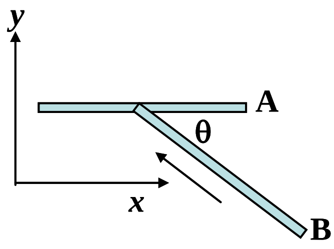
Pan Pearl River Delta Physics Olympiad 2005

Jan. 29, 2005

Morning Session (9 am – 12 pm)

Q1 (5 points)

Two identical worms of length L are lying on a smooth and horizontal surface. The mass of the worms is evenly distributed along their body length. The starting positions of the two worms are shown in the figure. The coordinate of the center of worm-A is (0, 0). Worm-B then starts to climb slowly over worm-A with their bodies always form an angle θ. After Worm-B has completely climbed over worm-A, what are the center positions of the two worms?



Q2 (13 points)

An air bubble of size 0.001m^3 and a rigid tank of the same volume and mass as the bubble are released at a depth of 2.0 km below the sea surface. Ignore friction. The temperature of the air bubble remains the same at any depth. Air density at sea level is 1.21 kg/m^3, and the atmosphere pressure is 1.0 x 10^5 N/m^2. (hint: $\int_a^b \frac{dx}{\alpha+\beta x} = \frac{1}{\beta}\ln\left(\frac{\alpha+\beta b}{\alpha+\beta a}\right)$)

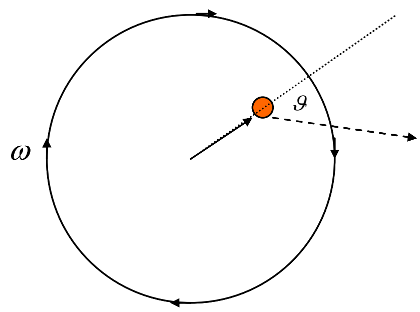
(a) What is the size of the bubble when it rises to the sea level? **(**3 points**)**

(b) Derive an expression for the net energy gained by the bubble and the tank at height h. **(**7 points**)**

(c) Find the final velocities of the bubble and the tank when they reach the sea level. **(**3 points**)**

Q3 (12 points)

A man with mass 0.5M is standing on a round table (disk shaped, uniform thickness) rotating at angular speed ω. The mass of the table is 0.5M and the friction between the table and the ground is negligible. The man carries with him 10 stones each with mass 0.01M. The radius of the table is R and the man is standing at a distance r (< R) from the center of the table.



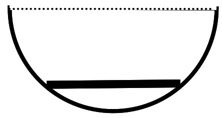
(a) Find the total angular momentum of the system. (4 points)

To slow down the rotation of the table, the man decides to throw the stones outward from the table, each at speed v (relative to him) and with angle ϑ relative to the radial direction (see figure).

(b) Determine the angular speed of the table after the man has thrown his first stone as a function of angle ϑ, and find the optimum angle ϑ to slow down the table. **(**4 points**)**

(c) What is the angular speed of the table after the man has thrown all his stones, each time at the optimum angle? (Leave your answer as the sum of multiple terms.) **(**4 points**)**

Q4 (8 points)



A uniform rod of length L and mass M is resting in a smooth hemisphere of radius R ($>0.5L$), as shown.

(a) Find the vibration frequency of the rod about its equilibrium position. **(**4 points**)**

(b) In the vibration motion, the maximum deviation angle of the rod from its equilibrium position is $\theta_{\max}$. Let the amplitude of the contact force from the hemisphere to the rod at each end be N. The difference between N when the rod is at $\theta_{\max}$ and when the rod is at its equilibrium position can be written as $\Delta N = \alpha M g \theta_{\max}^2$. Find α. **(**4 points**)**

Q5 (12 points)

The electric field of an electromagnetic (EM) wave is $\vec{E} = E_0 \vec{x}_0 e^{i(kZ-\omega t)}$, where E_0 is a real constant, and $\omega = \frac{c}{\tilde{n}} k$. Here ω is real, c is the speed of light in vacuum, and $\tilde{n}$ is the complex dielectric constant of the medium.

(a) Briefly discuss what will happen to the EM wave amplitude as it propagates in the medium if $\tilde{n}$ is real, imaginary, or complex. **(**4 points**)**

(b) Find the magnetic field $\vec{B}$, and the time-averaged (over one period) Poynting's vector $<\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})>$. **(**4 points**)**

(c) The quantity $q = \frac{d<\vec{S}>}{dz}$ describes the loss of EM wave energy to the medium. Calculate q and briefly discuss the physical meanings of the results if $\tilde{n}$ is real, imaginary, or complex. **(**3 points**)**

(d) With reference to the results above, does an EM wave that decreases in amplitude while propagating always loose energy to the medium? **(**1 points**)**

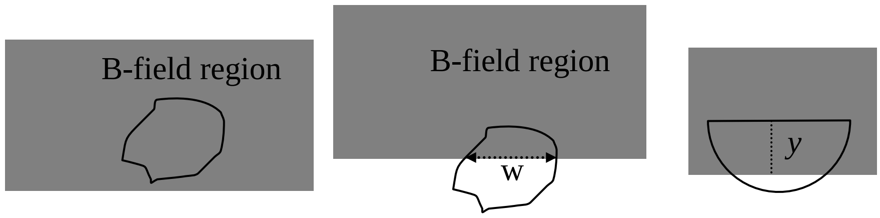
Pan Pearl River Delta Physics Olympiad 2005

Jan. 29, 2005

Afternoon Session (2 pm – 5 pm)

Q6 (12 points)

Consider a uniform magnetic field B within the shaded region and pointing out of the paper plane, as shown below.



(a) (b) (c)

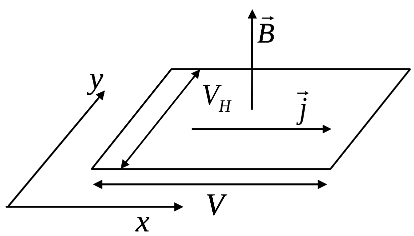
(a) Find the total force of the magnetic field on a closed thin wire coil carrying a steady electric current I, all of which is inside the field region. The coil plane is within the paper. (3 points)

(b) The total force of the magnetic field on the coil when part of it is outside the field region can be expressed as $F = \alpha wBI$, where w is the distance between the two points where the coil intersects the bottom edge of the field region, and its direction is either upward or downward depending on the direction of the current. Find the value of α. (3 points)

(c) A semicircle thin wire coil of radius r, resistance R, and mass m is falling down and out of the field region. The plane of the coil remains in the paper plane, and its straight edge remains parallel to the horizontal bottom edge of the field region. Ignore self-inductance of the coil. Derive the differential equation for the distance between the straight edge of the coil and the edge of the field region y ($< R$). If you have not found α in (b) you may take it as a known constant in solving this part of the problem. (6 points)

Q7 (15 points)

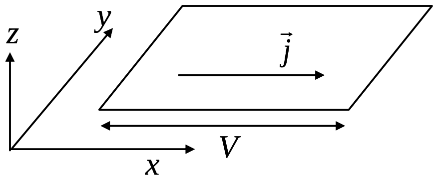
It is well known that when crossed electric and magnetic fields are applied to a piece of semiconductor, a voltage V_H perpendicular to the direction of charge motion will be induced (see figure). The phenomenon is called the *Hall Effect*.



(a) Assume that the semiconductor is a square sheet of size $W \times W$. The electric current is due to the motion of positive charge carriers each carrying charge e. The surface density of the carriers is n, and the conductivity of the semiconductor is σ. There is also a negative charge background so there is charge neutrality everywhere except at the side edges. The electric field is uniform in the semiconductor. A magnetic field B is applied in the direction perpendicular to the sheet. When a voltage V is applied a voltage V_H across the two edges parallel to the current along the x-direction will be induced, in addition to the electric current $\vec{j}$. When the steady state is reached, find the Hall Coefficient $R_H \equiv V_H / V$. (Note that $\vec{j}$ is a unknown quantity) (6 points)

Nowadays it is also known that for certain semiconductor structures, a *spin-Hall* effect will also occur. The effect is associated with the magnetic moment $\vec{m}$ of the charge carriers. For two dimensional structures it is known that an additional force $\vec{F}_R = \eta_R(\vec{m}\times\vec{v})$ (called Rashba force) will act on the carriers, where $\vec{v}$ is the velocity of the carriers on the 2-dimensional (X-Y) plane, and η_R is a constant. The magnetic moment $\vec{m}$ is restricted to point perpendicular to the plane, i.e. $\vec{m} = \pm m\hat{z}$. The external magnetic field is absence. Ignore the magnetic dipole interactions between the carriers.

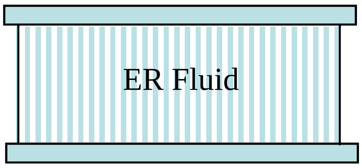
(b) Assume again that the electric field is uniform and its force along the *x*-axis is much stronger than $\vec{F}_R$, find the currents flowing in the *y*-direction in terms of the voltage *V* and other parameters given in (a). How are the currents related to $\vec{m}$? (6 points)



(c) Due to collision with the boundary, the carriers with particular magnetic moment will loose their sense of direction within a 'life time' τ after they reach the edge. In other words, each second there are n_m/τ of carriers loosing their moment direction within a unit length of edge, where n_m is the surface density of the carriers still maintaining their moment direction ($\pm\hat{z}$). Find the magnetization $\vec{M}$ near the edges. (3 points)

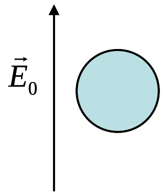
Q8 (23 points)

Electrorheological (ER) fluids, which are composed of small dielectric spheres suspended in an insulating liquid, such as silicone oil, are materials that can transform from liquid-like form to solid-like under an external electric field. A typical test setting of ER fluids is shown in the figure, where ER fluid is filled between two parallel conducting plates of area *A* separated by a distance *D*. When no voltage is applied between the plates the ER fluid is liquid-like so the plates can moved horizontally almost without friction.



When a voltage *V* is applied, the small spheres are polarized and aligned into vertical columns, and to move a conducting plate relative to the other by a small displacement δx requires a small force δf. The shear modulus η is defined as $\eta = \dfrac{D}{A}\dfrac{\delta f}{\delta x}$. The radius of the spheres is *R* (<< *D*), their dielectric constant is ε, and the volume fraction of spheres to fluid is *m*. The dielectric constant of the liquid without the spheres is 1. Ignore gravity. You are to find η in terms of the physical qualities given above.

(a) The first step is to find the polarization $\vec{P}$ of an isolated sphere in a uniform external electric field $\vec{E}_0$. This can be done by solving (a1) – (a3) below, and <u>utilizing the known fact</u> that under such circumstance the polarization is <u>uniform</u> in the sphere and parallel to $\vec{E}_0$.



(a1) Find the electric field <u>due to the polarization</u> $\vec{P}$ at the center of the sphere. (3 points)
(a2) Find the total electric field inside the sphere. (3 points)

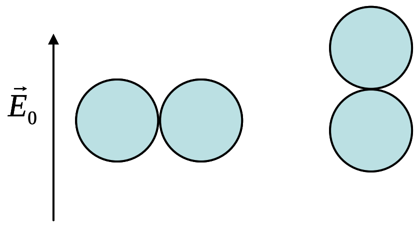
(a3) The total induced electric dipole moment of a sphere can be expressed as $\vec{p}_0 = \alpha \vec{E}_0$. Find the constant α. (3 points)

(b) Treat each sphere as an ideal electric dipole located at the center of the sphere, and assume that the dipole moment depends only on $\vec{E}_0$. If you have not found α in (a3) you may take it as a known constant in solving the following problems. (Hint: Keep the expansion terms up to d^2, where d is the length of the dipole.)

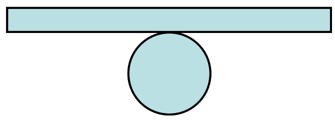
(b1) Find the electrostatic energies of two spheres in contact in the side-by-side and the top-bottom configurations, as shown in the figures below. (4 points)

(b2) Find the electrostatic force of the conductor plate on the sphere that is in contact with the plate. (3 points)

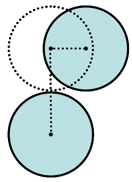
(b3) Find the restoring horizontal force between two spheres when the upper one in the top-bottom configuration is displaced horizontally by a small distance δa, as shown. (3 points)



(b1)

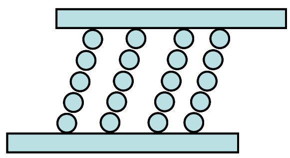


(b2)



(b3)

(c) Assume that under the applied electric field, <u>all</u> spheres form continuous, straight, and single file thin columns between the plates. According to your answers in (b1), do the columns like to bunch together? Consider only the force between adjacent spheres within a column, when the top plate is displaced by a small distance δx, the top sphere of each column remains stick to the plate and is moved by the same distance. As shown in the figure, each sphere in the column below is then displaced uniformly relative to the one just above. The bottom spheres remain fixed to the bottom plate. Find the shear modulus η. (4 points)



Pan Pearl River Delta Physics Olympiad 2005
Jan. 29th, 2005
Morning Session Marking Scheme

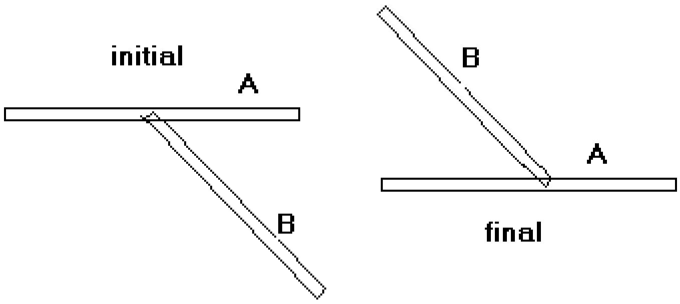
Q1. Original Position of A (center) A 的起始中心位置: (*0,0*)---- (1 分)

Original Position of B (center) A 的起始中心位置: (*L/2cos θ,-L/2sin θ*) --- (1 分)

Center-of-mass of A+B remains fixed A+B 的重心不变 ---- (1 分)

Final Position of A (center) A 的最终中心位置: (*L/2cos θ,-L/2sin θ*) ---- (1 分)

Final Position of B (center) B 的最终中心位置: (*0,0*) ------(1 分)



Q2.

a. According to the Boyle's Law 利用理想气体原理,

$P_1V_1 = P_2V_2$

$P_h = \rho_w gh + P_0 = [(1000\times 9.8\times 2\times 10^3) + 10^5]Nm^{-2}$ ---- (1 分)

$= 1.97\times 10^7\,Nm^{-2}$ ---- (1 分)

$V_0 = \dfrac{P_hV_h}{P_0} = \dfrac{1.97\times 10^7}{10^5}(10^{-3})m^3 = 0.197m^3$ ---- (1 分)

共 (3 分)

b. Buoyant Force 浮力,

$F = \Delta\rho gV \qquad \Delta\rho = \rho_w - \rho \qquad\qquad (\rho_w >> \rho, \Delta\rho \approx \rho_w)$

For the tank 钢瓶,

$\rho = \dfrac{\rho_0 V_0}{V_h} = \dfrac{(1.21)(0.197)}{10^{-3}}kgm^{-3} = 238.4kgm^{-3}$ ---- (1 分)

$E_t = \Delta\rho gV_h h = (1000 - 243.21)(9.8)(10^{-3})(2\times 10^3)J = 1.48\times 10^4 J$ --- (1 分)

For the bubble 气泡,

Energy gained = buoyant force part 浮力作功

$$\rho \propto P \Rightarrow \rho_b \frac{P_b}{P_0}\rho_0 = \frac{\rho_0}{P_0}(\rho_w gh + P_0)$$ ---- (1 分)

$$E_b = \int F dh$$

$$= \int_0^h (\rho_w - \frac{\rho_0}{P_0}(\rho_w gh + P_0))g(\frac{P_0 V_0}{\rho_w gh + P_0})dh$$

$$= \int_0^h (\frac{\rho_w g P_0 V_0}{\rho_w gh + P_0} - \rho_0 gV)dh$$

$$= P_0 V_0 \ln[\frac{P_0 + \rho_w gh}{P_0}] - \rho_0 g V_0 h$$

$$= [(1.97\times10^4)\ln[197] - (1.21)(9.8)(0.197)(2\times10^3)]J$$

$$= (1.041\times10^5 - 4247.3)J$$

$$= 0.998\times10^5 J$$ ---- (1 分)

(if assume 如果假设 $\Delta\rho \approx \rho_w$, we have the following modification 我们得到)

$$E_b = \int F dh$$

$$= \int_0^h \rho_w g(\frac{P_0 V_0}{\rho_w gh + P_0})dh$$

$$= P_0 V_0 \ln[\frac{P_0 + \rho_w gh}{P_0}]$$

$$= (1.97\times10^4)\ln[197]J$$

$$= 1.041\times10^5 J$$

共 (7 分)

c. For the tank 钢瓶,

$$\frac{1}{2}mv^2 = E_t$$

$$v = \sqrt{\frac{2E_t}{\rho_0 V_0}} = \sqrt{\frac{2(1.48\times10^4)}{(1.21)(0.197)}}ms^{-1} = 352.4ms^{-1}$$ ---- (1 分)

$$\frac{1}{2}mv^2 = E_b$$

$$v = \sqrt{\frac{2E_b}{\rho_0 V_0}} = \sqrt{\frac{2(0.998\times10^5)}{(1.21)(0.197)}}ms^{-1} = 915.2ms^{-1}$$

or $\quad v=\sqrt{\dfrac{2(1.041\times10^5)}{(1.21)(0.197)}}=934.5ms^{-1}$ ---- (2 分)

共（3 分）

Q3.

a.

$$I=\sum_i m_i r_i^2=(0.5\mathrm{M}+10(0.01\mathrm{M}))r^2+\frac{1}{2}\mathrm{M}R^2 \qquad \text{let 取} \qquad \frac{R}{r}=n>1$$

$$=(0.6+0.5n^2)\mathrm{M}r^2$$

$$L=I\omega=(0.6+0.5n^2)\mathrm{M}\omega r^2 \quad (4\text{ 分})$$

b. $\quad L=m\omega r^2 \qquad$ where $\qquad M=(0.6+0.5n^2)\mathrm{M}$ and $m=0.01\mathrm{M}$

In the 1st throw, by the conservation of angular momentum, 扔了一石子后，由角动量守恒

$$L=(M-m)\omega_1 r^2+mr^2(\frac{v}{r}\sin\theta+\omega_1) \quad \text{---- (2 分)}$$

$$\Rightarrow \omega_1=\frac{L-mvr\sin\theta}{Mr^2}$$

For the optimum angle to slow down,

$$\Rightarrow \sin\theta=1\Rightarrow\theta=90^0C$$

---- (1 分)

$$\Rightarrow \omega_1=\frac{L-mvr}{Mr^2}=\frac{L}{Mr^2}-\frac{mv}{r}(\frac{1}{M}) \quad \text{---- (1 分)}$$

共（4 分）

c. For the 2nd stone 扔第二颗石子后,

$$\omega_2=\frac{L_1-mvr}{Mr^2} \qquad \text{---- (1 分)}$$

where 其中 $\quad L_1=(M-m)\omega_1 r^2$ and $M_1=M-m$

$$=\frac{L}{Mr^2}-\frac{mv}{r}(\frac{1}{M}+\frac{1}{M-m}) \quad \text{---- (1 分)}$$

For the nth stone 扔第 n 颗石子后,

$$\omega_n = \frac{L}{Mr^2} - \frac{mv}{r}\sum_{i=1}^{n}\frac{1}{M-(i-1)m}$$

$$\omega_{10} = \frac{L}{Mr^2} - \frac{mv}{r}\sum_{i=1}^{10}\frac{1}{M-(i-1)m} \quad \text{---- (2 分)}$$

共（4 分）

Q4. (a) 长竿绕圆心运动。球面对长竿的力通过圆心，力矩为 0。---（1 分）

According to the Parallel Axis Theorem 根据平行轴定理,

$$I = I_0 + M(R^2 - \frac{1}{4}L^2) = \frac{1}{12}ML^2 + Mh^2 \quad \text{---（2 分）}$$

$$T = 2\pi\sqrt{\frac{I}{Mgh}} \qquad \text{where 其中} \qquad h = \sqrt{R^2 - \frac{1}{4}L^2}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{gh}{h^2 + \frac{1}{12}L^2}} \quad \text{---（1 分）}$$

共（4 分）

（b）长竿最大偏角时两端的力分别为 At maximum angle the forces on the ends are $N \pm \delta N$, respectively

$2N\sin\beta = Mg\cos\theta_{\max} = Mg(1-\frac{1}{2}\theta_{\max}^2)$， ---（1）

其中 where $\sin\beta \equiv \frac{\sqrt{R^2 - 1/4L^2}}{R}$ 。

$Mh\omega^2\theta_{\max} = Mg\theta_{\max} - 2\delta N\cos\beta$ ---（2）

$\delta NL\sin\beta = I_0\omega^2\theta_{\max}$ ---（3）

Putting(3) into(2) one gets the same frequency as in (a) 由(3)得 δN 。将 (3) 代入 (2) 可得频率，与(a)相同。

长竿水平时两端的力均为 At level position both end forces are *N'*，angular speed is 角速度为 $\omega^2 = Mgh\theta_{\max}^2 / I$ 。

$2N'\sin\beta = Mg + Mh\omega^2$ --- (4)

Finally 最后得 $\alpha = (\frac{6}{12+(L/h)^2} + \frac{1}{4})/\sin\beta$ 。

Q5.

a. $\vec{E} = E_0 \hat{x} e^{i(kz-\omega t)}$ where 其中 $k = \frac{\omega}{c}\tilde{n}$

Let 令 $k = \frac{\omega}{c}(a+ib)$ where 其中 $\tilde{n} = a+ib$

a and b are real, a 和 b 为实数

$$\vec{E} = E_0 \hat{x} e^{i\omega(\frac{az}{c}+i\frac{bz}{c}-t)} = E_0 e^{-\frac{b\omega z}{c}} \hat{x} e^{i\omega(\frac{az}{c}-t)}$$ ---（1 分）

if 如果 $k = \frac{\omega}{c}a$,

$\vec{E} = E_0 \hat{x} e^{i\omega(\frac{az}{c}-t)}$ 波幅不随传播而变 ---（1 分）

if 如果 $k = \frac{\omega}{c}(a+ib)$,

波幅随传播而变 ---（1 分）

if 如果 $k = i\frac{\omega b}{c}$,

$\vec{E} = E_0 e^{-\frac{b\omega z}{c}} x e^{-i\omega t}$ 波幅随传播而变 ---（1 分）

共（4 分）

b. $\vec{B} = \frac{1}{i\omega}\nabla\times\vec{E} = \frac{1}{i\omega}(ikE_0\hat{y}e^{i(kz-\omega t)})$

$= \frac{k}{\omega}E_0\hat{y}e^{i(kz-\omega t)}$ ---（1 分）

$= \frac{1}{c}(a+ib)E_0 e^{-\frac{b\omega z}{c}} \hat{y} e^{i\omega(\frac{az}{c}-t)}$ ---（1 分）

For complex k, 如 k 是复数.

$$\langle\vec{S}\rangle = \frac{1}{\mu_0}\mathrm{Re}(\vec{E}\times\vec{B}^*) = \frac{1}{2\mu_0}\mathrm{Re}[\frac{1}{c}(a-ib)E_0^2 e^{-\frac{2b\omega z}{c}}]$$

$= \frac{a}{2\mu_0 c}E_0^2 e^{-\frac{2b\omega z}{c}}$ ---（3 分）

共（5 分）

c. $q=\frac{d\langle\vec{S}\rangle}{dz}=\frac{a}{2\mu_0 c}(-\frac{2b\omega}{c})E_0^2 e^{-\frac{b\omega z}{c}}$

$$=-\frac{ab\omega}{\mu_0 c^2}E_0^2 e^{-\frac{2b\omega z}{c}}$$ ---（2 分）

if a or b = 0 当 a 或 $b=0$时 $q=0$

d. 不. 当 a = 0 但 $b\neq 0$时，波幅随传播而变，但$q=0$。

No. When a = 0 but $b\neq 0$, the wave amplitude changes but $q=0$.（2 分）

第一届泛珠三角物理奥林匹克竞赛

第二部分答案

Q6

（a） （3 分）答案一：把线圈看成无数个小的正方形的线圈叠加的总效果。小线圈的合力是零，因此总的合力是零。

答案二：将线圈在磁场里移动并不需要做功，因无电磁感应，因此总的合力是零。

答案三：用矢量投影。

（b） 将线圈在磁场边界分成两半，假想一正负电流。--- （1 分）

α＝1 --- （2 分）

共（3 分）

（c） 线圈在磁场中运动时，切割磁场的长度

$$w = 2\sqrt{r^2 - y^2} \quad \text{---} \quad （1\text{ 分}）$$

产生的电流

$$I = \frac{Bwv}{R} \quad \text{---} \quad （1\text{ 分}）$$

其中 $v = \dfrac{dy}{dt}$ --- （1 分）

磁场对线圈的作用力

$$F = BIw \quad \text{---} \quad （1\text{ 分}）$$

运动方程为 $m\dfrac{d^2y}{dt^2} = F - mg$ --- （1 分）

综合上面各式，化简为:

$$\frac{d^2y}{dt^2} + \frac{4B^2}{mR}(r^2 - y^2)\frac{dy}{dt} + g = 0 \quad \text{---} \quad （1\text{ 分}）$$

共（6 分）

Q7

(a)

$j = nev = \sigma E$, --- （1 分） $E = \dfrac{V}{W}$ --- （1 分）

$F_E = \dfrac{eV_H}{W}$, --- （1 分） $F_B = eBv$ --- （1 分）

由 $F_E = F_B$，--- （1 分）

可以求出 $\dfrac{V_H}{V} = \dfrac{B\sigma}{en}$ --- （1 分）

共（6 分）

（b）平均来讲电子有一半的自旋向上，一半的自旋向下。沿 X-方向运动的电子收 y 方

向的力，$j = j_y + j_{-y} = 0$，但 $j_{spin} \neq 0$。 j_{spin}实际上是自旋电流，而不是电荷电流。

$$j_y = \sigma E_y = \frac{\sigma F_Y}{e} = \frac{\sigma \eta_R m v_x}{e} = \frac{\sigma \eta_R m}{e} \cdot \frac{\sigma V}{neW} = \frac{\sigma^2 \eta_R m V}{nWe^2}$$

电流方向一左一右，由自旋是上还是下决定。共（6 分）

（c）

达到平衡时，退激化的电子等于电流补充进来的电子

$$\frac{n_m}{\tau} = \frac{j_y}{e} \quad \text{---}\ （2\ 分）$$

$$M = n_m m = \frac{\sigma^2 \eta_R m^2 \tau V}{Wne^3} \quad \text{---}\ （1\ 分）$$

共（3 分）

Q8

(a)部分

A1:

Surface charge density 电荷面密度$\sigma = P\cos\theta$ --- （1 分）

$$E_p = \frac{P}{4\pi\varepsilon_0 R^2} \times 2 \times \int_0^{\frac{\pi}{2}} d\theta \cos\theta \cdot R^2 \sin\theta \cdot \cos\theta \int_0^{2\pi} d\phi = \frac{P}{3\varepsilon_0} \quad \text{---}\ （2\ 分）$$

共（3 分）

A2:

$$E = E_0 - \frac{p}{3\varepsilon_0} = E_0 - \frac{\varepsilon - 1}{3} E \text{，} \quad \text{---}\ （2\ 分）\ 可得\ \vec{E} = \frac{3}{\varepsilon + 2}\overrightarrow{E_0} \quad \text{---}\ （1\ 分）$$

共（3 分）

A3:

$$\vec{P} = (\varepsilon - 1)\varepsilon_0 \vec{E} \ \text{---}\ （1\ 分）$$

$$= 3\varepsilon_0 \frac{(\varepsilon - 1)}{\varepsilon + 2}\overrightarrow{E_0} \ \text{---}\ （1\ 分）$$

$$\vec{p} = \frac{4}{3}\pi R^3 \vec{P} = 4\pi\varepsilon_0 R^3 \frac{\varepsilon - 1}{\varepsilon + 2}\overrightarrow{E_0}$$

$$\alpha = 4\pi\varepsilon_0 R^3 \frac{\varepsilon - 1}{\varepsilon + 2} \ \text{---}\ （1\ 分）$$

共（3 分）

(b) 部分

B1:

左右 $W=\frac{2Q^2}{4\pi\varepsilon_0}[\frac{1}{2R}-\frac{1}{\sqrt{4R^2+d^2}}]=\frac{p^2}{32\pi\varepsilon_0 R^3}$ ----（2 分）

上下 $W=-\frac{p^2}{16\pi\varepsilon_0 R^3}$ ----（2 分）

共（4 分）

B2：(电像法 image charge) ----（1 分）

$$W=\frac{p^2}{16\pi\varepsilon_0 R^3},\text{ ----（1 分）}\quad F=\frac{\partial W}{\partial(2R)}=\frac{3p^2}{32\pi\varepsilon_0 R^4}\text{ ----（1 分）}$$

共（3 分）

B3：

$$W=\frac{Q^2}{4\pi\varepsilon_0}(\frac{2}{\sqrt{4R^2+x^2}}-\frac{1}{\sqrt{(2R-d)^2+x^2}}-\frac{1}{\sqrt{(2R+d)^2+x^2}})=-\frac{3p^2}{128\pi\varepsilon_0 R^5}x^2$$

----（1 分）

$$F=-\frac{dW}{dx}=\frac{3p^2}{64\pi\varepsilon_0 R^5}\delta a\text{ ----（2 分）}$$

共（3 分）

（c）部分：左右排列时，能量为正，与距离三次方成反比，不易粘在一起。----（1 分）

细柱的体积：$\frac{4}{3}\pi R^3\times\frac{D}{2R}$ ，总体小球的体积：ADm

细柱的根数：$\dfrac{ADm}{\frac{4}{3}\pi R^3\times\frac{D}{2R}}$ ----（1 分）

$$\delta a=\frac{2R}{D}\delta x,\ \delta f=F\text{ ----（1 分）}$$

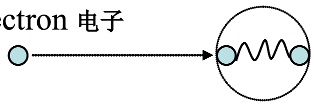
$$\eta=\frac{D\delta f}{A\delta x}=\frac{9m\varepsilon_0}{4}(\frac{\varepsilon-1}{\varepsilon+2})^2(\frac{V}{D})^2=\frac{9m\varepsilon_0}{4}(\frac{\varepsilon-1}{\varepsilon+2})^2E_0^2\text{ ----（1 分）}$$

共（4 分）

Pan Pearl River Delta Physics Olympiad 2006
2006 年泛珠江三角物理競賽
Part-1 卷-1
(9:00 am – 12:00 pm, 02-09-2006)

Q1 (8 points) 題 1 (8 分)

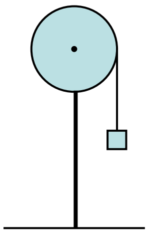
A meson is composed of two quarks with rather complicated interaction between them. To learn the physics inside the meson high energy electrons are usually employed to collide inelastically with the mesons. The process is very difficult to interpret, and to grasp the essential elements of it the Parton model was developed. The model assumes that the incoming electron collides elastically with only part of the meson, i. e., with only one quark. The quark then passes its energy to its parent meson. The essence of the model can be illustrated with the following drastically simplified model. The electron of mass m_1 and kinetic energy E collides elastically with a quark of mass m_2 in a meson. The other quark has mass m_3. The two quarks are connected by a weightless spring. Before the collision the two quarks are at rest and the spring is at its natural length. All motions are along one dimension only, and ignore all relativistic effects. Find the internal energy of the meson as represented by the vibration energy of the spring, and the kinetic energy of the meson as a whole after the collision.



介子由兩個誇克構成，而誇克之間的相互作用相當複雜。研究介子可通過用高能電子與之作非彈性碰撞來進行。由於碰撞過程難於分析，為掌握其主要內涵，人們發展了一種簡化了的‘分粒子’模型。其主要內容為：電子只和介子的某部分（比如其中一個誇克）作彈性碰撞。碰撞後的誇克再經過介子內的相互作用把能量和動量傳給整個介子。模型的主要精神可用下面的簡化模型來闡述：一電子質量為 m_1，動能為 E，與介子的一個誇克（質量 m_2）作彈性碰撞。介子裏另一個誇克的質量為 m_3。誇克間以一無質量彈簧相連。碰撞前誇克處於靜止狀態，彈簧處於自然長度。所有運動都是一維的。忽略一切相對論效應。求碰撞後以彈簧振動形式代表的介子內能，和介子作為一整體所具有的動能。

Q2 (8 points) 題 2 (8 分)

A uniform iron pulley of total mass M_1 and radius R is fixed above the ground. It can spin freely about its center. Additional weights in the form of many small magnets with total mass M_2 can be attached to the pulley. A long thread is winding around its edge, and at its end hangs a weight of mass M_3 at height H from the ground that can be released suddenly. Assume that there is no slip between the thread and the pulley. (a) How should the magnets be placed with circular symmetry on the pulley such that the final speed of M_3 is minimum when it reaches the ground? (b) Find the minimum speed.



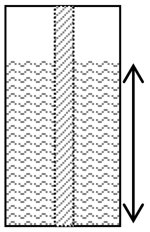
一均勻鐵滑輪質量為 M_1，半徑為 R，固定在地面上方。滑輪可以輪心自由轉動。另有總重量為 M_2 的許多小磁鐵塊可吸附在滑輪上。一長細繩繞著滑輪邊緣，終端掛一質量為 M_3 的小重塊，離地高度為 H。細繩與滑輪間無滑動。(a)問小磁鐵塊應怎樣圓對稱地分佈在滑輪上，才能使小重塊被放開後到達地面時的速度為最小？(b) 求該速度。

Q3 (10 points) 題 3 (10 分)

A cylindrical vessel of radius R is filled with water (density ρ) up to height H.

(a) Calculate the force of water exerting on a narrow vertical strip of width d ($<< R < H$) of the cylinder wall. (3 points)

(b) The vessel is then placed on a platform spinning at angular speed ω ($<\sqrt{gh/R}$). The rotation axis coincides with the central axis of the vessel. The water spins at the same angular speed as the platform. Find the shape of the water surface, and the extra force of water exerting on the strip due to spinning. (g is the gravity acceleration.) (7 points)



一圓柱容器半徑為 R，裝有高度達 H 的水（密度為ρ）。

(a) 求水對容器面上一寬度為 d ($<< R < H$) 的豎直面的壓力。(3 分)

(b) 將容器放在以角速度ω ($<\sqrt{gH/R}$) 轉動的轉臺上，轉動軸與容器中心軸重合。水的轉動和轉臺一致。求水面形狀和水對豎直面因轉動而帶來的附加壓力。(g 為重力加速度。) (7 分)

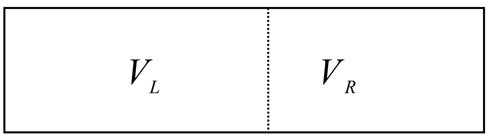
Q4 (12 points) 題 4 (12 分)

A cylinder of fixed volume V is partitioned into left and right compartments separated by a movable piston of negligible mass. The left and right compartments are filled with n_L and n_R moles of the same ideal gas, respectively, at initial temperatures T_L and T_R.

a) What are the volumes of the left and right compartments V_L and V_R initially when the system is at equilibrium? (2 points)

b) Suppose heat can be exchanged between the two compartments but there is no heat exchange between the cylinder and outside world. The amount of heat flow per unit time $\frac{\Delta Q_{L\to R}}{\Delta t}$ from left to right compartment is proportional to the temperature difference between the two compartments, i.e. $\frac{\Delta Q_{L\to R}}{\Delta t} = k(T_L - T_R)$. Find the temperature difference, $T_L - T_R$ as a function of time. (8 points)

c) Find out how V_L and V_R changes with time. (2 points)



一固定體積為 V 的汽室中間由一可左右無磨擦滑動的輕活塞分開。左 、右汽室分別充有 n_L 、n_R 摩爾初始溫度分別為 T_L 、 T_R 的相同理想氣體。

a) 求平衡時左 、右汽室的體積 V_L 、V_R 。(2 分)

b) 假設左 、右汽室之間可以作熱交換，但與外界無熱量交換，單位時間熱量的交換 $\frac{\Delta Q_{L\to R}}{\Delta t}$ 與兩邊的溫

差成正比，即$\frac{\Delta Q_{L\to R}}{\Delta t}=k(T_L-T_R)$。求$T_L-T_R$與時間的關係。(8 分)

c) 求V_L 、V_R與時間的關係。(2 分)

Q5 (12 points) 題 5 (12 分)

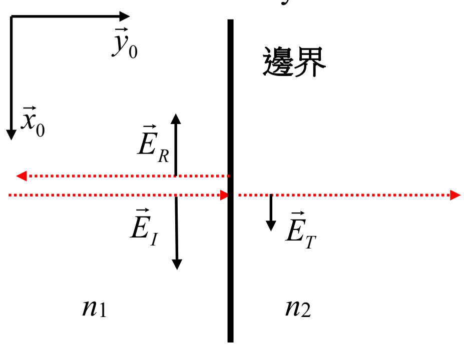
A planar boundary at $y = 0$ separates two non-magnetic media with refractive indices n_1 (real) and n_2 (can be complex). An electromagnetic wave $\vec{E}_I = E_0\vec{x}_0 e^{i(k_1 y-\omega t)}$ is incident onto the boundary, and its reflected wave and transmitted wave are given by $\vec{E}_R = E_r\vec{x}_0 e^{i(-k_1 y-\omega t)}$ and $\vec{E}_T = E_t\vec{x}_0 e^{i(k_2 y-\omega t)}$, respectively.

