}{\beta}x_3' + y_3' \tag{10c}$$
$$z_1 = z_3'. \tag{10d}$$

Plugging (10) into (9), we find

$$ct_3 \simeq ct_3' \tag{11a}$$
$$x_3 \simeq x_3' - (\gamma-1)\frac{\beta_y''}{\beta}y_3' \tag{11b}$$
$$y_3 \simeq (\gamma-1)\frac{\beta_y''}{\beta}x_3' + y_3' \tag{11c}$$
$$z_3 = z_3'. \tag{11d}$$

This is the equation for an infinitesimal rotation (keeping in mind that $\beta_y''/\beta \ll 1$). Thus, to go from S_3' to S_3, we must rotate about the $+z$ axis an angle

$$-(\gamma-1)\frac{\beta_y''}{\beta}.$$

To reiterate, β_y'' is the relative velocity in the y direction between frames S_3 and S_2, as viewed from the lab frame S_1.

We expect (x_3, y_3, z_3, t_3) to be the correct form. The Lorentz transformation is the most general tool. The velocity addition used in problem 2 is dependent on the Lorentz transformation.

Part 2

Problem 4

We are given

$$\vec{\mu} = -\frac{e}{m_e}\vec{L}. \tag{12}$$

Suppose the electron is orbiting counter-clockwise in the x-y plane. In the instantaneous rest frame of the electron, the proton travels with speed v a distance r away from the electron. By the law of Biot and Savart, the magnetic field at the location of the electron is

$$B = \frac{\mu_0 ev}{4\pi r^2}, \tag{13}$$

which points in the positive z direction with our choice of coordinates.

Since we are using the Bohr model, the acceleration of the electron is given by the equation

$$\frac{v^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2 m_e} \tag{14}$$

which gives

$$v = \left(\frac{e^2}{4\pi\epsilon_0 r m_e}\right)^{1/2}. \tag{15}$$

Now the electron will experience a torque as a result of this magnetic field because it possesses a spin magnetic moment. This torque is

$$\vec{\tau} = \vec{\mu} \times \vec{B} = -\frac{e}{m_e}\vec{L} \times \vec{B}. \tag{16}$$

Therefore,

$$\left|\frac{d\vec{L}}{dt}\right| = \frac{eLB\sin\theta}{m_e}. \tag{17}$$

θ is the angle between $\vec{L}$ and $\vec{B}$. Thinking in analogy to $v = r\omega$, we find that the angular velocity of the precession of $\vec{L}$ is

$$\omega_L = \frac{eB}{m}, \tag{18}$$

which will be in the positive z direction. (We have used the subscript L because this precession is often called the *Larmor precession.*) Applying (13) and (15), we find that

$$\omega_L = \frac{\mu_0 e^3}{4\pi m_e r^2}\left(\frac{1}{4\pi\epsilon_0 m_e r}\right)^{1/2}. \tag{19}$$

Problem 5

Once again, assume the electron is orbiting counter-clockwise in the x-y plane.

In analogy to Problem 1, S_1 is the lab frame. S_2 is the rest frame of the electron when it is travelling with velocity $\vec{v}$ at time t. S_3 is the rest frame of the electron when it is travelling with velocity $\vec{v} + \vec{a}dt$ at time $t + dt$. To keep the analogy, let $\vec{v}$ be in the x direction of S_1. Then $\vec{a}dt$ is in the y direction of S_1 and S_2. (Since the electron is orbiting counter-clockwise, it must also be the positive y direction.)

First, $\beta = v/c$ as defined in Problem 1. Second, note that $\vec{a}dt$ is the relative velocity between S_3 and S_2 as measured in the lab frame S_1. Therefore, $\beta_y'' = a\,dt/c$. Combining these facts, the infinitesimal rotation angle between S_2 and S_3 will be

$$d\theta = -(\gamma - 1)\frac{a\,dt}{v}, \tag{20}$$

in the positive z direction. Since we can also assume that $v/c \ll 1$, $\gamma - 1$ simplifies to $v^2/2c^2$. We can continue to repeat this process as the electron goes around in a circle. Therefore, this continuous infinitesimal rotational angle will manifest as an angular velocity: the angular frequency with which the frame S_3 rotates relative to the lab frame. This angular frequency is

$$\omega_T = -\frac{av}{2c^2} = -\frac{v^3}{2c^2 r} \tag{21}$$

in the positive z direction. (We have used the subscript T because this frequency is called the Thomas precession frequency.)

Using (15), (21) can be rewritten as

$$\omega_T = -\frac{e^3}{8\pi\epsilon_0 c^2 m_e r^2}\left(\frac{1}{4\pi\epsilon_0 m_e r}\right)^{1/2}. \tag{22}$$

Problem 6

The final precession frequency of the electron will be $\omega_L + \omega_t$. Note that $c = (1/\mu_0\epsilon_0)^{1/2}$. Thus, (22) can be rewritten as

$$\omega_T = -\frac{\mu_0 e^3}{8\pi m_e r^2}\left(\frac{1}{4\pi\epsilon_0 m_e r}\right)^{1/2}, \tag{23}$$

which is exactly half the result of Problem 4. Therefore, the precession frequency of the electron is

$$\omega = \frac{\mu_0 e^3}{8\pi m_e r^2}\left(\frac{1}{4\pi\epsilon_0 m_e r}\right)^{1/2}. \tag{24}$$

This relativistic correction of 1/2 because known as the *Thomas half* and finally matched the experimental determination of hydrogen's fine structure with the quantum mechanical prediction. Just a few year's later, Dirac would propose his Dirac Equation, which "came installed" with relativistic mechanics, thus eliminating the need for these kinds of calculations.

Grading Scheme

P1	For correct Lorentz transformation from S_1 to S_2	1 pt
P1	For correct Lorentz transformation from S_2 to S_3	2 pt
P1	For correct Lorentz transformation from S_1 to S_3	3 pt
P1	For not expressing the answer to first order with the given approximation. Half credit for algebraic mistakes	-2 pt
P2	For the correct velocity.	2 pt
P2	For the correct Lorentz transformation.	4 pt
P2	For not expressing to first order. Half credit for algebra mistakes.	-2 pt
P3	For correct magnitude.	2 pt
P3	For correct direction.	1 pt
P4	For the correct magnetic field at the electron's location within its rest frame.	0.5 pt
P4	For the correct torque applied on the electron due to its spin magnetic moment.	0.5 pt
P4	For the correct precession frequency. Half credit for algebra mistakes.	2 pt
P5	For the correct analogy (assigning values for β, β'_x, β'_y, etc) to parts 1, 2, and 3.	2 pt
P5	For the correct answer. Half credit for algebra mistakes.	4 pt
P5	For not assuming $v \ll c$.	-1 pt
P6	For the correct answer.	2 pt
P6	For not making the substitution $c = (\mu_0\varepsilon_0)^{-1/2}$.	-1 pt

T3: Moving Media

Interesting phenomena can arise in situations where there are 2 media that are moving with respect to each other. In particular, objects can move at much faster speeds than the relative speeds of the media, without using any energy. In this problem, we explore two such examples of this effect.

Moving Cylinders

Suppose we have three cylinders, two small cylinders and a large cylinder, of radii r and R. The frictionless pivots (centers) of the cylinders are attached to a massless triangular frame, such that the large cylinder is in contact with the two small cylinders but the two small cylinders are not touching each other. The small cylinders each have a thin groove along their circumferences (which does not affect the moment of inertia significantly), so that the large cylinder makes contact with the small cylinder at a point with radial distance αr from the center of the small cylinder. The axes of all cylinders are perpendicular to the plane of the triangular frame. The system is placed on a level ground and a long flat horizontal board is put on top of the large cylinder, with the two small cylinders touching the ground (making contact at their outer edge with radial distance r, not αr). Assume that the friction due to contact between all surfaces is large enough to prevent any slipping.

1. **(4 pts.)** The board is moved with speed v in a direction perpendicular to the axes of the cylinders. Find the speed of the cylinder system.

2. **(5 pts.)** The mass of the small and large cylinders are m and M, respectively. The mass of the board is m'. If at a moment in time the board is pushed with speed v and acceleration a, find the power P required to push the board. Assume the cylinders have uniform mass distribution.

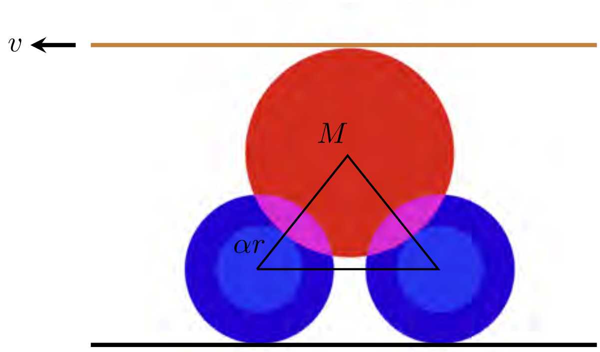


Figure 4: A visual of the three cylinder setup.

Windsurfing

In windsurfing, it is possible to sail faster than the wind without using any energy. Suppose we have a sailboat moving on a large, motionless body of water. The air of density ρ is moving at a speed v uniformly in one direction. If the sailboat is pointed in a certain direction and moves in that direction with velocity $\mathbf{u}$, the drag force from the water $\mathbf{F}$ satisfies $\mathbf{F} \cdot \mathbf{u} = -\gamma u^2$, where γ can be assumed to be a constant drag coefficient.

3. **(10 pts.)** If the wind is moving in the $\hat{x}$ direction, what is the maximum possible sustainable x-component of velocity for the sailboat? Assume that the sailboat can neither generate nor store energy in its interaction with the air. Also, the effective cross-sectional area of the sail is A (this is the component of cross-sectional area that is perpendicular to the wind in the reference frame of the sailboat).

4. **(3 pts.)** What is the power dissipated due to the interaction with the water?

5. **(3 pts.)** It seems that the law of conservation of energy is being violated, as the speed of the sailboat isn't changing despite heat generation in the water. Explain why energy is still conserved.

Moving Cylinders

(a) Consider the reference frame of the cylinder system. Since friction is large enough to prevent any slipping, if the speed of the edge of the large cylinder is u, then the speed of the edge of the small cylinder is $\frac{u}{\alpha}$. Thus, the difference in speeds $u(\frac{1}{\alpha}-1)=v$. The speed of the cylinder system is $\frac{u}{\alpha}=\frac{v}{1-\alpha}$.

(b) The power P is equal to the rate of change of kinetic energy of the system. The board increases kinetic energy at a rate $m'va$, the cylinders increase rotational kinetic energy at rates $I\omega\dot{\omega}$. The rate of increase of translational kinetic energy of the cylinder system is $\frac{M+2m}{(1-\alpha)^2}va$. Adding up all the contributions, one gets

$$\begin{aligned} P &= m'va + \frac{M+2m}{(1-\alpha)^2}va + \frac{1}{2}MR^2\left(\frac{\alpha}{(1-\alpha)R}\right)^2 va + 2\cdot\frac{1}{2}mr^2\left(\frac{1}{(1-\alpha)r}\right)^2 va \\ &= \left(m' + \frac{M+3m}{(1-\alpha)^2} + \frac{1}{2}\frac{M\alpha^2}{(1-\alpha)^2}\right)va. \end{aligned}$$

Windsurfing

(c) Suppose the sailboat is pointed at some angle from the direction of the wind. We work in the reference frame of the sailboat throughout this solution, and consider the system containing the air and the sailboat. The water does no work on this system, because it is only in contact with the sailboat, which is not moving in this frame. To achieve the optimal state, we want no energy to be lost (except the heat in the water). Thus, the only thing that the sailboat can do is change the direction of the velocity vector of the wind it intercepts, leaving the magnitude the same! Suppose the wind is coming in at a speed x, and the water is moving at a speed y (all speeds are relative to the sailboat). Let θ be the angle between these two velocities. Then by law of cosines, we get $\cos\alpha = \frac{x^2+y^2-v^2}{2xy}$. For the sailboat to get the maximum push from the air, the air must be thrown back directly parallel to y. Thus, the force is the rate of momentum transfer, which is $\rho A x^2(1-\cos\alpha) = \gamma y$. Simplifying, we get $\frac{\rho A}{\gamma}x(2xy - x^2 - y^2 + v^2) = 2y^2$. Taking differentials, we get $\frac{\rho A}{\gamma}((2x^2-2xy)dy + (4xy - 3x^2 - y^2 + v^2)dx) = 4ydy$. Also, by geometric relations, we get that the x-component speed of the boat relative to the water in excess of the wind speed (v) is $\frac{y^2-x^2-v^2}{2v}$. Thus, this is maximized when its derivative is zero, which is when $xdx = ydy$. Combining this with the previous differential, we get

$$\frac{\rho A}{\gamma}(2x^3 - 5x^2y + 4xy^2 - y^3 + v^2y) = 4xy$$

This gives us the following two equations and two variables (x and y):

$$(2x^2y - x^3 - xy^2 + xv^2) = 2 * \frac{\gamma}{\rho A v}y^2 v$$

$$2x^3 - 5x^2y + 4xy^2 - y^3 + v^2y = 4 * \frac{\gamma}{\rho A v}xyv$$

Using the numerical value $\frac{\gamma}{\rho A v} = 0.05$, we can graph the two equations on a graphing calculator (such as Desmos) and we obtain the solution $(x, y) = (6.17, 6.69)v$ that maximizes $y^2 - x^2$. Thus, the maximum x-component of the velocity of the sailboat is approximately $3.9v$.

(d) The power dissipated is given by $F\cdot v = \gamma y^2$. This is approximately $44.8\gamma v^2$.

(e) Energy conservation is not violated, because in the frame of the water, the air had a lot of kinetic energy to begin with, and during the process, the sailboat reduced the speed of part of the air, thus taking energy from the air.

Grading Scheme

A1	Correct ideas when finding the speed of smaller and larger cylinder	2.0
A2	Correct answer for the entire speed of the cylinder system	2.0
B1	Uses the fact that $\Delta E/\Delta t = P$	1.0
B2	Considers the increase in energy of E_{board}	1.0
B3	Considers the increase in energy of $E_{\text{translational}}$	1.0
B4	Considers the increase in energy of $E_{\text{rotational}}$	1.0
B5	Correct final answer	1.0
C1	Correct idea that the energy of the sail comes from wind	1.0
C2	Correct idea that wind velocity's direction can change, but not magnitude	4.0
C3	Expression for maximum force due to wind	2.0
C4	Expression for drag force from water	1.0
C5	Correct answer within 2% (1 pt removed if final equations are correct but not answer and vice versa)	2.0
D1	Correct formula for power ($F \cdot v = \gamma y^2$)	2.0
D2	Correct answer (within 4%)	1.0
E1	Correct explanation that energy comes from wind	3.0

T4: Missing Energy

Part A

1. **(3 pts.)** Consider a simple circuit with two parallel-plate capacitors of capacitance C_1 and C_2 connected to each other using purely conducting wires and a switch. One of the capacitors is initially charged to a voltage V_0, while the other one is completely uncharged. The circuit is kept in a square shaped figure of side length ℓ throughout the problem, while the diameter of the conducting wires is D. Find the initial total energy of the circuit when the switch is open, given by E_0, and a sufficiently long time after the switch is closed, given by E_∞. Calculate the remaining energy $E_\Delta = E_0 - E_\infty$. What is E_Δ for the case $C_2 \to \infty$?

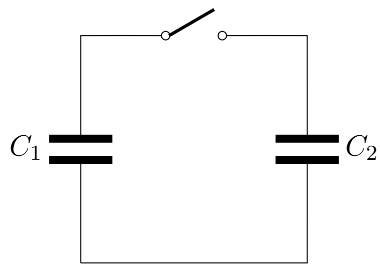


Figure 5: The two parallel plate capacitor-switch circuit.

It seems odd for there to be a difference in energy as the circuit is a closed system. Three young scientists Fermi, Jackson, and Feynman have created different theories to find and verify the correct source of this missing energy.

Thermal Losses: Fermi

To investigate the cause of this missing energy, Fermi assumes that there must be an ohmic resistive load r and a self-inductance L in the circuit responsible for E_Δ.

2. **(3 pts.)** Find the current in the circuit $I(t)$ as a function of time and E_Δ for the circuit.

3. **(1 pt.)** For small values of r, find the oscillation frequency Ω of $I(t)$.

4. **(1 pt.)** For $L = 0$, can Fermi's reasoning be correct for any value of r? If yes, what is this value of r?

Dipole Radiation Losses: Jackson

Jackson believes that the missing energy is dissipated in the form of dipole radiation losses due to the charges accelerating. He assumes that the electric dipole moment of the system remains constant during the process, but the magnetic moment is allowed to vary. Thus, he seeks to determine the maximal possible radiation losses. For this he uses Larmor's formula, which states that for small velocities relative to the speed of light c, total power radiated which the radiation power is defined as:

$$P_r = \frac{2}{3} \cdot \frac{\ddot{m}^2}{4\pi\epsilon_0 c^5}$$

where m is the magnetic moment of the circuit as a function of time. Ignore all relativistic effects and the possible charge accumulation in the wires compared to that on the capacitor plates. Moreover, note that he does not assume any resistance or self-inductance in the circuit in his model.

5. **(4 pts.)** Find the total energy dissipation E_r due to this radiation.

6. **(1 pt.)** For what value of time interval $\Delta\tau$ taken by the charges to move from one capacitor to another, can Jackson's theory be reasoned true?

Kinetic Energy: Feynman

Feynman has the following hypothesis:

> The missing energy goes into the kinetic energy of the charge carriers going from C_1 to C_2. Assume that the mean free path of collisions of the carriers is $\lambda > 2\ell$.

7. **(4 pts.)** Find the total kinetic energy ΔK gained by the carriers during a total charge transfer from C_1 to C_2.

8. **(1 pt.)** Could this be a valid hypothesis to explain the cause of the missing energy? When the charges get completely deposited on the plates of C_2, what happens to this kinetic energy?

Part B

Instead of charging one capacitor using the other, we take an ideal parallel-plate capacitor such that the surface charge density $\pm\sigma$ on its plates is uniform throughout both plates, and that the charges are 'fixed' to the surface as the plate expands. The dimensions of the plates are a, b and the plate separation distance is d. The plate is now stretched quasi-statically by a factor of κ in one of the dimensions such that the dimensions of the capacitor plates are now $\kappa a, b$ but the plate separation remains d.

9. **(6 pts.)** Calculate the work $\mathrm{d}W$ done during stretching the plates of this capacitor. Also write a simplified form of this expression for $d \ll \kappa a, b$.

10. **(1 pt.)** In this case, there are no resistive loads, and since the process of plate expansion is quasi-static, there is no gain in kinetic energy of the charge carriers. Where does the energy disappear in this case?

Solution

T4: Part A

Problem 1

When the switch is open, there is $q_0 = \pm C_1 V_0$ charge on one of the capacitors, while the other capacitor is uncharged. At this time, we have

$$E_0 = \frac{{q_0}^2}{2C_1} = \frac{1}{2}C_1{V_0}^2$$

A long time after the switch is switched on, the potential across both capacitors must be equal. Moreover, by charge conservation we have

$$q_0 = C_1V_0 = q_\infty = (C_1 + C_2)V_\infty \Rightarrow V_\infty = \frac{C_1V_0}{C_1 + C_2}$$

Hence the energy after a long time is

$$E_\infty = \frac{{q_\infty}^2}{2(C_1 + C_2)} = \frac{1}{2}(C_1 + C_2){V_\infty}^2 = \frac{1}{2}\frac{{C_1}^2}{C_1 + C_2}{V_0}^2$$

This means that

$$E_\Delta = E_0 - E_\infty = \frac{1}{2}\frac{C_1C_2}{C_1 + C_2}{V_0}^2 = \frac{1}{2}C_{\mathrm{eq}}{V_0}^2$$

where C_{eq} is the equivalent capacitance of the circuit. For $C_1 = C_2$, this loss is minimal and equal to $E_0/2$. For $C_2 \to \infty$, the whole E_0 energy is lost. In the following problems, we will investigate why there is a loss of energy and work out different explanations of it.

Problem 2

Let q_1 be the charge on the first capacitor at any time t. By conservation of charge, the charge on the second capacitor must be $q_0 - q$. Apply Kirchoff's Voltage Law on the circuit, we have

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + ir + \frac{q}{C_1} - \frac{q_0 - q}{C_2} = 0$$

and

$$i = -\frac{\mathrm{d}q}{\mathrm{d}t}$$

Differentiating the DE gives us

$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + \frac{r}{L}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{LC_{\mathrm{eq}}}i = 0$$

To solve this standard second order DE, we write its auxillary quadratic equation in z:

$$z^2 + \frac{r}{L}z + \frac{1}{LC_{\mathrm{eq}}} = 0$$

Upon solving this we get

$$z = -\frac{r}{2L} \pm j\sqrt{\frac{1}{LC_{\mathrm{eq}}} - \frac{r^2}{4L^2}}$$

Substituting back $i = e^{jz}$ we obtain

$$i(t) = e^{-\frac{r}{2L}t}\left(c_1 e^{j\sqrt{\frac{1}{LC_{\mathrm{eq}}} - \frac{r^2}{4L^2}}t} + c_2 e^{-j\sqrt{\frac{1}{LC_{\mathrm{eq}}} - \frac{r^2}{4L^2}}t}\right)$$

Using Euler's identity and imposing the boundary conditions $i(t=0) = 0$, $\frac{\mathrm{d}i(t=0)}{\mathrm{d}t} = \frac{V_0}{L}$ we get the current of the form

$$i(t) = \begin{cases} \dfrac{V_0}{L\sqrt{\frac{1}{LC_{\mathrm{eq}}} - \frac{r^2}{4L^2}}} e^{-\frac{r}{2L}t} \sin\left(\sqrt{\frac{1}{LC_{\mathrm{eq}}} - \frac{r^2}{4L^2}}t\right) & \text{for } \frac{1}{LC_{\mathrm{eq}}} > \frac{r}{2L} \\ \frac{V_0}{L} t e^{-\sqrt{\frac{1}{LC_{\mathrm{eq}}} - \frac{r^2}{4L^2}}t} & \text{for } \frac{1}{LC_{\mathrm{eq}}} = \frac{r}{2L} \\ \dfrac{V_0}{jL\sqrt{\frac{1}{LC_{\mathrm{eq}}} - \frac{r^2}{4L^2}}} e^{-\sqrt{\frac{1}{LC_{\mathrm{eq}}} - \frac{r^2}{4L^2}}t} \sinh\left(j\sqrt{\frac{1}{LC_{\mathrm{eq}}} - \frac{r^2}{4L^2}}t\right) & \text{for } \frac{1}{LC_{\mathrm{eq}}} < \frac{r}{2L} \end{cases}$$

for overdamping, critical damping, and underdamping. For small values of r and L, the system is likely to undergo underdamping. Also note that

$$E_\Delta = \int_0^\infty (i(t))^2 r \mathrm{d}t$$

Substituting value of $i(t)$ from any of the cases, we find that

$$E_\Delta = \frac{1}{2}\frac{C_1 C_2}{C_1 + C_2}{V_0}^2$$

as found in the first part. This means that Fermi's explanation may be a valid one, if the experimental values suggest the existence of non-zero r and L.

Problem 3

For small values of r, we must have

$$\frac{1}{LC_{\mathrm{eq}}} > \frac{r}{2L}$$

On a first degree of approximation for $r \to 0^+$, this gives us

$$\Omega = \sqrt{\frac{1}{LC_{\mathrm{eq}}} - \frac{r^2}{4L^2}} \approx \sqrt{\frac{1}{LC_{\mathrm{eq}}}}$$

Problem 4

Fermi's reasoning should hold true for all finite values of $r > 0$ for $L \to 0^+$. But note that for zero self-inductance in the circuit, it behaves as a simple RC-circuit with time constant $\tau = rC_{\text{eq}}$. Since relativistic effects are negligible, we must also simultaneously have $r \gg \frac{\ell}{cC_{\text{eq}}}$.

Problem 5

Due to discrepancy in Larmor's formula during the competition, the following solutions to this problem will be given full credit:

Solution 1:

Magnetic moment of the circuit is $m = iA = i\ell^2$, and thus $\ddot{m} = \ell^2 \frac{\mathrm{d}^2 i}{\mathrm{d}t^2}$. Since the capacitors are arranged symmetrically and the current in the wires flows uniformly, the electric dipole of the system must remain constant. Hence, radiation arises from the magnetic dipole radiation. Assume the voltage drop due to this radiation as V_r. We write

$$V_r = \frac{P_r(t)}{i(t)} = i(t)R_r = \frac{2}{3}\frac{\ddot{i}^2\ell^4}{4\pi\epsilon_0 c^5 i}$$

where $R_r = \frac{P_r}{i^2}$ is called the radiation resistance of the circuit and $\ddot{i} = \frac{\mathrm{d}^2 i}{\mathrm{d}t^2}$. From our assumption, since R_r is small, the characteristic time τ of the circuit must be large compared to the oscillation period. Thus we approximate as follows:

$$i = i_0 e^{-\frac{t}{\tau}} \cos(\Omega t)$$

$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} \approx i_0 e^{-\frac{t}{\tau}} \cos(\Omega t)$$

Thus $\ddot{i} = -\Omega^2 i$. The voltage drop is approximated to

$$V_r = \frac{2}{3}\frac{\Omega^4 i^2 \ell^4}{4\pi\epsilon_0 c^5 i} = \frac{\Omega^4 i \ell^4}{6\pi\epsilon_0 c^5}$$

The dissipated power is given by

$$P_r(t) = \frac{\Omega^4 i^2 \ell^4}{6\pi\epsilon_0 c^5}$$

and

$$R_r = \frac{\Omega^4 \ell^4}{6\pi\epsilon_0 c^5}$$

Noting that $\Omega = \frac{2\pi c}{\lambda}$, we have that $R_r = k\left(\frac{\ell}{\lambda}\right)^4$ where λ is the wavelength of radiation. In practice, R_r is very small.

Solution 2: The net resistance in the system is zero. We have

$$a = \frac{\Delta v}{\tau}$$

Hence, we write the work energy theorem as follows:

$$e\Delta V_{12} = e\left[V_0 - \frac{q}{C_{\text{eq}}}\right] = \frac{1}{2}m\Delta v^2$$

Substituting Δv into Larmor's formula, the power radiated by each charge is

$$\mathrm{d}P_r = \frac{2}{3}\frac{q^2 a^2}{4\pi\epsilon_0 c^3} = \frac{q^2\Delta v^2}{4\pi\epsilon_0 c^3 \tau^2} = \frac{e^2 \cdot \frac{2e\left[V_0 - \frac{q}{C_{\text{eq}}}\right]}{m}}{4\pi\epsilon_0 c^3\tau^2}$$

We obtain P_r by integrating the same from $q = 0$ to $q = q_{\text{total}}$.

Problem 6

For Jackson's hypothesis to be true:

$$\frac{1}{2}m_e\Delta v^2 = P_r\Delta\tau = \frac{2}{3}\frac{q^2a^2}{4\pi\epsilon_0 c^3}\tau$$

$$\tau = \frac{e^2}{3\pi\epsilon_0 m_e c^3} \approx \frac{r_e}{3\pi\epsilon_0 c}$$

where r_e is the radius of an electron. Thus, all of the missing energy $\frac{1}{2}C_{\text{eq}}{V_0}^2$ can appear as dipole radiation iff the charges move from one capacitor to the other in the time in which light travels $\frac{1}{3\pi\epsilon_0}$ times the radius of an electron. However, this is theoretically impossible. Thus, Jackson's hypothesis cannot be true.

Problem 7

Let the charge on the capacitor with capacitance C_1 be $q(t)$. At equilibrium, final charge on C_1 is given by $q_0\frac{C_1}{C_1+C_2}$. The potential difference between the plates after the charge has been transferred is given by:

$$\Delta V = \frac{q}{C_1} - \frac{q_0 - q}{C_2}$$

The kinetic energy gained by the charges is then simply

$$\Delta K = \int_{q_0}^{\frac{C_1}{C_1+C_2}q_0} \Delta V \mathrm{d}q = \frac{{q_0}^2 C_2}{2C_1(C_1+C_2)} = \frac{{q_0}^2}{2C_1}\left[\frac{C_2}{C_1+C_2}\right] = \frac{1}{2}C_{\text{eq}}{V_0}^2$$

Problem 8

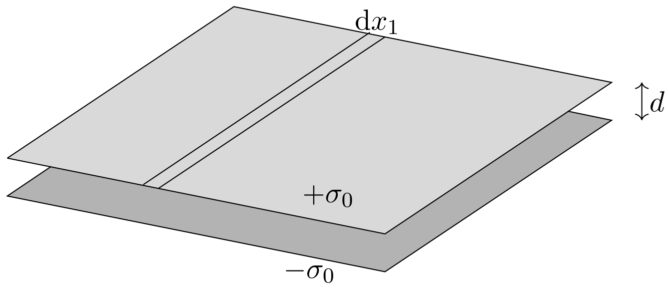
This could be a possible explanation of the missing energy since the energy gained by the charge carriers comes out to be the same as E_Δ. When the charge gets deposited on the plate of the second capacitor, it accumulates there and dissipates thermal energy as it interacts with the charges on the plate of this capacitor.

Grading Scheme

A1	Correct answers for E_∞, E_Δ and $C \to \infty$ case	2.5
A2	Correct analysis of $C_2 \to \infty$ case	0.5
B1	Correct expression for $i(t)$	1.5
B2	Considered underdamping, critical damping and over-damping	0.5
B3	Correct simplified expression of E_Δ	1.0
C1	Correct frequency approximated to the first order	1.0
D1	Noted that all non-zero finite values work	0.5
D2	Second constraint from characteristic time constant	0.5
E1	Correct idea of radiative voltage and resistance	1.0
E2	Correct approximation for large characteristic time constant τ	1.0
E3	Correct calculation of power and energy dissipated	2.0
F1	Correct use of work energy theorem	0.5
F2	Correct final expression for τ	0.5
G1	Exact expression of ΔV	1.0
G2	Correct limits on charge transferred	1.0
G3	Calculation of $\Delta K = \int \Delta V \mathrm{d}q$	1.0
G4	Correct simplified final expression for kinetic energy	1.0
H1	Correct explanation of why the Feynman scenario is possible	1.0

Problem 9

Let us assume that the capacitor plates are lying in the $x-y$ plane, and the electric field between the plates is in the z-direction. The initial potential energy of the system is given as $E_0 = 2\pi\sigma^2 abd$. There are repulsive forces on the charges of the plates parallel to its surface. Usually, these are ignored. But in this case, with the stretching of the plates, work will be performed by these forces. These forces are small near the centre of the plate surface, but as we will find out, appreciable as we go towards the plate edges performing a significant amount of work. The field is thus, fringing.



Since the expansion is done in one dimension, only the x-component of the forces is important. Take a line charge at $x = x_1$ with linear charge density $\sigma \mathrm{d}x_1$. At $x = x_2$ on both plates, we also consider line charges. Net force of repulsion due to both the plates is then given by

$$F_0 = \frac{\sigma^2}{2\pi\epsilon_0}\left[\int_0^{2x_1-a} \frac{\sqrt{b^2+(x_1-x_2)^2}-(x_1-x_2)}{x_1-x_2}\mathrm{d}x_2 \right.$$
$$\left. -\int_0^{2x_1-a} \frac{x_1-x_2}{\sqrt{d^2+(x_1-x_2)^2}}\mathrm{d}x_2 \cdot \frac{b^2+d^2+(x_1-x_2)^2}{\sqrt{d^2+(x_1-x_2)^2}}\sqrt{d^2+(x_1-x_2)^2}\right]$$

Now the plates are expanded quasi-statically such that the following transformations occur $\sigma \to \frac{\sigma}{\kappa}, a \to \kappa a$. As the expansion factor goes from κ to $\kappa + \mathrm{d}\kappa$, the line of charge at x_1 moves a distance $(x - \frac{\kappa a}{2})\frac{\mathrm{d}\kappa}{\kappa}\hat{x}$. The work done by the repulsive forces during expansion of both plates is then

$$\mathrm{d}W = 4 \times \frac{\sigma^2 \mathrm{d}\kappa}{4\pi\epsilon_0\kappa}\int_0^{\kappa a} (2x_1 - \kappa a)\cdot(\sqrt{{x_1}^2+b^2} - x_1 + \sqrt{{x_1}^2+d^2} - \sqrt{{x_1}^2+b^2+d^2}$$
$$- b\ln\left(\frac{x_1}{\sqrt{{x_1}^2+b^2}-b}\cdot\frac{\sqrt{{x_1}^2+d^2}-b}{\sqrt{{x_1}^2+d^2}}\right)\mathrm{d}x_1$$

Now we use the substitution $\Gamma = x_1 - x_2$, solve the indefinite integrals and simplify our result to obtain:

$$\begin{aligned}
\mathrm{d}W = \frac{\sigma^2 \mathrm{d}\kappa}{\pi\epsilon_0\kappa}\Bigg[& \frac{2d^2}{3}\left(\sqrt{\kappa^2a^2+d^2} - d + \sqrt{b^2+d^2} - \sqrt{\kappa^2a^2+b^2+d^2}\right) \\
& -b^2\left(\sqrt{b^2+d^2} + \sqrt{\kappa^2a^2+b^2} - b + \sqrt{\kappa^2a^2+b^2+d^2}\right) \\
& +\frac{\kappa^2a^2}{6}\left(\sqrt{\kappa^2a^2+d^2} + \sqrt{\kappa^2a^2+b^2} - \kappa a - \sqrt{\kappa^2a^2+b^2+d^2}\right) \\
& +\frac{\kappa a b^2}{2}\ln\left(\frac{b}{\sqrt{\kappa^2a^2+b^2}-\kappa a}\cdot\frac{\sqrt{\kappa^2a^2+b^2+d^2}-\kappa a}{\sqrt{b^2+d^2}}\right) \\
& -\frac{\kappa a d^2}{2}\ln\left(\frac{d}{\sqrt{\kappa^2a^2+d^2}-\kappa a}\cdot\frac{\sqrt{\kappa^2a^2+b^2+d^2}-\kappa a}{\sqrt{b^2+d^2}}\right) \\
& -bd^2\ln\left(\frac{d}{\sqrt{d^2+b^2}-b}\cdot\frac{\sqrt{\kappa^2a^2+b^2+d^2}-b}{\sqrt{\kappa^2a^2+d^2}}\right) \\
& +\kappa abd\arctan\left(\frac{\kappa ab}{d\sqrt{\sqrt{\kappa^2a^2+b^2+d^2}}}\right)\Bigg]
\end{aligned}$$

This can be written as

$$\mathrm{d}W = 4\sigma^2 abd\frac{\mathrm{d}\kappa}{\kappa^2}\left(\frac{\pi}{2} + \mathcal{O}(\frac{d}{\kappa a}, \frac{d}{b}, \frac{d}{\sqrt{\kappa^2a^2+b^2}})\right)$$

This is our answer for the general case.

Thus, for $d \ll \kappa a, b$, we have

$$\mathrm{d}W = 2\pi\sigma^2 abd\frac{\mathrm{d}\kappa}{\kappa^2} = E_0\frac{\mathrm{d}\kappa}{\kappa^2}$$

Problem 10

There is work done against the repulsive forces between the charges on a plate within each capacitor, and the missing energy is used up as work done against them as the plates are expanded quasi-statically.

This analysis is very similar to what we did in Part A. In this case, there was no thermal, radiative losses or gain in kinetic energy due to the quasi-static nature of the process. Instead, the repulsive forces on the capacitor plates were significant and proportional to the plate dimensions. If some other external agency such as "magic tweezers" had been used to pick up charges from one capacitor to the another quasi-statically, the source of missing energy would have been the work done using the magic tweezers.

Grading Scheme

A1	Realize that repulsive forces between charges on plates are significant	1.0
A2	Write forces between line charges considering both plates	1.5
A3	Correct calculation for the work done in expansion	2.5
A4	Correct final simplified expression of $\mathrm{d}W$ and answer for the $d \ll \kappa a, b$ case	1.0
B1	Correct explanation of work done against repulsive forces	1.0

2021 Online Physics Olympiad: Invitational Contest

Experimental Exam

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This competition could not be possible without the help of our sponsors, who are all doing great things in physics, math, and education.

General Instructions

The experimental examination consists of 1 long answer question worth 50 points over 1 full day from August 15, 0:01 am GMT.

- **The team leader should submit their final solution document in this google form. We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it.**
- If you wish to request a clarification, please use this form. To see all clarifications, view this document.
- Participants are given a google form where they are allowed to submit up-to 1 gigabyte of data for their solutions. It is recommended that participants write their solutions in LaTeX. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting.

Specific Rules

For any part of this paper, you are allowed to use online tools and resources to help you, as long as you are not requesting help from anyone outside of your team. Allowable resources include Wikipedia, research papers, Wolfram Alpha, Python, Excel, etc.

However, you must document every resource that you use and cite them when applicable. As a general rule of thumb, you should derive any results that cannot be found on Wikipedia. Therefore, solutions along the lines of: "By Wolfram Alpha, this is true." will not be accepted. Be reasonable please.

Every time you are asked to run an experiment, you must provide the input parameters and a screenshot of the output.

Accessing the Program

To access the Python notebook, follow this **link**. You will be able to perform all the code online, without downloading anything. If you cannot access the link, we will also provide the source code on our website.

Background Information

In this problem, you will use a computer simulation written in *Python* to complete a series of questions relating to slit interference patterns. While you do not need to understand how exactly the code produces the results, it may be beneficial to understand the algorithm the code takes.

The program uses a simple algorithm to determine the interference pattern.

1. An aperture pattern is created by specifying `slits`.
2. The slits are divided into uniform $\lambda \times \lambda$ segments.
3. For each position x on the screen, a wave is generated (and represented as a phasor) between the point x and the center of *each* segment created in the previous step.
4. These phasors are added up and squared to get the intensity.
5. The final intensity pattern is normalized such that the maximum value is 1.

For this entire problem, assume that the heights of the slits are λ.

Remark: This algorithm treats the $d = \lambda$ case as a thin slit (i.e. point source), and it is inaccurate for wide slits (try to see why). This only causes an issue for problem 1.2 & 1.3. Thin slits are used for all other experimental parts afterwards.

Part One

Let us first gain confidence in using the program. To do so, we will derive

$$\frac{\sin(x)}{x} = \cos(x/2)\cos(x/4)\cos(x/8)\cdots \tag{1}$$

via a series of questions.

Problem 1.1 (1 pt)

Suppose monochromatic, coherent light of wavelength λ falls down onto two slits of width w, with midpoints separated by a distance d. Find the amplitude function $A(\theta)$ for the interference pattern produced as a result, where θ is the deviation angle from the center. Assume the screen is far away. A proof is not necessarily necessary, but will help with partial credit in the event that your answer is wrong.

Problem 1.2 (4 pts)

Currently, the code simulates the interference patterns of two thin slits separated by a distance of $d = 8\lambda$. Modify the code to simulate:

- A double slit experiment with wide slits. You are free to choose the location and widths of the slits. Take into account the remark in the background information.

The program will output the intensity function. Make note of the minimum and maximum in the intensity in the experimental results, and compare it to the theoretical results (which you derived in the previous question). Do they agree?

If they do not agree, provide possible reasoning to why they do not agree.

Problem 1.3 (3 pts)

Suppose we have some pattern of slits with overall width w that produces an interference pattern with amplitude $A_0(\theta)$. Suppose we place two of these patterns with midpoints separated a distance $d \geq w$ apart (so that they do not overlap). Find, with proof, as a function of A_0 and other parameters, the amplitude function $A(\theta)$ for the interference pattern produced as a result.

Verify this result experimentally using the code.

Problem 1.4 (5 pts)

Complete the problem by showing

$$\frac{\sin(x)}{x} = \cos(x/2)\cos(x/4)\cos(x/8)\cdots \tag{2}$$

There is no experimental portion associated with this part.

Part Two

Problem 2.1 (8 pts)

Recall that the intensity of the interference pattern from two thin slits behaves like $\cos^2(x)$. Is it possible to have a series of thin slits such that the *amplitude* pattern behaves exactly like the intensity pattern from the two thin slits case? Specifically, the amplitude $A_1(x)$ from one pattern of slits behaves like the intensity $I_2(x)$ from another pattern of slits if:

$$\frac{I_2(x)}{I_{2,\max}} = \frac{A_1(x)}{A_{1,\max}} \tag{3}$$

What sort of aperture would create such an $A(x)$? Verify this experimentally. You will notice that the pattern will deviate towards the edge. At what angle does this deviation become significant? Note that "significant" is subjective, so you will need to provide justification for how you define significant.

Solve this problem using Fourier Optics (There are at least 2 separate ways to do so. As long as you borrow concepts from Fourier Optics, you will receive full points):

- Here are four potentially useful references from Wikipedia. **Any result you use that is not in these references must be derived. This holds for the following two problems as well.**
  - Fraunhofer Diffraction Equation
  - Fourier Optics
  - Convolution
  - Fourier Transform

  Remember that asking for help on public forums or seeking help from other students, other teams, professors, i.e. is strictly prohibited.

Problem 2.2 (4 pts)

In the previous question, you have constructed a series of thin slits such that the amplitude behaves like $\cos^2(x)$. As a result, the intensity behaves like $\cos^4(x)$.

Now, construct a series of slits such that the amplitude function behaves like the intensity pattern from the previous question, i.e. $\cos^4(x)$. You may choose to verify this experimentally, but it is not necessary to get full marks.

Problem 2.3 (4 pts)

Now generalize this to an arbitrary $\cos^n(x)$. If you want the amplitude function to behave like $\cos^n(x)$ where n is a non-negative integer, what should the slit pattern be?

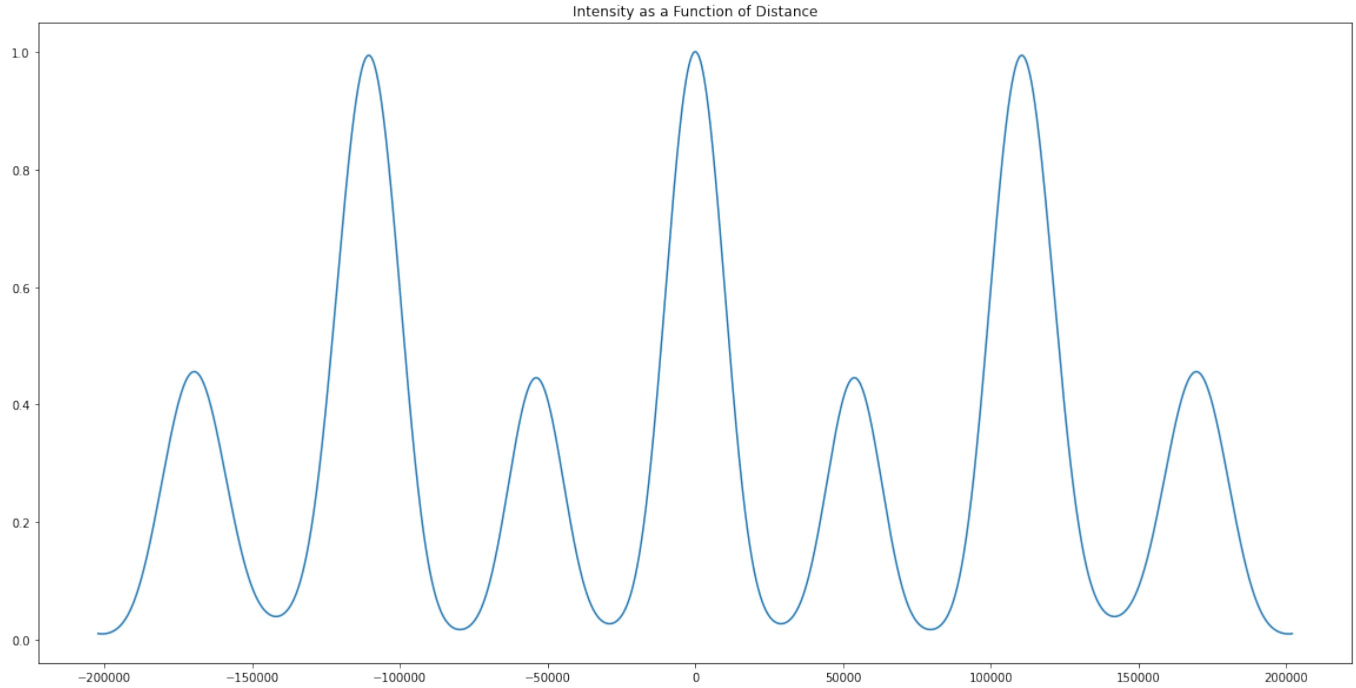
Hint: If you are stuck, try coming up with a conjecture based off of the $\cos^2(x)$ and $\cos^4(x)$ cases, list out how much amplitude they let in, and have everything share a common denominator. If you did the previous parts correctly, the numerators should follow a very familiar pattern.

Part Three

In the previous two parts, you have mostly been asked questions that can be solved analytically, and then used the code to double check your answer. Now, we will ask a few questions that requires data analysis.

Problem 3.1 (11 pts)

The following diagram shows the intensity of the interference pattern produced by a series of slits (where each slit can reduce the amplitude by some factor). The wall is 25 cm away and 500 nm light is used. The x-axis represents locations on the screen, with units of λ.

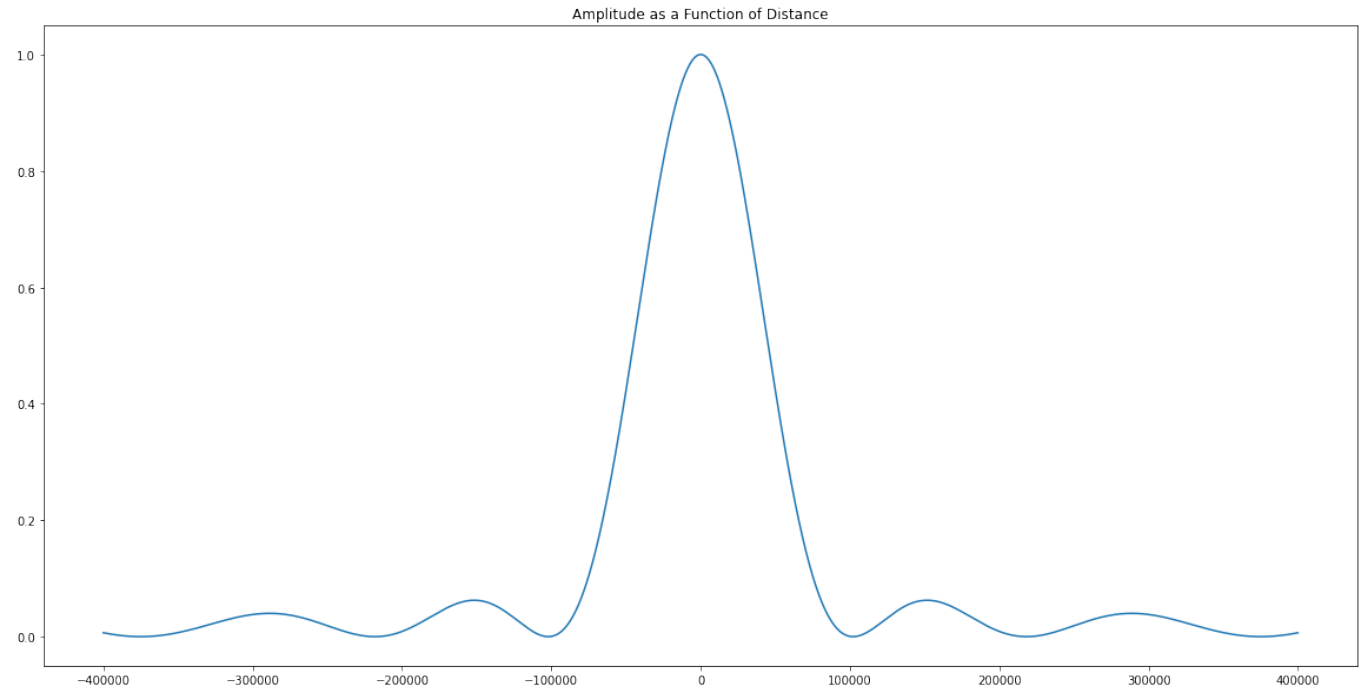


Determine the aperture pattern to the best of your ability. The raw data file is attached on the website.

Note: You may use any online tool and/or resource to do this problem, so long as you are not asking help from people outside your team.

Problem 3.2 (7 pts)

The following diagram shows the **amplitude** of the interference pattern produced by a series of 9 thin slits, centered at $-4\lambda, -3\lambda, \ldots, +4\lambda$. Each slit reduces the wavelength by some factor (this factor isn't necessarily the same for each slit).



Again, the screen is the same distance away as the previous problem and the same scale is used. Determine how much each slit reduces the amplitude of light that passes through.

Let f_x be the reduction factor of the slit centered at x, which has units of λ. For example, the rightmost slit is located at $x = 4$. Make a plot of (x, f_x) for $0 \leq x \leq 4$ and make note of patterns you see.

Problem 3.3 (3 pts)

The interference pattern from the previous problem can be approximated (at least in the small angle range) by a relatively simple function. Find this function.

Hint: Look at the pattern that was hinted at in the previous question

Solution

Part One

Problem 1.1

We have:

$$A(\theta) \propto \left(\frac{\sin\beta}{\beta}\right)\cos\left(\frac{\pi d}{\lambda}\theta\right) \tag{4}$$

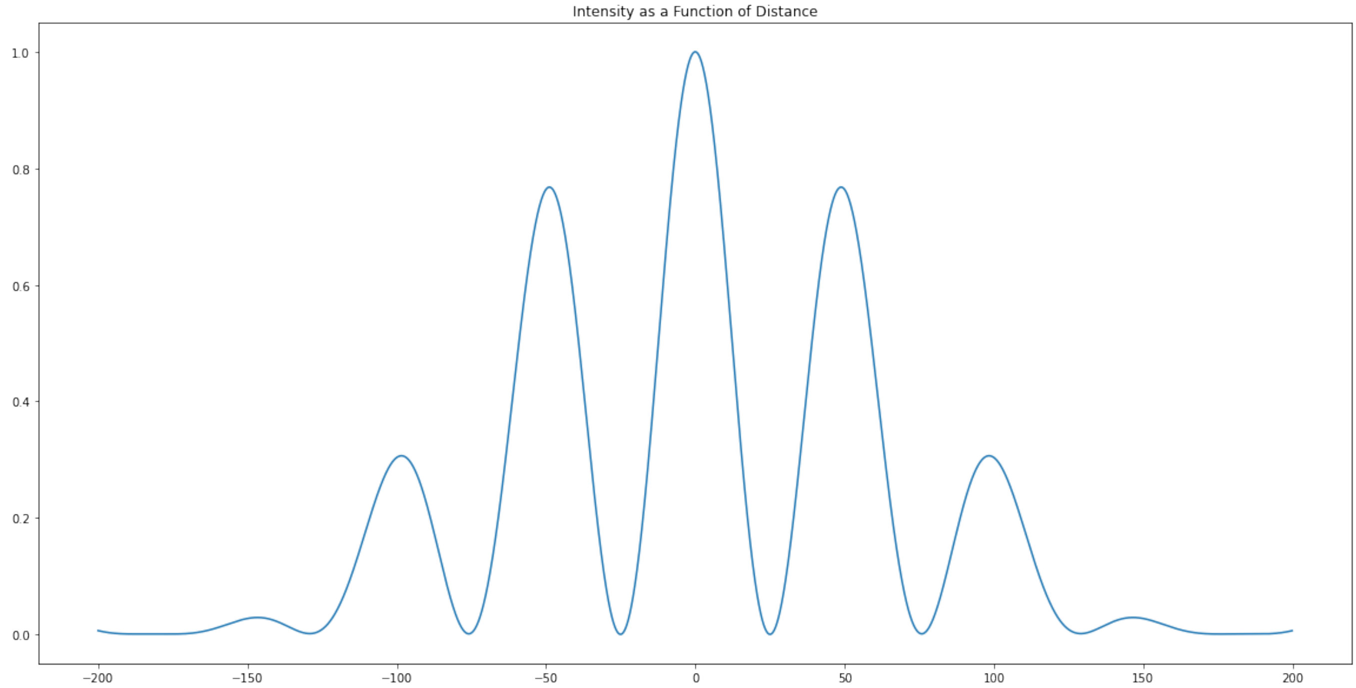
where $\beta = \dfrac{\pi w \sin\theta}{\lambda}$.

Problem 1.2

We will let $w = 3\lambda$ and $d = 10\lambda$, and let the distance between the slits and the wall be $D = 500\lambda$. This corresponds with setting the slits to be

```
slits = [
      [-5*wavelength, 3*wavelength, wavelength, wavelength, 1],
      [5*wavelength, 3*wavelength, wavelength, wavelength, 1]
    ]
```

and gives the following pattern:



The first minimum is at $(25.0 \pm 0.1)\lambda$. Note that the uncertainty comes from the fact that the for loop iterates through locations on the screen in intervals of 0.1λ.

Theoretically, the first minimum occurs when $\cos\left(\dfrac{\pi d}{\lambda}\theta\right) = 0$, which implies $\dfrac{\pi d\theta}{\lambda} = \dfrac{\pi}{2}$, which occurs when $\theta = \dfrac{\lambda}{2d}$. Using the approximation $\theta \approx \dfrac{x}{D}$, we get the first minimum to be:

$$x_{\text{min}} = \frac{D}{2d}\lambda = 25\lambda. \tag{5}$$

However, the second minimum occurs at $(75.9 \pm 0.1)\lambda$, which disagrees with the theoretical $x_{\text{min},2} = 3x_{\text{min}} = 75\lambda$. Even after accounting for non-small angles, i.e. $x_{3,\text{min}} = D\arctan\left(\dfrac{3\lambda}{2d}\right) = 74.4\lambda$, it still doesn't agree. This is because the code doesn't adjust for thick slits properly.

Problem 1.3

For an angle θ, phasors of the two patterns will each have an amplitude $A_0(\theta)$, but are an angle $\frac{2\pi d}{\lambda}\theta$ apart. Therefore:

$$A = A_0(\theta)\cos\left(\frac{\pi d}{\lambda}\theta\right) \tag{6}$$

Note that we have already verified this experimentally using the code in the previous question. We just need to show that the interference pattern of a single (wide) slit is given by:

$$A(\theta) \propto \frac{\sin\beta}{\beta}, \tag{7}$$

which we can do similarly to what we did above by looking at the minimum and/or maximum points.

Problem 1.4

The separations are $d, 2d, \ldots, 2^{k-1}d$. Therefore, using the previous result recursively, we get:

$$A = A_0(\theta)\cos\left(\frac{\pi d}{\lambda}\theta\right)\cos\left(\frac{2\pi d}{\lambda}\theta\right)\cdots\cos\left(\frac{2^{k-1}\pi d}{\lambda}\theta\right) \tag{8}$$

Let us consider the case where $d = w$ and take the limit $k \to \infty$ while keeping $2^k w$ constant. On the left hand side, we have a single slit of width $2^k d$, and we know the pattern is

$$A = \frac{\sin x}{x} \tag{9}$$

if we let $x = \frac{\pi(2^k d)\theta}{\lambda}$. On the right hand side, we have the same scenario as the previous part:

$$A = A_0(\theta)\cos(x/2)\cos(x/4)\cdots\cos\left(x/2^k\right).$$

As we take $k \to \infty$, we have $d \to 0$, implying A_0 goes to 1. Therefore, we get the desired relationship.

Part Two

Problem 2.1

Any aperture pattern can be represented by an aperture function, where a slit centered at x_0 that reduces the amplitude to f of the original can be represented by $f \cdot \delta(x - x_0)$ where $\delta(x)$ is the delta function.

It turns out that taking the Fourier transform of the aperture function can give the amplitude function. It may seem strange that we are transforming a variable with dimension of length x_1 (Which represents the location of the slits) to another dimension of length x_2 (which represents the location at the screen), when Fourier transforms typically take variables to their inverses (i.e. time to frequency). However, this is not the case. We are actually taking the transform from x_1 to $\frac{x_2}{\lambda D}$.

While we will not go into the details, it turns out taking the Fourier transform of the amplitude function will also get us the aperture function (i.e. transforming x_2 into x_1).

For a double slit, the intensity function is $\cos^2\left(\pi d\frac{x}{\lambda D}\right) = \cos^2(\pi d x_2)$. We can use the identity $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ to write:

$$\cos^2(\pi d \cdot x_2) = \frac{1}{2} + \frac{1}{2}\cos(2\pi d \cdot x_2). \tag{10}$$

Taking the Fourier Transform, we get:

$$\hat{f}\left[\frac{1}{2}+\frac{1}{2}\cos(2\pi d\cdot x_2)\right]=\hat{f}\left[\frac{1}{2}\right]+\hat{f}\left[\frac{1}{2}\cos(2\pi d\cdot x_2)\right] \tag{11}$$

$$=\frac{1}{2}\hat{f}[1]+\frac{1}{2}\hat{f}[\cos(2\pi d\cdot x_2)] \tag{12}$$

$$=\frac{1}{2}\delta(x_1)+\frac{1}{2}\left(\frac{\delta\left(x_1-\frac{2\pi d}{2\pi}\right)+\delta\left(x_1+\frac{2\pi d}{2\pi}\right)}{2}\right) \tag{13}$$

$$=\frac{1}{2}\delta(x_1)+\frac{1}{4}\delta\left(x_1-d\right)+\frac{1}{4}\delta\left(x_1+d\right) \tag{14}$$

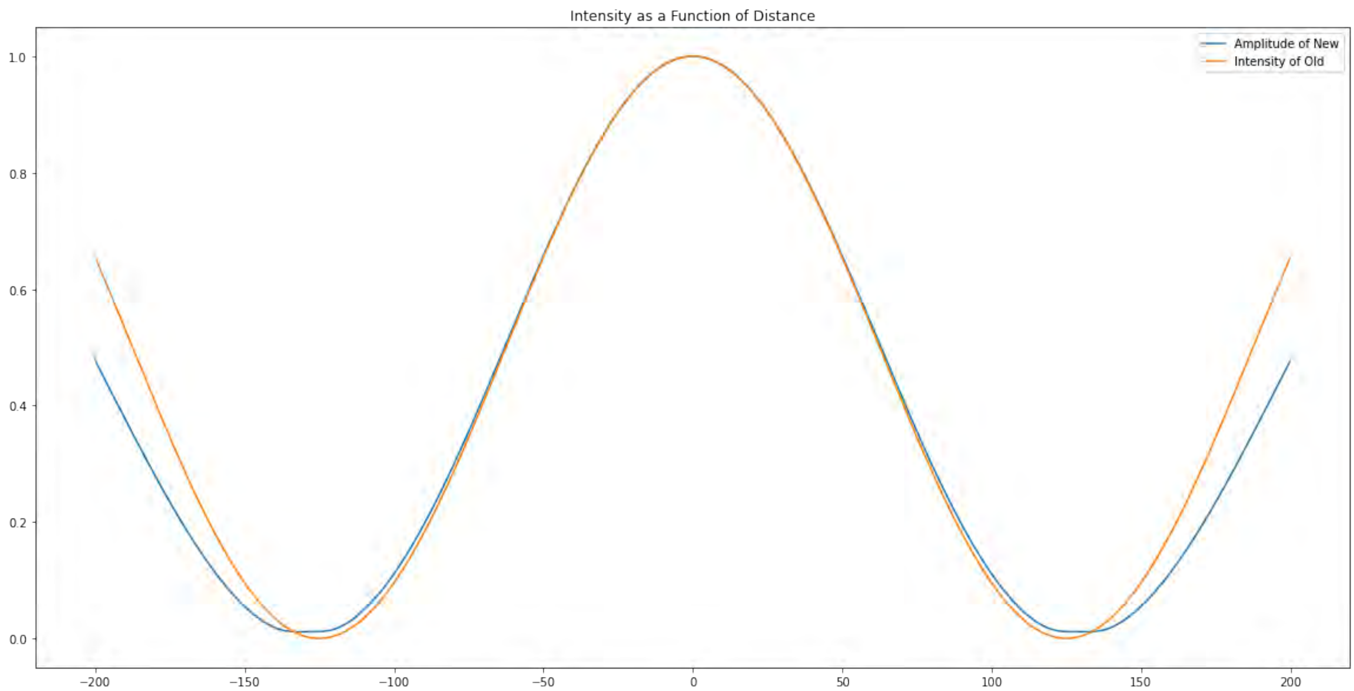
and to normalize it (since only the relative amplitudes matter), we're left with:

$$\delta(x_1)+\frac{1}{2}\delta(x_1-d)+\frac{1}{2}\delta(x_1+d). \tag{15}$$

We can verify this experimentally. Setting

```
slits = [
  [0*wavelength, 1*wavelength, wavelength, wavelength, 1],
  [-2*wavelength, 1*wavelength, wavelength, wavelength, 0.5],
  [2*wavelength, 1*wavelength, wavelength, wavelength, 0.5],
]
```

will give us the following plot, where we have plotted the intensity pattern of the previous aperture pattern for reference:



Notice that this is not perfect. This is because the Fourier Transform uses the far-field approximation, but the location of the first minimum isn't necessarily small enough.

Problem 2.2

We do a similar thing. Note that:

$$\cos^4(x)=\left(\frac{1}{2}(1+\cos(2x))\right)^2=\frac{1}{4}(1+2\cos(2x)+\cos^2(2x)) \tag{16}$$

Applying linearity of the fourier transform, we have

$$\begin{aligned}\hat{f}\left[\frac{1}{4}(1+2\cos(2ax)+\cos^2(2ax))\right] &= \frac{1}{4}\delta(x)+\frac{1}{4}\delta\left(x-\frac{2a}{2\pi}\right)+\frac{1}{4}\delta\left(x+\frac{2a}{2\pi}\right)+\frac{1}{4}\hat{f}\left[\cos^2(2ax)\right] \\ &= \frac{1}{4}\delta(x)+\frac{1}{4}\delta(x+d)+\frac{1}{4}\delta(x-d)+\frac{1}{4}\left(\frac{1}{2}\delta(x_1)+\frac{1}{4}\delta\left(x_1-2d\right)+\frac{1}{4}\delta\left(x_1+2d\right)\right) \\ &= \frac{3}{8}\delta(x)+\frac{1}{4}\delta(x\pm d)+\frac{1}{16}\delta(x\pm 2d)\end{aligned}$$

where we have let $a = \pi d$, and used the result from the previous problem. Normalizing this gives a slit in the middle that lets the full amplitude in, two slits located a distance d from the center that reduces the amplitude by $2/3$, and two slits located a distance $2d$ from the center that reduces the amplitude by $\frac{1}{6}$.

Problem 2.3

We want to find the Fourier transform:

$$\hat{f}[\cos^n(ax_2)] \tag{17}$$

However, multiplication in the x_2 domain is equivalent to convolution in the x_1 domain. We can apply the convolution theorem to say that:

$$\hat{f}[\cos^n(ax_2)] = \hat{f}[\cos(ax_2)]^n \tag{18}$$

where multiplication on the RHS is denoted by the convolution operator $\star$. We've seen that

$$\hat{f}[\cos(ax_2)] = \frac{1}{2}\left(\delta\left(x_1-\frac{a}{2\pi}\right)+\delta\left(x_1+\frac{a}{2\pi}\right)\right) \tag{19}$$

Note that convolution is associative and distributive. Using the fact that δ is the identity, we have the property:

$$\delta(x-a)\star\delta(x-b) = \delta(x-a-b) \tag{20}$$

Thus, we have:

$$\hat{f}[\cos(ax_2)]^n = \frac{1}{2^n}\left(\delta\left(x_1-\frac{a}{2\pi}\right)+\delta\left(x_1+\frac{a}{2\pi}\right)\right)^n$$

Note that $\frac{a}{2\pi} = \frac{d}{2}$. Expanding this using the distributive property, we see that each term contains k copies of $\delta(x-d/2)$ and $n-k$ copies of $\delta(x+d/2)$, which combines to give $\delta(x-kd/2+nd/2-kd/2) = \delta(x+(n/2-k)d)$ where $0 \le k \le n$. If we label the $n+1$ slits as S_k, then S_k is located at a location of $(k-n/2)d$ and reduces the amplitude by $\frac{1}{2^n}\binom{n}{k}$. Here the $\binom{n}{k}$ comes in because there are $\binom{n}{k}$ terms in the convolution expansion that has k copies of the first term and $n-k$ copies of the second term (which corresponds to a unique slit).

Part Three

Problem 3.1

First, we make a few observations:

- The intensity pattern is symmetrical and centered at $x = 0$: Therefore it probably consists of only cosines.
- The intensity never reaches 0, so the amplitude is always positive. We can then safely take the square root without worrying about reversibility issues.
- The number of peaks in between each period is 1, so it must mean that one frequency is exactly double the other. Looking at the period, we can conclude that two of the slits are located $\pm 5\lambda$ away from the center and $\pm 10\lambda$ frmo the center.

Using a data analysis tool such as Python or Excel, we find that the period is 102100.0λ.

Since the slits are at integer locations, we can write out a Fourier *series* by noting that each cosine would be in the form of:

$$\cos\left(d' \cdot \frac{2\pi}{\lambda D}x\right) \tag{21}$$

where d is an integer. Thus, let's clean up our data by:

- Taking the square root.
- Looking at only one period (to prevent edge effects from ruining the data)
- Plotting against $10\pi x/\lambda D$ instead of just x.

Taking a Fourier Series (i.e. by having $n = 2$), we get:

```
{y: a0 + a1*cos(w*x) + a2*cos(2*w*x) + b1*sin(w*x) + b2*sin(2*w*x)}

Parameter Value         Standard Deviation
a0        5.407222e-01 9.935952e-04
a1        1.419923e-01 1.439942e-03
a2        2.758724e-01 1.411563e-03
b1        -3.714775e-05 1.369781e-03
b2        -6.656304e-05 1.378570e-03
```

allowing us to reconstruct the sinusoidal wave. Taking a Fourier Transform, we get the aperture function. Here it is, for reference:

```
slits = [
      [0*wavelength, 1*wavelength, wavelength, wavelength, 3],
      [-5*wavelength, 1*wavelength, wavelength, wavelength, 0.45],
      [5*wavelength, 1*wavelength, wavelength, wavelength, 0.45],
      [-10*wavelength, 1*wavelength, wavelength, wavelength, 0.77],
      [10*wavelength, 1*wavelength, wavelength, wavelength, 0.77],
    ]
```

Problem 3.2

Similar to the previous problem, we "cut" off the interference pattern at a small angle (but big enough such that the distinctive behaviour of the curve is captured). Scaling x to be $\frac{2\pi}{\lambda D}x$, we can determine the Fourier coefficients to be:

```
Parameter Value         Standard Deviation
a0        2.045300e-01 1.285129e-04
a1        3.294887e-01 1.439479e-04
a2        2.371716e-01 8.086478e-05
a3        1.652102e-01 8.350614e-05
a4        6.081714e-02 2.142034e-04
```

Taking the Fourier Transform analytically, we get

$$0.0854\delta(x \pm 4) + 0.2070\delta(x \pm 3) + 0.2973\delta(x \pm 2) + 0.4130\delta(x \pm 1) + 0.5126\delta(x) \tag{22}$$

Plotting f_X against x forms a straight line ($r^2 = 0.998$).

Problem 3.3

Note that the aperture function here is actually a triangle:

$$\text{tri}(x_1/5) \tag{23}$$

The Fourier Transform from x_1 to x_2 gives:

$$\mathrm{sinc}^2\left(\frac{5x_2}{\lambda D}\right). \tag{24}$$

2022 Online Physics Olympiad: Open Contest

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This competition could not be possible without the help of our sponsors, who are all doing great things in physics, math, and education.

Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

- Use $g = 9.81$ m/s^2 in this contest, **unless otherwise specified**. See the constants sheet on the following page for other constants.
- This test contains 35 short answer questions. Each problem will have three possible attempts.
- The weight of each question depends on our scoring system found here. Put simply, the later questions are worth more, and the overall amount of points from a certain question decreases with the number of attempts that you take to solve a problem as well as the number of teams who solve it.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should contain **three** significant figures, unless otherwise specified. All answers within the 1% range will be accepted.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used. Technology and computer algebra systems like Wolfram Alpha or the one in the TI nSpire will not be needed or allowed. Attempts to use these tools will be classified as cheating.
- You are *allowed* to use Wikipedia or books in this exam. Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions.
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in SI units (meter, second, kilogram, watt, etc.) unless otherwise specified. Please input all angles in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put units in your answer on the submission portal! If your answer is "x meters", input only the value x into the submission portal.
- **Do not communicate information to anyone else apart from your team-members before June 13, 2022**.

List of Constants

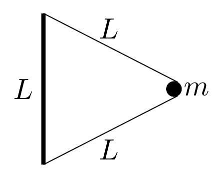
- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^8$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\mathrm{N \cdot m^2})/\mathrm{kg}^2$$
- Solar Mass
$$M_\odot = 1.988 \cdot 10^{30}\ \mathrm{kg}$$
- Acceleration due to gravity, $g = 9.81$ m/s^2
- 1 unified atomic mass unit,
$$1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15}\ \mathrm{eV \cdot s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9\ (\mathrm{N \cdot m^2})/\mathrm{C^2}$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T \cdot m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\mathrm{T \cdot m})/\mathrm{A}$$
- 1 atmospheric pressure,
$$1\ \mathrm{atm} = 1.01 \cdot 10^5\ \mathrm{N/m^2} = 1.01 \cdot 10^5\ \mathrm{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$$

Problems

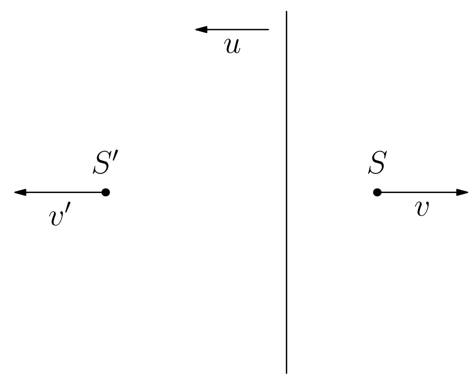
1. Two Projectiles A player throws two tennis balls on a level ground at $v = 20$ m/s in the same direction, once at an angle of $\alpha = 35°$ and once at an angle $\beta = 55°$ to the horizontal. The distance between the landing spots of the two balls is d. Find d in meters.

Assume the height of the player is negligble and ignore air resistance.

2. Bow and Arrow Consider the following simple model of a bow and arrow. An ideal elastic string has a spring constant $k = 10$ N/m and relaxed length $L = 1$ m which is attached to the ends of an inflexible fixed steel rod of the same length L as shown below. A small ball of mass $m = 2$ kg and the thread are pulled by its midpoint away from the rod until each individual part of the thread have the same length of the rod, as shown below. What is the speed of the ball in meters per seconds right after it stops accelerating? Assume the whole setup is carried out in zero gravity.



3. City Lights A truck (denoted by S) is driving at a speed $v = 2$ m/s in the opposite direction of a car driving at a speed $u = 3$ m/s, which is equipped with a rear-view mirror. Both v and u are measured from an observer on the ground. Relative to this observer, what is the speed (in m/s) of the truck's image S' through the car's mirror? Car's mirror is a plane mirror.



4. Springing Earth For this problem, assume the Earth moves in a perfect circle around the sun in the xy plane, with a radius of $r = 1.496 \times 10^{11}$ m, and the Earth has a mass $m = 5.972 \times 10^{24}$ kg. An alien stands far away from our solar system on the x axis such that it appears the Earth is moving along a one dimensional line, as if there was a zero-length spring connecting the Earth and the Sun.

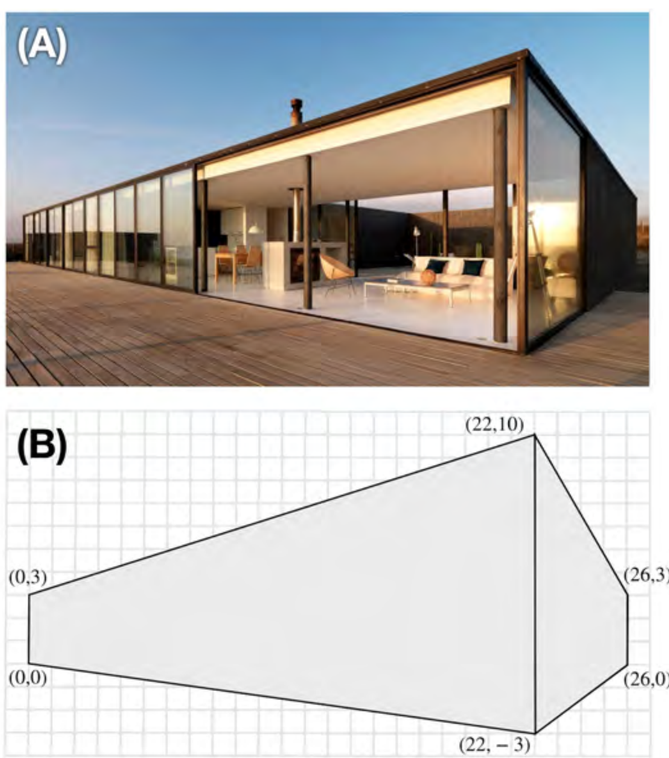
For the alien at this location, it is impossible to tell just from the motion if it's 2D motion via gravity or 1D motion via a spring. Let U_g be the gravitational potential energy ignoring its self energy if Earth moves via gravity, taking potential energy at infinity to be 0 and U_s be the maximum spring potential energy if Earth moves in 1D via a spring. Compute U_g/U_s.

5. Battle Ropes Battle ropes can be used as a full body workout (see photo). It consists of a long piece of thick rope (ranging from 35 mm to 50 mm in diameter), wrapped around a stationary pole. The athlete grabs on to both ends, leans back, and moves their arms up and down in order to create waves, as shown in the photo.

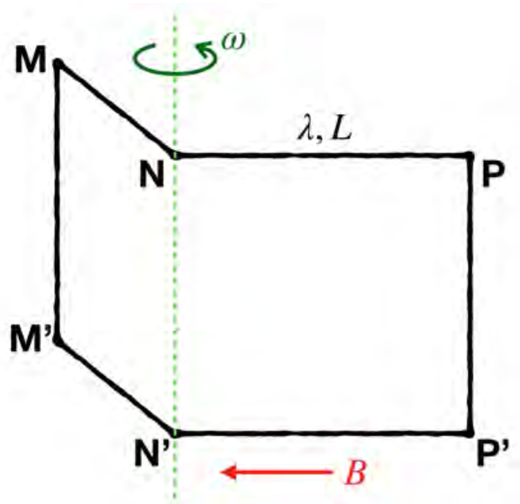
The athlete wishes to upgrade from using a 35 mm diameter rope to a 50 mm diameter rope, while keeping everything else the same (rope material, rope tension, amplitude, and speed at which her arms move back and forth). By doing so, the power she needs to exert changes from P_0 to P_1. Compute P_1/P_0.

6. Polarizers Given vertically polarized light, you're given the task of changing it to horizontally polarized light by passing it through a series of $N = 5$ linear polarizers. What is the maximum possible efficiency of this process? (Here, efficiency is defined as the ratio between output light intensity and input light intensity.)

7. Fatal Frame These days, there are so many stylish rectangular home-designs (see figure A). It is possible from the outline of those houses in their picture to estimate with good precision where the camera was. Consider an outline in one photograph of a rectangular house which has height $H = 3$ meters (see figure B for square-grid coordinates). Assume that the camera size is negligible, how high above the ground (in meters) was the camera at the moment this picture was taken?



8. The Wire Consider a thin rigid wire-frame MNPP'N'M' in which MNN'M' and NPP'N' are two squares of side L with resistance per unit-length λ and their planes are perpendicular. The frame is rotated with a constant angular velocity ω around an axis passing through NN' and put in a region with constant magnetic field B pointing perpendicular to NN'. What is the total heat released on the frame per revolution (in Joules)? Use $L = 1\text{m}$, $\lambda = 1\Omega/\text{m}$, $\omega = 2\pi\text{rad/s}$ and $B = 1\text{T}$.



9. Melting Iceberg In this problem, we explore how fast an iceberg can melt, through the dominant mode of forced convection. For simplicity, consider a very thin iceberg in the form of a square with side lengths $L = 100$ m and a height of 1 m, moving in the arctic ocean at a speed of 0.2 m/s with one pair of edges parallel to the direction of motion (Other than the height, these numbers are typical of an average iceberg). The temperature of the surrounding water and air is 2°C, and the temperature of the iceberg is 0°C. The density of ice is 917 kg/m^3 and the latent heat of melting is $L_w = 334 \times 10^3$ J/kg.

The heat transfer rate $\dot{Q}$ between a surface and the surrounding fluid is dependent on the heat transfer coefficient h, the surface area in contact with the fluid A, and the temperature difference between the surface and the fluid ΔT, via $\dot{Q} = hA\Delta T$.

In heat transfer, three useful quantities are the Reynold's number, the Nusselt number, and the Prandtl number. Assume they are constant through and given by (assuming laminar flow):

$$\text{Re} = \frac{\rho v_\infty L}{\mu}, \qquad \text{Nu} = \frac{hL}{k}, \qquad \text{Pr} = \frac{c_p \mu}{k}$$

where:

- ρ: density of the fluid
- v_∞: speed of the fluid with respect to the object (at a very far distance)
- L: length of the object in the direction of motion
- μ: dynamic viscosity of the fluid
- k: thermal conductivity of the fluid
- c_p : the specific heat capacity of the fluid

Through experiments, the relationship between the three dimensionless numbers is, for a flat plate:

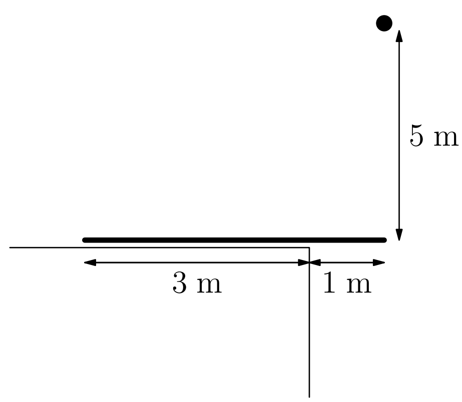
$$\mathrm{Nu} = 0.664\mathrm{Re}^{1/2}\mathrm{Pr}^{1/3}.$$

Use the following values for calculations:

	Air	Water
ρ (kg/m^3)	1.29	1000
μ (kg/(m · s))	1.729×10^{-5}	1.792×10^{-3}
c_p (J/(kg · K))	1004	4220
k (W/(m · K))	0.025	0.556

The initial rate of heat transfer is $\dot{Q}$. Assuming this rate is constant (this is not true, but will allow us to obtain an estimate), how long (in days) would it take for the ice to melt completely? Assume convection is only happening on the top and bottom faces. Round to the nearest day.

10. Scale A scale of uniform mass $M = 3$ kg of length $L = 4$ m is kept on a rough table (infinite friction) with $l = 1$ m hanging out of the table as shown in the figure below. A small ball of mass $m = 1$ kg is released from rest from a height of $h = 5$ m above the end of the scale. Find the maximum angle (in degrees) that the scale rotates by in the subsequent motion if ball sticks to the scale after collision. Take gravity $g = 10$ m/s^2.



11. Levitating In a galaxy far, far away, there is a planet of mass $M = 6 \cdot 10^{27}$ kg which is a sphere of radius R and charge $Q = 10^3$ C uniformly distributed. Aliens on this planet have devised a device for transportation, which is an insulating rectangular plate with mass $m = 1$ kg and charge $q = 10^4$ C. This transportation device moves in a circular orbit at a distance $r = 8 \cdot 10^6$ m from the center of the planet. The aliens have designated this precise elevation for the device, and do not want the device to deviate at all. In order to maintain its orbit, the device contains a relatively small energy supply. Find the power (in Watts) that the energy supply must release in order to sustain this orbit.

The velocity of the device can be assumed to be much smaller than the speed of light, so that relativistic effects can be ignored. The device can also be assumed to be small in comparison to the size of the planet.

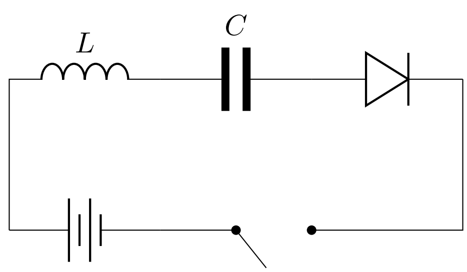
12. Singing in the Rain A raindrop of mass $M = 0.035$ g is at height $H = 2$ km above a large lake. The raindrop then falls down (without initial velocity), mixing and coming to equilibrium with the lake. Assume that the raindrop, lake, air, and surrounding environment are at the same temperature $T = 300$ K. Determine the magnitude of entropy change associated with this process (in J/K).

13. Rocket Launch A rocket with mass of 563.17 (not including the mass of fuel) metric tons sits on the launchpad of the Kennedy Space Center (latitude 28°31′27"N, longitude 80°39′03"W), pointing directly upwards. Two solid fuel boosters, each with a mass of 68415kg and providing 3421kN of thrust are pointed directly downwards.

The rocket also has a liquid fuel engine, that can be throttled to produce different amounts of thrust and gimbaled to point in various directions. What is the minimum amount of thrust, in kN, that this engine needs to provide for the rocket to lift vertically (to accelerate directly upwards) off the launchpad?

Assume $G = 6.674 \times 10^{-11} \frac{\mathrm{N \cdot m^2}}{\mathrm{s^3}}$, and that the Earth is a perfect sphere of radius 6370km and mass 5.972×10^{24}kg that completes one revolution every 86164s and that the rocket is negligibly small compared to the Earth. Ignore buoyancy forces.

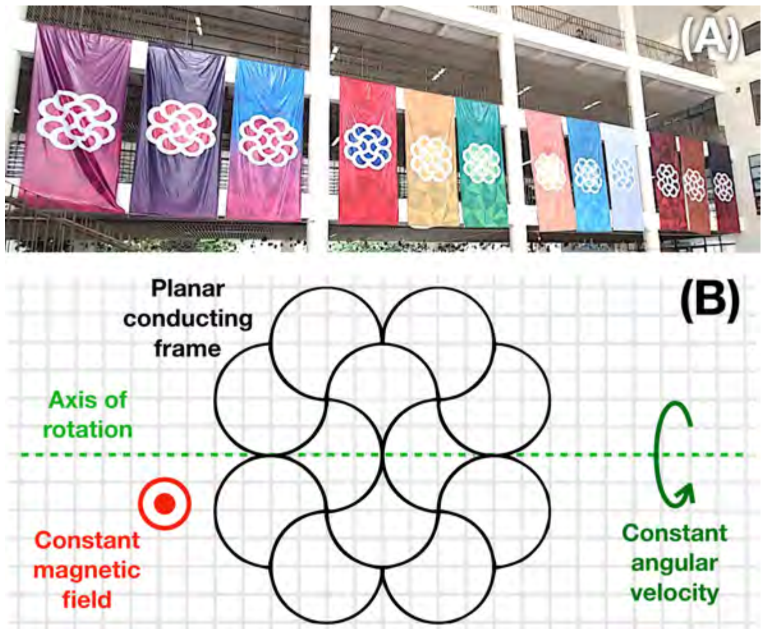
The following information applies for the next two problems. A circuit has a power source of $\mathcal{E} = 5.82\,\mathrm{V}$ connected to three elements in series: an inductor with $L = 12.5\,\mathrm{mH}$, a capacitor with $C = 48.5\,\mu\mathrm{F}$, and a diode with threshold voltage $V_0 = 0.65\,\mathrm{V}$. (Of course, the polarity of the diode is aligned with that of the power source.) You close the switch, and after some time, the voltage across the capacitor becomes constant. (*Note:* An ideal diode with threshold voltage V_0 is one whose IV characteristic is given by $I = 0$ for $V < V_0$ and $V = V_0$ for $I > 0$.)



14. LC-Diode 1 How much time (in seconds) has elapsed before the voltage across the capacitor becomes constant?

15. LC-Diode 2 What is the magnitude of final voltage (in volts) across the capacitor?

16. Raging Loop At Hanoi-Amsterdam High School in Vietnam, every subject has its own flag (see Figure A, taken by Tung X. Tran). While the flags differ in color, they share the same central figure. Consider a planar conducting frame of that figure rotating at a constant angular velocity in a uniform magnetic field (see Figure B). The frame is made of thin rigid wires with uniform curvature and resistance per unit length. What fraction of the total heat released is released by the outermost wires?

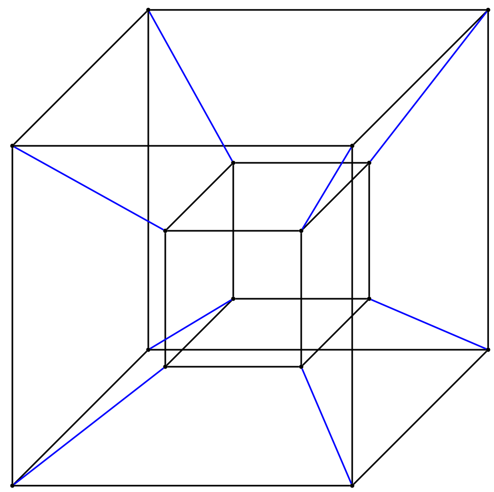


17. Moon Landing A spacecraft is orbiting in a very low circular orbit at a velocity v_0 over the equator of a perfectly spherical moon with uniform density. Relative to a stationary frame, the spacecraft completes a revolution of the moon every 90 minutes, while the moon revolves in the same direction once every 24 hours. The pilot of the spacecraft would like to land on the moon using the following process:

1. Start by firing the engine directly against the direction of motion.
2. Orient the engine over time such that the vertical velocity of the craft remains 0, while the horizontal speed continues to decrease.
3. Once the velocity of the craft relative to the ground is also 0, turn off the engine.

Assume that the engine of the craft can be oriented instantly in any direction, and the craft has a TWR (thrust-to-weight ratio, where weight refers to the weight at the moon's surface) of 2, which remains constant throughout the burn. If the craft starts at $v_0 = 500$ m/s, compute the delta-v expended to land, minus the initial velocity, i.e. $\Delta v - v_0$.

18. Tesseract Oscillations A tesseract is a 4 dimensional example of cube. It can be drawn in 3 dimensions by drawing two cubes and connecting their vertices together as shown in the picture below:



Now for the 3D equivalent. The lines connecting the vertices are replaced with ideal springs of constant $k = 10$ N/m (in blue in the figure). Now, suppose the setup is placed in zero-gravity and the outer cube is fixed in place with a sidelength of $b = 2$ m. The geometric center of the inner cube is placed in the geometric center of the outer cube, and the inner cube has a side-length $a = 1$ m and mass $m = 1.5$ kg. The inner cube is slightly displaced from equilibrium. Consider the period of oscillations

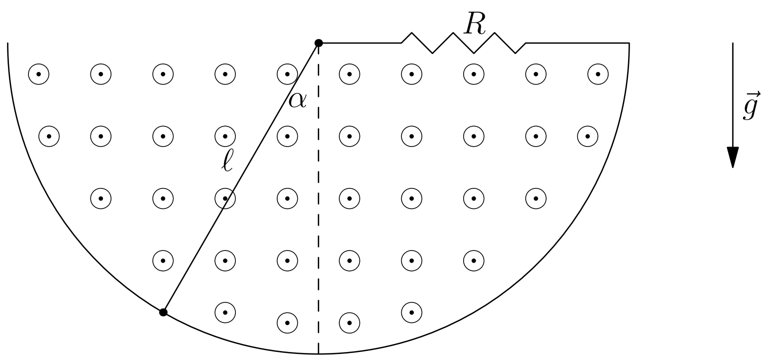
- T_1: when the springs have a relaxed length of 0;
- T_2: when the springs are initially relaxed before the inner cube is displaced.

What is $T_1 + T_2$?

19. The Room Consider two points S and S' randomly placed inside a D-dimensional hyper-rectangular room with walls that are perfect-reflecting $(D-1)$-dimensional hyper-plane mirrors. How many different light-rays that start from S, reflect N times on one of the walls and $N-1$ times on each of the rest, then go to S'? Use $D = 7$ and $N = 3$.

20. Two Rings Two concentric isolated rings of radius $a = 1$ m and $b = 2$ m of mass $m_a = 1$ kg and $m_b = 2$ kg are kept in a gravity free region. A soap film of surface tension $\sigma = 0.05\text{Nm}^{-1}$ with negligible mass is spread over the rings such that it occupies the region between the rings. The smaller ring is pulled slightly along the axis of the rings. Find the time period of small oscillation in seconds.

21. Pendulum Circuit An open electrical circuit contains a wire loop in the shape of a semi-circle, that contains a resistor of resistance $R = 0.2\Omega$. The circuit is completed by a conducting pendulum in the form of a uniform rod with length $\ell = 0.1$ m and mass $m = 0.05$ kg, has no resistance, and stays in contact with the other wires at all times. All electrical components are oriented in the yz plane, and gravity acts in the z direction. A constant magnetic field of strength $B = 2$ T is applied in the $+x$ direction.

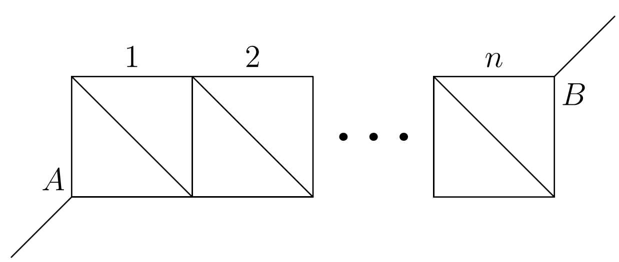


Ignoring self inductance and assuming that $\alpha \ll 1$, the general equation of motion is in the form of $\theta(t) = A(t)\cos(\omega t + \varphi)$, where $A(t) \geq 0$. Find ω^2.

22. Broken Table A table of unknown material has a mass $M = 100$ kg, width $w = 4$ m, length $\ell = 3$ m, and 4 legs of length $L = 0.5$ m with a Young's modulus of $Y = 1.02$ MPa at each of the corners. The cross-sectional area of a table leg is approximately $A = 1\text{ cm}^2$. The surface of the table has a coefficient of friction of $\mu = 0.1$. A point body with the same mass as the table is put at some position from the geometric center of the table. What is the minimum distance the body must be placed from the center such that it slips on the table surface immediately after? Report your answer in centimeters.

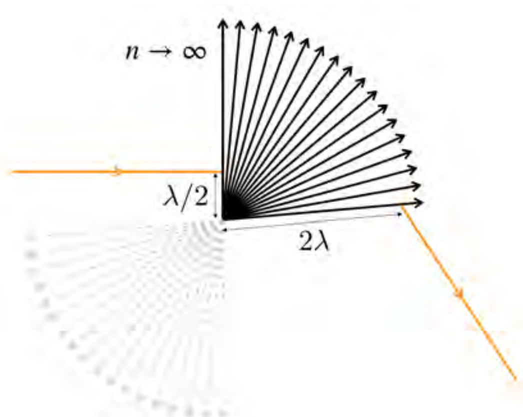
The table surface and floor are non-deformable.

23. Resistance box In the figure below, the resistance of each wire (side and diagonal) is 1Ω. Find the value of $p + q$ if $\lim_{n\to\infty} \frac{R_{AB}}{n} = \frac{p}{q}$ where p and q are co-prime integers.



24. Dipole Conductor An (ideal) electric dipole of magnitude $p = 1 \times 10^{-6}$ C·m is placed at a distance $a = 0.05$ m away from the center of an uncharged, isolated spherical conductor of radius $R = 0.02$ m. Suppose the angle formed by the dipole vector and the radial vector (the vector pointing from the sphere's center to the dipole's position) is $\theta = 20^o$. Find the (electrostatic) interaction energy between the dipole and the charge induced on the spherical conductor.

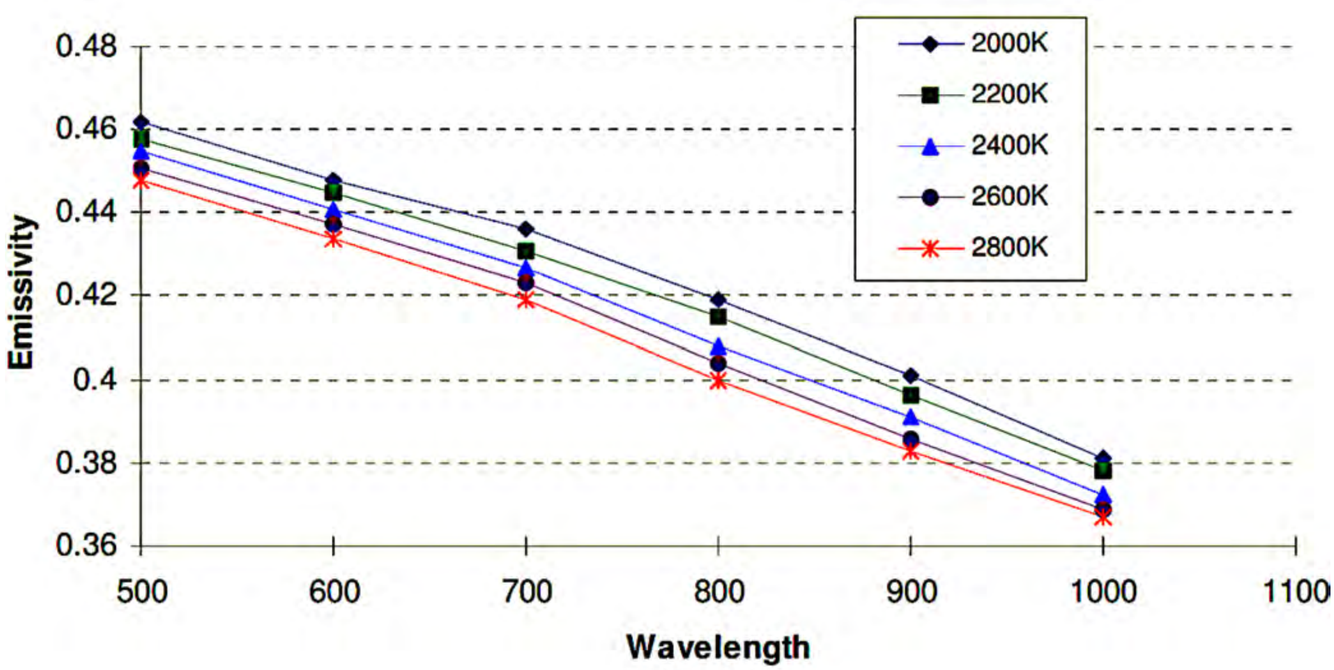
25. Dying Light Consider an optical system made of many identical ideal (negligible-thickness) half-lenses with focal length $f > 0$, organized so that they share the same center and are angular-separated equally at density n (number of lenses per unit-radian). Define the length-scale $\lambda = f/n$. A light-ray arrives perpendicular to the first lens at distance $\lambda/2$ away from the center, then leaves from the last lens at distance 2λ away from the center. Estimate the total deflection angle (in rad) of the light-ray by this system in the limit $n \to \infty$.



26. Tungsten For black body radiation, Wien's Displacement Law states that its spectral radiance will peak at

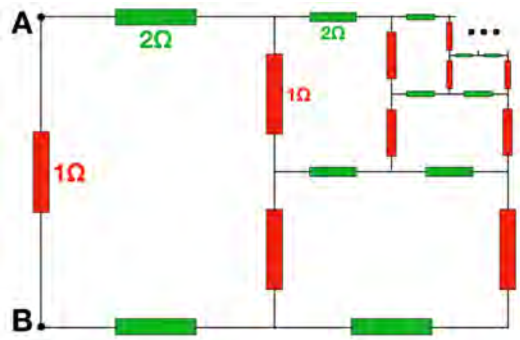
$$\lambda_{\text{peak}} = \frac{b}{T}.$$

where $b = 2.89777 \times 10^{-3}$ mK, and T is the temperature of the object. When QiLin tried to reproduce this in a lab, by working with a tungsten-filament lightbulb at 2800 K, he computed a different value for b by measuring the peak wavelength using a spectrometer and multiplying it with the temperature. He hypothesizes that this discrepancy is because tungsten is not an ideal black body. The graph below, courtesy of the CRC Handbook of Chemistry and Physics, shows the emissivity of tungsten at various conditions (the units for wavelength is nm).



Assuming QiLin's hypothesis is correct, and assuming there were no other errors in the experiment, how off was his value for b? Submit $\frac{|b_{\text{theory}}-b_{\text{experiment}}|}{b_{\text{theory}}}$ as a decimal number, to *one* significant digit (giving you room to estimate where the points are).

27. Bioshock Infinite The equivalent resistance (in Ω) between points A and B of the following infinite resistance network made of 1Ω and 2Ω resistors is $\overline{0.abcdefgh...}$ in decimal form. Enter $\overline{efg}$ into the answer box (It should be an integer in the range of 0-999).



28. Magnetic Ball A uniform spherical metallic ball of mass m, resistivity ρ, and radius R is kept on a smooth friction-less horizontal ground. A horizontal uniform, constant magnetic field B exists in the space parallel to the surface of ground. The ball was suddenly given an impulse perpendicular to magnetic field such that ball begin to move with velocity v without losing the contact with ground. Find the time in seconds required to reduce its velocity by half.

Numerical Quantities: $m = 2$ kg, $4\pi\epsilon_0 R^3 B^2 = 3$ kg, $\rho = 10^9 \Omega$m, $v = \pi$ m/s.

For the following two problems, this information applies. Assume $g = 9.8\text{m/s}^2$. On a balcony, a child holds a spherical balloon of radius 15 cm. Upon throwing it downwards with a velocity of 4.2 m/s, the balloon starts magically expanding, its radius increasing at a constant rate of 35 cm/s. Another child, standing on the ground, is holding a hula hoop, 4 m below the point where the center of the balloon was released.

29. Magical Balloon 1 If the minimum radius of the hoop such that the balloon falls completely through the hula hoop without touching it is r, compute the difference between r and the largest multiple of 5cm less than or equal to r. Answer in centimeters; your answer should be in the range $[0, 5)$.

30. Magical Balloon 2 Consider the horizontal plane passing through the center of the balloon at the start. If the total volume above this plane that the balloon falls through after it is thrown downwards is V, compute the difference between V and half the original volume of the balloon. Answer in milliliters; your answer should be nonnegative.

Note that when refer to the "volume an object falls through", it refers to the volume of the union of all points in space which the object occupies as it falls.

31. Hydrogen Magnetism In quantum mechanics, when calculating the interaction between the electron with the proton in a hydrogen atom, it is necessary to compute the following volume integral (over all space):

$$\mathbf{I} = \int \mathbf{B}(\mathbf{r})|\Psi(\mathbf{r})|^2 dV$$

where $\Psi(\mathbf{r})$ is the spatial wavefunction of the electron as a function of position $\mathbf{r}$ and $\mathbf{B}(\mathbf{r})$ is the (boldface denotes vector) magnetic field produced by the proton at position $\mathbf{r}$. Suppose the proton is located at the origin and it acts like a finite-sized magnetic dipole (but much smaller than a_0) with dipole moment $\mu_p = 1.41 \times 10^{-26}$ J/T. Let the hydrogen atom be in the ground state, meaning $\Psi(\mathbf{r}) = \frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}}$, where $a_0 = 5.29 \times 10^{-11}$ m is the Bohr radius. Evaluate the magnitude of the integral $|\mathbf{I}|$ (in SI units).

32. Relativistic Collision Zed is trying to model the repulsive interaction between 2 objects, A and B (with masses m_A and m_B, respectively), in a relativistic setting. He knows that in relativity, forces cannot act at a distance, so he models the repulsive force with a small particle of mass m that bounces elastically between A and B. Throughout this problem, assume everything moves on the x-axis. Suppose that initially, A and B have positions and velocities x_A, v_A and x_B, v_B, respectively, where $x_A < x_B$ and $v_A > v_B$. The particle has an initial (relativistic) speed v.

For simplicity, assume that the system has no total momentum. You may also assume that $v_A, v_B \ll v$, and that $p_m \ll p_A, p_B$, where p_m, p_A, p_B are the momenta of the particle, A, and B, respectively. Do NOT assume $v \ll c$, where c is the speed of light.

Find the position (in m) of A when its velocity is 0, given that $m_A = 1$ kg, $m_B = 2$ kg, $v_A = 0.001c$, $m = 1 \times 10^{-6}$ kg, $v = 0.6c$, $x_A = 0$ m, $x_B = 1000$ m.

Note: Answers will be tolerated within 0.5%, unlike other problems.

33. Microscope Consider an optical system consisting of two thin lenses sharing the same optical axis. When a cuboid with a side parallel to the optical axis is placed to the left of the left lens, its final image formed by the optical system is also a cuboid but with 500 times the original volume. Assume the two lenses are 10 cm apart and such a cuboid of volume 1 cm^3 is placed such that its right face is 2 cm to the left of the left lens. What's the maximum possible volume of the intermediate image (i.e., image formed by just the left lens) of the cuboid? Answer in cm^3.

34. Resistor grid Consider an infinite square grid of equal resistors where the nodes are exactly the lattice points in the 2D Cartesian plane. A current $I = 2.7$ A enters the grid at the origin $(0, 0)$. Find the current in Amps through the resistor connecting the nodes $(N, 0)$ and $(N, 1)$, where $N = 38$ can be assumed to be much larger than 1.

35. Strange Gas Suppose we have a non-ideal gas, and in a certain volume range and temperature range, it is found to satisfy the state relation

$$p = AV^{\alpha}T^{\beta}$$

where A is a constant, $\alpha = -\frac{4}{5}$ and $\beta = \frac{3}{2}$, and the other variables have their usual meanings. Throughout the problem, we will assume to be always in that volume and temperature range.

Assume that $\gamma = \frac{C_p}{C_V}$ is found to be constant for this gas (γ is independent of the state of the gas), where C_p and C_v are the heat capacities at constant pressure and volume, respectively. What is the minimum possible value for γ?

2022 Online Physics Olympiad: Open Contest

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Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

- Use $g = 9.81$ m/s^2 in this contest, **unless otherwise specified**. See the constants sheet on the following page for other constants.
- This test contains 35 short answer questions. Each problem will have three possible attempts.
- The weight of each question depends on our scoring system found here. Put simply, the later questions are worth more, and the overall amount of points from a certain question decreases with the number of attempts that you take to solve a problem as well as the number of teams who solve it.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should contain **three** significant figures, unless otherwise specified. All answers within the 1% range will be accepted.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used. Technology and computer algebra systems like Wolfram Alpha or the one in the TI nSpire will not be needed or allowed. Attempts to use these tools will be classified as cheating.
- You are *allowed* to use Wikipedia or books in this exam. Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions.
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in SI units (meter, second, kilogram, watt, etc.) unless otherwise specified. Please input all angles in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put units in your answer on the submission portal! If your answer is "x meters", input only the value x into the submission portal.
- **Do not communicate information to anyone else apart from your team-members before June 13, 2022**.

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}\ \mathrm{mol}^{-1}$
- Universal gas constant, $R = 8.31\ \mathrm{J/(mol \cdot K)}$
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$
- 1 electron volt, $1\ \mathrm{eV} = 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^8$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\mathrm{N \cdot m^2})/\mathrm{kg}^2$$
- Solar Mass
$$M_\odot = 1.988 \cdot 10^{30}\ \mathrm{kg}$$
- Acceleration due to gravity, $g = 9.81\ \mathrm{m/s^2}$
- 1 unified atomic mass unit,
$$1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15}\ \mathrm{eV \cdot s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9\ (\mathrm{N \cdot m^2})/\mathrm{C}^2$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T \cdot m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\mathrm{T \cdot m})/\mathrm{A}$$
- 1 atmospheric pressure,
$$1\ \mathrm{atm} = 1.01 \cdot 10^5\ \mathrm{N/m^2} = 1.01 \cdot 10^5\ \mathrm{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}\ \mathrm{m \cdot K}$
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$$

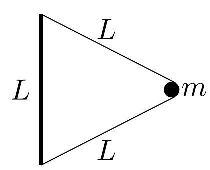
Problems

1. Two Projectiles A player throws two tennis balls on a level ground at $v = 20$ m/s in the same direction, once at an angle of $\alpha = 35°$ and once at an angle $\beta = 55°$ to the horizontal. The distance between the landing spots of the two balls is d. Find d in meters.

Assume the height of the player is negligble and ignore air resistance.

Solution 1: The range of a projectile is proportional as $R \propto \sin 2\theta$, or $R \propto \cos\theta \sin\theta$. As $\cos(90 - \theta) = \sin\theta$, and $\alpha + \beta = 90$, the distance travelled by both projectiles are the same.
$\boxed{0 \text{ m}}$

2. Bow and Arrow Consider the following simple model of a bow and arrow. An ideal elastic string has a spring constant $k = 10$ N/m and relaxed length $L = 1$ m which is attached to the ends of an inflexible fixed steel rod of the same length L as shown below. A small ball of mass $m = 2$ kg and the thread are pulled by its midpoint away from the rod until each individual part of the thread have the same length of the rod, as shown below. What is the speed of the ball in meters per seconds right after it stops accelerating? Assume the whole setup is carried out in zero gravity.



Solution 2: We can use conservation of energy. The bow string has its potential increased as

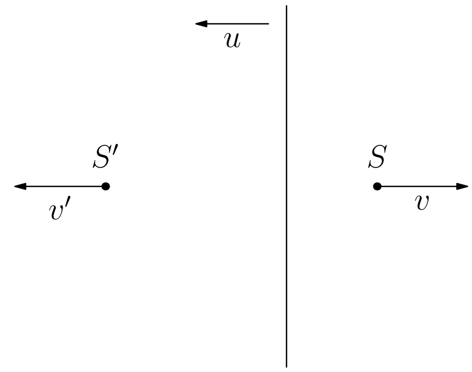
$$E_p = \frac{1}{2}k(2L - L)^2 = \frac{1}{2}kL^2.$$

This all turns into the kinetic energy of the ball $E_k = \frac{1}{2}mv^2$, so

$$E_p = E_k \implies \frac{1}{2}kL^2 = \frac{1}{2}mv^2 \implies v = L\sqrt{\frac{k}{m}}.$$

$\boxed{2.23 \text{ m/s}}$

3. City Lights A truck (denoted by S) is driving at a speed $v = 2$ m/s in the opposite direction of a car driving at a speed $u = 3$ m/s, which is equipped with a rear-view mirror. Both v and u are measured from an observer on the ground. Relative to this observer, what is the speed (in m/s) of the truck's image S' through the car's mirror? Car's mirror is a plane mirror.



Solution 3: In the mirror's frame of reference, the source speed and the image speed is both $u+v$ but in opposite direction. Now, go back the the observer's frame of reference, the image speed becomes v':

$$v' = (u+v) + u = 2u + v = 8\text{m/s} \tag{1}$$

$\boxed{8 \text{ m/s}}$

4. Springing Earth For this problem, assume the Earth moves in a perfect circle around the sun in the xy plane, with a radius of $r = 1.496 \times 10^{11}$ m, and the Earth has a mass $m = 5.972 \times 10^{24}$ kg. An alien stands far away from our solar system on the x axis such that it appears the Earth is moving along a one dimensional line, as if there was a zero-length spring connecting the Earth and the Sun.

For the alien at this location, it is impossible to tell just from the motion if it's 2D motion via gravity or 1D motion via a spring. Let U_g be the gravitational potential energy ignoring its self energy if Earth moves via gravity, taking potential energy at infinity to be 0 and U_s be the maximum spring potential energy if Earth moves in 1D via a spring. Compute U_g/U_s.

Solution 4: One naive idea is to directly compute U_g and U_s, but we can use the fact that their frequencies are the same, or:

$$\omega^2 = \frac{k}{m} = \frac{GM}{r^3} \implies kr^2 = \frac{GM}{r}$$

Then,

$$U_g = -\frac{GMm}{r} = -kr^2$$

and

$$U_s = \frac{1}{2}kr^2.$$

Therefore,

$$U_g/U_s = -2.$$

$\boxed{2}$

5. Battle Ropes Battle ropes can be used as a full body workout (see photo). It consists of a long piece of thick rope (ranging from 35 mm to 50 mm in diameter), wrapped around a stationary pole. The athlete grabs on to both ends, leans back, and moves their arms up and down in order to create waves, as shown in the photo.

The athlete wishes to upgrade from using a 35 mm diameter rope to a 50 mm diameter rope, while keeping everything else the same (rope material, rope tension, amplitude, and speed at which her arms move back and forth). By doing so, the power she needs to exert changes from P_0 to P_1. Compute P_1/P_0.

Solution 5: The power transmitted by a wave is given by

$$P = \frac{1}{2}\mu\omega^2 A^2 v,$$

where $\mu = \frac{m}{L}$ is the linear mass density, A is the amplitude, and v is the speed of the wave. The speed of a wave on a rope is given by

$$v = \sqrt{\frac{T}{\mu}}, \tag{2}$$

where T is the tension. Note that ω, A, T will all remain constant when changing the radius. Thus, $P \propto \sqrt{\mu} \propto \sqrt{m}$. As we increase the radius by a factor of $f = \frac{50}{35}$, we change the mass by f^2, so the power changes by a factor of f, giving us

$$P_1/P_0 = f = 1.43$$

$\boxed{1.43}$

6. Polarizers Given vertically polarized light, you're given the task of changing it to horizontally polarized light by passing it through a series of $N = 5$ linear polarizers. What is the maximum possible efficiency of this process? (Here, efficiency is defined as the ratio between output light intensity and input light intensity.)

Solution 6:
Let $\theta_0 = 0$ be the original direction of polarization and $\theta_5 = \pi/2$ the final direction of polarization. The 5 polarizers are directed along $\theta_1, \theta_2, \ldots, \theta_5$. Let $\delta_k = \theta_k - \theta_{k-1}$, so that the efficiency is

$$\eta = \prod_{k=1}^{5} \cos^2 \delta_k.$$

We wish to maximize η subject to the constraint that $\sum_k \delta_k = \pi/2$. Clearly, the $\delta_k's$ should be non-negative, implying that $0 \leq \delta_k \leq \pi/2$ and thus $\cos\delta_k \geq 0$ for all k.

We claim that the maximum is achieved when all δ_k are equal. If not, let $\delta_i \neq \delta_{i+1}$. Then

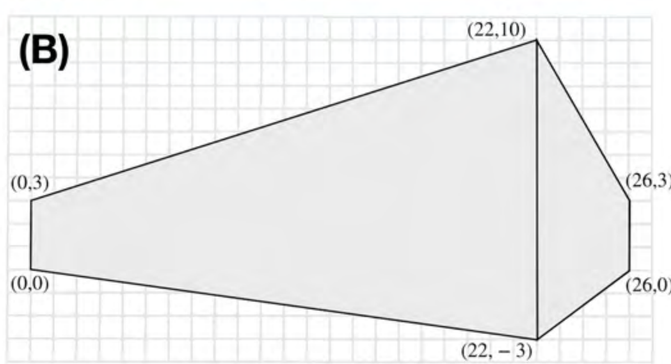
$$\begin{aligned}\cos\delta_i \cos\delta_{i+1} &= \frac{1}{2}[\cos(\delta_i+\delta_{i+1}) + \cos(\delta_i - \delta_{i+1})] \\ &< \frac{1}{2}[\cos(\delta_i+\delta_{i+1}) + 1] \\ &= \frac{1}{2}[\cos(\delta_i'+\delta_{i+1}') + \cos(\delta_i' - \delta_{i+1}')]\end{aligned}$$

where $\delta_i' = \delta_{i+1}' = \frac{\delta_i+\delta_{i+1}}{2}$. So replacing δ_i, δ_{i+1} with δ_i', δ_{i+1}' increases η.
So η is maximized when all δ_k are equal, i.e., $\delta_k^* = \frac{\pi}{10}$ for all k. Then

$$\eta^* = \cos^{10}\left(\frac{\pi}{10}\right) \approx 0.6054.$$

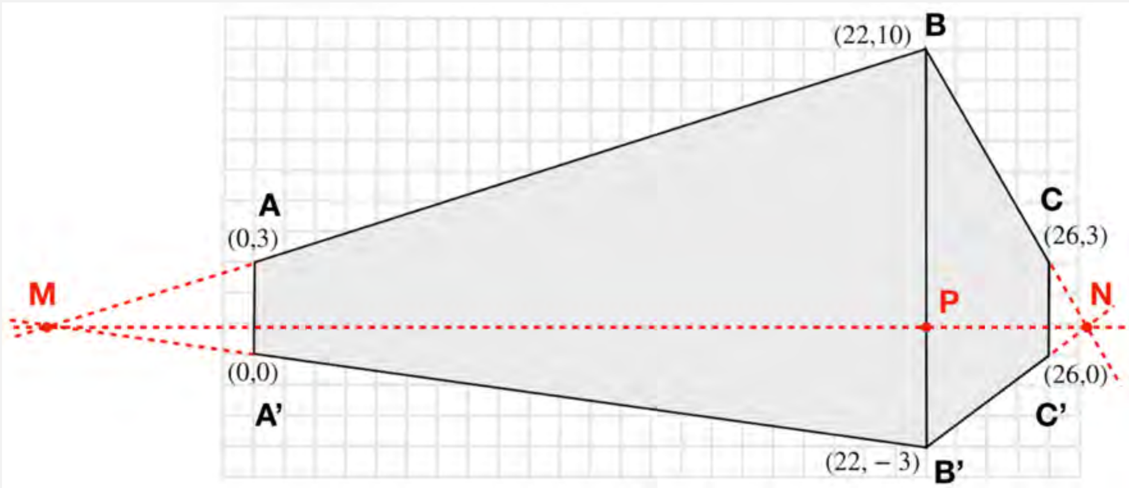
$\boxed{0.6054}$

7. Fatal Frame These days, there are so many stylish rectangular home-designs (see figure A). It is possible from the outline of those houses in their picture to estimate with good precision where the camera was. Consider an outline in one photograph of a rectangular house which has height $H = 3$ meters (see figure B for square-grid coordinates). Assume that the camera size is negligible, how high above the ground (in meters) was the camera at the moment this picture was taken?



Solution 7:
The formation of the house's image seen in the picture is due to pinhole principle, and note that the fish-eye effect here is weak (straight-lines stays straight). Define points A, B, C, A', B', C' as in the attached Fig., since AA', BB', CC' stays parallel we know that the camera looked horizontally

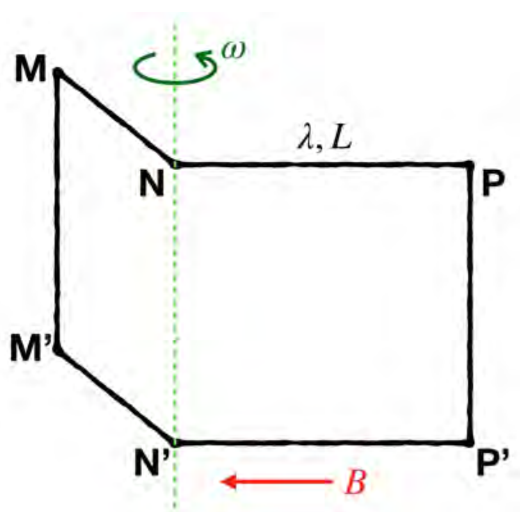
at the time this picture is taken.



To determine the height of the camera at the very same moment, we need to know the where is the horizontal plane passing through the camera in the picture which is collapsed into a line. That can be found by finding the intersection M of $AB \cap A'B'$ and the intersection N of $BC \cap B'C'$, then MN is the line of interests. MN intersects BB' at P, the position of P can be calculated too be $(22, 0.9)$, therefore the height of the camera is the length-ratio PB'/BB' times 3m, which equals to 0.9m.

$\boxed{0.9 \text{ m}}$

8. The Wire Consider a thin rigid wire-frame MNPP'N'M' in which MNN'M' and NPP'N' are two squares of side L with resistance per unit-length λ and their planes are perpendicular. The frame is rotated with a constant angular velocity ω around an axis passing through NN' and put in a region with constant magnetic field B pointing perpendicular to NN'. What is the total heat released on the frame per revolution (in Joules)? Use $L = 1\text{m}$, $\lambda = 1\Omega/\text{m}$, $\omega = 2\pi\text{rad/s}$ and $B = 1\text{T}$.



Solution 8: In this setting, for every orientation during rotation the total magnetic flux passing though MNPP'N'M' is tthe same as through MPP'M', which has area $S = \sqrt{2}L^2$.

The magnetic flux is:

$$\Phi(t) = BS\sin(\omega t) = \sqrt{2}BL^2\sin(\omega t)\ . \tag{3}$$

The emf running around the wire-frame is:

$$E(t) = \frac{d}{dt}\Phi(t) = \sqrt{2}BL^2\omega\cos(\omega t)\ . \tag{4}$$

The electrical current running around the wire-frame is:

$$I(t) = \frac{E(t)}{6\lambda L} = \frac{BL\omega\cos(\omega t)}{3\sqrt{2}\lambda}\ . \tag{5}$$

The heat released power is:

$$\frac{d}{dt}Q(t) = I^2(t)\times 6\lambda L = \frac{B^2L^3\omega^2\cos^2(\omega t)}{3\lambda}\ . \tag{6}$$

Thus the total heat released per revolution is:

$$Q = \int_0^{2\pi/\omega} dt\frac{d}{dt}Q(t) = \frac{B^2L^3\omega^2\int_0^{2\pi/\omega} dt\cos^2(\omega t)}{3\lambda} = \frac{\pi B^2L^3\omega}{3\lambda} \approx 6.58\text{J}\ . \tag{7}$$

$\boxed{6.58\text{ J}}$

9. Melting Iceberg In this problem, we explore how fast an iceberg can melt, through the dominant mode of forced convection. For simplicity, consider a very thin iceberg in the form of a square with side lengths $L = 100$ m and a height of 1 m, moving in the arctic ocean at a speed of 0.2 m/s with one pair of edges parallel to the direction of motion (Other than the height, these numbers are typical of an average iceberg). The temperature of the surrounding water and air is 2°C, and the temperature of the iceberg is 0°C. The density of ice is 917 kg/m^3 and the latent heat of melting is $L_w = 334\times 10^3$ J/kg.

The heat transfer rate $\dot{Q}$ between a surface and the surrounding fluid is dependent on the heat transfer coefficient h, the surface area in contact with the fluid A, and the temperature difference between the surface and the fluid ΔT, via $\dot{Q} = hA\Delta T$.

In heat transfer, three useful quantities are the Reynold's number, the Nusselt number, and the Prandtl number. Assume they are constant through and given by (assuming laminar flow):

$$\text{Re} = \frac{\rho v_\infty L}{\mu}, \qquad \text{Nu} = \frac{hL}{k}, \qquad \text{Pr} = \frac{c_p\mu}{k}$$

where:

- ρ: density of the fluid
- v_∞: speed of the fluid with respect to the object (at a very far distance)
- L: length of the object in the direction of motion

- μ: dynamic viscosity of the fluid
- k: thermal conductivity of the fluid
- c_p : the specific heat capacity of the fluid

Through experiments, the relationship between the three dimensionless numbers is, for a flat plate:

$$\mathrm{Nu} = 0.664\mathrm{Re}^{1/2}\mathrm{Pr}^{1/3}.$$

Use the following values for calculations:

	Air	Water
ρ (kg/m^3)	1.29	1000
μ (kg/(m $\cdot$ s))	1.729×10^{-5}	1.792×10^{-3}
c_p (J/(kg $\cdot$ K))	1004	4220
k (W/(m $\cdot$ K))	0.025	0.556

The initial rate of heat transfer is $\dot{Q}$. Assuming this rate is constant (this is not true, but will allow us to obtain an estimate), how long (in days) would it take for the ice to melt completely? Assume convection is only happening on the top and bottom faces. Round to the nearest day.

Solution 9: The heat transfer coefficient for water-ice and air-ice contact can be figured out with the relationship between the three dimensionless numbers

$$\mathrm{Nu} = 0.664\mathrm{Re}^{1/2}\mathrm{Pr}^{1/3} \implies h = 0.664\frac{k}{L}\left(\frac{\rho v_\infty L}{\mu}\right)^{1/2}\left(\frac{c_p \mu}{k}\right)^{1/3}.$$

As

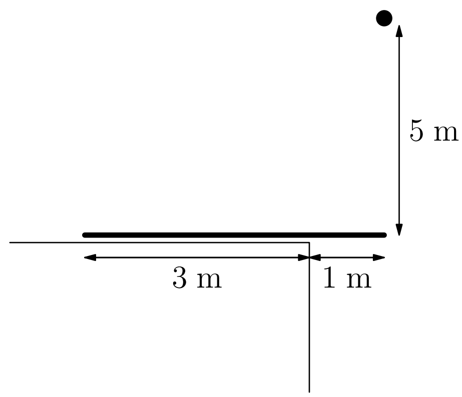
$$\frac{\mathrm{d}Q}{\mathrm{d}t} = hA\Delta T,$$

we then have

$$t(\dot{Q}_a + \dot{Q}_w) = (h_w A_w + h_a A_a)\Delta T \implies t = \frac{\rho L^2 H L_w}{\Delta T}\frac{1}{h_w L^2 + h_a L^2} = 59.84 \approx 60 \text{ days}.$$

$\boxed{\text{60 days}}$

10. Scale A scale of uniform mass $M = 3$ kg of length $L = 4$ m is kept on a rough table (infinite friction) with $l = 1$ m hanging out of the table as shown in the figure below. A small ball of mass $m = 1$ kg is released from rest from a height of $h = 5$ m above the end of the scale. Find the maximum angle (in degrees) that the scale rotates by in the subsequent motion if ball sticks to the scale after collision. Take gravity $g = 10$ m/s^2.



Solution 10: The ball falls with velocity $\sqrt{2gh} = 10$. Applying conservation of angular momentum about the end point of the table.

$$\begin{aligned} L_i &= mvx \\ &= 10 \\ L_f &= I_1\omega \\ I_1 &= 4+3+1=8 \\ &\boxed{\implies \omega = 1.25} \end{aligned}$$

Now applying energy conservation

$$\begin{aligned} E_i &= \frac{1}{2}I\omega^2 = 6.25 \\ E_f &= M_{total}gx_{com}\sin(\theta) = 20\sin\theta \\ \theta &= 18.21^\circ \end{aligned}$$

$\boxed{18.21^\circ}$

11. Levitating In a galaxy far, far away, there is a planet of mass $M = 6 \cdot 10^{27}$ kg which is a sphere of radius R and charge $Q = 10^3$ C uniformly distributed. Aliens on this planet have devised a device for transportation, which is an insulating rectangular plate with mass $m = 1$ kg and charge $q = 10^4$ C. This transportation device moves in a circular orbit at a distance $r = 8 \cdot 10^6$ m from the center of the planet. The aliens have designated this precise elevation for the device, and do not want the device to deviate at all. In order to maintain its orbit, the device contains a relatively small energy supply. Find the power (in Watts) that the energy supply must release in order to sustain this orbit.

The velocity of the device can be assumed to be much smaller than the speed of light, so that relativistic effects can be ignored. The device can also be assumed to be small in comparison to the size of the planet.

Solution 11: The centripetal force is given by

$$\frac{mv^2}{r} = \frac{GMm}{r^2} - \frac{qQ}{4\pi\epsilon_0 r^2},$$

which implies the centripetal acceleration is

$$a = \frac{v^2}{r} = \frac{GM}{r^2} - \frac{qQ}{4\pi\epsilon_0 mr^2}.$$

Now, the device loses energy due to its acceleration, as given by the Larmor formula. The power needed to sustain motion is

$$P = \frac{q^2a^2}{6\pi\epsilon_0 c^3} = 0.522 \text{ W}.$$

$\boxed{0.522 \text{ W}}$

12. Singing in the Rain A raindrop of mass $M = 0.035$ g is at height $H = 2$ km above a large lake. The raindrop then falls down (without initial velocity), mixing and coming to equilibrium with the lake. Assume that the raindrop, lake, air, and surrounding environment are at the same temperature $T = 300$ K. Determine the magnitude of entropy change associated with this process (in J/K).

Solution 12: The total heat gain is equal to the change in potential energy of the raindrop, which spreads through out the whole environment at thermally equilibrium temperature T (the environment is very large so any change in T is negligible). The entropy gain ΔS is thus generated by the dissipation of this potential energy MgH to internal energy ΔU in the environment (given that the specific volume of water doesn't change much, $\Delta U \approx MgH$). Hence the entropy change associated with this process can be estimated by:

$$S = \frac{\Delta U}{T} \approx \frac{MgH}{T} \approx 2.29 \times 10^{-3}\text{J/K} \tag{8}$$

$\boxed{2.29 \times 10^{-3} \text{ J/K}}$

13. Rocket Launch A rocket with mass of 563.17 (not including the mass of fuel) metric tons sits on the launchpad of the Kennedy Space Center (latitude 28°31′27"N, longitude 80°39′03"W), pointing directly upwards. Two solid fuel boosters, each with a mass of 68415kg and providing 3421kN of thrust are pointed directly downwards.

The rocket also has a liquid fuel engine, that can be throttled to produce different amounts of thrust and gimbaled to point in various directions. What is the minimum amount of thrust, in kN, that this engine needs to provide for the rocket to lift vertically (to accelerate directly upwards) off the launchpad?

Assume $G = 6.674 \times 10^{-11} \frac{\text{N}\cdot\text{m}^2}{\text{s}^3}$, and that the Earth is a perfect sphere of radius 6370km and mass 5.972×10^{24}kg that completes one revolution every 86164s and that the rocket is negligibly small compared to the Earth. Ignore buoyancy forces.

Solution 13: Note the additional information provided in the problem (latitude, Earth radius, revolution period), which makes it clear that the effect of the rotation of the Earth must also be considered.
We first compute local gravitational acceleration:

$$g = \frac{GM}{R^2} = 9.823\frac{\text{m}}{\text{s}^2}$$

And also acceleration due to the Earth's rotation:

$$a = (R\cos\theta)\omega^2 = 0.02976\frac{\mathrm{m}}{\mathrm{s}^2}$$

Then vertical acceleration is:

$$g - a\cos\theta = 9.7965\frac{\mathrm{m}}{\mathrm{s}^2}$$

And horizontal acceleration:

$$a\sin\theta = 0.0142\frac{m}{s^2}$$

The total mass of the craft is 700 metric tons, so the needed forces for liftoff are:

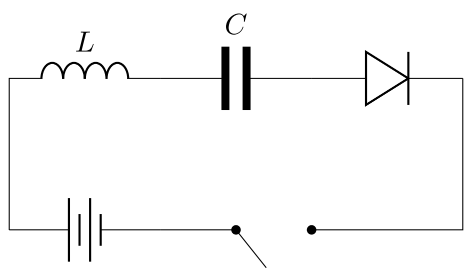
$$(F_x, F_y) = (6857527, 9948)\mathrm{N}$$

Subtracting out the effect of the solid fuel boosters, the force the liquid fuel engine needs to provide is:

$$(F_x, F_y) = (15527, 9948)\mathrm{N}$$

So the final answer is 18.44kN. $\boxed{18.44\mathrm{kN}}$

The following information applies for the next two problems. A circuit has a power source of $\mathcal{E} = 5.82\,\mathrm{V}$ connected to three elements in series: an inductor with $L = 12.5\,\mathrm{mH}$, a capacitor with $C = 48.5\,\mu\mathrm{F}$, and a diode with threshold voltage $V_0 = 0.65\,\mathrm{V}$. (Of course, the polarity of the diode is aligned with that of the power source.) You close the switch, and after some time, the voltage across the capacitor becomes constant. (*Note:* An ideal diode with threshold voltage V_0 is one whose IV characteristic is given by $I = 0$ for $V < V_0$ and $V = V_0$ for $I > 0$.)



14. LC-Diode 1 How much time (in seconds) has elapsed before the voltage across the capacitor becomes constant?

Solution 14:
When current is flowing clockwise, the circuit is equivalent to an LC circuit with a power source $\mathcal{E} - V_0$. Thus, the voltage U is sinusoidal about its equilibrium voltage $U_0 = \mathcal{E} - V_0$ with frequency $\omega = 1/\sqrt{LC}$.
When the switch is closed, $I = 0$ and $U = U_{min} = 0$. Afterwards, I increases to I_{max} and decreases back to 0, completing half a period of a sine wave. However, I can not go negative due to the presence of the diode. Instead, a reverse voltage builds up on the diode (so that the voltage across the inductor becomes 0), and I stays at 0. At this point, U becomes constant as well.
The time it took until the system became static was half a period of the LC circuit oscillation, i.e.,

$$\frac{\pi}{\omega} = \pi\sqrt{LC} = 2.446 \times 10^{-3}\ \mathrm{s}.$$

$\boxed{2.446 \times 10^{-3} \text{ s}}$

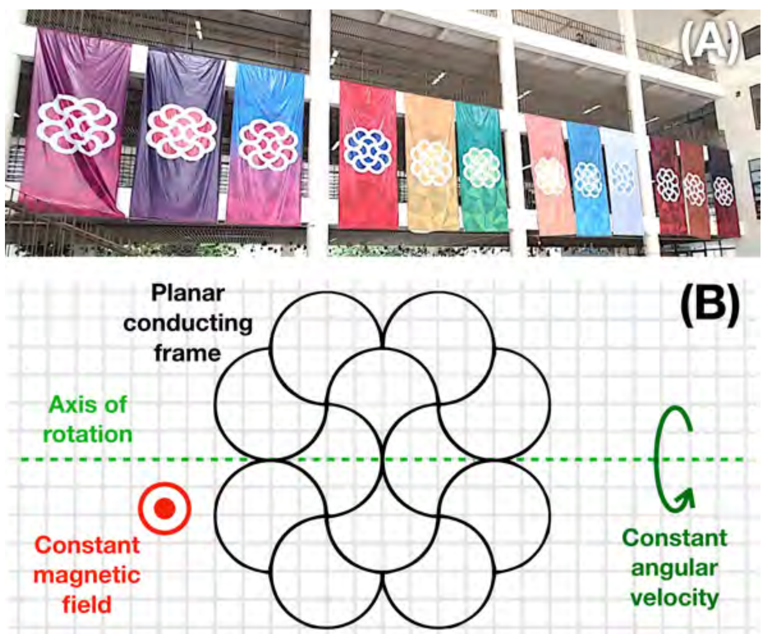
15. LC-Diode 2 What is the magnitude of final voltage (in volts) across the capacitor?

Solution 15:
As I increased from 0 to I_{max} and then decreased to 0 sinusoidally, U increased from $U_{min} = 0$ to U_{max} sinusoidally. Recall that the equilibrium $U_0 = \frac{U_{min}+U_{max}}{2} = \mathcal{E} - V_0$, so the final voltage on the capacitor is

$$U_{max} = 2U_0 = 2(\mathcal{E} - V_0) = 10.34 \text{ V}.$$

$\boxed{10.34 \text{ V}}$

16. Raging Loop At Hanoi-Amsterdam High School in Vietnam, every subject has its own flag (see Figure A, taken by Tung X. Tran). While the flags differ in color, they share the same central figure. Consider a planar conducting frame of that figure rotating at a constant angular velocity in a uniform magnetic field (see Figure B). The frame is made of thin rigid wires with same uniform curvature and same resistance per unit length. What fraction of the total heat released is released by the outermost wires?

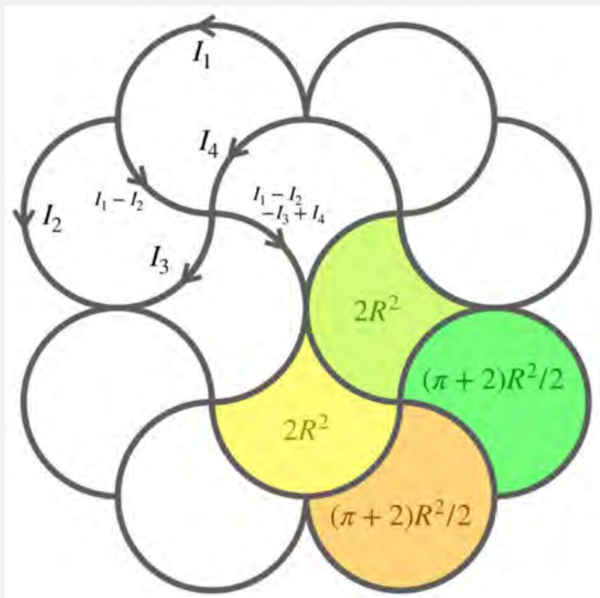


Solution 16:
We call the current looping in the wires I_1, I_2, I_3, I_4 as shown in the attached Fig., and define the resistance of every quarter-circular section (radius R) of the wires to be ρ, then considering the

EMF on every loop gives:

$$\begin{aligned}
\Big(2I_1 + (I_1 - I_2) - I_4\Big)\rho &= \frac{dB_\perp}{dt} \times \frac{\pi+2}{2}R^2 \ , \\
\Big(2I_2 - I_3 - (I_1 - I_2)\Big)\rho &= \frac{dB_\perp}{dt} \times \frac{\pi+2}{2}R^2 \ , \\
\Big(2I_4 + 2(I_1 - I_2 - I_3 + I_4)\Big)\rho &= \frac{dB_\perp}{dt} \times 2R^2 \ , \\
\Big(2I_3 - 2(I_1 - I_2 - I_3 + I_4)\Big)\rho &= \frac{dB_\perp}{dt} \times 2R^2 \ ,
\end{aligned} \tag{9}$$

in which $B_\perp$ is the perpendicular component of the magnetic field.



The set of equations in Eq. (9) can be solved to get the relation between currents:

$$I_1 = I_2 \ , \ I_3 = I_4 \ , \ \frac{I_1}{I_3} = \frac{\pi+4}{4} \ . \tag{10}$$

The fraction of heat released on the outermost wires can be calculated:

$$\frac{Q_{outermost}}{Q_{all}} = \frac{\left(8I_1^2 + 8I_2^2\right)\rho}{\left(8I_1^2 + 8I_2^2 + 4I_3^2 + 4I_4^2\right)\rho} = \frac{(\pi+4)^2}{(\pi+4)^2 + 8} \approx 0.864 \text{ J} \tag{11}$$

$\boxed{0.864 \text{ J}}$

17. Moon Landing A spacecraft is orbiting in a very low circular orbit at a velocity v_0 over the equator of a perfectly spherical moon with uniform density. Relative to a stationary frame, the spacecraft completes a revolution of the moon every 90 minutes, while the moon revolves in the same direction once every 24 hours. The pilot of the spacecraft would like to land on the moon using the following process:

1. Start by firing the engine directly against the direction of motion.
2. Orient the engine over time such that the vertical velocity of the craft remains 0, while the horizontal speed continues to decrease.

3. Once the velocity of the craft relative to the ground is also 0, turn off the engine.

Assume that the engine of the craft can be oriented instantly in any direction, and the craft has a TWR (thrust-to-weight ratio, where weight refers to the weight at the moon's surface) of 2, which remains constant throughout the burn. If the craft starts at $v_0 = 500$ m/s, compute the delta-v expended to land, minus the initial velocity, i.e. $\Delta v - v_0$.