(a) Use the boundary conditions of electric and magnetic fields to find $r \equiv E_r/E_0$ and $tr \equiv E_t/E_0$ in terms of n_1 and n_2. (8 points)

(b) For $n_1 = 1$, $n_2 = 2.0i$, where $i \equiv \sqrt{-1}$, find the reflection $R \equiv |r|^2$. (1 points)

(c) For $n_1 = 1$, $n_2 = 1.0 + 2.0i$, find the phase shift of the reflected wave relative to the incidence wave. (3 points)

Boundary



一位於$y=0$ 的平面的左、右分別是折射率為 n_1 (實數)、n_2 (可能是複數)的非磁性介質。一電磁波$\vec{E}_I = E_0\vec{x}_0 e^{i(k_1 y-\omega t)}$射向介面，其反射波為$\vec{E}_R = E_r\vec{x}_0 e^{i(-k_1 y-\omega t)}$，透射波為$\vec{E}_T = E_t\vec{x}_0 e^{i(k_2 y-\omega t)}$。

(a) 利用電磁場的邊界條件，導出$r \equiv E_r/E_0$, $tr \equiv E_t/E_0$。你的答案應只含 n_1 和 n_2。(8 分)

(b) 當 $n_1 = 1$, $n_2 = 2.0i$ 時, 其中$i \equiv \sqrt{-1}$, 求反射率$R \equiv |r|^2$。(1 分)

(c) 當 $n_1 = 1$, $n_2 = 1.0 + 2.0i$ 時, 求反射波相對於入射波的位相差。(3 分)

Pan Pearl River Delta Physics Olympiad 2006
2006 年泛珠江三角物理競賽
Part-2 卷-2
(2:00 pm – 5:00 pm, 02-09-2006)

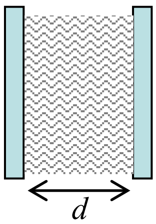
Q6 題 6 Casmir Effect 卡士米爾效應
(13 points **13 分)**

Part-A What is the pressure of ideal gases when the temperature goes to absolute zero? (2 points)

Part-B Casmir proposed in the 1950's that vacuum is actually filled with virtual electromagnetic waves, and the energies stored in the vacuum produces an observable force between two parallel metallic plates separated by distance d. The electromagnetic energy stored between the plates is given in Quantum Mechanics by $E=\frac{1}{2}\sum_{n<p}E_n$, where $E_n=\hbar ck_n$, c is the velocity of light and $\hbar$ is a fundamental constant called Planck's constant, p (>> 1) is a large number that depends on the material properties of the plates. $k_n \equiv 2\pi/\lambda_n$, where λ_n is the wavelength of a standing electromagnetic wave that fits into the space between the plates.

(a) Assume that the waves take the form of $\sin k_n x$, what are the possible values of k_n such that at the plate surfaces x = 0 and x = d the standing waves vanish? (2 points)
(b) Show that the vacuum electromagnetic force is of the form a/d^2. What is the constant a? (5 points)
(c) What is the force due to the vacuum electromagnetic waves outside the two plates? (2 point)
(d) Find the amplitude of the force when d = 1.0 mm. (Hint: If you cannot get a in (b), use dimensional analysis to estimate its order of magnitude.) (2 points)

($\hbar=1.05\times10^{-34}$ (Joule*second); $p=2000$; $c=3.0\times10^{8}$ m/s .)



（A）當理想氣體的溫度趨於零時，其壓強為多少？(2 分)
（B）卡士米爾在 50 年代提出，真空中其實充滿了'虛'電磁波。它們所具有的能量會對一對相隔距離為 d 的金屬平板產生作用力。 根據量子力學，兩板間儲有的能量為 $E=\frac{1}{2}\sum_{n<p}E_n$，其中 $E_n=\hbar ck_n$，c 為真空光速，$\hbar$ 為普朗克常數，p (>> 1)是個很大的數，和金屬板的材料性質有關。 $k_n\equiv 2\pi/\lambda_n$，其中 λ_n 是可在兩板之間形成的駐波的波長。

(a) 設電磁波的波動方程具有 $\sin k_n x$ 這樣的形式，在兩板面 $x=0$ 和 $x=d$ 處波必須消失，求 k_n 可取的值。(2 分)
(b) 證明真空電磁力具有 a/d^2 這樣的形式，並求常數 a。(5 分)
(c) 板外邊的真空電磁力為多少? (2 分)
(d) 求當 d = 1.0 mm 時力的數值。 (提示: 如果你沒有在（b）部分求得 a，你可用量綱分析來估計力的大小。) (2 分)

($\hbar=1.05\times10^{-34}$ (焦耳*秒); $p=2000$; $c=3.0\times10^{8}$ m/s .)

Q7 **題 7** Electric Current Driven Magnetic Domain Movement 電流驅動的磁疇運動
(15 points **15 分)**

Electrons and some atoms have permanent angular momentum called **spins** ($\vec{S}$) that are fixed in amplitude while their directions can be changed by an external torque. Associated with the spins are the magnetic moments that make them as individual magnetic dipoles $\vec{m} = \mu_B \vec{S}$, where μ_B is a constant called magneton. For simplicity, we assume that the electrons and the atoms here have the same spin amplitude S and magneton.

(a) Refer to Figure-A below. Dipole-A is at $\vec{r} = 0$ and along $\vec{z}_0$, and Dipole-B is at $\vec{r} = a\vec{x}_0$ and along $\vec{y}_0$. Find the torque of Dipole-A on Dipole-B. (3 points)

(b) Find the direction of Dipole-B after a short time Δt, assuming that the direction of Dipole-A remains unchanged. (5 points)

(c) Figure-B is a schematic of a magnetic domain in (x_0, $x_0 + 2b$) that contains many atoms (Dipole-B's). The directions of the dipoles are pointed in an ordered fashion, as shown. Dipole-B's are at the fixed sites of equal distance along the x--axis (within the space of $2b$ there are 10^4 dipole-B's). An electron (Dipole-A) flies by and its action on the Dipole-B <u>is the same as your answer in (b)</u>. Figure-C shows the initial angular profile of the dipoles. The dipoles at $x_0 + b$ are along $\vec{y}_0$, the dipoles at $x \le x_0$ are along $-\vec{x}_0$, and that at $x \ge x_0 + 2b$ are along $\vec{x}_0$. Dipoles along $-\vec{x}_0$ can no longer rotate. Dipoles along $\vec{x}_0$ cannot rotate either, except for the ones close to the domain edge which can rotate freely as the ones pointing in other directions. Given the electric current carried by the electrons as I, find the speed of the movement of the domain center when the current is just turned on. (Take the electron charge e as known) (5 points)

(d) Find the time it takes for the whole domain to pass the $x_0 + 2b$ point. (2 points)

(Hint: Magnetic field $\vec{B}(\vec{r})$ at position $\vec{r}$ due to a magnetic dipole $\vec{m}$ at coordinate origin is given by $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi r^3}\left[\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m}\right]$. Torque $\vec{N}$ of magnetic field $\vec{B}$ on a dipole $\vec{m}$ is $\vec{N} = \vec{m} \times \vec{B}$.)

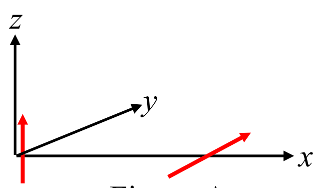


Figure-A

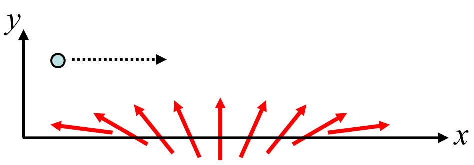


Figure-B

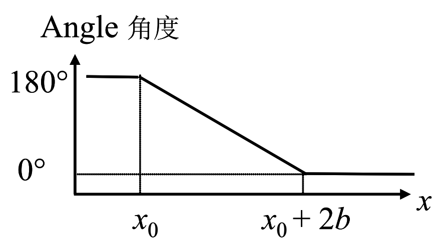


Figure-C

電子和某些原子具有叫做**自旋**的固定角動量 $\vec{S}$，其大小是不變的，但方向可由外力矩改變。 由自旋產生的磁矩為磁偶極子 $\vec{m} = \mu_B \vec{S}$，其中 μ_B 為常數。為簡單起見，我們假設電子和原子的自旋和磁偶極子都是一樣的。

(a) 參照上圖-A。磁偶極子-A 在 $\vec{r} = 0$ 處，沿方向 $\vec{z}_0$，磁偶極子-B 在 $\vec{r} = a\vec{x}_0$，沿方向 $\vec{y}_0$。求磁偶極子-A 對磁偶極子-B 的力矩。(3 分)

(b) 假設磁偶極子-A 的方向固定，求過了一短時間 Δt 後，磁偶極子-B 的方向。(5 分)

(c) 上圖-B 描述的是一個在(x_0, $x_0 + 2b$)空間裏的含有很多原子(磁偶極子-B)的磁疇。磁偶極子-B 占滿了 x--軸上的固定等距位置，它們的方向以一定的規律排列。(在 $2b$ 長度內可含有 10^4 個磁偶極子-B)。一電子(磁偶極子-A)飛過，<u>它對每個磁偶極子-B 的作用和(b)部分的答案相同</u>。上圖-C 為開始時磁偶極子方向分佈。在 $x_0 + b$ 的磁偶極子沿 $\vec{y}_0$ 方向，在 $x \ge x_0 + 2b$ 的磁偶極子沿 $\vec{x}_0$ 方向，在 $x \le x_0$ 的磁偶極子沿 $-\vec{x}_0$ 方向。沿 $-\vec{x}_0$ 的磁偶極子不能再轉動，沿 $\vec{x}_0$ 的磁偶極子除了靠近磁疇邊緣的那些以外也不

能轉動。指向其他方向的磁偶極子和磁疇外面一點點的沿 $\vec{x}_0$ 的磁偶極子可自由轉動。給定電子的電流為 I， 求電流剛開始時磁疇中心的速度。(當電子電荷 e 為已知) (5 分)

(d) 求整個磁疇通過 x_0+2b 點所需的時間。(2 分)

(提示: 在 $\vec{r}$ 處由位於座標原點的磁偶極子 $\vec{m}$ 產生的磁場為 $\vec{B}(\vec{r})=\dfrac{\mu_0}{4\pi r^3}\left[\dfrac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2}-\vec{m}\right]$。在磁場 $\vec{B}$ 裏磁偶極子 $\vec{m}$ 受的力矩 $\vec{N}$ 為 $\vec{N}=\vec{m}\times\vec{B}$ 。)

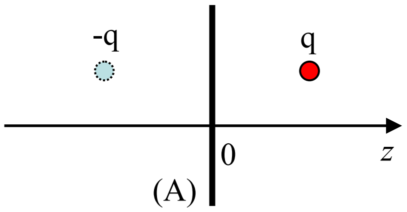
Q8 Force between a conductor sphere and a conductor plane 導體球與導體板之間的力
(22 points **22 分)**

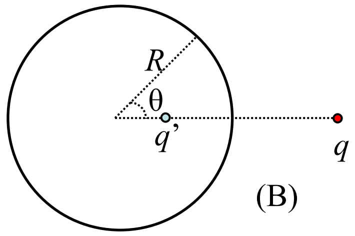
The uniqueness theorem of the theory of electric field ensures that virtual charges could be placed ***outside*** the closed space where the electric field is to be determined to mimic the real boundary conditions. For example, in Figure-A below, to find the electric field when a point charge q is placed in front of a large planar conductor at zero potential, a virtual, or 'image' charge $-q$ is placed at the mirror-image position so that the combined contribution of the two charges is to make the potential everywhere on the plane zero. The force on q by the plane is then the same as the force by the image charge.

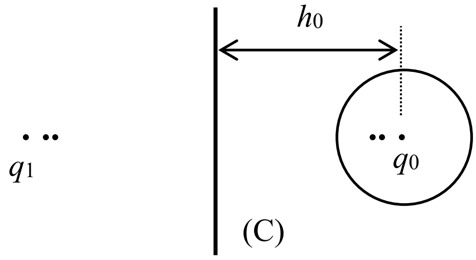
Part-A
Consider the force of a conductor sphere of radius R held at zero potential on a point charge q at a distance d (> R) from the sphere center. The image charge q' should be inside the sphere (outside the space outside the sphere) and on the line joining the sphere center and q, as shown in Figure-B. The combined contribution of q and q' is to make the potential everywhere on the sphere surface zero. Find the position and the value of q'. (4 points)

Part-B
As shown in Figure-C, a problem often encountered in atomic force microscopy is to determine the force of the sample on the probe tip, which is a conductor sphere of radius R held at electric potential V at distance h_0 from a large conducting sample surface at zero potential. In the following we will apply the 'image charge' method step by step to find the force.

(a) We start by putting a point charge q_0 inside the sphere such that the sphere surface is an equal-potential one with voltage V, while ignoring the effect of the conductor plate. Determine q_0 in terms of V and R. (1 point)
(b) Determine the value and position (h_1) of the image charge of q_0 and call it q_1, such that the combined contribution of them is to make the sample surface a zero-potential plane. (2 points)
(c) The presence of q_1 now makes the conductor sphere surface no longer equal-potential. Put another point charge q_2 inside the sphere such that the combined effect of q_0, q_1 and q_2 is to make the potential on the sphere surface equal to V again. Determine q_2 and its position h_2. (2 points)
(d) Repeat (b) to determine the image charge q_3 of q_2, and repeat (c) to find the image charge q_4 of q_3. Derive the general expression between h_{2n} and $h_{2(n+1)}$, q_{2n} and $q_{2(n+1)}$, and q_{2n+1} and $q_{2(n+1)+1}$, $n = 0, 1, 2...$ (3 points)
(e) Derive the total force of the sample on the sphere ***<u>in the form of a summation over an infinite series</u>***. (2 points)
(f) Suppose the force in (e) is 1.1×10^{-12} N when $V=V_0$, $R=1.0\times10^{-8}$ m, and $h_0=5.0\times10^{-8}$ m, find the force when $V=2V_0$, $R=1.0$ m, and $h_0=5.0$ m. (4 points)
(g) Given $R/h_0 = 1/51$, to determine the force in (e) up to ~1% accuracy, what are the terms that should be kept? (2 points)
(h) Estimate the order of magnitude of the relative error caused by the terms neglected in (g). (2 points)







利用電場理論的唯一性定理，我們可在一閉合空間<u>外</u>用一虛擬電荷分佈來產生和原來相同的電勢邊界條件，以便求出閉合空間<u>內</u>的電勢/電場。如圖-A 所示，一點電荷 q 放在零電勢的大導電平面前，為求電場，可以在平面左邊放一‘鏡像’電荷-q，兩電荷的合電勢使平面的電勢仍然為零。平面對 q 的力等於-q 對 q 的力。

（A）

我們先考慮一半徑為 R 的零電勢導體球對放在離球心距離為 $d\ (> R)$ 的點電荷 q 的力的問題。如圖-B 所示，q 的鏡像電荷 q' 在球裏面（球以外空間的外面）球心與 q 的連線上。q 和 q' 的合電勢使球面上處處電勢為零。求 q' 的值和位置。（4 分）

（B）

如圖-C 所示，原子力顯微鏡經常遇到的問題是要確定樣品對掃描探頭的力。掃描探頭為一半徑為 R 電勢為 V 的導體球，離導電樣品表面距離為 h_0。樣品電勢為零。下面我們利用鏡像電荷方法一步步地把力最終求出來。

(a) 首先，當樣品不在，放一點電荷 q_0 在球心。求使球面的電勢為 V 時 q_0 的值。(1 分)

(b) q_0 和它的鏡像電荷 q_1 必須使樣品的表面處處電勢為零。求 q_1 的值和位置 h_1。(2 分)

(c) q_1 的出現使導體球面不再是等勢面。在球裏放上點電荷 q_2，使 q_0, q_1 和 q_2 的合電勢場在球面處處為 V。求 q_2 的值和位置 h_2。 (2 分)

(d) 重複（b）求 q_2 的鏡像電荷 q_3，重複（c）求 q_3 的鏡像電荷 q_4。導出 h_{2n} 與 $h_{2(n+1)}$, q_{2n} 與 $q_{2(n+1)}$，以及 q_{2n+1} 與 $q_{2(n+1)+1}$ 之間的一般關係式。 $n = 0, 1, 2...$ (3 分)

(e) 求樣品對球的力。答案可<u>以無窮級數相加形式表達</u>。(2 分)

(f) 已知當 $V = V_0$, $R = 1.0 \times 10^{-8}$ m, $h_0 = 5.0 \times 10^{-8}$ m 時，樣品對球的力為 1.1×10^{-12} N, 求當 $V = 2V_0$, $R = 1.0$ m, and $h_0 = 5.0$ m 時的力。(4 分)

(g) 已知 $R/h_0 = 1/51$，如要求用(e)的答案來計算力時可達到大約 1%的精確度，需要保留哪幾項？(2 分)

(h) 估計(g)中因忽略其餘項所引起的相對誤差的數量級。(2 分)

END 完

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Note to graders:

- If the final answer is correct and there are sufficient steps to show the process, give full marks.
- If the final answer is wrong, find the step(s) where mistakes are made and deduct points there only. No repeat deduction of points for wrong answers being used for further steps. This is the principle of error non-progressiveness which we follow here.

Q1 (8 points)

Kinetic energy of meson $E_k = \dfrac{4m_1 m_2^{\ 2}}{(m_2+m_1)^2(m_3+m_2)}E$.

Internal energy of meson $E_i = \dfrac{4m_1 m_2 m_3}{(m_2+m_1)^2(m_3+m_2)}E$

Suggested steps:

(1) Use energy and momentum conservation to obtain the speed of m_2 after collision with m_1.

$v_2 = \dfrac{2m_1 m_2}{(m_1+m_2)}E$ (2 points)

The problem then becomes a pair of weights connected by a spring, with one mass suddenly acquires velocity.

(2) Find the center-of-mass speed V_c of m_2-m_3 system (m_3 is still at rest right after collision). The kinetic energy E_k of the m_2+m_3 system is then $(m_2+m_3)/2$ times the square of V_c. (3 points)

(3) The 'internal energy' E_i can then be found using energy conservation, counting both the meson and the electron after collision, or just counting the energy m_2 has after collision. (3 points)

Q2 (8 points) When the moment of inertia of the pulley is the largest, which means putting mass along the edge. Then use energy conservation that the initial potential energy is equal to the kinetic energy of mass m_3 and that of the pulley. (2 points)

Moment of inertia $I = \dfrac{1}{2}M_1R^2 + M_2R^2$ (2 points)

$v = R\omega$ (1 points)

$M_3gh = \dfrac{1}{2}I\omega^2 + \dfrac{1}{2}M_3v^2$ (2 points)

$v = \sqrt{\dfrac{4M_3gh}{M_1+2M_2+2M_3}}$ (1 points)

Q3 (10 points)

(a) Pressure is proportional to depth, and the total force is found by integrating the entire depth. The force acting on the strip is

$P = \rho gh$ (1 point)

$F = \int PdA = d\int_0^H \rho ghdh = \dfrac{1}{2}\rho gdH^2$ (2 points)

(b)

Step-1 Find the shape of the liquid surface (2 points)

Method-1: In the rotating reference frame the inertia force is $\omega^2 r$, and the associated potential is $U = \frac{1}{2}\omega^2 r^2$. The liquid surface is an equal-potential surface, and the gravitation potential is gh. So the shape is a rotating parabola determined by $gh + \frac{1}{2}\omega^2 r^2 =$ constant.

Method-2: The slope of the surface $\frac{dh}{dr}$ should be such that normal force should balance the gravity in the vertical direction, and give the concentric force for $\omega^2 r$ in the horizontal direction. That leads to the same answer as Method-1.

Method-3

$F_p = pA \Rightarrow \Delta W_p = -P\Delta V$ where $\Delta V = 2\pi r \Delta r \Delta h$

$$F_c = m\omega^2 r \Rightarrow \Delta W_c = m\omega^2 r \Delta r \qquad \text{where } m = 2\pi\rho h r \Delta r$$

$$\Delta W_t = \sum_i \Delta W_i = 0 \qquad (\ )$$

$$\Rightarrow (2\pi\rho h r \Delta r)\omega^2 r \Delta r = \rho g h (2\pi r \Delta r \Delta h)$$

$$\Rightarrow \frac{dh}{dr} \approx \frac{\Delta h}{\Delta r} = \frac{\omega^2}{g} r$$

$$\Rightarrow h = \frac{\omega^2}{2g} r^2 + C \qquad \text{where } C \text{ is a constant. } (\ )$$

Step-2 Determine C (2 points)

Consider the Volume during rotation,

$$2\pi \int_0^R h r dr = \pi R^2 H$$

$$2\pi \int_0^R \left(\frac{\omega^2}{2g} r^2 + C\right) r dr = \pi R^2 H$$

$$\Rightarrow C = H - \frac{\omega^2 R^2}{4g}$$

$$\left[h = \frac{\omega^2}{2g}\left(r^2 - \frac{R^2}{2}\right) + H \right] \qquad (\ )$$

For $r = R$, $h = \frac{\omega^2 R^2}{4g} + H$ (1 point)

$$F' = \frac{1}{2}\rho g d \left(\frac{\omega^2 R^2}{4g} + H\right)^2 \qquad \text{(1 point)}$$

The Amount of Extra Force is

$$\Delta F = F' - F = \frac{1}{2}\rho g d \left[\frac{\omega^2 R^2}{2g} H + \left(\frac{\omega^2 R^2}{4g}\right)^2 \right] \qquad \text{(1 point)}$$

Q4 (12 points)

a) Use the ideal gas law,

$$P = R\frac{(n_L T_L + n_R T_R)}{V}, \quad (V = V_L + V_R); \quad V_L = \frac{n_L T_L}{(n_L T_L + n_R T_R)} V \qquad \text{(2 points)}$$

b) Energy conservation $c \equiv n_L T_L(0) + n_R T_R(0) = n_L T_L(t) + n_R T_R(t)$ (1) (1 point)

and $0 = n_L dT_L + n_R dT_R$ (1a)

Equal pressure leads to $n_L T_L(t) V_R(t) = n_R T_R(t) V_L(t)$ (2) (1 point)

and $\frac{dT_L}{T_L(t)} + \frac{dV_R}{V_R(t)} = \frac{dT_R}{T_R(t)} + \frac{dV_L}{V_L(t)}$ (2a)

Heat transfer plus work done due to expansion

$$k(T_L(t) - T_R(t))dt = dQ = -\frac{3}{2}RdT_L - P(t)dV_L = -\frac{3}{2}RdT_L - n_L RT_L(t)\frac{dV_L}{V_L(t)} \quad (3)\ (1\ \text{point})$$

Finally we have $V = V_L(t) + V_R(t)$ (4)

From Eqs. (1), (2) and (4) we get $n_L RT_L(t)\frac{dV_L}{V_L(t)} = n_L RdT_L$ (5) (1 point)

Also from Eq. (1a) $d(T_L(t) - T_R(t)) = (1 + \frac{n_R}{n_L})dT_L$ (6) (1 point)

Put Eqs. (5) and (6) into Eq. (3) $\frac{d(T_L - T_R)}{dt} = -\frac{2k}{5R}\left(\frac{1}{n_L} + \frac{1}{n_R}\right)(T_L - T_R) = -\beta(T_L - T_R)$ (2 points)

Where $\beta \equiv \frac{2k}{5R}\left(\frac{1}{n_L} + \frac{1}{n_R}\right)$, and $(T_L(t) - T_R(t)) = (T_L(0) - T_R(0))e^{-\beta t}$. (2 points)

c) Find out how V_L and V_R changes with time.

The conservation of energy supplies another equation

Using Eq. (1)

$$T_R(t) = \frac{1}{n_R + n_L}\left(n_L T_L(0)(1 - e^{-\kappa t}) + T_R(0)(n_R + n_L e^{-\kappa t})\right),$$

$$T_L(t) = \frac{1}{n_R + n_L}\left(n_R T_R(0)(1 - e^{-\kappa t}) + T_L(0)(n_L + n_R e^{-\kappa t})\right)$$

(1 point)

and $V_{L(R)} = \frac{n_{L(R)} T_{L(R)}(t)}{(n_L T_L(t) + n_R T_R(t))} V$. (1 point)

Q5 (12 points)

(a) The E-fields in medium-1 and -2 are $\vec{E}_1 = \vec{E}_I + \vec{E}_R$, $\vec{E}_2 = \vec{E}_T$ (1 point)

$\omega\vec{B} = \vec{k} \times \vec{E}$, with $\vec{k} = k\vec{y}_0$ (1 point)

$\vec{B}_1 = \frac{k}{\omega}(E_0 - E_r)\vec{z}_0 e^{i(k_1 y - \omega t)}$ (1 point)

$\vec{B}_2 = \frac{k}{\omega}E_t \vec{z}_0 e^{i(k_2 y - \omega t)}$ (1 point)

Using the boundary conditions $\vec{E}_1^{//} = \vec{E}_2^{//}$, $\vec{B}_1^{//} = \vec{B}_2^{//}$ at $y = 0$, (1 point)

Dispersion relation $k = \frac{n\omega}{c}$, (1 point)

one gets $E_0 - E_R = E_T$, and $E_0 n_1 - E_R n_1 = E_T n_2$, (1 point)

Solving the equations, one obtains $r = \dfrac{n_1 - n_2}{n_2 + n_1}$ $\quad t = \dfrac{2n_1}{n_2 + n_1}$ (1 point).

(b) $R = 1$ (1 point)

(c) Use (a) and find the phase of r with the given n_1 and n_2. $r = \dfrac{1+i}{2}$ (2 point),

Phase shift = 45° (1 point).

Q6 (13 points)

Part-A $PV=nRT$ so the pressure goes to zero. (2 points)

Part-B

(a) $k_n d = 2\pi n$, where n are integers. (2 points)

(b) $E = \dfrac{\hbar c}{2} \sum_{n<p_c} k_n = \dfrac{\pi \hbar c}{d} \sum_{n<p_c} n = \dfrac{\pi \hbar c}{2d} n(n-1) \cong \dfrac{\pi \hbar c}{2d} n^2$. (2 points)

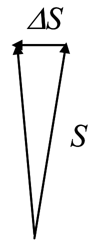
The force is given by $F = -\dfrac{\partial E}{\partial d} = \dfrac{\pi \hbar c}{2d^2} n^2$. (2 points)

The system energy decreases with increasing d so the force is pushing outwards. (1 point)

(c) Outside we have $d = \infty$ (1 point), so $F = 0$ (1 point)

(d) $F = \dfrac{(1.05\times10^{-34} Js)(3\times10^{8} ms^{-1})\pi[2000(2000-1)]}{4} \dfrac{1}{(1\times10^{-3} m)^2} = 9.89\times10^{-14} Nm^2$. (2 points)

Q7 (15 points)



(a) The torque is $\vec{\tau} = \dfrac{\mu_0}{4\pi a^3} \vec{m}_A \times \vec{m}_B = -\dfrac{\mu_0 (\mu_B S)^2}{4\pi a^3} \vec{x}_0$ which is perpendicular to $\vec{m}_B$ (3 points)

(b) The torque causes $\vec{S}_B$ to spin within the x-y plane, (2 points)

and the angular speed is given by $\tau = S\omega$. (2 point)

The angle of the y-axis is $\Delta\theta = \omega \Delta t = \dfrac{\mu_0}{4\pi a^3} \mu_b^2 S \Delta t$ (1 point)

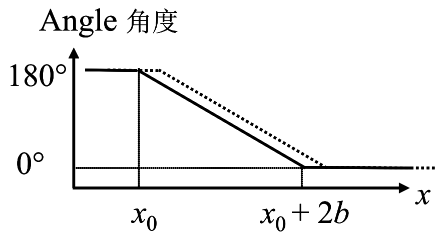


Figure-C

(c) Find how many electrons should pass to cause a particular spin to rotation from 90° - $d\theta$ to 90°and from θ find position of the spin. $(\dfrac{I}{e}\Delta\theta)dt = d\theta = \dfrac{\pi}{2b} dx$ (3 points)

$v = \dfrac{dx}{dt} = \dfrac{2bI}{\pi e} \Delta\theta = \dfrac{\mu_0}{2e\pi^2 a^3} bI\mu_b^2 S \Delta t$ (2 points)

(d) $t = 2b / v = \dfrac{4e\pi^2 a^3}{\mu_0 I \mu_b^2 S \Delta t}$. (2 points)

Q8 (22 points)

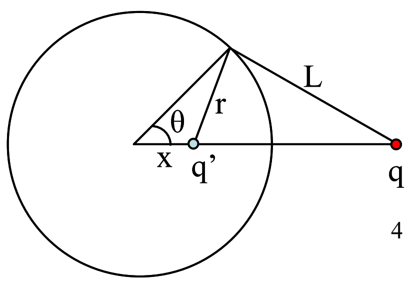
Part-A

Place a point charge q' at distance x from the center, which is at a distance r from a point on the sphere surface. The distance of this point to charge q is L.

Then $L^2 = R^2 + d^2 - 2Rd\cos\theta$ (1)

$r^2 = R^2 + x^2 - 2Rx\cos\theta$ (2)

(1 point)



Zero potential on sphere surface => $\frac{q}{L} = -\frac{q'}{r}$ (3) (1 point)

Putting Eq. (1) and (2) into (3) leads to
$q'^2 (R^2 + d^2 - 2Rd\cos\theta) = q^2 (R^2 + x^2 - 2Rx\cos\theta)$ (4)
Equation (4) must be true for all angle θ, so
$q'^2 (R^2 + d^2) = q^2 (R^2 + x^2)$ (5) and
$q'^2 Rd\cos\theta = q^2 Rx\cos\theta$ (6)
Solving Eq. (5) and (6), we get two sets of solutions.
Solution-1: $x = d$ and $q' = -q$. Not the right one because q' ends up outside the sphere.
Solution-2: $q' = -qR/d$, and $x = R^2/d < R$. Correct. (2 points)

Part-B
(a) $q_0 = 4\pi\varepsilon_0 RV$ (1 point)
(b) $-q_0$ and at a distance h_0 on the other side of the plane. (2 points)
(c) The contribution of q_1 and q_2 is to make the sphere surface zero potential. The answer in Part-A can be used here. $h_2 = h_0 - \frac{R^2}{h_0 + h_0}$, $q_2 = \frac{R}{2h_0} q_0$ (2 points)

(d) $h_{2(n+1)} = h_0 - \frac{R^2}{h_0 + h_{2n}}$, (1 point) $q_{2(n+1)} = \frac{R}{h_0 + h_{2n}} q_{2n}$, (1 point) $q_{2n+1} = -q_{2n}$ (1 point)

(e) Sum over all charges on both sides of the plane. $F = \frac{-1}{4\pi\varepsilon_0} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{q_{2m} q_{2n}}{(h_{2m} + h_{2n})^2}$ (2 points)

(f) Using (a) and (d), define $k_0 \equiv h_0 / R$,

$$k_{2(n+1)} \equiv \frac{h_{2(n+1)}}{R} = k_0 - \frac{1}{k_0 + k_{2n}}, \text{ (1 point)}$$

$$q_{2n} = \frac{1}{k_0 + k_{2n-2}} q_{2(n-1)} = \frac{1}{k_0 + k_{2(n-1)}} \frac{1}{k_0 + k_{2(n-2)}} \frac{1}{k_0 + k_{2(n-3)}} \ldots q_0 \quad \text{(1 point)}$$

Using (e) $F = -4\pi\varepsilon_0 V^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(k_0 + k_{2(n+1)})} \frac{1}{(k_0 + k_{2(m+1)})} \frac{1}{(k_{2m} + k_{2n})^2} \equiv -4\pi\varepsilon_0 V^2 g(k_0)$. (1 point)

The force is proportional to V^2, and depend only on the ratio of $k_0 = R/h_0$. So the answer is 4.4 x 10^{-12} N. (1 point)
(g) Since all the image charges above the surface must be inside the spheres, the distance between any charge outside the sample and those inside should be large than $2(h_0 - R)$. Then $h_0 + h_{2n-2} > 2(h_0 - R)$. So $|q_{2n}| = \frac{R}{h_0 + h_{2n-2}} |q_{2n-2}| < \frac{R}{2(h_0 - R)} |q_{2n-2}| = \frac{1}{100} |q_{2n-2}|$, and only the $n = 1$ terms should be kept. (1 point)
The total force is the sum of q_0q_1, q_0q_3, and q_2q_1, or (01), (03), and (12) for short. $F = -\frac{\pi\varepsilon_0 V^2}{51^2}(1 + \frac{1}{51})$. (1 point)

(h) The terms in the expansion that are of the order of $\left(\frac{R}{2(h_0 - R)}\right)^2 = (10^{-2})^2$ are (05), (23), (41),

$$|F_{error}| = 4\pi\varepsilon_0 V^2 \left[\frac{10^{-4}}{4} k_2^2 + \frac{10^{-4}}{4(1/k_0 + 1/k_4)^2} + \frac{10^{-4}}{4(1/k_1 + 1/k_5)^2} \right] < 3\pi\varepsilon_0 V^2 10^{-4} \frac{1}{(50)^2}$$

So the relative error is at most 3×10^{-4}. (2 points)

What are left in (g) in the force calculation is

$$|F_{error}| < 4\pi\varepsilon_0 V^2 \left[\frac{10^{-4}}{4h_2^2} + \frac{10^{-4}}{4(h_0+h_4)^2} + \frac{10^{-4}}{4(h_1+h_5)^2} + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{1}{(h_{2m}+h_{2n})^2} \left(\frac{R}{2(h_0-R)}\right)^{m+n} \right]$$

$$< 4\pi\varepsilon_0 V^2 10^{-4} \left[\frac{3}{4(h_0-R)^2} + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{1}{(2h_0-2R)^2} \left(\frac{R}{2(h_0-R)}\right)^{m+n} \right]$$

$$< 4\pi\varepsilon_0 V^2 10^{-4} \frac{1}{4(h_0-R)^2} \left(3 + \left(\sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n \right)^2 \cong 3\pi\varepsilon_0 V^2 10^{-4} \frac{1}{(h_0-R)^2}\right.$$

So the relative error is at most 3×10^{-4}.