Solution 17:
The trick in this question is to work in dimensionless units. Let v be the ratio of the craft's velocity to orbital velocity. Then, if the craft has horizontal velocity v, the acceleration downwards is the following:

$$a_V = \frac{v_0^2}{r} - \frac{(v_0 v)^2}{r} = \frac{v_0^2}{r}(1 - v^2) = g_m(1 - v^2)$$

As the TWR is 2, the total acceleration the engine provides is $a_0 = 2g_m$, where g_m is the surface gravity of the moon. As this total acceleration is the sum of horizontal and vertical components, and the vertical component cancels out the downwards acceleration:

$$a_H = \sqrt{(2g_m)^2 - (g_m(1 - v^2))^2} = g_m\sqrt{2^2 - (1 - v^2)^2}$$

And a_H is related to v (which is dimensionless!) by the following relation:

$$\frac{\mathrm{d}(v_0 v)}{\mathrm{d}t} = -a_H$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{g_m}{v_0}\sqrt{2^2 - (1 - v^2)^2}$$

$$\frac{\mathrm{d}v}{\sqrt{2^2 - (1 - v^2)^2}} = -\frac{g_m}{v_0}\mathrm{d}t$$

At the start, $v = 1$. However, at landing, velocity of the craft is 0 relative to the surface, not a stationary frame! Therefore, we use the orbital periods data to determine the final v to be $\frac{1.5}{24} = \frac{1}{16}$. Then integrating:

$$\int_{v=1}^{v=1/16} \frac{\mathrm{d}v}{\sqrt{2^2 - (1 - v^2)^2}} = -\frac{g_m}{v_0}t$$

And as delta-v is related to time by $\Delta v = a_0 t$:

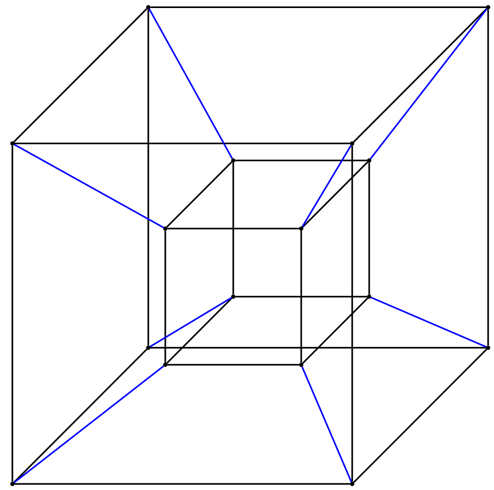
$$\Delta v = a_0 t = a_0 \frac{v_0}{g_m}\int_{1/16}^{1} \frac{\mathrm{d}v}{\sqrt{2^2 - (1 - v^2)^2}} = a_0\frac{v_0}{a_0}2(0.503) = 500 \cdot 1.006 = 503.06\frac{\mathrm{m}}{\mathrm{s}}.$$

And subtracting:

$$\Delta v - v_0 = 3.06\frac{\mathrm{m}}{\mathrm{s}}$$

$\boxed{3.06 \text{ m/s}}$

18. Tesseract Oscillations A tesseract is a 4 dimensional example of cube. It can be drawn in 3 dimensions by drawing two cubes and connecting their vertices together as shown in the picture below:

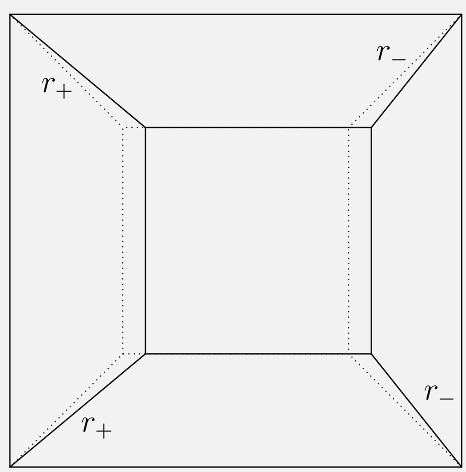


Now for the 3D equivalent. The lines connecting the vertices are replaced with ideal springs of constant $k = 10$ N/m (in blue in the figure). Now, suppose the setup is placed in zero-gravity and the outer cube is fixed in place with a sidelength of $b = 2$ m. The geometric center of the inner cube is placed in the geometric center of the outer cube, and the inner cube has a side-length $a = 1$ m and mass $m = 1.5$ kg. The inner cube is slightly displaced from equilibrium. Consider the period of oscillations

- T_1: when the springs have a relaxed length of 0;
- T_2: when the springs are initially relaxed before the inner cube is displaced.

What is $T_1 + T_2$?

Solution 18: First let us prove that there is a net external torque of $\vec{\tau} = 0$ on the cube for small displacements which means the inner cube behaves like a point mass. Consider a simple case when the cube is pushed to one side.



If we label the vertices of the cube from 1 to 4 clockwise, where 1 is the top left side, it is apparent that sides 1 and 2 provide a positive torque while sides 3 and 4 provide a negative torque. As the displacement is small, the angles created are small enough such that $\sin\theta \approx \theta$. As force is proportional to the extension of the spring as $F \propto x$, we can write that

$$\tau \propto \theta(r_+ + r_- - r_+ - r_-) \propto 0.$$

If torque is zero when the cube is displaced in the x-direction, then by symmetry, the torque is zero when the cube is displaced in the y-direction. Superposing both solutions implies that torque as a function of displacements in the x and y directions $\alpha\hat{x} + \beta\hat{y}$ is

$$\tau(\alpha x + \beta y) = \tau(\alpha x) + \tau(\beta y) = \alpha\tau(x) + \beta\tau(y) = 0.$$

1. Label the vertices of the outer cube as $1, 2, \ldots, 8$ and the vectors that point to these vertices from the inner cube as $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_2$. Consider when the inner cube deviates from equilibrium with a vector $\vec{r}$. The force as a function of $\vec{r}$ is

$$\begin{aligned} F(\vec{r}) &= k[(\vec{r}_1 - \vec{r}) + (\vec{r}_2 - \vec{r}) + \ldots (\vec{r}_8 - \vec{r})]. \\ &= k\left(\sum_{i=1}^{8} \vec{r}_i - 8\vec{r}\right) \\ &= -8k\vec{r} \end{aligned}$$

This implies the period of oscillations is

$$T_1 = 2\pi\sqrt{\frac{m}{8k}}.$$

2. Let the center of the inner cube be $(0, 0, 0)$. Consider the coordinates $(a/2, a/2, a/2)$ and $(b/2, b/2, b/2)$ which correspond to the vertex of the inner and larger cube respectively. Consider moving the cube in the x-direction. From defining $y = b/2 - a/2$, the compressional/extension of each spring $\pm\Delta\ell$ is then

$$\begin{aligned} \Delta\ell &= \pm\sqrt{(x+y)^2 + 2y^2} - \sqrt{3}y \\ &= \pm\sqrt{3}y\sqrt{1 + \frac{2x}{3y} + \mathcal{O}(x^2)} - \sqrt{3}y \\ &\approx \pm\sqrt{3}y\frac{x}{3y} \\ &= \pm\frac{x}{\sqrt{3}}. \end{aligned}$$

The total energy in all springs together are then

$$E = 8 \times \frac{1}{2}k\left(\frac{x}{\sqrt{3}}\right)^2 \implies F = -\frac{8k}{3}x \implies T_2 = 2\pi\sqrt{\frac{3m}{8k}}.$$

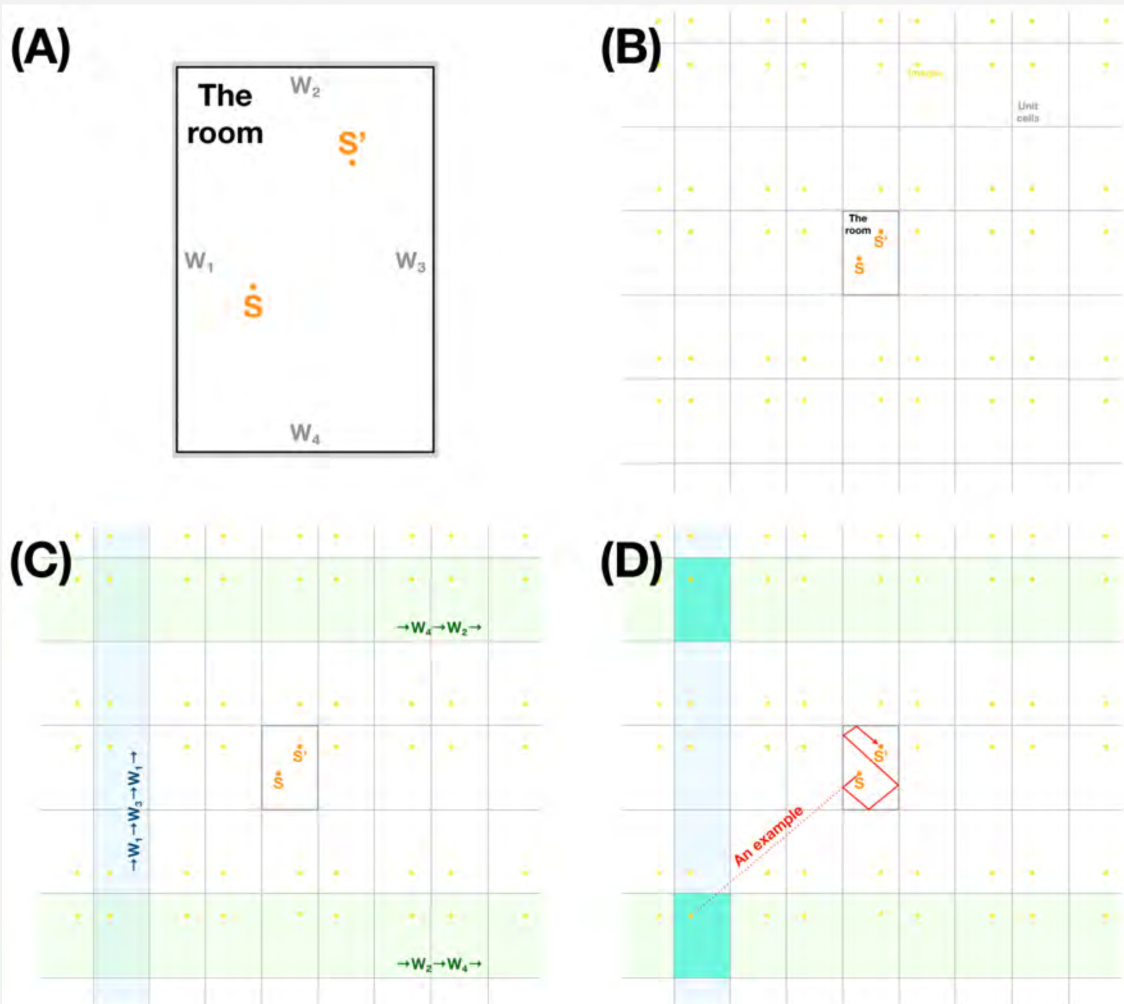
Hence, our total answer is

$$T_1 + T_2 = 2\pi(1 + \sqrt{3})\sqrt{\frac{m}{8k}}.$$

$\boxed{2.35 \text{ s}}$

19. The Room Consider two points S and S' randomly placed inside a D-dimensional hyper-rectangular room with walls that are perfect-reflecting $(D-1)$-dimensional hyper-plane mirrors. How many different light-rays that start from S, reflect N times on one of the walls and $N-1$ times on each of the rest, then go to S'? Use $D = 7$ and $N = 3$.

Solution 19:
Using the hyper-rectangular room as the fundamental unit-cell of an infinite hyper-grid in space, then find all possible positions of S'-images through reflections: we can realize that there is only one position of S'-image inside every unit-cell.



Consider two opposite hyper-plane mirrors, since the rest of the mirrors are perpendicular to them, the numbers of reflections on them for any light-path traveled from point S and point S' can only different by 1 or less. If the numbers are both equal and non-zero, then the available positions S'-image the light-path from S should reach are two unit-cell hyper-rows that parallel to the mirrors. If the numbers are not equal, then the available positions S'-image the light-path from S should reach are one unit-cell hyper-rows that that parallel to the mirrors.

Say, without loss of generality, pick one mirror to be reflected N times and the rest to be reflected $N-1$ times each, then the number of light-rays for that pick should equal to the number of unit-cell intersections between all relevant unit-cell hyper-rows, which is half the total number of vertices a hyper-rectangular has, thus 2^{D-1}. There are $2D$ walls, thus the total number of light-rays that satisfies the task given is $2D \times 2^{D-1} = D2^D$, independent of N for all values $N > 1$.

To illustrate the above explanation, let's take a look at the simple case of $D = 2$ and $N = 2 > 1$. Consider a rectangular room, with random points S, S' and $2D = 4$ walls W_1, W_2, W_3, W_4 (see Fig. A). Without loss of generality, we want to find light-rays that go from S, reflect $N = 2$ times on W_1 and $N - 1 = 1$ times on W_2, W_3, W_4 then come to S'. Each image of S' is an unique point in a unit-cell generated by the room (see Fig. B). Note that every light-ray from S that reach S'-images in the $\to W_2 \to W_4 \to$ and $\to W_4 \to W_2 \to$ unit-cell green-rows will satisfy the requirement of one reflection on each of W_2 and W_4, every light-ray from S that reach S'-images in the $\to W_1 \to W_3 \to W_1 \to$ unit-cell blue-rows will satisfy the requirement of two reflection on W_1 and one reflection on W_3 (see Fig. C). The intersection of these rows are two unit-cells, corresponds two possible images thuss two possible light-rays that satisfies the requirement (see Fig. D).

For $D = 7$ and $N = 3 > 1$, we get 896 light-rays.

$\boxed{\text{895 light-rays}}$

* This puzzle was created with helps from Long T. Nguyen.

20. Two Rings Two concentric isolated rings of radius $a = 1$ m and $b = 2$ m of mass $m_a = 1$ kg and $m_b = 2$ kg are kept in a gravity free region. A soap film of surface tension $\sigma = 0.05\text{Nm}^{-1}$ with negligible mass is spread over the rings such that it occupies the region between the rings. The smaller ring is pulled slightly along the axis of the rings. Find the time period of small oscillation in seconds.

Solution 20: The force on the two rings when they are a distance L apart follows as

$$F = 4\pi r\sigma \sin\theta$$

In small displacements, the change in θ is small. Therefore,

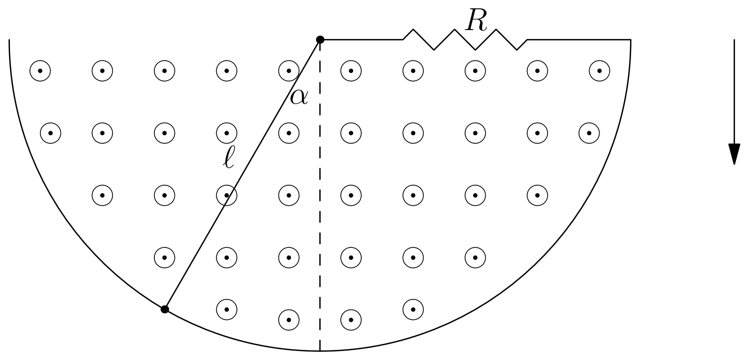
$$F = 4\pi r\sigma\theta$$
$$\frac{F}{4\pi r\sigma} = \frac{\mathrm{d}y}{\mathrm{d}r}$$
$$\int_a^b \frac{F}{4\pi\sigma r}\mathrm{d}r = \int_0^L \mathrm{d}y$$
$$\frac{F}{4\pi\sigma}\ln(b/a) = L$$

Now let a_1 and a_2 be acceleration of a and b respectively. We have that

$$a_{net} = a_1 + a_2 \tag{12}$$
$$a_{net} = F\left(\frac{m_1 + m_2}{m_1 m_2}\right) \tag{13}$$
$$\cancel{L}\omega^2 = \frac{4\pi\sigma}{\ln(b/a)}\cancel{L}\left(\frac{m_1 + m_2}{m_1 m_2}\right) \tag{14}$$
$$T = 2\pi\sqrt{\frac{\ln(b/a) m_1 m_2}{4\pi\sigma(m_1 + m_2)}} \tag{15}$$
$$T = 2\pi\sqrt{\frac{10\ln(2)}{3\pi}} = 5.388 \text{ s} \tag{16}$$

$\boxed{5.388 \text{ s}}$

21. Pendulum Circuit An open electrical circuit contains a wire loop in the shape of a semi-circle, that contains a resistor of resistance $R = 0.2\Omega$. The circuit is completed by a conducting pendulum in the form of a uniform rod with length $\ell = 0.1$ m and mass $m = 0.05$ kg, has no resistance, and stays in contact with the other wires at all times. All electrical components are oriented in the yz plane, and gravity acts in the z direction. A constant magnetic field of strength $B = 2$ T is applied in the $+x$ direction.



Ignoring self inductance and assuming that $\alpha \ll 1$, the general equation of motion is in the form of $\theta(t) = A(t)\cos(\omega t + \varphi)$, where $A(t) \geq 0$. Find ω^2.

Solution 21: The area enclosed by the wire loop is

$$A = \frac{1}{2}\ell^2\alpha + A_0$$

for small angles α, and A_0 is a constant number (which gets ignored since we really care about how this angle is changing). The flux is $\Phi = BA$ and from Lenz's Law, we have,

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{1}{2}B\ell^2\dot{\alpha}.$$

One can verify that if α is increasing, the current will flow in the clockwise direction, so we set the counterclockwise direction as positive. The current through the wire is thus,

$$i = \frac{\varepsilon}{R} = -\frac{B\ell^2}{2R}\dot{\alpha}.$$

The magnetic force acting on it is $F_B = iB\ell$ and the resulting torque is

$$\tau_B = F_B\frac{\ell}{2} = -\frac{B^2\ell^4}{4R}\dot{\alpha}.$$

Please verify that the sign is correct. The gravitational torque is $\tau_g = -mg\frac{\ell}{2}\alpha$, so the torque equation gives us

$$0 = \frac{1}{3}m\ell^2\ddot{\alpha} + \frac{B^2\ell^4}{4R}\dot{\alpha} + mg\frac{\ell}{2}\alpha$$
$$0 = \ddot{\alpha} + \frac{3}{4}\frac{B^2\ell^2}{mR}\dot{\alpha} + \frac{3}{2}\frac{g}{\ell}\alpha$$

Recall that for a damped harmonic oscillator in the form of $\ddot{\alpha} + \gamma\dot{\alpha} + \omega_0^2\alpha = 0$, the frequency of oscillations is $\omega^2 = \omega_0^2 - \gamma^2/4$, so in our case, we have

$$\omega^2 = \frac{3}{2}\frac{g}{\ell} - \frac{9}{64}\left(\frac{B^2\ell^2}{mR}\dot{\alpha}\right)^2 = 145 \text{ s}^{-1}$$

$\boxed{145 \text{ s}^{-1}}$

22. Broken Table A table of unknown material has a mass $M = 100$ kg, width $w = 4$ m, length $\ell = 3$ m, and 4 legs of length $L = 0.5$ m with a Young's modulus of $Y = 1.02$ MPa at each of the corners. The cross-sectional area of a table leg is approximately $A = 1$ cm^2. The surface of the table has a coefficient of friction of $\mu = 0.1$. A point body with the same mass as the table is put at some position from the geometric center of the table. What is the minimum distance the body must be placed from the center such that it slips on the table surface immediately after? Report your answer in centimeters.

The table surface and floor are non-deformable.

Solution 22: This problem requires some 3 dimensional reasoning. Suppose $\mathbf{s} = (s_x, s_y)$ is the gradient of the table. We can use this to calculate the additional torque from the displacement of the mass. The forces from each table leg are

$$F_i = \frac{YA}{L}\left(\pm s_x\frac{\ell}{2} \pm s_y\frac{w}{2}\right).$$

Taking the cross product as $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F_i}$ shows that torque is given as

$$\boldsymbol{\tau} = \frac{YA}{L}\begin{pmatrix} -s_y w^2 \\ s_x \ell^2 \end{pmatrix}$$

which must balance out the torque Mgd from a point mass. Hence, rewriting yields

$$d = \frac{YA}{MgL}\sqrt{s_y^2 w^4 + s_x^2 \ell^4}.$$

Furthermore, note that the angle required from slipping is given from a force analysis as

$$mg\sin\theta = \mu mg\cos\theta \implies \mu = \tan\theta = |\mathbf{s}| = \sqrt{s_x^2 + s_y^2}.$$

When $w > \ell$, we can rewrite

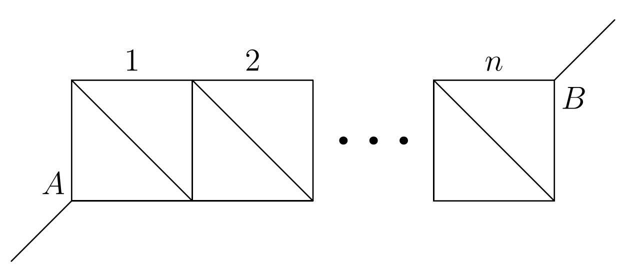
$$s_y^2 w^4 + s_x^2 \ell^4 = s_y^2(w^4 - \ell^4) + \ell^4\mu^2$$

which is minimized to $\ell^4\mu^2$ when $s_y = 0$. Hence, we obtain

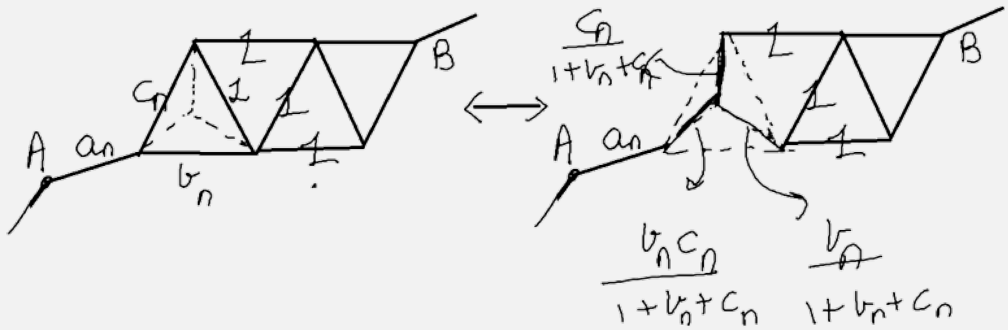
$$d = \frac{\mu\ell^2 YA}{MgL}.$$

$\boxed{18.71 \text{ m}}$

23. Resistance box In the figure below, the resistance of each wire (side and diagonal) is 1Ω. Find the value of $p + q$ if $\lim_{n\to\infty} \frac{R_{AB}}{n} = \frac{p}{q}$ where p and q are co-prime integers.



Solution 23: The circuit in the question can be redrawn after applying 2n times delta star



$$\Longrightarrow a_{n+1} = a_n + \frac{b_n c_n}{1+b_n+c_n} \tag{17}$$
$$b_{n+1} = 1 + \frac{c_n}{1+b_n+c_n}$$
$$c_{n+1} = \frac{b_n}{1+b_n+c_n}$$

For $n \to \infty$, $b_n = b_{n+1}$ and $c_n = c_{n+1}$

$$\Longrightarrow b = 1 + \frac{c}{1+b+c} \tag{18}$$
$$c = \frac{b}{1+b+c} \tag{19}$$

On solving above equations we get $b = \frac{1}{2}\left(1+\frac{3}{\sqrt{5}}\right)$ and $c = \frac{1}{\sqrt{5}}$. Therefore we get $a_{n+1} = a_n + \frac{2}{5}$ hence $\lim_{n\to\infty} \frac{R_{AB}}{n} = \frac{2}{5}$ $\boxed{\frac{2}{5}}$

24. Dipole Conductor An (ideal) electric dipole of magnitude $p = 1 \times 10^{-6}$ C·m is placed at a distance $a = 0.05$ m away from the center of an uncharged, isolated spherical conductor of radius $R = 0.02$ m. Suppose the angle formed by the dipole vector and the radial vector (the vector pointing from the sphere's center to the dipole's position) is $\theta = 20^o$. Find the (electrostatic) interaction energy between the dipole and the charge induced on the spherical conductor.

Solution 24:
We can use the fact that if a charge Q is placed at a distance a from a grounded, conducting sphere of radius R, as far as the field outside the sphere is concerned, there is an image charge of magnitude $-Q\frac{R}{a}$ at a position $\frac{R^2}{a}$ from the origin, on the line segment connecting the origin and charge Q. It is straightforward to check that indeed, the potential on the sphere due to this image charge prescription is 0. If the point charge is instead a dipole p, we can think of this as a superposition of 2 point charges, and use the fact above. In particular, there is one charge $-Q$ at point $(a, 0, 0)$ and another charge Q at point $(a + s\cos\theta, s\sin\theta, 0)$, where s is small and $Qs = p$. Note that the dipole points in the direction θ above the x-axis. Consequently, there will be an image charge at $(\frac{R^2}{a}, 0, 0)$ with magnitude $Q\frac{R}{a}$ and an image charge at $(\frac{R^2}{a+s\cos\theta}, \frac{R^2 s\sin\theta}{a(a+s\cos\theta)}, 0)$ with magnitude $-Q\frac{R}{a+s\cos\theta}$. The image charges are close to each other but do not cancel out exactly,

so they can be represented as a superposition of an image point charge Q' and an image dipole p'. The image point charge has magnitude $Q' = -QR(\frac{1}{a+s\cos\theta} - \frac{1}{a}) = \frac{QRs\cos\theta}{a^2}$. The image dipole has magnitude $p' = Q\frac{R}{a} * \frac{R^2 s}{a^2} = \frac{QR^3 s}{a^3}$ and points towards the direction θ below the positive x-axis. Finally, since the sphere in the problem is uncharged instead of grounded, to ensure the net charge in the sphere is 0, we place another image charge $-Q'$ at the origin.
Now we can calculate the desired interaction energy, which is simply the interaction energy between the image charges and the real dipole. Using the dipole-dipole interaction formula, the interaction between the image dipole and the real dipole is given by:

$$U_1 = \frac{kpp'}{(a - \frac{R^2}{a})^3}(\cos(2\theta) - 3\cos^2\theta)$$

The interaction between the image charge at the image dipole's position and the real dipole is given by:

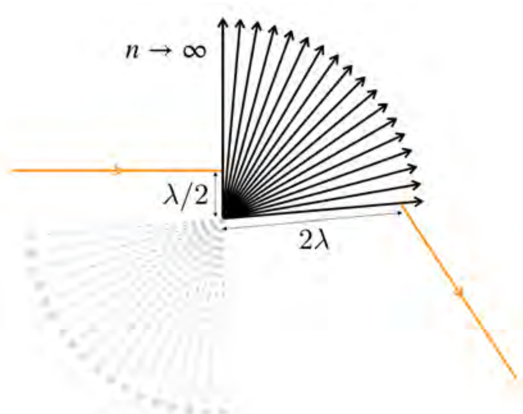
$$U_2 = -\frac{kpQ'\cos\theta}{(a - \frac{R^2}{a})^2}$$

The interaction between the image charge at the center and the real dipole is given by:

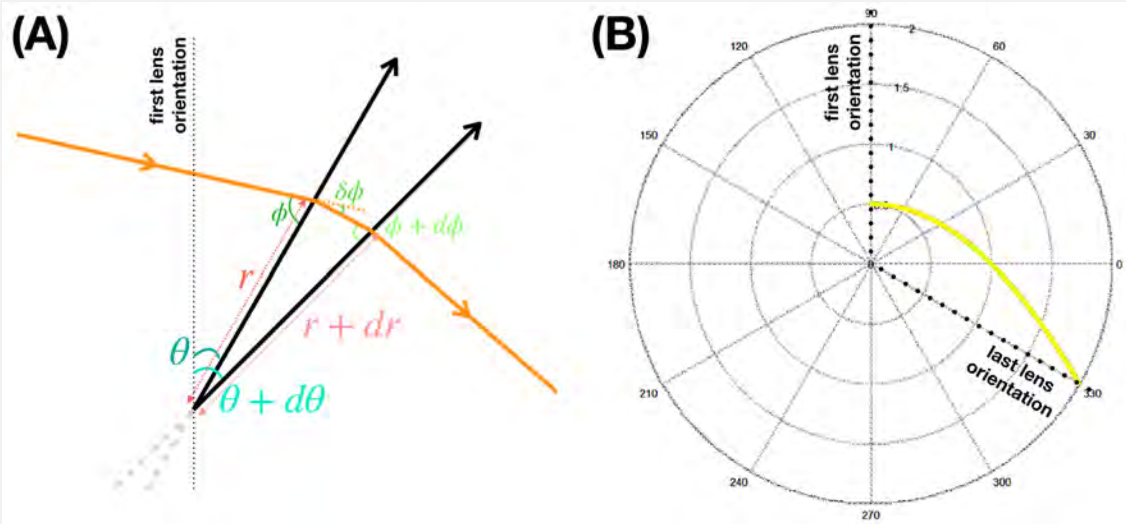
$$U_3 = \frac{kpQ'\cos\theta}{a^2}$$

The final answer is $U = U_1 + U_2 + U_3 = \frac{p^2 R}{4\pi\epsilon_0}(\frac{cos^2\theta}{a^4} - \frac{a^2\cos^2\theta + R^2}{(a^2-R^2)^3}) = -25.22$ J. $\boxed{-25.22 \text{ J}}$

25. Dying Light Consider an optical system made of many identical ideal (negligible-thickness) half-lenses with focal length $f > 0$, organized so that they share the same center and are angular-separated equally at density n (number of lenses per unit-radian). Define the length-scale $\lambda = f/n$. A light-ray arrives perpendicular to the first lens at distance $\lambda/2$ away from the center, then leaves from the last lens at distance 2λ away from the center. Estimate the total deflection angle (in rad) of the light-ray by this system in the limit $n \to \infty$.



Solution 25: We define the angles as in Fig. A. The light-path inside the optical system is $r(\theta)$, and the angle between the first and last lens is Θ (which is an unknown but can be uniquely determined from know information).



Consider two consecutive lens at angle θ and $\theta + d\theta$, in which $d\theta = 1/n \to 0$ in the continuum limit $n \to \infty$. From the ideal-lens' equation, using the approximation that f is very large compare to other relevant length-scales in this optical setting:

$$\frac{1}{f} = \frac{1}{r\tan\phi} + \frac{1}{r\tan(\pi - \phi - \delta\phi)} \approx \frac{\delta\phi}{r\sin^2\phi} \;\Rightarrow\; \delta\phi \approx \frac{r}{f}\sin^2\phi \;, \tag{20}$$

the differential equation for the angle of arrival ϕ can be written as:

$$d\phi = \delta\phi - d\theta \;\Rightarrow\; \frac{d\phi}{d\theta} = \frac{r}{f/n}\sin^2\phi - 1 = \frac{r}{\lambda}\sin^2\phi - 1 \;. \tag{21}$$

We also have the differential relation between radial position $r(\theta)$ of the light-path and the angle of arrival ϕ as followed:

$$\frac{dr}{d\theta} = r\cot\phi \;. \tag{22}$$

From Eq. (21) and Eq. (22), we arrive at:

$$\frac{d\phi}{dr} = \frac{\frac{r}{\lambda} - 1}{r\cot\phi} \;. \tag{23}$$

Define $\zeta = \tan\phi$, then Eq. (23) becomes:

$$\frac{d\phi}{dr} = \frac{1}{1+\zeta^2}\frac{d\zeta}{dr} = \frac{\frac{r}{\lambda}\frac{\zeta^2}{1+\zeta^2} - 1}{r/\zeta} \;\Rightarrow\; -\frac{d\zeta}{\zeta^3 dr} - \frac{1}{\zeta^2 r} = \frac{1}{r} - \frac{1}{\lambda} \;. \tag{24}$$

Define $\xi = 1/\zeta^2 = 1/\tan^2\phi$, then Eq. (23) gives:

$$\frac{d\zeta}{\zeta^2 dr} = -\frac{1}{2}\frac{d\xi}{dr} \;\Rightarrow\; \frac{d\xi}{dr} - \frac{2}{r}\xi = 2\left(\frac{1}{r} - \frac{1}{\lambda}\right) \;\Rightarrow\; \frac{d}{dr}\left(\frac{\xi}{r^2}\right) = \frac{2}{r^2}\left(\frac{1}{r} - \frac{1}{\lambda}\right) \;. \tag{25}$$

Integrating both sides, then up to a constant value C, Eq. (25) gives:

$$\frac{\xi}{r^2} = -\frac{1}{r^2} + \frac{2}{\lambda r} + C \;\Rightarrow\; \xi = -1 + 2\frac{r}{\lambda} + C\frac{r^2}{\lambda^2} \;. \tag{26}$$

At $\theta = 0$, $r = \lambda/2$ and $\phi = \pi/2$ (thus $\xi = 0$), we can determine $C = 0$. Hence:

$$\cot\phi = \sqrt{2\frac{r}{\lambda} - 1} \ . \tag{27}$$

Plug Eq. (27) into Eq. (22):

$$\frac{dr}{d\theta} = \frac{r}{\lambda}\sqrt{2\frac{r}{\lambda} - 1} \ \Rightarrow \ \theta = 2\arctan\sqrt{2\frac{r}{\lambda} - 1} \ . \tag{28}$$

At $\theta = \Theta$, $r = 2\lambda$ therefore we can use Eq. (28) to get:

$$\Theta = 2\arctan\sqrt{3} = \frac{2\pi}{3} \ . \tag{29}$$

Using Eq. (27), the deflection angle Δ can be calculated to be:

$$\Delta = \Theta - \phi\Big|_{r=\lambda/2} + \phi\Big|_{r=2\lambda} = \Theta - \frac{\pi}{2} + \operatorname{arccot}\sqrt{3} = \frac{2\pi}{3} - \frac{\pi}{2} + \frac{\pi}{6} = \frac{\pi}{3} \approx 1.05\text{rad} \ . \tag{30}$$

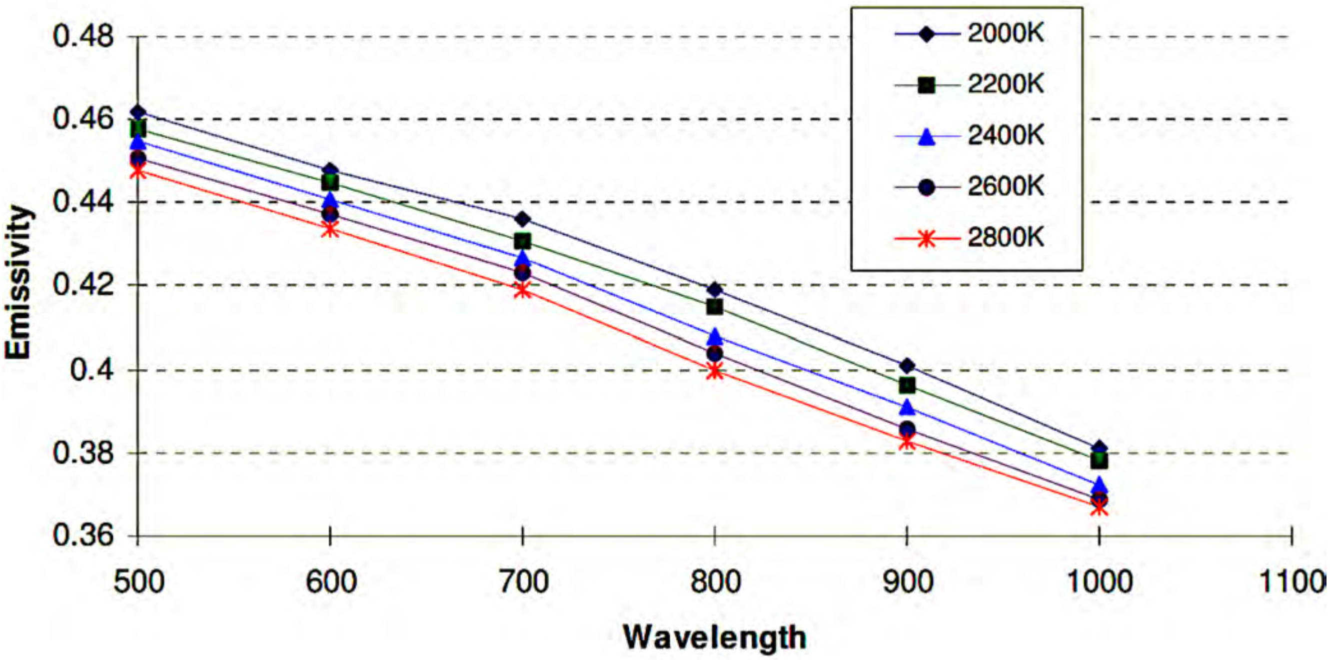
For the sake of completeness, we provide the simulated light-path inside the optical system where $n = 1000$ using MatLab (which is in great agreement with our theoretical analysis).
$\boxed{1.05 \text{ rad}}$

* This puzzle was created with helps from Long T. Nguyen.

26. Tungsten For black body radiation, Wien's Displacement Law states that its spectral radiance will peak at

$$\lambda_{\text{peak}} = \frac{b}{T}.$$

where $b = 2.89777 \times 10^{-3}$ mK, and T is the temperature of the object. When QiLin tried to reproduce this in a lab, by working with a tungsten-filament lightbulb at 2800 K, he computed a different value for b by measuring the peak wavelength using a spectrometer and multiplying it with the temperature. He hypothesizes that this discrepancy is because tungsten is not an ideal black body. The graph below, courtesy of the CRC Handbook of Chemistry and Physics, shows the emissivity of tungsten at various conditions (the units for wavelength is nm).



Assuming QiLin's hypothesis is correct, and assuming there were no other errors in the experiment, how off was his value for b? Submit $\frac{|b_{\text{theory}} - b_{\text{experiment}}|}{b_{\text{theory}}}$ as a decimal number, to *one* significant digit (giving you room to estimate where the points are).

Solution 26: Recall Planck's Law, which says the spectral radiance of a black body is given by

$$B_0(\lambda, T) = \frac{2hc^3}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}.$$

The regular Wien's Displacement Law can be derived by finding the peak by computing $\frac{\partial B_0}{\partial \lambda}$, to find the wavelength associated with the maximal radiance. For a nonideal body with emissivity $\epsilon(\lambda, T)$, we can write the radiance as

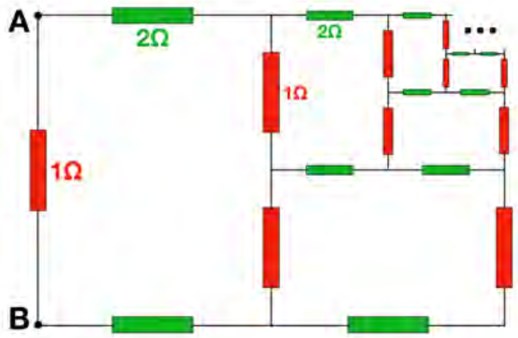
$$B(\lambda, T) = B_0(\lambda, T)\epsilon(\lambda, T).$$

We can estimate $\epsilon(\lambda, T)$ by looking at the given graph. The tungsten is at 2800K, so we will use the red line, and assuming it is near a black body, the peak wavelength should be around 1000 nm. Performing a linear approximation around 1000 nm, we get

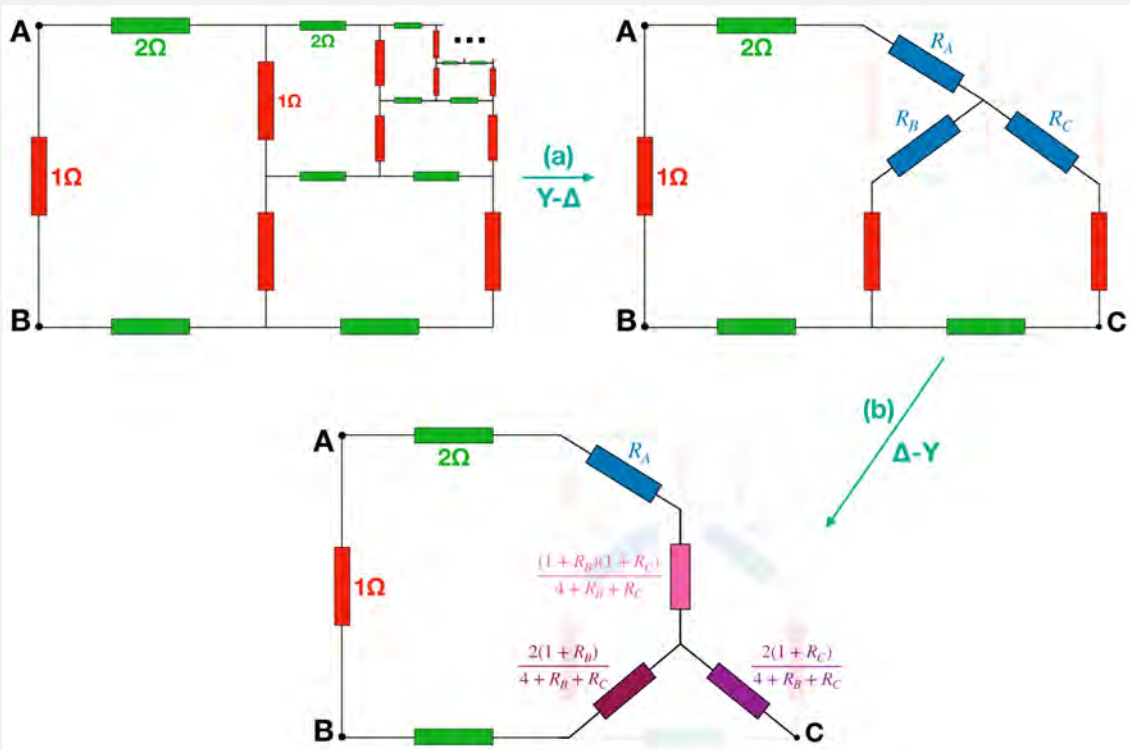
$$\epsilon(\lambda, T) = -173333(\lambda - 1000 \cdot 10^{-9}) + 0.366,$$

where λ is in meters. Numerically finding the maximum of $B(\lambda, T)$ with respect to λ (i.e. with a graphing calculator), we get the new peak wavelength to be $\lambda_{new} = 949$ nm, while the old peak wavelength (assuming a perfect blackbody) is $\lambda_{old} = 1035$ nm, and their percent difference (rounded to 1 significant digit) is 0.08. $\boxed{0.08}$.

27. Bioshock Infinite The equivalent resistance (in Ω) between points A and B of the following infinite resistance network made of 1Ω and 2Ω resistors is $\overline{0.abcdefgh...}$ in decimal form. Enter $\overline{efg}$ into the answer box (It should be an integer in the range of 0-999).



Solution 27: We consider two-steps: (a) Y-to-Δ transformation and (b) Δ-to-Y transformation on this resistance network (see attached figure), which R_A, R_B and R_C are unknowns to be determined (note these resistance values are always non-negative $R_A, R_B, R_C \geq 0$).



Stop at the transformed network after step (a), the equivalent resistance R_{AB} between points A and B can be calculated in two different ways:

$$R_{AB} = R_A + R_B = \frac{1}{\frac{1}{1} + \frac{1}{4+R_A+\frac{1}{\frac{1}{R_B}+\frac{1}{3+R_C}}}} . \tag{31}$$

Stop at the transformed network after step (b), the equivalent resistance R_{BC}between points B, C and R_{CA} between points C, A can also be calculated in two different ways:

$$R_{BC} = R_B + R_C = \frac{1}{\frac{1}{3+R_A+\frac{(1+R_B)(1+R_C)}{4+R_B+RC}} + \frac{1}{2+\frac{2(1+R_B)}{4+R_B+R_C}}} + \frac{2(1+R_C)}{4+R_B+R_C} , \tag{32}$$

$$R_{CA} = R_C + R_A = \frac{2(1+R_C)}{4+R_B+R_C} + \frac{1}{\frac{1}{2+R_A+\frac{(1+R_B)(1+R_C)}{4+R_B+RC}} + \frac{1}{3+\frac{2(1+R_B)}{4+R_B+R_C}}} \ . \tag{33}$$

Define the variable λ:

$$\lambda = R_A R_B + R_B R_C + 4R_A + 8R_B + 6R_C + 23 \geq 23 \ . \tag{34}$$

Expanding Eq. (31) gives:

$$R_A + R_B = \frac{(-4+\lambda) - R_B - R_C}{\lambda} \ \Rightarrow \ \lambda R_A + (1+\lambda)R_B + R_C = (-4+\lambda) \ , \tag{35}$$

while expanding Eq. (32) and Eq. (33) then dividing both their nominators and denominators with a non-zero value $(4 + R_B + R_C) \neq 0$ gives:

$$\begin{aligned} R_B + R_C &= \frac{(-48+4\lambda) - 4R_A - 16R_B - 4R_C}{\lambda} \\ &\Rightarrow \ 4R_A + (16+\lambda)R_B + (4+\lambda)R_C = (-48+4\lambda) \ , \end{aligned} \tag{36}$$

$$\begin{aligned} R_C + R_A &= \frac{(-72+5\lambda) - 4R_A - 25R_B - 9R_C}{\lambda} \\ &\Rightarrow \ (4+\lambda)R_A + 25R_B + (9+\lambda)R_C = (-72+5\lambda) \ . \end{aligned} \tag{37}$$

The system of three linear-equations with three unknowns, Eq. (35), Eq. (36), Eq. (37) can be solved by the ratios of 3×3 matrices' determinants as followed:

$$R_A = \frac{\det\begin{vmatrix} (-4+\lambda) & (1+\lambda) & 1 \\ (-48+4\lambda & (16+\lambda) & (4+\lambda) \\ (-72+5\lambda) & 25 & (9+\lambda) \end{vmatrix}}{\det\begin{vmatrix} \lambda & (1+\lambda) & 1 \\ 4 & (16+\lambda) & (4+\lambda) \\ (4+\lambda) & 25 & (9+\lambda) \end{vmatrix}} = \frac{-40+120\lambda-24\lambda^2+\lambda^3}{(2+\lambda+2)(4+\lambda^2)} \ , \tag{38}$$

$$R_B = \frac{\det\begin{vmatrix} \lambda & (-4+\lambda) & 1 \\ 4 & (-48+4\lambda) & (4+\lambda) \\ (4+\lambda) & (-72+5\lambda) & (9+\lambda) \end{vmatrix}}{\det\begin{vmatrix} \lambda & (1+\lambda) & 1 \\ 4 & (16+\lambda) & (4+\lambda) \\ (4+\lambda) & 25 & (9+\lambda) \end{vmatrix}} = \frac{2(-4-32\lambda+9\lambda^2)}{(2+\lambda+)(4+\lambda^2)} \ , \tag{39}$$

$$R_C = \frac{\det\begin{vmatrix} \lambda & (1+\lambda) & (-4+\lambda) \\ 4 & (16+\lambda) & (-48+4\lambda) \\ (4+\lambda) & 25 & (-72+5\lambda) \end{vmatrix}}{\det\begin{vmatrix} \lambda & (1+\lambda) & 1 \\ 4 & (16+\lambda) & (4+\lambda) \\ (4+\lambda) & 25 & (9+\lambda) \end{vmatrix}} = \frac{2(-12+52\lambda-39\lambda^2+2\lambda^3)}{(2+\lambda)(4+\lambda^2)} \ . \tag{40}$$

Plug Eq. (38), Eq. (39), Eq. (40) into Eq. (34), after an exhausting (but trivial) brute-force algebraic manipulation we arrive at:

$$\lambda = \frac{\lambda^2(220-365\lambda+55\lambda^2)}{(2+\lambda)^2(4+\lambda^2)} \ \Rightarrow \ 16 - 204\lambda + 364\lambda^2 - 51\lambda^3 + \lambda^4 = 0 \ . \tag{41}$$

This Eq. (41) is a quartic polynomal, which solutions have algebraic expressions (general formula is known, can be found in textbooks or Wikipedia). Since $\lambda \geq 23$, the physical solution is:

$$\lambda = \frac{1}{4}\left(51 + \sqrt{1177} + \sqrt{3714 + 102\sqrt{1177}}\right) . \tag{42}$$

From (31), replacing λ in Eq. (38) and (39) with the value given in Eq. (42):

$$\begin{aligned} R_{AB} = R_A + R_B &= \frac{-48 + 56\lambda - 6\lambda^2 + \lambda^3}{(2+\lambda)(4+\lambda^2)} \\ &= \frac{1}{356}\left(-482 - 8\sqrt{1177} + \sqrt{678070 + 12874\sqrt{1177}}\right) . \end{aligned} \tag{43}$$

With a handheld 8-digit calculator, it is possible with a good choice for order of arithmetic operations to obtain the numerical value with very high precision. Here's an example:

(1) Input 1177, $\sqrt{\ }$, $\times$, 12874, +, 678070, $\sqrt{\ }$ then memorize this value M.

(2) Input 1177, $\sqrt{\ }$, $\times$, 8, +, 482, $-$, M, / , 35.6 then the screen will show -8.4752823.

Get rid of the minus sign and move the decimal sign forward 1-digit, the numerical value for R_{AB} is approximately 0.847528223Ω. We then can use 282 as the answer for this physics puzzle, which is indeed in great agreement with better calculators.

$\boxed{282}$

* This puzzle was created with helps from Tuan K. Do.

28. Magnetic Ball A uniform spherical metallic ball of mass m, resistivity ρ, and radius R is kept on a smooth friction-less horizontal ground. A horizontal uniform, constant magnetic field B exists in the space parallel to the surface of ground. The ball was suddenly given an impulse perpendicular to magnetic field such that ball begin to move with velocity v without losing the contact with ground. Find the time in seconds required to reduce its velocity by half.

Numerical Quantities: $m = 2$ kg, $4\pi\epsilon_0 R^3 B^2 = 3$ kg, $\rho = 10^9\Omega$m, $v = \pi$ m/s.

Solution 28: WLOG assuming magnetic field to be into the plane (negative z-axis) and velocity of the block along x-axis. Let any time t ball has velocity v and surface charge $\sigma\cos\theta$, where θ is measured from Y-axis. As the ball is moving it will also generate and electric field E=vB along positive Y-axis. Which will be opposed by the electric field of ball. Also we know the electric filed generated by the ball is $\frac{\sigma}{3\epsilon_0}$ in negative Z-axis. As this charge distribution will arise form a vertical electric field and the subsequent induced charges will also produce only a vertical electric field only thus our assumption about charge distribution and net electric field must be true.

Now at any point on the surface of the sphere rate of increase in surface charge density is given by-

$$JdA\cos\theta = \frac{d(\sigma\cos\theta)}{dt}dA$$
$$J = \frac{d\sigma}{dt}$$
$$E - \frac{\sigma}{3\epsilon_0} = \rho\frac{d\sigma}{dt}$$
$$vB - \frac{\sigma}{3\epsilon_0} = \rho\frac{d\sigma}{dt} \tag{44}$$

As the magnetic field is uniform to calculate force path of the current will not matter. Hence assuming it to be straight line between two points located at θ and $-\theta$. So the force can be written as -

$$dF = B \times dI \times l$$
$$dF = B(2\pi(R\sin\theta) \times Rd\theta \times J)(2R\cos(\theta))$$
$$F = 4\pi R^3 B\frac{d\sigma}{dt}\int_0^{\pi/2}\sin\theta\cos^2\theta d\theta$$
$$F = \frac{4}{3}\pi R^3 B\frac{d\sigma}{dt} \tag{45}$$

Now writing force equation on the sphere we have

$$F = -m\frac{dv}{dt}$$
$$\frac{4}{3}\pi R^3 B\frac{d\sigma}{\cancel{dt}} = -m\frac{dv}{\cancel{dt}}$$
$$\frac{4}{3}\pi R^3 B\int_0^{\sigma} d\sigma = -m\int_{v_0}^{v} dv$$
$$\frac{4}{3}\pi R^3 B\sigma = mv_0 - mv \tag{46}$$

Solving equation 1 and 3 gives us

$$\left(m + 4\pi R^3 B^2\epsilon_0\right)v - mv_0 = -3m\rho\epsilon_0\frac{dv}{dt}$$

Integrating it from v_0 to $\frac{v_0}{2}$gives

$$t = \frac{3m\rho\epsilon_0}{(m + 4\pi R^3 B^2\epsilon_0)}\ln\left(\frac{8\pi R^3 B^2\epsilon_0}{4\pi R^3 B^2\epsilon_0 - m}\right) \tag{47}$$
$$t = \frac{6\ln(6)}{5}\rho\epsilon_0 = 0.019 \text{ s} \tag{48}$$

$\boxed{0.019 \text{ s}}$

For the following two problems, this information applies. Assume $g = 9.8\text{m/s}^2$. On a balcony, a child holds a spherical balloon of radius 15 cm. Upon throwing it downwards with a velocity of 4.2 m/s, the balloon starts magically expanding, its radius increasing at a constant rate of 35 cm/s. Another child, standing on the ground, is holding a hula hoop, 4 m below the point where the center of the balloon was released.

29. Magical Balloon 1 If the minimum radius of the hoop such that the balloon falls completely through the hula hoop without touching it is r, compute the difference between r and the largest multiple of 5cm less than or equal to r. Answer in centimeters; your answer should be in the range $[0, 5)$.

30. Magical Balloon 2 Consider the horizontal plane passing through the center of the balloon at the start. If the total volume above this plane that the balloon falls through after it is thrown downwards is V, compute the difference between V and half the original volume of the balloon. Answer in milliliters; your answer should be nonnegative.

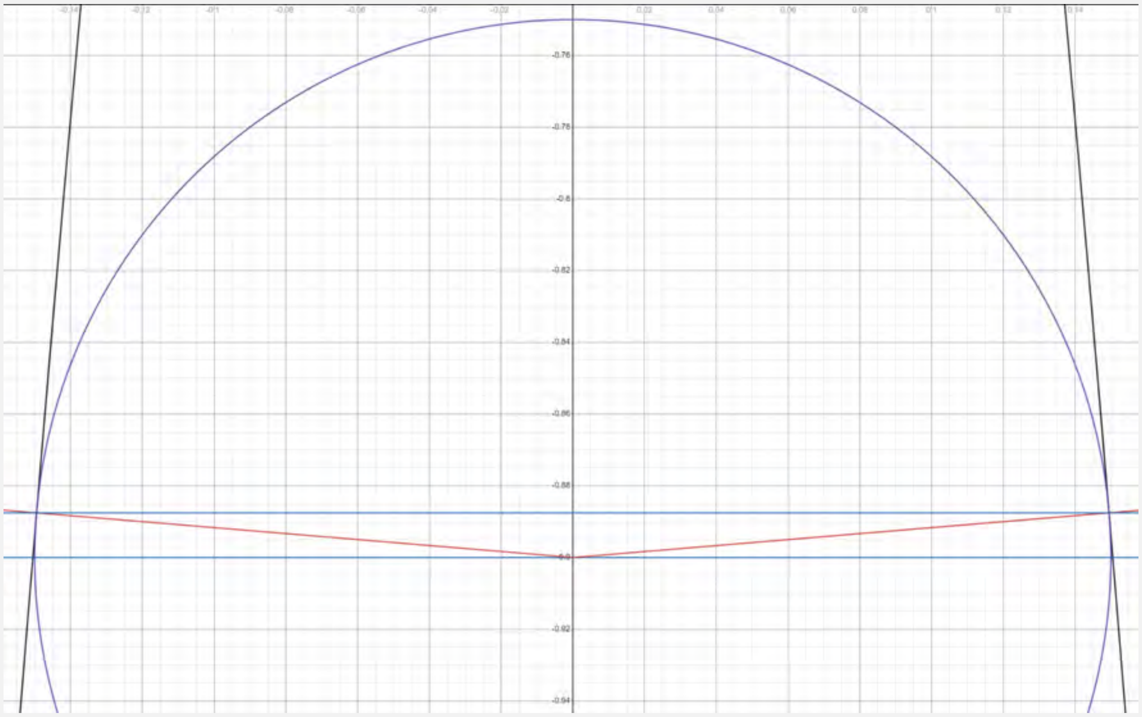
Note that when refer to the "volume an object falls through", it refers to the volume of the union of all points in space which the object occupies as it falls.

Solution 29: The first key insight comes by noting that the numbers in the problem are carefully chosen. If we go backwards in time, the balloon had radius 0 at a time $t_0 = -\frac{3}{7}$sec. However, then the balloon's downwards velocity at that point would have been $v_0 = 4.2 - gt = 0$m/s - also zero velocity! Therefore, we can consider the balloon as having been released from rest at a height $4 + \frac{v^2}{2g} = 4.9$m above the lower hula-hoop.
If we consider the balloon as a whole, it both expands and falls, making the overall volume occupied by it over time difficult to calculate. The next key insight comes from thinking about individual points on the balloon, relative to the center of mass. The center of the balloon falls under gravity; relative to the center of mass, a point on the balloon travels away from it at a constant rate. However, the motion of this point is identical to a projectile, launched from the starting point at a speed equal to the expansion rate, in the direction of expansion.
Therefore, the needed radius of the bottom hula hoop is the maximum distance a projectile launched at 35cm/s can travel horizontally, before falling 4.9m. This is sufficient to solve the problem, getting a distance of 35.0223cm and therefore an answer of 0.0223, but an easier solve can be obtained by making more insights, described below.

Solution 30: After turning the problem into a projectile distance problem, the third key insight comes from realizing that the distance to the edge of the volume fell through at any angle of elevation or depression is the same as the maximum distance that a projectile can travel up or down a slope of the same angle. Through either intuition or polar coordinate bashing, we can determine the shape of the surface, which is a paraboloid of revolution.
The easiest way to parametrize the paraboloid is with the zero-velocity balloon at $(0, 0)$. Three points on the parabola can easily be found by considering the maximum horizontal projectile distance and the maximum vertical projectile distance: $\left(0, \frac{v^2}{2g} = 0.00625\right)$ and $\left(\frac{v^2}{g} = 0.0125, 0\right)$.
Therefore, we get the parametrization of the paraboloid:

$$f(x) = \frac{v^2}{2g} - \frac{g}{2v^2}x^2 = 0.00625 - 40x^2$$

By finding x for $y = -4.9$m, this immediately provides a faster solution to the previous question.
Plotting the initial location of the ball in purple, as well as the paraboloid in black:



Notice, however, that the balloon does not exactly meet the edge of the paraboloid at the starting height (the lower blue line) - the intersection is actually a bit above, and the paraboloid is actually a bit wider at $y = -0.9$m. This slight discrepancy creates a difference in volume. We now seek to compute this difference.

The balloon falls at a rate of 4.2m/s at this point, while expanding at 35cm/s radially outwards, and this radially outwards vector is perpendicular to the slope of the paraboloid at the point where the paraboloid and balloon touch. Solving the triangle, the red line has a slope of $\frac{1}{\sqrt{143}}$. This line intersects with the balloon edge (and the paraboloid) at $y = -0.8875$m (upper blue line). It can be checked that this point is on both the paraboloid and the balloon.

We know the volume of a paraboloid is $V = r^2 h \int_0^1 2\pi r(1-r^2)dr = \frac{\pi}{2} r^2 h$. To compute the volume of the paraboloid between these two lines, we take the difference between the volume above the lower and upper lines. The upper line gives:

$$V_1 = \frac{\pi}{2}\frac{2v^2}{g}\left(0.8875 + \frac{v^2}{2g}\right)^2 = 31368.3731\text{mL}$$

While the lower line gives:

$$\frac{\pi}{2}\frac{2v^2}{g}\left(.9 + \frac{v^2}{2g}\right)^2 = 32251.9461\text{mL}$$

The difference between the two gives the volume between, which is 883.5729mL.

We now compute the volume of the original balloon between these two lines, which we split up into a portion of the sphere (below the red line) and a cone (above the red line). The proportion of the sphere is given by a solid angle integral:

$$f = \frac{1}{2}\int_0^{\sin^{-1} .35/4.2} \cos\theta d\theta = \frac{1}{24}$$

And the volume:

$$V_{\text{sphere}} = \frac{4}{3}\pi\left(.15\right)^3 f = 589.0486\text{mL}$$

While the volume of the cone is given by:

$$V_{\text{cone}} = \frac{1}{3}\pi\frac{2v^2}{g}\left(0.8875 + \frac{v^2}{2g}\right)(0.9 - 0.8875) = 292.4790\text{mL}$$

Summing gives 881.5276mL.
Finally, taking the difference of the two volumes gives our final result, 2.0453mL.

31. Hydrogen Magnetism In quantum mechanics, when calculating the interaction between the electron with the proton in a hydrogen atom, it is necessary to compute the following volume integral (over all space):

$$\mathbf{I} = \int \mathbf{B}(\mathbf{r})|\Psi(\mathbf{r})|^2 dV$$

where $\Psi(\mathbf{r})$ is the spatial wavefunction of the electron as a function of position $\mathbf{r}$ and $\mathbf{B}(\mathbf{r})$ is the (boldface denotes vector) magnetic field produced by the proton at position $\mathbf{r}$. Suppose the proton is located at the origin and it acts like a finite-sized magnetic dipole (but much smaller than a_0) with dipole moment $\mu_p = 1.41 \times 10^{-26}$ J/T. Let the hydrogen atom be in the ground state, meaning $\Psi(\mathbf{r}) = \frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}}$, where $a_0 = 5.29 \times 10^{-11}$ m is the Bohr radius. Evaluate the magnitude of the integral $|\mathbf{I}|$ (in SI units).

Solution 31:
First, note that the result of the integral will be a vector in the direction the dipole is pointing, call it the z-direction. Thus we can replace $\mathbf{B}$ in the integral with B_z. Note that for any $R > 0$, the integral over the space outside the sphere of radius R is 0. To show this, since $|\Psi|$ is exponentially decaying, we only need to show that the integral over a spherical shell is 0. To show this, we can show that the integral of $\mathbf{B}$ inside a sphere of radius R is independent of R. Indeed, this quickly follows from dimensional analysis (the only relevant quantities are μ_0, μ_p, and R, and one can check that $\mu_0\mu_p$ already gives the right dimensions, and there is no dimensionless combination of these 3 quantities. In fact, we will actually compute this integral at the end.)
Now, it suffices to compute the integral of $\mathbf{B}|\Psi|^2$ inside the sphere. Since R was arbitrary, we can make it very small, much smaller than a_0. Then we can replace $|\Psi(\mathbf{r})|^2$ with $|\Psi(0)|^2 = \frac{1}{\pi a_0^3}$, a constant that can be factored out. The problem reduces to computing the integral of $\mathbf{B}$ inside a sphere of radius R.
We can compute this integral by splitting the sphere up into many thin discs, all perpendicular to the z axis. We have to add up the $\mathbf{B}$ field integrated over the volume of each disc, which is equivalent to the magnetic flux through the disc times the thickness of the disc. The magnetic flux through each disc can be calculated using the mutual inductance reciprocity theorem. Suppose a current I goes around the boundary of the disc (a ring) with radius r. Then the mutual inductance M between the ring and the dipole is given by the flux through the dipole divided by I:

$$M = \frac{B * A}{I}$$

where B is the magnetic field produced by the ring's current at the dipole's position, and A is the area of the dipole. The dipole itself carries current $i = \frac{\mu_p}{A}$, so the flux through the ring is given by

$$\Phi = M * i = \frac{BiA}{I} = \frac{\mu_p B}{I}$$

where $B = \frac{\mu_0 I}{4\pi} * \frac{2\pi r^2}{(r^2+z^2)^{\frac{3}{2}}} = \frac{\mu_0 I r^2}{2(r^2+z^2)^{\frac{3}{2}}}$, where z is the z coordinate of the ring. Using $r = R\sin\theta$ and $z = R\cos\theta$, we obtain

$$\Phi(\theta) = \frac{\mu_0 \mu_p r^2}{2(r^2+z^2)^{\frac{3}{2}}} = \frac{\mu_0 \mu_p \sin^2\theta}{2R}$$

Finally, we integrate over the thickness of the disc $R\sin\theta d\theta$ to get :

$$\int_0^{\pi} \Phi(\theta) R \sin\theta d\theta = \frac{1}{2}\mu_0\mu_p \int_0^{\pi} \sin^3\theta d\theta = \frac{2}{3}\mu_0\mu_p$$

Thus, $|\mathbf{I}| = \frac{2}{3}\mu_0\mu_p * \frac{1}{\pi a_0^3} = \frac{2\mu_0\mu_p}{3\pi a_0^3} = 0.0254$ T.

32. Relativistic Collision Zed is trying to model the repulsive interaction between 2 objects, A and B (with masses m_A and m_B, respectively), in a relativistic setting. He knows that in relativity, forces cannot act at a distance, so he models the repulsive force with a small particle of mass m that bounces elastically between A and B. Throughout this problem, assume everything moves on the x-axis. Suppose that initially, A and B have positions and velocities x_A, v_A and x_B, v_B, respectively, where $x_A < x_B$ and $v_A > v_B$. The particle has an initial (relativistic) speed v.

For simplicity, assume that the system has no total momentum. You may also assume that $v_A, v_B \ll v$, and that $p_m \ll p_A, p_B$, where p_m, p_A, p_B are the momenta of the particle, A, and B, respectively. Do NOT assume $v \ll c$, where c is the speed of light.

Find the position (in m) of A when its velocity is 0, given that $m_A = 1$ kg, $m_B = 2$ kg, $v_A = 0.001c$, $m = 1 \times 10^{-6}$ kg, $v = 0.6c$, $x_A = 0$ m, $x_B = 1000$ m.

Note: Answers will be tolerated within 0.5%, unlike other problems.

Solution 32:
Since total momentum is 0, we have $m_A v_A + m_B v_B = 0$, so $v_B = -0.0005c$ By conservation of energy:

$$\frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 + \gamma_0 mc^2 = \gamma mc^2$$

where we define $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ to correspond to the final state of the particle and γ_0 to the initial state. This equation allows us to solve for γ. Since the speed of the particle is much larger than the speed of the masses A and B, we can imagine the particle moving in a infinite well potential where the walls are slowly moving. Applying the adiabatic theorem, we get that the adiabatic invariant px is conserved, where p is the particle's momentum, and x is the distance between A and B. Thus, $\gamma v x$ is conserved, so

$$\gamma v x = \gamma_0 v_0 (x_B - x_A)$$

We can solve for x, since we know γ and v. Finally, we realize that the center of mass stays at $x_c m = \frac{m_A x_A + m_B x_B}{m_A + m_B}$, so the final position of A is simply

$$x_c m - \frac{m_B}{m_A + m_B} x = 378 \text{ m}$$

33. Microscope Consider an optical system consisting of two thin lenses sharing the same optical axis. When a cuboid with a side parallel to the optical axis is placed to the left of the left lens, its final image formed by the optical system is also a cuboid but with 500 times the original volume. Assume the two lenses are 10 cm apart and such a cuboid of volume 1 cm^3 is placed such that its right face is 2 cm to the left of the left lens. What's the maximum possible volume of the intermediate image (i.e., image formed by just the left lens) of the cuboid? Answer in cm^3.

Solution 33:
First, note that the two lenses share a focal point. Here's why. For any cuboid with four edges parallel to the optical axis, consider the four parallel rays of light that these four edges lie on. The intermediate images formed by the left lens of these four edges lie on these same light rays after they've passed through the left lens, and the final images of the edges (images formed by the right lens of the intermediate images) lie on these same light rays after they've also passed through the right lens. Since the initial rays were parallel and the same goes for the final rays, the intermediate rays intersect at a focal point of both the left lens and the right lens at the same time.
Now, let f, f' be the focal lengths of the left and right lenses, respectively. Although they can be any nonzero real number, we will WLOG assume that they are positive. (The following derivation will still hold with either f or f' negative, as long as we have the right sign conventions for all the variables.)

For a point at a distance x_1 to the left of F, its image is at a distance $x_2' = -\left(\frac{f'}{f}\right)^2 x_1$ to the right of F'. This follows from applying Newton's formula twice:

$$x_2' = \frac{f'^2}{x_1'} = -\frac{f'^2}{x_2} = -\frac{f'^2}{f^2}x_1.$$

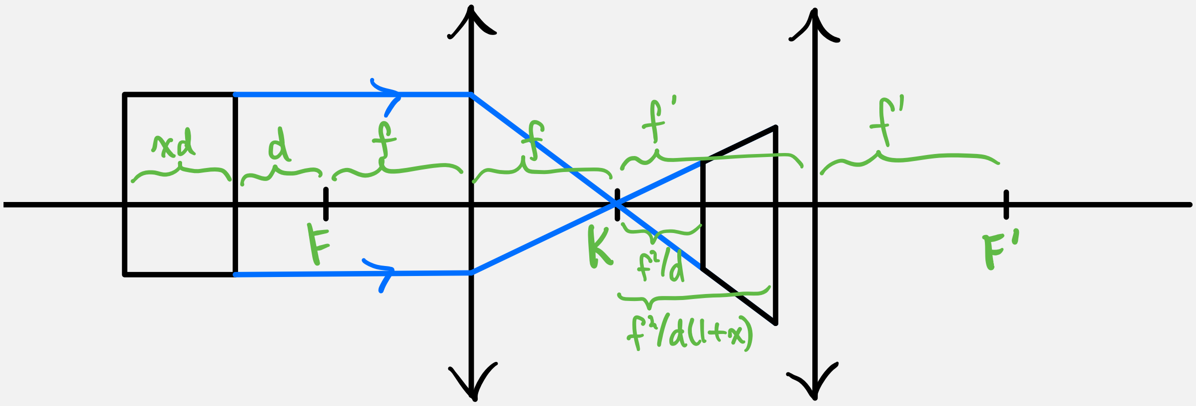
Thus, the optical system magnifies horizontal distances by $\left(\frac{f'}{f}\right)^2$.
On the other hand, for a point at height h_1 (relative to the optical axis), consider a horizontal light ray through the point. Then the final light ray (after it passes through both lenses) is at a height of

$$h_2 = -\frac{f'}{f}h_1,$$

which is the height of the final image of the point. Hence, the optical system magnifies transverse distances (i.e., distances perpendicular to the optical axis) by $\frac{f'}{f}$.
The two results above imply that volumes are magnified by

$$\left(\frac{f'}{f}\right)^2 \left(\frac{f'}{f}\right)^2 = \left(\frac{f'}{f}\right)^4.$$

(The second factor is squared because there are two transverse dimensions.) Given that volumes are magnified by 500 times, we obtain $\frac{f'}{f} = \pm 500^{1/4}$.
We now look at a cuboid with volume V whose right face is at a distance d to the left of F. Let it have width xd and transverse cross-sectional area $A = V/xd$. The intermediate image is a frustrum that results from truncating a pyramid with vertex located at K.



By Newton's formula, the bases of the frustrum are at distances $\frac{f^2}{d}$ and $\frac{f^2}{d(1+x)}$ to the right of K, and they have areas

$$\left(\frac{\frac{f^2}{d}}{f}\right)^2 A = \frac{Vf^2}{xd^3} \quad \text{and} \quad \left(\frac{\frac{f^2}{d(1+x)}}{f}\right)^2 A = \frac{Vf^2}{x(1+x)^2 d^3}.$$

Thus, the volume of the frustrum is

$$\frac{1}{3}\left(\frac{f^2}{d}\frac{Vf^2}{xd^3} - \frac{f^2}{d(1+x)}\frac{Vf^2}{x(1+x)^2d^3}\right) = \frac{1}{3}\frac{Vf^4}{xd^4}\left(1 - \frac{1}{(1+x)^3}\right) \leq \frac{1}{3}\frac{Vf^4}{xd^4}(1-(1-3x)) = \frac{Vf^4}{d^4},$$

where equality is approached as $x \to 0$.
Since $f + f' = 10$ cm and $\frac{f'}{f} = \pm 500^{1/4} \approx \pm 4.7287$, either $f = 1.7456$ cm and $d = 2\text{ cm} - f = 0.2544$ cm, which gives $V_{max} = \frac{Vf^4}{d^4} = 2216\text{ cm}^3$, or $f = -2.6819$ cm and $d = 2\text{ cm} - f = 4.6819$ cm, which gives $V_{max} = 0.1077\text{ cm}^3$. The former is larger, so the answer is 2216 cm^3.

34. Resistor grid Consider an infinite square grid of equal resistors where the nodes are exactly the lattice points in the 2D Cartesian plane. A current $I = 2.7\,\text{A}$ enters the grid at the origin $(0,0)$. Find the current in Amps through the resistor connecting the nodes $(N,0)$ and $(N,1)$, where $N = 38$ can be assumed to be much larger than 1.

Solution 34:
WLOG, let each resistor have unit resistance.
Kirchoff's current law says that the total current entering the node $(x,y) \neq (0,0)$ is zero:

$$[U(x+1,y) - U(x,y)] + [U(x-1,y) - U(x,y)] + [U(x,y+1) - U(x,y)] + [U(x,y-1) - U(x,y)] = 0$$
$$[U(x+1,y) - 2U(x,y) + U(x-1,y)] + [U(x,y+1) - 2U(x,y) + U(x,y-1)] = 0.$$

This is an approximation of the equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0,$$

which is Laplace's equation $\nabla^2 U = 0$ in 2D. Given $U \to 0$ as $r \to \infty$, this implies that the potential

field approximates that of a point charge at O in 2D. This approximation is valid far from the origin where changes in U over unit length are small.
The electric field corresponding to this potential field is

$$\mathbf{E} = -\nabla U = (-\partial_x U, -\partial_y U) \approx (U(x-1,y) - U(x,y), U(x,y-1) - U(x,y)) = (i_x(x,y), i_y(x,y))$$

far from the origin, where $i_x(x,y), i_y(x,y)$ are the horizontal and vertical currents passing through node (x,y). (Note that the current horizontal current is different to the left vs. to the right of the node, and similarly for the vertical current, but the difference is negligible for $N \gg 1$.)
The current $i = \sqrt{i_x^2 + i_y^2}$ at a distance $r \gg 1$ away from the origin is given by Gauss's law:

$$2\pi r i(r) \approx I,$$

so at $(N,0)$ we have

$$i_x(N,0) = i(N) \approx \frac{I}{2\pi N}.$$

The difference between the entering and exiting horizontal currents at $(N,0)$ is approximately

$$-\partial_x i_x(N,0) = \frac{I}{2\pi N^2}.$$

This difference is directed equally into the vertical resistors adjacent to $(N,0)$, so the final answer is

$$\frac{I}{4\pi N^2} = 1.488 \times 10^{-4} \text{ A}.$$

35. Strange Gas Suppose we have a non-ideal gas, and in a certain volume range and temperature range, it is found to satisfy the state relation

$$p = AV^{\alpha}T^{\beta}$$

where A is a constant, $\alpha = -\frac{4}{5}$ and $\beta = \frac{3}{2}$, and the other variables have their usual meanings. Throughout the problem, we will assume to be always in that volume and temperature range.

Assume that $\gamma = \frac{C_p}{C_V}$ is found to be constant for this gas (γ is independent of the state of the gas), where C_p and C_v are the heat capacities at constant pressure and volume, respectively. What is the minimum possible value for γ?

Solution 35:
We claim that the conditions given uniquely determine γ.
The fundamental thermodynamic relation gives:

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V$$

So

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

where we have used a Maxwell relation.

$$\mathrm{d}U = \left(\frac{\partial U}{\partial T}\right)_V \mathrm{d}T + \left(\frac{\partial U}{\partial V}\right)_T \mathrm{d}V = C_V \mathrm{d}T + \left(T\left(\frac{\partial p}{\partial T}\right)_V - p\right)\mathrm{d}V$$

We have

$$C_p = \left(\frac{\partial U}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = C_V + T\left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_p$$

Rearranging, gives

$$C_V = \frac{T\left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_p}{\gamma - 1}$$

From the symmetry of mixed second partial derivatives, we know

$$\left(\frac{\partial C_V}{\partial V}\right)_T = \left(\frac{\partial^2 U}{\partial T \partial V}\right) = \left(\frac{\partial}{\partial T}\left(\frac{\partial U}{\partial V}\right)_T\right)_V = \frac{\partial}{\partial T}\left(T\left(\frac{\partial p}{\partial T}\right)_V - p\right) = \left(\frac{\partial^2 p}{\partial T^2}\right)_V$$

Plugging our expression for C_V into here, and plugging in the equation of state, we can solve for γ to get

$$\gamma = \frac{\alpha + \beta}{\alpha(1 - \beta)} = \frac{7}{4}$$

2022 Online Physics Olympiad: Invitational Contest

Theoretical Examination

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Instructions for Theoretical Exam

The theoretical examination consists of 5 long answer questions and 110 points over 2 full days from July 30, 0:01 am GMT.

- **The team leader should submit their final solution document in this google form. We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it.**

- If you wish to request a clarification, please use this form. To see all clarifications, view this document.

- Participants are given a google form where they are allowed to submit up-to 1 gigabyte of data for their solutions. It is recommended that participants write their solutions in LaTeX. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting. If you wish a premade LaTeX template, we have made one for you here.

- Since each question is a long answer response, participants will be judged on the quality of your work. To receive full points, participants need to show their work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in the IPhO formula sheet) can be cited without proof.

- Remember to state any approximations made and which system of equations were solved after every step. Explicitly showing every step of algebra is not necessary. Participants may leave all final answers in symbolic form (in terms of variables) unless otherwise specified. Be sure to state all assumptions.

Problems

- **T1: Maxwell's Demon**

- **T2: Euler's Disk**

- **T3: Rocket**

- **T4: Magical Box**

- **T5: Quantum Computing**

List of Constants

Proton mass	$m_p = 1.67 \cdot 10^{-27}$ kg
Neutron mass	$m_n = 1.67 \cdot 10^{-27}$ kg
Electron mass	$m_e = 9.11 \cdot 10^{-31}$ kg
Avogadro's constant	$N_0 = 6.02 \cdot 10^{23}\ \mathrm{mol}^{-1}$
Universal gas constant	$R = 8.31\ \mathrm{J/(mol \cdot K)}$
Boltzmann's constant	$k_B = 1.38 \cdot 10^{-23}$ J/K
Electron charge magnitude	$e = 1.60 \cdot 10^{-19}$
1 electron volt	$1\ \mathrm{eV} = 1.60 \cdot 10^{-19}$ J
Speed of light	$c = 3.00 \cdot 10^{8}$ m/s
Universal Gravitational constant	$G = 6.67 \cdot 10^{-11}\ (\mathrm{N \cdot m^2})/\mathrm{kg}^2$
Acceleration due to gravity	$g = 9.81\ \mathrm{m/s^2}$
1 unified atomic mass unit	$1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$
Planck's constant	$h = 6.63 \cdot 10^{-34}\ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15}\ \mathrm{eV \cdot s}$
Permittivity of free space	$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}$
Coulomb's law constant	$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}\ (\mathrm{N \cdot m^2})/\mathrm{C}^2$
Permeability of free space	$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T \cdot m/A}$
Stefan-Boltzmann constant	$\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$

T1: Maxwell's Demon

Zed has a container divided by a wall into two chambers of equal volume V. The left chamber has N_1 molecules and the right chamber has N_2 molecules of some monatomic ideal gas ($N_1 < N_2$). Each gas molecule has mass m and can be treated as a point particle. The entire system is isolated and is at temperature T.

(a) **(5 pts.)** Let's say that he makes a hole in the wall. Then there will be a net flow of molecules from the right chamber to the left chamber. At equilibrium, let's say each chamber has $N = (N_1 + N_2)/2$ molecules. By how much has the entropy increased?

Zed now wants to revert the container back to its original state with N_1 and N_2 molecules in each chamber. He plans to achieve this by covering the hole with a door with area A that only opens towards the second chamber.

(b) **(5 pts.)** He thinks that any molecule in the left chamber incident on the door will enter the right chamber, and no molecules in the right chamber will enter the left one. Under such a model, what is the initial rate of change in entropy of the system?

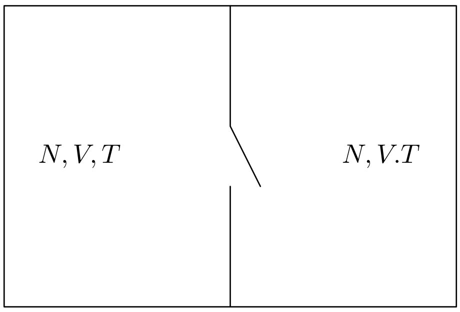


Figure 1: Parts (c) and (d)

Under the assumptions made by part (b), Zed's device violates the second law of thermodynamics. We'll now investigate why this actually does not happen for a particular kind of door. This door, of mass M, has a hinge that exerts a restoring torque $\tau = K\theta$ when the door is open at an angle θ, where θ is not necessarily small (Figure 1).

(c) **(5 pts.)** Explain in one or two sentences why this door behaves effectively like a hole in the wall with area A', and hence the second law of thermodynamics is not violated.

(d) **(10 pts.)** Estimate A' in terms of the variables given and fundamental constants. You may make appropriate simplifying assumptions.

T2: Euler's Disk

A thin, uniform disk of mass m and radius a is initially set at an angle α_0 to the horizontal, on a frictionless surface. It is given an initial angular velocity Ω_0 with respect to a vertical axis passing through its center.

(a) **(4 pts.)** Determine Ω_0 for the steady state case, where $\dot{\alpha} = \ddot{\alpha} = \dot{\Omega} = 0$.

(b) **(2 pts)** Write an expression for the total energy of the disk.

The disk is then moved onto a special surface with small bumps of height h spread over it – each bump is separated by δ. As the disk climbs over a bump and falls back down, its impact is absorbed by the surface, causing a net energy loss in the system. The disk is set in motion with the same initial conditions as before but with $\alpha_0 << 1$

(c) **(6 pts.)** Assuming that this is the only source of energy loss, write a differential equation for $\dot{\alpha}$ in first order to α.

(d) **(4 pts.)** Hence, write an approximate expression for Ω as a function of time.

(e) **(2 pts.)** Using this model, determine the time it takes for the frequency of the sound the disk makes against the surface to reach the maximum audible frequency f_0.

T3: Rocket

OPhO organizers have a "propulsionless" rocket, which for simplicity can be assumed to be a 2-dimensional rectangular box of mass $2M$ and horizontal length L. Assume that the horizontal sides of the box are massless while the vertical sides of the box each have mass M. The rocket is initially at rest. We will now explore the mechanism for how this rocket move. Suppose we have N particles of mass m/N each on the left and right sides of the box. At time $t = 0$, we launch the N particles on the left side of the box together to the right with velocity $\frac{v}{N}$. In addition, in intervals of time $\frac{L}{v}$, starting at $t = 0$, we launch a particle from the right side of the box to the left side with velocity v. Once a particle reaches the opposite side of the box, it is stopped. The particular mechanism to shoot and catch the particles can be ignored here. Assume that this mechanism can conserve energy. After time $t = \frac{NL}{v}$, there will be N particles on each side of the box, which is identical to the initial state.

Neglect relativistic effects in part (a) only.

(a) **(1 pt.)** According to classical (Newtonian) mechanics, what happens to the rocket? Does it move?

(b) **(5 pts.)** If $v \ll c$, how far does the rocket move? Answer in lowest nonzero order in v/c.

(c) **(10 pts.)** How far does the center of mass of the rocket system move? Once again, answer in lowest nonzero order in v/c. Justify your answer.

(d) **(6 pts.)** Explain why this process cannot continue indefinitely. If it could continue forever, we would able to move the rocket indefinitely with no propulsion.

(e) **(5 pts.)** Give an estimate for how long this process can continue. How far does the rocket move in this time?

T4: Magical Box

A cubical box of mass M and side length L sits on a horizontal, frictionless plane. The box is filled with an ideal gas of particle mass m, particle volume density n, and initial temperature T_0. One of the vertical walls inside the cube is made of a highly conductive material, kept at a constant temperature $T_b \gg T_0$. The wall is so conductive that the temperature of gas instantaneously changes to T_b after rebounding. All other walls are made of ideal insulators.

(a) **(1 pt.)** State, with a reasoning, the direction in which the box will start moving.

(b) **(7 pts.)** Approximate the initial acceleration a_0 of the box. For this question, make sure your equation is valid for $T_b = T_0$ as well.

(c) **(3 pts.)** The acceleration of the box then decreases from a_0 to a_f for a short time until $t = \tau_0$. Determine a_f.

(d) **(3 pts.)** If τ_1 is the time it takes for acceleration to level off for an identical box with the conductive wall at temperature $\frac{T_b}{3}$, calculate $\frac{\tau_1}{\tau_0}$.

T5: Quantum Computing

In this problem, you will learn the fundamentals of quantum computers, as well as the physics on how they can be constructed! We have tried to provide as much background information as necessary, but if you believe some part is missing or unclear, please fill out the clarifications form.

Introduction

Physicists use *braket* notation to describe vectors in quantum systems. When using a vector $\vec{v}$ to describe a quantum state, the *ket*, written as $|v\rangle$ can be used. Both notations below are equivalent:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to |v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The *bra*, on the other hand, is the conjugate transpose of the ket $\langle v| = (|v\rangle)^\dagger$. Given two vectors $|v\rangle$ and $|w\rangle$, the *braket* $\langle v|w\rangle = |v\rangle \cdot |w\rangle$ is the inner product of both vectors. This notation will be used throughout this problem.

In any digital device, information is communicated via 0s and 1s, or binary code. The simplest units of this information are called bits. Similar to a bit, the *qubit* can be represented as a linear combination of two orthogonal states: quantum-0 and quantum-1, which are typically $|0\rangle$ and $|1\rangle$. Here,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Typically, we write a single qubit state as

$$|\Psi\rangle = a\,|0\rangle + b\,|1\rangle\,,$$

where $a, b \in \mathbb{C}$, and $\langle \Psi \rangle \, \Psi = 1$.

(a) **(1 pt.)** A qubit is prepared in the state $a\,|0\rangle + b\,|1\rangle$.

(i) What is the probability of measuring the qubit in the state $|0\rangle$?

(ii) What is the probability of measuring the qubit in the state $|-\rangle = |0\rangle - |1\rangle$?

Hint: If you are still confused about measurement (it's tricky!), check out this qiskit article. You can ignore all the parts with code, we'll save those for the computer science students writing OCSO.

A quantum gate performs an unitary operator on a quantum state. Applying an operator (sometimes known as a gate) to a qubit state can be represented in the diagram below.

$$|\Psi\rangle \quad ——\boxed{\hat{U}}—— \quad \hat{U}\,|\Psi\rangle$$

where $\hat{U}$ is a local unitary since it only acts on a single qubit. There are five important gates:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Here, I, X, Y, Z form the four **Pauli matrices** and H is known as the **Hadamard** gate, which we will use later on when we talk about entanglement.

For example, if $|\Psi\rangle = 0.6\,|0\rangle + 0.8\,|1\rangle$ and apply the gate $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we end up with

$$X\,|\Psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix} = 0.8\,|0\rangle + 0.6\,|1\rangle\,.$$

(b) **(1 pt.)** A qubit is prepared in the state $|\Psi\rangle = a|0\rangle + b|1\rangle$. What is the probability of measuring the qubit $\hat{U}|\Psi\rangle$ in the state $|0\rangle$? Express your answer in terms of a, b, and properties of the unitary $\hat{U}$.

The heart of quantum information lies in what we can do with more than a single qubit. If one qubit has two dimensions ($|0\rangle$ and $|1\rangle$), then a two-qubit system can be represented in four dimensions. For a two qubit system, the state can be written as $a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$, where $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ can be seen as the "basis vectors." If we have two independent qubits, i.e. $|\Psi_1\rangle = a|0\rangle + b|1\rangle$ and $|\Psi_2\rangle = c|0\rangle + d|1\rangle$, we can represent their combined state using the **tensor product**, i.e.

$$\begin{aligned}|\Psi\rangle &= |\Psi_1\rangle \otimes |\Psi_2\rangle \\ &= (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) \\ &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle.\end{aligned}$$

Here, we can see that

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, |01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, |10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Note that not every two qubit state can be written as a tensor product. When this occurs, we say that they are **entangled.** We can immediately determine if a state is entangled by calculating its **concurrence,** defined by

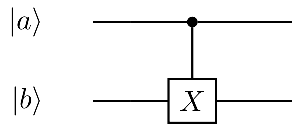
$$C = 2|a_0a_3 - a_1a_2|.$$

If $C = 0$, then the two qubits are separate and the system is **separable.** If $C = 1$, the system is maximally entangled, such as

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

Physically, this means that a measurement of one qubit directly leads to a "collapse" of the other qubit (this is the classic example shown in popular science media). Note that $0 \leq C \leq 1$.

We can change the concurrence using a **control operation.** For example,



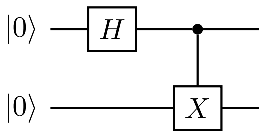
performs the **CNOT** gate. The unitary X is applied to $|b\rangle$ if $|a\rangle = 1$, otherwise nothing is done. That is, we have:

$$\begin{aligned}|00\rangle &\mapsto |00\rangle \\ |01\rangle &\mapsto |01\rangle \\ |10\rangle &\mapsto |11\rangle \\ |11\rangle &\mapsto |10\rangle.\end{aligned}$$

The CNOT gate is an example of a global unitary, since it acts on more than one qubit. Global unitaries for 2 qubit systems can be written as a 4×4 matrix. For example, we can write

$$CNOT = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},$$

where $|00\rangle, \ldots, |11\rangle$ form the 4 standard basis vectors. We can combine local and global unitaries to create entangled states. For example, consider the following circuit:



The initial state is $|\Psi_{\text{in}}\rangle = |00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$. After applying the Hadamard gate H, the state becomes

$$|\Psi_{\text{middle}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}.$$

After applying the CNOT gate, the state becomes:

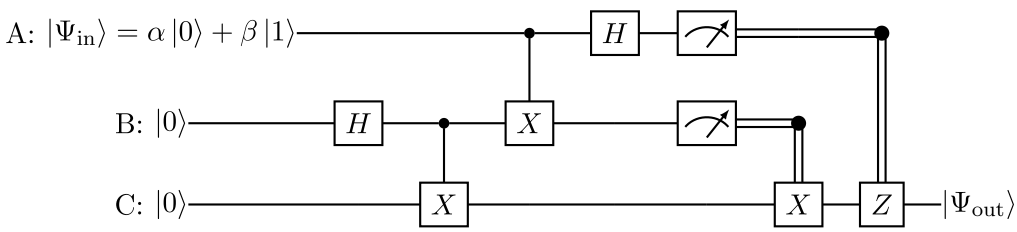
$$|\Psi_{\text{out}}\rangle = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\\frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

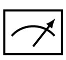
Note that we can avoid matrix multiplication in this last step by seeing what CNOT does on each term of $|\Psi_{\text{middle}}\rangle$. CNOT will not have an effect on $\frac{1}{2}|00\rangle$ since the first qubit is $|0\rangle$. CNOT will have an effect on $\frac{1}{2}|10\rangle$ since the first qubit is a $|1\rangle$, so it'll flip the second qubit to a $|1\rangle$, giving us the map $\frac{1}{2}|10\rangle \mapsto \frac{1}{2}|11\rangle$.

(c) **(1 pt.)** Construct a quantum circuit where the input state is $|00\rangle$ and the output state is $\frac{i}{\sqrt{2}}(|0\rangle - |1\rangle)$ using only $X, Y, Z, H, CNOT$ gates.

Quantum Teleportation

Quantum teleportation is the transfer of the quantum state of one qubit to another (not the actual physical qubit) using a shared entangled resource and two classical bits of information. It is performed using the following circuit.



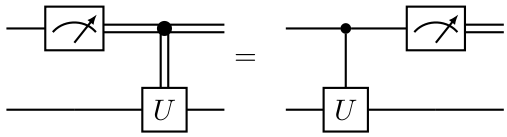
The  gate measures the qubit (returns either a 0 or a 1) and the wider wire represents that information that flows through this wire is a classical bit.

(d) **(1 pt.)** Verify that the above circuit does teleport the qubit from the top branch to the bottom branch by looking at the specific case of $\alpha = \beta = \frac{1}{\sqrt{2}}$

(e) **(3 pts.)** After the first operation is performed on branch C, the branch is brought a very far distance from the other two branches. By doing so, it appears we can create faster-than-light communication during the teleportation process, which is impossible! Explain why there is no contradiction. Justify rigorously.

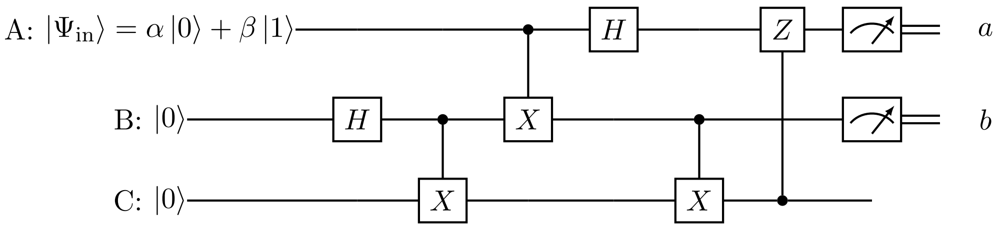
We can analyze this by performing matrix multiplication, but using a circuit-based approach is much cleaner. To do so, we need to use the **Griffiths-Niu Theorem.**

(f) **(2 pts.)** The following circuits, according to the Griffiths-Niu Theorem, are equivalent:

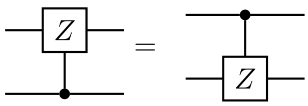


Prove the Griffiths-Niu Theorem.

Using this theorem, we can redraw our circuit as:

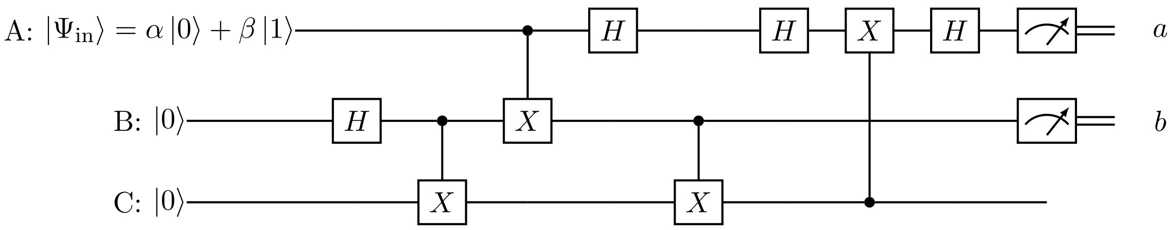


(g) **(1 pts.)** For a control-Z gate, it doesn't matter which branch is the control. In other words,

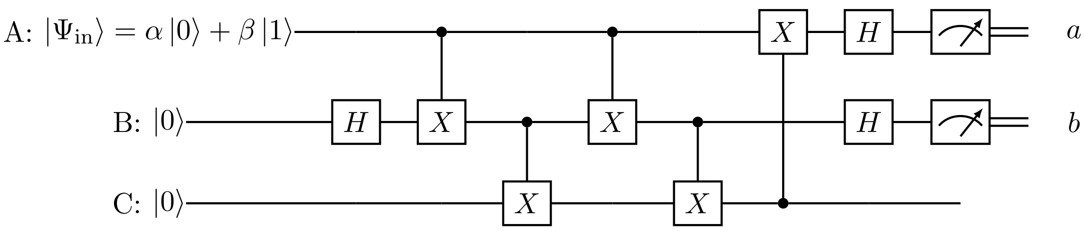


Prove this relationship.

Using the above problem, we can flip the control-Z gate. Then using the identity $Z = HXH$, we can reduce it further:

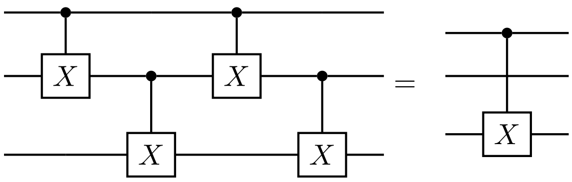


Since $H^2 = I$, we can simplify the top part. Furthermore, we can introduce another CNOT between the first and the second branch.

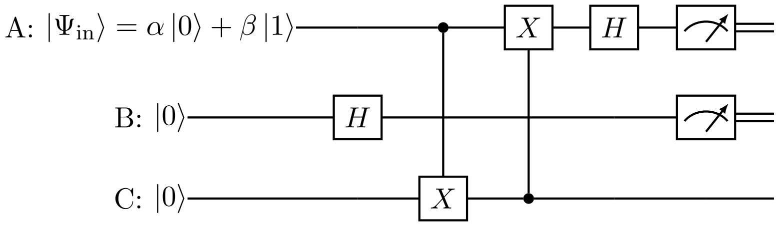


We were allowed to introduce this CNOT gate since $XH|0\rangle = H|0\rangle$. This actually makes it easier using the following problem:

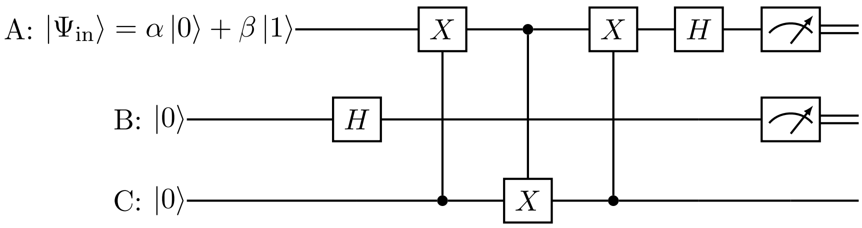
(h) **(2 pts.)** Prove that the below two circuits are equivalent.



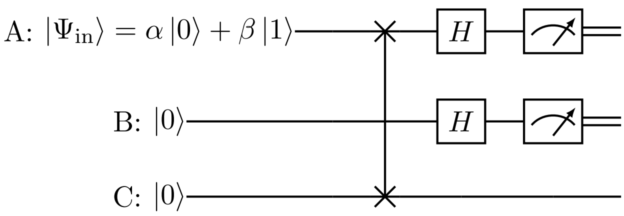
Using this substitution, we end up with:



We can now introduce another CNOT gate, which doesn't do anything since C will always be $|0\rangle$.



Three alternating CNOT gates is equivalent to the SWAP gate, so we can write:



where we clearly see a swapping that occurs between the top and bottom branch!

Building Quantum Computers

According to theoretical physcist David P. Divencenzo, there are five necessary (but not necessarily sufficient) criteria to build a quantum computer:

- A well-characterized qubit.
- The ability to initialize qubits.
- Long and relevant decoherence times.
- A "universal set" of quantum gates.
- The ability to measure qubits.

In this section, we will focus on how we can create qubits and how we can create a universal set of quantum gates. Consider two energy levels E_1, E_0 as the qubit states $\left|1\right\rangle, \left|0\right\rangle$ respectively. Assume that

$$E_1 = \frac{1}{2}\hbar\omega, \qquad E_0 = -\frac{1}{2}\hbar\omega.$$

Also assume that the qubit state is time varying, in the form of:

$$\left|\Psi(t)\right\rangle = A(t)\left|0\right\rangle + B(t)\left|1\right\rangle.$$

(i) **(14 pts.)** Using the above setup, show how we can implement the quantum gates X, Y, Z. *Hint:* The Schrodinger Equation tells us

$$i\hbar\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = \hat{\mathcal{H}}\left|\Psi(t)\right\rangle,$$

where $\hat{\mathcal{H}} = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}$.

2022 Online Physics Olympiad: Invitational Contest

Theoretical Examination

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Instructions for Theoretical Exam

The theoretical examination consists of 5 long answer questions and 110 points over 2 full days from July 30, 0:01 am GMT.

- **The team leader should submit their final solution document in this google form. We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it.**

- If you wish to request a clarification, please use this form. To see all clarifications, view this document.

- Participants are given a google form where they are allowed to submit up-to 1 gigabyte of data for their solutions. It is recommended that participants write their solutions in LaTeX. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting. If you wish a premade LaTeX template, we have made one for you here.

- Since each question is a long answer response, participants will be judged on the quality of your work. To receive full points, participants need to show their work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in the IPhO formula sheet) can be cited without proof.

- Remember to state any approximations made and which system of equations were solved after every step. Explicitly showing every step of algebra is not necessary. Participants may leave all final answers in symbolic form (in terms of variables) unless otherwise specified. Be sure to state all assumptions.

Problems

List of Constants

Proton mass	$m_p = 1.67 \cdot 10^{-27}$ kg
Neutron mass	$m_n = 1.67 \cdot 10^{-27}$ kg
Electron mass	$m_e = 9.11 \cdot 10^{-31}$ kg
Avogadro's constant	$N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
Universal gas constant	$R = 8.31$ J/(mol $\cdot$ K)
Boltzmann's constant	$k_B = 1.38 \cdot 10^{-23}$ J/K
Electron charge magnitude	$e = 1.60 \cdot 10^{-19}$
1 electron volt	1 eV $= 1.60 \cdot 10^{-19}$ J
Speed of light	$c = 3.00 \cdot 10^{8}$ m/s
Universal Gravitational constant	$G = 6.67 \cdot 10^{-11}$ (N $\cdot$ m^2)/kg^2
Acceleration due to gravity	$g = 9.81$ m/s^2
1 unified atomic mass unit	1 u $= 1.66 \cdot 10^{-27}$ kg $= 931$ MeV/c^2
Planck's constant	$h = 6.63 \cdot 10^{-34}$ J $\cdot$ s $= 4.41 \cdot 10^{-15}$ eV $\cdot$ s
Permittivity of free space	$\epsilon_0 = 8.85 \cdot 10^{-12}$ C^2/(N $\cdot$ m^2)
Coulomb's law constant	$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}$ (N $\cdot$ m^2)/C^2
Permeability of free space	$\mu_0 = 4\pi \cdot 10^{-7}$ T $\cdot$ m/A
Stefan-Boltzmann constant	$\sigma = 5.67 \cdot 10^{-8}$ W/m^2/K^4

T1: Maxwell's Demon

Zed has a container divided by a wall into two chambers of equal volume V. The left chamber has N_1 molecules and the right chamber has N_2 molecules of some monatomic ideal gas ($N_1 < N_2$). Each gas molecule has mass m and can be treated as a point particle. The entire system is isolated and is at temperature T.

(a) **(5 pts.)** Let's say that he makes a hole in the wall. Then there will be a net flow of molecules from the right chamber to the left chamber. At equilibrium, let's say each chamber has $N = (N_1 + N_2)/2$ molecules. By how much has the entropy increased?

Zed now wants to revert the container back to its original state with N_1 and N_2 molecules in each chamber. He plans to achieve this by covering the hole with a door with area A that only opens towards the second chamber.

(b) **(5 pts.)** He thinks that any molecule in the left chamber incident on the door will enter the right chamber, and no molecules in the right chamber will enter the left one. Under such a model, what is the initial rate of change in entropy of the system?

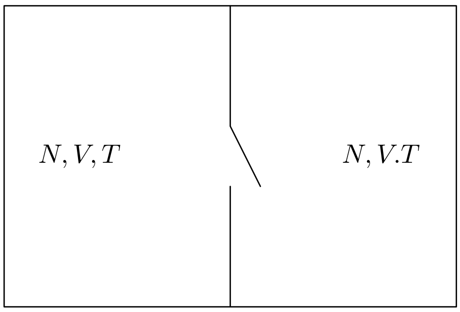


Figure 1: Parts (c) and (d)

Under the assumptions made by part (b), Zed's device violates the second law of thermodynamics. We'll now investigate why this actually does not happen for a particular kind of door. This door, of mass M, has a hinge that exerts a restoring torque $\tau = K\theta$ when the door is open at an angle θ, where θ is not necessarily small (Figure 1).

(c) **(5 pts.)** Explain in one or two sentences why this door behaves effectively like a hole in the wall with area A', and hence the second law of thermodynamics is not violated.

(d) **(10 pts.)** Estimate A' in terms of the variables given and fundamental constants. You may make appropriate simplifying assumptions.

Solution

(a) The internal energy of the system must remain the same as the system is isolated. This implies the temperature of each of the molecules remains as T even when the hole is created. As entropy is a state function and the temperature remains the same, the change in entropy is caused by isothermal expansion/compression. There are two contributions to entropy change: the one experienced by the particles in the first chamber and those in the second chamber.

Since each container as a volume of V, the total volume that each particle can occupy is now $2V$. The number density is thus $\nu = \frac{N_1+N_2}{2V} = \frac{N}{V}$. The particles in the first chamber increase in volume from V to $\frac{V}{\nu}$. This means that

$$\Delta S_1 = N_1 k_B \ln \frac{N_1/\nu}{V} = N_1 k_B \ln \frac{N_1}{N}.$$

Similarly, for particles in the second chamber, we have

$$\Delta S_2 = N_2 k_B \ln \frac{N_2}{N}.$$

The total entropy change is thus

$$\Delta S = \Delta S_1 + \Delta S_2 = (N_1 \ln N_1 + N_2 \ln N_2 - 2 \ln N) k_B$$

Note: Max 2 points for the approach that assumes particles in N_1 are distinguishable from particles in N_2. In this approach, the volume of each compartment doubles so the answer is $(N_1 + N_2)k \ln 2$.

(b) S is a function of N_1. Therefore, we can express $\dot{S}$ as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}N_1}\frac{\mathrm{d}N_1}{\mathrm{d}t}.$$

Right when the door is opened, the system is essentially in equilibrium, so $\frac{\mathrm{d}S}{\mathrm{d}N_1} = 0$. Hence, $\frac{\mathrm{d}S}{\mathrm{d}t} = 0$ as well.

Note: Even though initially $\dot{S} = 0$, it is negative for $t > 0$, and the entropy of the system decreases.

(c) The door is almost always open due to thermal contact with the wall at temperature T. How open the door is doesn't significantly increase as a particle hits it from the left, since its energy is on the same order as the energy of the door $\sim kT$. The effective size of the opening is thus the same for particles traveling in both directions, so particles pass through the opening in both directions at the same rate, hence keeping the entropy constant.

Note: This is just an intuitive explanation as to why we shouldn't find it surprising that there's an equal particle flow in the opposite direction. A rigorous argument (that is not cyclic) would be much more involved.

Marking scheme:

Door is generally open due to thermal motion of molecules in wall	3 pts
Door doesn't get much more open when a particle hits it from the left	2 pts

Explanations that get partial credit:

- When a particle hits the door from the left, the door opens and a particle from the right passes through: max. 3pts (only explains the existence of particle flow in the opposite direction)
- Due to the equipartition theorem, $\frac{1}{2}K\langle\theta^2\rangle = \frac{1}{2}kT > 0$, so the door is effectively open: max. 3pts (same reason as above)

(d) The equipartition theorem gives $\frac{1}{2}K\langle\theta^2\rangle = \frac{1}{2}kT$, so the typical $\theta \approx \sqrt{kT/K}$. When $\sqrt{kT/K}1$, the size of the opening is about $2A\sin\left(\frac{1}{2}\sqrt{\frac{kT}{K}}\right)$; when $\sqrt{kT/K} \gtrsim 1$, the size of the opening is better approximated by the size of the hole, A.

Note: In reality, there's a smooth transition from the $\sqrt{kT/K}1$ regime to the $\sqrt{kT/K} \gtrsim 1$ regime. A precise model would likely involve modeling the joint distribution $\rho(n, \mathbf{v}, \theta, \dot{\theta}|\mathbf{r})$ of the door's angle and its rate of change, and the particle density and velocity distribution of gas particles at each point in space.

T2: Euler's Disk

A thin, uniform disk of mass m and radius a is initially set at an angle α_0 to the horizontal, on a frictionless surface. It is given an initial angular velocity Ω_0 with respect to a vertical axis passing through its center.

(a) **(4 pts.)** Determine Ω_0 for the steady state case, where $\dot{\alpha} = \ddot{\alpha} = \dot{\Omega} = 0$.

(b) **(2 pts)** Write an expression for the total energy of the disk.

The disk is then moved onto a special surface with small bumps of height h spread over it – each bump is separated by δ. As the disk climbs over a bump and falls back down, its impact is absorbed by the surface, causing a net energy loss in the system. The disk is set in motion with the same initial conditions as before but with $\alpha_0 << 1$

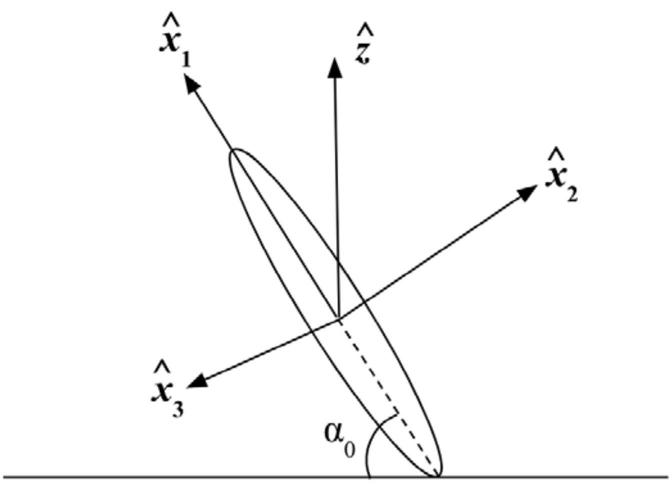
(c) **(6 pts.)** Assuming that this is the only source of energy loss, write a differential equation for $\dot{\alpha}$ in first order to α.

(d) **(4 pts.)** Hence, write an approximate expression for Ω as a function of time.

(e) **(2 pts.)** Using this model, determine the time it takes for the frequency of the sound the disk makes against the surface to reach the maximum audible frequency f_0.

Solution

(a) Let us use the following three axes:



Let's say the disk moves in a counter-clockwise direction when viewed from above. Since the disk is given an angular velocity $\Omega_0\hat{\mathbf{z}}$, it must roll without slipping with angular velocity $-\omega'\,\hat{\mathbf{x}}_2$. Since it rolls without slipping:

$$\frac{2\pi a\cos(\alpha_0)}{\omega' a} = \frac{2\pi}{\Omega_0}$$

Thus,

$$\omega' = \Omega_0\cos(\alpha_0)$$

Therefore, the net angular velocity vector is:

$$\boldsymbol{\omega} = \Omega_0\hat{\mathbf{z}} - \omega'\,\hat{\mathbf{x}}_2 = \Omega_0\hat{\mathbf{z}} - \Omega_0\cos(\alpha_0)\hat{\mathbf{x}}_2 = \Omega_0\sin(\alpha_0)\hat{\mathbf{x}}_1$$

Since the disk's point of contact with the ground and the COM is at rest, we can let the axis passing through both points – $\hat{\mathbf{x}}_\mathbf{1}$ – be the instantaneous axis of rotation. Note that this is validated by the net angular velocity vector lying along $\hat{\mathbf{x}}_\mathbf{1}$. By the perpendicular axes theorem, the moment of inertia around $\hat{\mathbf{x}}_\mathbf{1}$ is $I = \frac{1}{4}ma^2$. Thus, the angular momentum vector is:

$$\mathbf{L} = I\boldsymbol{\omega} = \frac{\Omega_0}{4}ma^2\sin(\alpha_0)\hat{\mathbf{x}}_\mathbf{1}$$

If we calculate torque $\boldsymbol{\tau}$ from the point of contact with the ground, the only force that contributes is weight. Thus,

$$\boldsymbol{\tau} = mga\cos(\alpha_0)\hat{\mathbf{x}}_\mathbf{3}$$

This should equal:

$$\frac{d\mathbf{L}}{dt} = \frac{\Omega_0}{4}ma^2\sin(\alpha_0)\frac{d\hat{\mathbf{x}}_\mathbf{1}}{dt}$$

Let's do some quick vector analysis to find $\frac{d\hat{\mathbf{x}}_\mathbf{1}}{dt}$. Note that, since $\hat{\mathbf{x}}_\mathbf{1}$ is the instantaneous axis of rotation, the unit vectors $\hat{\mathbf{x}}_\mathbf{1}$ and $\hat{\mathbf{x}}_\mathbf{3}$ spin on the plane of the disk at this given instant. Suppose $\hat{\mathbf{x}}_\mathbf{1}$ spins for an infinitesimal time t such that its coordinates on the plane of the disk is:

$$\hat{\mathbf{x}}_\mathbf{1} = \langle\sin(\Omega_0\cos\alpha_0\, t), \cos(\Omega_0\cos\alpha_0\, t)\rangle$$

Taking the time derivative,

$$\frac{d\hat{\mathbf{x}}_\mathbf{1}}{dt} = \Omega_0\cos\alpha_0\,\langle-\cos(\Omega_0\cos\alpha_0\, t), \sin(\Omega_0\cos\alpha_0\, t)\rangle = \Omega_0\cos\alpha_0\hat{\mathbf{x}}_\mathbf{3}$$

Plugging this into the equation above and solving for Ω_0,

$$\Omega_0 = \sqrt{\frac{4g}{a\sin(\alpha_0)}}$$

(b) As derived in the previous question, the total rotational kinetic energy of the disk can be written as:

$$\frac{1}{2}\left(\frac{1}{2}ma^2\right)(\Omega_0\sin\alpha_0)^2$$

again, since $\hat{\mathbf{x}}_\mathbf{1}$ is the IAR. Since the COM of the disk remains at height $H = a\sin\alpha_0$, we can write its total energy as:

$$\frac{1}{2}\left(\frac{1}{2}ma^2\right)(\Omega_0\sin\alpha_0)^2 + mga\sin\alpha_0 = \frac{3}{2}mga\sin\alpha_0$$

(c) We simply have to add a $\frac{1}{2}m\dot{H}^2$ term to the total energy above to take into account the disk's falling COM. Approximating the total energy E at the $\alpha \to 0$ limit,

$$E = \frac{1}{2}ma^2(\dot{\alpha}\cos\alpha)^2 + \frac{3}{2}mga\sin\alpha \approx \frac{1}{2}ma^2\dot{\alpha}^2 + \frac{3}{2}mga\alpha$$

Now, we'll find the power P dissipated by the bumps and equate $P = \frac{dE}{dt}$. As the disk climbs over a bump and falls back down, its gravitational potential energy mgh will be absorbed by the surface. In other words, the disk will lose mgh every $\frac{\delta}{\Omega\cos\alpha a} \approx \frac{\delta}{\Omega a}$. Hence, we get:

$$P = -\frac{mgha\Omega}{\delta}$$

Therefore,

$$ma^2\dot{\alpha}\ddot{\alpha} + \frac{3}{2}mga\dot{\alpha} = -\frac{mgha\Omega}{\delta}$$

Assuming the disk spends most of its time falling down from a bump, we can approximate its COM's acceleration to be $a\ddot{\alpha} \approx g$. Fully simplifying the equation above, we get:

$$\dot{\alpha} = -\frac{4h}{5\delta}\sqrt{\frac{g}{a}}\frac{1}{\sqrt{\alpha}}$$

(d) Solving the differential equation from the previous question gives

$$\alpha^{\frac{3}{2}} - \alpha_0^{\frac{3}{2}} = -\frac{6h}{5\delta}\sqrt{\frac{g}{a}}t$$

We can simply plug in α into $\Omega = \sqrt{\frac{4g}{a\alpha}}$ to find the answer.

(e) The disk will produce a "click" every time it falls off a bump, hitting the surface. The frequency f at any given time will therefore be:

$$f = \frac{\delta}{\Omega a}$$

Any attempt with this idea were given full marks.

T3: Rocket

OPhO organizers have a "propulsionless" rocket, which for simplicity can be assumed to be a 2-dimensional rectangular box of mass $2M$ and horizontal length L. Assume that the horizontal sides of the box are massless while the vertical sides of the box each have mass M. The rocket is initially at rest. We will now explore the mechanism for how this rocket move. Suppose we have N particles of mass m/N each on the left and right sides of the box. At time $t=0$, we launch the N particles on the left side of the box together to the right with velocity $\frac{v}{N}$. In addition, in intervals of time $\frac{L}{v}$, starting at $t=0$, we launch a particle from the right side of the box to the left side with velocity v. Once a particle reaches the opposite side of the box, it is stopped. The particular mechanism to shoot and catch the particles can be ignored here. Assume that this mechanism can conserve energy. After time $t=\frac{NL}{v}$, there will be N particles on each side of the box, which is identical to the initial state.

Neglect relativistic effects in part (a) only.

(a) **(1 pt.)** According to classical (Newtonian) mechanics, what happens to the rocket? Does it move?

(b) **(5 pts.)** If $v \ll c$, how far does the rocket move? Answer in lowest nonzero order in v/c.

(c) **(10 pts.)** How far does the center of mass of the rocket system move? Once again, answer in lowest nonzero order in v/c. Justify your answer.

(d) **(6 pts.)** Explain why this process cannot continue indefinitely. If it could continue forever, we would able to move the rocket indefinitely with no propulsion.

(e) **(5 pts.)** Give an estimate for how long this process can continue. How far does the rocket move in this time?

Solution

(a) At each moment, the particles from the left side of the rocket have momentum $N\frac{m}{N}\frac{v}{N}$ and the particles from the right side of the box have momentum $\frac{mv}{N}$. As these are equal, by conservation of momentum for the entire system, the rocket must have 0 momentum, so it does not move.

(b) Let $\beta = v/c$. The relativistic momentum of the particles from the left side of the rocket is

$$N\frac{1}{\sqrt{1-\frac{\beta^2}{N^2}}}\frac{m}{N}\frac{\beta c}{N} = \frac{\beta mc}{N\sqrt{1-\beta^2/N^2}}.$$

The momentum of the particle from the right side of the rocket is

$$\frac{1}{\sqrt{1-\beta^2}}\frac{m}{N}(\beta c) = \frac{\beta mc}{N\sqrt{1-\beta^2}}.$$

The difference in these quantities to lowest nonzero order in β is

$$\frac{\beta mc}{N}\left(1+\frac{\beta^2}{2}-1-\frac{\beta^2}{2N^2}\right) = \frac{\beta^3 mc(N^2-1)}{2N^3}.$$

This is the momentum of the rocket to the right. Thus, it is equal to $\gamma(2M+\frac{N-1}{N}m)V$ if V is the speed of the rocket. As the momentum is third order in β, we can assume that $\gamma \approx 1$ to find V to lowest nonzero order. We obtain

$$V = \frac{\beta^3 mc(N^2-1)}{2N^2(2MN+(N-1)m)}.$$

Now, the time is $t = \frac{NL}{v} = \frac{NL}{\beta c}$, so the total distance traveled is

$$Vt = \frac{(N^2-1)m}{2N(2MN+(N-1)m)}\frac{v^2}{c^2}L.$$

Valid solutions that assume $N \gg 1$ or $m \ll M$ are acceptable.

(c) The center of energy in a closed relativistic system with zero total momentum does not move, so the rocket's center of mass at the end is exactly where it was in the beginning.

For a proof of this fact, consider a system of particles. We wish to show the claim that $\sum_i x_i E_i$ is constant. Whenever there is a collision between particles i and j, note that just before and after the collision, both particles are at the same position x, so if E_i' and E_j' are the energies after the collision, then the change in $\sum_i x_i E_i$ during the collision is $xE_i' + xE_j' - xE_i - xE_j = x(E_i' + E_j' - E_i - E_j) = 0$. This is true as energy is conserved during the collision. We have the same conclusion when one particle decays for the same reason. Lastly, it suffices to show that $\sum_i x_i E_i$ is conserved for a set of particles moving freely with zero total momentum. In this case, E_i is constant and equal to $\gamma_i m_i c^2$, so the change in $\sum_i x_i E_i$ is $\sum_i (v_i t)\gamma_i m_i c^2 = c^2 t \sum_i \gamma_i m_i v_i = c^2 t \sum_i p_i = 0$. Here, we have assumed that the particles move in one dimension, but it is simple to extend this to multiple dimensions. Therefore, we can conclude that the center of energy does not move.

Half credit will be given for the answer and a classical explanation. For full credit, a satisfactory relativistic explanation must be provided.

(d) The center of mass does not move although the rocket system moves to the right. This means that through this process, mass is transferred from the right side of the rocket to the left side. Every time this process is repeated, the same amount of mass is transferred from the right to the left side, and since the right side only starts with mass M, the process cannot continue indefinitely. Unfortunately, we cannot construct a propulsionless rocket even with relativity.

(e) If a mass δm is transferred from the left side to the right side in each step, then the center of mass of the rocket moves a distance $\frac{\delta m}{2M}L$ to the left from the geometric center of the rocket. In order for the center of mass of the rocket to not move, we must have

$$\frac{\delta m}{2M}L = \frac{(N^2-1)m}{2N(2M+(N-1)m)}\frac{v^2}{c^2}L,$$

$$\delta m = \frac{(N^2-1)mM}{N(2MN+(N-1)m)}\frac{v^2}{c^2}.$$

The number of times that this can continue is roughly

$$\frac{M}{\delta m} = \frac{N(2MN+(N-1)m)}{(N^2-1)m}\frac{c^2}{v^2}.$$

The total distance that can be traveled is roughly at most $L/2$ (if all of the mass from the right goes to the left).

Here, we implicitly assumed $m \ll M$. Any solution with correct reasoning and a valid estimate is acceptable.

T4: Magical Box

A cubical box of mass M and side length L sits on a horizontal, frictionless plane. The box is filled with an ideal gas of particle mass m, particle volume density n, and initial temperature T_0. One of the vertical walls inside the cube is made of a highly conductive material, kept at a constant temperature $T_b \gg T_0$. The wall is so conductive that the temperature of gas instantaneously changes to T_b after rebounding. All other walls are made of ideal insulators.

(a) **(1 pt.)** State, with a reasoning, the direction in which the box will start moving.

(b) **(7 pts.)** Approximate the initial acceleration a_0 of the box. For this question, make sure your equation is valid for $T_b = T_0$ as well.

(c) **(3 pts.)** The acceleration of the box then decreases from a_0 to a_f for a short time until $t = \tau_0$. Determine a_f.

(d) **(3 pts.)** If τ_1 is the time it takes for acceleration to level off for an identical box with the conductive wall at temperature $\frac{T_b}{3}$, calculate $\frac{\tau_1}{\tau_0}$.

Solution

(a) Let us consider two particles of mass m, both starting at the center of the box. Particle A travels in the x-direction to the conductive wall, while particle B similarly goes to the opposite insulated one, both travelling at a speed v. We neglect the impact of the other walls, as due to symmetry, any effects will cancel out. Furthermore, we can take such an approximation because Maxwell's distribution shows that half the particles would travel to the opposite wall and vice versa.

Due to the given conditions, the momentum of the system must initially be zero. As the velocity of a particle is proportional to the square root of its temperature, $v \propto \sqrt{T}$, then the velocity of A after rebounding will be slightly greater as $v + \Delta v$ in the negative x-direction. The velocity of B after rebounding will be the same in the positive x-direction. As such, the net momentum of both particles is $-m\Delta v$. This means that the box must start moving in the positive x-direction, or in the direction of the conductive wall, to conserve the momentum of the system.

(b) Inside the box, the particles will be moving randomly at different speeds. We want to find the number of particles approaching a wall in the given speeds $[v, v + \mathrm{d}v]$ and angles $[\theta, \theta + \mathrm{d}\theta]$. If all molecules are moving in equal directions, the fraction of particles within a solid angle $\mathrm{d}\Omega$ is $\mathrm{d}\Omega/4\pi$. If we consider the angles between θ and $\theta + \mathrm{d}\theta$, we can relate Ω as

$$\mathrm{d}\Omega = 2\pi \sin\theta \mathrm{d}\theta \implies \frac{\mathrm{d}\Omega}{4\pi} = \frac{1}{2}\sin\theta \mathrm{d}\theta.$$

The number of particles in a unit volume is then

$$\rho = nf(v)\mathrm{d}v\frac{1}{2}\sin\theta \mathrm{d}\theta$$

where $f(v)$ is the Maxwell speed distribution function. For molecules approaching a wall of area A at angle θ, the volume encapsulated within a unit time $\mathrm{d}t$ is

$$\mathrm{d}V = Av\mathrm{d}t\cos\theta$$

where A is the area of the wall, or in this case, L^2. Therefore, the number of particles approaching the wall is

$$N = \rho \mathrm{d}V = L^2 v\mathrm{d}t\cos\theta nf(v)\mathrm{d}v\frac{1}{2}\sin\theta \mathrm{d}\theta$$

Suppose the velocity of particles after hitting the insulated wall is v_b. Then, by momentum conservation in the parallel direction of the wall, impulse on the box per collision is:

$$m\left(v\cos\theta + v_b\sqrt{1-\frac{v^2}{v_b^2}\sin^2\theta}\right) \approx m(v\cos\theta + v_b)$$

Thus, the net impulse imparted for collisions at speed v and at angle θ is:

$$\mathcal{I}^* = Nm(v\cos\theta + v_b) = \frac{1}{2}nL^2 vf(v)\mathrm{d}v(v_b\cos\theta + v)\sin\theta\cos\theta\mathrm{d}\theta\mathrm{d}t$$

We can then find the force, and hence our acceleration, imparted on the wall by integrating over impulse and dividing by unit time:

$$\begin{aligned}\mathcal{F}^* &= \frac{1}{2}nL^2\int_0^{\frac{\pi}{2}}\int_0^{\infty} v(v\cos\theta + v_b)f(v)\mathrm{d}v\sin\theta\cos\theta\mathrm{d}\theta \\ &= \frac{1}{2}nL^2\int_0^{\frac{\pi}{2}}(\langle v^2\rangle\cos\theta + \langle vv_b\rangle)\sin\theta\cos\theta\mathrm{d}\theta \\ &= \frac{1}{6}mnL^2\langle v^2\rangle + \frac{1}{4}mnL^2\langle vv_b\rangle\end{aligned}$$

For the insulated wall, $v_b = v$, so

$$\mathcal{F} = \frac{1}{3}mnL^2\langle v^2\rangle.$$

The net force imparted on the box is thus

$$\sum\mathcal{F} = \frac{1}{6}mnL^2\langle v^2\rangle\left[\frac{3}{2}\frac{\langle vv_b\rangle}{\langle v^2\rangle} - 1\right]$$

Since $\langle vv_b\rangle / \langle v^2\rangle \approx \frac{\sqrt{T_0T_b}}{T_0} = \sqrt{\frac{T_b}{T_0}}$, we can write our final answer as

$$a = \frac{1}{6}nL^2\frac{3k_BT_0}{m}\left(\frac{3}{2}\sqrt{\frac{T_b}{T_0}} - 1\right) = \frac{nL^2k_BT_0}{2m}\left(\frac{3}{2}\sqrt{\frac{T_b}{T_0}} - 1\right).$$

Remarks on the Net Momentum of the System

Though analyzing the impulse imparted at the conductive and insulated walls clearly indicates that the box moves from its initial position, most solvers, including the problem writer, overlooked the rather obvious fact that the net force on the system is zero. If we set the system to include the box, the gas, and the heating element, all interactions that occur in the consequent motion of the box are undoubtedly adiabatic; in other words, it doesn't make physical sense for the final momentum of the system to be nonzero – the box's COM **must** return to rest, obeying the laws of Newtonian mechanics. Let's divide the problem into two cases based on the size of L relative to the gas's mean free path λ.

$L \ll \lambda$

$L \ll \lambda$ implies that particles interact solely with the walls and not with themselves – once a particle collides with the conductive wall, gains momentum, and moves straight towards the insulated wall and collides with it. It's pretty easy to imagine that the box won't move all that much from its original position. In the first few collisions, the box will surely be accelerated towards the conductive wall, but it will come to a stop when the "hot" particles collide with the insulated wall, delivering their momentum. Once all particles collide with the conductive wall, the collision frequency and average impulse per collision at the two walls will be the same, resulting in zero net force.

$L \gg \lambda$

For this case, particles do collide with each other, and it is possible for the box to be accelerated for a sustained period of time. The temperature of the gas will gradually increase from the conductive side through gas particle collisions, eventually reaching thermal equilibrium at T_b.
The box will certainly accelerate from its initial position. There are multiple factors that causes the box to slow down, as it has been noted by several solvers. For instance, when the box's speed becomes greater than the average velocity of gas particles, there will be a rapid decline in the collision rate of gas particles at the conductive wall. This is supported by some quick calculations.

When the box is moving with velocity v_b, particles that collide with the conductive wall have a horizontal component of velocity greater than or equal to v_b. The number of particles per area that collide with the conductive wall can therefore be found as:

$$\frac{1}{2} n_b \int_0^{\frac{\pi}{2}} \left(\int_{v_b \sec\theta}^{\infty} v f(v) dv \right) \sin\theta \cos\theta d\theta$$

where f(v) is the Maxwell distributuon function of the gas in the vicinity of the conductive wall. If we evaluate the integral for different values of v_b, we find that the collision frequency per unit area decreases rapidly as v_b exceeds the average velocity of gas particles $\langle v \rangle$. For instance, collision frequency per unit area decreases from its initial value by a factor of 1000 when $v_b = 2\langle v \rangle$.

This effect is also accompanied by "weaker" collisions at the conductive wall when considering particle velocities relative to the box, ultimately causing the box to slow down. But *howmuch* does the box's speed decrease by? It would slow down until $v = v_{\text{critical}}$ and the pressure on the conductive wall overcomes than that on the insulated wall again. Then, the box will accelerate towards the conductive wall and the cycle repeats. However, this won't continue indefinitely. There are several ways to justify this as well. For instance, as the gas in the vicinity of the insulated wall heats up, the maximum pressure difference (so when the velocity of the box is at minimum) of the conductive and insulated walls $P_c - P_i$ will decrease. This means two things:

1. The maximum speed of the box between two consecutive "slow-downs" will get smaller

2. The $v_{critical}$ necessary for $P_c > P_i$ will also get smaller

Eventually, when the entire gas reaches T_b, v_{critical} will just be zero, so the box will eventually come to a full stop, as needed. From the two observations above, we can conclude that the box will eventually return to rest after its velocity-time curve undergoes something that resembles damped harmonic oscillations.

T5: Quantum Computing

In this problem, you will learn the fundamentals of quantum computers, as well as the physics on how they can be constructed! We have tried to provide as much background information as necessary, but if you believe some part is missing or unclear, please fill out the clarifications form.

Introduction

Physicists use *braket* notation to describe vectors in quantum systems. When using a vector $\vec{v}$ to describe a quantum state, the *ket*, written as $|v\rangle$ can be used. Both notations below are equivalent:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rightarrow |v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The *bra*, on the other hand, is the conjugate transpose of the ket $\langle v| = (|v\rangle)^\dagger$. Given two vectors $|v\rangle$ and $|w\rangle$, the *braket* $\langle v|w\rangle = |v\rangle \cdot |w\rangle$ is the inner product of both vectors. This notation will be used throughout this problem.

In any digital device, information is communicated via 0s and 1s, or binary code. The simplest units of this information are called bits. Similar to a bit, the *qubit* can be represented as a linear combination of two orthogonal states: quantum-0 and quantum-1, which are typically $|0\rangle$ and $|1\rangle$. Here,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Typically, we write a single qubit state as

$$|\Psi\rangle = a|0\rangle + b|1\rangle,$$

where $a, b \in \mathbb{C}$, and $\langle \Psi \rangle \, \Psi = 1$.

(a) **(1 pt.)** A qubit is prepared in the state $a|0\rangle + b|1\rangle$.

(i) What is the probability of measuring the qubit in the state $|0\rangle$?

(ii) What is the probability of measuring the qubit in the state $|-\rangle = |0\rangle - |1\rangle$?

Hint: If you are still confused about measurement (it's tricky!), check out this qiskit article. You can ignore all the parts with code, we'll save those for the computer science students writing OCSO.

A quantum gate performs an unitary operator on a quantum state. Applying an operator (sometimes known as a gate) to a qubit state can be represented in the diagram below.

$$|\Psi\rangle \qquad \text{———}\boxed{\hat{U}}\text{———} \qquad \hat{U}|\Psi\rangle$$

where $\hat{U}$ is a local unitary since it only acts on a single qubit. There are five important gates:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Here, I, X, Y, Z form the four **Pauli matrices** and H is known as the **Hadamard** gate, which we will use later on when we talk about entanglement.

For example, if $|\Psi\rangle = 0.6|0\rangle + 0.8|1\rangle$ and apply the gate $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we end up with

$$X|\Psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix} = 0.8|0\rangle + 0.6|1\rangle.$$

(b) **(1 pt.)** A qubit is prepared in the state $|\Psi\rangle = a|0\rangle + b|1\rangle$. What is the probability of measuring the qubit $\hat{U}|\Psi\rangle$ in the state $|0\rangle$? Express your answer in terms of a, b, and properties of the unitary $\hat{U}$.

The heart of quantum information lies in what we can do with more than a single qubit. If one qubit has two dimensions ($|0\rangle$ and $|1\rangle$), then a two-qubit system can be represented in four dimensions. For a two qubit system, the state can be written as $a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$, where $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ can be seen as the "basis vectors." If we have two independent qubits, i.e. $|\Psi_1\rangle = a|0\rangle + b|1\rangle$ and $|\Psi_2\rangle = c|0\rangle + d|1\rangle$, we can represent their combined state using the **tensor product**, i.e.

$$\begin{aligned} |\Psi\rangle &= |\Psi_1\rangle \otimes |\Psi_2\rangle \\ &= (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) \\ &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle. \end{aligned}$$

Here, we can see that

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, |01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, |10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Note that not every two qubit state can be written as a tensor product. When this occurs, we say that they are **entangled.** We can immediately determine if a state is entangled by calculating its **concurrence,** defined by

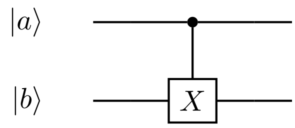
$$C = 2|a_0a_3 - a_1a_2|.$$

If $C = 0$, then the two qubits are separate and the system is **separable.** If $C = 1$, the system is maximally entangled, such as

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

Physically, this means that a measurement of one qubit directly leads to a "collapse" of the other qubit (this is the classic example shown in popular science media). Note that $0 \le C \le 1$.

We can change the concurrence using a **control operation.** For example,



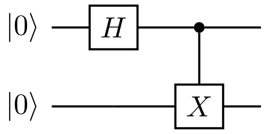
performs the **CNOT** gate. The unitary X is applied to $|b\rangle$ if $|a\rangle = 1$, otherwise nothing is done. That is, we have:

$$\begin{aligned} |00\rangle &\mapsto |00\rangle \\ |01\rangle &\mapsto |01\rangle \\ |10\rangle &\mapsto |11\rangle \\ |11\rangle &\mapsto |10\rangle. \end{aligned}$$

The CNOT gate is an example of a global unitary, since it acts on more than one qubit. Global unitaries for 2 qubit systems can be written as a 4×4 matrix. For example, we can write

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where $|00\rangle, \ldots, |11\rangle$ form the 4 standard basis vectors. We can combine local and global unitaries to create entangled states. For example, consider the following circuit:



The initial state is $|\Psi_{\text{in}}\rangle = |00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$. After applying the Hadamard gate H, the state becomes

$$|\Psi_{\text{middle}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}.$$

After applying the CNOT gate, the state becomes:

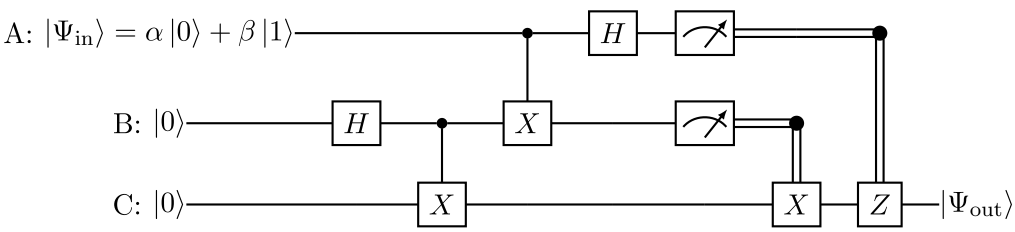
$$|\Psi_{\text{out}}\rangle = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\\frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

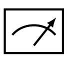
Note that we can avoid matrix multiplication in this last step by seeing what CNOT does on each term of $|\Psi_{\text{middle}}\rangle$. CNOT will not have an effect on $\frac{1}{2}|00\rangle$ since the first qubit is $|0\rangle$. CNOT will have an effect on $\frac{1}{2}|10\rangle$ since the first qubit is a $|1\rangle$, so it'll flip the second qubit to a $|1\rangle$, giving us the map $\frac{1}{2}|10\rangle \mapsto \frac{1}{2}|11\rangle$.