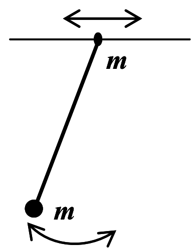
Pan Pearl River Delta Physics Olympiad 2007
2007 年泛珠三角及中華名校物理奧林匹克邀請賽
Part-1 (Total 7 Problems)　卷-1（共 7 題）
(9:30 am – 12:30 pm, 02-26-2007)

Q.1 (3 points) 題 1（3 分）

An airplane is initially rising up at speed v_0 at an angle θ to the horizon. Find the trajectory of the plane such that weightless condition can be achieved in the plane.

一架飛機以與水平面成θ角的初速度 v_0 上升。求飛機以什麼樣的軌跡飛行，能使飛機裏的物體處於失重狀態。

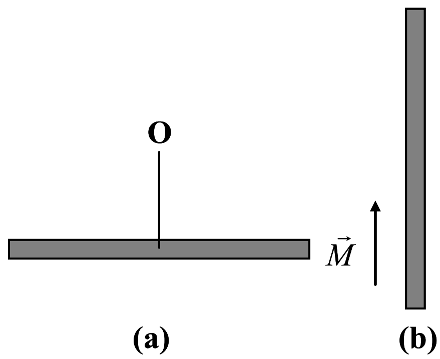
Q.2 (6 points) 題 2（6 分）



As shown, two identical weights are fixed on the two ends of a uniform rigid rod of length L. The upper weight is restricted to move on a smooth horizontal rail and the rod is free to swing along the rail. The masses of the weights and the rod are equal. Find the small angle vibration frequency of the system.

如圖所示，兩個質量為 m 的重塊分別固定在一根長度為 L 質量為 m 的均勻杆兩端。上面的重塊可以沿光滑的水平軌道滑行，杆可沿軌道方向自由擺動。求整個系統的小角度振動頻率。

Q.3 (6 points) 題 3（6 分）



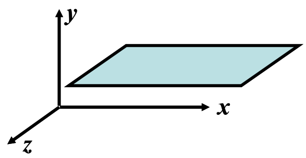
(a) A disc shaped medium block of radius R and thickness d ($<< R$) is uniformly magnetized with magnetization $\vec{M}$ perpendicular to the disc plane. Find the magnetic field at point-O on the central axis of the disk and at a distance h from the cavity center.

一半徑為 R，厚度為 d ($<< R$)的圓盤形均勻磁化介質，磁化強度為$\vec{M}$。盤的表面垂直於$\vec{M}$。求圓盤中心軸上到圓盤中心距離為 h 的點 O 的磁場。

(b) A long and thin cylindrical medium is uniformly magnetized with magnetization $\vec{M}$ along the cylinder long axis. Find the magnetic field inside and outside the medium.

一細長圓柱型介質沿柱軸方向均勻磁化，磁化強度為$\vec{M}$。求介質裏、外的磁場。

Q.4 (5 points) 題 4（5 分）



A large flat dielectric slab of thickness d and dielectric constant ε is moving along the x-direction at speed v. Its large surface plane is perpendicular to the y-axis. A magnetic field of strength B is applied along the z-direction. Find the surface bound charge density on the two large surfaces of the slab, and the electric field in the slab.

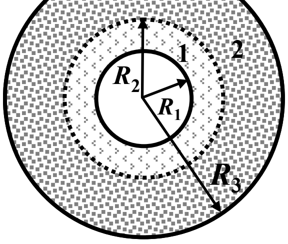
一個厚度為 d，介電常數為ε的大平板以速度 v 沿 X-方向運動。它的表面與 Y-軸垂直。Z-方向加有磁場 B。求平板兩表面上的束縛電荷密度，以及平板中的電場。

Q.5 (10 points) 題 5（10 分）

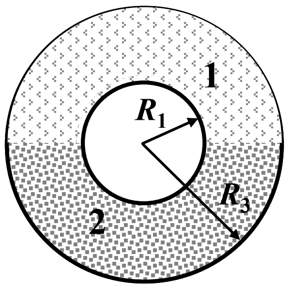
The space between two concentric conductor spherical shells of radii R_1 and R_3 is filled with two types of media. The dielectric constant and the conductivity of medium-1 and medium-2 are ε_1, σ_1 and ε_2, σ_2, respectively. The voltage difference between the two shells is V_0.

(a) In case-A, the media form two concentric shells with the conductor shells, and the radius of the boundary between the two media is R_2. Find the following: (i) total current from the inner shell to the outer shell; (ii) total free charge on the two conductor shells and on the boundary between the two media.

(b) In case-B, medium-1 fills the upper hemisphere and medium-2 fills the other half. Find the following: (i) total current from the inner shell to the outer shell; (ii) total free charge on the upper and lower halves of the two conductor shells.



Case-A



Case-B

如圖所示，兩個半徑分別為 R_1 和 R_3 的同心導電球殼之間充滿了兩種介質。球殼之間電勢差為 V_0。介質 1 和 2 的介電常數和導電率分別為 ε_1, σ_1 和 ε_2, σ_2。

(a) 兩種介質為與導電球殼同心的球殼，其界面為半徑為 R_2 的球面。(i) 求兩導電球殼間的總電流；(ii) 求兩導電球殼以及兩介質之間界面上的電荷。

(b) 介質-1 填充上半部分，介質-2 填充下半部分。(i) 求兩導電球殼間的總電流；(ii) 求兩導電球殼上、下部分的電荷。

Q.6 (12 points) 題 6（12 分）

(a) Assume that atmosphere is made of diatom ideal gas in adiabatic equilibrium. Determine air pressure P, temperature T and density ρ as a function of altitude h, provided that their values at $h = 0$ are known. (Hint: Set up a differential equation for a thin layer of air at some altitude. $\int x^{\alpha}dx = \frac{1}{\alpha+1}x^{\alpha+1}$, where $\alpha \neq 1$ is a constant.) (6 points)

(a) 大氣可看成絕熱平衡下的雙原子理想氣體。求空氣壓強 P、溫度 T 和密度ρ作為高度 h 的函數，假定它們在 $h = 0$ 處的值為已知。（提示：對某高度的一薄層氣體建立微分方程。$\int x^{\alpha}dx = \frac{1}{\alpha+1}x^{\alpha+1}$ ，$\alpha \neq -1$ ）(6 分)

(b) When the partial pressure of water vapor in air exceeds the saturated water vapor pressure (P_s) at a given temperature, the water vapor will condense into droplets which fall down as rain. P_s = 55.35 mmHg at 40°C, and P_s = 6.50 mmHg at 5 °C. The air/vapor mixture can be considered as diatom ideal gas and the mass of a water molecule is approximately the same as an 'air' molecule. In the humid air at sea level at 40°C the water vapor partial pressure is 90 % of P_s. The density of air is ρ_0 = 1.18 kg m^{-3} at 20 °C and 1.0 atm. The humid air then rises adiabatically to an altitude where the temperature is 5 °C. Ignore air pressure change due to the reduction of water vapor.

(b1) How much rain can one cubic meter of the humid air at sea level generate? (5 points)
(b2) Use the results in (a), find the altitude where the temperature is 5 °C. (1 point)

(b) 當空氣中水蒸汽的分壓強超過該溫度下的飽和水蒸汽壓(P_s)時，水蒸汽將凝聚成滴導致下雨。已知 40°C 時 P_s = 55.35 mmHg，5 °C 時 P_s = 6.50 mmHg。空氣/水蒸汽的混合物可當作是雙原子理想氣體，水分子的質量近似等於‘空氣’分子的質量。40°C 時海平面上的潮濕空氣中，水蒸汽分壓是 P_s的 90 %。已知 20 °C 時，1 個大氣壓下的空氣密度 ρ_0 = 1.18 kg m^{-3}。忽略由於水蒸汽的減少導致的氣壓改變。該潮濕空氣絕熱上升到某一高度，該處溫度為 5 °C。

（b1）一立方米海平面上的潮濕空氣能夠產生多少雨？(5 分)
（b2）用（a）的結果，求溫度為 5 °C 處的高度。(1 分)

Q7 (8 points) 題 7（8 分）

(i) Find the torque on an electric dipole $\vec{p}$ in a uniform electric field $\vec{E}$. (1 point)

(ii) A medium is uniformed polarized with polarization $\vec{P}$ by an electric field $\vec{E}$. Find the torque per volume on the medium exerted by the electric field. (1 point)

(iii) An electromagnetic wave $\vec{E} = E_0(\vec{x}_0 + \vec{y}_0)e^{i(kz-\omega t)}$ is propagating along the z-axis in an isotropic medium. In such medium the relation between the electric displacement $\vec{D}$ and $\vec{E}$ is given by $\vec{D} = \varepsilon_0\varepsilon\vec{E}$, so $\vec{D}$ and $\vec{E}$ are always pointing in the same direction. Find the torque per volume on the medium exerted by the electromagnetic wave. (1 point)

(iv) An electromagnetic wave $\vec{E} = E_0(\vec{x}_0 e^{ik_1 z} + \vec{y}_0 e^{ik_2 z})e^{-i\omega t}$ is propagating along the z-axis in an anisotropic medium. In such medium the electric displacement is $\vec{D} = \varepsilon_0 E_0(\varepsilon_x \vec{x}_0 e^{ik_1 z} + \varepsilon_y \vec{y}_0 e^{ik_2 z})e^{-i\omega t}$, so $\vec{D}$ is not parallel to $\vec{E}$. Note that $k_1 = \frac{\omega}{c}\sqrt{\varepsilon_x}$ and $k_2 = \frac{\omega}{c}\sqrt{\varepsilon_y}$, where c is the speed of light in vacuum. Find the time-averaged (over one period) torque per volume on the medium exerted by the electromagnetic wave. (3 points)

(v) Following (iv), find the time-averaged total torque on a section of cylindrical shaped medium of unit cross section area with its long axis along the z-direction from $z = 0$ to $z = d$, and the smallest value of d at which the total torque is maximum. (2 points)

(i) 求一個電偶極子 $\vec{p}$ 在電場 $\vec{E}$ 中受到的力矩。(1 分)

(ii) 某介質在電場 $\vec{E}$ 中均勻極化，極化強度為 $\vec{P}$。求單位體積介質在該電場中受到的力矩。(1 分)

(iii) 在一各向同性的介質中，電磁波 $\vec{E} = E_0(\vec{x}_0 + \vec{y}_0)e^{i(kz-\omega t)}$ 沿 z-軸傳播。在該介質中電位移矢量 $\vec{D}$ 和電場 $\vec{E}$ 的關係滿足 $\vec{D} = \varepsilon_0\varepsilon\vec{E}$，因此 $\vec{D}$ 和 $\vec{E}$ 總是保持同一方向。求單位體積介質在該電磁波中受到的力矩。(1 分)

(iv) 在一各向異性的介質中，電磁波 $\vec{E} = E_0(\vec{x}_0 e^{ik_1 z} + \vec{y}_0 e^{ik_2 z})e^{-i\omega t}$ 沿 z-軸傳播。在該介質中電位移矢量為 $\vec{D} = \varepsilon_0 E_0(\varepsilon_x \vec{x}_0 e^{ik_1 z} + \varepsilon_y \vec{y}_0 e^{ik_2 z})e^{-i\omega t}$，因此通常 $\vec{D}$ 和 $\vec{E}$ 不平行。這裏 $k_1 = \frac{\omega}{c}\sqrt{\varepsilon_x}$，$k_2 = \frac{\omega}{c}\sqrt{\varepsilon_y}$，$c$ 是真空中光速。求單位體積介質在該電磁波中受到的一個週期裏的平均力矩。(3 分)

(v) 根據 (iv), 求長軸平行於 z-軸，單位橫截面積的圓柱形介質中 $z = 0$ 到 $z = d$ 部分所受的一個週期的平均力矩，以及使力矩最大所需的 d 的最小值。（2 分）

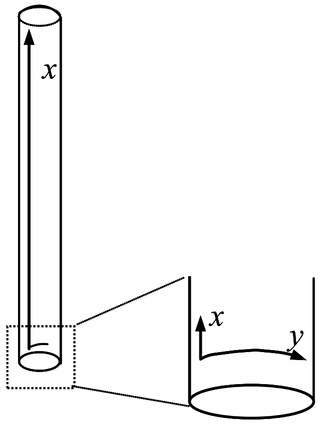
THE END 完

Pan Pearl River Delta Physics Olympiad 2007
2007 年泛珠三角及中華名校物理奧林匹克邀請賽
Part-2 (Total 3 Problems) 卷-2（共 3 題）

(2:30 pm – 5:30 pm, 02-26-2007)

Q1 Folded Space (6 points) 題 1 卷起的空間（6 分）

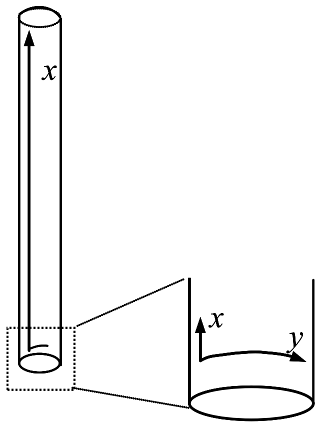
(a) Consider a one-dimensional standing electromagnetic wave in the form of $E(x) = A\sin(k_x x)$ along the x-direction confined within the space between $x = 0$ and $x = a$. The wave must vanish at these two end points. Find the allowed values of k_x. (1 point)



(b) The String Theory predicts that our space is more than three-dimension, and the additional hidden dimensions are folded up like the dimension y on the surface of a thin cylinder shown in the figure. Suppose the radius of the cylinder is b (<< a), and the electromagnetic wave on the surface now takes the form $E(x, y) = A\sin(k_x x)\cos(k_y y)$, where y is the coordinate of the folded space around the cylinder. Find the allowed values of k_y. (3 points)

(c) The photon energy is given by $W = \frac{hc}{2\pi}\sqrt{k_x^2 + k_y^2}$, and hc = 1239 (eV × nanometer), where eV stands for electron volt and 1 nanometer is 10^{-9} meters. The highest energy photons human can make so far is about 1.0×10^{12} eV. If this is sufficient to create a photon in the folded space, what should be the value of b? (2 points)

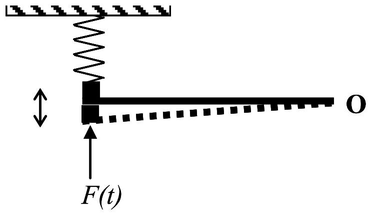
(a) 一維電磁駐波 $E(x) = A\sin(k_x x)$ 在 x-方向限制在 $x = 0$ 和 $x = a$ 之間。 在兩個端點處駐波消失。求 k_x的可能值。(1 分)



(b) 弦理論認為物理空間多於三維，多出的隱藏維空間象細圓柱的表面一樣卷了起來，如圖中 y 坐標所示。設圓柱的半徑為 b (<< a), 在圓柱面上電磁波的形式為 $E(x, y) = A\sin(k_x x)\cos(k_y y)$ ，其中 y 是繞圓柱的折疊空間的坐標。求 k_y的可能值。(3 分)

(c) 光子能量$W = \frac{hc}{2\pi}\sqrt{k_x^2 + k_y^2}$, 其中 hc = 1239 (eV × nm)，eV 表示 1 電子伏特, 1 nm 等於 10^{-9} 米。目前人類能產生的最高能量的光子大約為 1.0×10^{12} eV。如果該能量能夠產生一個折疊空間的光子，b 的值滿足什麼條件？(2 分)

Q2 Atomic Force Microscope (AFM) in thermal noise (22 points)
題 2 熱噪聲下的原子力顯微鏡 (22 分)

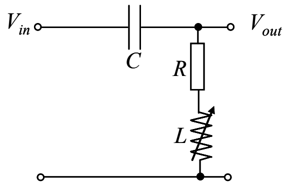


(i) An AFM is modeled as a uniform rigid rod of length l and mass m_1 with a point mass m_2 on one end (the tip), and the other end is fixed at point O around which the rod is free to rotate. A spring of force constant K is attached to the tip. Find the resonant frequency ω_0 of the AFM. (4 points)
原子力顯微鏡能夠簡化為一個長度為 l，質量為 m_1 的均勻硬杆，一端有一個質量為 m_2 的質點 (針尖)，另一端固定在點 O ，杆可繞點 O 自由轉動。一個彈性係數為 K 的彈簧連著針尖。求原子力顯微鏡的共振頻率ω_0。(4 分)

(ii) Given an external driving force $F(t) = F_1\cos(\omega_1 t)$, derive the differential equation for the small vertical displacement $x(t)$ of the tip from its equilibrium position, and solve it using a trial solution $x(t) = A_1\cos(\omega_1 t + \Phi_1)$ where the amplitude A_1 and phase Φ_1 are to be determined. (4 points)
給定一個外驅動力 $F(t) = F_1\cos(\omega_1 t)$, 推導針尖離平衡位置的小位移 $x(t)$的微分方程，並用試探解 $x(t) = A_1\cos(\omega_1 t + \Phi_1)$ 解它，其中振幅 A_1 和位相Φ_1 待定。(4 分)

(iii) Given two driving forces $F(t) = F_1\cos(\omega_1 t) + F_2\cos(\omega_2 t)$, find $x(t)$. (4 points)
給定兩個外驅動力 $F(t) = F_1\cos(\omega_1 t) + F_2\cos(\omega_2 t)$, 求 $x(t)$ 。 (4 分)

(iv) The driving force comes from thermal noise, which can be described as a sum of many harmonic driving forces $F_{thermal}(t) = \sum_n F_n\cos(\omega_n t)$ in the entire frequency range. Find $x(t)$ under the thermal noise driving force. (2 points)
驅動力來自於熱噪聲，它能夠寫成覆蓋所有頻率的許多簡諧驅動力的和 $F_{thermal}(t) = \sum_n F_n\cos(\omega_n t)$ 。求熱驅動力下的 $x(t)$ 。(2 分)



(v) Consider the electronic band pass filter as shown. Given the input voltage $V_{in}(t) = V_0 e^{i\omega t}$, find the value of inductance L such that the **denominator** of the absolute value of the output voltage is minimum. (2 points)
考慮一個如圖所示的電子帶通濾波器。輸入電壓為 $V_{in}(t) = V_0 e^{i\omega t}$ ，求使輸出電壓絕對值**分母最小**的電感 L 的值。(2 分)

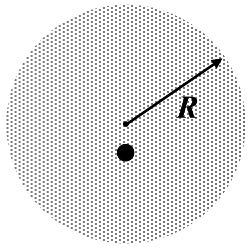
(vi) The AFM signal which is proportional to the solution $x(t)$ in (iv) is applied as the input signal to the filter. Assuming that only the signal with the frequency $\omega_n = \omega$, where ω makes the denominator of the output voltage amplitude minimum in (v), can pass through the filter, draw a sketch of the amplitude of the output voltage vs L if $F_n = 1$ for all n, and describe briefly how the AFM resonant frequency in (i) can be found experimentally. (6 points)
將正比於（iv）中 $x(t)$的原子力顯微鏡信號輸入到電子濾波器。假設僅有頻率 ω_n 等於（v）中使輸出電壓絕對值分母最小的ω 的信號能通過該濾波器，假定對所有 n，$F_n = 1$ ，試畫出輸出電壓的大小隨 L 變化的簡圖， 並簡單描述實驗上如何找到（i）中所述原子力顯微鏡的共振頻率。(6 分)

Q3 The Lorentz-Lorenz Relation (22 points) 題 3 洛倫茲-洛倫茲關係 (22 分)

The dielectric constant $\varepsilon(\omega)$ of a dielectric medium is given by the so called Lorentz-Lorenz Relation $\frac{\varepsilon(\omega)-1}{\varepsilon(\omega)+n}\cdot\varepsilon_0=\frac{1}{3}K(\omega)$, where n is a number and K is a material-related constant that depends explicitly on the frequency ω of the applied electric field. You are to derive the relation through the steps below.

介質的介電常數$\varepsilon(\omega)$滿足所謂的洛倫茲-洛倫茲關係$\frac{\varepsilon(\omega)-1}{\varepsilon(\omega)+n}\cdot\varepsilon_0=\frac{1}{3}K(\omega)$，其中 K 與所加電場的頻率ω以及介質的物質常數有關，n 是一數字。下面逐步推出這一關係。

(i) An atom can be approximately treated as consisting of a uniform spherical electron density (electron cloud) of radius R with total charge $-Ze$ and positive charge nucleus Ze at the center, where e is the charge of a positron. The nucleus mass is much larger than an electron mass m_e. In a uniform external electric field E_0 the electron cloud is displaced slightly from the nucleus while maintaining its spherical shape. Find the displacement. (4 points)



一個原子可以近似看成由總電荷為$-Ze$、均勻分佈成半徑為 R 的球形電子雲，和處於中心的帶電 Ze 的原子核組成，其中 e 是正電子的電荷。 核的質量遠大於電子質量 m_e。在均勻外電場 E_0 中電子雲輕微偏離核，但保持球形。求偏離的位移。(4 分)

(ii) The atom is placed in an oscillating uniform electric field $E(t)=A\cos(\omega t)$. Find the induced dipole moment of the atom. (6 points)

將原子放在震盪的均勻外場 $E(t)=A\cos(\omega t)$ 中， 求原子的電偶極矩。(6 分)

(iii) In a medium the number of atoms per unit volume is N, find the polarization P of the medium. (2 points)

已知介質的單位體積原子數是 N，求介質的極化矢量 P 。(2 分)

(iv) Note that in (iii) the electric field is the external field E_{ext}. Consider a small sphere containing many atoms in the large medium. The total field E_{total} in the sphere consists of two contributions, namely that from the medium inside the sphere E_{self} and the external field E_{ext}. Given that the electric fields are uniform inside the sphere, find the relation between E_{ext} and E_{total}, and determine n and K in the Lorentz-Lorenz Relation. (7 points)

在 (iii) 中的電場是外加電場 E_{ext}。考慮大介質中的一個包含很多原子的小球。球中的總電場 E_{total} 來源於兩部分，一是球中的介質產生的電場 E_{self} ，二是外加電場 E_{ext}。已知球中的電場是均勻的，求 E_{ext} 和 E_{total} 的關係，並求洛倫茲-洛倫茲關係中的 n 和 K。 (7 分)

(v) Use the result in (iv), briefly explain the Mirage phenomenon. (3 points)

用(iv)的結果簡單解釋海市蜃樓現象。（3 分）

THE END 完

Answers
Part I

Q1. The plane should follow the parabola 飛機須沿拋物線運動。

$x = v_0 t\cos\theta,\ y = v_0 t\sin\theta - \frac{1}{2}gt^2$ **(3 points)**

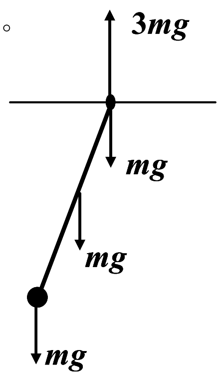
Q2 (6 points)

$I\ddot{\omega} = T$

The center of the rod will not move in the horizontal direction 杆中心在水平方向不動。

$$I = \frac{ml^2}{12} + 2m(\frac{l}{2})^2 = \frac{7}{12}ml^2 \qquad \text{(2 points)}$$

There are two ways to find the torque. 找力距的方法有兩種。



Method-1 方法-1

The forces acting upon the rod are shown. The torque to the center of the rod is 由如圖力的分析，可得

$$T = -(2mg + mg)\frac{l}{2}\theta = -mg\frac{3l}{2}\theta \ . \qquad \text{(2 points)}$$

Method-2 方法-2

Given a small angle deviation θ from equilibrium, the potential energy is 給定一個角度的小位移θ ，勢能為

$$U = mg\frac{3l}{2}(1-\cos\theta) \approx mg\frac{3l}{4}\theta^2 .$$

$$T = -\frac{\partial U}{\partial\theta} = -mg\frac{3l}{2}\theta \qquad \text{(2 points)}$$

Finally, 最後得 $\frac{7ml^2}{12}\ddot{\theta} = \frac{3mgl}{2}\theta \Rightarrow \omega = \sqrt{\frac{18g}{7l}}$.(2 points)

Q3 (6 points)

(a) The bound current density on the disk edge is 盤邊的束縛電流密度為

$K = -\vec{M} \times \vec{n} = -M$, (1 point)

The bound current is 束縛電流 $\Rightarrow I = Jd = -Md$, (1 point)

The B-field is 磁場為 $B(z=h) = \frac{\mu_0 R^2 I}{2(R^2+h^2)^{\frac{3}{2}}} = -\frac{\mu_0 R^2 Md}{2(R^2+h^2)^{\frac{3}{2}}}$ (1 point)

(b) The bound current density is $K = -\vec{M} \times \vec{n} = -M$, which is on the side wall of the cylinder. (1 point)

柱側面上的束縛電流密度為 $K = -\vec{M} \times \vec{n} = -M$

The problem is then the same as a long solenoid. Take a small Ampere loop we get $B = \mu_0 K = \mu_0 M$ inside;

為求一長線圈的磁場，取一小閉合路徑，得介質內 $B = \mu_0 K = \mu_0 M$ (1 point)

Outside 介質外 $B = 0$ (1 point)

Q4 (5 points)

Each unit charge in the slab experiences the Lorentz force $-vB\,\vec{y}_0$. (1 point)

The problem is then the same as a dielectric slab placed between two parallel conductor plates that carry surface charge density $\pm\sigma$, and $\frac{\sigma}{\varepsilon_0} = vB$. In such case, the electric displacement is $D = \sigma$. $P = D - \varepsilon_0 E = D - \frac{D}{\varepsilon} = \varepsilon_0 vB(\frac{\varepsilon-1}{\varepsilon})$. (2 point)

介質內單位電荷受力 $-vB\,\vec{y}_0$ 。問題變成兩電荷面密度為 $\pm\sigma$, and $\frac{\sigma}{\varepsilon_0} = vB$ 的導電板間充滿介質。因此 $D = \sigma$. $P = D - \varepsilon_0 E = D - \frac{D}{\varepsilon} = \varepsilon_0 vB(\frac{\varepsilon-1}{\varepsilon})$.

Finally, the bound surface charge is $\sigma_b = P = \varepsilon_0 vB(\frac{\varepsilon-1}{\varepsilon})$. The upper surface carries positive bound charge, and the lower surface carries negative charge. (1 point)

最後得束縛電荷密度 $\sigma_b = P = \varepsilon_0 vB(\frac{\varepsilon-1}{\varepsilon})$ ，上表面帶正電，下表面帶負電。

The electric field is $\vec{E} = \frac{\sigma_b}{\varepsilon_0}\vec{y}_0 = vB(\frac{\varepsilon-1}{\varepsilon})\vec{y}_0$, which is along the y-direction (opposite to the Lorentz force). (1 point)

電場為$\vec{E}=\frac{\sigma_b}{\varepsilon_0}\vec{y}_0=vB(\frac{\varepsilon-1}{\varepsilon})\vec{y}_0$，與 Lorentz 力方向相反。

Q5. (10 points)

(a) Because of the spherical symmetry, the E-field and the current density $\vec{J}$ are all along the radial direction. In steady condition, the electric current I through any spherical interfaces must be equal. Since the area of the sphere is proportional to r^2, $\vec{J}$ must be proportional to $1/r^2$. So let $\vec{J}=\frac{K}{r^2}\hat{r}$, where K is a constant to be determined, and the expression holds in both media. (1 point)

由對稱性可知，電場和電流密度$\vec{J}$須沿半徑方向。穩態時，流過每個包住球心的球面的電流相等，因此$\vec{J}$與 $1/r^2$ 成正比。設在兩介質裏$\vec{J}=\frac{K}{r^2}\hat{r}$，$K$為待定常數。

In medium-1 介質-1, $E_1=\frac{1}{\sigma_1}J=\frac{1}{\sigma_1}\frac{K}{r^2}$, and the voltage drop from R_1 to R_2 is

$V_1=\frac{1}{\sigma_1}K(\frac{1}{R_1}-\frac{1}{R_2})$ (1 point)

介質-1, $E_1=\frac{1}{\sigma_1}J=\frac{1}{\sigma_1}\frac{K}{r^2}$，從 R_1 到 R_2 的電壓為$V_1=\frac{1}{\sigma_1}K(\frac{1}{R_1}-\frac{1}{R_2})$

Likewise, in medium-2, $E_2=\frac{1}{\sigma_2}\frac{K}{r^2}$, and the voltage drop from R_2 to R_3 is

$V_2=\frac{1}{\sigma_2}K(\frac{1}{R_2}-\frac{1}{R_3})$. (1 point)

同樣，在介質-2, $E_2=\frac{1}{\sigma_2}\frac{K}{r^2}$，從 R_2 到 R_3 的電壓為$V_2=\frac{1}{\sigma_2}K(\frac{1}{R_2}-\frac{1}{R_3})$

The total voltage drop between R_1 and R_3 is 總電壓 $V=V_1+V_2$.

So 得 $\frac{1}{K}=\frac{1}{V}\left[\frac{1}{\sigma_1}(\frac{1}{R_1}-\frac{1}{R_2})+\frac{1}{\sigma_2}(\frac{1}{R_2}-\frac{1}{R_3})\right]$

The current is 電流為 $I=4\pi R_1^2 J(R_1)=4\pi K$. (1 point)

The electric displacement in media are $D_{1,2}=\frac{\varepsilon_{1,2}}{\sigma_{1,2}}J$, so the charge on the inner, outer, and boundary shells are $4\pi K\frac{\varepsilon_1}{\sigma_1}$, $-4\pi K\frac{\varepsilon_2}{\sigma_2}$, and $4\pi K\left(\frac{\varepsilon_2}{\sigma_2}-\frac{\varepsilon_1}{\sigma_1}\right)$, respectively.

介質-1 裏電位移 $D_{1,2}=\frac{\varepsilon_{1,2}}{\sigma_{1,2}}J$，因此在各面上的電荷為 $4\pi K\frac{\varepsilon_1}{\sigma_1}$, $-4\pi K\frac{\varepsilon_2}{\sigma_2}$, $4\pi K\left(\frac{\varepsilon_2}{\sigma_2}-\frac{\varepsilon_1}{\sigma_1}\right)$。

(b) Due to symmetry, the electric field is of the form $\vec{E}=\frac{K}{r^2}\hat{r}$,

so $V=K\left(\frac{1}{R_1}-\frac{1}{R_3}\right)$, and $K=V\cdot\frac{R_3-R_1}{R_1R_3}$. (2 point)

由對稱性可知，電場為 $\vec{E}=\frac{K}{r^2}\hat{r}$，因此電壓為 $V=K\left(\frac{1}{R_1}-\frac{1}{R_3}\right)$，得 $K=V\cdot\frac{R_3-R_1}{R_1R_3}$。

The current densities in the two hemispheres are $\vec{J}_1=\sigma_1\vec{E}$ and $\vec{J}_2=\sigma_2\vec{E}$. The total current is $I=2\pi R_1^2\cdot J_1(R_1)+2\pi R_1^2\cdot J_2(R_1)=2\pi K(\sigma_1+\sigma_2)$. (2 point)

上下半部的電流密度為 $\vec{J}_1=\sigma_1\vec{E}$，$\vec{J}_2=\sigma_2\vec{E}$。 總電流為

$$I=2\pi R_1^2\cdot J_1(R_1)+2\pi R_1^2\cdot J_2(R_1)=2\pi K(\sigma_1+\sigma_2).$$

The total free charge is $Q_1=2\pi\varepsilon_0\varepsilon_1 R_1^2\cdot E_1(R_1)=2\pi K\varepsilon_0\varepsilon_1$ on the upper half and $Q_2=2\pi K\varepsilon_0\varepsilon_2$ on the lower half of the inner shell. On the outer shell the charges are negative of the corresponding ones of the inner shell. (1 point)

內球面上半部總自由電荷為$Q_1 = 2\pi\varepsilon_0\varepsilon_1 R_1^2 \cdot E_1(R_1) = 2\pi K\varepsilon_0\varepsilon_1$，下半部總自由電荷為$Q_2 = 2\pi K\varepsilon_0\varepsilon_2$。外球面的電呵與內球面相反。

Q6. (12 points)

a) $P_h - P_{h+dh} = \rho g dh \Rightarrow \dfrac{dP}{dh} = -\rho g$ (1 point)

$PV^{\frac{7}{5}} = \text{Constant} \Rightarrow P = C\rho^{\frac{7}{5}} \Rightarrow \rho = (\dfrac{P}{C})^{\frac{5}{7}}, where (C = \dfrac{P_0}{\rho_0^{\frac{7}{5}}})$ (1 point)

Combine these two equations, 合併兩式得

$\dfrac{dP}{dh} = -(\dfrac{P}{C})^{\frac{5}{7}} g \Rightarrow P = P_0(1 - \dfrac{2\rho_0 gh}{7P_0})^{\frac{7}{2}}$ (2 point)

$\rho = \rho_0(1 - \dfrac{2\rho_0 gh}{7P_0})^{\frac{5}{2}}$, and $T = T_0(1 - \dfrac{2\rho_0 gh}{7P_0})$. (2 point)

b) $TV^{\frac{2}{5}} = \text{Constant}, \dfrac{T}{\rho^{\frac{2}{5}}} = \text{Constant}$ (1 point)

The density at 40°C is $\dfrac{\rho^{\frac{2}{5}}}{313} = \dfrac{1.18^{\frac{2}{5}}}{293} \Rightarrow \rho = 1.18 \times (\dfrac{313}{293})^{\frac{5}{2}}$ (1 point)

40°C 的空氣密度為$\dfrac{\rho^{\frac{2}{5}}}{313} = \dfrac{1.18^{\frac{2}{5}}}{293} \Rightarrow \rho = 1.18 \times (\dfrac{313}{293})^{\frac{5}{2}}$

The fraction of water vapor at 40°C at sea level 40°C 的水蒸汽分壓為,

$\eta_1 = \dfrac{55.35}{760} \times 90\%$ (1 point)

The fraction of water at 5°C at high altitude 5°C 的水蒸汽分壓為,

$\eta_2 = \dfrac{6.5}{760(\frac{278}{313})^{\frac{7}{2}}}$ (1 point)

Rain 下雨量

$$= (\eta_1-\eta_2)\times\rho\times V = (\frac{55.35}{760}\times 90\% - \frac{6.5}{760(\frac{278}{313})^{\frac{7}{2}}})\times 1.18\times(\frac{313}{293})^{\frac{5}{2}} = 0.07kg \quad \text{(1 point)}$$

高度： $278 = 313(1-\frac{2\times 1.18\times 9.8\times h}{7\times 1.03\times 10^5}) \Rightarrow h = 3500m$ (1 point)

Q7. **(8 points)**

i) $\vec{p}\times\vec{E}$ (1 point)

ii) $\vec{P}\times\vec{E}$ (1 point)

iii) $\vec{D} = \varepsilon_0\vec{E}+\vec{P} \Rightarrow \vec{P} = (\varepsilon-1)\varepsilon_0\vec{E} \Rightarrow \vec{P}\times\vec{E} = 0$ (1 point)

iv) $\vec{P} = \vec{D}-\varepsilon_0\vec{E} = \varepsilon_0[(\varepsilon_X-1)E_X\vec{X}_0+(\varepsilon_Y-1)E_Y\vec{Y}_0]$ (1 point)

$\overline{T} = \frac{1}{2}\mathrm{Re}(\vec{P}\times\vec{E}^*) = \frac{1}{2}\mathrm{Re}(\vec{D}\times\vec{E}^*) = \frac{1}{2}\varepsilon_0 E_0^2(\varepsilon_X-\varepsilon_Y)\cos(\Delta kz)$, (1 point)

Where 其中 $\Delta k \equiv (\sqrt{\varepsilon_X}-\sqrt{\varepsilon_Y})\frac{\omega}{c}$. (1 point)

v) $\overline{T} = \frac{1}{2}\varepsilon_0(\varepsilon_X-\varepsilon_Y)E_0^2\int_0^d\cos(\Delta kz)dz = \frac{1}{2\Delta k}\varepsilon_0(\varepsilon_X-\varepsilon_Y)E_0^2\sin(\Delta kd)$. (1 point)

Maximum occurs when 最大值在 $\Delta kd = \frac{\pi}{2} \Rightarrow d = \frac{\pi}{2\Delta k}$. (1 point)

Part II

Q1. (6 points)

a) $k_x a = n\pi \Rightarrow k_x = \dfrac{n\pi}{a}, n = 1,2,3,.......$ (1 point)

b) $k_y 2b\pi = 2m\pi \Rightarrow k_x = \dfrac{m}{b}, m = 1,2,3,......$ (3 points)

c) $W = \dfrac{1239}{2\pi}\sqrt{k_x{}^2 + k_y{}^2} \Leftrightarrow \dfrac{1239}{2\pi}\sqrt{\dfrac{n^2\pi^2}{a^2} + \dfrac{m^2}{b^2}} = 10^{12}$

$\Rightarrow \dfrac{1239}{2\pi}\dfrac{m}{b} < 10^{12} \Rightarrow b > \dfrac{1239}{\pi} \times 10^{-12} \approx 2\times10^{-10} nm$ (2 points)

Q2. (22 points)

i) $I = m_2 l^2 + \dfrac{m_1}{3} l^2$ (2 points)

$$I\dot{\omega} = F(t)\cdot l \Leftrightarrow (m_2 l^2 + \frac{m_1}{3} l^2)\ddot{\theta} = -\theta k l^2 \Rightarrow \omega_0 = \sqrt{\frac{k}{m_2 + \frac{m_1}{3}}}$$ (2 points)

ii) $(m_2 l + \dfrac{m_1 l}{3})\ddot{x} = F_1 \cos(\omega_1 t) l - xkl$ (2 points)

$x = A_1 \cos(\omega_1 t + \phi_1)$

$\Rightarrow -(\dfrac{m_1}{3} + m_2) A_1 \omega_1{}^2 \cos(\omega_1 t + \phi_1) = F_1 \cos(\omega_1 t) - A_1 \cos(\omega_1 t + \phi_1) k$

$\Rightarrow \phi_1 = 0, A_1 = \dfrac{F_1}{k - (\dfrac{m_1}{3} + m_2)\omega_1{}^2}$ (2 points)

iii) $(m_2 + \dfrac{m_1}{3})\ddot{x} = F_1 \cos(\omega_1 t) + F_2 \cos(\omega_2 t) - xk$ (2 points)

$x = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t$

$\Rightarrow A_1 = \dfrac{F_1}{k - (\dfrac{m_1}{3} + m_2)\omega_1{}^2}, A_2 = \dfrac{F_2}{k - (\dfrac{m_1}{3} + m_2)\omega_2{}^2}$ (2 points)

iv) $x = \sum^{n} A_n \cos \omega_n t \quad \Rightarrow A_n = \dfrac{F_n}{k - (\dfrac{m_1}{3} + m_2)\omega_n{}^2}$ (2 points)

v) $V_{out} = V_0 e^{i\omega t} = \dfrac{R + iL\omega}{\dfrac{1}{i\omega C} + R + iL\omega} V_0 e^{i\omega t} = \dfrac{iR\omega C - L\omega^2 C}{1 - L\omega^2 C + iR\omega C} V_0 e^{i\omega t}$

$$|V_{out}|^2 = V_0^2 \frac{(R\omega C)^2 + (L\omega^2 C)^2}{(1 - L\omega^2 C)^2 + (R\omega C)^2} \qquad \text{(1 point)}$$

To make the denominator minimum, we should have 使分母最小

$1 - LC\omega^2 = 0 \Rightarrow L = \dfrac{1}{C\omega^2}$ (1 point)

(vi) For a given L, only the signal with $\omega_n = \sqrt{\dfrac{1}{CL}}$ in the answer of (iv) can pass through the filter, (1 point) and the output is proportional to A_n. (1 point) By varying L one selects different ω_n, and the output is proportional to the selected A_n. (1 point) From (iv), the maximum A_n is the one when $k = (\dfrac{m_1}{3} + m_2)\omega_n^2$. (1 point) So as L is varied, one finds a particular L_{max} at which the signal peaks, and $k = (\dfrac{m_1}{3} + m_2)CL_{max}$. (1 point)

給定 L, 則只有頻率為 $\omega_n = \sqrt{\dfrac{1}{CL}}$ 的信號可通過濾波器, (1 point) 其大小正比於 A_n. (1 point) L 的變化等於選擇不同的 ω_n 的信號. (1 point) 由(iv)知, $k = (\dfrac{m_1}{3} + m_2)\omega_n^2$ 時 A_n 最大. (1 point) 因此可得使輸出信號最大的電感 L_{max}，並有 $k = (\dfrac{m_1}{3} + m_2)CL_{max}$.