(c) **(1 pt.)** Construct a quantum circuit where the input state is $|00\rangle$ and the output state is $\frac{i}{\sqrt{2}}(|0\rangle - |1\rangle)$ using only $X, Y, Z, H, CNOT$ gates.

Quantum Teleportation

Quantum teleportation is the transfer of the quantum state of one qubit to another (not the actual physical qubit) using a shared entangled resource and two classical bits of information. It is performed using the following circuit.



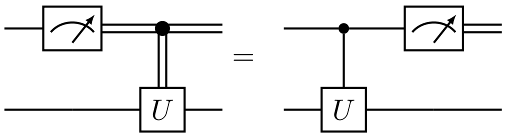
The  gate measures the qubit (returns either a 0 or a 1) and the wider wire represents that information that flows through this wire is a classical bit.

(d) **(1 pt.)** Verify that the above circuit does teleport the qubit from the top branch to the bottom branch by looking at the specific case of $\alpha = \beta = \frac{1}{\sqrt{2}}$

(e) **(3 pts.)** After the first operation is performed on branch C, the branch is brought a very far distance from the other two branches. By doing so, it appears we can create faster-than-light communication during the teleportation process, which is impossible! Explain why there is no contradiction. Justify rigorously.

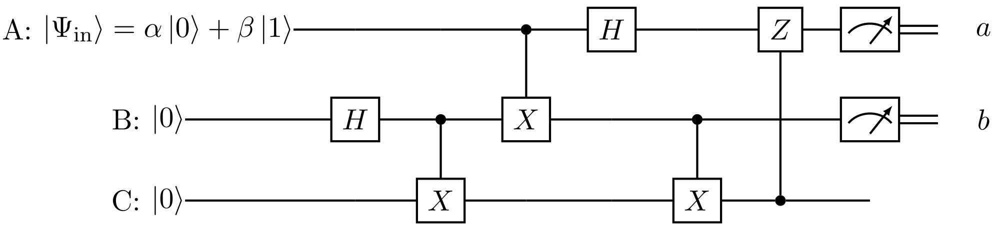
We can analyze this by performing matrix multiplication, but using a circuit-based approach is much cleaner. To do so, we need to use the **Griffiths-Niu Theorem.**

(f) **(2 pts.)** The following circuits, according to the Griffiths-Niu Theorem, are equivalent:

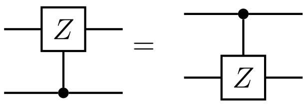


Prove the Griffiths-Niu Theorem.

Using this theorem, we can redraw our circuit as:

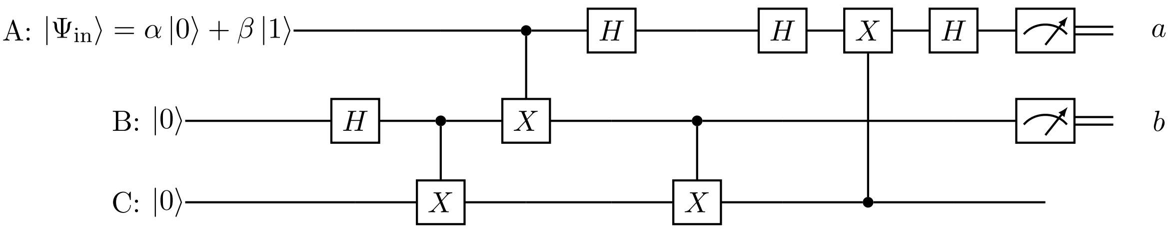


(g) **(1 pts.)** For a control-Z gate, it doesn't matter which branch is the control. In other words,

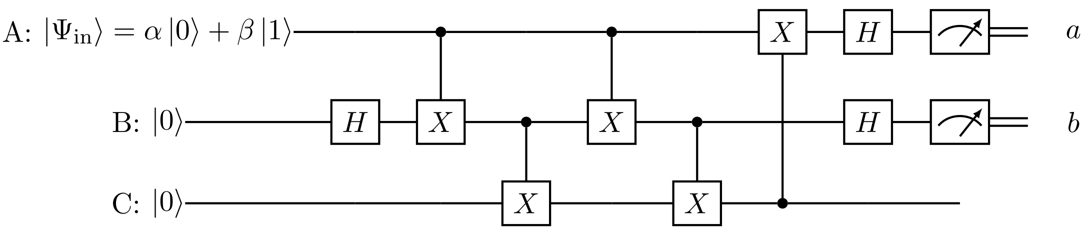


Prove this relationship.

Using the above problem, we can flip the control-Z gate. Then using the identity $Z = HXH$, we can reduce it further:

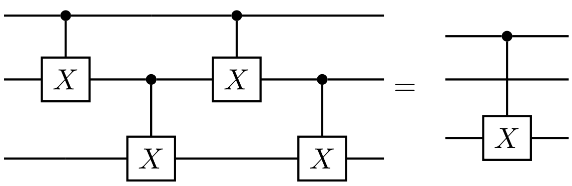


Since $H^2 = I$, we can simplify the top part. Furthermore, we can introduce another CNOT between the first and the second branch.

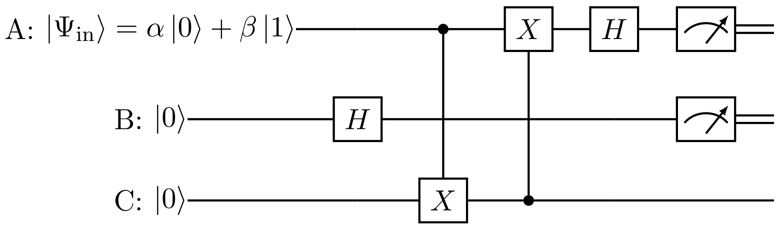


We were allowed to introduce this CNOT gate since $XH|0\rangle = H|0\rangle$. This actually makes it easier using the following problem:

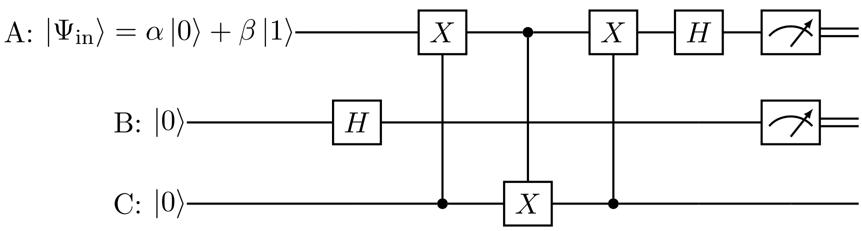
(h) **(2 pts.)** Prove that the below two circuits are equivalent.



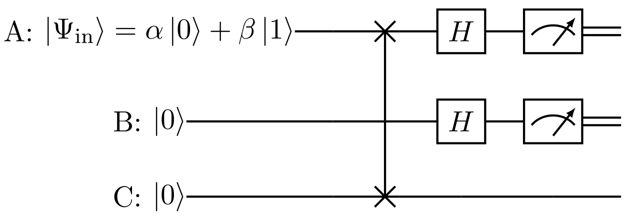
Using this substitution, we end up with:



We can now introduce another CNOT gate, which doesn't do anything since C will always be $|0\rangle$.



Three alternating CNOT gates is equivalent to the SWAP gate, so we can write:



where we clearly see a swapping that occurs between the top and bottom branch!

Building Quantum Computers

According to theoretical physcist David P. Divencenzo, there are five necessary (but not necessarily sufficient) criteria to build a quantum computer:

- A well-characterized qubit.
- The ability to initialize qubits.
- Long and relevant decoherence times.
- A "universal set" of quantum gates.
- The ability to measure qubits.

In this section, we will focus on how we can create qubits and how we can create a universal set of quantum gates. Consider two energy levels E_1, E_0 as the qubit states $|1\rangle, |0\rangle$ respectively. Assume that

$$E_1 = \frac{1}{2}\hbar\omega, \qquad E_0 = -\frac{1}{2}\hbar\omega.$$

Also assume that the qubit state is time varying, in the form of:

$$|\Psi(t)\rangle = A(t)\,|0\rangle + B(t)\,|1\rangle\,.$$

(i) **(14 pts.)** Using the above setup, show how we can implement the quantum gates X, Y, Z. *Hint:* The Schrodinger Equation tells us

$$i\hbar\frac{\partial}{\partial t}\,|\Psi(t)\rangle = \hat{\mathcal{H}}\,|\Psi(t)\rangle\,,$$

where $\hat{\mathcal{H}} = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}$.

Grading Scheme

(a) (0.5 pts) $|a|^2$

(b) Three answers were acceptable due to question ambiguity:

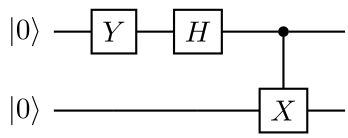
(i) (0.5 pts) 0

(ii) (0.5 pts) $\frac{1}{2}|a-b|^2$

(iii) (0.5 pts) $|a-b|^2$

(c) (1 pts) $|aU_{11} + bU_{12}|^2$

(d) (1 pts) Drawing/describing the below or equivalent:



To check equivalent, use website.

(e) (1 pts) Via either matrix multiplication or circuit analysis (or equivalent)

Note: The last step is the trickiest (where you have to deal with measurement, resulting in 4 cases). Make sure the participant checks all 4 cases or uses a clever method to cirumvent checking the cases. If they skip over the measurement step (or it's not clearly written), deduct 0.5 pts.

(f) (1 pts) Recognizing that classical bits need to be communicated via control wires so the qubit isn't teleported faster than light.

(1 pts) Stating that *nothing* can be recovered from the state before classical bits are communicated.

(1 pts) Proving the above statement. (This actually turns out to be much harder, so if the team attempts to prove the above, then give the point. The main idea is to recognize that simply stating you need control wires to prevent FTL communication is not sufficient)

(g) (2 pts) Complete proof. Partial proofs will receive 1 point.

(h) (1 pts) Complete proof. Partial proofs will receive 0.5 points.

(i) (1 pts) Complete proof. Partial proofs will receive 0.5 points.

(j)
- (1 pts) Sets up the system of ODE
- (2 pts) Solves the ODE
- (6 pts) Recognizes that the solution can be written in terms of Pauli matrices,

$$|\Psi(t)\rangle = \left(\cos(\omega t/2)\hat{I} + i\sin(\omega t/2)\hat{Z}\right)|\Psi(0)\rangle$$

- (5 pts) Shows how using the above form, we can create the X, Y, Z gates.

2023 Online Physics Olympiad: Open Contest

Sponsors

This competition could not be possible without the help of our sponsors, who are all doing great things in physics, math, and education.

Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

- Use $g = 9.8 \text{ m/s}^2$ in this contest, **unless otherwise specified**. See the constants sheet on the following page for other constants.
- This test contains 35 short answer questions. Each problem will have three possible attempts.
- The weight of each question depends on our scoring system found here. Put simply, the later questions are worth more, and the overall amount of points from a certain question decreases with the number of attempts that you take to solve a problem as well as the number of teams who solve it.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should contain **three** significant figures, unless otherwise specified. All answers within the 1% range will be accepted.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used. Technology and computer algebra systems like Wolfram Alpha or the one in the TI nSpire will not be needed or allowed. Attempts to use these tools will be classified as cheating.
- You are *allowed* to use Wikipedia or books in this exam. Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions.
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in base SI units (meter, second, kilogram, watt, etc.) unless otherwise specified. Please input all angles in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put units in your answer on the submission portal! If your answer is "x meters", input only the value x into the submission portal.
- **Do not communicate information to anyone else apart from your team-members before July 25, 2023**.

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$ C
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^{8}$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\mathrm{N} \cdot \mathrm{m}^2)/\mathrm{kg}^2$$
- Solar Mass
$$M_\odot = 1.988 \cdot 10^{30}\ \mathrm{kg}$$
- Acceleration due to gravity, $g = 9.8$ m/s^2
- 1 unified atomic mass unit,
$$1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15}\ \mathrm{eV \cdot s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}\ (\mathrm{N \cdot m^2})/\mathrm{C^2}$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T \cdot m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\mathrm{T \cdot m})/\mathrm{A}$$
- 1 atmospheric pressure,
$$1\ \mathrm{atm} = 1.01 \cdot 10^{5}\ \mathrm{N/m^2} = 1.01 \cdot 10^{5}\ \mathrm{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$$

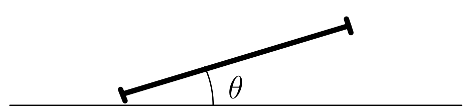
Problems

1. Coin Flip 1 The coin flip has long been recognized as a simple and unbiased method to randomly determine the outcome of an event. In the case of an ideal coin, it is well-established that each flip has an equal 50% chance of landing as either heads or tails.

However, coin flips are not entirely random. They appear random to us because we lack sufficient information about the coin's initial conditions. If we possessed this information, we would always be able to predict the outcome without needing to flip the coin. For an intriguing discussion on why this observation is significant, watch this video by Vsauce.

Now, consider a scenario where a coin with uniform density and negligible width is tossed directly upward from a height of $h = 0.75$ m above the ground. The coin starts with its heads facing upward and is given an initial vertical velocity of $v_y = 49$ m/s and a positive angular velocity of $\omega = \pi$ rad/s. What face does the coin display upon hitting the ground? **Submit 0 for heads and 1 for tails.** You only have one attempt for this problem. Assume the floor is padded and it absorbs all of the coin's energy upon contact. The radius of the coin is negligible.

2. Coin Flip 2 A coin of uniform mass density with a radius of $r = 1$ cm is initially at rest and is released from a slight tilt of $\theta = 8°$ onto a horizontal surface with an infinite coefficient of static friction. The coin has a thicker rim, allowing it to drop and rotate on one point. With every collision, the coin switches pivot points on the rim, and energy is dissipated through heat so that $k = 0.9$ of the coin's prior total energy is conserved. How long will it take for the coin to come to a complete stop?



A cross-sectional view of the coin before release. The rim can be seen on the edges of the coin.

3. Highway Suppose all cars on a (single-lane) highway are identical. Their length is $l = 4$m, their wheels have coefficients of friction $\mu = 0.7$, and they all travel at speed v_0. Find the v_0 which maximizes the flow rate of cars (i.e. how many cars travel across an imaginary line per minute). Assume that they need to be able to stop in time if the car in front instantaneously stops. Disregard reaction time.

4. Spinning Around Here is a Physoly round button badge, in which the logo is printed on the flat and rigid surface of this badge. Toss it in the air and track the motions of three points (indicated by cyan circles in the figure) separated a straight-line distance of $L = 5$ mm apart. At a particular moment, we find that these all have the same speed $V = 4$ cm/s but are heading to different directions which form an angle of $\theta = 30°$ between each pair. Determine the then angular velocity of the badge (in rad/s).

5. Born To Try In a resource-limited ecological system, a population of organisms cannot keep growing forever (such as lab bacteria growing inside culture tube). The effective growth rate g (including contributions from births and deaths) depends on the instantaneous abundance of resource $R(t)$, which in this problem we will consider the simple case of linear-dependency:

$$\frac{\mathrm{d}}{\mathrm{d}t}N = g(R)N = \alpha RN \ ,$$

where $N(t)$ is the population size at time t. The resources is consumed at a constant rate β by each organism:

$$\frac{\mathrm{d}}{\mathrm{d}t}R = -\beta N \ .$$

Initially, the total amount of resources is R_0 and the population size is N_0. Given that $\alpha = 10^{-9}$ resource-unit^{-1}s^{-1}, $\beta = 1$ resource-unit/s, $R_0 = 10^6$ resource-units and $N_0 = 1$ cell, find the total time it takes from the beginning to when all resources are depleted (in hours).

6. Lightbulb [This problem is removed from the test.]

7. Hyperdrive In hyperdrive, Spaceship-0 is relativistically moving at the velocity $\frac{1}{3}c$ with respect to reference frame R_1, as measured by Spaceship-1. Spaceship-1 is moving at $\frac{1}{2}c$ with respect to reference frame R_2, as measured by Spaceship-2. Spaceship-k is moving at speed $v_k = \frac{k+1}{k+3}c$ with respect to reference frame R_{k+1}. The speed of Spaceship-0 with respect to reference frame R_{20} can be expressed as a decimal fraction of the speed of light which has only x number of 9s following the decimal point (i.e., in the form of $0.\underbrace{99\ldots9}_{x \text{ times}}c$). Find the value of x.

8. Asteroid The path of an asteroid that comes close to the Earth can be modeled as follows: neglect gravitational effects due to other bodies, and assume the asteroid comes in from far away with some speed v and lever arm distance r to Earth's center. On January 26, 2023, a small asteroid called 2023 BU came to a close distance of 3541 km to Earth's surface with a speed of 9300 m/s. Although BU had a very small mass estimated to be about 300,000 kg, if it was much more massive, it could have hit the Earth. How massive would BU have had to have been to make contact with the Earth? Express your answer in scientific notation with 3 significant digits. Use 6357 km as the radius of the Earth. The parameters that remain constant when the asteroid mass changes are v and r, where v is the speed at infinity and r is the impact parameter.

9. Spaceship IK Pegasi and Betelgeuse are two star systems that can undergo a supernova. Betelgeuse is 548 light-years away from Earth and IK Pegasi is 154 light-years away from Earth. Assume that the two star systems are 500 light-years away from each other.

Astronomers on Earth observe that the two star systems undergo a supernova explosion 300 years apart. A spaceship, the *OPhO Galaxia Explorer* which left Earth in an unknown direction before the first supernova observes both explosions occur simultaneously. Assume that this spaceship travels in a straight line at a constant speed v. How far are the two star systems according to the *OPhO Galaxia Explorer* at the moment of the simultaneous supernovae? Answer in light-years.

Note: Like standard relativity problems, we are assuming intelligent observers that know the finite speed of light and correct for it.

10. Drag 1 A ball of mass 1 kg is thrown vertically upwards and it faces a quadratic drag with a terminal velocity of 20 m/s. It reaches a maximum height of 30 m and falls back to the ground. Calculate the energy dissipated until the point of impact (in J).

11. Drag 2 In general, we can describe the quadratic drag on an object by the following force law:

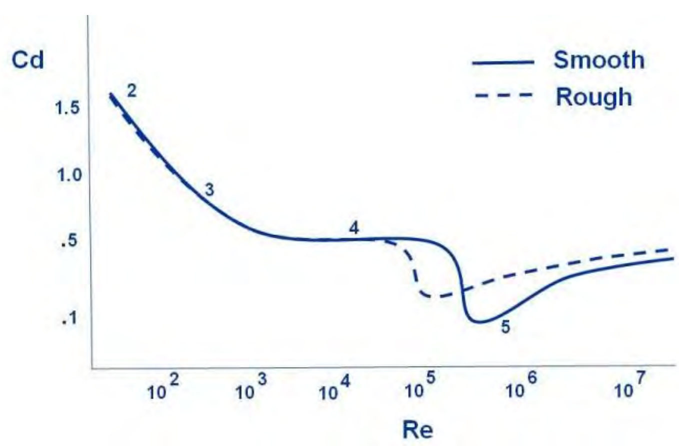
$$F_D = \frac{1}{2} C_D \rho A v^2$$

where A is the cross-sectional area of the object exposed to the airflow, v is the speed of the object in a fluid, and C_D is the drag coefficient, a dimensionless quantity that varies based on shape.

Another useful quantity to know is the Reynold's number, a dimensionless quantity that helps predict fluid flow patterns. It is given by the formula:

$$\text{Re} = \frac{\rho v L}{\mu}$$

where ρ is the density of the surrounding fluid, μ is the dynamic viscosity of the fluid, and L is a reference length parameter that varies based on each object. For a smooth [1] sphere traveling in a fluid, its diameter serves as the reference length parameter.



A logarithmic graph of C_D vs Re of a sphere from the NASA Glenn Research Center.

The relationship between the drag coefficient and the Reynold's number holds significant importance. Due to the complexity of fluid dynamics, empirical data is commonly used, as depicted in the figure provided above. Notably, the figure indicates a significant decrease in the drag coefficient around $\text{Re} \approx 4 \times 10^5$. This phenomenon, known as the drag crisis, occurs when a sphere transitions from laminar to turbulent flow, resulting in a broad wake and high drag. The table in the link below presents a range of C_d versus Re values of a smooth sphere.

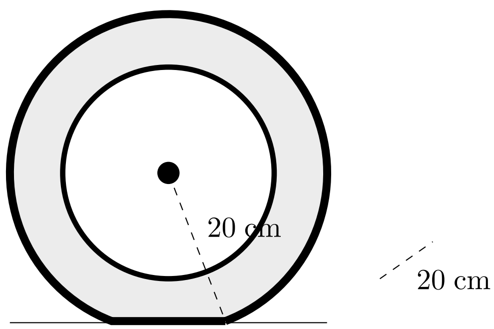
[1] meaning a smooth surface.

Desmos table: https://www.desmos.com/calculator/wnpkg5wnt0

Let's consider a smooth ball with a radius of 0.2 m and a mass of 0.1 kg dropped in air with a constant density of $\rho = 1.255$ kg/m^3. It is found that at velocity 5 m/s, the Reynold's number of the ball is $3.41 \cdot 10^5$. If the ball is dropped from rest, it approaches a *stable* terminal velocity v_1. If the ball is thrown downwards with enough velocity, it will experience turbulence, and approach a *stable* terminal velocity v_2. Find $\Delta v = v_2 - v_1$. Ignore any terminal velocities found for Reynold numbers less than an order of magnitude 10^{-1}.

Note: This problem is highly idealized as it assumes the atmosphere has air of constant density and temperature. In reality, this is not true!

The following information applies for the next two problems. Pictured is a wheel from a 4-wheeled car of weight 1200kg. The absolute pressure inside the tire is 3.0×10^5Pa. Atmospheric pressure is 1.0×10^5Pa. Assume the rubber has negligible "stiffness" (i.e. a negligibly low Sheer modulus compared to its Young's modulus).



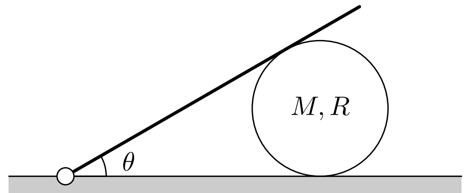
12. Hysteresis 1 The rubber on the bottom of the wheel is completely unstretched. The rubber has a thickness of 7 mm. Based on this information, find the Young's Modulus of the rubber. Is this answer reasonable?

13. Hysteresis 2 The rubber experiences a phenomena known as *hysteresis* – it takes more force to stretch the rubber than allow it to return to equilibrium. Specifically, assume that the Young's Modulus when the rubber is stretched is the answer to 12, and is 1/2 of that when the rubber returns to equilibrium. Compute the power the car's engines has to deliver to overcome the hysteresis losses, if the car moves at 20 m/s. Remember that there are 4 tires!

14. Marbles Two identical spherical marbles of radius 3 cm are placed in a spherical bowl of radius 10 cm. The coefficient of static friction between the two surfaces of the marble is 0.31 and the coefficient of static friction between the surfaces of the marbles and the bowl is 0.13. Find the maximum elevation from the bottom of the bowl that the center of one of the marbles can achieve in equilibrium. The bowl is fixed in place and will neither rotate nor translate. Equilibrium refers to stable equilibrium.

15. Fringe effect approximation Two parallel square plates of side length 1 m are placed a distance 30 cm apart whose centers are at $(-15 \text{ cm}, 0, 0)$ and $(15 \text{ cm}, 0, 0)$ have uniform charge densities -10^{-6} C/m^2 and 10^{-6} C/m^2 respectively. Find the magnitude of the component of the electric field perpendicular to axis passing through the centers of the two plates at $(10 \text{ cm}, 1 \text{ mm}, 0)$.

16. Sliding Along A hollow sphere of mass M and radius R is placed under a plank of mass $3M$ and length $2R$. The plank is hinged to the floor, and it initially makes an angle $\theta = \frac{\pi}{3}$ rad to the horizontal.



A not-to-scale picture of the sphere-plank setup.

Under the weight of the plank, the sphere starts rolling without slipping across the floor. What is the sphere's initial translational acceleration? Assume the plank is frictionless.

The following information applies for the next two problems. A space elevator consists of a heavy counterweight placed near geostationary orbit, a thread that connects it to the ground (assume this is massless), and elevators that run on the threads (also massless). The mass of the counterweight is 10^7 kg. Mass is continuously delivered to the counterweight at a rate of 0.001 kg/s. The elevators move upwards at a rate of 20 m/s. Assume there are many elevators, so their discreteness can be neglected. The elevators are massless. The counterweight orbits the Earth.

17. Space Elevator 1 Find the minimum possible displacement radially of the counterweight. Specify the sign.

18. Space Elevator 2 Assuming a radial displacement that is 10 times what you found in the previous part, find the displacement tangentially of the counterweight. Does it lag or lead Earth's motion?

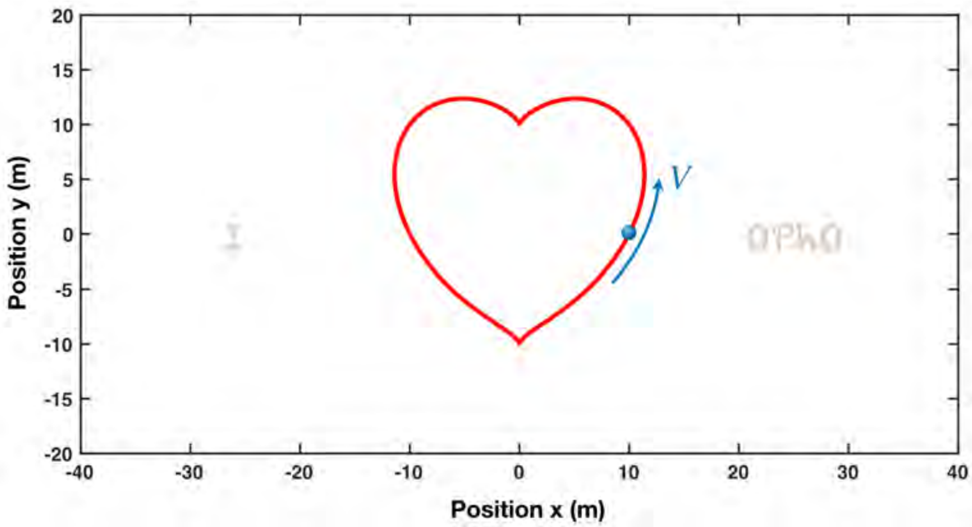
19. Laser Power Consider a spherical shell of thickness $\delta = 0.5$cm and radius $R = 5$cm made of an Ohmic material with resistivity $\rho = 10^{-7}\Omega$m. A spherical laser source with a tuned frequency of $f_0 = 3 \times 10^{12}$Hz and intensity $I_0 = 10^5$ W/m^2 is placed at the center of the shell and is turned on. Working in the limit $\frac{c}{f_0} \ll \delta \ll R$, approximate the initial average power dissipated by the shell. Neglect inductance.

20. Twinkle Twinkle. A stable star of radius R has a mass density profile $\rho(r) = \alpha(1 - r/R)$. Here, "stable" means that the star doesn't collapse under its own gravity. If the internal pressure at the core is provided solely by the radiation of photons, calculate the temperature at the core. Assume the star is a perfect black body and treat photons as a classical ideal gas. Use $R = 7 \times 10^5$ km and $\alpha = 3$ g/cm^3. Round your answer to the nearest kilokelvin. We treat photons as a classical gas here to neglect any relativistic effects.

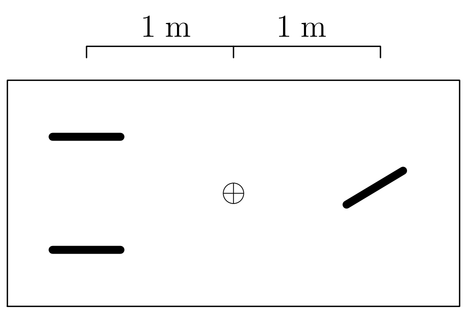
21. My Heart Will Go On On a flat playground, choose a Cartesian Oxy coordinate system (in unit of meters). A child running at a constant velocity $V = 1$m/s around a heart-shaped path satisfies the following order-6 algebraic equation:

$$\left(x^2 + y^2 - L^2\right)^3 - Lx^2y^3 = 0 \ , \ L = 10 \ .$$

When the child is at the position $(x, y) = (L, 0)$, what is the magnitude of their acceleration?



22. Tricycle A boy is riding a tricycle across along a sidewalk that is parallel to the x-axis. This tricycle contains three identical wheels with radius 0.5 m. The front wheel is free to rotate while the last two wheels are parallel to each other and to the main body of the tricycle. See the diagram.



The front wheel is rotating at a constant angular speed of $\omega = 3$ rad/s. The child is controlling the tricycle such that the front wheel is making an angle of $\theta(t) = 0.15\sin((0.1 \text{ rad/s})t)$ with the main body of the tricycle. Determine the maximum lateral acceleration in m/s^2. Assume a massless frame. The marked plus sign implies CoM. The degree is in radians.

23. Sonic Fryer In this problem, we consider a simple model for a thermoacoustic device. The device uses heavily amplified sound to provide work for a pump that can then extract heat. Sound waves form standing waves in a tube of radius 0.25 mm that is closed on both sides, and a two-plate stack is inserted in the tube. A temperature gradient forms between the plates of the stack, and the parcel of gas trapped between the plates oscillates sinusoidally between a maximum pressure of 1.03 MPa and a minimum of 0.97 MPa. The gas is argon, with density 1.78 kg/m^3 and adiabatic constant 5/3. The speed of sound is 323 m/s. The heat pump itself operates as follows:

The parcel of gas starts at minimum pressure. The stack plates adiabatically compress the parcel of gas to its maximum pressure, heating the gas to a temperature higher than that of the hotter stack plate. Then, the gas is allowed to isobarically cool to the temperature of the hotter stack plate. Next, the plates adiabatically expand the gas back to its minimum pressure, cooling it to a temperature lower than that of the colder plate. Finally, the gas is allowed to isobarically heat up to the temperature of the colder stack plate.

Find the power at which the thermoacoustic heat pump emits heat.

The following information applies for the next two problems. For your mass spectroscopy practical you are using an apparatus consisting of a solenoid enclosed by a uniformly charged hollow cylinder of charge density $\sigma = 50\ \mu\text{C/m}^2$ and radius $r_0 = 7$ cm. There exists an infinitesimal slit of insular material between the cylinder and solenoid to stop any charge transfer. Also, assume that there is no interaction between the solenoid and the cylinder, and that the magnetic field produced by the solenoid can be easily controlled to a value of B_0.

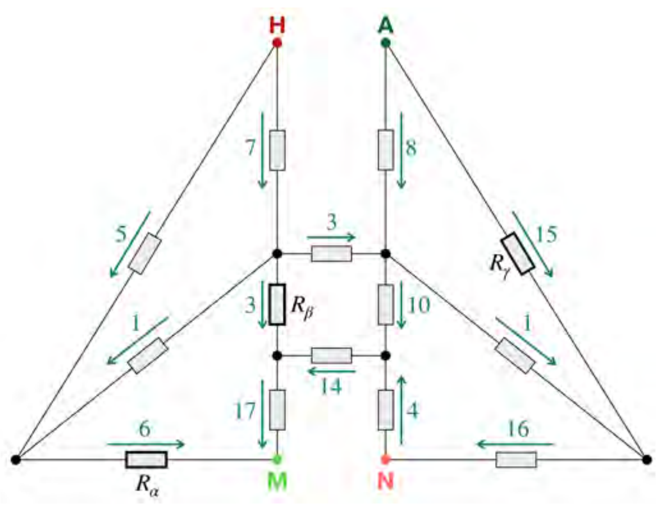
An electron is released from rest at a distance of $R = 10$ cm from the axis. Assume that it is small enough to pass through the cylinder in both directions without exchanging charge. It is observed that the electron reaches a distance R at different points from the axis 7 times before returning to the original position.

24. Isotope Separator 1 Calculate B_0 under the assumption that the path of the electron does not self-intersect with itself.

25. Isotope Separator 2 Calculate the time it took for the particle to return to original position. Answer in milliseconds.

> **Hint:** You may find interest in the Gaussian error function:
>
> $$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \text{d}t.$$
>
> Specific values of the error function can be calculated on desmos.

26. Somos El Barco For any circuit network made of batteries and resistors, if we know the voltages of all the batteries and the resistance values of all the resistors, we can calculate all the electrical currents. However, if we know the voltages of all the batteries and all the currents, it is still not enough to uniquely determine the resistance values of all the resistors. Consider a sail-shape circuit network, in which we connect points H and N with a $\mathcal{E}_{\text{HN}} = 10$V battery, points A and M with a $\mathcal{E}_{\text{AM}} = 20$V battery. The electrical currents in this network have directions and magnitudes (in mA) as shown the figure. The possible resistance values of resistors R_α, R_β, R_γ is not a single point (corresponds to an unique solution) but a confined region in the three-dimensional $(R_\alpha, R_\beta, R_\gamma)$-space. Determine the volume of this region (in Ω^3).



The title of this problem means "we are the boat".

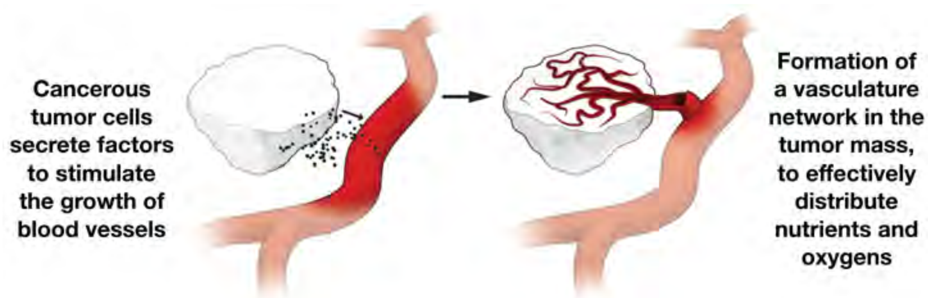
27. The Final Countdown A model of cancer tumor dynamics under a low-dose chemotherapy consists of three non-negative variables (P, Q, R), in which P represents the cancer tumor size, Q represents the (normalized) carrying capability of the tumor vasculature network, and R represents the local (normalized) activity of the immunology system:

$$\frac{\mathrm{d}}{\mathrm{d}t}P = \xi P \ln \frac{Q}{P} - \theta PR - \varphi_1 PC \quad ,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}Q = bP - \left(\mu + dP^{2/3}\right) Q - \varphi_2 QC \quad ,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}R = \alpha \left(P - \beta P^2\right) R + \gamma - \delta R + \varphi_3 RC \quad .$$

Here, C is the local concentration of chemotherapeutic agent at the tumor site, which we can assume to follow by a simple pharmacokinetics model:

$$\frac{\mathrm{d}}{\mathrm{d}t}C = -\frac{1}{\tau}C + U \quad ,$$

where U is the rate of chemotherapy drug administrated to the patient body. Let us assume an unchanging rate U, and treat it as another parameter of the model. All other unmentioned symbols are positive constant parameters, which values can be measured and should depends on the particular kinds of cancers and treatments. In total, there are 14 parameters – such a high-degree of complexity is very common in biophysical models. For each set of parameters, there can be many possible stationary states, which can be associated with various levels of malignancy. What is the maximum number of stationary non-zero tumor sizes (including both stable and unstable ones) for a set of parameters in this model? **For this problem, you can only submit your answer once.**



28. Magnetic Carts Two carts, each with a mass of 300 g, are fixed to move on a horizontal track. As shown in the figure, the first cart has a strong, tiny permanent magnet of dipole moment 0.5 $\mathrm{A} \cdot \mathrm{m}^2$ attached to it, which is aligned along the axis of the track pointing toward the other cart. On the second cart, a copper tube of radius 7 mm, thickness 0.5 mm, resistivity $1.73 \cdot 10^{-8}\Omega$, and length 30 cm is attached. The masses of the magnet and coil are negligible compared to the mass of the carts. At the moment its magnet enters through the right end of the copper tube, the velocity of the first cart is 0.3 m/s and the distance between the two ends of each cart is 50 cm, find the minimum distance achieved between the two ends of the carts in centimeters. While on the track, the carts experience an effective coefficient of static friction (i.e., what it would be as if they did not have wheels) of 0.01. Neglect the self-inductance of the copper tube.

A picture of the two cart setup. The black rectangle represents the magnet while the gold rectangle represents the copper tube.

Hint 1: The magnetic field due to a dipole of moment $\vec{\mu}$, at a position $\vec{r}$ away from the dipole can be written as

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{(3\hat{r}(\vec{r}\cdot\vec{\mu}) - \vec{\mu})}{r^3}\hat{r}$$

where $\hat{r}$ is the unit vector in the direction of $\vec{r}$.

Hint 2: The following mathematical identity may be useful:

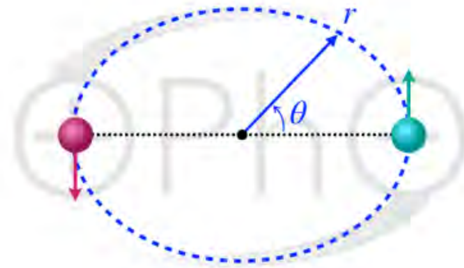
$$\int_{-\infty}^{\infty}\frac{u^2du}{(1+u^2)^5} = \frac{5\pi}{128}.$$

29. De-Terraforming In the far future, the Earth received an enormous amount of charge as a result of Mad Scientist ecilA's nefarious experiments. Specifically, the total charge on Earth is $Q = 1.0 \times 10^{11}$C. (compare this with the current 5×10^5C).
Estimate the maximum height of a "mountain" on Earth that has a circular base with diameter $w = 1.0$ km, if it has the shape of a spherical sector. You may assume that $h_{\max} \ll w$. The tensile strength of rock is 10 MPa.

30. All Around The World The logo of OPhO describes two objects travelling around their center-of-mass, following the same oval-shape trajectory. For simplicity, we assume these objects are point-like, have identical mass, and interacts via an interacting potential $U(d)$ depends on the distance d between them. Choose the polar coordinates (r, θ) as shown in the figure, where the origin is located at the center of the logo, then the shared trajectory obeys the equation:

$$r(\theta) = \frac{L}{2}\left[1 - \epsilon\cos(2\theta)\right]^{-(1+\gamma)} \ ,$$

in which we consider $\epsilon = 0.12$ and $\gamma = 0.05$. Here the smallest and largest separation between the objects are $d_{\min}$ and $d_{\max}$. Since the interacting potential $U(d)$ is defined up to a constant, let us pick $U(L) = 0$. Find the ratio $U(d_{\min})/U(d_{\max})$.

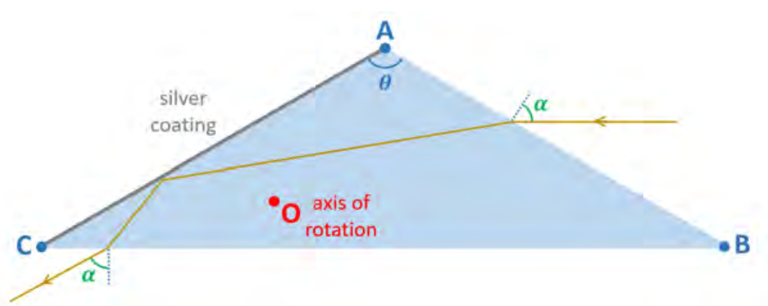


31. Electrostatic Pendulum 1 Follin is investigating the electrostatic pendulum. His apparatus consists of an insulating Styrofoam ball with a mass of 14 mg and radius $r = 0.5$ cm suspended on a uniform electrically-insulating string of length 1 m and mass per unit length density of $1.1 \cdot 10^{-5}$ kg/m between two large metal plates separated by a distance 17 cm with a voltage drop of $10kV$ between them, such that when the ball is in equilibrium, its center of mass is exactly equidistant to the two plates. Neglect the possibility of electrical discharge throughout the next two problems.

Follin then gives the ball a charge 0.15 nC. Assuming that the charge is distributed evenly across the surface of the ball, find the subsequent horizontal deflection of the pendulum bob's center of mass from its hanging point at equilibrium.

32. Electrostatic Pendulum 2 Hoping to get a larger deflection, Follin replaces the insulating Styrofoam ball with a conducting pith ball of mass 250 mg and 2 cm and daisy chains 4 additional 10 kV High Voltage Power Supplies to increase the voltage drop across the plates to 50 kV. Leaving the plate separation and the string unchanged, he repeats the same experiment as before, but forgets to measure the charge on the ball. Nonetheless, once the ball reaches equilibrium, he measures the deflection from the hanging point to be 5.6 cm. Find the charge on the ball.

The following information applies for the next two problems. Consider a uniform isosceles triangle prism ABC, with the apex angle $\theta = 110°$ at vertex A. One of the sides, AC, is coated with silver, allowing it to function as a mirror. When a monochrome light-ray of wavelength λ approaches side AB at an angle of incidence α, it first refracts, then reaches side AC, reflects, and continues to base BC. After another refraction, the ray eventually exits the prism at the angle of emergence which is also equal to the angle of incidence.



33. Man In The Mirror 1 What is the relative refractive index of the prism for that particular wavelength λ with respect to the outside environment, given that $\alpha = 70°$.

34. Man In The Mirror 2 Consider keeping the incident ray fixed while changing the monochrome color to a different wavelength λ', so that by rotating the prism around an axis of rotation O it can follows the above description (approaches side AB at an angle of incidence α', refracts, then reaches side AC, reflects, and continues to base BC, then another refraction, the ray eventually exits the prism at the angle of emergence which is also equal to the angle of incidence). We observe then the emergent ray remains unchanged for any value of λ'. Find the maximum possible length-ratio between the distance from the axis to the one of the vertices $\max(\mathrm{OA}, \mathrm{OB}, \mathrm{OC})$ and the base BC.

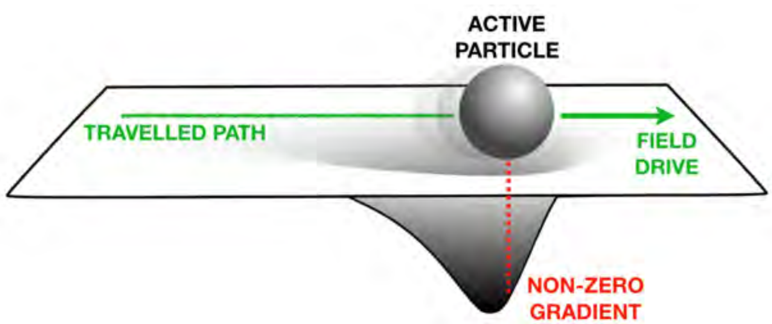
35. Funiculì, Funiculà Field-drive is a locomotion mechanism that is analogous to general relativistic warp-drive. In this mechanism, an active particle continuously climbs up the field-gradient generated by its own influence on the environment so that the particle can bootstrap itself into a constant non-zero velocity motion. Consider a field-drive in one-dimensional (the Ox axis) environment, where the position of the particle at time t is given by $X(t)$ and its instantaneous velocity follows from:

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = \kappa\frac{\partial}{\partial x}R(x,t)\Big|_{x=X(t)} ,$$

in which κ is called the guiding coefficient and $R(x,t)$ is the field-value in this space. Note that, the operation $...|_{x=X(t)}$ means you have to calculate the part in ... first, then replace x with $X(t)$. For a biological example, the active particle can be a cell, the field can be the nutrient concentration, and the strategy of climbing up the gradient can be chemotaxis. The cell consumes the nutrient and also responses to the local nutrient concentration, biasing its movement toward the direction where the concentration increases the most. If the nutrient is not diffusive and always recovers locally (e.g. a surface secretion) to the value which we defined to be 0, then its dynamics can usually be approximated by:

$$\frac{\partial}{\partial t}R(x,t) = -\frac{1}{\tau}R(x,t) - \gamma\exp\left\{-\frac{[x-X(t)]^2}{2\lambda^2}\right\} ,$$

where τ is the timescale of recovery, γ is the consumption, and λ is the characteristic radius of influence. Before we inoculate the cell into the environment, $R = 0$ everywhere at any time. What is the smallest guiding coefficient κ (in $\mu\mathrm{m}^2/\mathrm{s}$) for field-drive to emerge, if the parameters are $\tau = 50\mathrm{s}$, $\gamma = 1\mathrm{s}^{-1}$, and $\lambda = 10\mu\mathrm{m}$.



The title of this problem means "funicular up, funicular down".

2023 Online Physics Olympiad: Open Contest

Sponsors

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Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

- Use $g = 9.8$ m/s^2 in this contest, **unless otherwise specified**. See the constants sheet on the following page for other constants.
- This test contains 35 short answer questions. Each problem will have three possible attempts.
- The weight of each question depends on our scoring system found here. Put simply, the later questions are worth more, and the overall amount of points from a certain question decreases with the number of attempts that you take to solve a problem as well as the number of teams who solve it.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should contain **three** significant figures, unless otherwise specified. All answers within the 1% range will be accepted.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used. Technology and computer algebra systems like Wolfram Alpha or the one in the TI nSpire will not be needed or allowed. Attempts to use these tools will be classified as cheating.
- You are *allowed* to use Wikipedia or books in this exam. Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions.
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in base SI units (meter, second, kilogram, watt, etc.) unless otherwise specified. Please input all angles in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put units in your answer on the submission portal! If your answer is "x meters", input only the value x into the submission portal.
- **Do not communicate information to anyone else apart from your team-members before July 25, 2023**.

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}\ \text{mol}^{-1}$
- Universal gas constant, $R = 8.31$ J/(mol · K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$ C
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^{8}$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\text{N} \cdot \text{m}^2)/\text{kg}^2$$
- Solar Mass
$$M_\odot = 1.988 \cdot 10^{30}\ \text{kg}$$
- Acceleration due to gravity, $g = 9.8\ \text{m/s}^2$
- 1 unified atomic mass unit,
$$1\ \text{u} = 1.66 \cdot 10^{-27}\ \text{kg} = 931\ \text{MeV/c}^2$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \text{J} \cdot \text{s} = 4.41 \cdot 10^{-15}\ \text{eV} \cdot \text{s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \text{C}^2/(\text{N} \cdot \text{m}^2)$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}\ (\text{N} \cdot \text{m}^2)/\text{C}^2$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7}\ \text{T} \cdot \text{m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\text{T} \cdot \text{m})/\text{A}$$
- 1 atmospheric pressure,
$$1\ \text{atm} = 1.01 \cdot 10^{5}\ \text{N/m}^2 = 1.01 \cdot 10^{5}\ \text{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m · K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \text{W/m}^2/\text{K}^4$$

Problems

We thank contestants (in no order) Baiyu Zhu, David Lee, Ong Zhi Zheng, Guangyuan Chen, and Nathan Zhao for contributing some solutions.

1. Coin Flip 1 The coin flip has long been recognized as a simple and unbiased method to randomly determine the outcome of an event. In the case of an ideal coin, it is well-established that each flip has an equal 50% chance of landing as either heads or tails.

However, coin flips are not entirely random. They appear random to us because we lack sufficient information about the coin's initial conditions. If we possessed this information, we would always be able to predict the outcome without needing to flip the coin. For an intriguing discussion on why this observation is significant, watch this video by Vsauce.

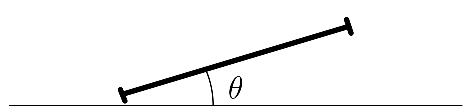
Now, consider a scenario where a coin with uniform density and negligible width is tossed directly upward from a height of $h = 0.75$ m above the ground. The coin starts with its heads facing upward and is given an initial vertical velocity of $v_y = 49$ m/s and a positive angular velocity of $\omega = \pi$ rad/s. What face does the coin display upon hitting the ground? **Submit 0 for heads and 1 for tails.** You only have one attempt for this problem. Assume the floor is padded and it absorbs all of the coin's energy upon contact. The radius of the coin is negligible.

Solution 1: We have the following quadratic:

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$
$$0 = 0.75 + 49t - 4.9t^2$$
$$t = -0.01, 10.01$$

The first solution is extraneous so $t = 10.01$ is correct. Now, $\theta = \omega t \approx 10\pi$. As one full rotation is $\phi = 2\pi$, then the coin performs 5 full rotations before landing on the ground. This means the answer is $\boxed{0}$, or heads.

2. Coin Flip 2 A coin of uniform mass density with a radius of $r = 1$ cm is initially at rest and is released from a slight tilt of $\theta = 8°$ onto a horizontal surface with an infinite coefficient of static friction. The coin has a thicker rim, allowing it to drop and rotate on one point. With every collision, the coin switches pivot points on the rim, and energy is dissipated through heat so that $k = 0.9$ of the coin's prior total energy is conserved. How long will it take for the coin to come to a complete stop?



A cross-sectional view of the coin before release. The rim can be seen on the edges of the coin.

Solution 2: By the parallel axis theorem, the moment of inertia of the coin around the pivot can be expressed as

$$I = I_x + m\ell^2 = \frac{1}{4}mr^2 + mr^2 = \frac{5}{4}mr^2.$$

As θ is small, that means the perpendicular force of gravity is $mg\cos\theta \approx mg$. Hence, Newton's second law to find the angular acceleration as

$$I\alpha = \tau \implies \alpha = \frac{4g}{5r} \cdot (1 \text{ rad})$$

Using rotational kinematics, the time for the first collision will be $\theta = \frac{1}{2}\alpha t^2 \implies t_0 = \sqrt{\frac{5r\theta_0}{2g}}$. By conservation of energy, the angular velocity at the time of collision is $\frac{1}{2}I\omega_0^2 = mgr\sin\theta \approx mgr$, meaning that $\omega_0 = \sqrt{\frac{8g\theta_0}{5r}}$. Now consider k. This is the ratio of the initial energy E_{n-1} and next energy E_n. Then, we can say that

$$k = \frac{\frac{1}{2}I\omega_n^2}{\frac{1}{2}I\omega_{n-1}^2} \implies \omega_n = \sqrt{k}\omega_{n-1}.$$

By recurrence, $\omega_n = k^{n/2}\omega_0$. The time of flight for each cycle will be $t_n = \frac{2\omega_n}{\alpha}$. Therefore,

$$\begin{aligned}
T &= t_0 + \sum_{n=1}^{\infty} t_n \\
&= \sqrt{\frac{5r\theta_0}{2g}} + \sum_{n=1}^{\infty}\left(2k^{n/2}\sqrt{\frac{8g\theta_0}{5r}} \cdot \frac{5r}{4g}\right) \\
&= \sqrt{\frac{5r\theta_0}{2g}} + \sqrt{\frac{10r\theta_0}{g}}\frac{k^{1/2}}{1-k^{1/2}}
\end{aligned}$$

Converting $8°$ to $\frac{2\pi}{45}$ radians, $k = 0.9$, $g = 9.8$ m/s^2, $r = 0.01$ m, we find that $\boxed{T = 0.716 \text{ s}}$ which is a reasonable estimate.

Alternate: This problem can also be solved by creating a differential equation for the height h of the center of mass of the coin and finding the time for $h \to 0$.

3. Highway Suppose all cars on a (single-lane) highway are identical. Their length is $l = 4$m, their wheels have coefficients of friction $\mu = 0.7$, and they all travel at speed v_0. Find the v_0 which maximizes the flow rate of cars (i.e. how many cars travel across an imaginary line per minute). Assume that they need to be able to stop in time if the car in front instantaneously stops. Disregard reaction time.

Solution 3: Suppose the maximum speed is v'. Notice that $a = \mu g$ so it takes a time $t' = \frac{v'}{\mu g}$ amount of time to stop. This means the car will travel a distance of

$$d = \frac{1}{2}\mu g\left(\frac{v'}{\mu g}\right)^2 = \frac{v'^2}{2\mu g}.$$

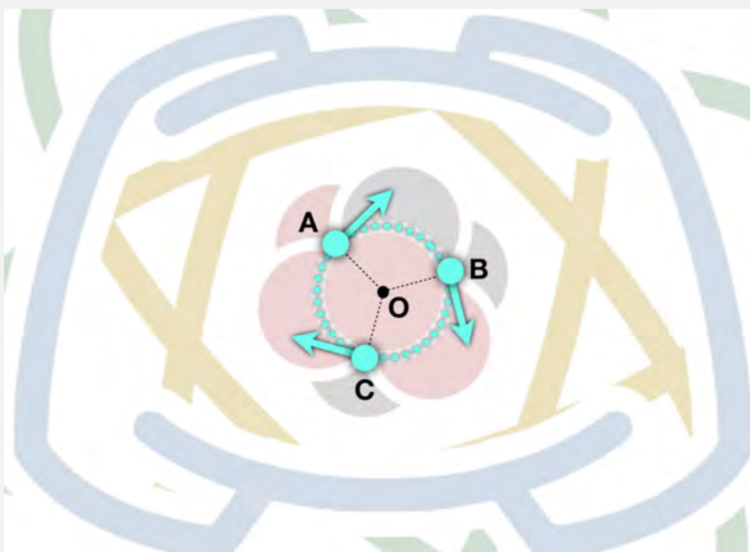
Thus for every distance $d + l = l + \frac{v'^2}{2\mu g}$ there is a car. This means in unit time t, there will be $N = \frac{d+l}{v}$ cars that passes through the line. To maximize the flow rate, we need to maximize

$$f(v') = \frac{l}{v'} + \frac{v'}{2\mu g}$$

$$f' = \frac{-l}{v'^2} + \frac{1}{2\mu g} = 0 \Rightarrow v' = \sqrt{2\mu g l} = \boxed{7.41 \text{ m/s}}$$

4. Spinning Around Here is a Physoly round button badge, in which the logo is printed on the flat and rigid surface of this badge. Toss it in the air and track the motions of three points (indicated by cyan circles in the figure) separated a straight-line distance of $L = 5$ mm apart. At a particular moment, we find that these all have the same speed $V = 4$ cm/s but are heading to different directions which form an angle of $\theta = 30°$ between each pair. Determine the then angular velocity of the badge (in rad/s).

Solution 4: Call the three tracking points on the Physoly badge A, B, C, and their geometrical center O. The distance from O to these three points are the same and equal to $L/\sqrt{3}$.



Due to symmetry, the velocity vector of O has to be perpendicular to the ABC plane. In the reference frame of O, the points A, B, C both have the same speed $2V\sin(\theta/2)/\sqrt{3}$ but are heading to different directions which form an angle of 120° between each pair. Also due to symmetry, the axis of rotation has to be perpendicular to the ABC plane, thus the velocity vectors of points A, B, C in O reference frame looks like described in the attached figure. For $L = 5$mm, $V = 4$cm/s, $\theta = 30° = \pi/12$, the angular velocity of the badge can be calculated as:

$$\Omega = \frac{2V\sin(\theta/2)/\sqrt{3}}{L/\sqrt{3}} = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)\frac{V}{L} \approx \boxed{4.1411\text{rad/s}}$$

5. Born To Try In a resource-limited ecological system, a population of organisms cannot keep growing forever (such as lab bacteria growing inside culture tube). The effective growth rate g (including contributions from births and deaths) depends on the instantaneous abundance of resource $R(t)$, which in this problem we will consider the simple case of linear-dependency:

$$\frac{\mathrm{d}}{\mathrm{d}t}N = g(R)N = \alpha RN \ ,$$

where $N(t)$ is the population size at time t. The resources is consumed at a constant rate β by each organism:

$$\frac{\mathrm{d}}{\mathrm{d}t}R = -\beta N \ .$$

Initially, the total amount of resources is R_0 and the population size is N_0. Given that $\alpha = 10^{-9}$ resource-unit^{-1}s^{-1}, $\beta = 1$ resource-unit/s, $R_0 = 10^6$ resource-units and $N_0 = 1$ cell, find the total time it takes from the beginning to when all resources are depleted (in hours).

Solution 5:
We can find the analytical solution for the following set of two ODEs describing the population-resource dynamics:

$$\frac{dN}{dt} = \alpha RN \ , \tag{1}$$

$$\frac{dR}{dt} = -\beta N \ . \tag{2}$$

Divide Eq.(1) by Eq.(2) on both sides, we get a direct relation between the population size $N(t)$ and the amount of resource $R(t)$:

$$\frac{dN}{dR} = -\frac{\alpha}{\beta}R \implies N = N_0 + \frac{\alpha}{2\beta}\left(R_0^2 - R^2\right) \ .$$

Plug this in Eq.(2), we obtain the total time T it takes from the beginning to when the resource is depleted:

$$\frac{dR}{dt} = -\frac{\alpha}{2}\left[\left(\frac{2\beta}{\alpha}N_0 + R_0^2\right) - R^2\right] \implies t\Big|_{R=0} = \frac{2}{\alpha}\int_0^{R_0} dR\left[\left(\frac{2\beta}{\alpha}N_0 + R_0^2\right) - R^2\right]^{-1}$$

$$= \frac{2}{\alpha\sqrt{\frac{2\beta}{\alpha}N_0 + R_0^2}}\operatorname{arctanh}\left(\frac{R_0}{\sqrt{\frac{2\beta}{\alpha}N_0 + R_0^2}}\right) \ .$$

Use the given numerical values, we arrive at $t\Big|_{R=0} \approx 7594.3\text{s} \approx \boxed{2.1095\text{hrs}}$.

6. Lightbulb An incandescent lightbulb is connected to a circuit which delivers a maximum power of 10 Watts. The filament of the lightbulb is made of Tungsten and conducts electricity to produce light. The specific heat of Tungsten is $c = 235$ J/(K $\cdot$ kg). If the circuit is alternating such that the temperature inside the lightbulb fluctuates between $T_0 = 3000°$ C and $T_1 = 3200°$ C at a frequency of $\omega = 0.02$ s^{-1}, estimate the mass of the filament.

Solution 6: This problem was voided from the test because we gave no method for energy dissipation. Therefore, there was ambiguity and energy would be constantly fed to the lightbulb making it hotter and hotter.

However, the problem can still be solved. Here is a solution written by one of our contestants

Guangyuan Chen. The circuit continuously delivers sinusoidal power to the lightbulb. In order to maintain a quasi-equilibrium state, there must be some form of heat loss present in the system. We will assume that the main source of heat loss is through radiation, though we will see that the precise form of heat loss is largely inconsequential.
Let the power delivered by the circuit be in the form $P_d = P_0 \sin^2 \Omega t$. Let the power loss due to the radiation be $P_l = -kT^4$ where k is some constant and T is the temperature of the filament. We can write:

$$mc\frac{\mathrm{d}T}{\mathrm{d}t} = P_0 \sin^2 \Omega t - kT^4$$

This is a differential equation that cannot be solved by hand. However, since the deviation in temperature from the equilibrium of roughly 100 °C is much smaller than the equilibrium temperature of roughly 3100 °C, we can do a first order approximation of the differential equation. The temperature thus varies sinusoidally with equilibrium temperature $T_{equi} = 3100\,^\circ\mathrm{C}$ and amplitude $T_a = 100\,^\circ\mathrm{C}$. Since the average power delivered is $\frac{1}{2}P_0$, at equilibrium temperature, we have:

$$\frac{1}{2}P_0 = kT_{equi}^4$$

$$k = \frac{P_0}{2T_{equi}^4}$$

We rewrite equation 1 with the approximation $T = T_{equi} + \Delta T$ and with the substitution from equation 3.

$$mc\frac{\mathrm{d}\Delta T}{\mathrm{d}t} = P_0 \sin^2 \Omega t - \frac{P_0}{2T_{equi}^4}(T_{equi} + \Delta T)^4$$

$$mc\frac{\mathrm{d}\Delta T}{\mathrm{d}t} \approx P_0 \sin^2 \Omega t - \frac{P_0}{2}\left(1 + 4\frac{\Delta T}{T_{equi}}\right)$$

$$mc\frac{\mathrm{d}\Delta T}{\mathrm{d}t} \approx -\frac{P_0}{2}\left(\cos 2\Omega t + 4\frac{\Delta T}{T_{equi}}\right)$$

At this point we can see that regardless of the physical mechanism of heat loss, a linear approximation can always be made for sufficiently small variations in temperature which results in an equation of the same form as equation 6. To solve this equation, we can guess the following solution:

$$\Delta T = A \sin 2\Omega t + B \cos 2\Omega t$$

Strictly speaking, we also need to include a term $Ce^{-\lambda t}$. However, for sufficiently long times approaching quasi-equilibrium, this term will go to zero, so we need not include it here. Substituting equation 7 into equation 6:

$$2mc\Omega(A \cos 2\Omega t - B \sin 2\Omega t) = -\frac{P_0}{2}\left(\cos 2\Omega t + \frac{4}{T_{equi}}(A \sin 2\Omega t + B \cos 2\Omega t)\right)$$

Equating the coefficients of the sin and cos:

$$2mc\Omega A = -\frac{P_0}{2}\left(1 + \frac{4B}{T_{equi}}\right)$$

$$-2mc\Omega B = -\frac{2P_0 A}{T_{equi}}$$

Equations 9 and 10 can be solved simultaneously to give

$$A = -\frac{mc\Omega T_{equi} P_0}{4((mc\Omega T_{equi})^2 + P_0^2)} T_{equi}$$

$$B = -\frac{P_0^2}{4((mc\Omega T_{equi})^2 + P_0^2)} T_{equi}$$

Equation 7 can be written in the form $T_a \sin(2\Omega t + \phi)$. To find T_a, we simply have:

$$T_a = \sqrt{A^2 + B^2} = \frac{P_0}{4\sqrt{(mc\Omega T_{equi})^2 + P_0^2}} T_{equi}$$

Rearranging for m, we have:

$$m = \frac{P_0}{c\Omega T_{equi}} \sqrt{\left(\frac{T_{equi}}{4T_a}\right)^2 - 1}$$

The final step is to relate Ω and ω. Since T oscillates with angular frequency 2Ω, its period of oscillation is $\frac{2\pi}{2\Omega}$. This is equal to $\frac{1}{\omega}$, hence we have:

$$\Omega = \pi\omega$$

Substituting into equation 14, we get our final numerical answer.

$$m = \frac{P_0}{\pi c\omega T_{equi}} \sqrt{\left(\frac{T_{equi}}{4T_a}\right)^2 - 1} = \boxed{1.68 \times 10^{-3}\,\text{kg}}$$

7. Hyperdrive In hyperdrive, Spaceship-0 is relativistically moving at the velocity $\frac{1}{3}c$ with respect to reference frame R_1, as measured by Spaceship-1. Spaceship-1 is moving at $\frac{1}{2}c$ with respect to reference frame R_2, as measured by Spaceship-2. Spaceship-k is moving at speed $v_k = \frac{k+1}{k+3}c$ with respect to reference frame R_{k+1}. The speed of Spaceship-0 with respect to reference frame R_{20} can be expressed as a decimal fraction of the speed of light which has only x number of 9s following the decimal point (i.e., in the form of $0.\underbrace{99\ldots9}_{x \text{ times}}c$). Find the value of x.

Solution 7: Let us define the rapidity as

$$\tanh\phi \equiv \beta = \frac{v}{c}$$

where tanh is the hyperbolic tangent function. Let $\beta_1 = \tanh\phi_1$ and $\beta_2 = \tanh\phi_2$. If we add β_1 and β_2 using the relativistic velocity addition formula, we find that

$$\beta = \frac{\beta_1 + \beta_2}{1 - \beta_1\beta_2} = \frac{\tanh\phi_1 + \tanh\phi_2}{1 + \tanh\phi_1 \tanh\phi_2} = \tanh(\phi_1 + \phi_2).$$

We can then rewrite the problem as

$$v_f = \tanh\left(\text{arctanh}\frac{1}{3} + \text{arctanh}\frac{2}{4} + \cdots + \text{arctanh}\frac{20}{22}\right).$$

Using the fact that $\text{arctanh}(\phi) = \frac{1}{2}\ln\left(\frac{1+\phi}{1-\phi}\right)$, we can find that

$$v_f = \tanh\left(\frac{1}{2}\sum_{k=0}^{19}\ln\left(\frac{1+\frac{k+1}{k+3}}{1-\frac{k+1}{k+3}}\right)\right) = \tanh\left(\frac{1}{2}\sum_{k=0}^{19}\ln(k+2)\right)$$
$$= \tanh\left(\ln\sqrt{2\cdot 3\cdot 4\cdots 21}\right) = \tanh\left(\ln\sqrt{21!}\right)$$

As $\tanh\phi = (e^{\phi} - e^{-\phi})/(e^{\phi} + e^{-\phi})$, then

$$\tanh(\ln(\phi)) = \frac{\phi - \frac{1}{\phi}}{\phi + \frac{1}{\phi}} = 1 - \frac{2}{\phi^2 + 1} \implies v_f = 1 - \frac{2}{21! + 1}$$

This implies $\boxed{19}$ zeros, but you can also use Stirlings approximation to further approximate the factorial.

Alternate 1: Define u_k as Spaceship 0's velocity in frame R_k, given $c = 1$. The recurrence relation from the relativistic velocity addition formula is: $u_{k+1} = \frac{u_k(k+3)+(k+1)}{u_k(k+1)+(k+3)}$, starting with $u_1 = \frac{1}{3}$. The relation can be simplified as: $u_{k+1} = 1 + \frac{2(u_k-1)}{u_k(k+1)+(k+3)}$. Introducing $v_k = u_k - 1$, $v_{k+1} = v_k\frac{2}{v_k(k+1)+(2k+4)}$. Further simplifying with $w_k = \frac{1}{v_k}$, we get $w_{k+1} = w_k(k+2) + \frac{k+1}{2}$. By setting $x_k = w_k + c$, we find $c = -\frac{1}{2}$ simplifies to $x_{k+1} = x_k(k+2)$. This gives $x_k = (k+1)(k)(k-1)...(4)(3)x_1$ and using the initial condition $x_1 = \frac{1}{u_1-1} + \frac{1}{2} = 1$, we obtain $x_k = \frac{(k+1)!}{2}$. Consequently, $u_k = \frac{(k+1)!-1}{(k+1)!+1}$ and substituting $k = 20$, $u_{20} \approx 1 - 3.9\times 10^{-20}$, yielding $\boxed{19}$ significant digits.

Alternate 2: Let $l = k + 2$, then $u_l = \frac{l-1}{l+1}$. Let $u_m = \frac{m-1}{m+1}$. Then you can find that the velocity addition of any l, m will be $\frac{ml-1}{ml+1}$. Using this identity, we can use recurrence to find that $u_k = \frac{(k+1)!-1}{(k+1)!+1}$.

8. Asteroid The path of an asteroid that comes close to the Earth can be modeled as follows: neglect gravitational effects due to other bodies, and assume the asteroid comes in from far away with some speed v and lever arm distance r to Earth's center. On January 26, 2023, a small asteroid called 2023 BU came to a close distance of 3541 km to Earth's surface with a speed of 9300 m/s. Although BU had a very small mass estimated to be about 300,000 kg, if it was much more massive, it could have hit the Earth. How massive would BU have had to have been to make contact with the Earth? Express your answer in scientific notation with 3 significant digits. Use 6357 km as the radius of the Earth. The parameters that remain constant when the asteroid mass changes are v and r, where v is the speed at infinity and r is the impact parameter.

Solution 8:
Let $v_1 = 9300$ m/s, $d = 3541$ km, and $m = 300,000$ kg, and let M and R be the Earth's mass and radius.
First we find v and r. We use the reference frame of the Earth, where the asteroid has reduced mass $\mu = \frac{Mm}{M+m}$ and the Earth has mass $M + m$. Then by energy and angular momentum conservation, we have

$$\mu v r = \mu v_1(R + d)$$

and

$$\frac{1}{2}\mu v^2 = \frac{1}{2}\mu v_1^2 - \frac{GMm}{R+d}.$$

We solve for

$$v = \sqrt{2G(M+m) \cdot \frac{R+d}{r^2-(R+d)^2}},$$

so

$$v_1 = \sqrt{\frac{2G(M+m)}{R+d} \cdot \frac{r^2}{r^2-(R+d)^2}},$$

and we compute $r = 37047$ km and $v = 2490$ m/s.
Now we consider when the asteroid is massive enough to touch the Earth. We let m' and μ' be the mass of the asteroid and its reduced mass, and using a similar method to above, we arrive at

$$v = \sqrt{2G(M+m') \cdot \frac{R}{r^2-R^2}},$$

so we can solve for $m' = 3.74 \times 10^{24}$ kg.

9. Spaceship IK Pegasi and Betelgeuse are two star systems that can undergo a supernova. Betelgeuse is 548 light-years away from Earth and IK Pegasi is 154 light-years away from Earth. Assume that the two star systems are 500 light-years away from each other.

Astronomers on Earth observe that the two star systems undergo a supernova explosion 300 years apart. A spaceship, the *OPhO Galaxia Explorer* which left Earth in an unknown direction before the first supernova observes both explosions occur simultaneously. Assume that this spaceship travels in a straight line at a constant speed v. How far are the two star systems according to the *OPhO Galaxia Explorer* at the moment of the simultaneous supernovae? Answer in light-years.

Note: Like standard relativity problems, we are assuming intelligent observers that know the finite speed of light and correct for it.

Solution 9: For any inertial observer, define the 4-distance between two events as $\Delta s^\mu = (c\Delta t, \Delta\mathbf{x})$, where Δt and $\Delta\mathbf{x}$ are the temporal and spatial intervals measured by the observer. By the properties of 4-vectors, the following quantity is Lorentz-invariant:

$$\Delta s^\mu \Delta s_\mu = c^2\Delta t^2 - \|\Delta\mathbf{x}\|^2$$

In the spaceship's frame, this is equal to $\Delta s^\mu \Delta s_\mu = -\|\Delta\mathbf{x}'\|^2$, since the two supernovas are simultaneous; hence

$$\|\Delta\mathbf{x}'\| = \sqrt{\|\Delta\mathbf{x}\|^2 - c^2\Delta t^2}$$

Since the observers have already taken into account the delay due to the nonzero speed of light, $c\Delta t = 300\text{ly}$, $\|\Delta\mathbf{x}\| = 500\text{ly}$, and

$$\|\Delta\mathbf{x}'\| = \sqrt{\|\Delta\mathbf{x}\|^2 - c^2\Delta t^2} = 400\text{ly}$$

10. Drag 1 A ball of mass 1 kg is thrown vertically upwards and it faces a quadratic drag with a terminal velocity of 20 m/s. It reaches a maximum height of 30 m and falls back to the ground. Calculate the energy dissipated until the point of impact (in J).

Solution 10: If we suppose that the magnitude of quadratic drag is $F_{\text{drag}} = cv^2$ for some constant c, then when the ball is falling downwards at terminal velocity v_t this upwards drag force must cancel gravity to provide zero net force:

$$mg = cv_t^2 \implies c = \frac{mg}{v_t^2}$$

so we can rewrite our equations of motion in terms of terminal velocity. In the ascending portion of the ball's trajectory, suppose that the ball's speed changes from v to $v + dv$ after traveling an infinitesimal distance from h to $h + dh$. Then:

$$d(\text{kinetic energy}) = -d(\text{potential energy and energy lost to drag}) \implies mv\,dv = -\frac{mgv^2}{v_t^2}\,dh - mg\,dh$$

where we used chain rule and $dW = F \cdot dx$. This is a separable differential equation for velocity in terms of height (!), which we can solve:

$$\int \frac{v\,dv}{1 + \frac{v^2}{v_t^2}} = \int -g\,dh + \text{constant} = \text{constant} - gh$$

We u-sub the entire denominator ($u = 1 + v^2/v_t^2 \implies du = 2v/v_t^2\,dv$), which nicely cancels the numerator. Our left-hand integral is

$$\frac{v_t^2}{2}\int \frac{du}{u} = \frac{v_t^2}{2}\ln\left|1 + \frac{v^2}{v_t^2}\right| = \text{constant} - gh$$

Now to find the constant! At our maximum height $h_0 = 30$ m, the speed and left-hand side are zero, so the constant must be gh_0. Then we can find the initial speed of the projectile by substituting $h = 0$, from which:

$$v_{\text{initial}} = v_t\sqrt{e^{\frac{2gh_0}{v_t^2}} - 1}$$

We can use a similar argument for the downwards trajectory, albeit with drag pointing upwards. We should get that the final speed immediately before impact is

$$v_{\text{final}} = v_t\sqrt{1 - e^{\frac{-2gh_0}{v_t^2}}}$$

from which the dissipated energy is the difference in final and initial kinetic energies, or

$$\frac{1}{2}m\left(v_{\text{final}}^2 - v_{\text{initial}}^2\right) \approx 515.83 \text{ joules}$$

11. Drag 2 In general, we can describe the quadratic drag on an object by the following force law:

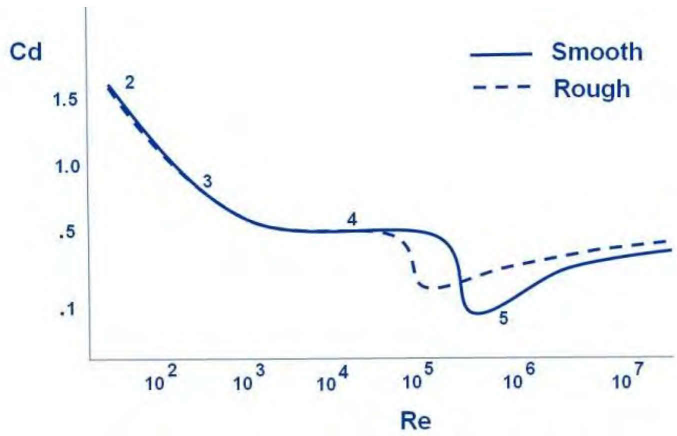
$$F_D = \frac{1}{2}C_D\rho A v^2$$

where A is the cross-sectional area of the object exposed to the airflow, v is the speed of the object in a fluid, and C_D is the drag coefficient, a dimensionless quantity that varies based on shape.

Another useful quantity to know is the Reynold's number, a dimensionless quantity that helps predict fluid flow patterns. It is given by the formula:

$$\mathrm{Re} = \frac{\rho v L}{\mu}$$

where ρ is the density of the surrounding fluid, μ is the dynamic viscosity of the fluid, and L is a reference length parameter that varies based on each object. For a smooth [1] sphere traveling in a fluid, its diameter serves as the reference length parameter.



A logarithmic graph of C_D vs Re of a sphere from the NASA Glenn Research Center.

The relationship between the drag coefficient and the Reynold's number holds significant importance. Due to the complexity of fluid dynamics, empirical data is commonly used, as depicted in the figure provided above. Notably, the figure indicates a significant decrease in the drag coefficient around $\mathrm{Re} \approx 4 \times 10^5$. This phenomenon, known as the drag crisis, occurs when a sphere transitions from laminar to turbulent flow, resulting in a broad wake and high drag. The table in the link below presents a range of C_d versus Re values of a smooth sphere.

Desmos table: https://www.desmos.com/calculator/wnpkg5wnt0

Let's consider a smooth ball with a radius of 0.2 m and a mass of 0.1 kg dropped in air with a constant density of $\rho = 1.255$ kg/m^3. It is found that at velocity 5 m/s, the Reynold's number of the ball is $3.41 \cdot 10^5$. If the ball is dropped from rest, it approaches a *stable* terminal velocity v_1. If the ball is thrown downwards with enough velocity, it will experience turbulence, and approach a *stable* terminal velocity v_2. Find $\Delta v = v_2 - v_1$. Ignore any terminal velocities found for Reynold numbers less than an order of magnitude 10^{-1}.

Note: This problem is highly idealized as it assumes the atmosphere has air of constant density and temperature. In reality, this is not true!

Solution 11: Terminal velocity exists when the net force is 0. Using $v = \frac{\mu \cdot \mathrm{Re}}{2\rho r}$ where $L = 2r$, we find that

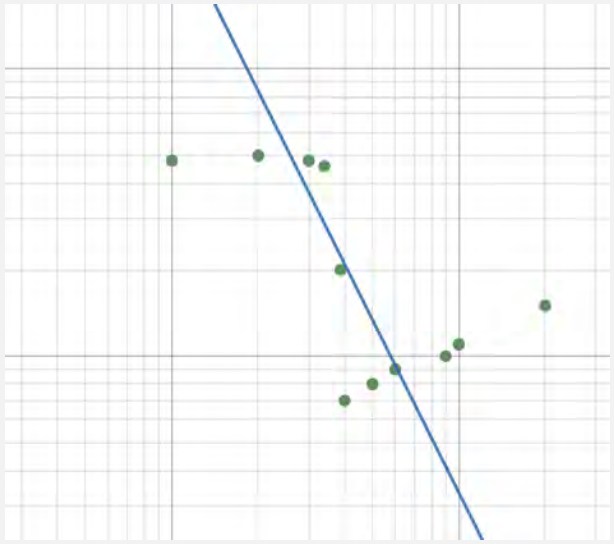
$$\frac{1}{2}\rho_a C_D(\pi r^2)\left(\frac{\mu \cdot \mathrm{Re}}{2\rho r}\right)^2 = mg - \rho_a g\left(\frac{4}{3}\pi r^3\right).$$

[1]meaning a smooth surface.

Since $\rho = \frac{m}{4\pi r^3/3} = 2.98$ kg/m^3 is on the same order as $\rho_a = 1.255$ kg/m^3, the buoyant force must be accounted for and is non-negligible. We can rearrange to find that

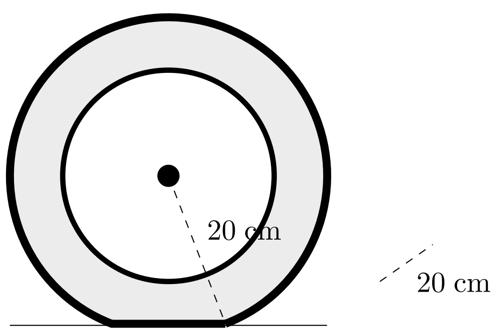
$$C_D \ \mathrm{Re}^2 = \frac{8\rho_a}{\pi\mu^2}\left(mg - \frac{4}{3}\rho_a g\pi r^3\right)$$

Using x as C_D and y as Re, we can plot an equation $xy^2 = \text{const}$ on the C_D vs Re graph. There, we can find three intersections.



The intersection in the middle is not stable. So we find the intersections of the other two to be $\mathrm{Re}_1 \approx 2.6\times10^5$ and $\mathrm{Re}_2 = 6\times10^5$. Hence, $v_1 = 3.81$ m/s and $v_2 = 8.79$ m/s, meaning $\Delta v = 4.98$ m/s. A complete working on desmos can be found here.

The following information applies for the next two problems. Pictured is a wheel from a 4-wheeled car of weight 1200kg. The absolute pressure inside the tire is 3.0×10^5Pa. Atmospheric pressure is 1.0×10^5Pa. Assume the rubber has negligible "stiffness" (i.e. a negligibly low Sheer modulus compared to its Young's modulus).



12. Hysteresis 1 The rubber on the bottom of the wheel is completely unstretched. The rubber has a thickness of 7 mm. Based on this information, find the Young's Modulus of the rubber. Is this answer reasonable?

Solution 12: Let $P = 2.0 \times 10^5$ Pa be the gauge pressure of the tire, $w = 20$ cm be the width of the tire, $r = 20$ cm be the radius of the tire, $t = 7$ mm be the thickness of the rubber, $M = 1200$ kg be the mass of the car, Y be the Young's modulus, and θ be the angle subtended by the portion of the tire in contact with the ground. We have

$$\frac{Mg}{4} = 2rwP\sin\left(\frac{\theta}{2}\right)$$

$$\theta = 2\arcsin\left(\frac{Mg}{8RwP}\right) = 0.3696$$

Let T be the tension in the stretched portion of the rubber. Balance forces on a small section of the tire subtending an angle $d\theta$.

$$Prwd\theta = 2T\sin\left(\frac{d\theta}{2}\right)$$

Using the approximation $\sin\theta = \theta$,

$$T = Prw$$

The stress on the rubber is

$$\sigma = \frac{T}{wt} = \frac{Pr}{t}$$

The strain is

$$\epsilon = \frac{\theta}{2\sin\left(\frac{\theta}{2}\right)} - 1$$

The Young's modulus is

$$Y = \frac{\sigma}{\epsilon} = \frac{\frac{Pr}{t}}{\frac{b}{2\sin\left(\frac{b}{2}\right)} - 1} = \boxed{1.00 \times 10^9 \text{ Pa}}$$

13. Hysteresis 2 The rubber experiences a phenomena known as *hysteresis* – it takes more force to stretch the rubber than allow it to return to equilibrium. Specifically, assume that the Young's Modulus when the rubber is stretched is the answer to 12, and is 1/2 of that when the rubber returns to equilibrium. Compute the power the car's engines has to deliver to overcome the hysteresis losses, if the car moves at 20 m/s. Remember that there are 4 tires!

Solution 13: From the previous part, we have the tension $T = Prw$ and strain $\epsilon = \frac{\theta}{2\sin\left(\frac{\theta}{2}\right)} - 1$ where $\theta = 2\arcsin\left(\frac{Mg}{8RwP}\right)$ is the angle subtended by the portion of the tire in contact with the road. The work required to stretch a portion of tire of length dx is

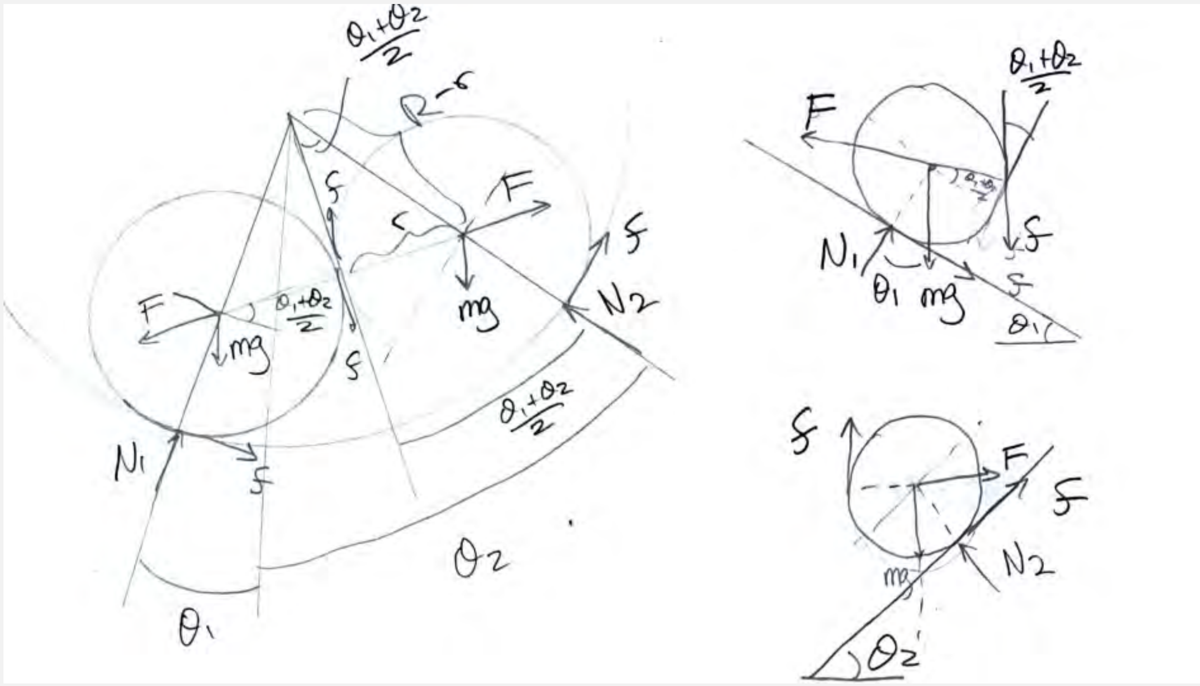
$$W = \frac{1}{2}T\epsilon dx$$

Dividing both sides by dt, multiplying by a factor of $\frac{1}{2}$ to account for half the power being restored when the rubber returns to equilibrium, and multiplying by 4 to account for all four tires, we have

$$\frac{dW}{dt} = T\epsilon\frac{dx}{dt} = T\epsilon v = \boxed{914 \text{ W}}$$

14. Marbles Two identical spherical marbles of radius 3 cm are placed in a spherical bowl of radius 10 cm. The coefficient of static friction between the two surfaces of the marble is 0.31 and the coefficient of static friction between the surfaces of the marbles and the bowl is 0.13. Find the maximum elevation from the bottom of the bowl that the center of one of the marbles can achieve in equilibrium. The bowl is fixed in place and will neither rotate nor translate. Equilibrium refers to stable equilibrium.

Solution 14:



In the figure above, $r = 3cm$ and $R = 10cm$.
By torque balance about the center of both marbles, all frictions as shown in the FBD above must be equal. WLOG, let $\theta_2 > \theta_1$ and consider the elevation of the 2nd marble for the final answer.
By geometry, we can see that $\sin\frac{\theta_1+\theta_2}{2} = \frac{r}{R-r}$. Solving for equilibrium on both marbles give the following set of equations:

$$mg\sin\theta_1 + f\sin\frac{\theta_1+\theta_2}{2} + f - F\cos\frac{\theta_1+\theta_2}{2} = 0$$

$$N_1 - mg\cos\theta_1 - F\sin\frac{\theta_1+\theta_2}{2} - f\cos\frac{\theta_1+\theta_2}{2} = 0$$

$$-mg\sin\theta_2 + F\cos\frac{\theta_1+\theta_2}{2} + f + f\sin\frac{\theta_1+\theta_2}{2} = 0$$

$$N_2 - mg\cos\theta_2 - F\sin\frac{\theta_1+\theta_2}{2} + f\cos\frac{\theta_1+\theta_2}{2} = 0$$

Note that $N_1 - mg\cos\theta_1 - f\cos\frac{\theta_1+\theta_2}{2} = N_2 - mg\cos\theta_2 + f\cos\frac{\theta_1+\theta_2}{2}$ and therefore $N_2 < N_1$.
These equations can be simplified to get:

$$f = \frac{1}{2}mg\frac{\sin\theta_1 - \sin\theta_2}{1+\sin\frac{\theta_1+\theta_2}{2}}$$

$$F = \frac{1}{2}mg\frac{\sin\theta_1 + \sin\theta_2}{\cos\frac{\theta_1+\theta_2}{2}}$$

$$N_2 = \frac{1}{2}mg(2\cos\theta_2 + (\sin\theta_1 + \sin\theta_2)\cdot\tan\frac{\theta_1+\theta_2}{2} - \frac{\sin\theta_1 - \sin\theta_2}{1+\sin\frac{\theta_1+\theta_2}{2}}\cos\frac{\theta_1+\theta_2}{2})$$

There are two cases to consider. The first is that the friction between the two surfaces of the marbles is limiting, in which $f = \mu_m F$ where $\mu_m = 0.31$. The second is that the friction between the marbles and the bowl is limiting, in which $f = \mu_b F$ where $\mu_b = 0.13$. θ_2 in each case can be solved numerically, as seen here on desmos. It turns out that μ_b is the limiting factor, in which $\theta_2 = 0.6294rad$ and the elevation reached by the center of the second marble is $R-(R-r)\cos\theta_2 = 0.0434$ m.

The problem was intended to be solved with the value of $\mu_b = 0.17$. However, due to a typographical error in the problem statement, 0.13 was displayed instead. In the case of which $\mu_b = 0.17$, θ_2 has an analytical solution while it does not for 0.13. The error tolerance on this problem was adjusted such that those who submitted answers whose results derive from either a value of 0.13 and 0.17 for μ_b were accepted, in order to accommodate those less fluent with numerical software such as desmos.

When $\mu_b = 0.17$ is used instead:

$$\theta_2 = \arcsin\frac{r}{R-r} + \arctan\frac{\mu_m r}{R-2r}$$

and

$$R-(R-r)\cos\theta_2 = 0.0452 \text{ m}$$

.

15. Fringe effect approximation Two parallel square plates of side length 1 m are placed a distance 30 cm apart whose centers are at $(-15 \text{ cm}, 0, 0)$ and $(15 \text{ cm}, 0, 0)$ have uniform charge densities -10^{-6} C/m^2 and 10^{-6} C/m^2 respectively. Find the magnitude of the component of the electric field perpendicular to axis passing through the centers of the two plates at $(10 \text{ cm}, 1 \text{ mm}, 0)$.

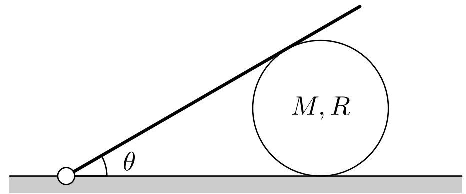
Solution 15: By symmetry, the electric field due to the portion of the plates between $y = 50$ cm and $y = -49.8$ cm will cancel out. We only need to consider the electric field from two 2 mm wide strips of charge, which are small enough to be approximated as wires with charge per unit length $\lambda = \sigma w = \pm 2\times 10^{-9}$ C/m^2 at a distance $y = 50$ cm away. The y-component of the electric field from these wires is then

$$E_y = \frac{0.5\lambda}{4\pi\epsilon_0}\int_{-0.5}^{0.5}\left(\frac{1}{(z^2+0.5^2+0.05^2)^{\frac{3}{2}}} - \frac{1}{(z^2+0.5^2+0.25^2)^{\frac{3}{2}}}\right)\mathrm{d}z$$

$$E_y = \frac{0.5\lambda}{4\pi\epsilon_0}\left(\frac{z}{(0.5^2+0.05^2)\sqrt{z^2+0.5^2+0.05^2}} - \frac{z}{(0.5^2+0.25^2)\sqrt{z^2+0.5^2+0.25^2}}\right)$$

$$E_y = \frac{0.5\lambda}{4\pi\epsilon_0}\left(\frac{1}{(0.5^2+0.05^2)\sqrt{0.5^2+0.5^2+0.05^2}} - \frac{1}{(0.5^2+0.25^2)\sqrt{0.5^2+0.5^2+0.25^2}}\right) = \boxed{11.9 \text{ N/C}}$$

16. Sliding Along A hollow sphere of mass M and radius R is placed under a plank of mass $3M$ and length $2R$. The plank is hinged to the floor, and it initially makes an angle $\theta = \frac{\pi}{3}$ rad to the horizontal. Under the weight of the plank, the sphere starts rolling without slipping across the floor. What is the sphere's initial translational acceleration? Assume the plank is frictionless.



A not-to-scale picture of the sphere-plank setup.

Solution 16: Let N be the normal force acting from the ball on the plank. We can use torque at a distance $x = R\cot\frac{\theta}{2}$ from the hinge (by geometry) to write (using $m = 3M$)

$$mg\frac{l}{2}\cos\theta - NR\cot\frac{\theta}{2} = \frac{ml^2}{3}\alpha$$

You can also find N from

$$N\sin\theta - f = Ma_0$$
$$fR = \frac{2}{3}MR^2\alpha$$
$$a_0 = R\alpha$$

You can relate the acceleration at the point of contact $a_p = \vec{a}_0 + \vec{a}_0 \times \vec{r} = \alpha_0(1-\cos\theta)$. Hence combining all equations gives $a = \frac{9}{32}g \approx \boxed{2.76\ \text{m/s}^2}$.

Alternate: To avoid internal contact forces, we consider Lagrangian mechanics with generalized coordinate θ. The potential energy of the system is

$$U(\theta) = 3MgR\sin\theta + MgR,$$

and so the θ-generalized force is $F_\theta = \left.\frac{\partial V}{\partial\theta}\right|_{\theta=\frac{\pi}{3}} = \frac{3}{2}MgR$. The kinetic energy of the system is

$$K(\dot{\theta}) = \frac{1}{2}\left(\frac{1}{3}\cdot 3M(2R)^2\right)\dot{\theta}^2 + \frac{1}{2}\left(1+\frac{2}{3}\right)Mv^2,$$

where the extra $\frac{2}{3}$ accounts for the rotational kinetic energy of the sphere. Note that the component of velocity normal to the board is $R\dot{\theta}\sqrt{3}$, so trigonometry gives

$$v = \frac{R\dot{\theta}\sqrt{3}}{\sin\theta} = 2R\dot{\theta}.$$

Substituting,

$$K(\dot{\theta}) = 2MR^2\dot{\theta}^2 + \frac{10}{3}MR^2\dot{\theta}^2 = \frac{16}{3}MR^2\dot{\theta}^2,$$

but this is also equal to $\frac{1}{2}m_\theta\dot{\theta}^2$, so the θ-generalized mass is $m_\theta = \frac{32}{3}MR^2$. By the Euler-Lagrange equation,

$$\ddot{\theta} = \frac{F_\theta}{m_\theta} = \frac{9}{64}\frac{g}{R},$$

and we also have $a = 2R\ddot{\theta}$, so $a = \frac{9}{32}g \approx 2.76\ \text{m/s}^2$.

The following information applies for the next two problems. A space elevator consists of a heavy counterweight placed near geostationary orbit, a thread that connects it to the ground (assume this is massless), and elevators that run on the threads (also massless). The mass of the counterweight is 10^7 kg. Mass is continuously delivered to the counterweight at a rate of 0.001 kg/s. The elevators move upwards at a rate of 20 m/s. Assume there are many elevators, so their discreteness can be neglected. The elevators are massless. The counterweight orbits the Earth.

17. Space Elevator 1 Find the minimum possible displacement radially of the counterweight. Specify the sign.

18. Space Elevator 2 Assuming a radial displacement that is 10 times what you found in the previous part, find the displacement tangentially of the counterweight. Does it lag or lead Earth's motion?

Solution 17: As the orbit is geostationary, we can balance forces to write that

$$\frac{GM_E}{R_{\mathrm{GS}}^2} = \omega^2 R_{\mathrm{GS}} \implies R_{\mathrm{GS}} = \left(\frac{GM_E T^2}{4\pi}\right)^{1/3}$$

The total mass of masses = 1790 kg $\ll 10^6$ kg which means the displacements are small. The total gravity of masses is

$$\int_{R_E}^{R_{Es}} \frac{GM_E}{x^2}(\lambda dx) = \mathrm{GME}\,\lambda\left(\frac{1}{R_E} - \frac{1}{R_{\mathrm{cos}}}\right) = 2650 \text{ N}.$$

where $\lambda = 5 \cdot 10^{-5}$ kg/m. The total centrifugal of masses is

$$\int_{R_E}^{R_E} \frac{4\pi^2}{T^2} \times (\lambda dx) = \frac{2\pi^2}{T^2}\lambda\left(R_{Es}^2 - R_E^2\right) = 230 \text{ V}$$

Hence, the total force is 2420 N. The outwards force is just

$$F_{\mathrm{out}} = \omega^2 aM - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{GM_E}{x^2}\right) aM$$

where a is the horizontal displacement which is much less than R_E, R_{GS}. Hence, this can be rewritten as

$$F_{\mathrm{out}} = \left(\frac{4\pi^2}{T^2} - 2\frac{GM_E}{R_{\mathrm{GS}}^2}\right) aM = 2420 \text{ N}$$

This implies that $a = \boxed{15.21 \text{ km}}$.

Solution 18: The tension at the ground is $2420 \cdot 9 = 21780$ N. The rate of angular momentum delivered to the masses is then

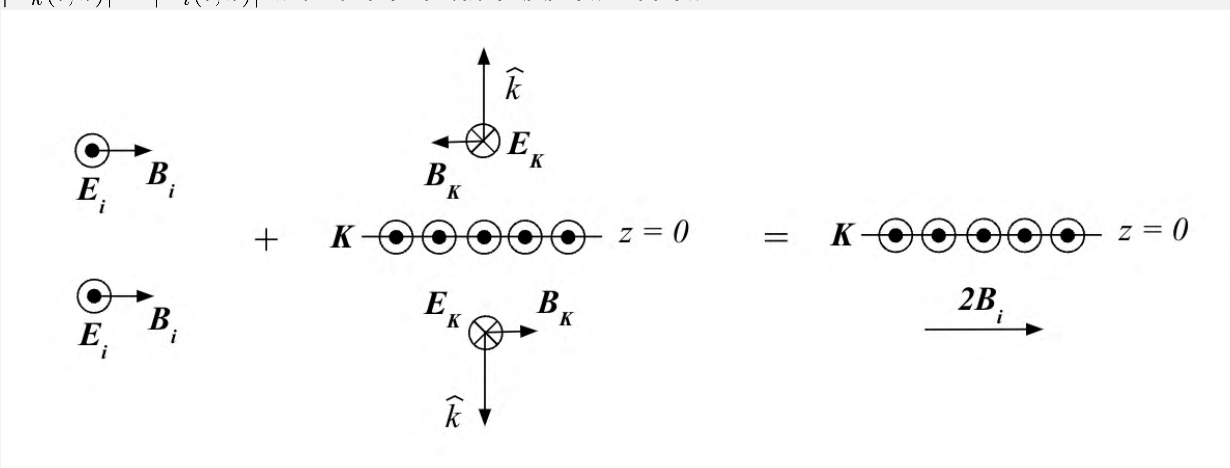
$$\omega(0.001 \text{ kg/s})(R_{\mathrm{GS}}^2 - R_E^2) = 1.27 \cdot 10^8 \text{ kgm}^2/\text{s}^2.$$

This is also the torque acting on the Earth. The horizontal force is 19.9 N which means that $\theta = 19.9/21800$. This tells us that the horizontal displacement is then

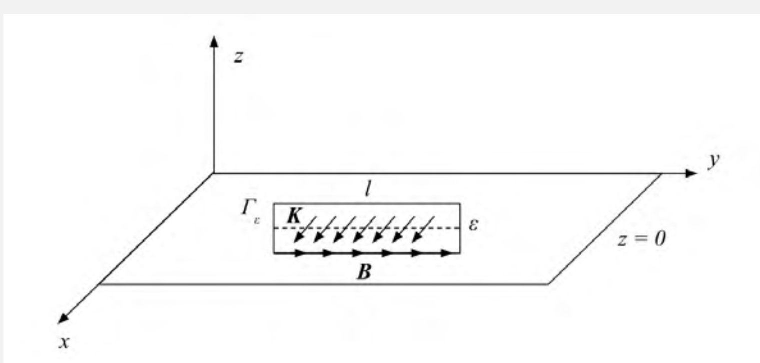
$$b = (R_{\mathrm{GS}} - R_E)\theta = \boxed{32.7 \text{ km}}$$

19. Laser Power Consider a spherical shell of thickness $\delta = 0.5$cm and radius $R = 5$cm made of an Ohmic material with resistivity $\rho = 10^{-7}\Omega$m. A spherical laser source with a tuned frequency of f_0 and intensity $I_0 = 10^5$ W/m^2 is placed at the center of the shell and is turned on. Working in the limit $\delta \ll \frac{c}{f_0} \ll R$, approximate the initial average power dissipated by the shell. Neglect inductance.

Solution 19: Note that the proper treatment of this problem requires more advanced electromagnetism. The following might seem sketchy, but this will do. We first consider the case of a thin plane made of an ideal conductor placed at $z = 0$. A monochromatic electromagnetic wave with an E field: $\vec{E}_i(t, z) = E_0 \cos(kz - \omega t)\hat{x}$ is incident on the plane. The corresponding B field is of course $\vec{B}_i(t, z) = \frac{E_0}{c} \cos(kz - \omega t)\hat{y}$. We expect a current density $\hat{K}(t) = K(t)\hat{x}$ to form in the direction of the E field, as the electric field accelerates the charges on the sheet – $K(t)$ will oscillate with the E field. We know that perfect conductors completely reflect electromagnetic waves, so we expect no EM field in the $z > 0$ region. The total EM field is a combination of the incident wave and the wave generated by the oscillating surface charges, so we are to believe that the latter provides just the right EM wave to cancel out $\vec{B}_i$ and $\vec{E}_i$ in the $z > 0$ region. Consider an instance where the EM waves have an orientation shown in the figure below. It's easy to verify that the magnetic field generated by a uniform sheet of current pointing out of the page is given as shown. Since the electric wave generated by $\vec{K}(t)$ must cancel the $\vec{E}_i$ pointing out of page in $z > 0$, we get a poynting vector (for the wave generated by $\vec{K}$) in the $\hat{z}$ direction in $z > 0$. By symmetry, this means that the poynting vector points in the $-\hat{z}$ direction in $z < 0$. Such simple symmetry arguments fully determine the wave generated by an oscillating sheet of charge: $|\vec{E}_k(t, z)| = |\vec{E}_i(t, z)|$, $|\vec{B}_k(t, z)| = |\vec{B}_i(t, z)|$ with the orientations shown below.



We therefore get that the total magnetic field in the $z < 0$ region is $\vec{B} = 2\vec{B}_i(t, z)$. Let us now take an Ampere loop as shown in the figure below:



We have:

$$T_\epsilon \equiv \oint_{\Gamma_\epsilon} \vec{B}(t, -\epsilon/2) \cdot d\vec{l} = \mu_0 I_{enc} + \frac{1}{c^2}\frac{\partial}{\partial t}\int_{S_\epsilon} \vec{E}(t, z) \cdot d\vec{A}$$

where S_ϵ is the surface bounded by Γ_ϵ. Since $\vec{E} = 0$ everywhere and $I_{enc} = K(t)l$,

$$\lim_{\epsilon \to 0} T_\epsilon = \frac{2E_0 l}{c}\cos(\omega t) = \mu_0 K(t) l$$

Hence, we have $\vec{K}(t) = \frac{2E_0}{\mu_0 c}\cos(\omega t)\hat{x}$. The key find in this long introduction is that an oscillating sheet of current $\vec{K}(t) = K_0 \cos(\omega t)\hat{x}$ generates an EM wave with an electric field $\vec{E}_K(t, 0) = -\frac{\mu_0 c}{2}\vec{K}(t)$ near the sheet (at $z = 0$). We finally return to the original problem.

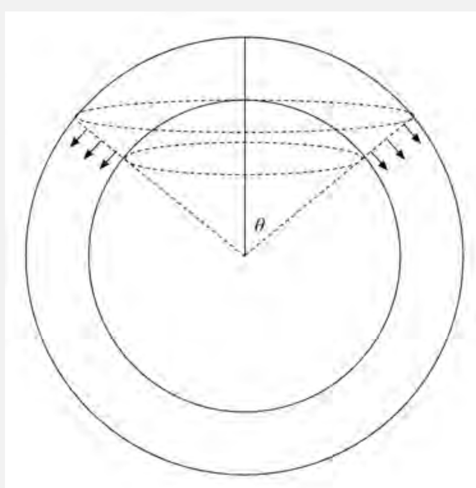
All laser beams have poynting vectors pointing radially outwards, so the E_i field must point in the tangential direction to the sphere at all normal incidence. Say the E_i field points in the $\hat{\theta}$ direction, so that the current in the shell flows from the north pole to the south pole. The condition $\delta \ll \frac{c}{f_0} \ll R$ allows us to neglect attenuation due to skin effects – essentially, the electric field, thus the surface current, is approximately uniform in the shell. The condition $\frac{c}{f_0} \ll R$ allows us to treat the incidence of the laser beams as an EM wave hitting an Ohmic plane of thickness δ, so we consider that problem first. Denote the *total* electric field inside the the plane at time t as $\vec{E}(t)$. The current density is given by Ohm's law: $\vec{J}(t) = \vec{E}(t)/\rho$. The corresponding surface current is of course $\vec{K}(t) = \delta\vec{E}/\rho(t)$. Carefully note the distinction between $\vec{E}$ and $\vec{E}_i$. Since the source wave $\vec{E}_i(t)$ is sinusoidal, we expect $\vec{E}(t)$ to be sinusoidal as well so that $\vec{E}_K(t) = -\frac{\mu_0 c}{2}\vec{K}(t)$ applies. The total electric field inside the sheet is a sum of $\vec{E}_K$ and $\vec{E}_i$, thus:

$$\vec{E}(t) = \vec{E}_i(t) - \frac{\mu_0 c\delta}{2\rho}\vec{E}(t)$$

Rearranging for $\vec{E}(t)$, we get:

$$\vec{E}(t) = \frac{\vec{E}_i(t)}{1 + \mu_0 c\delta/2\rho}$$

On the sphere, $\vec{E}(t)$ points in the $\hat{\theta}$ direction and has the same magnitude everywhere on the sphere. The relevant cross-section will be the lateral surface of a truncated cone, as shown below:



The current through this surface is therefore:

$$I(\theta,t) = \frac{\pi\delta(2R+\delta)|\vec{E}_i(t)|}{\rho+\mu_0 c\delta/2}\sin\theta$$

The power dissipated by a volume generated by $\theta \sim \theta + d\theta$ is

$$dP(t) = I^2(\theta,t)\frac{\rho(R+\delta/2)d\theta}{\pi\delta(2R+\delta)\sin\theta}$$

We integrate this expression from $\theta = 0$ to $\theta = \pi$:

$$P(t) = \frac{\pi\delta\rho(2R+\delta)^2}{(\rho+\mu_0 c\delta/2)^2}|\vec{E}_i(t)|^2$$

Now, the incident electric field has the form $|\vec{E}_i(t)|^2 = \frac{2I_0}{\epsilon_0 c}\cos^2(2\pi f_0 t)$. The average of this function from $t = 0$ to $t = \frac{1}{f_0}$ is 1/2, so our final answer is:

$$\bar{P} = \frac{\pi\delta\rho(2R+\delta)^2}{(\rho+\mu_0 c\delta/2)^2}\frac{I_0}{\epsilon_0 c}$$

which turns out to be $2.39078 \times 10^{-15} W$

20. Twinkle Twinkle. A stable star of radius R has a mass density profile $\rho(r) = \alpha(1 - r/R)$. Here, "stable" means that the star doesn't collapse under its own gravity. If the internal pressure at the core is provided solely by the radiation of photons, calculate the temperature at the core. Assume the star is a perfect black body and treat photons as a classical ideal gas. Use $R = 7 \times 10^5$ km and $\alpha = 3$ g/cm^3. Round your answer to the nearest kilokelvin. We treat photons as a classical gas here to neglect any relativistic effects.

Solution 20: For a star that doesn't collapse in on itself, there must be some source of pressure $p(r)$ that balances out the pressure due to gravity. Define the function:

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr$$

inside the star. The gravitational pressure on an infinitesimal shell of radius r and thickness dr is given by:

$$dp_g = -\frac{Gm(r)\rho(r)4\pi r^2 dr}{r^2 4\pi r^2} = -\frac{Gm(r)\rho(r)}{r^2}dr$$

Hence, the pressure source must provide a pressure gradient $\frac{dp}{dr}$ given by:

$$\frac{dp}{dr} = \frac{Gm(r)\rho(r)}{r^2}$$

If $\rho(r) = \alpha(1 - r/R)$, then $m(r) = \frac{1}{3}\pi\alpha r^3(3r/R - 4)$. We need the pressure gradient:

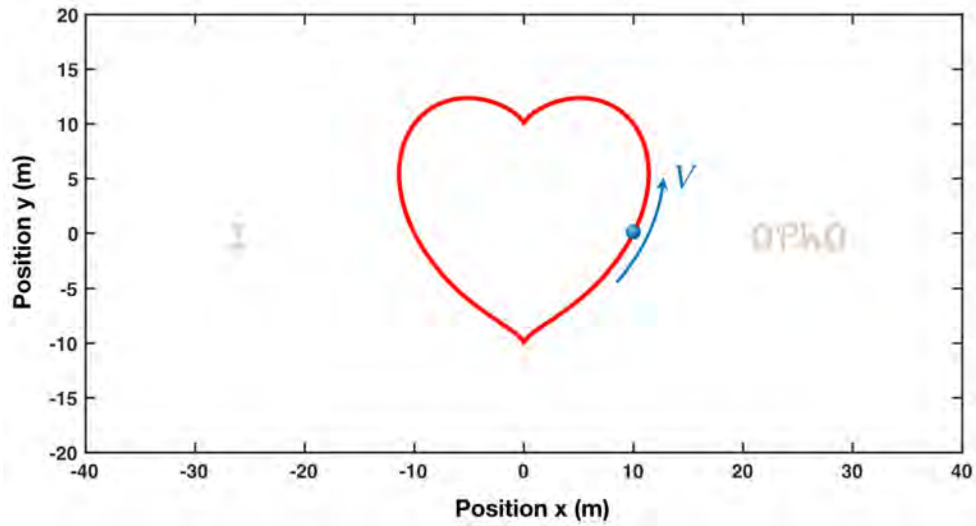
$$\frac{dp}{dr} = \frac{1}{3}\pi G\alpha^2 r(3r/R - 4)(r/R - 1)$$

We integrate this equation from $r = 0$ to R with the obvious boundary condition $p(R) = 0$. We find that $p(0) = \frac{5}{36}\pi G\alpha^2 R^2$. From the Stefan-Boltzmann law, along with elementary kinetic theory, the pressure due to local radiation is given by $\frac{4\sigma}{3c}T_c^4$. Plugging in values gives $T_c = 26718$kK

21. My Heart Will Go On On a flat playground, choose a Cartesian Oxy coordinate system (in unit of meters). A child running at a constant velocity $V = 1\text{m/s}$ around a heart-shaped path satisfies the following order-6 algebraic equation:

$$\left(x^2 + y^2 - L^2\right)^3 - Lx^2y^3 = 0 \ , \ L = 10 \ .$$

When the child is at the position $(x, y) = (L, 0)$, what is the magnitude of their acceleration?



Solution 21:
The acceleration can be found from the local geometry of the curves, thus let us study small deviations around the position of interests $(x, y) = (L, 0)$:

$$x = L + \delta_x \quad , \quad y = 0 + \delta_y \quad , \quad |\delta_x|, |\delta_y| \ll L \quad .$$

Consider the 2nd-order approximation in δ_x of δ_y with quadratic coefficients α and β:

$$\delta_y \approx \alpha\delta_x + \frac{\beta}{L}\delta_x^2 \sim \delta_x \quad .$$

To find these coefficients, we look at the algebraic equation of our heart-shape path up to the two lowest-orders of expansions (which are the 3rd and 4th):

$$\begin{aligned} 0 = \left(x^2 + y^2 - L^2\right)^3 - Lx^2y^3 &\approx L^2\left[8L\delta_x^3 + 12\delta_x^4 + 12\delta_x^2\delta_y^2 - 2\delta_x\delta_y^3 - L\delta_y^3 + \mathcal{O}(\delta_x^5)\right] \\ &\approx L^2\left[8L\delta_x^3 + 12\delta_x^4 + 12\alpha^2\delta_x^4 - 2\alpha^3\delta_x^4 - \left(\alpha^3 L\delta_x^3 + 3\alpha^2\beta\delta_x^4\right) + \mathcal{O}\left(\delta_x^5\right)\right] \\ &\propto \left(8 - \alpha^3\right)L\delta_x^3 + \left(12 + 12\alpha^2 - 2\alpha^3 - 3\alpha^2\beta\right)\delta_x^4 + \mathcal{O}\left(\delta_x^5\right) \quad . \end{aligned}$$

Thus, α and β can be found by solving:

$$8 - \alpha^3 = 0 \quad , \quad 12 + 12\alpha^2 - 2\alpha^3 - 3\alpha^2\beta = 0 \quad \Longrightarrow \quad \alpha = 2 \ , \ \beta = \frac{11}{3} \quad . \tag{3}$$

We can find the relations between velocities $(\dot{x}, \dot{y}) = \dot{\delta}_x, \dot{\delta}_y)$ and accelerations $(\ddot{x}, \ddot{y}) = \ddot{\delta}_x, \ddot{\delta}_y)$ evaluated at the position $(x, y) = (1, 0) \to (\delta_x, \delta_y) = (0, 0)$ by taking the time-derivatives:

$$\dot{\delta}_y = \alpha\dot{\delta}_x + 2\frac{\beta}{L}\delta_x\dot{\delta}_x = \left(\alpha + 2\frac{\beta}{L}\delta_x\right)\dot{\delta}_x = \alpha\dot{\delta}_x \quad , \tag{4}$$

$$\ddot{\delta}_y = \alpha\ddot{\delta}_x + 2\frac{\beta}{L}\dot{\delta}_x^2 + 2\frac{\beta}{L}\delta_x\ddot{\delta}_x = \left(\alpha + 2\frac{\beta}{L}\delta_x\right)\ddot{\delta}_x + 2\frac{\beta}{L}\dot{\delta}_x^2 = \alpha\ddot{\delta}_x + 2\frac{\beta}{L}\dot{\delta}_x^2 \quad . \tag{5}$$

For a constant running speed V, we get:

$$V = \left(\dot{x}^2 + \dot{y}^2\right)^{1/2} \implies \dot{\delta}_x = \left(1+\alpha^2\right)^{-1/2} V \ , \ \dot{\delta}_y = \alpha\left(1+\alpha^2\right)^{-1/2} V \quad ,$$

which we obtain by applying Eq. (4). Also, the temporal-constraint of constant speed means that the acceleration vector (if non-zero) should be perpendicular to the velocity vector:

$$\frac{d}{dt}V = 0 = \frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2\right)^{1/2} \propto \dot{\delta}_x\ddot{\delta}_x + \dot{\delta}_y\ddot{\delta}_y = 0 \implies \ddot{\delta}_x + \alpha\ddot{\delta}_y = 0 \quad .$$

Using Eq. (5), we can arrive at:

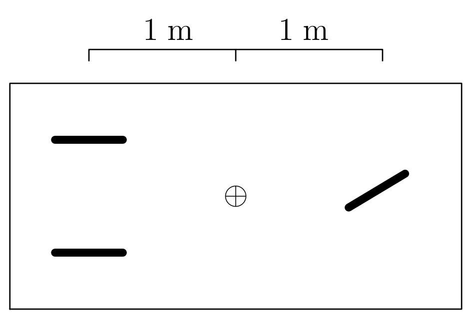
$$\ddot{\delta}_x + \alpha\left(\alpha\ddot{\delta}_x + 2\frac{\beta}{L}\dot{\delta}_x^2\right) = 0 \implies \ddot{\delta}_x = -2\frac{\beta}{L}\alpha\left(1+\alpha^2\right)^{-1}\dot{\delta}_x^2 = -2\beta\alpha\left(1+\alpha^2\right)^{-2}\frac{V^2}{L} \quad ,$$

$$\ddot{\delta}_y = -\alpha^{-1}\dot{\delta}_x = 2\beta\left(1+\alpha^2\right)^{-2}\frac{V^2}{L} \quad .$$

The quadratic coefficients are found in Eq. (3), and given that $V = 1\text{m/s}$, $L = 10\text{m}$, the magnitude of the total acceleration can be calculated:

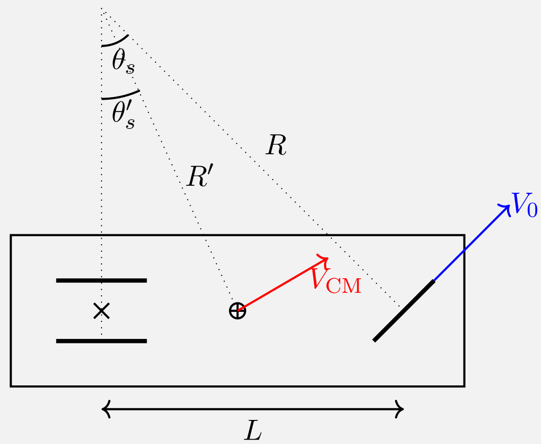
$$a = \left(\ddot{\delta}_x^2 + \ddot{\delta}_y^2\right)^{1/2} = 2\beta\left(1+\alpha^2\right)^{3/2}\frac{V^2}{L} = \frac{22}{15\sqrt{5}}\frac{V^2}{L} \approx \boxed{0.066591\text{m/s}^2} \quad .$$

22. Tricycle A boy is riding a tricycle across along a sidewalk that is parallel to the x-axis. This tricycle contains three identical wheels with radius 0.5 m. The front wheel is free to rotate while the last two wheels are parallel to each other and to the main body of the tricycle. See the diagram.



The front wheel is rotating at a constant angular speed of $\omega = 3$ rad/s. The child is controlling the tricycle such that the front wheel is making an angle of $\theta(t) = 0.15\sin((0.1 \text{ rad/s})t)$ with the main body of the tricycle. Determine the maximum lateral acceleration in m/s^2. Assume a massless frame. The marked plus sign implies CoM. The degree is in radians.

Solution 22: Assuming the wheels roll without slipping, the cart will instantaneously rotate about a fixed point. This fixed point can be constructed by drawing perpendicular lines from all the wheels and seeing where they intersect. Consider the below diagram,



We have:

$$\sin\theta_s = \frac{L}{R}, \qquad \sin\theta'_s = \frac{L}{2R'}. \tag{6}$$

The angular frequency of every point on the robot is the same $\omega_n = \frac{V_0}{R}$. Therefore,

$$\begin{aligned} V_{CM} &= \omega_n R' \qquad (7)\\ &= \frac{V_0}{R}\frac{L}{2\sin\theta'_s} \qquad (8)\\ &= \frac{V\sin\theta_s}{2\sin\theta'_s}. \qquad (9)\end{aligned}$$

The naive idea here is to say the lateral velocity is $V_{CM}\sin\theta'_s$, but note that the robot could already be rotated. Let the angle of rotation with respect to the horizontal line be α, so we have:

$$V_{CM,lat} = V_{CM}\sin\big(\theta'_s + \alpha\big). \tag{10}$$

Using the sine addition formula, we can simplify this to

$$\begin{aligned} V_{CM,lat} &= V_{CM}\sin\theta'_s\cos\alpha + V_{CM}\cos\theta'_s\sin\alpha \qquad (11)\\ &= \frac{V_0}{2}\sin\theta_s\cos\alpha + \frac{V_0\sin\theta_s}{2\tan\theta'_s}\sin\alpha. \qquad (12)\end{aligned}$$

To simplify this, note that we can write:

$$\frac{\sin\theta_s}{\tan\theta'_s} = 2, \tag{13}$$

where we used the fact that

$$\sin\theta_s \approx \tan\theta_s = 2\tan\theta'_s \approx 2\sin\theta'_s. \tag{14}$$

Then,

$$\dot{y}_{CM} = \frac{V_0}{2}\theta_s + V_0\alpha. \tag{15}$$

At the center of the two back wheels (marked with an X), the speed is

$$V_{back} = R\cos\theta_s\omega_n = V_0\cos\theta_s \approx V_0 \tag{16}$$

so its lateral velocity is

$$\dot{y}_{back} = V_0 \sin\alpha \approx V_0\alpha. \tag{17}$$

Note that the angle α can be written as

$$\alpha \approx \sin\alpha = \frac{y_{CM} - y_{back}}{(L/2)} \implies \dot{\alpha} = \frac{2}{L}\left(\dot{y}_{CM} - \dot{y}_{back}\right) = \frac{V_0}{L}\theta_s. \tag{18}$$

Taking higher derivatives of $\dot{y}_{CM}$, we have

$$\ddot{y}_{CM} = \frac{V_0}{2}\dot{\theta}_s + V_0\dot{\alpha} \tag{19}$$

$$= \frac{V_0}{2}\dot{\theta}_s + \frac{V_0^2}{L}\theta_s. \tag{20}$$

Plugging in $\theta(t) = 0.15\sin(0.1t)$ and $V_0 = 1.5$ m/s, we can optimize $\ddot{y}_{\mathrm{CM}}$ to get 0.16912 m/s.

23. Sonic Fryer In this problem, we consider a simple model for a thermoacoustic device. The device uses heavily amplified sound to provide work for a pump that can then extract heat. Sound waves form standing waves in a tube of radius 0.25 mm that is closed on both sides, and a two-plate stack is inserted in the tube. A temperature gradient forms between the plates of the stack, and the parcel of gas trapped between the plates oscillates sinusoidally between a maximum pressure of 1.03 MPa and a minimum of 0.97 MPa. The gas is argon, with density 1.78 kg/m^3 and adiabatic constant 5/3. The speed of sound is 323 m/s. The heat pump itself operates as follows:

The parcel of gas starts at minimum pressure. The stack plates adiabatically compress the parcel of gas to its maximum pressure, heating the gas to a temperature higher than that of the hotter stack plate. Then, the gas is allowed to isobarically cool to the temperature of the hotter stack plate. Next, the plates adiabatically expand the gas back to its minimum pressure, cooling it to a temperature lower than that of the colder plate. Finally, the gas is allowed to isobarically heat up to the temperature of the colder stack plate.

Find the power at which the thermoacoustic heat pump emits heat.

Solution 23:

The efficiency of the heat engine is $\epsilon = 1 - \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} = 0.0237$. The parcel oscillates between pressures P_1 and P_2 sinusoidally with amplitude $P_0 = \frac{P_1 - P_2}{2}$. For a sound wave, the pressure amplitude is $\rho s_0 \omega v$, where s_0 is the position amplitude and v is the speed of sound.

Then the average power with which the sound wave does work on the plates is

$$\langle P\rangle = \frac{1}{2}\rho\omega^2 s_0^2 A v = \frac{P_0^2}{2\rho v}A,$$

where A is the area of each plate. From this, the heat power generated by the engine is $\langle P\rangle/\epsilon = \boxed{6.47\text{ W.}}$

The following information applies for the next two problems. For your mass spectroscopy practical you are using an apparatus consisting of a solenoid enclosed by a uniformly charged hollow cylinder of charge density $\sigma = 50\ \mu\text{C/m}^2$ and radius $r_0 = 7$ cm. There exists an infinitesimal slit of insular material between the cylinder and solenoid to stop any charge transfer. Also, assume that there is no interaction between the solenoid and the cylinder, and that the magnetic field produced by the solenoid can be easily controlled to a value of B_0.

An electron is released from rest at a distance of $R = 10$ cm from the axis. Assume that it is small enough to pass through the cylinder in both directions without exchanging charge. It is observed that the electron reaches a distance R at different points from the axis 7 times before returning to the original position.

24. Isotope Separator 1 Calculate B_0 under the assumption that the path of the electron does not self-intersect with itself.

Solution 24: By Gauss' law, we have

$$\varepsilon_0 \oint \vec{E} \cdot \mathrm{d}\vec{A} = q_{\text{enc}} \implies \varepsilon_0(2\pi r h)E(r) = (2\pi r_0 h)\sigma \implies E(r) = \frac{\sigma r_0}{\varepsilon_0 r}\hat{r}.$$

We find the work done on the electron from a distance r to R is

$$W = -\int_R^r qE(r)\mathrm{d}r = \int_r^R \frac{q\sigma r_0}{\varepsilon_0 r}\mathrm{d}r = \frac{q\sigma r_0}{\varepsilon_0}\ln\frac{R}{r}.$$

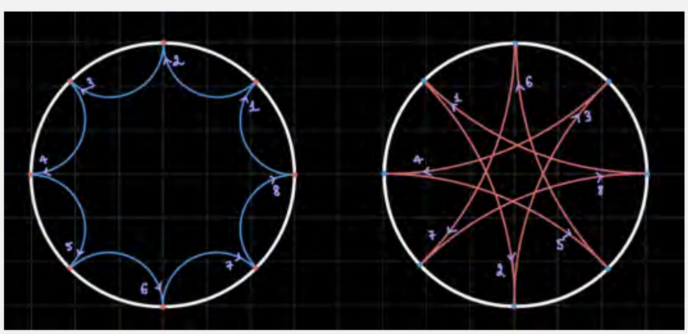
Therefore, by conservation of energy:

$$\frac{1}{2}mv(r)^2 = \frac{q\sigma r_0}{\varepsilon_0}\ln\frac{R}{r} \implies v(r) = \sqrt{\frac{2q\sigma r_0}{m}\ln\frac{R}{r}}.$$

Since the electron is moving in a magnetic field, the radius of the electron follows

$$\frac{mv^2}{a} = qvB \implies a = \frac{mv}{qB} = \sqrt{\frac{2q\sigma r_0}{qB_0^2}\ln\frac{R}{r_0}}$$

This means that $B = \sqrt{\frac{2\sigma m}{q\varepsilon_0 r_0}\ln\frac{R}{r_0}}$. The trajectory of the electron can follow the following paths



As the path is specified to be non-intersecting, we analyze the first path. We can say that the radius of each "circle" the electron travels through is $s = r\tan\frac{\pi}{8} = \frac{mv}{qB}$. Hence,

$$B = \cot\frac{\pi}{8}\sqrt{\frac{2\sigma m}{q\varepsilon_0 r_0}\ln\frac{R}{r_0}}$$

25. Isotope Separator 2 Calculate the time it took for the particle to return to original position. Answer in milliseconds.

Hint: You may find interest in the Gaussian error function:

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \text{d}t.$$

Specific values of the error function can be calculated on desmos.

Solution 25: From the previous part, we have the velocity of the electron is

$$v = \sqrt{\frac{2q_e \sigma r_0}{m_e \epsilon_0} \ln\left(\frac{R}{r}\right)}$$

Let T_0 represent the time from when the electron is released to when it enters the cylinder. Rewriting as a differential equation, separating variables, and integrating, we have

$$\frac{dr}{dt} = -\sqrt{\frac{2q_e \sigma r_0}{m_e \epsilon_0} \ln\left(\frac{R}{r}\right)}$$

$$\int_R^{r_0} \frac{dr}{\sqrt{\ln\left(\frac{R}{r}\right)}} = -\sqrt{\frac{2q_e \sigma r_0}{m_e \epsilon_0}} T_0$$

$$T_0 = \sqrt{\frac{\pi m_e \epsilon_0}{2q_e \sigma r_0}} R \,\text{erf}\left(\sqrt{\ln\left(\frac{R}{r_0}\right)}\right)$$

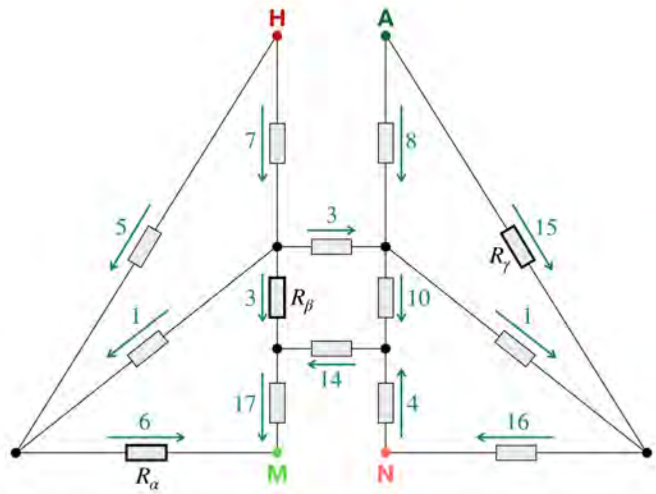
From the previous part, we know that the electron undergoes $\frac{3\pi}{4}$ radians of circular motion with radius of curvature $r_0 \tan\left(\frac{\pi}{8}\right)$ and velocity $v = \sqrt{\frac{2q_e r_0 \sigma}{m_e \epsilon_0} \ln\left(\frac{R}{r_0}\right)}$ Let T_1 be the time from when the electron enters the cylinder to when it exits the cylinder.

$$T_1 = \frac{3\pi}{4} r_0 \tan\left(\frac{\pi}{8}\right) \sqrt{\frac{m_e \epsilon_0}{2q_e r_0 \sigma \ln\left(\frac{R}{r_0}\right)}}$$

The total time is

$$T = 16T_0 + 8T_1 = 5.39 \times 10^{-9} \text{ s} = \boxed{5.39 \times 10^{-6} \text{ milliseconds}}$$

26. Somos El Barco For any circuit network made of batteries and resistors, if we know the voltages of all the batteries and the resistance values of all the resistors, we can calculate all the electrical currents. However, if we know the voltages of all the batteries and all the currents, it is still not enough to uniquely determine the resistance values of all the resistors. Consider a sail-shape circuit network, in which we connect points H and N with a $\mathcal{E}_{\text{HN}} = 10\text{V}$ battery, points A and M with a $\mathcal{E}_{\text{AM}} = 20\text{V}$ battery. The electrical currents in this network have directions and magnitudes (in mA) as shown the figure. The possible resistance values of resistors R_α, R_β, R_γ is not a single point (corresponds to an unique solution) but a confined region in the three-dimensional $(R_\alpha, R_\beta, R_\gamma)$-space. Determine the volume of this region (in Ω^3).



The title of this problem means "we are the boat".

Solution 26:
Let us call the nodes on this sail-shape circuit network as in figure A. Since the electrical current always flow toward lower potential, we have the diagram in figure B for the relation between the electrical potential of the nodes (the arrow indicate high-to-low). Use the notation $\mathcal{V}(\bullet)$ to indicate the electrical potential of the point $\bullet$.

The trick for this problem is noticing that the Ohm's laws in our circuit network can always be trivially satisfied for every possible solution following the comparison relationship in figure B, the voltage differences $\mathcal{V}(\mathrm{H}) - \mathcal{V}(\mathrm{N}) = 10\mathrm{V}$, $\mathcal{V}(\mathrm{A}) - \mathcal{V}(\mathrm{M}) = 20\mathrm{V}$ and the equality of in-out currents (in H $5 + 7$ equals out N $-4 + 16$, in A $8 + 15$ equals out M $6 + 17$) of the batteries. That indeed responsible to why the resistance values of the resistors in this network are not an unique point but rather a region in the 14-dimensional space.

Define:

$$X = \mathcal{V}(\mathrm{P}) - \mathcal{V}(\mathrm{M}) > 0 \ , \ Y = \mathcal{V}(\mathrm{Q}) - \mathcal{V}(\mathrm{U}) > 0 \ , \ Z = \mathcal{V}(\mathrm{A}) - \mathcal{V}(\mathrm{S}) > 0 \ , \ a = \mathcal{V}(\mathrm{U}) - \mathcal{V}(\mathrm{M}) > 0 \ .$$

We want to find the volume in the 3-dimensional subspace of possible $(R_\alpha, R_\beta, R_\gamma)$:

$$\int dR_\alpha dR_\beta dR_\gamma = \frac{\int dX dY dZ}{I_\alpha I_\beta I_\gamma} \ . \tag{21}$$

From figure B and the fixed voltage differences $\mathcal{V}(\mathrm{H}) - \mathcal{V}(\mathrm{N}) = 10\mathrm{V}$, $\mathcal{V}(\mathrm{A}) - \mathcal{V}(\mathrm{M}) = 20\mathrm{V}$, we can write down the inequality equations to specify the boundary of $\{(X, Y, Z)\}$:

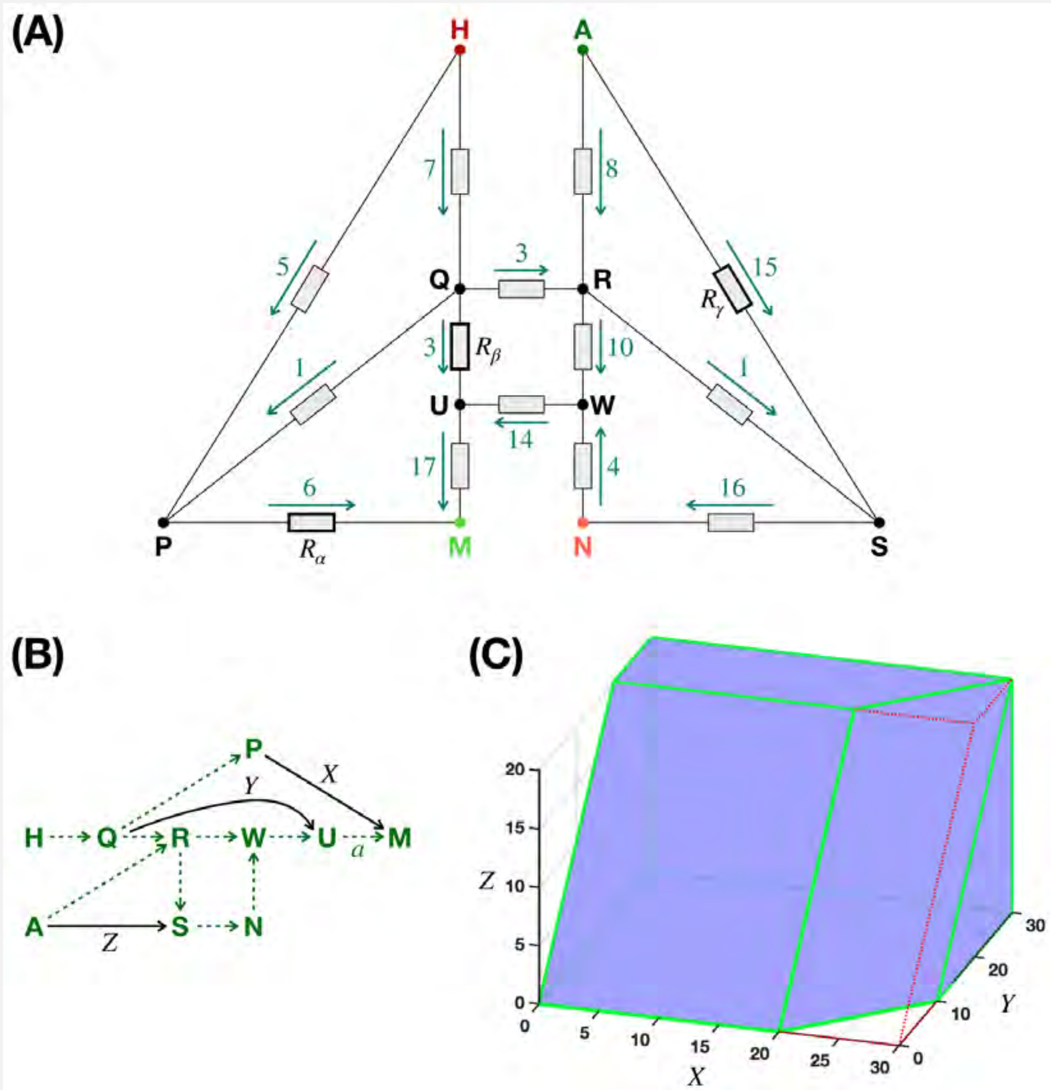
$$a < 20 \ , \ Y < 30 - a \ , \ a < X < a + Y \ , \ Z < \min\left[(20 - a), y\right] \ ,$$

which we can solve to obtain a polygonal-region as shown in figure C (basically, just carving out from the $30 \times 30 \times 20$ rectangle representing maximum bounds of X, Y, Z). Thus:

$$\int dX dY dZ = 30 \times 30 \times 20 - 30 \times \left(\frac{1}{2} \times 20 \times 20\right) - 20 \times \left(\frac{1}{2} \times 10 \times 10\right) = 1.1 \times 10^4 \mathrm{V}^3 \ .$$

Plug this answer back in Eq. (21) and use $I_\alpha = 6\mathrm{mA}$, $I_\beta = 3\mathrm{mA}$, $I_\gamma = 15\mathrm{mA}$:

$$\int dR_\alpha dR_\beta dR_\gamma = \frac{\int dX dY dZ}{I_\alpha I_\beta I_\gamma} = \frac{1.1 \times 10^4 \mathrm{V}^3}{6\mathrm{mA} \times 3\mathrm{mA} \times 15\mathrm{mA}} \approx \boxed{4.0741 \times 10^{10} \Omega^3} \ .$$



* We thank Long T. Nguyen for many useful discussions during the creation of this puzzle.