Correct sketch is a flat line with a peak at L_{max} 正確的簡圖是一平線，在 L_{max} 有一尖峰 (1 point)

Q3. (22 points)

i) Only the charge $q = \dfrac{r^3}{R^3}Ze$ that is inside the sphere r will have a non-zero net force on the nucleus. (2 points)

只有 $q = \dfrac{r^3}{R^3}Ze$ 這麼多的負電荷對原子核的合力不為零。

$$\frac{1}{4\pi\varepsilon_0}\frac{Zeq}{r^2} = ZeE_0 \Rightarrow \frac{1}{4\pi\varepsilon_0}\frac{rZe}{R^3} = E_0 \Rightarrow r = \frac{4\pi\varepsilon_0 E_0 R^3}{Ze} \qquad \text{(2 points)}$$

ii) $Zm_e\ddot{r} = ZeE(t) - \frac{Ze}{4\pi\varepsilon_0} \times \frac{rZe}{R^3}$ (2 points)

$E(t) = A\cos(\omega t), r = B\cos(\omega t + \varphi)$ (1 point)

$$\Rightarrow -Zm_e B\omega^2 \cos(\omega t + \varphi) = ZeA\cos(\omega t) - \frac{Z^2e^2}{4\pi\varepsilon_0} \times \frac{B\cos(\omega t + \varphi)}{R^3}$$

$$\Rightarrow \varphi = 0, B = \frac{eA}{\frac{Ze^2}{4\pi\varepsilon_0} \cdot \frac{1}{R^3} - m_e\omega^2}$$ (2 points)

$$\Rightarrow p = Zer = \frac{Ze^2 A}{\frac{Ze^2}{4\pi\varepsilon_0} \cdot \frac{1}{R^3} - m_e\omega^2} \cos(\omega t)$$ (1 point)

iii) $P(t) = Np = \frac{NZe^2}{\frac{Ze^2}{4\pi\varepsilon_0} \cdot \frac{1}{R^3} - m_e\omega^2} E(t)$ (2 points)

iv) $P = CE_{ext} = (\varepsilon - 1)\varepsilon_0 E_{total} = (\varepsilon - 1)\varepsilon_0 (E_{ext} + E_{self})$ (2 points)

The electric field E in a uniform polarized ball with polarization vector P can be calculated by considering the surface bound charge $P\cos\theta$ distributed on the ball.

均勻極化球內的電場 E 可由球面的束縛電荷 $P\cos\theta$ 求得。

$$E_{self} = \frac{1}{4\pi\varepsilon_0} \iint \frac{P\cos\theta(-\cos\theta)}{R^2} R^2 \sin\theta d\theta d\varphi = -\frac{P}{3\varepsilon_0}$$ (2 points)

$E_{self} = -\frac{P}{3\varepsilon_0} = \frac{1}{3} CE_{ext}$ (1 point)

$$\Rightarrow CE_{ext} = (\varepsilon - 1)\varepsilon_0 (E_{ext} - \frac{CE_{ext}}{3\varepsilon_0})$$

$$\Rightarrow 3\varepsilon_0 \frac{\varepsilon - 1}{\varepsilon + 2} = C = \frac{NZe^2}{\frac{Ze^2}{4\pi\varepsilon_0} \cdot \frac{1}{R^3} - m_e\omega^2}$$

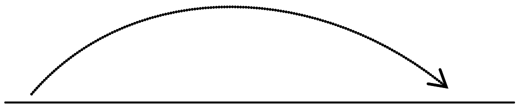
$$\Rightarrow K(\omega) = C = \frac{NZe^2}{\frac{Ze^2}{4\pi\varepsilon_0} \cdot \frac{1}{R^3} - m_e\omega^2} = \frac{N(Ze)^2}{Zm_e(\omega_0^2 - \omega^2)},$$

where $\omega_0^2 \equiv \dfrac{4\pi\varepsilon_0 R^3}{Zm_e(Ze)^2}$ is the resonant frequenc of an atom (2 points)

and $n = 2$

$\omega_0^2 \equiv \dfrac{4\pi\varepsilon_0 R^3}{Zm_e(Ze)^2}$ 是原子的共振頻率。

v) For air K is very small and $\varepsilon(\omega)$ is close to 1.



So the L-L relation becomes

$\varepsilon(\omega) - 1 = \dfrac{1}{\varepsilon_0} K(\omega)$ (1 point)

Air density decreases with height. So refractive index decreases with height. (1 point)

Light rays are bent in such condition. (1 point)

空氣的 $\varepsilon(\omega)$ 接近 1，K 很小，上述結果簡化為 $\varepsilon(\omega) - 1 = \dfrac{1}{\varepsilon_0} K(\omega)$。空氣的密度隨高度減小，因此折射率也減小，形成如圖所示的光線彎曲。

Pan Pearl River Delta Physics Olympiad 2008
2008 年泛珠三角及中华名校物理奥林匹克邀请赛
Part-1 (Total 6 Problems) 卷-1（共 6 题）
(9:00 am – 12:00 pm, 02-14-2008)

Q.1 (4 points) 题 1（4 分） A nucleus-A with mass m_A and initial velocity v_0 along the *x*-axis collides with a nucleus-B with mass m_B at rest. Some kinetic energy E is absorbed by nucleus-B and converted into nuclear energy during the collision. Since $E << m_B c^2$ where c is the speed of light in vacuum, the change of mass of nucleus-B can be neglected. After the collision nucleus-A moves at an angle $\theta = 90^\circ$ to the *x*-axis. Find the speed of nucleus-A and the velocity of nucleus-B.

一质量为 m_A 的原子核-A 以沿 X-轴方向的初速度 v_0 与一质量为 m_B 的静止原子核-B 相撞。碰撞中一部分机械能 E 被原子核-B 吸收而转变为核能。由于 $E << m_B c^2$，其中 c 是真空光速，所以可忽略原子核-B 质量的变化。碰撞后原子核-A 沿与 X-轴成 $\theta = 90^\circ$ 的方向飞出。求原子核-A 的速率，以及原子核-B 的速度。

Q.2 (6 points) 题 2（6 分）

Two large parallel conductor plates are held at voltage different V at distance d apart. A large dielectric slab of thickness $d/3$ and dielectric constant ε is placed midway in the gap between the plates and is moving parallel to the plates at speed v (<< speed of light).

(a) In the reference frame where the plates are stationary, find the magnetic field in the middle of the upper air gap, in the middle of the slab, and in the middle of the lower air gap.

(b) In the reference frame where the dielectric slab is stationary, repeat (a).

两大导电板间距 d，之间的电压差 V。一厚 $d/3$ 的大介质板，介电常数ε，在两板中间以速度 v（<<光速）沿与板平行的方向运动。

(a) 在导电板静止的参照系，求在上、下空隙中间位置以及介质板中间位置的磁场。

(b) 在介质板静止的参照系，求在上、下空隙中间位置以及介质板中间位置的磁场。

Q.3 (7 points) 题 3（7 分）

An electron has intrinsic angular momentum I called spin, and a permanent magnetic dipole moment $\vec{M} = -\frac{ge}{2m}\vec{I}$ associated with the spin, where e is the positive electron charge, m is the electron mass, and g is a number called g-factor. An electron with its spin aligned along its initial velocity in the *x*-direction enters a region of uniform magnetic field in the *z*-direction. Show that if g is exactly 2, then the spin is always in the same direction as the velocity of the electron. (The real $g = 2.00232...$. The deviation from 2 can be calculated precisely by quantum electrodynamics.)

电子具有固有的角动量 I，称为自旋，和与自旋相关的磁偶极矩 $\vec{M} = -\frac{ge}{2m}\vec{I}$。其中 e 为正电子电荷，m 为电子质量，g 为 g -因子。一电子初始速度和自旋都沿 X-方向，进入一均匀沿 Z-方向的磁场。证明若 g 刚好等于 2，则电子之后的速度和自旋始终在同一方向。（实际的 $g = 2.00232...$。和 2 的差可用量子电动力学准确计算。）

Q.4 (12 points) 题 4（12 分）

The atmosphere could be considered as ideal gas at constant temperature T = 300 K. The mass per mole of ‘air’ molecule is m = 0.029 kg. The gas constant is R = 8.31 J/(K*mol).

(a) For stationary atmosphere, set up a differential equation that relates the pressure $p(h)$ at height h with the gravity acceleration g, the atmosphere mole number density $\rho(h)$, and mass per mole m. **(1 point)**

(b) Using the ideal gas law and the result in (a), set up a differential equation for the pressure $p(h)$ as a function of height h. **(1 point)**

(c) Assuming p_0 is the pressure at $h = 0$, solve the differential equation in (b), and find the height where the pressure is $\frac{p_0}{2}$. (Hint: $\int \frac{dx}{x} = \ln(x)$) **(4 points)**

(d) In the case of wind with constant velocity v blowing in the atmosphere at all height, the differential equation is approximately $\frac{dp}{dh} + \frac{mv^2}{2}\frac{d\rho}{dh} = -m\rho g$. Find the pressure $p(h)$ at height h. **(3 points)**

(e) In (d) the effect of Earth spinning is ignored. In general, any object moving at velocity $\vec{v}$ on Earth will experience an additional ‘inertia force field’ $\vec{f}_{\text{int}} = 2\vec{\Omega}\times\vec{v}$, where $\vec{\Omega}$ is the angular velocity of the spinning Earth, in addition to the gravitational field g. Verify that the inertia force field can indeed be ignored even in the case of typhoons where the wind speed is up to 500 km/hr. **(1 point)**

(f) Using the wind speed in (e), find the height where the pressure is $\frac{p_0}{2}$. **(2 points)**

大气层可当作温度为 T = 300 K 的等温理想气体。 每 mole 的大气质量为 m = 0.029 kg。 理想气体常数 R = 8.31 J/(K*mol)。

(a) 对于静止的大气层，建立微分方程，把在高度 h 处的压强 $p(h)$ 、重力加速度 g、 大气分子 mole 密度 $\rho(h)$、和每 mole 的大气质量 m 这些物理量联系起来。**(1 分)**

(b) 利用理想气体方程和(a)的结果, 建立 $p(h)$ 为 h 的函数的微分方程。 **(1 分)**

(c) 设在 $h = 0$ 处压强为 p_0 ，解(b)的微分方程, 并找出 $\frac{p_0}{2}$ 处的高度 (提示: $\int \frac{dx}{x} = \ln(x)$) **(4 分)**

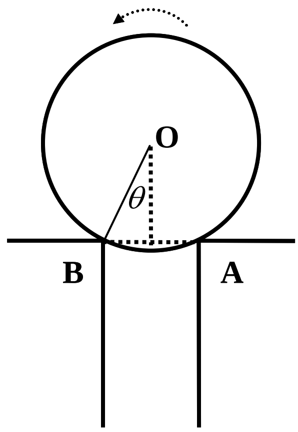
(d) 有风时，设大气在所有高度的速度均为 v, 则经过简化的微分方程为 $\frac{dp}{dh} + \frac{mv^2}{2}\frac{d\rho}{dh} = -m\rho g$ 。求在高度 h 处的压强 $p(h)$. **(3 分)**

(e) 在(d)中没有考虑地球的自转。一般说来，任何以速度 $\vec{v}$ 在地球表面运动的物体，除了重力 g 外，均受到 ‘惯性力场’ $\vec{f}_{\text{int}} = 2\vec{\Omega}\times\vec{v}$ 的作用，其中 $\vec{\Omega}$ 为地球自转的角速度。 验证甚至在刮 500 km/hr 的台风时地球自转的效应仍可忽略。 **(1 分)**

(f) 用(e)的风速， 求压强为 $\frac{p_0}{2}$ 处的高度。**(2 分)**

Q.5 (11 points) 题 5（11 分）

A uniform solid sphere with mass M, radius R, and moment of inertia $I=\frac{2}{5}MR^2$ around its center is initially rolling without slipping at center speed v on a horizontal surface. It then encounters a ditch of width d such that $\sin\theta=\frac{d}{2R}$, as shown in the figure. For convenience you may use θ and R to replace d in the following calculations. The initial speed v is smaller than a value $v_{\max}$ such that when the sphere arrives at the near edge of the ditch at point-A, it falls off while keeping in touch with point-A without slipping, until it hits the other edge at point-B.



(a) Find the angular speed of the sphere right before it hits point-B. **(3 points)**

(b) Find the maximum initial speed $v_{\max}$ that the sphere can keep in touch with point-A without slipping before it hits point-B. **(2 points)**

(c) Assuming no slipping when the sphere hits point-B, find the minimum initial speed $v_{\min}$ such that the sphere can get over the ditch. **(4 points)**

(d) To satisfy both the conditions in (b) and (c), the angle θ must satisfy $f(\theta)>0$. Determine $f(\theta)$. **(2 points)**

一均匀刚球质量 M，半径 R，绕球心的转动惯量 $I=\frac{2}{5}MR^2$，在平面上作纯滚动，球心速度 v。之后遇到一沟，沟宽 d，如图所示 $\sin\theta=\frac{d}{2R}$。为方便起见，在以下的解答你可用 θ 和 R 来代替 d。球的初速度 v 比 $v_{\max}$ 小，因此当球到达沟的一边 A 点处时,球一直与 A 点保持无滑动的接触，直到球碰到沟另一边的 B 点。

(a) 求碰到 B 点前的瞬间球的角速度。 **(3 分)**

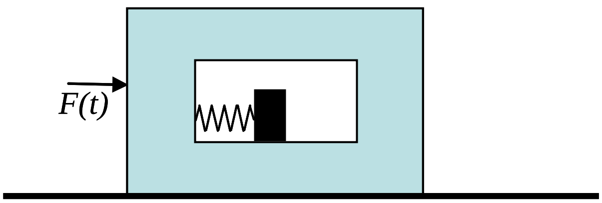
(b) 求球碰到沟另一边的 B 点前一直与 A 点保持无滑动接触的最大初速度 $v_{\max}$。 **(2 分)**

(c) 设球碰到 B 点后无滑动，求能使球滚过沟的最小初速度 $v_{\min}$ 。**(4 分)**

(d) 要使(b) 和 (c)的条件都成立，角度 θ 必须满足 $f(\theta)>0$。 求 $f(\theta)$。**(2 分)**

Q.6 (10 points) 题 6（10 分）

Consider a big block of mass M_1 placed on a smooth horizontal surface with a hollow rectangular cave carved out in the interior, as shown in the figure. Inside the cave are a spring of force constant K with one end attached to the wall, and a smaller block of mass M_2 attached to the other end of the spring which can move on the horizontal smooth cave surface.

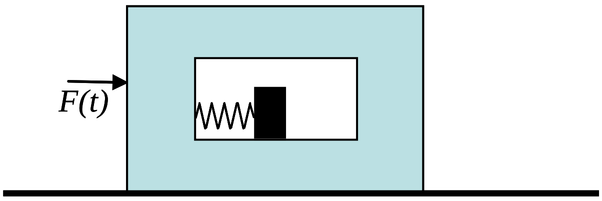


The natural length of the spring is about half the length of the cave. When the small block is moving relative to the big block it never hits the wall of the cave. Both blocks are confined to the motion along the x – axis. A periodic external force $F(t) = F_0 \cos(\omega t)$ is applied to the big block which can push and pull the block. Let the position of the big block be $X_1(t)$, and since the interior of the big block is hidden from the external observers, the effective mass of the system is then $M_{eff} = \frac{F(t)}{\ddot{X}_1(t)}$, where $\ddot{X}_1(t) \equiv \frac{d^2 X_1}{dt^2}$ is the second derivative of $X_1(t)$ to time.

(a) Find the effective mass. **(7 points)**
(b) Find the range of frequencies in which the effective mass is negative. **(3 points)**

如图，一质量 M_1 的大物块放在光滑平面上，大块内有一长方空腔，空腔内有一力常数 K 的弹簧，一端固定在空腔壁上，另一端系有一质量 M_2 的小物块，放在光滑空腔底平面上。弹簧的自然长度约为空腔长度的一半，小物块运动时不会碰到两边的腔壁。所有运动都是沿 x – 轴的一维运动。一周期性外力 $F(t) = F_0 \cos(\omega t)$ 加在大物块上，将它推前、拉后。设大物块的位置为 $X_1(t)$, 因大物块的内部是看不见的，所以系统的有效质量为 $M_{eff} = \frac{F(t)}{\ddot{X}_1(t)}$, 其中 $\ddot{X}_1(t)$ 为 $X_1(t)$对时间的二次导数， $\ddot{X}_1(t) \equiv \frac{d^2 X_1}{dt^2}$ 。

(a) 求有效质量。 **(7 分)**
(b) 求有效质量为负数的频率范围。**(3 分)**



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Pan Pearl River Delta Physics Olympiad 2008
2008年泛珠三角及中华名校物理奥林匹克邀请赛
Part-2 (Total 3 Problems) 卷-2（共3题）
(2:30 pm – 5:30 pm, 02-14-2008)

Q1 Point charge in a magnetic field (16 points) 题1磁场中的带电粒子（16分）

A point charge $-q$ ($q > 0$) with mass m moves without friction inside a region of magnetic field given by $\mathbf{B} = B_0 \frac{a}{r}\hat{\mathbf{z}}$ ($B_0 > 0$), where $r = \sqrt{x^2 + y^2}$ is the distance from the z-axis. At $t = 0$, the initial position and velocity of the charge is ($x = a$, $y = 0$, $z = 0$) and ($v_x = 0$, $v_y = v_0$, $v_z = 0$), where a, $v_0 > 0$.

(a) Show that the particle always stays on the xy plane. **(1 point)**
(b) Find the initial speed at which the charge should be launched so that it can perform circular motion around the origin. **(2 point)**
(c) Now the initial speed does not equal to the answer in (b). Set up a differential equation, in the form of $\frac{dL}{dr} = \boldsymbol{constant}$, for angular momentum L of the charge with respect to the z-axis, and solve the differential equation. If you cannot determine the ***constant***, just assume it is known and solve (d) and (e).
(Hint: $\vec{A}\times(\vec{C}\times\vec{B}) = (\vec{A}\cdot\vec{B})\vec{C} - (\vec{A}\cdot\vec{C})\vec{B}$) **(5 points)**
(d) From the result of (c), find the distance of the charge from the origin when it is moving in the tangential direction ($\vec{v}\perp\vec{r}$). Find the minimum v_0 above which the charge can never move in the tangential direction after it is initially launched. **(5 points)**
(e) Find the distance from the origin when the charge is moving in the radial direction ($\vec{v}$ parallel to $\vec{r}$). Find the minimum v_0 above which the charge can never move in the radial direction after it is initially launched. **(3 points)**

一质量 m 的点电荷 $-q$ ($q > 0$) 无阻力地在磁场 $\mathbf{B} = B_0 \frac{a}{r}\hat{\mathbf{z}}$ ($B_0 > 0$)中运动，其中 $r \equiv \sqrt{x^2 + y^2}$ 为电荷离 z-轴的距离。

在 $t = 0$ 时, 电荷的位置和速度分别为($x = a$, $y = 0$, $z = 0$) 和 ($v_x = 0$, $v_y = v_0$, $v_z = 0$), 其中 a, $v_0 > 0$。
(a) 证明粒子的运动始终在 xy 平面上。 **(1分)**
(b) 若粒子以原点作圆周运动，求初速度 。 **(2分)**
(c) 设粒子的初速度不等于(b)的答案。建立粒子绕 z-轴的角动量所满足的微分方程，其形式应为 $\frac{dL}{dr} = $ ***常数***，并解之。若无法确定***常数***，你可假设***常数***已知，并用来解答(d)和(e)。(提示： $\vec{A}\times(\vec{C}\times\vec{B}) = (\vec{A}\cdot\vec{B})\vec{C} - (\vec{A}\cdot\vec{C})\vec{B}$) **(5分)**
(d) 从(c)的结果, 求当 $\vec{v}\perp\vec{r}$ 时粒子离原点的距离。若粒子进入磁场后永远不再有 $\vec{v}\perp\vec{r}$ 的情形出现，求 v_0 的最小值。 **(5分)**
(e) 求当 $\vec{v}$ 平行于 $\vec{r}$ 时粒子离原点的距离。若粒子进入磁场后永远不再有 $\vec{v}$ 平行于 $\vec{r}$ 的情形出现，求 v_0 的最小值。 **(3分)**

Q2 Electric Fan (16 points) 题 2 电扇（16 分）

Consider an electric fan with moment of inertia *I* around its central axis initially rotating about the axis at constant speed ω_0 driven by a motor. The fan will slow down once the motor is turned off and finally stops due to the friction forces from two sources. One is the fixed torque τ due to the friction between its rotating central axis and its holder. The other is the air resistance on the fan blades which is proportional to the instantaneous rotating speed $\omega(t)$, so the air resistance torque is $\gamma\omega(t)$, where γ is a constant.

(a) Theory

(a.1) Write down the differential equation for the instantaneous rotating speed $\omega(t)$ of the fan. **(2 points)**

(a.2) Given the initial speed ω_0, the time it takes for the fan to stop (t_s) can be expressed as $t_s = A\cdot\ln(1+B\omega_0)$. Determine the constant *A* and *B* in terms of *I*, τ, and γ. **(4 points)**

(b) Design of experiment

Suppose the initial speed of the fan can be so slow that $\ln(1+B\omega_0)\approx B\omega_0$, or can be so fast that $B\omega_0 >> 1$. The initial speed of the fan can be set and read from the meter on the fan controller. You are also given the following items: a ruler, a stop watch, and several pairs of small known mass blocks (about 1/10 the mass of the blades) that can be firmly attached to the blades of the fan, but their air resistance can be neglected. Design an experiment to determine τ, γ, and *I*. You should state clearly what data are to be collected and processed, what plot(s) should be drawn, and how to extract the parameters from the plot(s) to reach the final answers. **(10 points)**

一电扇绕其中心轴的转动惯量 *I*，初始时在马达驱动下以角速度ω_0绕中心轴转动。马达关了后，由于两方面的摩擦阻力电扇的转动会慢下来，直到停止。阻力之一为中心轴与外套之间的固定摩擦阻力力矩τ。另一个是电扇叶片的空气阻力力矩，与瞬时角速度 $\omega(t)$成正比，因此可表达为$\gamma\omega(t)$, 其中 γ 为常数。

(a) 理论

(a.1) 写出电扇瞬时角速度的微分方程 **(2 分)**

(a.2) 给定初始角速度 ω_0, 从停止驱动到电扇停下的时间 t_s 可表达成 $t_s = A\cdot\ln(1+B\omega_0)$。用 *I*、 τ、γ 来表达系数 *A*、*B* 。**(4 分)**

(b) 实验设计

设电扇的初始角速度很慢，使$\ln(1+B\omega_0)\approx B\omega_0$得以成立。或很快，使$B\omega_0 >> 1$得以成立。电扇的初始角速度可由马达的控制器设置并读出。另有标尺一把，计时器一个, 和几对已知质量（约为电扇叶质量的 1/10 ）、可粘在电扇叶上的小重块。小重块的空气阻力可忽略。设计实验步骤以确定 τ、γ、和 *I* 。 你必须清楚表述需要测量哪些实验数据，如何作数据处理，如何作图，从图上得到哪些参数，最后得到结果。**(10 分)**

Q3 Negative Resistance Instability & Zero Resistance State (18 points)
题 3 负电阻不稳定性和零电阻状态 (18 分)

Two years ago it was discovered that the resistance of a semiconductor device containing two dimensional electron gas becomes negative when it is in a constant magnetic field and under strong microwave radiation. Because a system including the device and other circuit elements such as normal resistors, capacitors, inductors, and voltage sources will be unstable (violating the second law of thermodynamics) if the *total* resistance of the system R_{sys} is negative, the system reorganizes itself into a new state with zero total resistance. The resistance of the device can be expressed as a current and charge dependent resistance

$$R_{NR} \equiv R(i,q) = R_0\left(\left(\frac{i}{i_o}\right)^2 + \left(\frac{q}{q_o}\right)^2 - 1\right),$$

where $R_0 > 0$ so that the resistance is negative when the current passing through the system i or charge storing in the system q is small, and becomes positive when there is large enough current or charge building up in the system. Here R_0, i_o, q_o are constants that depend only on the internal structure of the device, the strength and frequency of microwave radiation, and the strength of the magnetic field. When R_{NR} is connected to other normal circuit elements like capacitor, normal resistor, inductor, and voltage source, the usual Kirchhoff's circuit laws still apply.

两年前人们发现在微波照射下，在直流磁场中的由二维电子气构成的半导体器件的电阻可成负值。由于含有这种器件和其它常规电阻、电感、电容、电源的系统的总电阻 R_{sys} 不能为负，否则会违反热力学第二定律，所以系统会自动调节，以达到一新的零电阻状态。器件的电阻与电流、电荷有关的表达式为

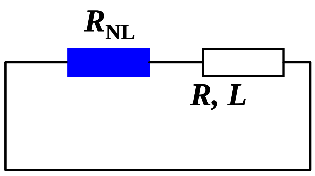
$$R_{NR} \equiv R(i,q) = R_0\left(\left(\frac{i}{i_o}\right)^2 + \left(\frac{q}{q_o}\right)^2 - 1\right),$$

其中 $R_0 > 0$。因此当系统的电流 i 或电荷 q 太小时，其电阻为负。而当系统的电流或电荷够大时，其电阻变为正值。R_0, i_o, q_o 为常数，只与器件的本身结构，以及磁场、微波频率和照射强度有关。设对于 R_{NR} 和其它常规电阻、电感、电容、电源形成的电路，常规 Kirchhoff 电路原理仍然适用。

a) We start with simple systems. 我们从简单的系统开始。

(i) When R_{NR} is in series with a normal resistor with resistance R, find the DC current and the voltage drop across R and R_{NR}. (Consider both cases when $R > R_0$ and $R < R_0$.) **(2 points)**

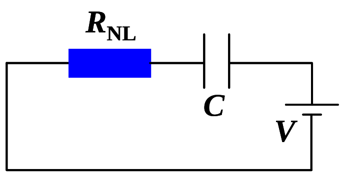
R_{NR} 和一常规电阻 R 串连，求电路的直流电流和 R 、R_{NR} 上的电压。(须考虑 $R > R_0$ 和 $R < R_0$ 这两种情况。) **(2 分)**



(ii) When R_{NR} is in series with a normal inductor with inductance L, find the DC current and the voltage drop across L and R_{NR}. **(1 point)**

R_{NR} 和一常规电感 L 串连，求电路的直流电流和 L 、R_{NR} 上的电压。**(1 分)**

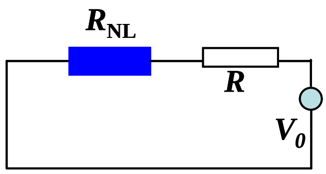
(iii) When R_{NR} is in series with a normal capacitance C and a DC voltage source V, find the minimum voltage V needed to make $R_{sys} = R_{NR} = 0$. Also find the DC current and the voltage drop across C and R_{NR}. (Hint: The charge q on the capacitor is the charge stored in the system that determines the value of R_{NR}.) **(2 points)**



R_{NR} 和一常规电容 C、直流电压源 V 串连。求使 $R_{sys}=R_{NR}=0$ 所需的最小电压，并求这时的直流电流和 C、R_{NR} 上的电压。（提示：电容上的电量 q 就是决定 R_{NR} 的储存于系统的电荷量）**(2 分)**

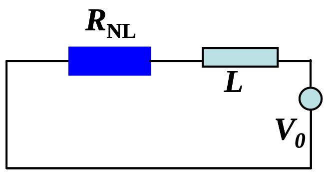
b) Now a small AC or DC voltage source V_0 is added in the above circuits. In the following calculations of the contribution of V_0 to the additional current and/or charge on top of the original ones, keep only their first order. In the cases when the small voltage source is AC, you should set up differential equations first.
现在以上的电路中加一直流或交流的小电源 V_0。在以下的计算由 V_0 所带来的额外电流或电荷的过程中，只需保留它们的一次项。当小电源为交流时，先建立微分方程。

(i) When R_{NR} is in series with a normal resistor with resistance R and a small DC voltage source V_0, find the DC current. You should consider both the cases $R>R_0$ and $R<R_0$, separately. **(3 points)**



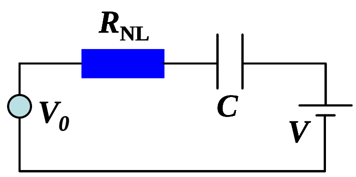
当 R_{NR} 和一常规电阻 R、小直流电压源 V_0 串连，求直流电流。（分别考虑 $R>R_0$，和 $R<R_0$ 这二种情况。）**(3 分)**

(ii) When R_{NR} is in series with a normal inductor with inductance L and a small AC voltage source with amplitude V_0 and frequency ω, find the current through L and R_{NR}. **(3 points)**

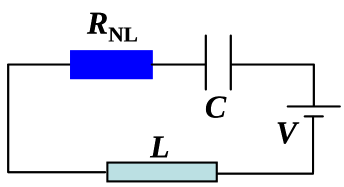


当 R_{NR} 和一常规电感 L、频率为ω幅度为 V_0 的小交流电压源串连，求电流。 **(3 分)**

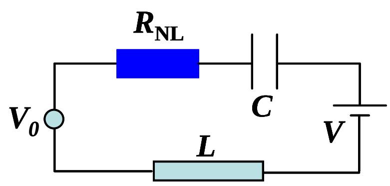
(iii) A small AC voltage source with amplitude V_0 and frequency ω is added to the circuit in part-a(iii), find the charge on the capacitor. **(2 points)**
a(iii)部分的电路里加上频率为ω幅度为 V_0 的小交流电压源，求电容上电荷。 **(2 分)**



(iv) Consider an LCR circuit with R_{NR} as the resistor in the presence of DC voltage source V, find the current in the circuit. **(2 points)**
在 LCR 电路中将常规电阻用 R_{NR} 代替，并加上直流电压源 V，求电流。**(2 分)**



(v) A small AC voltage source with amplitude V_0 and frequency ω is added to the circuit in (iv). Find the current in the circuit. **(3 points)**
在以上(iv)的电路中加上频率为ω幅度为 V_0 的小交流电压源，求电流。**(3 分)**



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Q1: (4 points)
In the process energy and momentum are conserved, so
碰撞前后动量和能量守恒，因此

$m_A v_0 = m_B v_{Bx} \quad 0 = m_B v_{By} + m_A v_A$ (1 **point**)

$m_A v_0^2 = m_A v_A^2 + m_B v_{Bx}^2 + m_B v_{By}^2 + 2E$ (1 **point**)

Where, v_A is in the y-direction, we can get 其中v_A是沿 Y-方向的速度。由此得

$v_A = \sqrt{\dfrac{m_A(m_B - m_A)v_0^2 - 2Em_B}{m_A(m_B + m_A)}}$, (1 **point**)

$v_{Bx} = \dfrac{m_A}{m_B} v_0$, $v_{By} = -\dfrac{m_A}{m_B} v_A$. (1 **point**)

Q2: (6 **points**)
From Gauss theorem, the area density of the free charge at the surfaces of the conductor plates is $\sigma_f = D$. (0.5 **points**) 根据高斯定理，导电板上自由电荷面密度与板间电位移的关系为$\sigma_f = D$。

In the air gap $E_1 = E_3 = D/\varepsilon_0$, while in the dielectric $E_2 = D/\varepsilon\varepsilon_0$. (0.5 **points**)
在空气中 $E_1 = E_3 = D/\varepsilon_0$，在介质内 $E_2 = D/\varepsilon\varepsilon_0$，电压为

$$V = \frac{d}{3}E_1 + \frac{d}{3}E_2 + \frac{d}{3}E_3 = \frac{dD}{3\varepsilon_0}(2 + \frac{1}{\varepsilon}) = \frac{d\sigma_f}{3\varepsilon_0}(2 + \frac{1}{\varepsilon}),$$

由此得

$\sigma_f = \dfrac{V}{d}\dfrac{3\varepsilon_0\varepsilon}{2\varepsilon + 1}$. (1 **point**)

The bound charge is $\sigma_b = \mp\varepsilon_0(E_1 - E_2) = \mp\dfrac{3V}{d}\dfrac{\varepsilon_0(\varepsilon - 1)}{2\varepsilon + 1}$. (1 **point**)

介质上、下界面的束缚电荷面密度为$\sigma_b = \mp\varepsilon_0(E_1 - E_2) = \mp\dfrac{3V}{d}\dfrac{\varepsilon_0(\varepsilon - 1)}{2\varepsilon + 1}$。

When the slab is moving with speed *v*, the electric currents at two sides are $K = \pm\sigma_b v$. From Ampere law, $B_1 = B_3 = 0$ (1 **point**)
介质板运动时，上、下界面的束缚电流面密度为$K = \pm\sigma_b v$。由安培定理得在空隙间的磁场 $B_1 = B_3 = 0$

Inside the dielectric slab the magnetic field is 在介质板里的磁场

$B_2 = \mu_0 K = \mu_0\sigma_b v = \mu_0 v\dfrac{3V}{d}\dfrac{\varepsilon_0(\varepsilon - 1)}{2\varepsilon + 1}$. (1 **point**)

When the parallel conductor plates are moving with speed -*v*, the electric currents at two plates are $K' = \mp\sigma_f v = \mp v\dfrac{V}{d}\dfrac{3\varepsilon_0\varepsilon}{2\varepsilon + 1}$. From the Ampere law, $B_1 = B_2 = B_3 = -\mu_0\sigma_f v$. (1 **point**)

当导电板运动时，上、下板的面电流密度为$K' = \mp\sigma_f v = \mp v\dfrac{V}{d}\dfrac{3\varepsilon_0\varepsilon}{2\varepsilon + 1}$。

由安培定理得 $B_1 = B_2 = B_3 = -\mu_0\sigma_f v$

Q3: (7 points)

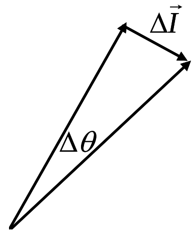
The electron spin (angular momentum) is I, and the associated magnetic dipole moment is $\vec{M} = -\dfrac{ge}{2m}\vec{I}$.