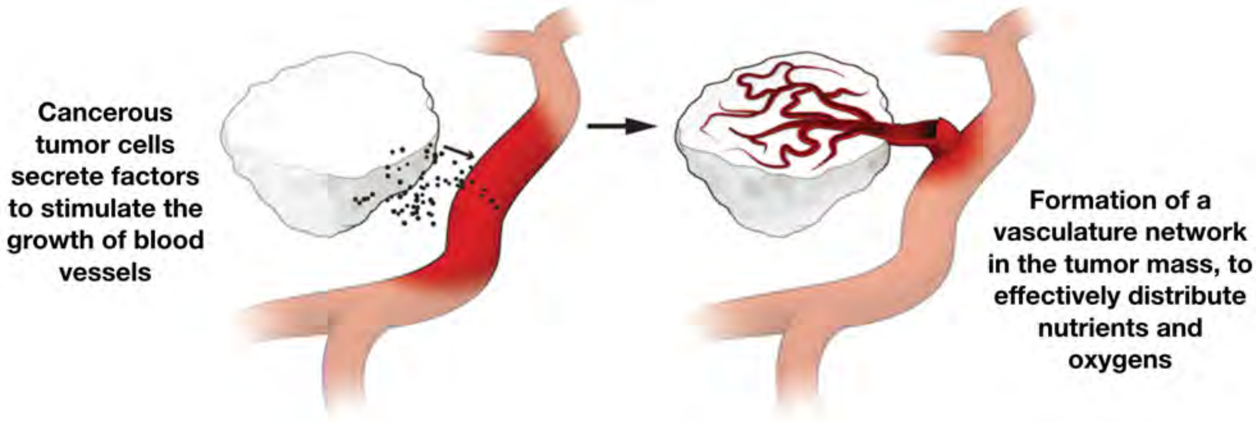
27. The Final Countdown A model of cancer tumor dynamics under a low-dose chemotherapy consists of three non-negative variables (P, Q, R), in which P represents the cancer tumor size, Q represents the (normalized) carrying capability of the tumor vasculature network, and R represents the local (normalized) activity of the immunology system:

$$\frac{\mathrm{d}}{\mathrm{d}t}P = \xi P \ln\frac{Q}{P} - \theta PR - \varphi_1 PC \quad ,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}Q = bP - \left(\mu + dP^{2/3}\right)Q - \varphi_2 QC \quad ,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}R = \alpha\left(P - \beta P^2\right)R + \gamma - \delta R + \varphi_3 RC \quad .$$

Here, C is the local concentration of chemotherapeutic agent at the tumor site, which we can assume to follow by a simple pharmacokinetics model:

$$\frac{\mathrm{d}}{\mathrm{d}t}C = -\frac{1}{\tau}C + U \quad ,$$

where U is the rate of chemotherapy drug administrated to the patient body. Let us assume an unchanging rate U, and treat it as another parameter of the model. All other unmentioned symbols are positive constant parameters, which values can be measured and should depends on the particular kinds of cancers and treatments. In total, there are 14 parameters – such a high-degree of complexity is very common in biophysical models. For each set of parameters, there can be many possible stationary states, which can be associated with various levels of malignancy. What is the maximum number of stationary non-zero tumor sizes (including both stable and unstable ones) for a set of parameters in this model? **For this problem, you can only submit your answer once.**



Solution 27:
For a constant drug-rate U, at stationary states the drug-concentration has to be $C = \tau U$. If (P_*, Q_*, R_*) are stationary solutions of this model, then set $C = \tau U$ and all the time-derivatives $d/dt...$ to 0, we obtain:

$$0 = \xi P_* \ln \frac{Q_*}{P_*} - \theta P_* R_* - \varphi_1 \tau U P_* \quad , \tag{22}$$

$$0 = bP_* - \left(\mu - dP_*^{2/3}\right) Q_* - \varphi_2 \tau U Q_* \quad , \tag{23}$$

$$0 = \alpha \left(P_* - \beta P_*^2\right) + \gamma - \delta R_* + \varphi 3 \tau U R_* \quad . \tag{24}$$

From Eq. (22) and Eq. (23), we arrive at the relations:

$$R_* = -\frac{1}{\theta}\left[\xi \frac{P_*}{Q_*} + \varphi_1 \tau U\right] \quad , \quad Q_* = \frac{bP_*}{\mu + \varphi_2 \tau U + dP_*^{2/3}} \quad .$$

Plug these into Eq. (24), we have an equality:

$$\xi \ln\left(\frac{\mu + \varphi_2 \tau U + dP_*^{2/3}}{b}\right) + \varphi_1 \tau U = -\frac{\theta\gamma}{\alpha\beta P_*^2 - \alpha P_* + \delta - \varphi_3 \tau U} \quad .$$

Define the LHS to be $\Phi(P_*)$ and the RHS to be $\Psi(P_*)$. The number of positive solutions P_* for this equation is the number of stationary states that we need to find. Between any two solutions of $\Psi(P_*) = \Phi(P_*)$, there should be a solution of the equation for the derivatives $\Psi'(P_*) = \Phi'(P_*)$, where we use the notation $' = d/dP_*$. We can calculate that:

$$\Phi'(P_*) = \frac{2}{3}\frac{\xi P_*^{-1/3}}{\mu + \varphi_2 \tau U + dP_*^{2/3}} \quad , \quad \Psi'(P_*) = \frac{\theta\gamma\alpha\left(2\beta P_* - 1\right)}{\left(\alpha\beta P_*^2 - \alpha P_* + \delta - \varphi_3 \tau U\right)^2} \quad ,$$

and since $\Phi'(P_*)$ is always positive, therefore the solutions of the derivatives equality can only exist for $P_* > 1/2\beta$. After some algebraic manipulation, $\Psi'(P_*) = \Phi'(P_*)$ becomes:

$$0 = \frac{1}{3}\frac{\xi d}{\theta\gamma}\left[\left(P_* - \frac{1}{2\beta}\right) + \Xi\right]^2 \left(P_* - \frac{1}{2\beta}\right)^{-1} - \left[\left(\mu + \varphi_2 \tau U\right) P_*^{1/3} + dP_*\right] \quad , \tag{25}$$

where:

$$\Xi = \frac{4\beta\left(\delta - \varphi_3 \tau U\right) - \alpha}{4\alpha\beta^2} \quad .$$

But the RHS of Eq.(25) is a convex function for $P_* > 1/2\beta$, which can be shown by looking at its 2nd-derivative d^2/dP_*^2 in this range:

$$\frac{1}{3}\frac{\xi d}{\theta\gamma}\left[6\left(P_* - \frac{1}{2\beta}\right) + 2\Xi^2\left(P_* - \frac{1}{2\beta}\right)^{-3}\right] + \frac{2}{9}\left(\mu + \varphi_2 \tau U\right) P_*^{-5/3}\Bigg|_{P_* > \frac{1}{2\beta}} > 0 \ ,$$

therefore it can have at most two zeros. Thus, back to the original problem $\Phi(P_*) = \Psi(P_*)$, the maximum number of non-zero stationary tumor sizes is $2 + 1 = \boxed{3}$.

28. Magnetic Carts Two carts, each with a mass of 300 g, are fixed to move on a horizontal track. As shown in the figure, the first cart has a strong, tiny permanent magnet of dipole moment 0.5 $\mathrm{A \cdot m^2}$ attached to it, which is aligned along the axis of the track pointing toward the other cart. On the second cart, a copper tube of radius 7 mm, thickness 0.5 mm, resistivity $1.73 \cdot 10^{-8}\Omega$, and length 30 cm is attached. The masses of the magnet and coil are negligible compared to the mass of the carts. At the moment its magnet enters through the right end of the copper tube, the velocity of the first cart is 0.3 m/s and the distance between the two ends of each cart is 50 cm, find the minimum distance achieved between the two ends of the carts in centimeters. While on the track, the carts experience an effective coefficient of static friction (i.e., what it would be as if they did not have wheels) of 0.01. Neglect the self-inductance of the copper tube.

A picture of the two cart setup. The black rectangle represents the magnet while the gold rectangle represents the copper tube.

Hint 1: The magnetic field due to a dipole of moment $\vec{\mu}$, at a position $\vec{r}$ away from the dipole can be written as

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{(3\hat{r}(\vec{r}\cdot\vec{\mu}) - \vec{\mu})}{r^3}\hat{r}$$

where $\hat{r}$ is the unit vector in the direction of $\vec{r}$.

Hint 2: The following mathematical identity may be useful:

$$\int_{-\infty}^{\infty} \frac{u^2 du}{(1+u^2)^5} = \frac{5\pi}{128}.$$

Solution 28:

We analyze the thin dz of copper which center is at distance z away from the magnet. Notice that the thickness of the copper tube ($w = 0.5mm$) is much smaller than its radius. Thus, we may model the copper tube as an infinitesimally thin current loop of radius $a_0 = a + (w/2)$ (note that $\frac{w}{2a}$ is significant!) with resistance $\rho\frac{2\pi a_0}{wdz}$ where $a = 7mm$. Using the formula for the magnetic field due to the dipole moment, it can be found that the B-field along the current loop is:

$$B = \frac{\mu_0\mu}{4\pi}\frac{(2z^2 - r^2)\hat{z} + 3rz\hat{r}}{(r^2 + z^2)^{5/2}}$$

where $\hat{z}$ is the unit vector parallel to the direction of motion of the magnet and $\hat{r}$ is the unit vector pointing radially outward from the center of the current loop. The flux through the loop is then:

$$\Phi_B = \int_0^{a_0} B_z(2\pi r)dr = \frac{1}{2}\mu_0\mu\frac{a_0^2}{(z^2 + a_0^2)^{3/2}}$$

in which Lenz law can be used to get the current of the loop, using the fact $v = -\frac{dz}{dt}$, where v is the relative velocity of the right cart to the left.

$$I = -\frac{d\Phi_B}{dt}(\rho\frac{2\pi a_0}{wdz})^{-1} = \frac{3\mu_0\mu}{4\pi}\frac{a_0wvzdz}{\rho(a_0^2 + z^2)^{5/2}}$$

The lorentz force on the current loop due to the magnet then obeys:

$$F = I(2\pi a_0)B_r\hat{z} = \frac{9}{8\pi}\mu_0^2\mu^2\frac{a_0^3wvz^2dz}{\rho(a_0^2 + z^2)^5}\hat{z}$$

which is repulsive as expected. By Newton's third law, the force on the magnet due to the current in the copper tube is:

$$F = -\frac{9}{8\pi\rho}\mu_0^2\mu^2a_0^3wv\int_{-x}^{L-x}\frac{z^2dz}{\rho(a_0^2 + z^2)^5}\hat{z}$$

where x is the depth of the magnet into the copper tube. However, when $x, L - x >> a_0$,

$$\int_{-x}^{L-x}\frac{z^2dz}{\rho(a_0^2 + z^2)^5}\hat{z} \approx \frac{a_0^3}{a_0^{10}}\int_{-\infty}^{\infty}\frac{u^2du}{(1 + u^2)^5} = \frac{5\pi}{128a_0^7}$$

Therefore, for as long as the magnet is sufficiently within the tube, the force is instead:

$$F = -\frac{45}{1024\rho a_0^4}\mu_0^2\mu^2wv\hat{z} = -\gamma v\hat{z}$$

where $\gamma = 0.1815kg/s$.
The velocity of the second cart starts moving when $F > \mu_sMg$ with $\mu_s = 0.01$ and $M = 0.3kg$. It turns out that $\frac{0.01Mg}{\gamma v_0} \approx 0.5$, so this static friction is overcome very quickly. Once both carts are in motion, and since there is no kinetic friction, momentum is in fact conserved! As we'll see soon, this results in the carts performing an (infinitely-long) perfectly inelastic collision. In addition, it allows us to use a reduced mass of $M_{red} = M/2$ to calculate the closest distance achieved between the two carts. While $x >> a$ is obeyed:

$$-\gamma v_{rel} = \frac{1}{2}M\frac{dv_{rel}}{dt}$$

giving:

$$v_{rel} = v_0e^{-\frac{2\gamma}{M}t}$$

in which v_{rel} goes to 0 as t goes to ∞, resulting in a perfectly inelastic collision. Since $v_{rel} = -\frac{dx}{dt}$

$$x_{min} = x_0 - \int_0^\infty v_{rel} dt = x_0 - \frac{mv_0}{2\gamma} = 0.252m$$

We see that $x_{min} >> a$, so while messy stuff occurs when $x \lesssim a_0$, accounting for this can only give rise to an error on the order of a_0. A 5% tolerance was given in this problem so that this would be insignificant.

29. De-Terraforming In the far future, the Earth received an enormous amount of charge as a result of Mad Scientist ecilA's nefarious experiments. Specifically, the total charge on Earth is $Q = 1.0 \times 10^{11}$C. (compare this with the current 5×10^5C).
Estimate the maximum height of a "mountain" on Earth that has a circular base with diameter $w = 1.0$ km, if it has the shape of a spherical sector. You may assume that $h_{\max} \ll w$. The tensile strength of rock is 10 MPa.

Solution 29: You can approximate the mound as a conducting sphere of radius r connecting by a wire to the Earth of radius R. We can therefore say that

$$\sigma_{\text{Mound}} \sim \frac{R}{r}\sigma_{\text{Earth}}$$

as voltages are equal and proportional to Q/R. The electrostatic pressure can be given as $P = \frac{\sigma E}{2}$, where $E/2$ comes from the fact that the section does not contribute a force on itself. This can be rewritten as

$$P = \left(\frac{R}{r}\frac{Q}{4\pi R^2}\right) \cdot \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2} = \frac{Q^2}{32\pi\varepsilon_0 R^3 r}.$$

By using Pythagorean theorem, we can find that $r^2 - (r-h)^2 = w^2$ which means that $h = \frac{w^2}{2r}$. We can hence write the tensile strength as

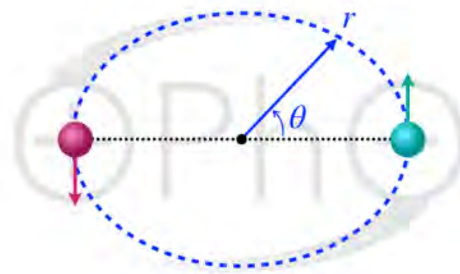
$$Y = \frac{Q^2 h}{16\pi\varepsilon_0 R^3 w^2} \implies h = \frac{16\pi\varepsilon_0 R^3 w^2 Y}{G^2} = \boxed{115 \text{ m}}.$$

Note: As this is an estimate problem, there were many models that could be taken. Some competitors thought that the way electrostatic pressure was calculated was incorrect because it assumes that the force exerted on an element on the mountain is due to the rest of the earth alone. By the same logic, an isolated conducting sphere with a very large surface charge density can exist on its own irrespective of its tensile strength (it would break due to internal free charge repulsions). This argument has validity, but we are considering the fracture surface with the earth while this competitor considered the rocks flying off the mountain itself due to the mountain. It's a grey spot since the problem is an estimate one, so the range was later increased.

30. All Around The World The logo of OPhO describes two objects travelling around their center-of-mass, following the same oval-shape trajectory. For simplicity, we assume these objects are point-like, have identical mass, and interacts via an interacting potential $U(d)$ depends on the distance d between them. Choose the polar coordinates (r, θ) as shown in the figure, where the origin is located at the center of the logo, then the shared trajectory obeys the equation:

$$r(\theta) = \frac{L}{2}\left[1 - \epsilon\cos(2\theta)\right]^{-(1+\gamma)} ,$$

in which we consider $\epsilon = 0.12$ and $\gamma = 0.05$. Here the smallest and largest separation between the objects are d_{min} and d_{max}. Since the interacting potential $U(d)$ is defined up to a constant, let us pick $U(L) = 0$. Find the ratio $U(d_{\text{min}})/U(d_{\text{max}})$.



Solution 30:
Define the dimensionless inverse-radius $u = \frac{L}{d}$ and the angular-derivative $' = \frac{d}{d\theta}$, then for any central interacting potential we have Binet equation (can be derived directly from the equations of motion in polar coordinates):

$$\frac{d}{du}U(u) \propto -\left(u'' + u\right) \quad . \tag{26}$$

For the given oval-shape trajectory:

$$d = L\left[1 - \epsilon\cos(2\theta)\right]^{-(1+\gamma)} \implies u = (1 - \epsilon\cos\phi)^{(1+\gamma)} \;,\; \cos\phi = \epsilon^{-1}\left[1 - u^{\left(\frac{1}{1+\gamma}\right)}\right] \quad , \tag{27}$$

where $\phi = 2\theta$. The angular 2nd-derivative of the dimensionless inverse-radius u can be calculated:

$$\begin{aligned} u'' = 4\frac{d^2}{d\phi^2}u &= 4\epsilon(1+\gamma)\left[\frac{\gamma\epsilon\left(1-\cos^2\phi\right)}{1-\epsilon\cos\phi} + \cos\phi\right](1-\epsilon\cos\phi)^{\gamma} \\ &= 4\epsilon(1+\gamma)\left\{\gamma\epsilon\left(1 - \epsilon^{-2}\left[1 - u^{\left(\frac{1}{1+\gamma}\right)}\right]^2\right)u^{-\left(\frac{1}{1+\gamma}\right)} + \epsilon^{-1}\left[1 - u^{\left(\frac{1}{1+\gamma}\right)}\right]\right\}u^{\left(\frac{\gamma}{1+\gamma}\right)} \\ &= 4(1+\gamma)\left\{\gamma\epsilon^2 u^{-\left(\frac{1-\gamma}{1+\gamma}\right)} - \gamma u^{-\left(\frac{1-\gamma}{1+\gamma}\right)}\left[1 - 2u^{\left(\frac{1}{1+\gamma}\right)} + u^{\left(\frac{2}{1+\gamma}\right)}\right] + \left[u^{\left(\frac{\gamma}{1+\gamma}\right)} - u\right]\right\} \\ &= -4\gamma(1+\gamma)(1-\epsilon^2)u^{-\left(\frac{1-\gamma}{1+\gamma}\right)} + 4(1+\gamma)(1+2\gamma)u^{\left(\frac{\gamma}{1+\gamma}\right)} - 4(1+\gamma)^2 u \quad . \end{aligned}$$

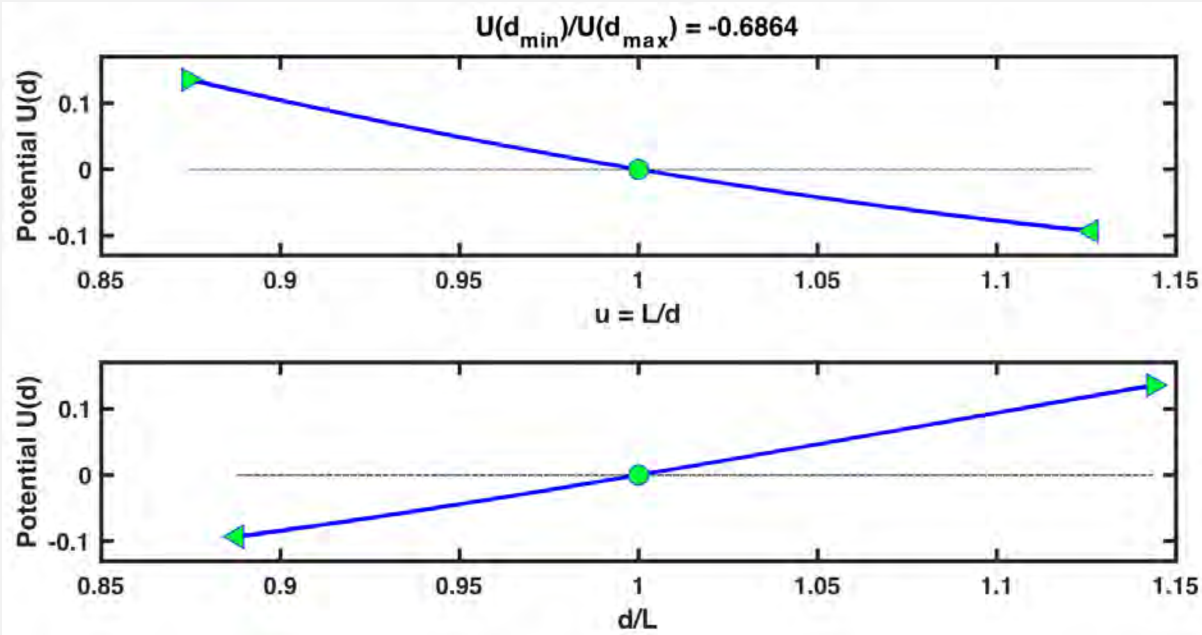
Plug this back into Eq. (26), we can arrive at:

$$\begin{aligned} &\frac{d}{du}U(u) \propto \left[4(1+\gamma)^2 - 1\right]u + 4\gamma(1+\gamma)(1-\epsilon^2)u^{-\left(\frac{1-\gamma}{1+\gamma}\right)} - 4(1+\gamma)(1+2\gamma)u^{\left(\frac{\gamma}{1+\gamma}\right)} \\ \implies\; &U(u) \propto \left(2 - \frac{1}{2(1+\gamma)^2}\right)u^2 + 2(1-\epsilon^2)u^{\left(\frac{2\gamma}{1+\gamma}\right)} - 4u^{\left(\frac{1+2\gamma}{1+\gamma}\right)} + W \quad , \end{aligned} \tag{28}$$

where W is a constant so that we can assign $U(u=1) = 0$. For $\epsilon = 0.12$ and $\gamma = 0.05$, we get $W = 0.48231$. Following that, we use Eq. (27), the maximum distance corresponds to $\theta = 0$ and thus $u_{\triangleright} = 0.87439$, the minimum distance corresponds to $\theta = \pi/2$ and thus $u_{\triangleleft} = 1.1264$. The ratio of interests, therefore, can be found directly:

$$\frac{U(u = u_{\triangleleft})}{U(u = u_{\triangleright})} \approx \boxed{-0.6864} \quad .$$

We plot the interacting potential, up to a pre-factor as given in Eq. (28), in the figure below.



31. Electrostatic Pendulum 1 Follin is investigating the electrostatic pendulum. His apparatus consists of an insulating Styrofoam ball with a mass of 14 mg and radius $r = 0.5$ cm suspended on a uniform electrically-insulating string of length 1 m and mass per unit length density of $1.1 \cdot 10^{-5}$ kg/m between two large metal plates separated by a distance 17 cm with a voltage drop of $10kV$ between them, such that when the ball is in equilibrium, its center of mass is exactly equidistant to the two plates. Neglect the possibility of electrical discharge throughout the next two problems.

Follin then gives the ball a charge 0.15 nC. Assuming that the charge is distributed evenly across the surface of the ball, find the subsequent horizontal deflection of the pendulum bob's center of mass from its hanging point at equilibrium.

32. Electrostatic Pendulum 2 Hoping to get a larger deflection, Follin replaces the insulating Styrofoam ball with a conducting pith ball of mass 250 mg and 2 cm and daisy chains 4 additional 10 kV High Voltage Power Supplies to increase the voltage drop across the plates to 50 kV. Leaving the plate separation and the string unchanged, he repeats the same experiment as before, but forgets to measure the charge on the ball. Nonetheless, once the ball reaches equilibrium, he measures the deflection from the hanging point to be 5.6 cm. Find the charge on the ball.

Solution 31: The force on the Styrofoam ball due to the electric field from the two plates is $F = \frac{QV}{d}$. Since the plates are conducting, the ball also experiences an attraction toward the closer plate. This force can be neglected because it is much smaller than $\frac{QV}{d}$ (an assumption that will be justified at the end of the solution). The mass of the string, however, needs to be considered. Consider the forces on an infinitesimal segment of the string. The horizontal component of the tension must balance out the electric force on the ball and the vertical component must balance out the weight of everything below it (string and ball). This gives us

$$\frac{dx}{dh} = \frac{F}{mg + \lambda gh}$$

where h is the height above the ball, x is the horizontal displacement from equilibrium, F is the electrostatic force, λ is the mass density of the string, and m is the mass of the ball. Separating variables, we have

$$x = F\int_0^L \frac{dh}{mg+\lambda gh}$$

where L is the length of the string. Integrating, we get

$$x = \frac{F}{\lambda g}\ln\left(\frac{m+\lambda L}{m}\right) = \boxed{0.0475 \text{ m}}$$

Now, let's justify our assumption that the attraction of the ball towards the plates is negligible. Using the method of image charges, the force due to the closer plate is equivalent to a charge $-Q$ a distance of $d-2x$ away. Infinitely many image charges exist due to the other plate but they have a smaller effect so they will not be considered. The force due to the image charge is

$$F' = \frac{Q^2}{4\pi\epsilon_0 (d-2x)^2} = 3.59\times 10^{-8} \text{ N}$$

which is indeed much less than $F = \frac{QV}{d} = 8.82\times 10^{-6}$ N

Solution 32: When a conducting ball is placed in a uniform electric field, the charges separate so that the sphere becomes a dipole with electric dipole moment

$$p = 4\pi\epsilon_0 r^3 E_0$$

Let Q be the charge on the ball. The ball can be approximated as a point dipole with dipole moment p and a point charge of magnitude Q at the center of the ball. Since the plates are conducting and must be equipotential, the charge and dipole will experience an attractive force to the plates. Using the method of image charges, the force caused by the closer plate is equivalent to that of a point charge $-Q$ and a dipole moment p at a distance $d-2x$ from the center of the ball, where $d = 17$ cm is the separation between the plates and $x = 5.6$ cm is the deflection of the ball from the hanging point. The forces from the farther plate are negligible compared to the force due to the electric field, QE_0. The force between a point charge Q and a point dipole moment p a distance r apart is

$$F = \frac{pQ}{2\pi\epsilon_0 r^3}$$

The force between two dipoles with dipole moments of magnitude p in the same direction as the vector from one dipole to the other is

$$F = \frac{3p^2}{2\pi\epsilon_0 r^4}$$

Adding up the charge-image charge, charge-image dipole, dipole-image charge, and dipole-image dipole interactions, we get that the total electrostatic force on the ball is

$$F = QE_0 + \frac{Q^2}{4\pi\epsilon_0 (d-2x)^2} + \frac{pQ}{\pi\epsilon_0 (d-2x)^3} + \frac{3p^2}{2\pi\epsilon_0 (d-2x)^4}$$

From the previous problem, we know that the displacement x of the ball from equilibrium as a function of the electrostatic force F is

$$x = \frac{F}{lg}\ln\left(\frac{m+l}{m}\right)$$

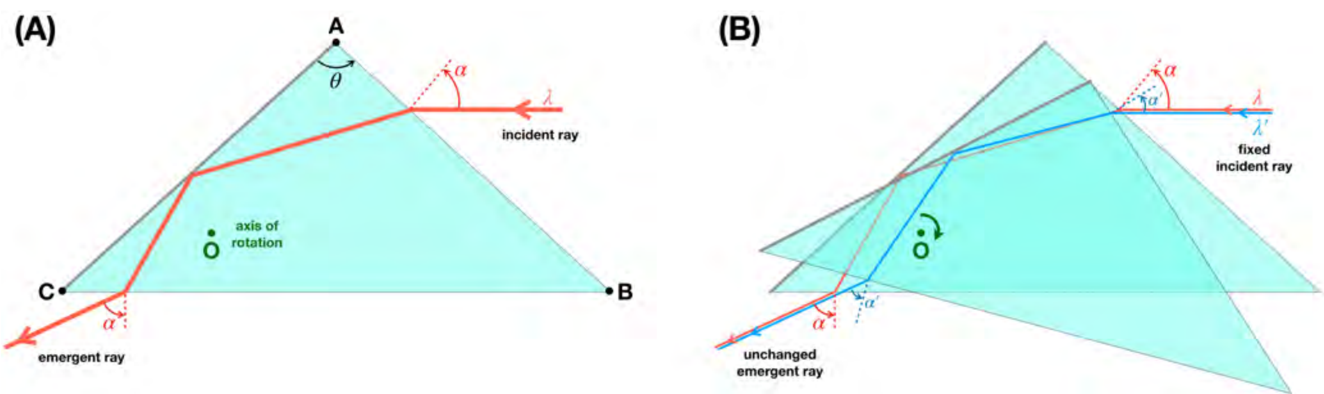
Rearranging this as a quadratic in Q and solving, we have

$$\frac{1}{4\pi\epsilon_0 (d-2x)^2}Q^2 + \left(\frac{p}{\pi\epsilon_0 (d-2x)^3} + E_0\right)Q + \left(\frac{3p^2}{2\pi\epsilon_0 (d-2x)^4} - \frac{lgx}{\ln\left(\frac{m+l}{m}\right)}\right) = 0$$

$$Q = \frac{-\left(\frac{p}{\pi\epsilon_0(d-2x)^3} + E_0\right) + \sqrt{\left(\frac{p}{\pi\epsilon_0(d-2x)^3} + E_0\right)^2 - 4\left(\frac{1}{4\pi\epsilon_0(d-2x)^2}\right)\left(\frac{3p^2}{2\pi\epsilon_0(d-2x)^4} - \frac{lgx}{\ln\left(\frac{m+l}{m}\right)}\right)}}{2\frac{1}{4\pi\epsilon_0(d-2x)^2}}.$$

Plugging gives $Q = \boxed{4.48 \times 10^{-10}\ \mathrm{C}}$.

The following information applies for the next two problems. Consider a uniform isosceles triangle prism ABC, with the apex angle $\theta = 110°$ at vertex A. One of the sides, AC, is coated with silver, allowing it to function as a mirror. When a monochrome light-ray of wavelength λ approaches side AB at an angle of incidence α, it first refracts, then reaches side AC, reflects, and continues to base BC. After another refraction, the ray eventually exits the prism at the angle of emergence which is also equal to the angle of incidence (see Fig. A).



33. Man In The Mirror 1 What is the relative refractive index of the prism for that particular wavelength λ with respect to the outside environment, given that $\alpha = 70°$.

34. Man In The Mirror 2 Consider keeping the incident ray fixed while changing the monochrome color to a different wavelength λ', so that by rotating the prism around an axis of rotation O it can follows the above description (approaches side AB at an angle of incidence α', refracts, then reaches side AC, reflects, and continues to base BC, then another refraction, the ray eventually exits the prism at the angle of emergence which is also equal to the angle of incidence). We observe then the emergent ray remains unchanged for any value of λ' (see Fig. B). Find the maximum possible length-ratio between the distance from the axis to the one of the vertices max(OA, OB, OC) and the base BC.

Solution 33: The light-path refracts on side AB at point M, reflects on side AC at point N and refracts on base BC at point P (see Fig. A). Define the angle of refraction inside the prism to be β, then from Snell's law:

$$\sin\alpha = n\sin\beta\ . \tag{29}$$

From the law of reflection and the 180^o-sum of three interior angles inside any triangles:

$$\widehat{\mathrm{MNA}} = 180^o - \widehat{\mathrm{NAM}} - \widehat{\mathrm{AMN}} = 180^o - \theta - (90^o - \beta)$$
$$= \widehat{\mathrm{PNC}} = 180^o - \widehat{\mathrm{NCP}} - \widehat{\mathrm{CPN}} = 180^o - \left(\frac{180^o - \theta}{2}\right) - (90^o + \beta) \ ,$$

we obtain the refraction angle β to be:

$$\beta = \frac{3\theta - 180^o}{4} \ .$$

Plug this finding into Eq. (29), we get the relative refraction index of the prism with respect to the outside environment:

$$n = \frac{\sin\alpha}{\sin\beta} = \frac{\sin\alpha}{\sin\left(\frac{3\theta - 180^o}{4}\right)}\Bigg|_{\alpha=70^o,\theta=110^o} \approx \boxed{1.5436} \ .$$

Solution 34: Another wavelength λ' corresponds to a new relative refractive index n' and therefore a new angle angle of incidence and emergence α'. Say, the new light-path after rotating the prism around the O axis is through a refraction at point M' on side AB, a reflection at point N' on side AC, and a refraction P' on base BC. Choose a parallelogram Bac coordinates where Ba pointing in the same direction as vector $\overrightarrow{\mathrm{BA}}$ and Bc pointing in the same direction as vector $\overrightarrow{\mathrm{BC}}$ (see Fig. B). Define the angle:

$$\gamma = \widehat{\mathrm{ABC}} = \widehat{\mathrm{BCA}} = \frac{180^o - \theta}{2} \ .$$

Due to symmetry (equal angles of incidence and emergence), the refraction angle inside the prism is always $\beta = 90^o - 3\gamma/2$ and the deviation angle between the incident and emergent rays is always γ, regardless of α'. The vector $\overrightarrow{\mathrm{P'N'}}$ made an angle $\gamma - 90^o - \beta = 5\gamma/2 - 180^o$ with Ba and therefore the line equation of M'N' in Bac is given by:

$$\frac{c - c_{\mathrm{P'}}}{a - a_{\mathrm{P'}}} = \frac{c - c_{\mathrm{P'}}}{a} = \frac{\sin(5\gamma/2 - 180^o)}{\sin[\gamma - (5\gamma/2 - 180^o)]} = -\frac{\sin(5\gamma/2)}{\sin(3\gamma/2)} \ , \tag{30}$$

where here we denote $(a_{\mathrm{X}}, c_{\mathrm{X}})$ as the position of point X in the parallelogram Bac coordinates. Similarly, the line equation of CA (which vector $\overrightarrow{\mathrm{CA}}$ makes an angle $2\gamma - 180^o$ with Ba) Bac is given by:

$$\frac{c - c_{\mathrm{C}}}{a - a_{\mathrm{C}}} = \frac{c - c_{\mathrm{C}}}{a} = \frac{\sin(2\gamma - 180^o)}{\sin[\gamma - (2\gamma - 180^o)]} = -\frac{\sin(2\gamma)}{\sin(\gamma)} \ . \tag{31}$$

Since N' is a point on line CA, the position N' $(a_{\mathrm{N'}}, c_{\mathrm{N'}})$ must satisfy both Eq. (30) and Eq. (31), which we can solve to obtain:

$$a_{\mathrm{N'}} = \frac{(c_{\mathrm{C}} - c_{\mathrm{P'}})\sin(3\gamma/2)}{\sin(\gamma/2)} \ , \ c_{\mathrm{N'}} = \frac{c_{\mathrm{P'}}\sin(3\gamma/2)\sin(2\gamma) - c_{\mathrm{C}}\sin(\gamma)\sin(5\gamma/2)}{\sin(\gamma/2)\sin(\gamma)} \ . \tag{32}$$

Next, we consider the line M'N'. Vector $\overrightarrow{\mathrm{N'M'}}$ makes an angle $-90^o - \beta = 3\gamma/2 - 180^o$ with Ba, thus, in Bac the line N'M' should obeys the equation:

$$\frac{c - c_{\mathrm{N'}}}{a - a_{\mathrm{N'}}} = \frac{\sin(3\gamma/2 - 180^o)}{\sin[\gamma - (3\gamma/2 - 180^o)]} = -\frac{\sin(3\gamma/2)}{\sin(\gamma/2)} \ .$$

Since point M' in also on side AB, therefore $c_{\mathrm{M'}} = 0$ and this leads to:

$$a_{\mathrm{M'}} = \frac{a_{\mathrm{N'}} \sin(3\gamma/2) + c_{\mathrm{N'}} \sin(\gamma/2)}{\sin(3\gamma/2)} = \frac{2c_{\mathrm{C}} \sin\gamma \cos(\gamma/2) - c_{\mathrm{P'}} \sin(3\gamma/2)}{\sin(3\gamma/2)} , \tag{33}$$

where for the final algebra manipulation we use Eq. (32).

Now let us choose another parallelogram Oxy coordinates where Ox pointing in the same direction as the incident ray and Oy pointing in the same direction as the emergent ray (see Fig. B). Note that $\widehat{x\mathrm{O}y} = \widehat{a\mathrm{B}c} = \gamma$, and the angle between Ba and Ox is $\alpha' - 90^o$. The coordinate transformation $(a, c) \to y$ due to pure translation and rotation:

$$y - y_{\mathrm{B}} = \frac{a \sin(\alpha' - 90^o) + c \sin[\gamma + (\alpha' - 90^o)]}{\sin\gamma} = -\frac{a\cos\alpha' + c\cos(\gamma + \alpha')}{\sin\gamma} , \tag{34}$$

where here we denote $(x_{\mathrm{X}}, y_{\mathrm{X}})$ as the position of point X in the parallelogram Bxy coordinates. Hence the y-coordinates of point M' is given by:

$$y_{\mathrm{M'}} = y_{\mathrm{B}} - \frac{a_{\mathrm{M'}}\cos\alpha'}{\sin\gamma} = y_{\mathrm{B}} - \frac{[2c_{\mathrm{C}} \sin\gamma \cos(\gamma/2) - c_{\mathrm{P'}} \sin(3\gamma/2)]\cos\alpha'}{\sin(3\gamma/2)\sin\gamma} , \tag{35}$$

in which we utilize the result found in Eq. (33). Similarly, for $(a, c) \to x$:

$$x - x_{\mathrm{B}} = \frac{a\cos(\gamma - \alpha') + c\cos\alpha'}{\sin\gamma} , \tag{36}$$

consider point P' on the x-axis we also arrive at the following relationship with Eq. (36):

$$x_{\mathrm{P'}} = x_{\mathrm{B}} - \frac{c_{\mathrm{P'}}\cos\alpha'}{\sin\gamma} ,$$

and express $(x_{\mathrm{B}}, y_{\mathrm{B}})$ in terms of $(a_{\mathrm{O}}, c_{\mathrm{O}})$ with Eq. (36) and Eq. (34):

$$\begin{aligned} x_{\mathrm{O}} - x_{\mathrm{B}} &= \frac{a_{\mathrm{O}}\cos(\gamma - \alpha') + c_{\mathrm{O}}\cos\alpha'}{\sin\gamma} \quad \Longrightarrow \quad x_{\mathrm{B}} = -\frac{a_{\mathrm{O}}\cos(\gamma - \alpha') + c_{\mathrm{O}}\cos\alpha'}{\sin\gamma} , \\ y_{\mathrm{O}} - y_{\mathrm{B}} &= -\frac{a_{\mathrm{O}}\cos\alpha' + c_{\mathrm{O}}\cos(\gamma + \alpha')}{\sin\gamma} \quad \Longrightarrow \quad y_{\mathrm{B}} = \frac{a_{\mathrm{O}}\cos\alpha' + c_{\mathrm{O}}\cos(\gamma + \alpha')}{\sin\gamma} . \end{aligned}$$

Using these to replace $c_{\mathrm{P'}}$ and $(x_{\mathrm{B}}, y_{\mathrm{B}})$ in Eq. (35) and rearrange the terms, we end up with:

$$x_{\mathrm{P'}} = y_{\mathrm{M'}} - (a_{\mathrm{O}} - c_{\mathrm{O}})\sin\alpha' - \left[(a_{\mathrm{O}} + c_{\mathrm{O}}) - \frac{c_{\mathrm{C}}\sin\gamma}{\cos(\gamma/2)\sin(3\gamma/2)}\right]\cot(\gamma/2)\cos\alpha' , \tag{37}$$

where the emergent ray position $x_{\mathrm{P'}}$ is a function of the incident ray position $x_{\mathrm{P'}}$, the angle of incidence/emergence α', and constants of prism rotation: γ, $(a_{\mathrm{O}}, a_{\mathrm{O}})$.

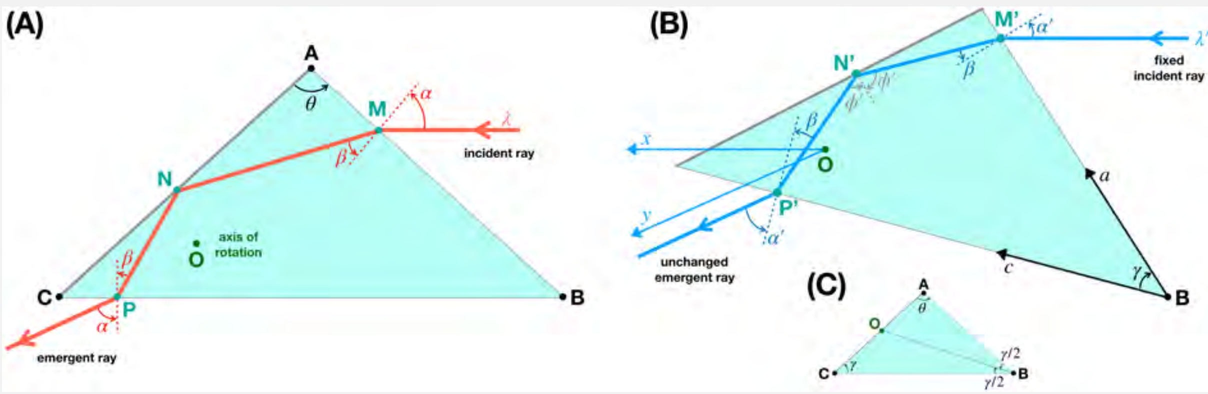
If we keep the incident ray fixed $y_{\mathrm{M'}} = \text{const}$ and rotate the prism after changing the monochromatic wavelength λ' (varying α'), to have the emergent ray stays the same $x_{\mathrm{P'}} = \text{const}$, followed from Eq. (37) we need:

$$\begin{aligned} 0 = a_{\mathrm{O}} - c_{\mathrm{O}} &= (a_{\mathrm{O}} + c_{\mathrm{O}}) - \frac{c_{\mathrm{C}}\sin\gamma}{\cos(\gamma/2)\sin(3\gamma/2)} \\ \Longrightarrow \quad a_{\mathrm{O}} = c_{\mathrm{O}} &= \frac{c_{\mathrm{C}}\sin\gamma}{2\cos(\gamma/2)\sin(3\gamma/2)} . \end{aligned} \tag{38}$$

We have found the position of the rotational axis O, independence of the position where the incident ray hits the side AB! The constraint $a_{\mathrm{O}} = c_{\mathrm{O}}$ means it is on the line bisecting the angle between the incidence and the emergence faces of the prism $\widehat{\mathrm{ABC}}$. We also note that the solution found in Eq. (38) satisfy the line equation of CA which is given in Eq. (31).

In summary, the rotational axis O is the intersection between the bisector of angle $\widehat{\mathrm{ABC}}$ and line CA (see Fig. C). After some geometrical analysis, we get the result:

$$\frac{\max\left(\mathrm{OA},\mathrm{OB},\mathrm{OC}\right)}{\mathrm{BC}} = \frac{\mathrm{OB}}{\mathrm{BC}} = \frac{\sin\widehat{\mathrm{OCB}}}{\sin\widehat{\mathrm{BOC}}} = \frac{\sin\gamma}{\sin\left(180^o - \frac{3}{2}\gamma\right)} = \frac{\sin\left(\frac{180^o-\theta}{2}\right)}{\sin\left(\frac{3\theta+180^o}{4}\right)}\Bigg|_{\theta=110^o} \approx \boxed{0.72300}\,.$$



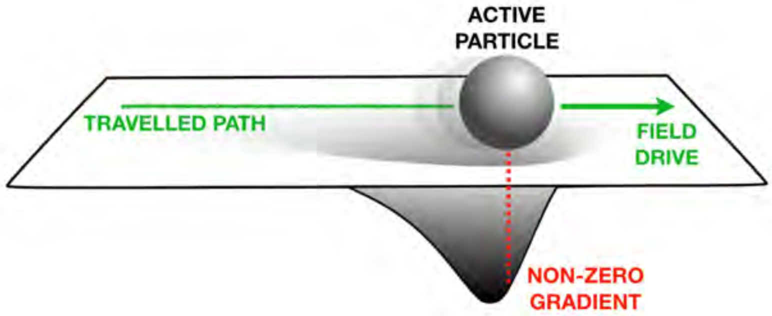
35. Funiculì, Funiculà Field-drive is a locomotion mechanism that is analogous to general relativistic warp-drive. In this mechanism, an active particle continuously climbs up the field-gradient generated by its own influence on the environment so that the particle can bootstrap itself into a constant non-zero velocity motion. Consider a field-drive in one-dimensional (the Ox axis) environment, where the position of the particle at time t is given by $X(t)$ and its instantaneous velocity follows from:

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = \kappa\frac{\partial}{\partial x}R(x,t)\Big|_{x=X(t)}\,,$$

in which κ is called the guiding coefficient and $R(x,t)$ is the field-value in this space. Note that, the operation $...|_{x=X(t)}$ means you have to calculate the part in ... first, then replace x with $X(t)$. For a biological example, the active particle can be a cell, the field can be the nutrient concentration, and the strategy of climbing up the gradient can be chemotaxis. The cell consumes the nutrient and also responses to the local nutrient concentration, biasing its movement toward the direction where the concentration increases the most. If the nutrient is not diffusive and always recovers locally (e.g. a surface secretion) to the value which we defined to be 0, then its dynamics can usually be approximated by:

$$\frac{\partial}{\partial t}R(x,t) = -\frac{1}{\tau}R(x,t) - \gamma\exp\left\{-\frac{[x-X(t)]^2}{2\lambda^2}\right\}\,,$$

where τ is the timescale of recovery, γ is the consumption, and λ is the characteristic radius of influence. Before we inoculate the cell into the environment, $R = 0$ everywhere at any time. What is the smallest guiding coefficient κ (in μm^2/s) for field-drive to emerge, if the parameters are $\tau = 50$s, $\gamma = 1$s^{-1}, and $\lambda = 10\mu$m.



The title of this problem means "funicular up, funicular down".

Solution 35:
Assume that we inoculate the cell into the environment at position $x = 0$ and $t = 0$. The field dynamics at $t > 0$ can be rewritten as:

$$\begin{aligned}\frac{\partial}{\partial t}R(x,t) + \frac{1}{\tau}R(x,t) = \exp\left(-\frac{t}{\tau}\right)\partial_t\left[\exp\left(+\frac{t}{\tau}\right)R(x,t)\right] &= -\gamma\exp\left\{-\frac{[x-X(t)]^2}{2\lambda^2}\right\} \\ \implies \exp\left(+\frac{t}{\tau}\right)R(x,t) = \int_0^t dt'\exp\left(+\frac{t'}{\tau}\right)&\left(-\gamma\exp\left\{-\frac{[x-X(t')]^2}{2\lambda^2}\right\}\right) \\ \implies R(x,t) = -\gamma\int_0^t dt'\exp&\left\{-\frac{t-t'}{\tau} - \frac{[x-X(t')]^2}{2\lambda^2}\right\} .\end{aligned} \tag{39}$$

If the cell can field-drive at a constant velocity $W > 0$, then after a very long time $t \to +\infty$ we expect the cell will be in a steady-state, moving at this velocity. For consistency, this field-drive velocity W should related to the field gradient evaluated at $x = X(t)$ such that:

$$W = \kappa\partial_x R(x,t)\Big|_{x=X(t)} . \tag{40}$$

From Eq. (39) we obtain:

$$\begin{aligned}W &= \kappa\partial_x\left(-\gamma\int_0^t dt'\exp\left\{-\frac{t-t'}{\tau} - \frac{[x-X(t')]^2}{2\lambda^2}\right\}\right)\Bigg|_{x=X(t)} \\ &= \frac{\kappa\gamma}{\lambda^2}\int_0^t dt'\,[x-X(t')]\exp\left\{-\frac{t-t'}{\tau} - \frac{[x-X(t')]^2}{2\lambda^2}\right\}\Bigg|_{x=X(t)} \\ &= \frac{\kappa\gamma}{\lambda^2}\int_0^t dt'\,[X(t)-X(t')]\exp\left\{-\frac{t-t'}{\tau} - \frac{[X(t)-X(t')]^2}{2\lambda^2}\right\} .\end{aligned}$$

We then use the steady field-drive condition $X(t) - X(t') = W(t-t')$ at $t \to +\infty$ and define $t'' = t - t'$, so that the temporal integration $\int dt''$ will run from 0 to $+\infty$:

$$\begin{aligned}W &= \frac{\kappa\gamma}{\lambda^2}\int_0^t dt'\,[W(t-t')]\exp\left\{-\frac{t-t'}{\tau} - \frac{[W(t-t')]^2}{2\lambda^2}\right\} \\ &= \frac{\kappa\gamma}{\lambda^2}\int_0^{+\infty} dt''(Wt'')\exp\left[-\frac{t''}{\tau} - \frac{(Wt'')^2}{2\lambda^2}\right] .\end{aligned} \tag{41}$$

For the set of parameter values $(\kappa, \tau, \gamma, \lambda)$ when the field-drive mechanism start to emerge, we can treat the field-drive velocity as infinitesimal small $W = 0^+$. Thus, divide both sides of Eq.(41) by W, we can arrive at:

$$1 = \frac{\kappa\gamma}{\lambda^2}\int_0^{+\infty} dt' t'' \exp\left[-\frac{t''}{\tau} - \frac{(Wt'')^2}{2\lambda^2}\right]\Bigg|_{W=0^+} = \frac{\kappa\gamma}{\lambda^2}\int_0^{+\infty} dt' t'' \exp\left(-\frac{t''}{\tau}\right) = \frac{\kappa\gamma\tau^2}{\lambda^2} \quad .$$

Hence, the smallest guiding coefficient that give us field-drive, for $\tau = 50\text{s}$, $\gamma = 1\text{s}^{-1}$, $\lambda = 10\mu\text{m}$:

$$\kappa = \frac{\lambda^2}{\gamma\tau^2} = \boxed{4 \times 10^{-2}\mu\text{m/s}} \quad .$$

2023 Online Physics Olympiad: Invitational Contest

Theoretical Examination

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Instructions for Theoretical Exam

The theoretical examination consists of 3 long answer questions and 160 points over 2 full days from August 5, 0:01 am GMT.

- **The team leader should submit their final solution document in this google form.**
- If you wish to request a clarification, please use this form. To see all clarifications, view this document.
- Participants are given a google form where they are allowed to submit up-to 100 megabytes of data for their solutions. It is recommended that participants write their solutions in LaTeX. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting. If you wish a premade LaTeX template, we have made one for you here.
- Since each question is a long answer response, participants will be judged on the quality of your work. To receive full points, participants need to show their work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in the IPhO formula sheet) can be cited without proof.
- Remember to state any approximations made and which system of equations were solved after every step. Explicitly showing every step of algebra is not necessary. Participants may leave all final answers in symbolic form (in terms of variables) unless otherwise specified. Be sure to state all assumptions.

Problems

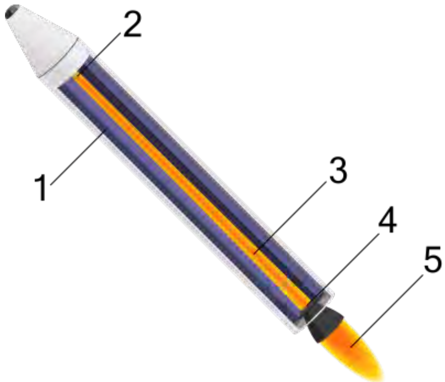
- **T1: Booster**
- **T2: The Complex Potential**
- **T3: General Relativity**

List of Constants

Proton mass	$m_p = 1.67 \cdot 10^{-27}$ kg
Neutron mass	$m_n = 1.67 \cdot 10^{-27}$ kg
Electron mass	$m_e = 9.11 \cdot 10^{-31}$ kg
Avogadro's constant	$N_0 = 6.02 \cdot 10^{23}\ \mathrm{mol}^{-1}$
Universal gas constant	$R = 8.31\ \mathrm{J/(mol \cdot K)}$
Boltzmann's constant	$k_B = 1.38 \cdot 10^{-23}$ J/K
Electron charge magnitude	$e = 1.60 \cdot 10^{-19}$
1 electron volt	$1\ \mathrm{eV} = 1.60 \cdot 10^{-19}$ J
Speed of light	$c = 3.00 \cdot 10^{8}$ m/s
Universal Gravitational constant	$G = 6.67 \cdot 10^{-11}\ (\mathrm{N \cdot m^2})/\mathrm{kg}^2$
Acceleration due to gravity	$g = 9.81\ \mathrm{m/s^2}$
1 unified atomic mass unit	$1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$
Planck's constant	$h = 6.63 \cdot 10^{-34}\ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15}\ \mathrm{eV \cdot s}$
Permittivity of free space	$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}$
Coulomb's law constant	$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}\ (\mathrm{N \cdot m^2})/\mathrm{C}^2$
Permeability of free space	$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T \cdot m/A}$
Stefan-Boltzmann constant	$\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$

T1: Booster

In this problem, we explore a simplified model of Solid Rocket Boosters (SRBs). SRBs are supplements to liquid rockets and provide enormous thrust at liftoff at the expense of lower specific impulse. They have a pretty simple design – a tube containing solid propellant with a central cavity that acts as a combustion chamber. As the solid fuel burns (deflagrates), gasses are forced out of the combustion chamber through a nozzle, producing thrust. It is desirable to choose a combustion chamber design which produces constant thrust (to reduce structural load on the spacecraft) and also have constant internal pressure and temperature (to reduce stress on the SRB). Below a diagram, along with a typical shape of the combustion chamber.



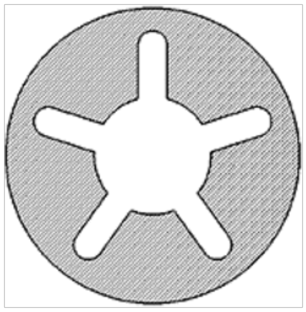
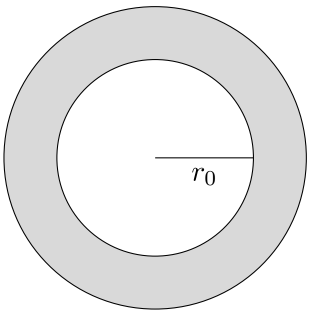


Figure 1: The structure of an SRB (left) and the shape of a typical chamber (right)

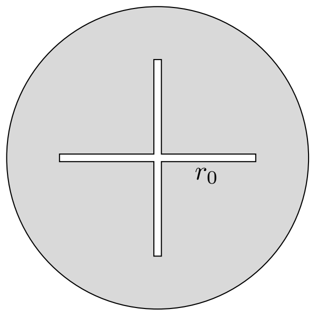
Data:
Rate of vaporization of surface of fuel: $v = 9 \times 10^{-3} \frac{\text{m}}{\text{s}}$
Length of combustion chamber: $l = 45\text{m}$
Radius of combustion chamber: $r_0 = 0.6\text{m}$
Density of propellant: $\rho = 1500 \frac{\text{kg}}{\text{m}^3}$
Molar mass of exhaust gas: $M = 0.040 \frac{\text{kg}}{\text{mol}}$
Temperature inside chamber: $T = 4000$ K
Ideal gas constant: $R = 8.31 \frac{\text{J}}{\text{mol}\cdot\text{K}}$

(a) Find the burn rate of fuel (in terms of $\frac{\text{kg}}{\text{s}}$) as a function of time for the following designs for the combustion chamber. Assume that the combustion chamber never reaches the walls of the SRB.

(i)



(ii)



Answer the following for design i.

(b) Compute the thrust as a function of time. Assume the temperature and pressure of the chamber stay constant, and the density of the gas is ρ_g.

(c) Is the assumption of constant internal conditions for this design valid?

(d) Now assume that the temperature is constant, but the pressure changes. Estimate to within a factor of 2 the pressure inside the chamber as a function of time.

T2: The Complex Potential

A Conformal Transformation

A complex number can be thought of as a mathematical object that "stores" two real numbers. An element z of the set of complex numbers $\mathbb{C}$ can be written as:

$$z = x + iy$$

where $i \equiv \sqrt{-1}$ is called the imaginary unit, and $x, y \in \mathbb{R}$. With this, one can think of constructing complex-valued functions – namely, a function $f : \mathbb{C} \to \mathbb{C}$:

$$f(z) = f(x + iy) = w(x, y) + iu(x, y) \quad (1) \tag{1}$$

Since complex numbers encode two real numbers x and y, we can consider complex-valued functions yielding two real-valued functions $w(x, y)$ and $u(x, y)$ that depend on x, y. w and u, being real-valued multivariable functions, allow us to extend calculus on $\mathbb{R}^2$ to $\mathbb{C}$. You are given that the composition of two complex differentiable functions yields another complex differentiable function on the appropriate domain.

(a) Starting from the definition of a derivative in single-variable calculus, show that if f given in (1) is complex differentiable, w and u satisfy:

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial w}{\partial y} = -\frac{\partial u}{\partial x}$$

(b) Show that $w(x, y)$ and $u(x, y)$ are solutions to the 2D Laplace equation:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{2}$$

(c) A parametrized curve γ on the complex plane can be thought of as a function that takes a real number t and yields a point $\gamma(t)$ on the complex plane. Two parametrized curves γ_1 and γ_2 intersect at some point $p = \gamma_1(t_1) = \gamma_2(t_2)$; tangent lines of the two curves at p form an angle α. Show that the tangent lines of the curves $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(p)$ form the same angle α, where $\circ$ is the composition function.

A Conformal Transformed Electrostatic System

We know from electrostatics that in a charge-free region, the electrostatic potential ϕ satisfies the Laplace equation $\nabla^2 \phi = 0$. As you've shown above, this motivates the idea of defining a complex differentiable function whose real part is the electrostatic potential – complex differentiability encodes the fact that the electrostatic potential satisfies (2). Given a 2D electrostatics problem in a charge-free region, we define the *complex potential* $f(z)$ as:

$$f(z) = \phi(x, y) + i\psi(x, y)$$

where ψ is an appropriate real-valued function that satisfies the conditions derived in question 1, making f a complex differentiable function. You are given the uniqueness theorem: the 2D Laplace equation has a unique solution that satisfies specified boundary conditions.

(d) Given a 2D electrostatics problem and an appropriate complex potential f, write the derivative $\frac{\mathrm{d}f}{\mathrm{d}z}$ in terms of the x and y components of the electric field: $\mathbf{E}(x, y) = E_x(x, y)\hat{x} + E_y(x, y)\hat{y}$.

(e) Show that the function $\phi(x, y) = \frac{\phi_0}{\pi} \arctan \frac{y}{x}$ satisfies the 2D Laplace equation with the boundary condition $\phi(x > 0, 0) = 0$ while $\phi(x < 0, 0) = \phi_0$.

(f) The complex logarithm function is defined at $z = re^{i\theta}$ as $\log(z) = \log|r| + i\theta$ for $r \neq 0$, $0 < \theta < 2\pi$. Show that the upper-half plane $\mathcal{H}$ (points on the complex plane with a positive imaginary part), under the logarithm map, turns into a strip bounded by $\mathrm{Im}(z) = 0$ and $\mathrm{Im}(z) = \pi$. You may assume that the logarithm function is complex differentiable on $\mathcal{H}$ and that its inverse is e^z for this problem.

(g) Starting from the electrostatic potential in (e), find the electrostatic potential in the region between two infinitely large capacitor plates separated by distance d. Carefully argue why this method yields the correct solution.

(h) Show that the function $f(z) = \frac{i(1-z)}{(1+z)}$ maps the unit circle $\mathcal{C}$, centered at the origin, to the real line and the interior of the circle to $\mathcal{H}$. Determine the inverse g of the function f and explain why there's a one-to-one correspondence between points bounded by $\mathcal{C}$ and $\mathcal{H}$.

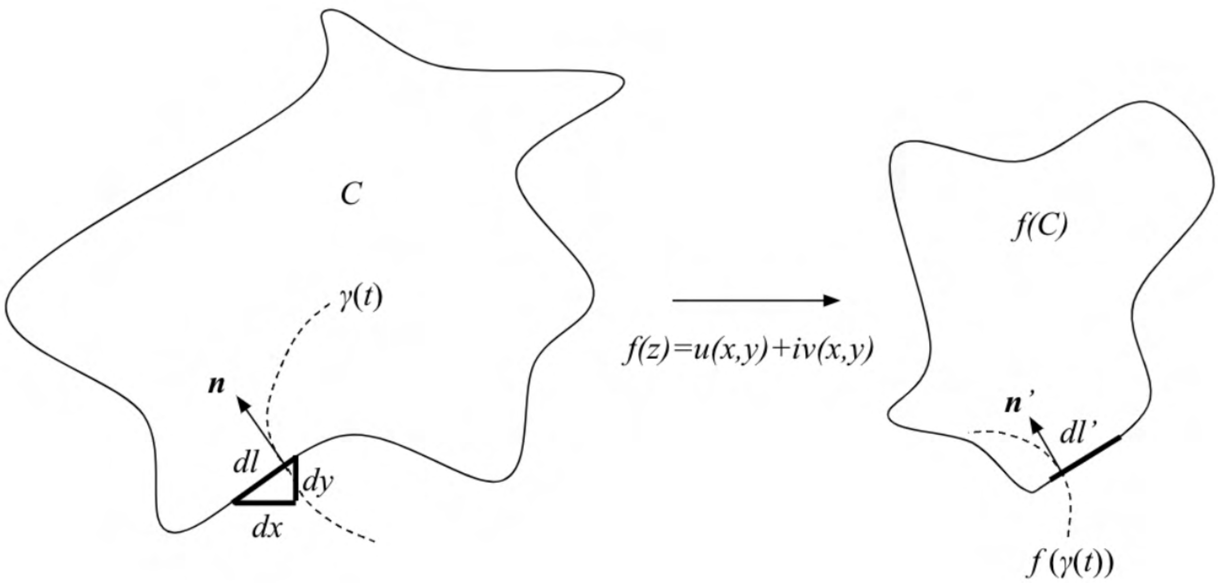
(i) Find the electrostatic potential in the region between an infinite rod of charge density λ and a grounded concentric conducting cylindrical shell of radius R.

(j) Now consider the same infinite rod suspended above a grounded conducting plate at a height h. Use your results for questions (h) and (i) to derive the electrostatic potential ϕ in the region above the plate.

(k) Consider the curve C shown in the figure below. You are given that the image of dl is dl'. If we take some line element dl on the curve C, find the scaling factor to its image dl' on $f(C)$. This is known as a conformal transformation in which the mapping is a holomorphic function.

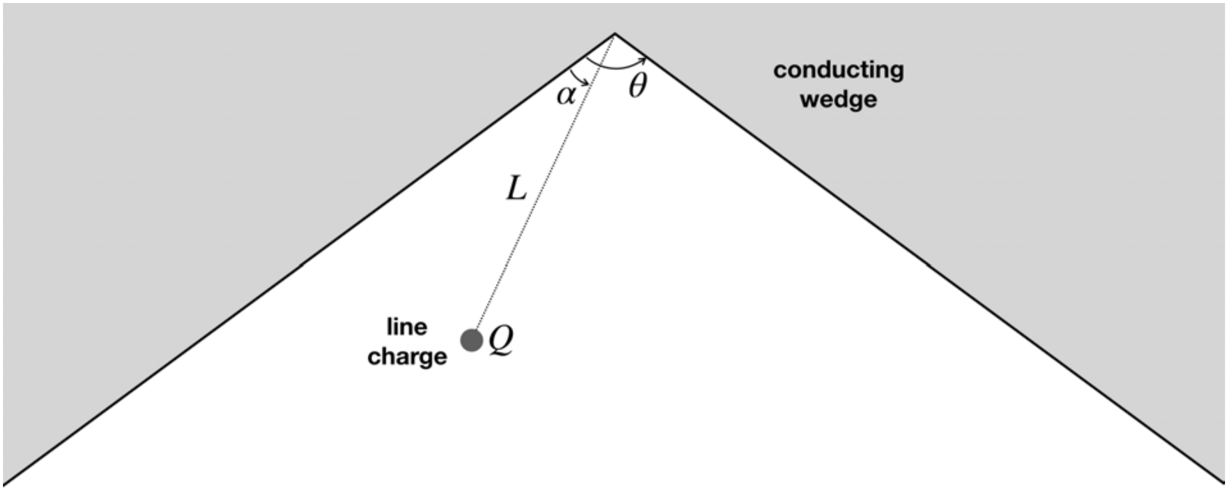
(l) Take C to represent a conductive surface. If ϕ is an appropriate solution to the 2D Laplace equation satisfying $\phi|_C = 0$, find expressions for the surface at dl, the electrostatic potential ϕ' satisfying $\phi'|_{f(C)} = 0$, and the surface charge at dl' from ϕ'.

(m) Verify that your results for questions (k) and (l) are consistent.



On the Electrostatic Interaction between a Line Charge and a Conducting Wedge

We can apply the knowledge we have gained here to solve a familiar physical puzzle. This puzzle has been widely known and popular in specific cases (e.g. easily solvable using the image method), but it lacks generality. Consider a grounded, infinitely large, conducting wedge with an opening angle θ. Inside the wedge, there is a thin, infinitely long, straight line with a uniform charge density of Q per unit length, positioned parallel to the edge of the wedge. The distance between the line and the edge is L, the plane that pass through both the line and the edge make an angle α with a face of the wedge.



(n) Find the direction and the magnitude of the electrical force acting on the line per unit length. Solve for the general case of $2\pi > \theta > \alpha$, then evaluate your answer in the unit of kQ^2/L, for $\theta = 120^o$ and $\alpha = 30^o$.

T3: General Relativity

The Alphabet of Spacetime Metrics

In multivariable calculus, we study the generalization of line elements to various coordinate systems. A coordinate system in $\mathbb{R}^3$ can be described by a function ψ that takes a point $(x, y, z) \in \mathbb{R}^3$ and outputs the three coordinates describing the same point in the coordinate system. For example, the ψ describing spherical coordinates would be defined as $\psi(x, y, z) = (r, \theta, \phi)$. As there must be a unique representation of each point in $\mathbb{R}^3$ in the ψ coordinates (ex. $r = 3, \theta = \frac{\pi}{2}, \phi = \pi$ cannot correspond to two points on the 3D space) there must be a one-to-one correspondence between points in $\mathbb{R}^3$ and points described in ψ coordinates. Hence, there's a well defined inverse function ψ^{-1}. In spherical coordinates, for instance, $\psi^{-1}(r, \theta, \phi) = (x, y, z)$.

Suppose we have a coordinate system $\psi(x, y, z) = (q_1, q_2, q_3)$. We ask a very important question: if we make an increment of dq_i in each q_i in the ψ coordinates, what's the physical distance between the starting and the finishing points in $\mathbb{R}^3$? In other words, what's the distance between the points $\psi^{-1}(q_1 + dq_1, q_2 + dq_2, q_3 + dq_3)$ and $\psi^{-1}(q_1, q_2, q_3)$? This distance is known as the *line element* generated by a coordinate system ψ. If the coordinate system is *orthogonal*, meaning the basis vectors $\hat{q_1}$, $\hat{q_2}$, $\hat{q_3}$ are all mutually perpendicular, we know that the physical displacements in taking $q_i \to q_i + dq_i$ are in perpendicular directions in $\mathbb{R}^3$, so we can find the line element by Pythagoras's theorem.

(a) Show that for an orthogonal coordinate system ψ in an arbitrary $\mathbb{R}^n$, the square of the line element ds^2 has the form $\sum_{i=1}^n g_i(q_1, ..., q_n)dq_i^2$ where each g_i is a function of $q_1, ..., q_n$. Find an expression for g_i in terms of the derivatives of ψ or ψ^{-1}.

We can think of physical observers as coordinate systems. If an arbitrary observer moving in space-time measures an event (t, x, y, z), the coordinates seen by the observer can be expressed as $\psi(t, x, y, z)$ by some function ψ. It turns out that special relativity can be described with the math we've developed so far. In particular, we can consider measuring space-time lengths by integrating the line element along paths in space-time. You are given that for *any* observer with the coordinate system $\psi \equiv (t, x, y, z)$, the line element is given by $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$

(b) It's a well-known result of special relativity that space-time lengths are measured to be the same regardless of the frame. Resolve the twin paradox by integrating the line element along the spaceship twin's world-line.

It turns out that physics in weak gravitational fields can also be described with the math we've developed so far. Consider a 1 dimensional system with a time-independent gravitational potential $\Phi(x)$ everywhere.

(c) An important postulate of general relativity states that observers that move solely under the influence of gravity take the path with the longest space-time length. Suppose an observer of this system takes a path parameterized by a variable $\sigma \in [0, \sigma_f]$: $(t(\sigma), x(\sigma), y(\sigma), z(\sigma))$. Show that the line element $ds^2 = -(1+2\Phi(x))dt^2+(1-2\Phi(x))dx^2$ generates equations of motion that are consistent with Newton's laws (neglecting special relativity).

Hint: Refer to the Euler-Lagrange equation, but you do NOT have to perform the derivation to solve this problem.

(d) Derive a statement of time dilation at a point (x, y, z).

The Spacetime Metric of a Black Hole

As we saw in part 1, the Minkowski metric describes geometry of spacetime in special relativity and is given as

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2.$$

In general relativity, the result is more complex, and we describe spacetime via the Schwarzchild metric, a solution of Einstein's equations under spherical symmetry. The spacetime interval in spherical coordinates can be expressed in terms of proper time τ and the speed of light c as

$$ds^2 = -c^2d\tau^2 = -\left(1 - \frac{r_s}{r}\right)c^2dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where t is the coordinate time, r is the radial coordinate, θ is the polar angle, ϕ is the azithumal coordinate, and r_s is the Schwarzchild radius of a massive body. The Schwarzchild radius is related to the body's mass M by $r_s = \frac{2GM}{c^2}$ where G is the universal gravitational constant.

General relativity describes gravity as the consequence of curving spacetime. Geodesics generalize straight lines to curved spacetime and are very useful as a result. In the following problem, the radial geodesic equation may be helpful:

$$\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} + \Gamma^r_{\mu\nu}\frac{\mathrm{d}\mu}{\mathrm{d}\tau}\frac{\mathrm{d}\nu}{\mathrm{d}\tau} = 0$$

where Γ represents a Christoffel symbol. The relevant terms of Γ are Γ^r_{tt} , Γ^r_{rr} , $\Gamma^r_{\theta\theta}$, and $\Gamma^r_{\phi\phi}$. Define $B = 1 - \frac{r_s}{r}$, then we can get:

$$\begin{aligned}
\Gamma^r_{tt} &= \frac{c^2B\frac{\mathrm{d}B}{\mathrm{d}r}}{2} \\
\Gamma^r_{rr} &= -\frac{B^{-1}\frac{\mathrm{d}B}{\mathrm{d}r}}{2} \\
\Gamma^r_{\theta\theta} &= -rB \\
\Gamma^r_{\phi\phi} &= -Br\sin^2\theta
\end{aligned}$$

The Photon Sphere

In 2019, the first image of a black hole was taken in 2019 by the Event Horizon Telescope (EHT) at the center of the galaxy M87. The historic image of the black hole showcased a glowing ring of light surrounding the dark abyss known as the "photon ring." In this region, light rays follow closed circular paths due to the strong gravitational pull of the black hole. When a photon follows a closed circular path on the photon ring, it can effectively orbit the black hole multiple times before either escaping to infinity or being captured by the event horizon. This phenomenon is known as the "photon sphere."

Figure 2: Event Horizon Telescope Collaboration

You may recall the massive outburst of media to the images of M87. Many were already talking about it before the images came out, like Veritasium. Take 10 minutes to watch the video.

In this problem, we will be analyzing the optical effects for an observer next to a black hole. Assume for an idealized case that the black hole has a mass $M = 15M_\odot$ and is uncharged and not rotating.

(e) Prove that the minimum radius of the black hole's photon ring follows the relationship $r_p = \frac{3}{2}r_s$.

(f) Bill is near the black hole a distance r_p away. Assume Bill's width is $w = 2\delta$ where $\delta = 0.25$ m. Sketch light ray trajectories showing how Bill could see his own back. Determine the range for the angle of view α enabling this.

(g) Approximately what is the orbital period (in miliseconds) of a singular photon in coordinate time t around the black hole. At what mass of a black hole would this time be greater than the visual reaction time of a human $t_r = 150$ ms (meaning if you closed your eyes, it would take some time for you to visualize your back again)? Is this mass possible?

(h) Propose a method (realistic or not) for Bill to safely get close to the black hole without getting ripped apart by gravity. If you believe there is no possible method, then give your reasoning behind why. You might want to consult various sources for information. To simplify the process and avoid teams searching for data such as "shear stress of a human", assume that Bill is a cylindrical object made of a material of your choosing that has a uniform density, a radius δ, and a length of $L = 1.75$ m. Bill possesses exceptional engineering skills, enabling him to construct virtually anything, provided it doesn't contravene the principles of physics.

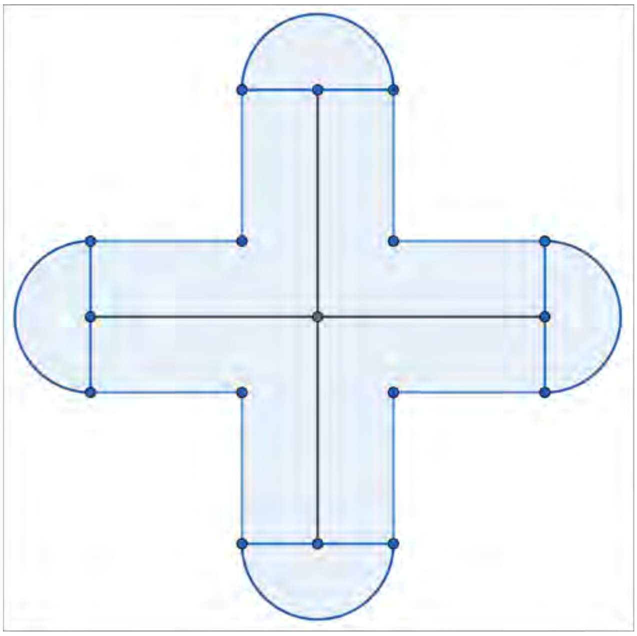
(i) Let's assume that the galaxy has N_{star} stars, each emitting light isotropically with the same total power L. Assume the stars are distributed uniformly throughout a spherical volume of radius R. For an ideal case where the path of the light from the stars is not obstructed by any other celestial objects, estimate the expected number of photons from starlight that would contribute to the photon ring or fall in the event horizon per unit time.

T1: Booster

Solution 1:
All point values should be multiplied by 3 (for a total of 30 points)

(a) (i) The shape of the chamber after time t is a circle with radius $r_0 + vt$. Thus the volume burned is $V = \pi(r_0 + vt)^2 l - \pi r_0^2 l$, and $\dot{M} = \dot{V}\rho_s = \boxed{2\pi\rho_s l v(r_0 + vt)}$

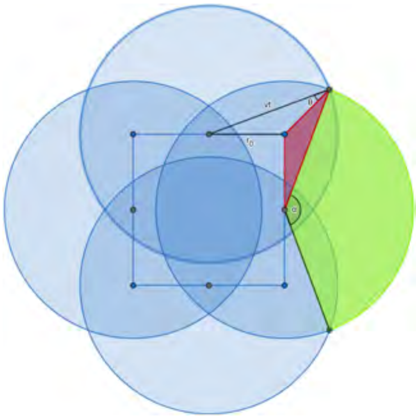
(ii) The volume of fuel that is burned after some time ($t \leq r_0/v$) looks like this:



The total area of the semicircular endcaps is $2\pi v^2 t^2$, and the area of the remaining cross-like figure is $4r_0^2 - 4(r_0 - vt)^2$. So the total area is $A = 2\pi v^2 t^2 + 4r_0^2 - 4(r_0 - vt)^2$. The mass flow rate is $\boxed{\dot{M} = l\rho_s \dot{A} = 4l\rho_s((\pi - 2)v^2 t + 2r_0 v)}$.

The intent of the problem was for $t \leq r_0/v$, but that wasn't clearly specified. For fairness to everyone who spent time on it, we solve the $t > r_0/v$ case-https://www.overleaf.com/project/64cc0b0dcfb587f85f42416c below.

This case looks like the following:



In the figure, by Law of Sines, $\theta = \arcsin\frac{r_0}{\sqrt{2}vt}$. Then $\alpha = \pi - 2(\pi/4 - \arcsin\frac{r_0}{\sqrt{2}vt}) = \pi/2 + 2\arcsin\frac{r_0}{\sqrt{2}vt}$.

The total area of the circular sectors (colored green) is $2(vt)^2(\frac{\pi}{2} + 2\arcsin\frac{r_0}{\sqrt{2}vt})$. The total area of the triangles (colored red) is $4vtr_0\sin\left(\pi/4 - \arcsin\frac{r_0}{\sqrt{2}vt}\right) = 4vtr_0(\frac{1}{\sqrt{2}}\sqrt{1 - \frac{r_0^2}{2v^2t^2}} -$

$\frac{r_0}{2vt}) = 4vtr_0(\frac{1}{\sqrt{2}}\sqrt{1-\frac{r_0^2}{2v^2t^2}} - \frac{r_0}{2vt})$. The area inside the square is constant, so differentiation will get rid of it.
We want:

$$\dot{M} = \rho_s l \frac{\mathrm{d}}{\mathrm{dt}}\left[2(vt)^2(\frac{\pi}{2} + 2\arcsin\frac{r_0}{\sqrt{2}vt}) + 4vtr_0(\frac{1}{\sqrt{2}}\sqrt{1-\frac{r_0^2}{2v^2t^2}} - \frac{r_0}{2vt})\right]$$
$$= \rho_s l(4v^2t(\pi/2 + 2\arcsin\frac{r_0}{\sqrt{2}vt}) - 4v^2t^2\frac{r_0}{vt^2\sqrt{2-\frac{r_0^2}{v^2t^2}}}$$
$$+ 4vr_0(\sqrt{2-\frac{r_0^2}{v^2t^2}} - \frac{r_0}{2vt}) + 4vtr_0(\frac{r_0^2}{2v^2t^3\sqrt{2-\frac{r_0^2}{v^2t^2}}} + \frac{r_0}{2vt^2}))$$

Grading Scheme

- **1 pts** for correct answer for (i)
- **2 pts** for correct answer for $t < r_0/v$ for (ii). 1 pt should be given if only a small mistake (e.g. neglecting the endcaps) was made in the figure.
- **1 pt** for something resembling the correct answer for $t > r_0/v$ for (ii)

(b) The gas is produced at a (volumetric) rate of $2\pi\frac{\rho_s}{\rho_g}lv(r_0+vt)$, and the area of the opening is $\pi(r_0+vt)^2$, so the velocity is

$$v_g = 2\frac{\rho_s}{\rho_g}\frac{lv}{r_0+vt}$$

Multiplying by the mass flow rate gives

$$\boxed{F = 4\pi\frac{\rho_s}{\rho_g}\rho_s l^2 v^2}$$

Which is time-independent.
Grading Scheme

- **2 pts** for correct final answer.

(c) The assumption is false. If the pressure and temperature stay constant, then by conservation of energy, the velocity of the exhaust gasses will be constant. Thus the thrust will be proportional to area and not constant in time, which contradicts part (b).
Grading Scheme

- **1 pts** for stating the assumption is false and a valid explanation.

(d) The speed of sound is $v = \sqrt{\frac{3RT}{M}}$ as only 3 of the degrees of freedom are translational (the constant doesn't actually matter that much). We can use this as an estimate for the speed of the exhaust. Thus the gas is expelled at a volumetric rate of $\dot{V} = \pi(r_0+vt)^2\sqrt{\frac{3RT}{M}}$. The density of the gas is $\frac{MP}{RT}$, so the rate of mass expulsion is $\dot{M} = \pi(r_0+vt)^2 P\sqrt{\frac{3M}{RT}}$.

This should equal the result from (a) i, yielding $\boxed{P \approx \rho_s\sqrt{\frac{4RT}{3M}}\frac{lv}{r_0+vt}}$

Grading Scheme

- **1 pt** for any dimensionally correct final answer.
- **+2 pts** for a final answer of the correct form (not considering the constant factor).

T2: The Complex Potential

Solution 2:

Part A

Has hinted in the problem, we define differentiability of complex functions just as we would in single variable calculus on the reals. A function $f : \mathbb{C} \to \mathbb{C}$, is complex-differentiable or *holomorphic* at a point $z_0 \in \mathbb{C}$ if the following limit converges to a number $f'(z_0) \in \mathbb{C}$:

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = f'(z_0) \tag{1}$$

We call the number $f'(z_0)$ the derivative of f at z_0. Note that the limit $h \to 0$ is taken by any sequence of points *in the complex plane* that converges to the origin. As in multivariable calculus on $\mathbb{R}^2$, h can approach the origin in two different ways: along the real axis and along the imaginary axis – both limits should yield the same result if the limit in (1) indeed converges. If h moves along the real axis, $h = \delta x \in \mathbb{R}$. The limit in (1) becomes:

$$\lim_{\delta x\to 0} \frac{f(x_0+\delta x+iy_0)-f(x_0+iy_0)}{\delta x}$$

where $z_0 = x_0 + iy_0$. If we write $f(x + iy) = w(x, y) + iu(x, y)$, the limit then becomes:

$$\lim_{\delta x\to 0}\left[\frac{w(x_0+\delta x,y_0)-w(x_0,y_0)}{\delta x}+i\frac{u(x_0+\delta x,y_0)-u(x_0,y_0)}{\delta x}\right]$$

but we see that these are just partial derivatives evaluated at z_0:

$$\left[\frac{\partial w}{\partial x}+i\frac{\partial u}{\partial x}\right]_{z_0}$$

Now, if we move along the imaginary axis, $h = i\delta y \in i\mathbb{R}$. The limit in (1) becomes:

$$\lim_{\delta y\to 0} \frac{f(x_0+i(y_0+\delta y))-f(x_0+iy_0)}{i\delta y}$$

Following a similar procedure from before:

$$\lim_{\delta y\to 0}\left[\frac{w(x_0,y_0+\delta y)-w(x_0,y_0)}{i\delta y}+i\frac{u(x_0,y_0+\delta y)-u(x_0,y_0)}{i\delta y}\right]$$

Since $\frac{1}{i} = -i$, the limit can be written as:

$$\left[\frac{\partial u}{\partial y}-i\frac{\partial w}{\partial y}\right]_{z_0}$$

As discussed, if f is complex-differentiable at z_0, these limits should be equal. Equating real and imaginary parts, we find:

$$\frac{\partial u}{\partial x} = -\frac{\partial w}{\partial y}$$
$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial y} \tag{2}$$

Part B

From the conditions in (2), we observe that:

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}$$
$$= \frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y\partial x} = 0$$

It can similarly be shown that $\nabla^2 u = 0$. We therefore find that a holomorphic function has real and imaginary parts that satisfy the 2D Laplace equation.

Part C

The 'tangent vectors' along γ_1 at t_1 and γ_2 at t_2 are represented by the complex numbers $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$, respectively. Let us denote the 'dot product' of two complex numbers z and w as: $\langle z, w\rangle \equiv \text{Re}[z]\text{Re}[w] + \text{Im}[z]\text{Im}[w]$. The cosine of the angle α is therefore:

$$\cos\alpha = \frac{\langle \gamma_1'(t_1), \gamma_2'(t_2)\rangle}{\|\gamma_1'(t_1)\|\|\gamma_2'(t_2)\|}$$

The transformed curves $f(\gamma_1(t))$ and $f(\gamma_2(t))$, at the intersection $f(p)$, will have tangent vectors $w_1 = \frac{d}{dt}f(\gamma_1(t))\big|_{t_1}$ and $w_2 = \frac{d}{dt}f(\gamma_2(t))\big|_{t_2}$, respectively. Let us write $\gamma_i(t) = x_i(t) + y_i(t)$ for $i = 1, 2$ and $\tilde{p} = (x_1(t_1), y_1(t_1)) = (x_2(t_2), y_2(t_2)) \in \mathbb{R}^2$. Using the chain rule,

$$\text{Re}[w_1] = \frac{\partial w}{\partial x}\bigg|_{\tilde{p}} \frac{dx_1}{dt}\bigg|_{t_1} + \frac{\partial w}{\partial y}\bigg|_{\tilde{p}} \frac{dy_1}{dt}\bigg|_{t_1}$$
$$\text{Im}[w_1] = \frac{\partial u}{\partial x}\bigg|_{\tilde{p}} \frac{dx_1}{dt}\bigg|_{t_1} + \frac{\partial u}{\partial y}\bigg|_{\tilde{p}} \frac{dy_1}{dt}\bigg|_{t_1}$$

And similarly for w_2:

$$\text{Re}[w_2] = \frac{\partial w}{\partial x}\bigg|_{\tilde{p}} \frac{dx_2}{dt}\bigg|_{t_2} + \frac{\partial w}{\partial y}\bigg|_{\tilde{p}} \frac{dy_2}{dt}\bigg|_{t_2}$$
$$\text{Im}[w_2] = \frac{\partial u}{\partial x}\bigg|_{\tilde{p}} \frac{dx_2}{dt}\bigg|_{t_2} + \frac{\partial u}{\partial y}\bigg|_{\tilde{p}} \frac{dy_2}{dt}\bigg|_{t_2}$$

Now, let us consider the cosine of the angle formed between w_1 and w_2:

$$\frac{\langle w_1, w_2\rangle}{\|w_1\|\|w_2\|}$$

Denoting the t derivatives with dots and omitting the input points/parameter values,

$$\text{Re}[w_1]\text{Re}[w_2] = \left(\frac{\partial w}{\partial x}\right)^2 \dot{x}_1\dot{x}_2 + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}(\dot{x}_2\dot{y}_1 + \dot{y}_2\dot{x}_1) + \left(\frac{\partial w}{\partial y}\right)^2 \dot{y}_1\dot{y}_2$$
$$\text{Im}[w_1]\text{Im}[w_2] = \left(\frac{\partial u}{\partial x}\right)^2 \dot{x}_1\dot{x}_2 + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}(\dot{x}_2\dot{y}_1 + \dot{y}_2\dot{x}_1) + \left(\frac{\partial u}{\partial y}\right)^2 \dot{y}_1\dot{y}_2$$

Hence,

$$\langle w_1, w_2\rangle = \left\|\frac{\partial f}{\partial x}\right\|^2 \dot{x}_1\dot{x}_2 + \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right](\dot{x}_2\dot{y}_1 + \dot{y}_2\dot{x}_1) + \left\|\frac{\partial f}{\partial y}\right\|^2 \dot{y}_1\dot{y}_2$$

Note that since f is complex-differentiable, we require $\left\|\frac{\partial f}{\partial x}\right\|^2 = \left\|\frac{\partial f}{\partial y}\right\|^2 = \|f'(p)\|^2$. Applying the condition found in (a) to the second term, we find that it vanishes. Hence,

$$\langle w_1, w_2\rangle = \left\|f'(p)\right\|^2 (\dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2) = \left\|f'(p)\right\|^2 \langle \gamma_1'(t_1), \gamma_2'(t_2)\rangle$$

Now, observe that $w_1 = f'(p)\gamma_1'(t_1)$ and $w_2 = f'(p)\gamma_2'(t_2)$

$$\frac{\langle w_1, w_2\rangle}{\|w_1\|\|w_2\|} = \frac{\langle \gamma_1'(t_1), \gamma_2'(t_2)\rangle}{\|\gamma_1'(t_1)\|\|\gamma_2'(t_2)\|}$$

As was to be shown.

Part D

Note that for a holomorphic complex potential $f = \phi + i\psi$, taking the derivative in the real direction:

$$\frac{df}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}$$

ψ was chosen such that $\partial_x\psi = -\partial_y\phi = E_y$ so

$$\frac{df}{dz} = -E_x + iE_y$$

Part E

There was a domain issue in the original problem statement – π should be added/subtracted depending on the domain, but this shouldn't matter in verifying the 2D Laplace equation since it involves derivatives. A better statement would be $\phi(r,\theta) = \frac{\phi_0}{\pi}\theta$ for $0 \le \theta \le \pi$, which clearly satisfies the given boundary conditions $\phi(x > 0, 0) = 0$, $\phi(x < 0, 0) = \phi_0$. In spherical coordinates, the Laplacian operator is

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Hence,

$$\nabla^2\phi(r,\theta) = 0$$

By the uniqueness theorem, the true, unique electric field respecting the boundary conditions is modeled by $-\nabla\phi$.

Part F

A radial line on the upper half plane at angle $\theta \in (0,\pi)$ is the set of points $L_\theta = \{re^{i\theta} \in \mathcal{H}, r \in (0,\infty)\}$. Under the logarithm map, this set is mapped to $\log(L_\theta) = \{\log|r| + i\theta, r \in (0,\infty)\}$ which is clearly the horizontal line at $\text{Im}[z] = \theta$ extending $-\infty < \text{Re}[z] < \infty$. Since the upper half plane can be thought as the collection of the sets L_θ for $\theta \in (0,\pi)$, $\log\mathcal{H}$ is clearly the strip bounded by $\text{Im}[z] = 0$ and $\text{Im}[z] = \pi$. Let us call this strip $\mathcal{S}$. We are given that $\log(z)$ is holomorphic on $\mathcal{H}$ and that $e^z : \mathcal{S} \to \mathcal{H}$

Part G

Note that the boundary condition in (e), under the logarithm map, becomes the desired boundary conditions of two infinite capacitor plates. Let us crudely denote the 'potential space' as Φ – the space to which complex potentials map. For example, if f is an appropriate complex potential to (e), we may write $f : \mathcal{H} \to \Phi$ with the boundary condition $\phi(x > 0, 0) = 0$ and $\phi(x < 0, 0) = \phi_0$ defined on the upper half plane $\mathcal{H}$. We observe that taking a map $f \circ e^z : \mathcal{S} \to \mathcal{H} \to \Phi$ defines a complex potential with the boundary conditions $\phi(x, 0) = 0$ and $\phi(x, \pi) = \phi_0$ defined on $\mathcal{S}$. For an arbitrary separation d, we can consider $f(e^{\frac{\pi}{d}z})$. Since $e^{\frac{\pi}{d}z}$ and f are holomorphic, so is $f(e^{\frac{\pi}{d}z}) = \phi(e^{\frac{\pi}{d}z}) + i\psi(e^{\frac{\pi}{d}z})$. As verified in (b), this means that the real part, $\phi(e^{\frac{\pi}{d}z})$, satisfies the 2D Laplace equation, as well as the boundary conditions $\phi(x, 0) = 0$ and $\phi(x, d) = \phi_0$, hence a valid potential for this problem. We have:

$$\phi_c(x,y) = \phi\left(e^{\frac{\pi}{d}r\cos\theta}, \frac{\pi}{d}r\sin\theta\right) = \frac{\phi_0}{d}r\sin\theta = \frac{\phi_0}{d}y$$

for $y \in (0, d)$. This is indeed the expected result from basic electrostatics.

Part H

This problem really isn't approachable without rigorous mathematical reasoning. The unit disc is the set defined by $\{z \in \mathbb{C}, |z| < 1\}$. The upper half plane is defined by the set $\{z \in \mathbb{C}, \text{Im}[z] \ge 0\}$. We first show that $f(\mathcal{D}) \subseteq \mathcal{H}$. For any $\xi \in \mathcal{D}$, we observe that

$$\begin{aligned}
\text{Im}\left[i\frac{1-\xi}{1+\xi}\right] &= \text{Re}\left[\frac{1-\xi}{1+\xi}\right] \\
&= \text{Re}\left[\frac{(1-\xi)(1+\xi^*)}{\|1+\xi\|^2}\right] \\
&= \text{Re}\left[\frac{1+\xi^* - \xi - \|\xi\|^2}{\|1+\xi\|^2}\right] \\
&= \text{Re}\left[\frac{1 - 2i\text{Im}[\xi] - \|\xi\|^2}{\|1+\xi\|^2}\right] \\
&= \frac{1-\|\xi\|^2}{\|1+\xi\|^2}
\end{aligned}$$

Clearly, if $|\xi| < 1$, we have

$$\text{Im}\left[i\frac{1-\xi}{1+\xi}\right] > 0$$

implying $f(\xi) \in \mathcal{H}$. Since for an arbitrary $f(\xi) \in f(\mathcal{D})$ we have $f(\xi) \in \mathcal{H}$, $f(\mathcal{D}) \subseteq \mathcal{H}$. Now we'd like to show that $\mathcal{H} \subseteq f(\mathcal{D})$. Consider an arbitrary $\omega \in \mathcal{H}$. We'd like to show that there exists a number $\xi \in \mathcal{D}$ such that $f(\xi) = \omega$, which would imply $\omega \in f(\mathcal{D})$. Observe that:

$$\begin{aligned} i\frac{1-\xi}{1+\xi} &= \omega \\ i - i\xi &= \omega + \omega\xi \\ i - \omega &= \xi(\omega + i) \\ \xi &= \frac{i-\omega}{i+\omega} \end{aligned}$$

Now, note that $\text{Re}[i-\omega] = -\text{Re}[i+\omega] = \text{Re}[\omega]$, so when comparing $|i-\omega|$ and $|i+\omega|$, we need only compare the imaginary parts. Denoting $\omega_i \equiv \text{Im}[\omega] > 0$,

$$|\text{Im}[i+\omega]| = |1+\omega_i| \geq |1 - |\omega_i||$$

since $\omega_i > 0$, we can get rid of the absolute value symbol – $|\omega_i| = \omega_i$ – and make the above a strict inequality. This suggests that $|\text{Im}[i-\omega]| < |\text{Im}[i+\omega]|$, which implies $|i-\omega| < |i+\omega|$, and we conclude:

$$|\xi| = \left|\frac{i-\omega}{i+\omega}\right| < 1$$

so for all $\omega \in \mathcal{H}$, there exists a $\xi \in \mathcal{D}$ with $f(\xi) = \omega$. We've shown that $f(\mathcal{D}) \subseteq \mathcal{H}$ and that $f(\mathcal{H}) \subseteq \mathcal{D}$, implying $f(\mathcal{D}) = \mathcal{H}$. The way we found ξ above hints that the $\omega \in \mathcal{H}$, $\xi \in \mathcal{D}$ pair is defined uniquely – that there is a one-to-one correspondence such that f is bijective. To show this, we can first suppose that f has an inverse h defined by:

$$h(w) = \frac{i-w}{i+w}$$

h is clearly single-valued, and it's easy to verify that $h(f(z)) = z$ and $f(h(w)) = w$. This would be a direct proof that f is a bijection between $\mathcal{D}$ and $\mathcal{H}$. Now we observe the mapping of the circle $C = \{|z| = 1\}$. Note that there should be problems with f around $z = -1 \in C$ since f is not holomorphic at $z = -1$. For an arbitrary point $e^{i\theta} \in C$ with $\theta \in [0, 2\pi)$, it's easy to simplify expressions down to:

$$f(e^{i\theta}) = \frac{\sin\theta}{1+\cos\theta}$$

Graphing this on desmos suggests that the arc $(0, \pi)$ maps to the positive real line and that $(\pi, 2\pi)$ maps to the negative real line. Notice there's a jump from ∞ to $-\infty$ as we move from the top arc to the bottom arc at π – this is where f "breaks".

Part I

This is a standard electrostatics problem. The potential ϕ_I in the region between the rod and the cylindrical shell can be found by integrating the appropriate electric field $\boldsymbol{E}$.

$$\begin{aligned} \boldsymbol{E}(r) &= \frac{\lambda}{2\pi\epsilon_0 r}\hat{r} \\ \phi_I(r) &= -\frac{\lambda}{2\pi\epsilon_0}\ln(r/R) \end{aligned}$$

where R is the radius of the concentric cylindrical shell.

Part J

From our results in (h), let us define a map $g : \mathcal{D}_R \to \mathcal{H}$, where $\mathcal{D}_R$ is the disc of radius R centered at the origin, as:

$$g(z) = ih\frac{1 - z/R}{1 + z/R}$$

Under this map, the set up in (i) is mapped to the boundary conditions on $\mathcal{H}$: $\phi = 0$ on the real line, infinite rod placed at $z = ih$. These are the exact boundary conditions that we require for this problem. The inverse map of g, denoted H, is given by:

$$H(w) = R\frac{i - w/h}{i + w/h}$$

H is holomorphic everywhere on the complex plane except at $-ih$, which isn't included in $\mathcal{H}$, so H holomorphic on the upper half plane. Since the boundary condition satisfied by ϕ_I has the geometry of $\mathcal{D}_R$, we can consider a complex potential $f = \phi_I + i\psi_I$ that maps $f : \mathcal{D}_R \to \Phi$. As argued in (g), $f \circ H : \mathcal{H} \to \mathcal{D}_R \to \Phi$ is a holomorphic function on $\mathcal{H}$ that respects all necessary boundary conditions. Since ϕ_I is only dependent on r, we'd like to consider $|H(w)|$:

$$\|H(\omega)\|^2 = R^2\frac{1 - \frac{2}{h}\,\mathrm{Im}[w] + \frac{\|w\|^2}{h^2}}{1 + \frac{2}{h}\,\mathrm{Im}[w] + \frac{\|w\|^2}{h^2}}$$

If we identify $\mathrm{Im}[w] = y$ and $\|w\|^2 = x^2 + y^2$, the desired potential function is:

$$\phi(x, y) = \frac{\lambda}{2\pi\epsilon_0}\ln\sqrt{\frac{1 + \frac{2}{h}y + \frac{x^2+y^2}{h^2}}{1 - \frac{2}{h}y + \frac{x^2+y^2}{h^2}}}$$

A method of images results in the same expression.

Part K

If we parameterize the curve C as $\gamma(t) = x(t) + iy(t)$, the length element dl at $\gamma(t_0)$ can be written as:

$$dl = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = |\gamma'(t_0)|dt$$

with all derivatives evaluated at t_0. We can use the calculations in (c) to write dl' in terms of dt:

$$dl' = \left|\frac{d}{dt}f(\gamma(t))\Big|_{t_0}\right|dt = |f'(\gamma(t_0))||\gamma'(t_0)|dt$$

Hence, we find the scaling factor is $|f'(\gamma(t_0))|$.

Part L

If $\hat{n}$ is orthogonal to dl, the surface charge at dl is proportional to $\nabla\phi \cdot \hat{n}$. Since the surface is a conductor, $\nabla\phi$ is parallel to $\hat{n}$ near the surface. From previous problems, we know that $\phi' = \phi \circ f^{-1}$ is the appropriate electrostatic potential in $f(C)$, and $\nabla\phi'$ is parallel to the mapping of $\hat{n}$, which remains orthogonal to the curve, as shown in (c). There's a theorem that states that if f is a bijective holomorphic map, f' never vanishes, f^{-1} is holomorphic, and the derivative of f^{-1} is given by $\frac{1}{f'}$. However, since we deal with the individual real and imaginary components of complex functions in this question, we may crudely rely on calculus on $\mathbb{R}^2$. The surface charge at dl' is proportional to $\nabla\phi' \cdot \hat{n}' = |\nabla\phi'|$. Carefully applying the chain rule results in:

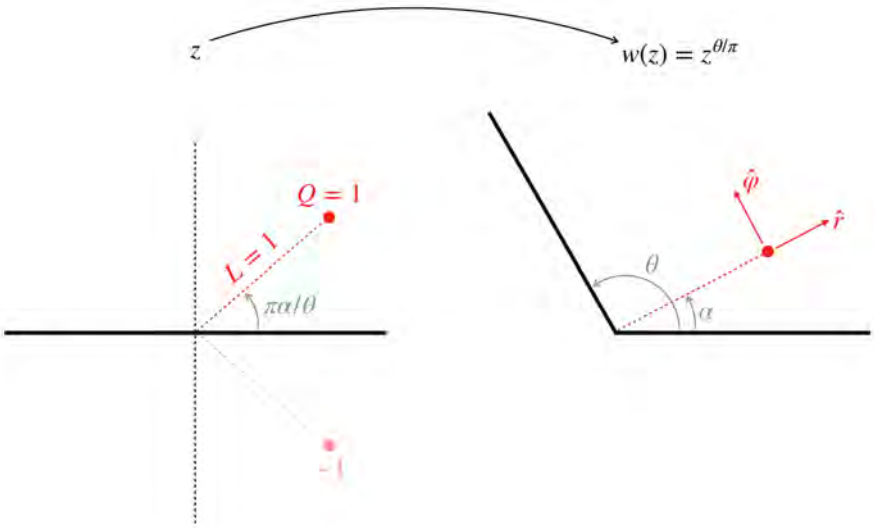
$$|\nabla\phi'| = \frac{|\nabla\phi|}{|f'(z)|}$$

Part M

If we take the charge at points to remain the same even after conformal transformations (think about stretching charged surfaces with the charges glued in place) we see that the scale factor found in (j) results in the scaling of the surface charge found in (l). Hence, given that functions behave nicely at boundaries, we can determine the surface charge of other surfaces simply by looking at the form of f and its derivative.

Part N

The electrostatic system of interested is effectively two-dimensional. For simplicity, set $L = 1$ and $Q = +1$. We can recover the general answer with dimensional analysis later on.
Consider a configuration in z-space in which the grounded conductor fills the area $\mathrm{Re}(z) < 0$ and a charge is placed at position $e^{+i\pi\alpha/\theta}$. This configuration can be mapped to our puzzle in w-space by the conformal transformation $w(z) = z^{\theta/\pi}$ as shown in the figure below.



In z-space, the electric potential in open space is as if created by the original charge and the image charge -1 located at position $e^{-i\pi\alpha/\theta}$. Thus, the complex potential is given by:

$$\phi(z) = \left[-2k\ln\left(z - e^{+i\pi\alpha/\theta}\right)\right] - \left[-2k\ln\left(z - e^{-i\pi\alpha/\theta}\right)\right] = 2k\ln\frac{z - e^{-i\pi\alpha/\theta}}{z - e^{+i\pi\alpha/\theta}}\ .$$

After the transformation $z \to w(z)$, in w-space it becomes $\phi(z) \to \phi'(w)$:

$$\phi'(w) = \phi(z)\Big|_{z\to w} = 2k\ln\frac{w - e^{-i\pi\alpha/\theta}}{w - e^{+i\pi\alpha/\theta}}\ .$$

Choose a polar coordinate $w = (r, \varphi)$, then we can obtain the real potential as:

$$\mathrm{Re}\left[\phi'(r,\varphi)\right] = k\ln\frac{1 + r^{2\pi/\theta} - 2r^{\pi/\theta}\cos\left[\frac{\pi}{\theta}(\varphi+\alpha)\right]}{1 + r^{2\pi/\theta} - 2r^{\pi/\theta}\cos\left[\frac{\pi}{\theta}(\varphi-\alpha)\right]}\ .$$

Subtract this by the electric potential created by the original charge in the w-space, we have a regularized complex potential which is not singular at the location of that charge:

$$\begin{aligned}\mathrm{Re}\left[\phi'_{\mathrm{reg}}(r,\varphi)\right] &= \mathrm{Re}\left\{\phi'(r,\varphi) - \left[-2k\ln\left(w - e^{i\alpha}\right)\right]\Big|_{w=re^{i\varphi}}\right\} \\ &= k\ln\frac{\left\{1 + r^{2\pi/\theta} - 2r^{\pi/\theta}\cos\left[\frac{\pi}{\theta}(\varphi+\alpha)\right]\right\}\left[1 + r^2 - 2r\cos(\varphi-\alpha)\right]}{1 + r^{2\pi/\theta} - 2r^{\pi/\theta}\cos\left[\frac{\pi}{\theta}(\varphi-\alpha)\right]}\ .\end{aligned}$$

We can then calculate the electrical field created at the location of the original charge by the induced charge distributed on the conducting wedge:

$$\vec{E} = -\left\{\partial_r \mathrm{Re}\left[\phi'_{\mathrm{reg}}(r,\varphi)\right]\hat{r} + \frac{1}{r}\partial_\varphi \mathrm{Re}\left[\phi'_{\mathrm{reg}}(r,\varphi)\right]\hat{\varphi}\right\}\Bigg|_{r=1,\varphi=\alpha} = -k\left[\hat{r} + \frac{\pi}{\theta}\cot\left(\frac{\pi}{\theta}\alpha\right)\hat{\varphi}\right] \ .$$

The total electrostatic force acting on the original charge is therefore $\vec{F} = \vec{E}$, and after putting back Q and L, we arrive at the expression:

$$\vec{F} = -k\frac{Q^2}{L}\left[\hat{r} + \frac{\pi}{\theta}\cot\left(\frac{\pi}{\theta}\alpha\right)\hat{\varphi}\right] \ .$$

For $\theta = 120^o$ and $\alpha = 30^o$, the magnitude of this force in the unit of kQ^2/L is $\sqrt{13}/2 \approx 1.8028$.

* This part was created with helps from Quy C. Tran and Nam H. Nguyen.

Grading Scheme

(a) 1.5 pts
(b) 1.5 pts
(c) 5 pts
(d) 1.5 pts
(e) 1 pt
(f) 1.5 pts
(g) 7.5 pts
(h) 7.5 pts
(i) 1.5 pts
(j) 7.5 pts
(k) 5 pts
(l) 5 pts
(m) 1.5 pts
(n) 15 pts

T3: General Relativity

Solution 3:

(a) By the multivariable chain rule, we can write the relationship between x_i and q_i as

$$\mathrm{d}x_i = \frac{\partial x_i}{\partial q_1}\mathrm{d}q_1 + \frac{\partial x_i}{\partial q_2}\mathrm{d}q_2 + \cdots + \frac{\partial x_n}{\partial q_n}\mathrm{d}x_n = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j}\mathrm{d}q_j.$$

We know that in original coordinates, the infinitesimal line element can be written by the Pythagorean theorem as

$$\mathrm{d}s = \sqrt{\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cdots + \mathrm{d}x_n^2} \implies \mathrm{d}s^2 = \sum_{i=1}^{n} \mathrm{d}x_i^2$$

Hence, we can find that

$$\mathrm{d}s^2 = \sum_{i}^{n} \left(\frac{\partial \vec{x}}{\partial q_i}\right)^2 \mathrm{d}q_i.$$

This means that $g_i = \left(\frac{\partial \psi^{-1}}{\partial q_i}\right)^2$.

Grading Scheme

- **(1 pt)** Uses multivariable chain rule or equivalent to find infintesimal length $\mathrm{d}x_i$.
- **(1 pt)** Uses Pythagorean theorem to find formula for line element $\mathrm{d}s$.
- **(1 pt)** Finds out that $g_i = \left(\frac{\partial \psi^{-1}}{\partial q_i}\right)^2$

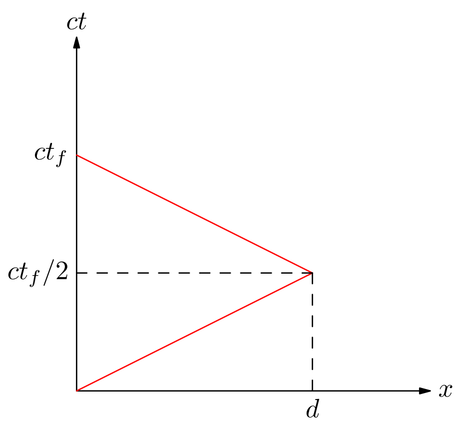
Notes

- We do not expect competitors to provide the most mathematically rigorous solution. A solution with true rigor would be able to prove why $\mathrm{d}s^2$ takes the given form.
- Some competitors were able to provide a solution explaining this. See the solution by $|\text{Enloe}\rangle$ on our website.

(b) Let us take a look at the Minkowski metric:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

Without loss of generalization, let us assume the rocket is moving only in the x-direction, meaning that $\mathrm{d}y = \mathrm{d}z = 0$. We can draw a spacetime diagram as shown below for the twin moving in the rocket:



Note that $\mathrm{d}x = v\mathrm{d}t$ where v is the constant velocity of the twin. Hence, we can rewrite as

$$\mathrm{d}s^2 = (v^2 - c^2)\mathrm{d}t^2.$$

Since $\mathrm{d}s^2$ is invariant, we can say that the time experienced by the twin on the spaceship is τ. As the other twin is not moving, then (and by noting that v changes sign at $ct_f/2$)

$$-c^2\mathrm{d}\tau^2 = (v^2 - c^2)\mathrm{d}t^2 \implies \Delta\tau = \int_0^{t_f} \sqrt{1 - \frac{v^2}{c^2}}\mathrm{d}t \implies \tau = \frac{t_f}{\gamma}.$$

Hence the twin on the spaceship ages less.

Grading Scheme

- **(0.5 pts)** Writes the relation $\mathrm{d}x = v\mathrm{d}t$.
- **(1 pt)** Uses invariance to find τ in terms of t. (-0.5 points if they do not acknowledge the change in sign of v at $ct_f/2$)
- **(0.5 pts)** Concludes that the spaceship twin ages less.

(c) We recall that the action of a system along a path $\vec{q}(t)$ between two times t_1 and t_2 is given as

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i, t)\mathrm{d}t.$$

We are given that the line element $\mathrm{d}s^2 = -(1 + 2\Phi(x))\mathrm{d}t^2 + (1 - 2\Phi(x))\mathrm{d}x^2$ (using units where $c = 1$). We can write an expression for the maximal spacetime length as

$$\begin{aligned}\mathrm{d}s &= \sqrt{-(1 + 2\Phi(x))\mathrm{d}t^2 + (1 - 2\Phi(x))\mathrm{d}x^2} \\ &= \sqrt{-(1 + 2\Phi(x))\left(\frac{\mathrm{d}t}{\mathrm{d}\sigma}\right)^2 + (1 - 2\Phi(x))\left(\frac{\mathrm{d}x}{\mathrm{d}\sigma}\right)^2}\mathrm{d}\sigma \\ S = \int \mathrm{d}s &= \int \sqrt{-(1 + 2\Phi(x)) + (1 - 2\Phi(x))\dot{x}^2}\mathrm{d}\sigma\end{aligned}$$

We notice that the expression inside the integral is the Lagrangian. Now we use the Euler-Lagrange equation $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}\dot{x}}$ using the approximation $\Phi(x) \ll 1$ and $\dot{x} \ll 1$. Without much detail on calculations, one can recover that $\ddot{x} = -\Phi'(x)$. This resembles Newton's second law $F = -m\frac{\mathrm{d}\Phi}{\mathrm{d}x}$.

Grading Scheme

- **(2 pts)** Is able to recover the Lagrangian. (-0.5 points if they write the integral with $\mathrm{d}t$ instead of $\mathrm{d}\sigma$.)
- **(2 pts)** Is able to recover Newton's second law by using the Euler-Lagrange equations. (-1 point if significant progress is achieved, but the final answer is not correct. -0.5 points if they don't show how they used approximations)

(d) To get rid of special relativistic effects, we assume zero velocity, and hence $\mathrm{d}s^2 = -(1+2\Phi(x))\mathrm{d}t^2$. As the factor for time dilation is $\gamma = \sigma/t$, we can write using $\mathrm{d}s^2 = -\mathrm{d}\sigma^2$ that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sqrt{1 + 2\Phi(x)} \approx 1 + \Phi(x)$$

Grading Scheme

- **(1 pt)** Writes expression assuming $\mathrm{d}s^2$ for general relativistic effects.
- **(1 pt)** Finds expression for time dilation.

(e) We apply the radial geodesic. We note that $\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} = \frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 0$. Thus, we can add the geodesic equations for Γ^r_{tt} and $\Gamma^r_{\phi\phi}$, so that

$$\Gamma^r_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \Gamma^r_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0$$

$$\left(\frac{d\phi}{dt}\right)^2 = -\frac{\Gamma^r_{tt}}{\Gamma^r_{\phi\phi}} = -\frac{\frac{c^2B\frac{\mathrm{d}B}{\mathrm{d}r}}{2}}{-Br\sin^2\theta} = \frac{c^2\frac{\mathrm{d}B}{\mathrm{d}r}}{2r} = \frac{c^2r_s}{2r^3} = \frac{GM}{r^3}$$

Thus,

$$\left(r\frac{d\phi}{dt}\right)^2 = c^2 - \frac{2GM}{r} = \frac{GM}{r},$$

which yields $r_p = \frac{3GM}{c^2}$.

Grading Scheme

- **(2 pts)** Recombines the radial geodesic equation to find a formula for $\frac{\mathrm{d}\phi}{\mathrm{d}t}$.
- **(2 pts)** Finds the radius to be $r_p = \frac{3}{2}r_s$. (-1 pt if significant progress is achieved but the final answer is not correct)

(f) Divide the Schwarzschild metric by the path parameter to obtain:

$$0 = -\left(1-\frac{r_s}{r}\right)c^2t^2 + \frac{\dot{r}^2}{1-\frac{r_s}{r}} + r^2\dot{\phi}^2$$

Where $ds^2 = 0$ for null geodesics. From this, we can identify the following conserved quantities:

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(r^2\dot{\phi}\right) \Rightarrow L = r^2\dot{\phi} \quad \text{(angular momentum)}$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\left(1-\frac{r_s}{r}\right)c^2\dot{t}\right) \Rightarrow E = \left(1-\frac{r_s}{r}\right)c^2\dot{t} \quad \text{(energy)}$$

Here, $b = \frac{L}{E}$ is the impact parameter. The Schwarzschild metric, for $\theta = \frac{\pi}{2}$, can be rewritten as:

$$0 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \frac{dr^2}{1-\frac{r_s}{r}} + r^2d\phi^2$$

This leads to:

$$E^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + \frac{L^2}{r^2}\left(1-\frac{r_s}{r}\right)$$
$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = -\left(1-\frac{r_s}{r}\right) + \frac{r^4}{b^2}$$

Differentiating with respect to ϕ and substituting $1/r$ shows:

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right) = \frac{r_p}{r^2} - \frac{1}{r}$$

Considering a small perturbation $r = r_p \pm \delta$ leads to:

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r+\delta}\right) = \frac{r_p}{(r+\delta)^2} - \frac{1}{r+\delta}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\left(1+\frac{\delta}{r}\right)^{-1}\right) = \frac{r_p}{r^2}\left(1+\frac{\delta}{r}\right)^{-2} - \frac{1}{r}\left(1+\frac{\delta}{r}\right)$$
$$-\frac{1}{r^2}\frac{\mathrm{d}^2\delta}{\mathrm{d}\phi^2} = \frac{1}{r}\left[\left(1-2\frac{\delta}{r}\right) - \left(1+\frac{\delta}{r}\right)\right]$$

Finally, we can retrieve a simple differential equation:

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}\phi^2} = \delta \Rightarrow \delta = Ae^{\phi} + Be^{-\phi}$$

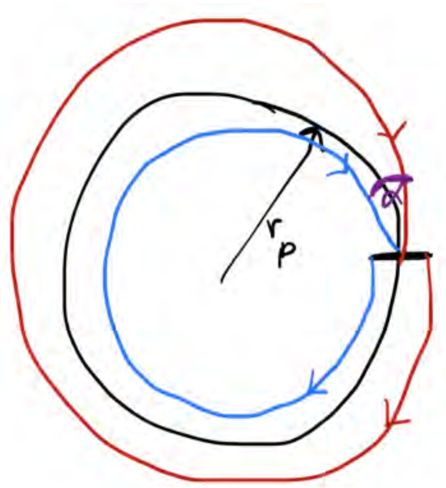
Given $\delta(0) = \delta_0$ (e.g., $\delta = 0.25$ m) and $\delta'(0) = r_p\beta$ (where β is an arbitrary constant representing the angle), one can find A and B as

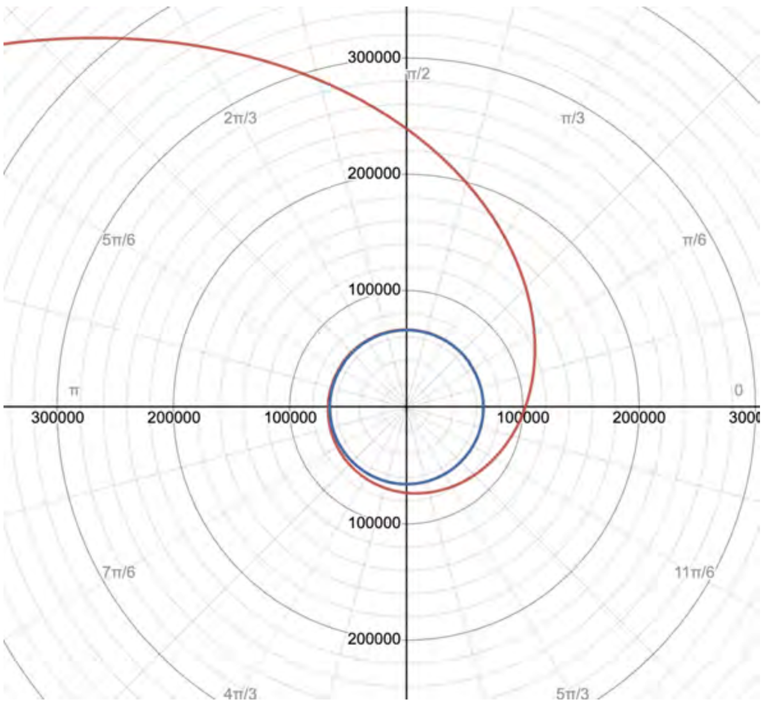
$$\begin{aligned}\delta &= \delta_0\left(\frac{e^{\phi}+e^{-\phi}}{2}\right) + r_p^{\alpha}\left(\frac{e^{\phi}-e^{-\phi}}{2}\right)\\ &= \delta_0\cosh\phi + r_p\beta\sinh\phi\end{aligned}$$

Thus, for the maximum case, we need $\delta(2\pi) = \delta_0$. To find α, consider the symmetry and multiply the result by 2. Therefore,

$$\delta_0 = \delta_0\cosh(2\pi) + \frac{r_p\alpha}{2}\sinh(2\pi) \implies \alpha = \frac{2\delta_0(1-\cosh(2\pi))}{r_p\sinh(2\pi)}$$

A drawing can be created on desmos.





Notice how the angle k must be extremely small for the effect to occur. In the above picture we set $k = 0.00000002$ and yet, the photon trajectory misses by about 3000 meters when we require the full deviation to be 0.25 m. Hence, it is extremely unlikely for one to be able to visualize this effect in real life. Additionally, it is worth noticing how these curves take form of spirals.

Grading Scheme

- **(2 pts)** For recovering expressions for angular momentum and energy.

- **(2 pts)** For finding a differential equation in terms of r_p, r, and ϕ.
- **(4 pts)** For simplifying the differential equation and using proper approximation to find an expression for $\frac{\mathrm{d}^2\delta}{\mathrm{d}\phi^2}$.
- **(2 pts)** For solving the differential equation and interpreting the results.
- **(1 pts)** For creating a proper labelled diagram that represents the scenario.

(g) The Schwarzchild Metric is represented by:

$$(ds)^2 = \left(cdt\sqrt{1-\frac{2GM}{rc^2}}\right)^2 - \left(\frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}}\right)^2 - (r\ d\theta)^2 - (r\sin\theta d\phi)^2$$

as we are only worried about the ϕ direction, $dr = d\theta = 0$ and $\theta = \pi/2$. Therefore, we have

$$c\sqrt{1-\frac{2GM}{rc^2}} = r\ \frac{d\phi}{dt}$$

To find the period, we need to calculate the angular speed of light in the ϕ-direction at $\frac{3GM}{c^2}$. Therefore,

$$\frac{d\phi}{dt} = \frac{c}{r}\sqrt{1-\frac{2GM}{rc^2}} = \frac{c^3}{3GM\sqrt{3}}.$$

Therefore, because angular speed is constant,

$$T = \frac{2\pi}{\frac{d\phi}{dt}} = \frac{2\pi}{\frac{c^3}{3GM\sqrt{3}}} = \frac{6\pi GM\sqrt{3}}{c^3}$$

Grading Scheme

- **(2 pts)** For simplifying the Schwarzschild metric and finding a formula for $\frac{\mathrm{d}\phi}{\mathrm{d}t}$.
- **(2 pts)** Getting the right answer.

(h) **Grading Scheme**

- **(4 pts)** Solid explanation proving point.

(i) **Grading Scheme**

- **(3 pts)** Finds the radius of the shadow disc is $\frac{3\sqrt{3}}{2}r_s$.
- If part (f) was not attempted, +2 points for recovering expressions for angular momentum and energy.
- **(3 pts)** +1.5 Uses solid angle Ω to find portion of light shining on disc. + 1.5 Integrates using number density to find total number of contributing photons. (-1 pt if correct answer is not obtained.)

OPhO Invitational: Fluid Dynamics

August 1, 2022

1 Introduction

Fluid dynamics is an important area in physics and engineering, and being able to simulate fluids properly is a challenging task that is applicable for rocketry, airplanes, cars, and even in designing pipes for air conditioning! The most popular technique is **computational fluid dynamics** (CFD), which numerically solves the Navier-Stokes equation. However, it is very difficult to set up and is very slow.

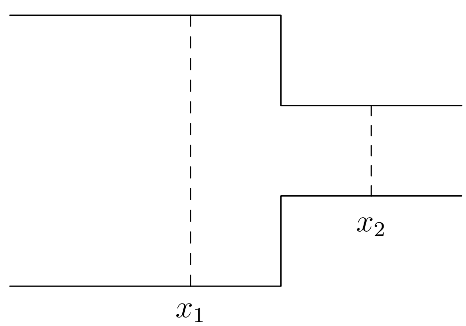
In this experiment, we will use a very simple simulation software that simulates fluids in two dimensions through the Lattice-Boltzmann algorithm (and is thus, *not* CFD), which you can find here.

Please take time to read the description on the page. **Make sure to keep all settings as default.** By dragging your cursor around, you can create different types of barriers, and clicking start will run the simulation. The first dropdown menu tells us the number of pixels. While a higher resolution will lead to more accurate results, it will be more strenous for the computer and harder to create boundaries. We recommend you to draw the barriers on a lower resolution but run the simulation on a higher resolution.

On the last row, we recommend checking **Flowlines** and **Sensor**. Enabling sensor will result in a cursor to appear on the screen that you can drag around, where the density ρ, horizontal speed u_x, and vertical speed u_y will all be shown. If for some reason the simulation is unstable, you may also check the **Data** button, which will record the data on the cursor as a function of time.

2 Mass Flow Conservation (15 pts)

We will first investigate if this implementation of the Lattice-Boltzmann algorithm correctly simulates the conservation of mass.



1. Create a tube that either opens up or closes down very abruptly such as in the diagram above. Pick two locations x_1 and x_2 such that one is in the wider section and the other is in the more narrow section. Compute the flow $\frac{dm}{dt}$ past both these points. (10 pts)
2. Do the two mass flow rates agree with each other? If not, suggest a possible reason why. (5 pts)

Note:

- You are free to pick the shape and dimensions of the tube and it does not need to be perfect.
- Please report your figures with uncertainties and make clear how you estimated and/or obtained them! You do not need to show your computations.
- Provide a screenshot of the boundaries and the parameters.

3 Law of the Wall (25 pts)

The Law of the Wall provides an emperical relationship between the average velocity of a turbulent flow at some point and the distance to a wall (or fluid boundary). Namely,

$$u_x = \frac{u_T}{\kappa} \ln\left(\frac{\rho u_T}{\mu} y\right) + C,$$

where

- u_x is the component of velocity parallel to the wall
- u_T is the shear velocity defined by $u_T = \sqrt{\tau_W/\rho}$,
- ρ is the density
- μ is the viscosity (specifically, the dynamic viscosity)
- κ, C are constants.

Note that $\tau_w = \mu \frac{\partial u_x}{\partial y}\Big|_{y=0}$ is the wall shear stress.

Studies have shown that κ ranges from 0.35 to 0.42. Using a straight line to model a wall, answer the following questions:

1. Is the relationship between u_x and y logarithmic? (10 pts)
2. What are the values of κ and C? (10 pts)
3. Does the value of κ agree with experiment? If it does not, suggest a possible reason why. (5 pts)

You are encouraged to add graphs.

Note:

- You are free to pick the length and direction of the boundary.
- Please report your figures with uncertainties and make clear how you estimated and/or obtained them! You do not need to show your computations.
- Provide a screenshot of the boundaries and the parameters.

2023 Online Physics Olympiad: Invitational Contest

Experimental Exam

Sponsors

This competition could not be possible without the help of our sponsors, who are all doing great things in physics, math, and education.

General Instructions

The experimental examination consists of 1 long answer question worth 25 points over 1 full day from August 6, 0:01 am GMT.

- **The team leader should submit their final solution document in this google form.**
- If you wish to request a clarification, please use this form. To see all clarifications, view this document.
- Participants are given a google form where they are allowed to submit up-to 100 megabytes of data for their solutions. It is recommended that participants write their solutions in LaTeX. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting. If you wish a premade LaTeX template, we have made one for you here.

Specific Rules

For any part of this paper, you are allowed to use online tools and resources to help you, as long as you are not requesting help from anyone outside of your team. Allowable resources include Wikipedia, research papers, Wolfram Alpha, Python, Excel, etc.

However, you must document every resource that you use and cite them when applicable. As a general rule of thumb, you should derive any results that cannot be found on Wikipedia. Therefore, solutions along the lines of: "By Wolfram Alpha, this is true." will not be accepted. Be reasonable please.

Every time you are asked to run an experiment, you must provide the input parameters and a screenshot of the output.

Accessing the Program

To access the Python notebook, follow this **link**. You will be able to perform all the code online, without downloading anything. If you cannot access the link, we will also provide the source code on our website.

Background Information

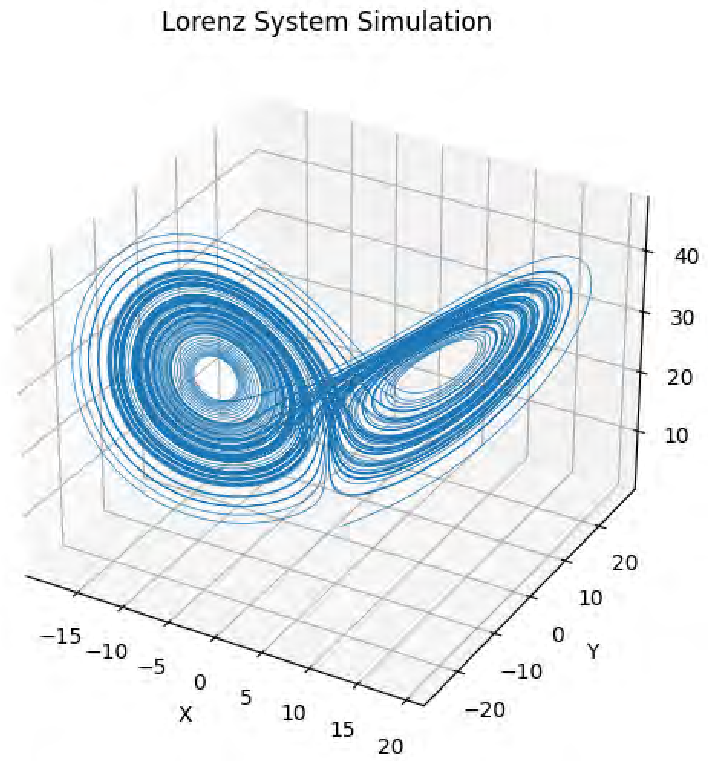
In this problem, you will be running a computer simulation written in *Python* to complete a series of questions relating to the Lorenz system, a system of ordinary differential equations. While you do not need to fully understand how exactly the code operates, it could be beneficial to grasp the process the code follows.

The script employs a simple algorithm to simulate and illustrate the Lorenz system.

1. Define constants for the Lorenz system equations, define the system of equations as a function, and establish the initial state. Set simulation parameters including time-step, maximum time, and the number of steps.
2. Implement Fourth-order and Second-order Runge-Kutta methods as functions. These will be used for solving the Lorenz system equations.
3. Run a loop for each time step where the system's state is updated using the Fourth-order Runge-Kutta method and stored in the state trajectory.
4. Plot the state trajectory of the Lorenz system in a 3D graph to visualize the system's behavior over time.

The Lorenz System

The Lorenz system is a system of ordinary differential equations (ODEs) that was first studied by Edward Lorenz when investigating the physical properties of a convective fluid flow. It is notable for having chaotic solutions for certain parameter values and initial conditions. In particular, the system shows sensitive dependence on initial conditions, which is a key property of chaotic systems. In other words, trajectories starting at slightly different initial conditions can end up at vastly different states. For more information, see this video on the Butterfly effect by Veritasium.



The Lorenz system is defined as a coupled differential equation for the quantities (x, y, z) where x is proportional to the rate of convection, y to the horizontal temperature variation, and z to the vertical temperature variation:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

where σ, ρ, and β are system parameters. The parameter σ is the Prandtl number, ρ is the Rayleigh number, and β is a geometric factor related to the shape of the fluid layer. The dot represents a time derivative.

The Lorenz system exhibits a variety of behaviors as the parameters σ, ρ, and β vary, including fixed points, limit cycles, and chaos. For certain parameter values, the system has chaotic solutions. The most famous example of this is the case where $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$. This set of parameter values gives rise to the Lorenz attractor, a fractal structure that is the path traced by the system in the phase space.

To simulate the Lorenz system, we use a special technique of approximation for differential equations called the Runge-Kutta methods. The basic idea is to compute the derivative at several points within a time step and then combine these derivatives in a weighted average to estimate the state of the system at the next time step. This approach provides a more accurate prediction than simply extrapolating the derivative at the beginning of the time step, as done in the simpler first-order Euler method.

For this problem, you do not have to write much code. We have written most of the needed functions. All you have to do is paste them in the Google Colab.

- `system(state, t)`: This function represents the Lorenz system. It takes in a state vector $[x, y, z]$ and a time t, and outputs the derivatives $[\dot{x}, \dot{y}, \dot{z}]$.
- `rk4_step(state, t, dt, system)`: This function implements the fourth-order Runge-Kutta (RK4) method for numerical integration. It takes in a state vector $[x, y, z]$, a time t, a time step $\mathrm{d}t$, and a system function. It outputs the state at time $t + \mathrm{d}t$.
- `rk2_step(state, t, dt, system)`: Similar to `rk4_step`, this function implements the second-order Runge-Kutta (RK2) method for numerical integration.
- `get_distance(state1, state2)`: This function takes in two state vectors and returns the Cartesian distance between them.

Problem 1

What condition needs to be satisifed for σ, ρ, β in order for the ODE to be stable? (i.e. bounded orbits). Where are the attractors located?

Problem 2

Suppose you change the parameter ρ slightly above its original value, and keep σ and β at their original values. How does the system behavior change? In particular, how does the position of the attractors change, and what happens to the system's stability?

Now suppose you keep ρ at its original value, and change σ and β to values slightly different from their original ones. Again, describe the changes in the system behavior, the position of the attractors, and the system's stability.

To analyze how chaotic a system is, we often use the Lypaunov exponent. Let's consider two trajectories of a dynamical system that start at slightly different initial conditions. We denote the state of the trajectory at time t as $\mathbf{r}(t) = [x(t), y(t), z(t)]$, and the state of the second trajectory as $\mathbf{r}(t) + \delta\mathbf{r}(t)$, where δ is a small perturbation.

The distance between both trajectories can be represented as $d(t) = ||\delta\mathbf{r}(t)||$. If the system is sensitive to initial conditions, then over time, $d(t)$ will grow or shrink at an exponential rate. Therefore, we can rewrite as

$$d(t) \approx d(0)e^{\lambda t}$$

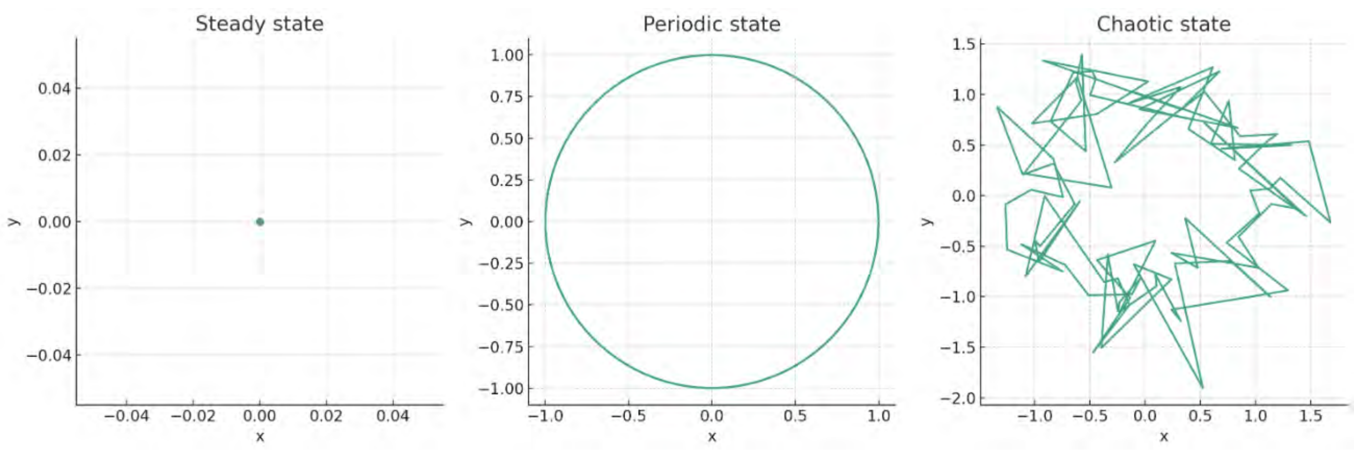
where λ is the Lypaunov exponent. Here, we assume that $d(0)$ is small enough so that an exponential approximation can be taken. Then, by taking natural logarithms, we find that the Lypaunov exponent can be written as

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{d(t)}{d(0)}\right)$$

As $r \in \mathbb{R}^3$ for the Lorenz system, there will be 3 Lypaunov exponents $\{\lambda_1, \lambda_2, \lambda_3\}$ that characterize the system in all 3 directions. We are typically most interested in the Maximum Lypaunov Exponent (MLE) as that tells us a lot about the system itself.

- If the Lypaunov exponent is positive ($\lambda > 0$), the trajectories are diverging on average (as $e^{\lambda t}$ approaches infinity) and the system is chaotic.
- If it is negative ($\lambda < 0$), the trajectories are converging on average (as $e^{-\lambda t}$ approaches 0) and the system is stable.
- If it is zero ($\lambda = 0$), the trajectories neither converge nor diverge on average (as $e^0 = 1$), indicating a neutral or marginally periodic stable system.

Below, we show a plot for an unrelated system and its behavior for corresponding MLE.



Problem 3

Compute the Maximal Lyapunov Exponent (MLE) for a Lorenz system of these parameters:

- $\sigma = 15.6$
- $\rho = 35.4$
- $\beta = 3.13$

What does this imply about the system? See if you can estimate uncertainties!

Problem 4

Call $T_{\max}$ the maximal time for when a simulation is accurate to 99% of reality. A simulation is characterized by the specific numerical solver and the time step dt.

Using `rk4_step`, estimate $T_{\max}$ at various values of dt, given the initial point of (1,1,1). Make a plot.

Then do the same thing with `rk2_step`. What differences do you notice?

Problem 5

Let $\sigma = 10, \rho = 28, \beta = 8/3$. What is the average angular frequency ω_0? Report with uncertainty. How does this change with position?