In the B-field, the torque on the spin is $\vec{M}\times\vec{B}$ and perpendicular to $\vec{M}$. (1 **point**)

The procession frequency is then determined by $MB\Delta t = \Delta I$, (1 **point**)

$MB\Delta t = I\Delta\theta$ (1 **point**)

$\Rightarrow MB = I\dfrac{\Delta\theta}{\Delta t} = I\omega$ (1 **point**)



which leads to $\omega = \dfrac{ge}{2m}B$. (1 **point**)

The negative sign in $\vec{M} = -\dfrac{ge}{2m}\vec{I}$ ensures that the spin turns in the same direction as the electron trajectory under Lorentz force. $\dfrac{mv^2}{R} = eBv \Rightarrow \omega' = \dfrac{eB}{m_e}$ (2 **points**)

电子的自旋（角动量）为 I，与其相关的磁偶极子为$\vec{M} = -\dfrac{ge}{2m}\vec{I}$.

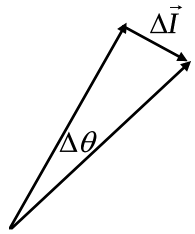
在磁场里磁偶极子受的力矩为$\vec{M}\times\vec{B}$与$\vec{M}$ 垂直。 (1 **point**)

$\vec{M}$ 进动的频率由下式可求：

$MB\Delta t = \Delta I$, (1 **point**)

$MB\Delta t = I\Delta\theta$ (1 **point**)

$\Rightarrow MB = I\dfrac{\Delta\theta}{\Delta t} = I\omega$ (1 **point**)



得 $\omega = \dfrac{ge}{2m}B$. (1 **point**)

空间的圆轨迹为： $\dfrac{mv^2}{R} = eBv \Rightarrow \omega' = \dfrac{eB}{m_e}$ (2 **points**)

$\vec{M} = -\dfrac{ge}{2m}\vec{I}$ 中的负号保证了自旋的旋转方向与它在空间的圆轨迹一致。

Q4 (12 points)

(a) Consider a thin layer of air at rest, the pressure difference balances the gravity,
取一薄层空气，上、下面上的压强差刚好与重力平衡，

$p(h+\Delta h) - p(h) = -m\rho g\Delta h$

$\Rightarrow \dfrac{dp}{dh} = -m\rho g$ (1 **point**)

(b) Put in the ideal gas law $p = \rho RT$, the differential equation is then
代入理想气体方程 $p = \rho RT$， 得微分方程

$\dfrac{dp}{dh} = -m\rho g = -\dfrac{mg}{RT}p$. (1 **point**)

(c) From (b), 由(b)解得

$$\frac{dp}{p}=-\frac{mg}{RT}dh\Rightarrow \ln p-\ln p_0=-\frac{mg}{RT}h\Rightarrow p(h)=p_0e^{-\frac{mg}{RT}h}.$$ (3 **point**s)

Put in the numbers, 代入数值，

$$\frac{mg}{RT}=\frac{0.029\times 9.8}{8.31\times 300}=\frac{1}{8.8}km^{-1}.$$

So the height is 得高度为

$8.8\times\ln 2=8.8\times 0.693=6.1km$. (1 **point**)

(d) With a constant wind with velocity v we replace the pressure equation by the Bernoulli's equation

有风时，微分方程为

$$\frac{dp}{dh}+\frac{mv^2}{2}\frac{d\rho}{dh}=-mg\rho.$$

Using the ideal gas law, we obtain 代入 $p=\rho RT$ 理想气体方程，得

$$\frac{dp}{dh}(1+\frac{mv^2}{2RT})=-\frac{mg}{RT}p.$$ (1 **point**)

Therefore 解得 $p(h)=p_0e^{-\frac{mg}{\left(RT+\frac{mv^2}{2}\right)}h}$. (2 **point**)

(e) $\vec{\Omega}\times\vec{v}\approx\Omega\times v=2\pi\times 500000/(24\times 60\times 60\times 60\times 60)\approx 0.01\ m/s^2<<g$. (1 **point**)

(f) The decay length in this case is 代入数值

$$h_0=\ln 2(\frac{RT+\frac{mv^2}{2}}{mg})=0.693\times\frac{8.31\times 300+0.5\times 0.029\times 2.5\times 10^4/3.6}{0.029\times 9.8}=6.1+0.24=6.3\ km$$

(2 **points**)

Q5 (11 points):

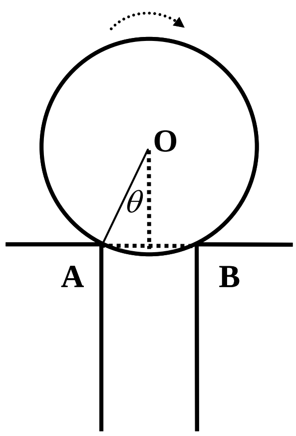
(a) When the ball arrives at point A, it begins to drop down. In this process the potential energy transforms into kinetic energy.

球到 A 点后下落，到 B 点时，势能变化为

$\Delta P=MgR(1-\cos\theta)$ (1 **point**)

The moment of inertia about the edge is 绕球边的转动惯量为

$$\tilde{I}=I+MR^2=\frac{7}{5}MR^2.$$



Kinetic energy is 动能为

$$K_1=\frac{1}{2}\tilde{I}\omega^2=\frac{1}{2}\times\frac{7}{5}MR^2\omega^2=\frac{7}{10}Mv^2$$ (0.5 **point**s)

So 因此 $K_2=\frac{7}{10}Mv'^2=MgR(1-\cos\theta)+\frac{7}{10}Mv^2$ (0.5 **point**s)

$$\Rightarrow v'^2=\frac{10}{7}gR(1-\cos\theta)+v^2$$

$$\Rightarrow \omega'^2 = \frac{10}{7R} g(1-\cos\theta) + \left(\frac{v}{R}\right)^2$$ (1 **point**)

(b) The critical condition for the ball to keep contact with point A before it touches point B is that: At the moment it touches point B, the centrifugal force equals the gravity component.

要保持与 A 点接触，即球以 A 点作圆周运动，其向心力全由重力提供。

$$\frac{Mv'^2}{R} = Mg\cos\theta$$ (1 **point**)

$$\Rightarrow \frac{10}{7} gR(1-\cos\theta) + v_{\max}^2 = gR\cos\theta \qquad \Rightarrow v_{\max}^2 = \frac{gR}{7}(17\cos\theta - 10)$$ (1 **point**)

where $\cos\theta = \dfrac{\sqrt{R^2 - \dfrac{d^2}{4}}}{R} = \sqrt{1 - \dfrac{d^2}{4R^2}}$

(c) In the process of the ball collide with point B, the angular momentum of the ball around point B is unchanged. Before the collision, the angular momentum of the ball around point B is $I\omega' + MR\tilde{v}' = I\omega' + MR\left(R\omega'\cos 2\theta\right) = MR^2\omega'\left(\frac{2}{5} + \cos 2\theta\right)$.

After the collision, it is $\tilde{I}\omega'' = \frac{7}{5}MR^2\omega''$.

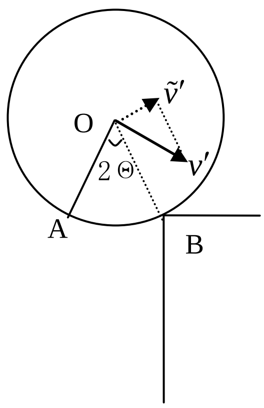
在与 B 点碰撞过程中，球相对于该点的角动量守恒。碰撞前的角动量为 $I\omega' + MR\tilde{v}' = I\omega' + MR\left(R\omega'\cos 2\theta\right) = MR^2\omega'\left(\frac{2}{5} + \cos 2\theta\right)$，碰撞后的角动量为 $\tilde{I}\omega'' = \frac{7}{5}MR^2\omega''$。

Hence 因此

$$\frac{7}{5}MR^2\omega'' = MR^2\omega'\left(\frac{2}{5} + \cos 2\theta\right) \Rightarrow \omega'' = \frac{2 + 5\cos 2\theta}{7}\omega'.$$ (2 **points**)

So the total energy after collision is 碰撞后的动能为

$$K_3 = \frac{7}{10}MR^2\omega''^2 = \frac{1}{70}\left(2 + 5\cos 2\theta\right)^2 Mv'^2$$ (1 **point**)



The requirement for the ball to get over the ditch:

要翻上沟边，动能要克服的势能为

$$K_3 = \frac{1}{70}\left(2 + 5\cos 2\theta\right)^2 Mv'^2 > MgR(1-\cos\theta)$$ (1 **point**)

$$\Rightarrow v^2 > 10gR\left[\frac{7}{\left(2 + 5\cos 2\theta\right)^2} - \frac{1}{7}\right]\left(1 - \cos\theta\right)$$

$$\Rightarrow v_{\min}^2 = 10gR\left[\frac{7}{\left(2 + 5\cos 2\theta\right)^2} - \frac{1}{7}\right]\left(1 - \cos\theta\right)$$ (2 **points**)

(d) We must have 根据题意，$v_{\min}^2 < v_{\max}^2$

既 $\dfrac{1}{10}\cos\theta - \dfrac{7\left(1 - \cos\theta\right)}{\left(2 + 5\cos 2\theta\right)^2} > 0$.

（Numerical result of Max θ: 数值计算得最大角为 0.597797 弧度，或 34°。）

Q6 (10 points)

Suppose the spring is extended, and choose the natural length as the origin of the coordinate of the small block $X_2 = 0$. The Dynamic equation of this system is

设弹簧是拉长的，选弹簧在自然长度时小物块的坐标 $X_2 = 0$。系统的运动方程为

$F + KX_2 = M_1\ddot{X}_1$ (1) (1 **point**) $-KX_2 - M_2\ddot{X}_1 = M_2\ddot{X}_2$ (2) (1 **point**)

Note that 由于 $\ddot{X}_{1,2} = -\omega^2 X_{1,2}$, (1 **point**)

From (2) we get 由(2)得 $X_2 = \dfrac{\omega^2 M_2}{K - \omega^2 M_2} X_1$. (1 **point**)

Then 因此 $F = -M_1\omega^2 X_1 - K\dfrac{\omega^2 M_2}{K - \omega^2 M_2} X_1 = -\omega^2 X_1 (M_1 + \dfrac{KM_2}{K - \omega^2 M_2})$ (2 **points**)

(a) Finally 最后得 $M_{eff} = \dfrac{F}{-\omega^2 X_1} = M_1 + \dfrac{KM_2}{K - \omega^2 M_2} = M_1 - \dfrac{KM_2}{\omega^2 M_2 - K}$ (1 **point**)

(b) Negative effective mass 负有效质量:

For negative M_{eff}, we get, after some algebra, $\omega^2 < K(\dfrac{1}{M_1} + \dfrac{1}{M_2})$. However, the term $\dfrac{KM_2}{\omega^2 M_2 - K}$ must be positive. So the final answer is

$\dfrac{K}{M_2} < \omega^2 < K(\dfrac{1}{M_1} + \dfrac{1}{M_2})$ (3 **points**) Missing $\dfrac{K}{M_2} < \omega^2$, (-1 **point**)

要使 $M_{eff} < 0$，经过简单代数运算，得 $\omega^2 < K(\dfrac{1}{M_1} + \dfrac{1}{M_2})$。但是 $\dfrac{KM_2}{\omega^2 M_2 - K}$ 必须是正的。因此得

$\dfrac{K}{M_2} < \omega^2 < K(\dfrac{1}{M_1} + \dfrac{1}{M_2})$ (3 **points**) 漏掉 $\dfrac{K}{M_2} < \omega^2$, (-1 **point**)

Part-II

Q1 (16 points):

(a) By Lorentz force law, $\mathbf{F} = -q\mathbf{v}\times\mathbf{B}$, F has no z-component when $\mathbf{B}$ is along the z-direction. Hence if $v_z = 0$ initially, it always remains zero. (1 **point**)

磁场的力为$\mathbf{F} = -q\mathbf{v}\times\mathbf{B}$，与 XY 面平行。由于初速度的 Z-分量 $v_z = 0$，所以粒子保持在 XY 面上运动。

(b) The B-field at $r = a$ is just right to keep the particle on a circular orbit of radius a.

在 $r = a$ 处的磁场刚好可以维持粒子以 a 为半径的圆周运动。

$\frac{mv_0^2}{a} = qv_0B_0 \Rightarrow v_0 = \frac{qB_0a}{m}$. (2 **points**)

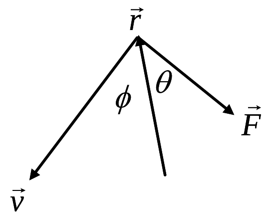
(c) Let the angular momentum of the charge about the origin be L. Then $L(t=0) = mav_0$.

令粒子绕原点的角动量为 L. 则 $L(t=0) = mav_0$。

$\frac{dL}{dt} = -rF\sin\theta$ (1 **point**)

$= -qrvB\cos\phi = -qrBv_r$ (1 **point**). $v_r = \frac{dr}{dt}$ is the radial velocity 径向速度分量 $v_r = \frac{dr}{dt}$.

$= qBr\frac{dr}{dt}$ (1 **point**)



$\Rightarrow \frac{dL}{dr} = qBr$ (1 **point**)

One can also obtain the same differential equation by

也可用下列公式

$\frac{d\mathbf{L}}{dt} = \mathbf{r}\times\mathbf{F} = -q\mathbf{r}\times(\mathbf{v}\times\mathbf{B}) = -q\left[(\mathbf{r}\cdot\mathbf{B})\mathbf{v} - (\mathbf{r}\cdot\mathbf{v})\mathbf{B}\right] = qrv_r\mathbf{B} = qr\frac{dr}{dt}\mathbf{B}$ (2 **points**)

$\frac{dL}{dt} = qBr\frac{dr}{dt}$ (1 **point**)

$\Rightarrow \frac{dL}{dr} = qBr$ (1 **point**)

$L(r) - L(a) = \int_a^r qBrdr = qB_0a(r-a)$. (1 **point**)

(d) Because B field does not do work, the speed of the charge is always v_0. Note that the angular momentum can be $L = \pm mrv_0$

由于磁场不做功，粒子的速率一直为 v_0。注意角动量可以是 $L = \pm mrv_0$。

When $L = mrv_0$, $r = a$, no tangential motion occurs afterwards. (1 **point**)

当 $L = mrv_0$， $r = a$。这只有在初始时有，之后就不再出现了。

When 当 $L = -mrv_0$， $r = a\frac{qB_0a - mv_0}{qB_0a + mv_0}$. (1 **point**)

When $qB_0a < mv_0$, the only solution is $r = a$, which corresponds to the initial condition. No tangential motion occurs afterwards. (1 **point**)

当 $qB_0a < mv_0$， 只有初始的 $r = a$, 之后就不再出现了。

当 $qB_0a > mv_0$, (1 **point**) $r = a\dfrac{qB_0a - mv_0}{qB_0a + mv_0} < a$. (1 **point**)

It is possible. 这是个可以出现的情形。

(e) When the charge is moving in the radial direction, $L = 0$. (1 **point**)
粒子沿径向运动时，$L = 0$。
Therefore, 因此

$$r = a\left(1 - \frac{mv_0}{qB_0a}\right). \qquad (1\ \textbf{point})$$

When $qB_0a > mv_0$, $r > 0$, the motion can be radial. (0.5 **points**)
当 $qB_0a > mv_0$, $r > 0$, 粒子可沿径向运动
When $qB_0a < mv_0$, $r < 0$, the motion never becomes radial. (0.5 **points**)
当 $qB_0a < mv_0$, $r < 0$, 粒子不可沿径向运动。

Q2 (16 points)

(A.1) The differential equation 微分方程为: $-I\dot{\omega} = \tau + \gamma\omega$ (2 **points**)

(A.2)

$$\frac{d\omega}{\omega + \frac{\tau}{\gamma}} = -\frac{\gamma}{I}dt \Rightarrow \int_{\omega_0}^{0}\frac{d\omega}{\omega + \frac{\tau}{\gamma}} = -\frac{\gamma}{I}\int_0^{t_s}dt \Rightarrow \ln\frac{\frac{\gamma}{\tau}}{\omega_0 + \frac{\tau}{\gamma}} = -\frac{\gamma}{I}t_s \Rightarrow t_s = \frac{I}{\gamma}\ln(1 + \frac{\gamma\omega_0}{\tau}) \quad (2\ \textbf{points})$$

So $A = \dfrac{I}{\gamma}$, (1 **point**); $B = \dfrac{\gamma}{\tau}$, (1 **point**)

(B)

- First, add each pair of blocks at equal distance to the center in order to avoid warbling, keep ω_0 fixed and measure stop time t_{sn}

$t_{sn} = \dfrac{I + n\Delta I}{\gamma}\ln(1 + \dfrac{\gamma\omega_0}{\tau}) = \dfrac{n\Delta I}{\gamma}\ln(1 + \dfrac{\gamma\omega_0}{\tau}) + \dfrac{I}{\gamma}\ln(1 + \dfrac{\gamma\omega_0}{\tau})$. Here n = 1, 2, .., and $\Delta I = mr^2$, where m is the mass of the small block, and r is the distance to the center of the fan measured by the ruler. Plot $t_{sn} \sim n\Delta I$, one gets the slope $K = \dfrac{1}{\gamma}\ln(1 + \dfrac{\gamma\omega_0}{\tau})$ and the interception $b = \dfrac{I}{\gamma}\ln(1 + \dfrac{\gamma\omega_0}{\tau})$. Then $I = b/K$. (4 **points**)

- 首先，保持 ω_0 为常数，将每对小重物放在离轴等距离的两边的叶片上，测量停止时间 t_{sn}。

$t_{sn} = \dfrac{I + n\Delta I}{\gamma}\ln(1 + \dfrac{\gamma\omega_0}{\tau}) = \dfrac{n\Delta I}{\gamma}\ln(1 + \dfrac{\gamma\omega_0}{\tau}) + \dfrac{I}{\gamma}\ln(1 + \dfrac{\gamma\omega_0}{\tau})$，其中 n = 1, 2, .., $\Delta I = mr^2$, m 是每块小重物的质量，r 是小重物离轴的距离。将 $t_{sn} \sim n\Delta I$ 作图，得直线的斜率 $K = \dfrac{1}{\gamma}\ln(1 + \dfrac{\gamma\omega_0}{\tau})$，与 Y 轴的交点 $b = \dfrac{I}{\gamma}\ln(1 + \dfrac{\gamma\omega_0}{\tau})$。求得转动惯量 $I = b/K$。

- Secondly, let the initial angular ω_0 be very slow such that

$$t_s = A\ln(1+B\omega_0) \simeq AB\omega_0 = \frac{I}{\tau}\omega_0 .$$

From the slope of the $t_s \sim \omega_0$ line we can get the value of τ . (3 **point**s)

第二，将初始转速调小，使$t_s = A\ln(1+B\omega_0) \simeq AB\omega_0 = \frac{I}{\tau}\omega_0$成立。测$t_s \sim \omega_0$并作图。直线的斜率为$\frac{I}{\tau}$。由此得$\tau$。

- Finally, let the initial angular ω_0 be very fast so that

$$t_s = A\ln(1+B\omega_0) \simeq A\ln(B\omega_0) = \frac{I}{\gamma}[\ln(\frac{\gamma}{\tau}) + \ln(\omega_0)] .$$

Change ω_0 and plot $t_s \sim \ln(\omega_0)$ which forms a straight line. The slope is I/γ . Since I is already known, γ can be readily obtained. (3 **point**s)

最后，将初始转速调大，使$t_s = A\ln(1+B\omega_0) \simeq A\ln(B\omega_0) = \frac{I}{\gamma}[\ln(\frac{\gamma}{\tau}) + \ln(\omega_0)]$成立。测$t_s \sim \ln(\omega_0)$并作图。直线的斜率为$I/\gamma$。由此得$\gamma$。

Q3 (18 points**)**

Ans:

a) (i) For $R < R_0$ there must be a constant current i is flowing in the system to increase R_{NR} so the total resistance of the system will not be negative.

当$R < R_0$，需有电流使R_{NR}增加，使系统的总电阻不为负。

$$\left(R - R_0\left(1-\left(\frac{i}{i_o}\right)^2\right)\right)i = 0$$ (Kirchhoff's law) (0.5 **point**s)

$$\Rightarrow i = \pm\sqrt{\frac{R_0 - R}{R_0}}\, i_o$$ (0.5 **point**s)

Voltage drop across $R_{NR} = -R_0\left(1-\left(\frac{i}{i_o}\right)^2\right)i = -Ri$ = - voltage drop across R. (0.5 **point**s)

电压为 $iR_{NR} = -R_0\left(1-\left(\frac{i}{i_o}\right)^2\right)i = -Ri$

For $R > R_0$, the total resistance > 0, so the solution is i = 0. There is no voltage drop. (0.5 **point**s)

若$R > R_0$，则系统的总电阻为正，电流为零。

(ii) L has no resistance. So a constant current $i = \pm i_o$ flows through the system and $R_{NR} = 0$. There is no voltage drop anywhere. (1 **point**)

电感无直流电阻，所以$R_{NR} = 0$。电流为$i = \pm i_o$，但无电压。

(iii) In this case i = 0 and charge $q = \pm q_o$ is needed to maintain the circuit in equilibrium (R_{NR} = 0). (1 **point**)
Therefore a minimum voltage $V_o = \pm q_o / C$ is needed to maintain the system at equilibrium. There is no voltage drop across R_{NR}. (1 **point**)
直流电流 i = 0。电容上的电荷为 $q = \pm q_o$，R_{NR} = 0。所需最小电压为 $V_o = \pm q_o / C$。R_{NR} 上无电压。

b) (i) For $R < R_0$ the Kirchhoff's Law becomes 当 $R < R_0$，有

$$\left(R - R_0 \left(1 - \left(\frac{i}{i_o} \right)^2 \right) \right) i = V_0$$

Writing $i = \sqrt{\dfrac{R_0 - R}{R_0}} i_o + j = i' + j$ where j is small,

代入 $i = \sqrt{\dfrac{R_0 - R}{R_0}} i_o + j = i' + j$，其中 j 为一级小量，

we obtain to linear order in j 保持 j 的一级小量，得

$$V_0 = \left[R - R_0 \left(1 - \left(\frac{i' + j}{i_o} \right)^2 \right) \right] (i' + j) \simeq 2R_0 \sqrt{\frac{R_0 - R}{R_0}} \frac{j}{i_0} (i' + j) \simeq 2(R_0 - R) j \text{, (1 } \textbf{point}\text{)}$$

so $j = \dfrac{V_0}{2(R_0 - R)}$ for both AC and DC. (1 **point**)

最后的（无论是 AC 或 DC）$j = \dfrac{V_0}{2(R_0 - R)}$。

For $R > R_0$, the original current is zero. $R_0 \left(\left(\dfrac{j}{i_o} \right)^2 - 1 \right) \simeq -R_0$.

So we obtain $j = \dfrac{V_0}{(R - R_0)}$. (1 **point**)

当 $R > R_0$，原来的电流为零。因此 $R_0 \left(\left(\dfrac{j}{i_o} \right)^2 - 1 \right) \simeq -R_0$，得 $j = \dfrac{V_0}{(R - R_0)}$。

<<Notice that for $R = R_0$, we obtain 当 $R = R_0$ 时，得

$$R_0 \left(\frac{j}{i_o} \right)^2 j = V \Rightarrow j = \left(\frac{V}{R_0} i_o^2 \right)^{\frac{1}{3}} .$$>>

(ii) In this case we obtain 在此情形，我们有

$$-R_0\left(1-\left(\frac{i_o+j}{i_o}\right)^2\right)(i_o+j)+L\frac{d(i_o+j)}{dt}=V$$, (1 **point**)

Note that 但是 $\frac{di_0}{dt}=0$.

$$-R_0\left(1-\left(\frac{i_o+j}{i_o}\right)^2\right)(i_o+j)=R_0\left(1+\frac{2j}{i_o}-1\right)(i_o+j)=2R_0 j$$

We then obtain 由此得微分方程 $2R_0 j+L\frac{dj}{dt}=V$. (1 **point**)

For AC voltage $V(t)=V_0\sin\omega t$, the solution of this equation is $j=j_o\sin(\omega t+\delta)$ where $\tan\delta=-\frac{L\omega}{2R_o}$ and $j_o=\frac{V_0}{2R_o\cos\delta-L\omega\sin\delta}$. (1 **point**)

(Or $\frac{dj}{dt}=-i\omega j$, and $j_o=\frac{V_0}{2R_o-iL\omega}$)

令 $V(t)=V_0\sin\omega t$ ，解为 $j=j_o\sin(\omega t+\delta)$ ，代入微分方程得 $\tan\delta=-\frac{L\omega}{2R_o}$ ，$j_o=\frac{V_0}{2R_o\cos\delta-L\omega\sin\delta}$。

（或用 $\frac{dj}{dt}=-i\omega j$，代入微分方程得 $j_o=\frac{V_0}{2R_o-iL\omega}$。两种解等价。）

(iii) For small additional voltage source there is additional small amount of charge *q'*. The equation becomes

有小电源时，原来的电荷会增加一小量 *q'*，原来的方程变为

$$-R_0\left(1-\left(\frac{q_o+q'}{q_o}\right)^2\right)\left(\frac{d(q_o+q')}{dt}\right)+\frac{q_o+q'}{C}=V+V'$$ (1 **point**)

Note that $\frac{dq_o}{dt}=0$, and $\frac{q_o}{C}=V$. The above equation then becomes

$$R_0(1+\frac{2q'}{q_0}-1)\frac{dq'}{dt}+\frac{q'}{C}=V'\Rightarrow\frac{q'}{C}=V'.$$

It is true for both AC and DC. (1 **point**)

由于 $\frac{dq_o}{dt}=0$，$\frac{q_o}{C}=V$，上述方程简化成 $R_0(1+\frac{2q'}{q_0}-1)\frac{dq'}{dt}+\frac{q'}{C}=V'\Rightarrow\frac{q'}{C}=V'$。

AC 或 DC 都适用。

(iv) For DC voltage at equilibrium *i* = 0 and the situation is same as (b(ii)). A minimum voltage $V_o=\pm q_o/C$ is needed to maintain the system at equilibrium. (2 **points**)

和(b(ii))相同。$V_o = \pm q_o / C$

(v) In the presence of an additional AC voltage, Kirchhoff's Law becomes

多一个 AC 电源，方程为

$$R_0\left(\left(\frac{q_o+q'}{q_o}\right)^2 + i_o^{-2}\left(\frac{d(q_o+q')}{dt}\right)^2 - 1\right)\left(\frac{d(q_o+q')}{dt}\right) + \frac{q_o+q'}{C} + L\frac{d^2(q_o+q')}{dt^2}$$ (1 **point**)

$$= V + V'$$

Note $\frac{q_o}{C} = V$, and $\frac{dq_o}{dt} = 0$, the first term becomes

由于 $\frac{q_o}{C} = V$ ，$\frac{dq_o}{dt} = 0$ ，上述方程第一项简化成

$$R_0\left(\left(\frac{q_o+q'}{q_o}\right)^2 + i_o^{-2}\left(\frac{d(q_o+q')}{dt}\right)^2 - 1\right)\left(\frac{d(q_o+q')}{dt}\right)$$

$$= R_0\left(\frac{2q'}{q_o} + i_o^{-2}\left(\frac{dq'}{dt}\right)^2\right)\left(\frac{dq'}{dt}\right) = 0$$

So we obtain 最终得 $\frac{q'}{C} + L\frac{d^2q'}{dt^2} = V'$, (1 **point**)

Therefore, for $V'(t) = V_0 \sin\omega t$. We obtain $q'(t) = \frac{V_0}{\left(C^{-1} - L\omega^2\right)}\sin\omega t$. (0.5 **points**)

And $j(t) = \frac{\omega V_0}{\left(C^{-1} - L\omega^2\right)}\cos\omega t$ (0.5 **points**)

用交流形式解代入，得 $q'(t) = \frac{V_0}{\left(C^{-1} - L\omega^2\right)}\sin\omega t$ ，$j(t) = \frac{\omega V_0}{\left(C^{-1} - L\omega^2\right)}\cos\omega t$ 。

Or 或

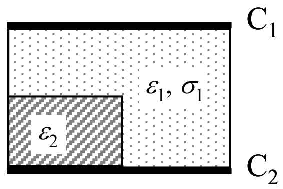
<< $q' = \frac{V_0}{\left(C^{-1} - L\omega^2\right)}$ ，$j = \frac{dq'}{dt} = \frac{-i\omega V_0}{\left(C^{-1} - L\omega^2\right)}$ >>

Pan Pearl River Delta Physics Olympiad 2009
2009 年泛珠三角及中華名校物理奧林匹克邀請賽
Part-1 (Total 5 Problems) 卷-1（共 5 題）
(9:00 am – 12:00 pm, 02-04-2009)

Math hints 數學提示: $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$; $\int \frac{dx}{x} = \ln x + C$.

Q.1 (8 points) 題 1（8 分）

Two large parallel square conductor plates C_1 and C_2 of side length a are separated by two types of media. The cross section is shown in the figure. The distance between the plates is d. Medium-2 is a rectangular insulator slab with dielectric constant ε_2, thickness $d/2$, and side lengths a and $a/2$, respectively. The rest of the space between the plates is filled with medium-1 with conductivity σ_1 and dielectric constant ε_1. Find the resistance and capacitance between the two plates.



兩塊邊長爲 a 的正方形平行導電板 C_1 和 C_2 之間夾著兩種介質，其截面如圖所示。導電板間的距離爲 d。介質-2 爲一厚度爲 $d/2$，邊長分別爲 a 和 $a/2$ 的絕緣長方形板，其介電常數爲 ε_2。導電板之間其餘的空間充滿了介質-1，其介電常數爲 ε_1，電導率爲 σ_1。求兩導電板之間的電阻和電容。

Q.2 (10 points) 題 2（10 分）

A galaxy is made of many stars revolving around its center on circular orbits. One way scientists study galaxies is to measure the relation between the revolving speed v of a star in the galaxy and its distance r from the center. Expressing the relation as $v \propto r^n$, scientists pay special attention to the exponential power n.

星系由很多繞中心作圓形軌道運行的恒星組成。科學家研究星系的一個方法是測量恒星在星系中的運行速度 v 和離星系中心的距離 r。用 $v \propto r^n$ 這樣的關係來表達，科學家們特別關心指數 n。

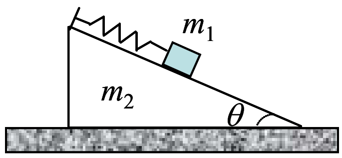
(a) If the gravity force on the star under investigation is mainly from a massive black hole at the galaxy center, derive the value of n. (4 points)
若作用于恒星的引力主要來自星系中心的巨型黑洞，求 n 的值。（4 分）

(b) Suppose there is no black hole at the center and the stars in the galaxy are evenly distributed such that the mass of the galaxy is evenly distributed in a flat disk. Consider a star in the disk at distance r from the center. The total gravity force on the star from the stars enclosed by its circular orbit can be treated as if they are concentrated at the center. The total gravity force on the star from the stars outside the orbit is zero. (This is like the case of a uniform density sphere.) Derive the value of n. (4 points)
若星系中心無黑洞，星系的恒星分布均勻，整個星系可當作是一個質量均勻分布的扁圓盤。對于一個離中心爲 r 的恒星來講，來自軌道內的恒星的引力與把它們集中在星系中心的一樣，來自軌道外的恒星的引力合力爲零。（類似于均勻質量球的情形）求 n 的值。（4 分）

(c) What scientists found from real galaxies is that for most of them, n is larger than 1. Give your brief explanation to this scientific puzzle. (2 points)
科學家們發現，很多實際星系的 n >1。請簡單解釋。（2 分）

Q.3 (10 points) 題 3（10 分）

As shown, a block of mass m_1 is connected to a spring of force constant k on the smooth slope (inclination angle θ) of a wedge of mass m_2 placed on a smooth floor. Given a small disturbance to the block and the system starts to oscillate. During the oscillation motion the block keeps in touch with the slope, and the wedge maintains contact with the floor. Find the oscillation frequency, and check your answer for two special cases of $\theta = 0$ and $\theta = 90°$.



如圖所示，一放在光滑地板上質量爲 m_2 的大物塊的光滑斜面上（傾角爲 θ）有一力常數爲 k 的彈簧，彈簧上系有質量爲 m_1 的小物塊。給小物塊一輕微擾動後，系統開始振蕩。在振蕩期間小物塊始終與斜面保持接觸，大物塊始終與地板保持接觸。求振蕩頻率，并用 $\theta = 0$ 和 $\theta = 90°$ 這兩個特殊情形檢驗你的答案。

Q.4 (10 points) 題 4（10 分）

An electromagnetic (EM) wave is propagating along the $\vec{z}_0$ direction in a non-magnetic conducting medium with conductivity σ and dielectric constant ε. Both constants are real numbers. Its electric field is $\vec{E}(z,t) = E_0 \vec{x}_0 e^{i(\tilde{k}z - \omega t)}$, where E_0 and ω are real constants while

$$\tilde{k} = \frac{\omega}{c}\sqrt{\varepsilon} + i\frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon\varepsilon_0}}.$$

一電磁波在非磁性導電介質中沿 $\vec{z}_0$ 方向傳播。介質的導電率爲實數 σ，介電常數爲實數 ε。電磁波的電場爲 $\vec{E}(z,t) = E_0 \vec{x}_0 e^{i(\tilde{k}z - \omega t)}$，其中 $\tilde{k} = \frac{\omega}{c}\sqrt{\varepsilon} + i\frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon\varepsilon_0}}$，$E_0$ 和 ω 爲實數。

(a) Find the magnetic field of the EM wave. (2 points)
求電磁波的磁場。（2 分）

(b) Find the time-averaged (over one period) Poynting's vector $<\vec{S}> = \frac{1}{\mu_0} < (\vec{E}\times\vec{B}) >$. (2 points)
求經過一個周期的時間平均的 Poynting 矢量 $<\vec{S}> = \frac{1}{\mu_0} < (\vec{E}\times\vec{B}) >$。（2 分）

(c) The quantity $q = -\frac{d<S>}{dz}$ describes the loss of EM wave energy to a unit volume of the medium. Calculate q. (2 points)
$q = -\frac{d<S>}{dz}$ 代表電磁波輸送給單位體積的介質的能量。求 q。（2 分）

(d) Find the power consumed by a unit volume of the medium by directly calculating the Joule Heat averaged over one period of the wave $<\vec{J}\cdot\vec{E}>$, where $\vec{J}$ is the current density. (2 points)
求單位體積的介質以焦耳熱形式經過一個周期的時間平均的能量消耗 $<\vec{J}\cdot\vec{E}>$，其中 $\vec{J}$ 爲電流密度。（2 分）

(e) Compare the answers in (c) and (d) and explain why they are equal or not equal. (2 points)
比較(c)和(d)的答案，解釋它們相等或不相等的原因。（2 分）

Q.5 (12 points) 題 5（12 分）

Consider a **non-ideal** gas with internal energy U given by $U = 3PV$, where P and V are the pressure and volume of the gas, respectively.

一**非理想**氣體的內能爲 $U = 3PV$ ，其中 P 爲氣體的壓强，V 爲氣體的體積。

(a) Find the relation between P and V in an adiabatic process. (2 points)
求絕熱過程 P 與 V 的關係式。（2 分）

Experiments show that the temperature T of the gas depends only on P, irrespective of its volume. The scale of the temperature can be fixed by setting $T = 1$ when $P = 1$. Consider the following Carnot Process

實驗證明該氣體的溫度只與壓强有關，而與體積無關。溫度的標度可用當 $P = 1$ 時 $T = 1$ 來定。考慮以下的卡諾過程

$$(P_1,V_1) \overbrace{\rightarrow}^{\text{Isothermal}} (P_1,V_2) \overbrace{\rightarrow}^{\text{Adiabatic}} (P_2,V_3) \overbrace{\rightarrow}^{\text{Isothermal}} (P_2,V_4) \overbrace{\rightarrow}^{\text{Adiabatic}} (P_1,V_1)$$

（Isothermal = 等溫；Adiabatic = 絕熱）

(b) Calculate the heat Q_1 absorbed by the gas in the first isothermal process. (1 point)
求第一個等溫過程氣體吸收的熱量 Q_1。（1 分）

(c) Calculate the heat Q_2 absorbed by the gas in the second isothermal process. (1 point)
求第二個等溫過程氣體吸收的熱量 Q_2。（1 分）

(d) The temperature T can be defined as $\frac{T_1}{T_2} = -\frac{Q_1}{Q_2}$. Find the $P \sim T$ relation. (4 points)
溫度可定義爲 $\frac{T_1}{T_2} = -\frac{Q_1}{Q_2}$。求 P 與 T 的關係式。（4 分）

(e) Find the heat capacity at constant volume of the gas. (2 points)
求氣體的恒定體積熱容量。（2 分）

(f) Find the entropy S of the gas in terms of P and V, given that $S = 0$ when $T = 0$. (2 points)
已知氣體的熵 S 在溫度 $T = 0$ 時爲零。求氣體的熵，幷以 P 和 V 表達。（2 分）

THE END 完

Pan Pearl River Delta Physics Olympiad 2009
2009 年泛珠三角及中華名校物理奧林匹克邀請賽
Part-2 (Total 3 Problems) 卷-2（共 3 題）
(2:30 pm – 5:30 pm, 02-04-2009)

Math hints 數學提示: $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$; $\int \frac{dx}{x} = \ln x + C$.

Q1 Solar System in the Early Days (16 points) 題 1 早期的太陽系（16 分）

In the early days of our solar system when the planets were not yet formed, the sun was surrounded by a big ball of gas at rest. The mass of the sun is M_s. Assume for simplicity that the gas is ideal and is made of single type of molecules of molecular mass m, and the total mass of the gas is much less than M_s, you are to find the distribution of the gas at a distance r (>> diameter of the sun) from the sun.

早期的太陽系行星還沒有形成時，太陽被一大團氣體包圍著。已知太陽的質量爲 M_s。爲簡單起見設氣體是理想的，幷由單種分子質量爲 m 的分子組成。氣體的總質量遠小于 M_s。你需要求出離太陽距離 r（>> 太陽半徑）處氣體的分布。

(a) Suppose the gas is all at the same temperature T_0. The mass density distribution of the gas can be expressed as $\rho = \rho_0 e^{\alpha/r}$. Find α. (3 points)

假設氣體的溫度爲 T_0 幷且是均勻的，氣體的質量密度分布可表達成 $\rho = \rho_0 e^{\alpha/r}$ ，求α。（3 分）

(b) There is a major flaw in the mass density expression in (a). Point out the flaw. (2 points)

上述氣體的質量分布表達式有一重大缺陷。指出缺陷。（2 分）

(c) In an improved model, suppose the sun emits J_0 amount of thermal energy per second, and there is no energy loss when the thermal energy flows from the sun out to the gas via thermal conduction, find the energy current density (energy through a unit area per second) at a distance r from the sun. (3 points)

在改良的模型裏，假設太陽每秒釋放出的熱能爲 J_0，當熱能以熱傳導方式從太陽傳給氣體時無損失，求離太陽距離 r 處熱能量流强度（每秒穿過單位面積的能量）。（3 分）

(d) The thermal energy current density $I(r)$ in (c) is proportional to the temperature gradient. That is, $I(r) = -\sigma \frac{dT}{dr}$, where σ is a positive constant called heat conductivity. The minus sign comes from the fact that heat always flows from high T region to low T region. Find the temperature at distance r from the sun. (3 points)

上述熱能量流强度 $I(r)$與溫度梯度成正比，既 $I(r) = -\sigma \frac{dT}{dr}$ ，其中 σ 爲熱導率，是正數，式中的負號是因爲熱流總是從高溫區域流向低溫區域的。求離太陽距離 r 處溫度。（3 分）

(e) The pressure can now be expressed as $P = P_0 \left(r / r_0 \right)^{-\beta}$. Find β and the mass density distribution. (3 points)

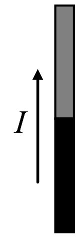
氣體的壓强可表述爲 $P = P_0 \left(r / r_0 \right)^{-\beta}$ ，求 β 以及質量密度分布。（3 分）

(f) From (d) one can see that beyond certain distance r_0 from the sun the temperature is below the ice temperature. Estimate r_0 in terms of the radius of a planet orbit of the present solar system. (2 points)

由(d)可知超過某距離 r_0 氣體的溫度會低于水的冰點。r_0 大概與現有的哪個行星的軌道半徑相近？（2 分）

Q2 Spin Torque (16 points) 題 2 自旋力矩（16 分）

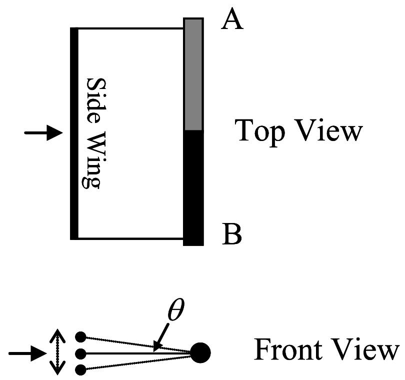
An electron has permanent angular momentum called **spin** ($\vec{S}$) that is fixed in amplitude $h/4\pi$, where h is the Planck Constant, while its direction can be changed by external interactions. Consider a thin wire with electric current I, the lower half of which is made of a magnetized conductor and the upper half non-magnetic conductor. Due to quantum mechanics, the electron spins can only point to one of the two directions, namely in the electric current direction or opposite to it. In the magnetized section of the wire there are α (> 50%) portion of the electrons with spins in the current direction, while in the non-magnetic section there are equal numbers of electrons with spins in the two directions. The extra number of electrons with spins in the current direction in the magnetic section will instantly flip to the opposite direction once they cross the boundary and enter the non-magnetic section. As a result, a net torque is exerted on the wire.



電子具有固有的角動量$\vec{S}$，叫做自旋，其方向可在與外界的相互作用下改變，其大小爲固定的$h/4\pi$，其中 h 爲普朗克常數。一導綫載有電流 I，導綫的下半部分由磁化了的導體組成，上半部分由非磁性導體組成。由量子力學得知，電子的自旋只能與電流同方向，或與電流反方向。磁化導體內有α（> 50%）部分的電子的自旋與電流同方向。非磁性導體內自旋與電流同方向和反方向的電子具有相同數量。磁化導體內多出的自旋與電流同方向的電子流進非磁性導體後會立刻改變它們的方向，從而產生力矩作用在導綫上。

(a) Find the torque in terms of h, I, α, and electron charge e. (4 points)
以 h, I, α, 和電子電荷 e 來表達力矩。（4 分）

(b) To measure the torque, a conducting side wing is attached rigidly to the wire but is electrically insulated from it, as shown in the figure. The wire is suspended between two fixed points A and B. Under constant current I_0 the spin torque will twist the entire wire along the AB axis up to a small angle θ_0, until it is balanced by the elastic restoring torque $\tau_e = -\kappa\theta_0$, where κ is a positive constant, at the two fixed points. Given that the moment of inertia of the entire structure with respect to the AB axis is J, and there is also a friction torque of $-\eta\frac{d\theta}{dt}$, where η is a positive constant, find the angle θ as a function of time t after the current is suddenly turned off at $t = 0$. (6 points)



爲了測量力矩，我們在導綫上裝上一導電邊翼(side wing) ，如俯視圖(top view)和正視圖(front view)所示。邊翼與導綫的連接是剛性的，但相互間不導電。導綫的兩端固定在 A、B 兩點，中間懸空。有恒定電流 I_0時，自旋力矩將導綫以 AB 爲軸扭轉一小角θ_0，直到被 A、B 點的彈性力矩$\tau_e = -\kappa\theta_0$抵消爲止，其中κ是正常數。已知整個結構以 AB 爲軸的轉動慣量爲 J，摩擦力矩爲$-\eta\frac{d\theta}{dt}$，其中η爲正常數。在時間 $t = 0$ 時電流被突然切斷，求θ之後的演化。（6 分）

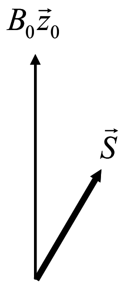
(c) When an alternating electric current $I = I_0 e^{i\omega t}$ is applied, the wire will twist back and forth. The side wing of length L at a distance d from the wire follows the twisting motion of the wire. A magnetic field B is applied perpendicular to the side wing, as shown by the solid arrow in the figure. The field will not change the spins. Find the electromotive potential between the two ends of the side wing. (6 points)

將導綫通上交流電 $I = I_0 e^{i\omega t}$ ，則導綫會來回扭動，帶動邊翼上下擺動。已知邊翼的長度爲 L，離導綫的距離爲 d，外加恒穩磁場場强爲 B，方向與邊翼垂直，如圖中實綫箭頭所示。該磁場不會改變電子的自旋。求邊翼兩端之間的電動勢。（6 分）

Q3 Nuclear Magnetic Resonance (18 points) 題 3 核磁共振 (18 分)

Like the electrons, most nuclei have permanent angular momentum called **spins** ($\vec{S}$) that are fixed in amplitude while their directions can be changed by an external torque. Associated with the spins are the magnetic moments that make them as individual magnetic dipoles $\vec{m} = \mu\vec{S}$, where μ is a constant. Here we will study the spin dynamics using classical mechanics.

大多數原子核象電子那樣具有叫作自旋的固有角動量 $\vec{S}$ ，其大小固定，方向隨外力矩的作用而改變。由自旋産生的磁偶極子爲 $\vec{m} = \mu\vec{S}$ ，其中 μ 爲常數。下面我們用經典力學研究自旋的運動。



(a) A nucleus with spin $\vec{S}$ is placed in a static magnetic field $B_0\vec{z}_0$ in a laboratory, as shown in the figure. Find the motion of the spin of the nucleus as seen in the laboratory reference frame. (4 points)
如圖，一自旋爲 $\vec{S}$ 的原子核處于實驗室的靜磁場 $B_0\vec{z}_0$ 中。求在實驗室參照系觀察到的自旋的運動。（4 分）

(b) In a rotating reference frame with angular velocity $\vec{\omega}_0$ the nucleus spin appears stationary. Find such reference frame. (2 points)
在一個以角速度 $\vec{\omega}_0$ 旋轉的參照系觀察到的自旋是固定不動的。找出該參照系。（2 分）

(c) The correct answer to (b) indicates that in a rotating reference frame with angular velocity $\vec{\omega}$ there is an additional magnetic field $\vec{B}_\omega$ seen by the spin. Find the additional magnetic field. (2 points)
上述(b)的正確答案說明，在一個以角速度 $\vec{\omega}$ 旋轉的參照系裏，自旋會感受到一個額外的磁場 $\vec{B}_\omega$ 。求額外磁場。（2 分）

(d) Back in the laboratory reference frame a second magnetic field is applied. The second field with strength B_1 is always perpendicular to $\vec{z}_0$ but its direction rotates in the XY plane at constant angular speed $-\omega_1$. Switching to the reference frame that rotates with the second magnetic field, find the total magnetic field seen by the nucleus. (3 points)
在實驗室參照系，我們加上另一個磁場。該磁場强度爲 B_1，其方向與 $\vec{z}_0$ 垂直，幷在 XY 平面內以均勻角速度 $-\omega_1$ 旋轉。在隨著該磁場旋轉的轉動參照系裏，求自旋感受到的總磁場。（3 分）

(e) When $\omega_0 = \omega_1$, find the minimum time it takes to flip the spin from $+\vec{z}_0$ to $-\vec{z}_0$. (This situation is called the Nuclear Magnetic Resonance) (4 points)
當 $\omega_0 = \omega_1$ 時，求將自旋從 $+\vec{z}_0$ 方向轉到 $-\vec{z}_0$ 方向所需的最短時間。（這情形稱爲核磁共振）（4 分）

(f) When $\omega_0 \neq \omega_1$, find the motion of the spin observed in the rotating frame. (3 points)
當 $\omega_0 \neq \omega_1$ 時，求在轉動參照系裏觀察到的自旋運動。（3 分）

THE END 完

Part-I 第一卷

Q1 題 1 **(8 points 8 分)** Solution 解:

For resistance R, only half of the area is conducting, as the other half is blocked by medium-2. Let the voltage between the plates be V, then the electric field

先求電阻 R。介質-2 不導電，所以只有一半的導電板導電。令兩板之間的電壓爲 V，則電場爲

$E = V/d$, (1 point 1 分)

The current density is 電流密度爲 $J = \sigma_1 E$, (1 point 1 分)

The current is 電流爲 $I = \frac{1}{2} J a^2 = \frac{a^2}{2} \sigma_1 \frac{V}{d}$. (1 point 1 分)

So the resistance is 因此電阻爲 $R = \frac{V}{I} = \frac{2d}{\sigma_1 a^2}$. (1 point 1 分)

For capacitance, it can be treated as two capacitors in parallel. The capacitance on the right is

再求電容。總電容可當作是左右兩個電容幷聯。右邊的電容爲

$C_1 = \frac{\varepsilon_0 \varepsilon_1 a^2}{2d}$. (1 point 1 分)

For the left half, let the electric displacement be D_2 which is the same throughout the region.

在左半邊，令電位移爲 D_2，電位移處處相同。 (1 point 1 分)

The total free charge on the left half is

左半邊的總自由電荷爲

$Q = \frac{a^2}{2} D_2$. (1 point 1 分)

The total voltage between the two plates is

兩導電板之間的總電壓爲

$V = \frac{1}{2} E_1 d + \frac{1}{2} E_2 d = \frac{d}{2\varepsilon_0} D_2 (\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2})$. (1 point 1 分)

So 因此 $C_2 = \frac{\varepsilon_0 a^2}{d} \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2}$,

$C = C_1 + C_2 = \frac{\varepsilon_0 a^2}{d} \left(\frac{\varepsilon_1}{2} + \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \right)$. (1 point 1 分)

Q2 題 2 (10 points 10 分) solution 解:

(a) Let the mass of the blackhole be M, then

令黑洞的質量爲 M，則

$\frac{GM}{r^2} = \frac{v^2}{r}$, (2 points 2 分)

So 因此 $\sqrt{\frac{GM}{r}} = v$, (1 point 1 分)

$n = -\frac{1}{2}$. (1 point 1 分)

(b) Let the areal mass density be σ, then

令質量面密度爲σ，則

$\frac{G\pi\sigma r^2}{r^2} = \frac{v^2}{r}$, (2 points 2 分)

So 因此 $\sqrt{G\pi\sigma r} = v$, (1 point 1 分)

$n = \frac{1}{2}$. (1 point 1 分)

(c) Anything reasonable is fine. It DOES NOT have to be dark matter.

任何有一定理由的解釋都行，不一定非暗物質不可。(2 points 2 分)

Q3 題 3 (10 points 10 分) Solution 解:

Method 1: Using conservation of energy

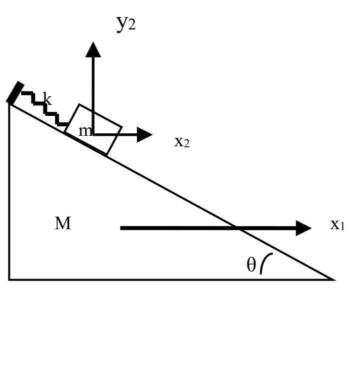
方法-1：利用能量守恒

Equations 方程:

$Mx_1 = -mx_2$, $y_2 = (x_2 - x_1)\tan\theta = (1 + \frac{m}{M})x_2 \tan\theta$

(1 point 1 分)

All coordinates are in the rest frame. 所有坐標取自靜止參照系。



The total kinetic energy of the system is

總動能爲

$$T=\frac{1}{2}M\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2+\frac{1}{2}m\dot{y}_2^2 \quad \text{(1 point 1 分)}$$

$$=\frac{1}{2}M\frac{m^2}{M^2}\dot{x}_2^2+\frac{1}{2}m\dot{x}_2^2+\frac{1}{2}m\tan^2\theta\bullet\dot{x}_2^2(1+\frac{m}{M})^2 \quad \text{(1 point 1 分)}$$

The total potential energy 總勢能爲:

$$V=\frac{1}{2}k[(x_2-x_1)^2+y_2^2]-mgy_2 \quad \text{(1 point 1 分)}$$

$$=\frac{1}{2}k[(1+\frac{m}{M})^2x_2^2+(1+\frac{m}{M})^2x_2^2\tan^2\theta]-mg(1+\frac{m}{M})x_2\tan\theta$$

$$=\frac{1}{2}kx_2^2(1+\frac{m}{M})^2\bullet\frac{1}{\cos^2\theta}-mg(1+\frac{m}{M})x_2\tan\theta \quad \text{(1 point 1 分)}$$

Using 利用 $ax^2-bx=a(x-\frac{b}{2a})^2-\frac{b^2}{4a}$, we get 得

$$V=\frac{1}{2}k(1+\frac{m}{M})^2\square\frac{1}{\cos^2\theta}(x_2-\frac{mMg\sin2\theta}{k(M+m)})^2-\frac{(mg\sin\theta)^2}{2k} \quad \text{(1 point 1 分)}$$

Make a transfer of coordinate 取坐標變換

$$x\equiv x_2-\frac{mMg\sin2\theta}{k(M+m)}, \text{(1 point 1 分)}$$

We reach an expression for the total energy that is of the form of $T+V=ax^2+b\dot{x}^2+c$ which must be constant by energy conservation. Let $x=A\cos(\omega t+\phi)$, we get

$$T+V=aA^2\cos^2(\omega t+\phi)+bA^2\omega^2\sin^2(\omega t+\phi)+c=A^2(a-b\omega^2)\cos^2(\omega t+\phi)+bA^2\omega^2+c.$$

For the total energy to be constant the $\cos^2(\omega t+\phi)$ term must be zero all the time, which leads to $\omega^2=a/b$.

我們得到總能量的表達式爲 $T+V=ax^2+b\dot{x}^2+c$ 。因能量守恒總能量應爲常數。令 $x=A\cos(\omega t+\phi)$ ，得

$$T+V=aA^2\cos^2(\omega t+\phi)+bA^2\omega^2\sin^2(\omega t+\phi)+c=A^2(a-b\omega^2)\cos^2(\omega t+\phi)+bA^2\omega^2+c$$

作爲常數，上式中 $\cos^2(\omega t+\phi)$ 項必須爲零。因此 $\omega^2=a/b$ 。

Therefore, the oscillation frequency ω in this case is

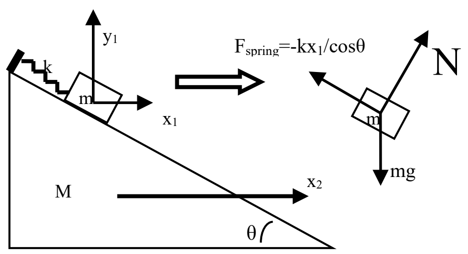
由上式得系統的頻率ω爲

$$\omega^2 = \frac{k(1+\frac{m}{M})^2/\cos^2\theta}{\frac{m}{M^2}+m+m(1+\frac{m}{M})\tan^2\theta} = \frac{k}{m}\frac{1+\frac{m}{M}}{\cos^2\theta+(1+\frac{m}{M})\sin^2\theta} = \frac{k}{m}\left(\frac{M+m}{M+m\sin^2\theta}\right).$$ (1 point 1 分)

For $\theta = 0$, we have 當$\theta = 0$，得$\omega^2 = k\left(\frac{1}{m}+\frac{1}{M}\right)$ (1 point 1 分),

For $\theta = 90°$ we have 當$\theta = 90°$，得$\omega^2 = \frac{k}{m}$. (1 point 1 分)

Method 2: Analytical Mechanics 方法-2：分析力學



Force figure 力圖 (2 points 2 分)

Equations 方程

$$m\ddot{x}_1 = -\frac{kx_1}{\cos\theta}\cos\theta - N\sin\theta + m\ddot{x}_2,$$

x_1 is in the frame on the slope. x_1是相對與斜面的橫坐標。 (1 point 1 分)

$$y_1 = x_1\tan\theta$$

$$m\ddot{x}_1\tan\theta = N\cos\theta - mg - kx_1\tan\theta$$ (1 point 1 分)

$$M\ddot{x}_2 = -(N\sin\theta + kx_1)$$ (1 point 1 分)

Process 解方程過程

Step 1: Eliminate $\ddot{x}_2$ 步驟-1：消去$\ddot{x}_2$

$$m\ddot{x}_1 = -kx_1 - N\sin\theta - \frac{m}{M}(N\sin\theta + kx_1)$$

$$m\ddot{x}_1\tan\theta = N\cos\theta - mg - kx_1\tan\theta$$

Step 2: Eliminate N　　步驟-2：消去 N

$$m\ddot{x}_1 + (1+\frac{m}{M})kx_1 = -N\sin\theta(1+\frac{m}{M})$$

$$m\ddot{x}_1\tan\theta + mg + kx_1\tan\theta = N\cos\theta$$

Which means 整理後得

$$m\cos\theta\,\ddot{x}_1 + \cos\theta(1+\frac{m}{M})kx_1 = -\sin\theta(1+\frac{m}{M})[m\tan\theta\,\ddot{x}_1 + mg + kx_1\tan\theta] \quad \text{(2 points 2 分)}$$

By assuming $x_1 = Ae^{i\omega t}$, the oscillation frequency ω is obtained

設解 $x_1 = Ae^{i\omega t}$，得頻率ω

$$\omega^2 = \frac{k(1+\frac{m}{M})}{m}\frac{\cos\theta + \frac{\sin^2\theta}{\cos\theta}}{\cos\theta + (1+\frac{m}{M})\frac{\sin^2\theta}{\cos\theta}} = \frac{k}{m}\left(\frac{M+m}{M+m\sin^2\theta}\right). \quad \text{(1 point 1 分)}$$

For $\theta = 0$, we have 當 $\theta = 0$，得 $\omega^2 = k\left(\frac{1}{m}+\frac{1}{M}\right)$ (1 point 1 分),

For $\theta = 90°$ we have 當 $\theta = 90°$，得 $\omega^2 = \frac{k}{m}$. (1 point 1 分)

Q4 題 4 (10 points 10 分)　　Solution 解:

(a) Let the magnetic field be $\vec{B}(z,t) = \vec{B}_0 e^{i(\tilde{k}z-\omega t)}$. The k-vector of the wave is $\vec{k} = \tilde{k}\vec{z}_0$.

Using the equation $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$,

令磁場的表達式爲 $\vec{B}(z,t) = \vec{B}_0 e^{i(\tilde{k}z-\omega t)}$ 。k-矢量爲 $\vec{k} = \tilde{k}\vec{z}_0$ 。利用方程 $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

we get 得 $\vec{B}(z,t) = \frac{\vec{k}\times\vec{E}(z,t)}{\omega}$, (1 point 1 分)

$$= \frac{\tilde{k}}{\omega}E_0(\vec{z}_0\times\vec{x}_0)e^{i(\tilde{k}z-\omega t)} = \frac{\tilde{k}}{\omega}E_0\vec{y}_0 e^{i(\tilde{k}z-\omega t)}$$

So 因此 $\vec{B}_0 = \dfrac{\tilde{k}}{\omega} E_0 \vec{y}_0$. (1 point 1 分)

(b) Note that 利用 $\tilde{k} = \dfrac{\omega}{c}\sqrt{\varepsilon} + i\dfrac{\sigma}{2}\sqrt{\dfrac{\mu_0}{\varepsilon\varepsilon_0}} \equiv k_R + ik_I$.

So 得 $<\vec{S}> = \dfrac{1}{\mu_0} < (\vec{E}\times\vec{B}) > = \dfrac{1}{2\mu_0}\mathrm{Re}(\vec{E}\times\vec{B}^*)$ (1 point 1 分)

$= \dfrac{E_0^2}{2\mu_0\omega}\vec{z}_0\,\mathrm{Re}(k_R - ik_I)e^{-2k_I z} = \dfrac{1}{2\mu_0}\dfrac{k_R}{\omega}E_0^2\vec{z}_0 e^{-2k_I z} = \dfrac{1}{2\mu_0 c}E_0^2\vec{z}_0 e^{-2k_I z}$ (1 point 1 分)

(c) $q = -\dfrac{d<S>}{dz} = \dfrac{k_I k_R}{\mu_0\omega}E_0^2 e^{-2k_I z} = \dfrac{\sigma}{2}\dfrac{\frac{\omega}{c}\sqrt{\varepsilon}\sqrt{\frac{\mu_0}{\varepsilon\varepsilon_0}}}{\mu_0\omega}E_0^2 e^{-2k_I z} = \dfrac{\sigma}{2}E_0^2 e^{-2k_I z}$. (2 points 2 分)

(d) Joule Heat 焦爾熱 $= <\vec{J}\cdot\vec{E}> = \dfrac{1}{2}\mathrm{Re}(\vec{J}\cdot\vec{E}^*) = \dfrac{\sigma}{2}\mathrm{Re}(\vec{E}\cdot\vec{E}^*) = \dfrac{\sigma}{2}E_0^2 e^{-2k_I z}$ (2 points 2 分)

(e) The energy loss of EM wave is equal to the Joule Heat.

電磁波能量的損失等于焦爾熱。(2 points 2 分)

Q5 (12 points) 題 5（12 分） Solution 解：

(a) First law 熱力學第一定律: $dU = -PdV + \delta Q$.

The equation of adiabatic processes is 絕熱過程的方程式爲

$\delta Q = dU + PdV = d(3PV) + PdV = 4PdV + 3VdP = 0$ (1 point 1 分)

$\Rightarrow PV^{4/3} = \text{Constant}$

$\Rightarrow PV^{4/3} =$ 常數 (1 point 1 分)

(b) In the Carnot cycle 在卡諾循環過程中:

$$(P_1,V_1)\overbrace{\rightarrow}^{\text{Isothermal}}(P_1,V_2)\overbrace{\rightarrow}^{\text{Adiabatic}}(P_2,V_3)\overbrace{\rightarrow}^{\text{Isothermal}}(P_2,V_4)\overbrace{\rightarrow}^{\text{Adiabatic}}(P_1,V_1).$$

Isothermal = 等溫；Adiabatic = 絕熱

The heat supplied to the gas during the first isothermal process is

第一個等溫過程中吸收的熱量爲

$$Q_1 = 3P_1V_2 - 3P_1V_1 + \int_{V_1}^{V_2} P_1 dV = 4P_1\left(V_2 - V_1\right). \text{ (1 point 1 分)}$$

(c) Similarly, the heat supplied to the gas during the second isothermal process is

第二個等溫過程中吸收的熱量爲

$$Q_2 = 4P_2\left(V_4 - V_3\right). \quad \text{(1 point 1 分)}$$

(d) From (a) we have 由(a)得:
$$\begin{aligned} P_1V_2^{4/3} &= P_2V_3^{4/3} \\ P_2V_4^{4/3} &= P_1V_1^{4/3} \end{aligned} \quad \text{(1 point 1 分)}$$

By definition 由定義

$$\frac{T_1}{T_2} = -\frac{Q_1}{Q_2} = -\frac{P_1\left(V_2 - V_1\right)}{P_2\left(V_4 - V_3\right)} = -\frac{P_1^{1/4}P_2^{3/4}\left(V_3 - V_4\right)}{P_2\left(V_4 - V_3\right)} = \frac{P_1^{1/4}}{P_2^{1/4}}. \quad \text{(2 points 2 分)}$$

Therefore, one may define the absolute temperature by $T = AP^{1/4}$, where A is an arbitrary constant. Since $T = 1$ when $P = 1, T = P^{1/4}$.

因此我們可以定義溫度爲$T = AP^{1/4}$，其中 A 爲任意常數。由 $P = 1$ 時 $T= 1$，得$T = P^{1/4}$ 。

(1 point 1 分)

(e) The internal energy is then 內能爲$U = 3T^4V$. (1 point 1 分)

Hence the heat capacity is 因此熱容量爲$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 12T^3V$. (1 point 1 分)

(f) The entropy is 熵爲$S = \int_0^T C_V \frac{dT}{T} = 12V\int_0^T T^2 dT = 4T^3V = 4P^{3/4}V$. (2 points 2 分)

Part-II 第二卷

Q1 (16 points) 題 1（16 分） Solution 解：

(a) Consider a thin layer of gas of unit area and thickness dr. The pressure at r should be larger than the pressure at $r + dr$ in order to balance the gravity $\frac{GMmn}{r^2}$. So we have $\frac{dP}{dr} = -\frac{GM_s mn}{r^2}$, where n is the molecular number density of the gas.

考慮離太陽 r 處一厚度爲 dr 的單位面積氣體。在 r 處的氣壓應比在 $r + dr$ 處的大一點，從而平衡太陽的引力 $\frac{GMmn}{r^2}dr$ 。因此有 $\frac{dP}{dr} = -\frac{GM_s mn}{r^2}$ ，其中 n 爲氣體的分子數密度。(1 point 1 分)

We also have the ideal gas law $P = nkT_0$. Replace P with n we get $\frac{dn}{n} = -\frac{GM_s m}{kT_0}\frac{dr}{r^2}$.

另有理想氣體方程 $P = nkT_0$。將 P 用 n 代入，得 $\frac{dn}{n} = -\frac{GM_s m}{kT_0}\frac{dr}{r^2}$ 。(1 point 1 分)

Finally, $\rho = \rho_0 e^{\alpha/r}$, where $\alpha = \frac{GM_s m}{kT_0}$.

最後得 $\rho = \rho_0 e^{\alpha/r}$ ，其中 $\alpha = \frac{GM_s m}{kT_0}$ 。(1 point 1 分)

(b) When $r -> \infty$, $\rho -> \rho_0$ instead of zero. That means the gas ball is infinitely large, which is unphysical.

當 $r -> \infty$ ，$\rho -> \rho_0$ 而不是零。這意味著氣體球是無限大的，不符合實際情況。(2 points 2 分)

(c) The amount of energy per second through any concentric sphere shells should be constant. That is, $J_0 = 4\pi r^2 I$.

每秒鐘穿過任意一個同心圓殼的能量應爲常數。所以 $J_0 = 4\pi r^2 I$ 。(1 point 1 分)

Then 從而得 $\frac{J_0}{4\pi r^2} = I$. (2 points 2 分)

(d) $I(r) = -\sigma\frac{dT}{dr}$, so 因此 $\frac{dT}{dr} = -\frac{J_0}{4\pi\sigma r^2}$, (1 point 1 分)

Then 由此得$T = \dfrac{J_0}{4\pi\sigma r}$.

The integral constant should be zero, as T should be zero at large distance.

因爲在無窮遠處溫度須爲零，所以由積分産生的常數必須爲零。(2 points 2 分)

(e) Again 重力和壓力平衡,$\dfrac{dP}{dr} = -\dfrac{GM_s mn}{r^2}$.

But now $P = nkT(r) = \dfrac{J_0}{4\pi\sigma r}kn$. Replace n by P in the first equation, we get $\dfrac{dP}{dr} = -\dfrac{4\pi GM_s m\sigma P}{kJ_0 r}$,

現在$P = nkT(r) = \dfrac{J_0}{4\pi\sigma r}kn$ 。將 P 代入 n，得$\dfrac{dP}{dr} = -\dfrac{4\pi GM_s m\sigma P}{kJ_0 r}$ (1 point 1 分)

which leads to $P = P_0\left(\dfrac{r}{r_0}\right)^{-\beta}$, where $\beta = \dfrac{4\pi GM_s m\sigma}{kJ_0}$.

從而得$P = P_0\left(\dfrac{r}{r_0}\right)^{-\beta}$，其中$\beta = \dfrac{4\pi GM_s m\sigma}{kJ_0}$ 。(1 point 1 分)

$\rho = \dfrac{4\pi\sigma mrP_0}{kJ_0}\left(\dfrac{r}{r_0}\right)^{-\beta}$. (1 point 1 分)

This time P and ρ go to zero at large r. 現在的 P 和ρ在無窮遠處爲零。

(f) From the surface temperatures of the planets we know today we estimate that r_0 is about the radius of the orbit of Mars.

由現在各行星的表面溫度我們可以推測大概和火星的軌道相近。(2 points 2 分)

Q2 (16 points) 題 2（16 分） Solution 解：

(a) The number of electrons crossing the junction per second is I/e.

每秒鐘通過界面的電子數爲 I/e。(1 point 1 分)

On average, there are $(\alpha - 0.5)I/e$ electrons flip their spins.

平均有$(\alpha - 0.5)I/e$的電子的自旋反轉。(1 point 1 分)

The net angular momentum change per second is then $(\alpha - 0.5)\dfrac{Ih}{4\pi e}\times 2$.

每秒鐘角動量的淨變化爲 $(\alpha - 0.5)\dfrac{Ih}{4\pi e}\times 2$ 。(1 point 1 分)

This is equal to the torque so $\tau = (\alpha - 0.5)\dfrac{h}{2\pi e}I$.

角動量的淨變化等于力矩 $\tau = (\alpha - 0.5)\dfrac{h}{2\pi e}I$ 。(1 point 1 分)

(b) The equation of motion is $J\dfrac{d^2\theta}{dt^2} = -\eta\dfrac{d\theta}{dt} - \kappa\theta$.

導綫扭擺的運動方程爲 $J\dfrac{d^2\theta}{dt^2} = -\eta\dfrac{d\theta}{dt} - \kappa\theta$ 。(1 point 1 分)

Let $\theta(t) = \theta_0 e^{i\tilde{\omega}t}$, where $\tilde{\omega}$ is complex,

令 $\theta(t) = \theta_0 e^{i\tilde{\omega}t}$ ，其中 $\tilde{\omega}$ 是複數，we get 得(1 point 1 分)

$\tilde{\omega}^2 - i\gamma\tilde{\omega} - \omega_0^2 = 0$, where 其中 $\omega_0^2 \equiv \kappa / J$, $\gamma \equiv \eta / J$. (1 point 1 分)

Let $\tilde{\omega} = \omega_R + i\omega_I$ and solving the equation, we get $\theta = \theta_0 e^{-\omega_I t} e^{i\omega_R t}$,

令 $\tilde{\omega} = \omega_R + i\omega_I$ ，并解上述方程，得 $\theta = \theta_0 e^{-\omega_I t} e^{i\omega_R t}$ ，(1 point 1 分)

Where 其中 $\omega_I = \gamma / 2$, (1 point 1 分) ， $\omega_R = \sqrt{\omega_0^2 + \gamma^2 / 4}$. (1 point 1 分)

(c) This is a forced oscillation with the force given by $\tau(t) = (\alpha - 0.5)\dfrac{h}{2e}I_0 e^{i\omega t} = \tau_0 e^{i\omega t}$.

這是個受迫振動問題，驅使力矩爲 $\tau(t) = (\alpha - 0.5)\dfrac{h}{2e}I_0 e^{i\omega t} = \tau_0 e^{i\omega t}$ 。(1 point 1 分)

The equation of motion is 運動方程爲 $J\dfrac{d^2\theta}{dt^2} = -\eta\dfrac{d\theta}{dt} - \kappa\theta + \tau(t)$. (1 point 1 分)

Let 令 $\theta(t) = \theta_0 e^{i\omega t}$, (1 point 1 分)

we get the oscillation amplitude $\theta_0 = \dfrac{\tau_0 / J}{\omega_0^2 - \omega^2 + i\gamma\omega}$.

得振動幅度爲 $\theta_0 = \dfrac{\tau_0 / J}{\omega_0^2 - \omega^2 + i\gamma\omega}$ 。(1 point 1 分)

The speed of the side wing is 邊翼的速度爲 $v(t) = i\omega d\theta_0 e^{i\omega t}$, (1 point 1 分)

and the electromotive potential is 電動勢爲 $\xi(t) = BLv(t) = i\omega dBL\theta_0 e^{i\omega t}$ 。(1 point 1 分)

Q3 (18 points)　題 3（18 分）　Solution 解：

(a) The magnetic dipole experiences a torque $\vec{\tau} = \vec{m} \times \vec{B}_0$ which is always perpendicular to the $\vec{S} \sim \vec{z}_0$ plane. The torque will turn the direction of $\vec{S}$ so $\vec{S}$ rotates around $\vec{B}_0$ at constant angular speed.

磁偶極子受到的力矩爲 $\vec{\tau} = \vec{m} \times \vec{B}_0$，其方向始終與 $\vec{S} \sim \vec{z}_0$ 平面垂直。力矩改變 $\vec{S}$ 的方向，因此 $\vec{S}$ 繞著 $\vec{B}_0$ 以勻角速度旋轉。(1 point 1 分)

Let the angle between $\vec{S}$ and $\vec{z}_0$ be θ. The torque is $mB_0 \sin\theta = \mu S B_0 \sin\theta$, while the change of angular momentum over time δt is $\delta S = S \sin\theta \delta\phi$.

令 $\vec{S}$ 與 $\vec{z}_0$ 之間的夾角爲 θ。則力矩的大小爲 $mB_0 \sin\theta = \mu S B_0 \sin\theta$，而角動量的變化爲 $\delta S = S \sin\theta \delta\phi$。 (1 point 1 分)

Since 既然 $S \sin\theta \delta\phi = \delta S = \mu S B_0 \sin\theta \delta t$ (1 point 1 分)

We have 我們得 $\omega_0 = \dfrac{\delta\phi}{\delta t} = \mu B_0$. (1 point 1 分)

(b) In the reference frame rotating at angular velocity $\omega_0 \vec{z}_0 = -\mu B_0 \vec{z}_0$, the spin appears stationary.

在以角速度 $\omega_0 \vec{z}_0 = -\mu B_0 \vec{z}_0$ 旋轉的參照系裏，自旋是不動的。(2 points 2 分)

(c) The effective B-field is 有效磁場爲 $\vec{B}_\omega = -\dfrac{\vec{\omega}}{\mu}$. (2 points 2 分)

(d) In the rotating frame of $-\omega_1 \vec{z}_0$, $\vec{B}_1$ also appears static.

在以角速度 $-\omega_1 \vec{z}_0$ 旋轉的參照系裏，$\vec{B}_1$ 是不動的。(1 point 1 分)

Let it be along the X' axis in the rotating frame, the total B-field is

$$\vec{B} = (B_0 - \frac{\omega_1}{\mu})\vec{z}_0^{'} + B_1 \vec{x}_0^{'}$$

令 $\vec{B}_1$ 在旋轉參照系裏沿 X' 方向，則總磁場爲 $\vec{B} = (B_0 - \dfrac{\omega_1}{\mu})\vec{z}_0^{'} + B_1 \vec{x}_0^{'}$。 (2 points 2 分)

(e) In this case, only $B_1\vec{x}_0^{'}$ remains. The spin will rotate around the $\vec{x}_0^{'}$ axis at angular speed $\omega_1 = \mu B_1$,

這時的磁場只剩下 $B_1\vec{x}_0^{'}$。自旋繞其以角速度 $\omega_1 = \mu B_1$ 旋轉， (2 points 2 分)

and the time to flip the spin is 倒轉自旋所需的時間爲 $t = \frac{1}{2}\frac{2\pi}{\omega_1} = \frac{\pi}{\mu B_1}$. (2 points 2 分)

(f) In this case the spin will rotate around the total B-field given by (c) at angular speed $\omega = \mu\sqrt{\left(B_0 - \frac{\omega}{\mu}\right)^2 + B_1^2}$.

這時的磁場由(c)給出。自旋的角速度爲 $\omega = \mu\sqrt{\left(B_0 - \frac{\omega}{\mu}\right)^2 + B_1^2}$ (3 points 3 分)

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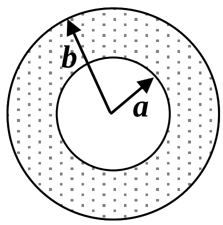
Pan Pearl River Delta Physics Olympiad 2010
2010年泛珠三角及中华名校物理奥林匹克邀请赛
Part-1 (Total 6 Problems) 卷-1（共6题）
(9:00 am – 12:00 pm, 02-19-2010)

Math hints 数学提示: $\int x^n dx = \frac{1}{n+1}x^{n+1}$, $\int x\sqrt{A^2-x^2}dx = -\frac{1}{3}\left(A^2-x^2\right)^{\frac{3}{2}}$

Q1 Solve the following short problems. (11 points)
题1 解下列各题。（11分）

(i) Nucleus-A of mass M_A and kinetic energy E_0 ($<<M_Ac^2$) collides with nucleus-B (mass M_B) at rest. After collision nucleus-A moves at 90° relative to its initial velocity with kinetic energy E_1. Find the kinetic energy that has been converted to nuclear energy in the process. (Both nuclei can be viewed as free particles.) (2 points)
质量为M_A的原子核-A，初始动能E_0 ($<<M_Ac^2$)，与初始静止的原子核-B（质量M_B）碰撞。碰撞后原子核-A以与初始速度成90°的方向飞出，动能为E_1。求在碰撞过程中由动能转变成核能的能量。（原子核可当作是自由粒子。）（2分）

(ii) Find the resistance and the capacitance between two concentric spherical conductor shells of radii a and b ($> a$). The space between the shells is filled with medium with dielectric constant ε and conductivity σ. (2 points)
两同心圆壳导体，半径分别为a、b ($> a$)。两壳间充满介电常数为ε、导电率为σ的介质。求两导体之间的电阻和电容。（2分）



(iii) (a) The electric field of an electromagnetic wave is $\vec{E} = E_0\vec{x}_0e^{i(kz-\omega t)}$. What is its state of polarization? (1 point)
(b) Write down the expression in the same way as in (a) for the electric field of a left circularly polarized electromagnetic wave. (1 point)
(a)一电磁波的电场为$\vec{E} = E_0\vec{x}_0e^{i(kz-\omega t)}$。问电磁波的偏振状态是什么？（1分）
(b)以与(a)相同的格式给出左旋圆偏振电磁波的电场。（1分）

(iv) The electric field of an electromagnetic wave in vacuum is $\vec{E} = E_0\vec{x}_0e^{i(kz-\omega t)}$. Find the magnetic field and the Poynting vector. (2 points)
真空中一电磁波的电场为$\vec{E} = E_0\vec{x}_0e^{i(kz-\omega t)}$。求电磁波的磁场和Poynting矢量。（2分）

(v) Use the Lorentz transformation to derive the relativistic Doppler Effect when the relative motion between the source and the receiver is (a) parallel or (b) perpendicular to the wave propagation direction. (3 points)
利用洛伦兹变换，推出下列情形的相对论多普勒效应：(a)波的传播方向与波源、接收器的相对速度平行；(b) 波的传播方向与波源、接收器的相对速度垂直。（3分）

Q2 (5 points) 题 2 （5 分）

(a) A spacecraft of mass m moves in a circular orbit at a distance r from the center of Earth (mass M_E). What is its kinetic energy? (1 point)
一质量为 m 的飞船，在离地球中心距离为 r 的圆形轨道上运行。地球质量为 M_E。求飞船的动能。（1 分）

(b) Suppose the spacecraft fires a rocket for a short time, speeding it up in the forward direction. After this short firing time, its kinetic energy increases by 30%. What is the furthest distance of the spacecraft from the center of Earth? Express your answer as a multiple of r. (4 points)
飞船启动火箭，在短时间内将飞船向前加速，使它的动能增加了 30%。问飞船离地球最远为多少 r？（4 分）

Q3 (9 points) 题 3 （9 分）

As shown below, a point light source is inside a glass block of refractive index n_G at distance y from the optical axis and distance T from the surface. A plano-convex lens made of transparent plastic material of refractive index n is placed near the source. The thickness of the lens is d, and the radius of its convex surface is R. The width of the air gap between the glass block and the lens is t.

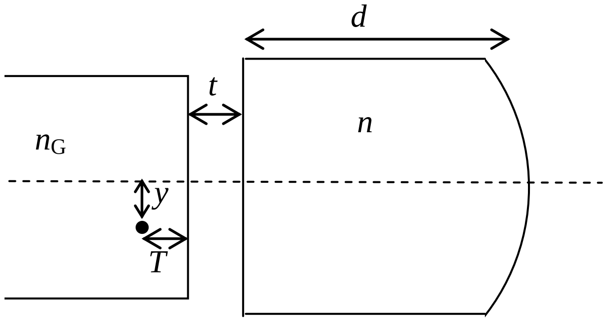
如下图所示，一点光源在玻璃内，与光轴的距离为 y，与玻璃平面的距离为 T。玻璃的折射率为 n_G。一透明塑料平凸透镜折射率为 n，厚度为 d，凸面的曲率半径为 R，与玻璃的距离为 t。

(a) Determine the width of the air gap t so that the light from the point source forms a parallel beam after passing through the lens. (3 points)
若点光源的光经过透镜后成为平行光束，求 t 的表达式。（3 分）

(b) Find the maximum thickness of the lens d_{max} such that it is still possible to adjust the width of the air gap to produce a parallel beam. (1 point)
当 d 大于 d_{max} 时，无论怎样调节 t 都无法得到平行光束，求 d_{max} 的表达式。（1 分）

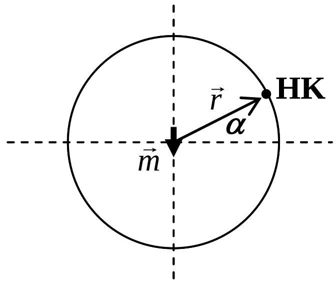
(c) Find the angle θ between the parallel beam and the optical axis. (3 points)
求平行光束与光轴的夹角θ。（3 分）

(d) If d is allowed to change by a small amount δd, and R is allowed to change by δR, find the angular change $\delta\theta$. (2 points)
若 d 改变一小值δd，R 改变一小值δR，求夹角的变化$\delta\theta$。（2 分）



Q4 (5 points) 题 4 （5 分）

The Earth's magnetic field is approximately a dipole field. One can imagine there is a magnetic dipole $\vec{m}$ at the center of the Earth. For simplicity, assume that $\vec{m}$ lies on the rotation axis of the Earth, pointing from north to south, as shown in the figure. Hong Kong is at the latitude of $\alpha = 22°$. The magnetic field at distance $\vec{r}$ from a dipole is $\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right]$.



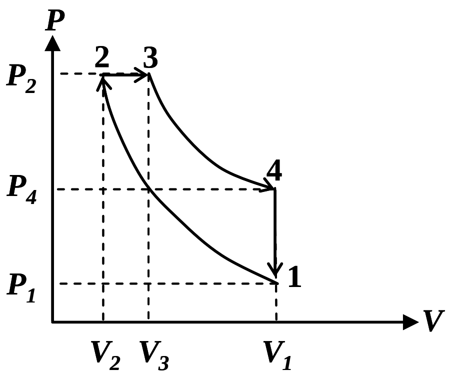
如图所示，地球的磁场可当作为一个磁偶极子的场，偶极子 $\vec{m}$ 在地球中心，与地球自转轴同轴，由北指向南。香港在北纬 $\alpha = 22°$。离磁偶极子 $\vec{r}$ 处的磁场为 $\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right]$。

(a) Find the direction of the magnetic field in Hong Kong in terms of east, west, north, south, and the angle to the horizontal plain. (2 points)
求在香港的地球磁场方向（以东、南、西、北和水平面为基准）。（2 分）

(b) A horizontal electric wire of 10 meters in length carrying 100 A of current is in the north-south direction. Find the magnetic force on the wire. (You need to recall roughly the strength of Earth magnetic field on Earth surface.) (3 points)
一段南北走向的 10 米长的电缆，载有 100A 的电流。求地球磁场对电缆的力。（你需自己给出地球磁场的大约强度。）（3 分）

Q5 (8 points) 题 5 （8 分）

Shown in the figure is the Diesel ideal-gas cycle. All processes are quasi-static and the gas is monatomic. The two processes 1→2 and 3→4 are both adiabatic. Find the efficiency of the cycle in terms of $\alpha = V_3/V_2$ and $r = V_1/V_2$.



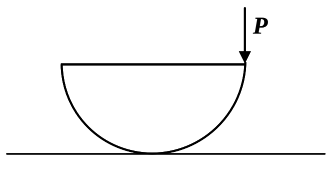
右图所示的是一个单原子理想气体的 Diesel 循环。循环中的每个过程都是准稳态的。1→2、3→4 为绝热过程。求循环的效率（用 $\alpha = V_3/V_2$ 、 $r = V_1/V_2$ 来表达）。

Q6 (12 points) 题 6 （12 分）

A uniform half disk of finite thickness with radius *R* and mass *M* is resting on a rough horizontal floor.

(a) Find the moment of inertia relative to the axis perpendicular to the half disk and through the contact point on the floor. (6 points)

(b) An impulse *P* is suddenly applied to the edge of the half disk as shown. The half disk rolls without slipping afterwards and maintains contact with the floor. Find the minimum impulse to flip the half disk. (6 points)



一个有一定厚度的均匀半圆盘，半径 *R*，质量 *M*，放在粗糙水平地面上。

(a) 一轴线与盘面垂直，并穿过半圆盘在平衡时与地面的接触点。求半圆盘相对于该轴线的转动惯量。（6 分）

(b) 一冲量 *P* 突然作用于盘的边缘。之后半圆盘作纯滚动，并一直保持与地面接触。求能使半圆盘翻转的最小冲量。（6 分）

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2010
2010 年泛珠三角及中华名校物理奥林匹克邀请赛
Part-2 (Total 3 Problems) 卷-2（共 3 题）
(2:30 pm – 5:30 pm, 02-19-2010)

Math hints 数学提示: $\int x^n dx = \frac{1}{n+1} x^{n+1}$; $\sum_{n=0}^{N} a^n = \frac{a^{N+1}-1}{a-1},\ N>1,\ a \neq 1$.

Q1 Chain Reactions (10 points)
题 1 连锁反应 （10 分）

(a) The mass of a proton is roughly equal to that of a neutron and is 1.67×10^{-27} Kg. Find the number of nuclei in 1000 Kg of ^{235}U. (3 points)
一个质子的质量和一个中子的质量相若，均为 1.67×10^{-27} Kg。求 1000 公斤 ^{235}U 所含的原子核数目。（3 分）

(b) When a ^{235}U nucleus is hit by a neutron, it captures the neutron, splits into several fragments, and releases 3 neutrons. These neutrons will hit 3 ^{235}U to create 9 neutrons, … The time between a neutron is released to the time it hits the next ^{235}U is 1.0×10^{-8} s. Estimate the total time needed to consume 1000 Kg of ^{235}U. (7 points)
当一个中子撞到一个 ^{235}U 原子核时，原子核会将中子俘获，然后裂变成数个碎块和 3 个中子。那 3 个中子分别撞向 3 个 ^{235}U 原子核，产生 9 个中子，… 中子从产生到撞到下一个 ^{235}U 原子核的时间间隔为 1.0×10^{-8} s。估计将 1000 公斤 ^{235}U 裂变完所需的时间。（7 分）

Q2 Quantum Gravity (20 points)
题 2 量子引力学（20 分）

(a) <u>The Planck Length</u>
The smallest length scale one can resolve is called the Planck Length L_P, which can be estimated in the following way. To get information within the length scale L, one needs a photon with the wavelength λ roughly the same as the length scale. But any photon creates a (tiny) blackhole within which no information can get out. When the photon wavelength is equal to 2π times the radius of the blackhole (=L_P), then we obtain the smallest length scale L_P. Note that a photon with wavelength λ has energy $E = hc/\lambda$ and effective mass $m = E/c^2$. (Planck Constant $h = 6.63\times10^{-34}\, J\cdot s$, speed of light in vacuum $c = 3.0\times10^{8}\, m/s$, universal gravity constant $G = 6.67\times10^{-11}\, N\cdot m^2/kg^2$)

(i) Determine the Planck Length. (4 points)
(ii) Determine the Planck Energy E_P which is the energy of a photon with its wavelength equal to 2π times the Planck Length, in terms of electron volt (eV). You may find the constant $hc = 1240\ eV\cdot nm$ useful. ($1\ nm = 10^{-9} m$) (2 point)

(b) <u>The Hubble's Law</u>
(i) The light spectra of a distant galaxy exhibit a red shift of $Z \equiv (\lambda' - \lambda)/\lambda = 1.03$, where λ is the wavelength if the source is at rest relative to us, and λ' is the observed wavelength. Find the speed of the galaxy relative to us, and determine whether the galaxy is moving away or towards us. (7 points)
(ii) According to Hubble's Law, the receding speed of a galaxy v is proportional to its distance D from us, i. e., $v = H_0 D$, where H_0 = 21.7 (km/s)/Mly, and Mly represents a million light years. Find the distance of the galaxy. (2 point)

(c) <u>Change of Speed of Light</u>

Several quantum gravity models predict that photons with higher energies move slower, i .e., the speed of photon in vacuum with energy E is $v = c(1 - E / E_P)$, where E_P is the Planck Energy in (a). It can be tested by observing the light pulses from gamma-ray bursts, where a low energy photon pulse and a high energy photon pulse are emitted simultaneuously. A gamma ray burst was observed recently in the galaxy in (b). A low energy (~10^3 eV) photon pulse and a high energy photon pulse with energy of 34×10^9 eV was observed.

(i) According to the models, what would be the time delay between the two pulses when they reached us? (4 points)

(ii) The actual delay observed was 0.9×10^{-3} s. Does the evidence support the models' prediction? (1 point)

(a) <u>普朗克长度</u>

我们可分辨的最短长度称为普朗克长度 L_P。下面我们来估计它的值。要获得长度为 L 的空间内的信息，我们需要的光子的波长λ要和 L 差不多短。但光子也有（很微小的）黑洞，黑洞内的信息在外面是无法得到的。令光子的波长等于 2π 乘以黑洞的半径(=L_P)，我们就得到最小的长度。已知光子的能量为$E = hc/\lambda$，有效质量为$m = E/c^2$。（普朗克常数 $h = 6.63\times10^{-34}\ J\cdot s$，真空光速$c = 3.0\times10^8\ m/s$，引力常数$G = 6.67\times10^{-11}\ N\cdot m^2/kg^2$）

(i) 求普朗克长度 。（4 分）

(ii) 求波长等于 2π 乘以普朗克长度的光子的能量，并将结果用电子伏特(eV)作单位。这便是普朗克能量 E_P。下面的常数你会觉得有用。（$hc = 1240\ eV\cdot nm$，$1\ nm = 10^{-9} m$）（2 分）

(b) <u>哈勃定理</u>

(i) 某一个遥远的星系的光谱的红移为$Z \equiv (\lambda' - \lambda)/\lambda = 1.03$。其中$\lambda$为静止光源的波长，$\lambda'$为我们实际测到的波长。求星系相对我们的速率，并确定星系是离我们而去，还是向我们飞来。（7 分）

(ii) 根据哈勃定理，星系飞离我们的速率 v 与离我们的距离 D 成正比，即$v = H_0 D$，其中 H_0 = 21.7 (km/s)/Mly，Mly 为一百万光年。求星系离我们的距离。（2 分）

(c) <u>光速的变化</u>

目前有些量子引力学的理论预测高能量的光子的速度会慢一点，即真空中能量为 E 的光子的速度为$v = c(1 - E/E_P)$，其中 E_P 为(a)中的普朗克能量。要验证这一预测，我们需要观测伽玛射线的爆发。伽玛射线爆发时，光源会同时射出一个低能量和一个高能量光子的脉冲。最近，在(b)的星系我们观测到一次伽玛射线爆发。我们接收到一个低能量(~10^3 eV)光子的脉冲，和一个能量为 34×10^9 eV 的高能量光子脉冲。

(i) 根据理论预测，这两个脉冲到达地球的时间差是多少？（4 分）

(ii) 实际观测到的时间差为 0.9×10^{-3} s。这结果支持那些量子引力学的理论吗？（1 分）

Q3 The Sound of Bubbles (20 points)

题 3 气泡之声（20 分）

Consider an air bubble of radius R submerged in water of mass density ρ. In equilibrium the pressure inside and outside the bubble is P_0, if surface tension of water is ignored for the time being. Air can be treated as diatomic ideal gas. Given a small disturbance the bubble then vibrates radially at a characteristic harmonic frequency. The following steps can help you to find the frequency. The approach is to express the potential energy of the vibrating bubble in terms of Ax^2, and the kinetic energy in terms of $B\left(\frac{dx}{dt}\right)^2$, where x is the small displacement of the radius. The frequency is then $\omega = \sqrt{A/B}$.

(a) Suppose the radius of a bubble is changed adiabatically from R to $R + x$, ($x << R$), find the restoring force acting on the bubble surface which is proportional to x. (2 points)

(b) Find the energy required for x to change from 0 to a small value x_0. (2 points)

(c) The radial velocity of water at the bubble surface of radius R is $\frac{dx}{dt}$. Given that water is incompressible, what is the radial velocity of water at distance r (> R) from the bubble center? (3 points)

(d) Find the total kinetic energy of water. (2 points)

(e) The kinetic energy in (d) is that of the vibrating bubble, because the mass of air can be ignored. Find the vibration frequency. (1 point)

Now we consider the effect of surface tension of water on the vibration frequency. The surface tension comes from the fact that it takes energy γS to create a water surface of area S, where γ is the surface tension coefficient.

(f) Find the surface tension force on one of the four sides of a flat and square water membrane of side length a. (2 points)

(g) Find the surface tension energy needed to create an air bubble of radius R in water. (1 point)

(h) Find the pressure difference between the inside and the outside of the bubble in equilibrium. (2 points)

(i) Modify the answer to (e) to include the effect of surface tension of water. (5 points)

一半径为 R 的空气泡浸没在质量密度为ρ的水里。若暂时不考虑水的表面张力，则平衡时泡内、外的压强相同。设泡外压强为 P_0。空气可当作是双原子理想气体。因一小扰动，气泡开始以一特征频率作径向简谐振动。以下步骤能帮助你求出振动频率。其主要方法是将气泡振动的势能表达成 Ax^2，将动能表达成 $B\left(\frac{dx}{dt}\right)^2$，其中 x 为半径的微小位移量，然后可得特征频率 $\omega = \sqrt{A/B}$ 。

(a) 令气泡的半径由 R 绝热地变为 $R + x$, ($x << R$)，求作用于气泡面上的恢复力。该力应与 x 成正比。（2 分）

(b) 求将 x 从 0 变到 x_0 所需的能量。（2 分）

(c) 在半径为 R 的气泡表面，水的径向速率为 $\frac{dx}{dt}$ 。已知水是不可压缩的，求在离气泡中心 r (> R)处水的径向速率。（3 分）

(d) 求水的总动能。（2 分）

(e) 因空气的质量可忽略，(d)中的动能就是气泡振动的动能。求特征频率。（1 分）

现在让我们考虑水的表面张力对振动频率的影响。产生一面积为 S 的液体表面需要用能量γS，其中γ为表面张力系数。这就是表面张力的来源。

(f) 一四方平面液体薄膜边长为 a，求其中一边所受的表面张力。（2 分）

(g) 求水中一半径为 R 的气泡所含的表面张力能量。（1 分）

(h) 求该气泡在平衡时内、外压强之差。（2 分）

(i) 求包括水的表面张力效应的气泡特征振动频率。（5 分）

《THE END 完》

第六届泛珠物理竞赛简单解答

Part-1 卷-1

Q1. 题-1

(i) Conservation of energy 能量守恒

$$E_n = E_0 - E_1 - \frac{1}{2}M_B v_{B1}^2$$

Conservation of momentum 动量守恒

$$M_A v_{A0}\hat{x} = M_A v_{A1}\hat{y} + M_B \vec{v}_{B1}$$

$$\vec{v}_{B1} = \frac{M_A}{M_B}\left(\sqrt{\frac{2E_0}{M_A}}\hat{x} - \sqrt{\frac{2E_1}{M_A}}\hat{y}\right)$$ (1 point)

So we have 得

$$E_n = E_0 - E_1 - \frac{M_A}{M_B}(E_0 + E_1)$$ (1 point)

(ii) For the capacitance: Gauss law 求电容，利用高斯定理

$$\int \nabla \cdot \vec{E}\, d^3x = \frac{Q}{\epsilon}$$

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\hat{r}$$

$$V = -\int_a^b \vec{E}\cdot\hat{r}dr = \frac{Q}{4\pi\epsilon}\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$C = \frac{Q}{V} = 4\pi\epsilon\frac{ab}{b-a}$$ (1 point)

On the resistance: Ohm's law 求电阻，利用 Ohm 定理

$$R = \frac{1}{\sigma}\int_a^b \frac{dr}{4\pi r^2} = \frac{1}{4\pi\sigma}\frac{b-a}{ab}$$ (1 point)

(iii) Linearly polarized along $\hat{x}_0$ 沿$\hat{x}_0$方向的线偏振 (1 point)

Left-handed circularly polarized: $\vec{E} = (\hat{x}_0 + i\hat{y}_0)E_0 e^{ikz - i\omega t}$ 左旋圆偏振 (1 point)

(iv)

$\vec{B} = \sqrt{\mu\epsilon}\hat{z}_0 \times \vec{E} = \sqrt{\mu\epsilon}E_0\hat{y}_0 e^{ikz-i\omega t}$ (1 point)

$\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^* = \frac{1}{2}\hat{z}_0\sqrt{\frac{\epsilon}{\mu}}E_0^2$ (1 point)

Replace μ by μ_0, ε by ε_0 for vacuum. 在真空中μ用μ_0代, ε用ε_0代。

(v):

$\phi = \omega t - \vec{k}\cdot\vec{x} = \omega' t' - \vec{k}'\cdot\vec{x}'$, $\quad \omega = ck_0 \quad \omega' = ck_0'$

$$\begin{cases} k_0' = \gamma\left(k_0 - \vec{\beta}\cdot\vec{k}\right) \\ k_P' = \gamma\left(k_P - \beta k_0\right) \quad \vec{\beta} = \frac{\vec{v}}{c},\ \gamma = \left(1-\beta^2\right)^{-1/2} \\ k_\perp' = k_\perp \end{cases}$$ (1 point)

$\left|\vec{k}\right| = k_0$, $\omega = ck_0$ $\left|\vec{k}'\right| = k_0'$, $\omega' = ck_0'$

$\omega' = \gamma\omega\left(1-\beta\cos\theta\right)$

For $\theta = 0$, $\omega' = \gamma\omega(1-\beta)$ (1 point)

And for $\theta = \pi/2$, $\omega' = \gamma\omega$. (1 point)

Q2

(a) Using Newton's second law, 利用牛顿第二定律

$\frac{GMm}{r^2} = m\frac{v^2}{r}$, $\rightarrow$ $v = \sqrt{\frac{GM}{r}}$

Kinetic energy 动能: $K = \frac{1}{2}mv^2 = \frac{GMm}{2r}$ (1 point)

(b) Let r_f be the further distance of the elliptical orbit from Earth. Let v_f be the velocity of this orbit at this distance. After the firing of the rocket, the velocity of the spacecraft becomes $\sqrt{1.3}v$.

令 r_f为椭圆轨道离地球的最远点，v_f为在该点飞船的速率。

Using the conservation of angular momentum,角动量守恒

$mr\sqrt{1.3}v = mr_f v_f$ (1) (1 point)

Initial total energy 初始总能量:

$K + U = 1.3\frac{GMm}{2r} - \frac{GMm}{r} = -0.35\frac{GMm}{r}$

Final total energy 现在总能量:

$$K + U = \frac{1}{2}mv_f^2 - \frac{GMm}{r_f}$$

Using the conservation of energy, 能量守恒

$$\frac{1}{2}mv_f^2 - \frac{GMm}{r_f} = -0.35\frac{GMm}{r} \qquad (2) \qquad \text{(1 point)}$$

Using (1) to eliminate v_f, 用(1)将 v_f代掉,

$$\frac{1}{2}mv^2\frac{1.3r^2}{r_f^2} - \frac{GMm}{r_f} = -0.35\frac{GMm}{r}$$

From the result of (a) 由(a)得,

$$\frac{GMm}{2r}\frac{1.3r}{r_f^2} - \frac{GMm}{r_f} = -0.35\frac{GMm}{r}$$

$$0.65\left(\frac{r}{r_f}\right)^2 - \frac{r}{r_f} + 0.35 = 0 \qquad \text{(1 point)}$$

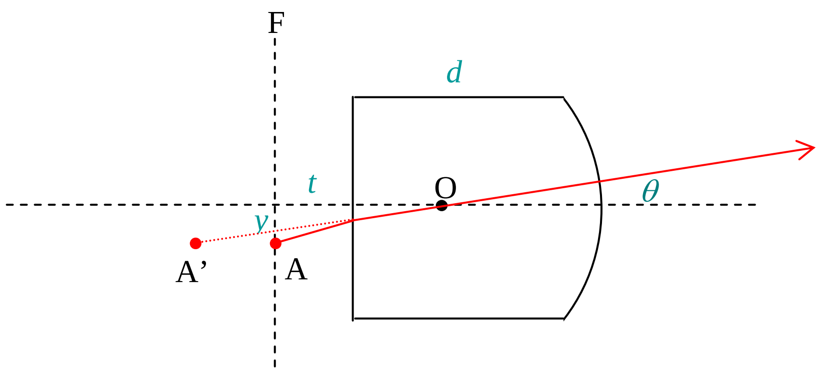
$$\frac{r}{r_f} = \frac{1 \pm \sqrt{1-4(0.65)(0.35)}}{1.3} = 1 \text{ or } \frac{7}{13}$$

$$r_f = 1.86r \qquad \text{(1 point)}$$

Q3

利用最原始的单球面成像公式：

$$\frac{n_1}{S_O} + \frac{n_2}{S_I} = \frac{n_2 - n_1}{R} \qquad （1）$$



(a)

<u>先考虑平面的成像</u>： $S_O = t$， （t是发光体到平面的距离）

若考虑发光体是在厚度为 T 的玻璃后面（上图没有给出），则 $S_O = t + T/n_G$，

n_G 是玻璃的折射率。平面的 R 为无穷大，利用式（1）， $n_1 = 1$， $n_2 = n$ （透镜

的折射率），得像距$S_I = -n(t+T/n_G)$，像在平面的左边。此像为平面右边的球面的物，距离球面$n(t+T/n_G)+d$。(1 point)

球面的成像：$S_O = n(t+T/n_G)+d$，$S_I = \infty$，$n_1 = n$，$n_2 = 1$，R为负数，利用式（1），得 $R=(n-1)(t+T/n_G+d/n)$ （2）(1 point)

$$t=\frac{1}{n-1}R-\frac{d}{n}-\frac{T}{n_G} \quad \text{(1 point)}$$

(b)

若 R 固定，则最大厚度$d_{\max}$为（另式（2）中的$t=0$）$d_{\max}=\frac{nR}{(n-1)}-\frac{nT}{n_G}$。若膜板厚度小于此值，则可调节空气间隔 t 来使发光体位于焦平面上，大于此值则无法使发光体位于焦平面上。(1 point)

(c)

出射角：

如图，物 A 经平面成的像为 A’。从 A’射出的光经过球心 O 穿过球面无折射，由几何关系得$\theta=\frac{y}{nt+Tn/n_G+(d-R)}=\frac{(n-1)y}{R}$。 （3）(3 points)

(d)

由 t 和 R 的偏差引起的误差可由式（3）微分后得

$$\frac{\delta a}{a}=\frac{\delta\theta}{\theta}=\frac{|\delta d|+|\delta R|}{nt+nT/n_G+(d-R)} \quad （4）\text{(2 points)}$$

Q4

(a)

$$\mathbf{B}=\frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\left(\mathbf{m}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\mathbf{m}\right]=\frac{\mu_0}{4\pi}\frac{1}{R^3}\left[3\left(\left(-m\hat{\mathbf{z}}\right)\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\left(-m\hat{\mathbf{z}}\right)\right]$$
$$=\frac{\mu_0}{4\pi}\frac{m}{R^3}\left[-3\sin\alpha\hat{\mathbf{r}}+\left(\sin\alpha\hat{\mathbf{r}}-\cos\alpha\hat{\boldsymbol{\theta}}\right)\right]=\frac{\mu_0}{4\pi}\frac{m}{R^3}\left[-2\sin\alpha\hat{\mathbf{r}}-\cos\alpha\hat{\boldsymbol{\theta}}\right]$$

From north to south, at an angle $\beta=\arctan(2\tan\alpha)=38.9^\circ$ pointing downwards. (2 points)

由北向南，与水平面成夹角$\beta=\arctan(2\tan\alpha)=38.9^\circ$

(b)

$$\left|\vec{B}\right|\cong 0.5\times10^{-4}T \quad \text{(1 point)}$$

$$\vec{F}=I\vec{l}\times\vec{B}=IBl\sin\theta=100\times10\times0.63\times5\times10^{-5}=0.0315N \quad \text{(2 points)}$$

Q5

No heat exchange during 1→2 and 3→4. 过程 1→2、3→4 无热交换。 (2 points)

Heat absorbed in 2→3 2→3 过程吸热:

$$Q_h = \frac{3}{2}\left(P_2V_3 - P_2V_2\right) + P_2\left(V_3 - V_2\right) = \frac{5}{2}P_2\left(V_3 - V_2\right).\ \text{(2 points)}$$

Heat released in 4→1 4→1 过程放热:

$$Q_c = \frac{3}{2}\left(P_4V_1 - P_1V_1\right) = \frac{3}{2}\left(P_4 - P_1\right)V_1.\ \text{(2 points)}$$

$$e = 1 - \frac{Q_c}{Q_h} = 1 - \frac{3\left(P_4 - P_1\right)V_1/2}{5P_2\left(V_3 - V_2\right)/2} = 1 - \frac{3}{5}\frac{\left(P_4 - P_1\right)V_1}{P_2\left(V_3 - V_2\right)}$$

$$= 1 - \frac{3}{5}\frac{P_4 - P_1}{P_2}\frac{V_1}{V_3 - V_2} = 1 - \frac{3}{5}\left(\frac{P_4}{P_2} - \frac{P_1}{P_2}\right)\frac{V_1/V_2}{V_3/V_2 - 1}$$

$$= 1 - \frac{3}{5}\left(\left(\frac{V_3}{V_1}\right)^{5/3} - \left(\frac{V_2}{V_1}\right)^{5/3}\right)\frac{V_1/V_2}{V_3/V_2 - 1}$$

$$= 1 - \frac{3}{5}\left(\left(\frac{\alpha}{r}\right)^{5/3} - \left(\frac{1}{r}\right)^{5/3}\right)\frac{r}{\alpha - 1} = 1 - \frac{3}{5}\frac{\left(\alpha^{5/3} - 1\right)}{r^{2/3}\left(\alpha - 1\right)} \qquad \text{(2 points)}$$

Q6

First find the distance of CM from the center 首先求质心离盘心的距离:

$$y_{CM} = \frac{1}{M}\int_0^R \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \frac{M}{\pi R^2/2}\, y\,dx\,dy$$

$$= \frac{2}{\pi R^2}\int_0^R 2y\sqrt{R^2 - y^2}\,dy \ \text{(or start from here) (1 point)}$$

$$= \frac{2}{\pi R^2}\int_R^0 \sqrt{R^2 - y^2}\,d\left(R^2 - y^2\right) = \frac{2}{\pi R^2}\left.\frac{\left(R^2 - y^2\right)^{3/2}}{3/2}\right|_R^0 = \frac{4}{3\pi R^2}R^3 = \frac{4}{3\pi}R.\ \text{(1 point)}$$

The moment of inertia about the center is $\frac{1}{2}MR^2$. So the moment of inertia about CM is 相对于盘心的转动惯量为 $\frac{1}{2}MR^2$。因此，相对于质心的转动惯量为

$$I_{CM} = \frac{1}{2}MR^2 - M\left(\frac{4}{3\pi}R\right)^2 = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right)MR^2.\ \text{(2 points)}$$

The moment of inertia about the point of contact is 相对于与地板的接触点的转动惯量为

$$I_C = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right)MR^2 + Md^2,\ \text{(1 point)}$$

$d = \left(1 - \frac{4}{3\pi}\right) R = 0.5756R$.

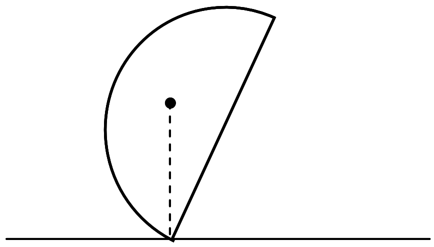
Hence 因此

$$I_C = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right) MR^2 + M\left(1 - \frac{4}{3\pi}\right)^2 R^2 = \left(\frac{3}{2} - \frac{8}{3\pi}\right) MR^2 = 0.65MR^2 .$$ (1 point)

(b)

$I\omega = PR \Rightarrow \omega = \frac{PR}{I}$ (1 point)

$U_0 = Mgd$, (1 point)



$T_0 = \frac{1}{2} I\omega^2 = \frac{1}{2}\frac{P^2R^2}{I}$, (1 point)

$U_f = Mg\sqrt{R^2 + d^2}$, $T_f = 0$ (1 point)

$U_0 + T_0 = U_f \Rightarrow$

$P^2 = 1.3M^2 g\left(\sqrt{R^2 + d^2} - d\right) = 1.3M^2 gR\left(\sqrt{1 + 0.5756^2} - 0.5756\right) = 0.752M^2 gR$ (1 point)

Thus, 得 $P_{\min} = 0.867M\sqrt{gR}$ (1 point)

Part-2 卷-2

Q1

(a)

$N = \frac{1000}{235 \times m} = \frac{1000}{235 \times 1.67 \times 10^{-27}} = 2.548 \times 10^{27}$ (3 points)

(b)

$N = 1 + 3 + 9 + \cdots = \sum_{n=0}^{m} 3^n = \frac{1}{2}(3^{m+1} - 1)$ (5 points)

$m = \frac{\log(2N+1)}{\log 3} - 1 \approx 58$, $T = mt = 5.8 \times 10^{-7} s$ (2 points)

Q2

(a)

$\frac{hc}{2\pi L_p} = E_p = m_p c^2$ (1) (1 point)

$\frac{c^2}{L_p} = \frac{Gm_p}{L_p^2}$ (2) (2 points)

Solving (1) and (2) we get $L_p = \sqrt{\frac{Gh}{2\pi c^3}} = 1.6\times10^{-35}m = 1.6\times10^{-26}nm$ (1 point)

(b)

$$E_p = \frac{hc}{2\pi L_p} = \frac{1240(eV\cdot nm)}{2\pi\times1.6\times10^{-26}(nm)} = 1.23\times10^{28}eV$$ (2 points)

(c)

$$Z \equiv \frac{\lambda'-\lambda}{\lambda} = \frac{\omega}{\omega'} - 1$$ (1 point)

$$(Z+1)^{-1} = \frac{\omega'}{\omega} = \gamma(1-\beta) = \sqrt{\frac{1-\beta}{1+\beta}},$$ (3 points)

with Z = 1.03, we get $\beta = \frac{(Z+1)^2-1}{(Z+1)^2+1} = 0.61$. (2 points)

Away from us 离我们而去. (1 point)

(d)

$$D = v/H_0 = \frac{0.61\times3.0\times10^5}{21.7}MLY = 8.4\times10^9 LY$$ (2 points)

(e)

$$\Delta t = \frac{D}{c}\cdot\frac{\Delta E}{E_p} = 8.4\times10^9\times86400\times365\times\frac{3.0\times10^{10}}{1.23\times10^{28}} = 8.4\times8.64\times3.65\times\frac{3.0}{1.23}$$ (4 points)

$$= 0.646\ s$$

No. (1 point)

Q3

(a) According to the state equation of air in the adiabatic process 绝热过程(C_p/C_v = 7/5)

$$p_0 R_0^{3\times\frac{7}{5}} = p'(R+x)^{3\times\frac{7}{5}}$$ (1 point)

➔ $P' = P_0(1-\frac{21}{5}\frac{x}{R})$, that is $P_{out} - P_{in} = \frac{21}{5}\frac{x}{R}P_0$

➔ So the force 力为 $F = 4\pi R^2\Delta P = -\frac{84}{5}\pi RP_0 x$ (1 point)

(b) The energy 势能为 $E = \int_0^{x_0} Fdx = \frac{84}{5}\pi RP_0\int_0^{x_0} xdx = \frac{42}{5}\pi RP_0 x_0^{\ 2}$ (2 points)

(c) According to the continuity equation 利用水流连续性,

$$4\pi R^2\frac{dx}{dt} = 4\pi r^2 v(r)$$ (1 point)

➔ Therefore 因此 $v(r)=\frac{R^2}{r^2}\frac{dx}{dt}$

(d) The kinetic energy of water 水的动能为

$$K=\int_R^\infty \frac{1}{2}(4\pi r^2\rho dr)(\frac{R^2}{r^2}\frac{dx}{dt})^2=2\pi R^3\rho(\frac{dx}{dt})^2$$ (2 points)

(e) $\omega=\sqrt{\frac{\frac{42}{5}\pi RP_0}{2\pi R^3\rho}}=\frac{1}{R}\sqrt{\frac{21P_0}{5\rho}}$ (1 point)

(f) Note that a membrane has two surfaces, so its surface tension energy is $E=2\gamma a^2$

薄膜有两个表面，因此表面能量为 $E=2\gamma a^2$ (1 point)

Let one side increase by dx, the energy change is $dE=2\gamma adx$, so $F_{tension}=2\gamma a$.令其中一边外移一小段 dx，则能量的改变为 $dE=2\gamma adx$，因此 $F_{tension}=2\gamma a$ (1 point)

(g) $E=4\gamma\pi R^2$, (1 point)

(h) ➔ $8\gamma\pi RdR=dE=PdV=P(4\pi R^2dR)$ ➔ $P_{tension}=2\gamma/R$. (1 point)

平衡时 $P_0=P_{gas}=P_{atm}+P_{tension}=P_{atm}+\frac{2\gamma}{R}$ at equilibrium. (1 point)

The net change of pressure when $R->R+x$ is

当 $R->R+x$ 时总的压强的变化为

$$dP=dP_{gas}-dP_{tension}=-3\kappa P_0\frac{x}{R}+\frac{2\gamma x}{R^2}$$ (2 points)

$$dE=dP\cdot 4\pi R^2x=-4\pi\left(3\kappa P_0R-2\gamma\right)x$$ (1 point)

$E=2\pi\left(3\kappa P_0R-2\gamma\right)x_0^2$, $\kappa=\frac{7}{5}$, so $E=\pi\left(\frac{42}{5}P_0R-4\gamma\right)x_0^2$ (1 point)

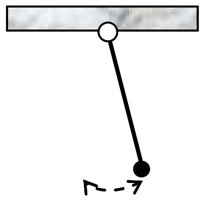
$$\omega=\sqrt{\frac{\frac{42}{5}\pi RP_0-4\pi\gamma}{2\pi R^3\rho}}=\frac{1}{R}\sqrt{\frac{1}{\rho}\left(\frac{21P_0}{5}-\frac{2\gamma}{R}\right)}$$ (1 point)

Pan Pearl River Delta Physics Olympiad 2011
2011 年泛珠三角及中华名校物理奥林匹克邀请赛
Part-1 (Total 6 Problems) 卷-1（共 6 题）
(9:00 am – 12:00 pm, 02-10-2011)

Math hints 数学提示: $(1+x)^{\alpha} \approx 1+\alpha x$ for $x << 1$, $\int x^n dx = \frac{1}{n+1}x^{n+1}, n \neq -1$.

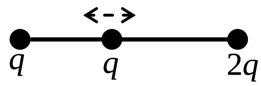
Q1 (8 points) 题 1（8 分）

(a) A compound pendulum consists of a uniform rigid rod of mass m and length L and a small ball of mass m attached to the end of the rod, as shown in the figure. The upper end of the rod is attached to the joint on the ceiling about which the rod can rotate freely. Find the simple harmonic oscillation frequency of the pendulum. (4 points)



如图，一均匀硬杆，质量为 m，长度为 L，上端以光滑铰链系在天花板上，下端有一质量也为 m 的小球。求系统的简谐振动频率。（4 分）

(b) Two point charges q and $2q$ are fixed on a smooth horizontal rail separated by a distance L. A small cart carrying charge q and with mass m can move freely on the rail, as shown in the figure. Find the simple harmonic oscillation frequency of the cart. (4 points)



如图，两个点电荷 q 和 $2q$ 分别固定在长为 L 的水平光滑导轨的两端。一质量为 m 的小车，带电 q，可在导轨上自由滑行。求小车的简谐振动频率。（4 分）

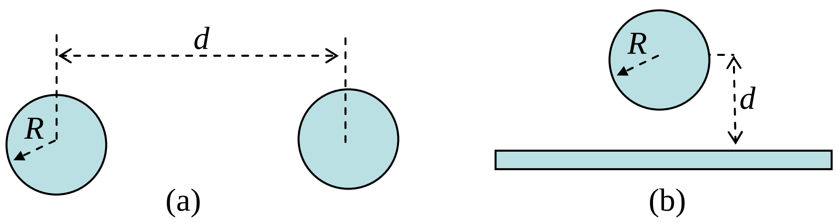
Q2 (9 points) 题 2（9 分）

The mass of an electron at rest, in the unit of energy using the mass-energy equivalence relation, is 0.511×10^6 eV, or 0.511 MeV, and eV is electron-Volt. Find the momentum of an electron, in the unit of MeV/c (c is the speed of light in vacuum) and keep only two digits, when its kinetic energy is (i) 1.0×10^{-6} MeV, (ii) 1.0 MeV, and (iii) 1.0×10^6 MeV.

一静止电子的质量，若以质量-能量的等价关系而用能量为单位，为 0.511×10^6 eV, 或是 0.511 MeV。eV 是电子伏特的缩写。求电子有以下动能时的动量。（以 MeV/c 为单位， c 是真空中的光速。保留 2 位有效数字。）(i) 1.0×10^{-6} MeV, (ii) 1.0 MeV, and (iii) 1.0×10^6 MeV。

Q3 (8 points) 题 3（8 分）

(a) Find the capacitance per unit length between a pair of parallel cylindrical conductors with radius R and separated by a distance d (>> R), as shown in Fig. a. (b) A cylindrical conductor with radius R is placed at a distance d (>> R) above a grounded conductor plate, as shown in Fig. b. Find the capacitance per unit length between the two conductors.



(a) 如图a，两平行圆柱形导体半径为R，间距为d (>> R)。求单位长度两导体之间的电容。 (b) 如图b，一圆柱形导体半径为R，离一接地导体板距离为d (>> R)。求单位长度两导体之间的电容。

Q4 (10 points) 题 4（10 分）

This problem demonstrates why the energy density of gravitational field is negative.
本题演示为何重力场的能量密度是负的。

(a) First, consider two large parallel flat sheets separated by a distance D, one carrying uniform electric charge area density σ, and the other carrying –σ.
首先考虑两块平行大薄板，板之间的间距为 D，一板带均匀面电荷σ，另一板带均匀面电荷–σ。

(i) Find the electric force from one sheet exerted on a unit area of the other sheet. (1 point)
求一板对另一板单位面积的作用力。（1 分）

(ii) Find the electric field E near the sheets. (1 point)
求板附近的电场 E。（1 分）

(iii) Assume the total area of a plate is A. Let the separation distance D decrease by a small amount δD. Find the work done by the electric force. (1 point)
设板的面积为 A。令板的间距 D 减少一小量 δD，求电场作的功。（1 分）

(iv) Using the answer in (iii), find the energy density of the electric field and express the energy density in terms of E. (2 point)
利用(iii)的答案， 求电场的能量密度，并以电场 E 来表达。（2 分）

(b) Now, consider two neutral parallel flat sheets of uniform area mass density σ separated by a distance D.
现在考虑两块中性平行大薄板，板之间的间距为 D，质量面密度为σ。

(i) Find the gravitational force from one sheet exerted on a unit area of the other sheet. (1 point)
求一板对另一板单位面积的引力。（1 分）

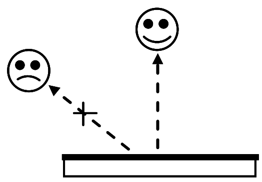
(ii) Find the gravitational field g near the sheets. (1 point)
求板附近的引力场 g。（1 分）

(iii) Assume the total area of a plate is A. Let the separation distance D decrease by a small amount δD. Find the work done by the gravitational force. (1 point)
设板的面积为 A。令板的间距 D 减少一小量 δD，求引力场作的功。（1 分）

(iv) Using the answer in (iii), find the energy density of the gravitational field and express the energy density in terms of g. (2 points)
利用(iii)的答案， 求引力场的能量密度，并以引力场 g 来表达。（2 分）

Q5 (5 points) 题 5（5 分）

A 'magic' transparent sheet can be placed onto the screen of a notebook computer to prevent peeping at large angles. Give a brief explanation of a possible working principle of such sheet.
将一'神奇'透明膜贴在手提电脑的屏幕上，就可防止旁边的人偷看。试给出一个'神奇'透明膜的工作原理。



Please turn to the next page 请转下页

Q6 (10 points) 题 6（10 分）

The operation of an internal combustion engine in automobiles can be modeled by the following four-stroke cycle in the pressure-volume (P-V) diagram as shown in the figure:

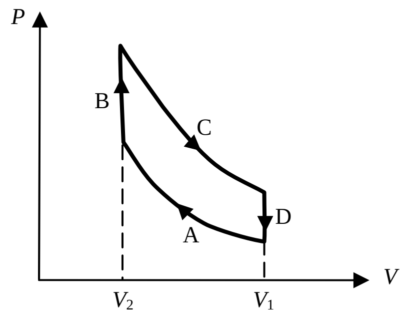
A) Adiabatic compression from V_1 to V_2.
B) Heating raising the temperature from T_1 to T_2 at constant volume.
C) Adiabatic expansion from V_2 to V_1, producing mechanical work.
D) Cooling at constant volume lowering the temperature from T_3 to T_4.

Let n be the number of moles of gas operated in a cycle, C_V and C_P be the molar specific heat of the gas at constant volume and constant pressure, respectively. Also let $\gamma = C_P/C_V$.

(a) Express the temperature ratios T_2/T_3 and T_1/T_4 in terms of the volume ratio V_1/V_2. (2 points)

(b) Let Q be the heat absorbed by the gas in the heating stroke, and W_1 and W_3 be the work done by the gas in the compression and expansion strokes respectively. Calculate the ratios W_1/Q and W_3/Q. (6 points)

(c) Find the efficiency of the engine in terms of V_1, V_2, and γ. (2 points)



汽车内燃机的运转过程可用以上的四冲程过程在压强-体积(P-V)图上表述:

A) 从 V_1 到 V_2 的绝热压缩过程。
B) 等体积燃烧过程将气体温度从 T_1 提升到 T_2 。
C) 从 V_2 到 V_1 的绝热膨胀过程，并作功。
D) 等体积冷却过程将气体温度从 T_3 降到 T_4。

设 n 为气体的摩尔量，C_V 和 C_P 分别为单位摩尔的等体积和等压热容量， $\gamma = C_P/C_V$。

(a) 将温度比 T_2/T_3 和 T_1/T_4 以体积比 V_1/V_2 来表达。（2 分）

(b) 设 Q 为等体积燃烧过程中气体吸的热量，W_1 和 W_3 分别为气体在压缩和膨胀过程中作的功，求功热比 W_1/Q 、W_3/Q。（6 分）

(c) 求内燃机的效率，并以 V_1、V_2 、γ 来表达。（2 分）

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2011

2011 年泛珠三角及中华名校物理奥林匹克邀请赛

Part-2 (Total 3 Problems) 卷-2（共 3 题）

(2:30 pm – 5:30 pm, 02-10-2011)

Math hints 数学提示: $\int \cos(x)dx = \sin(x)$ **;** $(1+x)^{\alpha} \approx 1+\alpha x$ for $x << 1$**.**

Q1　Neutrino Oscillation (15 points)　题 1 中微子振荡（15 分）

(a) Neutrinos are very light particles with masses below 10 eV. Estimate the speed of a neutrino with total energy of 10^8 eV in the unit of the speed of light in vacuum *c*. (3 points)
中微子是具有很小质量（10 电子伏以下）的粒子。估算一个总能量为 10^8 电子伏的中微子的速度（以真空中光速 *c* 为单位）。（3 分）

(b) Neutrinos have one fascinating property that no other particles have, namely they keep changing continuously from one type to another and changing back as time goes by. For example, an electron neutrino (state-1) will change to a muon neutrino (state-2) and change back over a time period of *T*. Its mass will change from m_1 in state-1 to m_2 in state-2. This phenomenon is called ***neutrino oscillation***. Quantum mechanics says that the period *T* is determined by $T = \dfrac{h}{|E_1 - E_2|}$, where E_1 and E_2 are the total energy of the neutrino in state-1 and state-2, respectively, $|E_2 - E_1|$ is the absolute value of the energy difference, and *h* is the Planck constant. For a neutrino of **fixed momentum** *p*, find *T* in terms of m_1, m_2, *h*, *p*, and *c*. (7 points)
中微子有个很有趣的独特性质。随着时间的流逝，它可以逐渐从一种中微子变成另一种中微子，再逐渐变回原来的中微子，周而复始，永不停止。比如，电子中微子（状态-1）可以逐渐变成渺子中微子（状态-2），再逐渐变回成电子中微子。这一现象称为中微子振荡。一个循环所需的时间，即振荡周期，为 *T*。根据量子力学，振荡周期为 $T = \dfrac{h}{|E_1 - E_2|}$，其中 E_1 和 E_2 分别为中微子在状态-1 和状态-2 的总能量，$|E_2 - E_1|$ 是两者之差的绝对值，*h* 为普郎克常数。**若中微子的动量 *p* 为恒定值**，m_1、m_2、*h*、*p*、*c* 为已知，求振荡周期 *T*。（7 分）

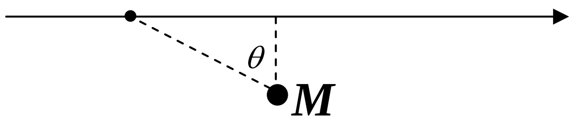
(c) Suppose electron neutrinos with energy of 10^8 eV change to muon neutrinos after traveling through the Earth, (You need to recall roughly the order of magnitude of the Earth diameter.) estimate the minimum value $|m_1^2 - m_2^2|$ in the unit of $(\text{eV})^2$. You may find the constant $hc = 1.24\times10^{-6}\ eV\cdot m$ useful. (5 points)
若能量为 10^8 电子伏的电子中微子在穿过地球的过程中变成了渺子中微子，估求 $|m_1^2 - m_2^2|$ 的最小值（以(电子伏)^2 为单位）。你需要自己给出地球直径的近似值，并可能会用到 $hc = 1.24\times10^{-6}\ eV\cdot m$。（5 分）

Q2　Bending Light with Gravity (15 points)　题 2　引力场里弯曲的光线（15 分）

(a) The path of light bent by gravitational field is an effect of General Relativity. However, one can analysis such phenomenon and obtain an approximate answer using ***Newtonian mechanics*** by assuming that a light beam is made of photons traveling at speed *c*, with the effective mass of each photon being equal to $m = E/c^2$, where *E* is the energy of the photon. Let us first consider the case of a classical particle flying by a fixed point mass *M* at high initial speed *v*. The trajectory of the particle will be bent due to the gravity of the point mass. Because of the high speed of the particle, the bending angle is very small and can be approximately calculated in the following way.

根据广义相对论，光线在引力场里会弯曲。若把光束看作是以光速 c 运动的具有能量 E 的光子，并赋予光子有效质量 $m = E/c^2$，则此现象也可以用***牛顿力学***来解释，并得到近似的答案。首先，我们考虑一初始速度 v 很高的经典粒子，飞过一质量为 M 的固定质点。由于引力作用，粒子的轨迹会弯曲。但由于粒子的初速度很高，弯曲的角度很小，因此可用以下的近似方法求得。

(i) Assume that the trajectory of the particle remains straight, as if the point mass was not there. The distance from the trajectory to the fixed point mass is b, as shown in the figure. Calculate the total impulse the particle receives from the point mass as it travels from the far left to the far right in terms of M, b, G the universal gravity constant, and m the mass of the particle. You may find it easier to use the angle θ as the integration variable. (4 points)



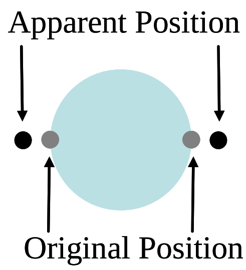
如图所示，假设粒子的轨迹一直为直线，不受引力的影响，轨迹离固定质点的距离为 b，求粒子从左到右飞过固定质点的过程中固定质点对粒子的冲量。在冲量的表达式里可能要用到引力常数 G 和粒子质量 m。用角度 θ 作变量可能简化积分运算。（4 分）

(ii) Determine the bending angle of the particle. (1 point)
求粒子轨迹的弯曲角度。（1 分）

(b) The angular distance of two stars in the sky happens to be the same as the sun's angular diameter.
天空中有两颗星的角距离刚好和太阳的角直径一样。

(i) The sun-earth distance is 1.5×10^{11} m, and the sun's diameter is 1.4×10^{9} m. Calculate the sun's angular diameter in the sky. (1 point)
已知日-地距离为 1.5×10^{11} 米，太阳的直径为 1.4×10^{9} 米。求太阳的角直径。（1 分）

(ii) When the sun happens to be right between the two stars in the sky, the stars appear to be 'pushed outwards' by the sun, as shown in the figure. Draw a diagram to briefly explain the phenomenon. (2 points)
当太阳刚好位于这两颗星的中间时，两颗星好象从它们原来的位置(Original Position) 被推到现在观察到的视觉位置(Apparent Position)。作简图解释此现象。（2 分）



(iii) Calculate the apparent angular distance between the two stars in (ii). The mass of the sun is 2.0×10^{30} kg. The speed of light in vacuum is $c = 3.0 \times 10^{8}\, m/s$. The universal gravity constant is $G = 6.67 \times 10^{-11}\, N \cdot m^2/kg^2$. (5 points)
求(ii)中观察到的两星之间的视觉角距离。已知太阳质量为 2.0×10^{30} kg，真空光速为 $c = 3.0 \times 10^{8}\, m/s$，引力常数为 $G = 6.67 \times 10^{-11}\, N \cdot m^2/kg^2$。（5 分）

(iv) With the present technology, why the phenomenon in (ii) can only be observed during a total solar eclipse? (1 point)
为何上述现象用现有的技术只有在日全食时才能观察到？（1 分）

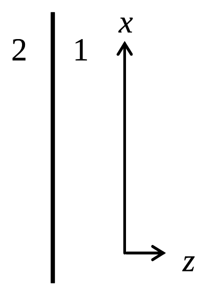
(v) If the moon's orbit size were doubled, could we still observe total solar eclipse? (1 point)
若月球的轨道半径为现在的两倍，还会有日全食现象吗？（1 分）

Q3 Magnetic monopoles (20 points) 题 3 磁单极子（20 分）

The uniqueness theorem of the theory of electric and magnetic fields ensures that virtual (image) charges could be placed ***outside*** the closed space in which the fields are to be determined to mimic the original boundary conditions. The fields ***in*** the closed space are then the same as the ones generated by the real charges inside the space and the virtual charges ***outside*** the space. For an interface without free charge and free electric current, the boundary conditions for electric field $\vec{E}$, electric displacement $\vec{D}$, magnetic field $\vec{B}$, and auxiliary field $\vec{H}$ are $\vec{E}_1^{//} = \vec{E}_2^{//}$, $\vec{D}_1^{\perp} = \vec{D}_2^{\perp}$, $\vec{B}_1^{\perp} = \vec{B}_2^{\perp}$, $\vec{H}_1^{//} = \vec{H}_2^{//}$. Here '⊥'means the components perpendicular to the interface, and '//' means the components parallel to the interface.

利用电、磁场理论的唯一性定理，我们可在一闭合空间外用虚拟电荷分布来产生和原来相同的边界条件，并用这些虚拟电荷分布和空间内原来的电荷分布求出闭合空间内的电、磁场。在一个无自由电荷、自由电流的界面，电场 $\vec{E}$、电位移矢量 $\vec{D}$、磁场 $\vec{B}$、磁辅助场 $\vec{H}$ 的边界条件分别为 $\vec{E}_1^{//} = \vec{E}_2^{//}$，$\vec{D}_1^{\perp} = \vec{D}_2^{\perp}$，$\vec{B}_1^{\perp} = \vec{B}_2^{\perp}$，$\vec{H}_1^{//} = \vec{H}_2^{//}$。这里'⊥'、'//'分别代表垂直、平行于界面的分量。

(a) Consider the case of a point charge q inside medium-1 at a distance d from the interface between two ordinary dielectric media with dielectric constants ε_1 and ε_2, respectively. The Z-axis is perpendicular to the interface, the point charge q is on the Z-axis, and the interface is at z = 0. Two image charges q_1 and q_2 are used to solve this problem. In medium-2, the electric field is equal to what is generated by a point charge of $(q_1 + q/\varepsilon_1)$ at position (0, 0, d). The field in medium-1 is equal to the total field by a point charge q/ε_1 at position (0, 0, d) *and* a point charge q_2 at position (0, 0, –d).



已知介质-1、介质-2 为介电常数分别为 ε_1、ε_2 的普通绝缘体。一点电荷 q 在介质-1 内，离两介质之间的界面的距离为 d。坐标 Z 轴与界面垂直，点电荷 q 在 Z 轴上，界面位置为 z = 0。求解本题需要用到两个虚拟电荷 q_1、q_2。在介质-2 内的电场由在(0, 0, d)的点电荷 $q_1 + q/\varepsilon_1$ 产生；在介质-1 内的电场由在(0, 0, d)的点电荷 q/ε_1 和在(0, 0, –d)的点电荷 q_2 共同产生。

(i) Apply the above boundary conditions to determine q_1 and q_2. (3 points)
利用上述边界条件，求 q_1、q_2。（3 分）

(ii) Determine the ***total*** surface charge density at the interface. (1 point)
求界面上的**总**电荷面密度。（1 分）

(iii) Verify your answer with the special condition $\varepsilon_1 = \varepsilon_2$. (1 point)
用特殊情况 $\varepsilon_1 = \varepsilon_2$ 来验证你的答案。（1 分）

(iv) Verify your answer with the special condition $\varepsilon_1 << \varepsilon_2$. (1 point)
用特殊情况 $\varepsilon_1 << \varepsilon_2$ 来验证你的答案。（1 分）

(b) A magnetic dipole is made of a south pole and a north pole. Interestingly, so far no magnetic monopoles, namely the objects carrying only a south pole or a north pole, have ever been discovered. If a magnetic monopole of 'magnetic charge' g ever exists, it will generate a magnetic field in the same way as an electric point charge generates an electric field. Recently, pseudo particles behaving like magnetic monopoles generated by the collective motions of electrons in a particular type of materials, called the 'topological insulators', was theoretically predicted. Such particle can be induced by an external electric point charge. Similar to part-a, a point electric charge q is placed at a distance d from the interface between an ordinary dielectric medium (medium-1) with dielectric constant ε_1 and magnetic permeability μ_1, and a topological insulator (medium-2) with dielectric constant ε_2, magnetic permeability μ_2, and a magneto-electric coupling constant β. The relations between various fields in medium-1 are $\vec{D}_1 = \varepsilon_1 \vec{E}_1$, and $\vec{H}_1 = \frac{\vec{B}_1}{\mu_1}$.Those in medium-2 are $\vec{D}_2 = \varepsilon_2 \vec{E}_2 - \beta \vec{B}_2$, and $\vec{H}_2 = \frac{\vec{B}_2}{\mu_2} + \beta \vec{E}_2$. The electric field and the magnetic field in both media can again be found by using image charges as in part-a. For simplicity, let us use a set of specially chosen units such that the electric field at position $\vec{r}$ generated by a point electric charge q at position $\vec{r}_0$ is $\vec{E}(\vec{r}) = q\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}$, and the magnetic field produced by a magnetic monopole of magnetic charge g in a similar spatial setting is $\vec{B}(\vec{r}) = g\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}$. In medium-2, the electric field is generated by a point electric charge of $(q_1 + q/\varepsilon_1)$ at position (0, 0, d), and the magnetic field is generated by a magnetic monopole with charge g_1 at position (0, 0, d). In medium-1, the electric field is generated by a point electric charge of q/ε_1 at position (0, 0, d) *and* a point charge q_2 at position (0, 0, $-d$), and the magnetic field is generated by a magnetic monopole with charge g_2 at position (0, 0, $-d$).

磁偶极子由一对南、北极组成。有趣的是，到现在为止我们还没能找到只带南极或北极的所谓磁单极子。一个带‘磁荷’ g 的磁单极子产生的磁场和一个点电荷产生的电场的表达式是一样的。最近，有理论预言，在一种叫拓扑绝缘体（topological insulator）的物质里，电子的运动可以产生象磁单极子这样的准粒子。这种粒子可由点电荷感应产生。与(a)部分类似，一点电荷 q 在普通的介质-1 内，离介质-1 与拓扑绝缘体（介质-2）的界面的距离为 d。介质-1 的介电常数为 ε_1，磁化率为 μ_1。介质-2 的介电常数为 ε_2，磁化率为 μ_2，电磁偶合系数为 β。介质-1 内各个场之间的关系为 $\vec{D}_1 = \varepsilon_1 \vec{E}_1$，$\vec{H}_1 = \frac{\vec{B}_1}{\mu_1}$。介质-2 内各个场之间的关系为 $\vec{D}_2 = \varepsilon_2 \vec{E}_2 - \beta \vec{B}_2$，$\vec{H}_2 = \frac{\vec{B}_2}{\mu_2} + \beta \vec{E}_2$。与(a)部分类似，介质里的电、磁场也能用虚拟电、磁荷来求解。为方便起见，我们选用一套特殊的单位，使一位于 $\vec{r}_0$ 的点电荷 q 在 $\vec{r}$ 处的电场为 $\vec{E}(\vec{r}) = q\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}$，而磁荷 g 产生的磁场为 $\vec{B}(\vec{r}) = g\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}$。在介质-2 内的电场由位于(0, 0, d)的电荷 $q_1 + q/\varepsilon_1$ 产生，磁场由位于(0, 0, d)的磁荷 g_1 产生。在介质-1 内的电场由位于(0, 0, d)的电荷 q/ε_1 和位于(0, 0, $-d$)的电荷 q_2 共同产生，磁场由位于(0, 0, $-d$)的磁荷 g_2 产生。

(i) Use the boundary conditions to determine q_1, q_2, g_1, and g_2. (10 points)
利用边界条件，求 q_1、q_2、g_1、g_2。（10 分）

(ii) Find the **total** surface electric charge density at the interface. To shorten the answers, you may treat q_1 and g_1 as known. (1 point)
求界面上的**总**电荷面密度。为简化起见，可把 q_1、g_1 当作已知。（1 分）

(iii) Find the **total** electric current density at the interface. As the electric current density is a vector, you should find both the components parallel and perpendicular to the interface. To shorten the answers, you may treat q_1 and g_1 as known. (3 points)
求界面上的**总**电流密度。由于电流密度是矢量，你必须给出电流密度垂直和平行于界面的分量。为简化起见，可把 q_1、g_1 当作已知。（3 分）

《THE END 完》

Note: The marking scheme is intended for grading when the final answer is wrong. If the final answer is correct, full mark should be given regardless of the procedure.

PART I

Q1 (8 points)

In a general variable q, if the potential energy $U_p = \frac{1}{2}aq^2$, and the kinetic energy $U_k = \frac{1}{2}b\dot{q}^2$,

then we have frequency $\omega = \sqrt{\frac{a}{b}}$.

(a)

$U_p = mgL(1-\cos\theta) + mg\frac{L}{2}(1-\cos\theta) = \frac{3}{2}mgL(1-\cos\theta) = \frac{1}{2}(\frac{3}{2}mgL)\theta^2$, (1 point)

$U_k = \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}(\frac{4}{3}mL^2)\dot{\theta}^2$, here $I = \frac{1}{3}mL^2$. (1 point)

Then $a = \frac{3}{2}mgL$, and $b = \frac{4}{3}mL^2$, then $\omega = \sqrt{\frac{a}{b}} = \frac{3}{2\sqrt{2}}\sqrt{\frac{g}{L}}$. (2 points)

Alternative solution:

$I = mL^2 + \frac{1}{3}mL^2 = \frac{4}{3}mL^2$. (1 point)

Torque $\tau = mgL\theta + \frac{1}{2}mgL\theta = \frac{3}{2}mgL\theta = K\theta$. (1 point)

$\omega = \sqrt{\frac{K}{I}} = \sqrt{\frac{3\cdot 3mgL}{2\cdot 4mL^2}} = \frac{3}{2\sqrt{2}}\sqrt{\frac{g}{L}}$. (2 points)

(b)

$$U_p = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{x+\delta x} + \frac{2}{L-x-\delta x}\right)$$

$$\simeq \frac{q^2}{4\pi\epsilon_0}\left[\left(\frac{1}{x}+\frac{2}{L-x}\right) - \left(\frac{1}{x^2}-\frac{2}{(L-x)^2}\right)\delta x + \left(\frac{1}{x^3}+\frac{2}{(L-x)^3}\right)\delta x^2\right]. \text{ (1 point)}$$

By properly choosing zero-energy as $\frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{x_0}+\frac{2}{L-x_0}\right)$, and making coefficient of δx zero

to find the balance point x_0, i. e., $\frac{1}{x_0^2} - \frac{2}{(L-x_0)^2} = 0 \Rightarrow x_0 = (\sqrt{2}-1)L$, we have

$$U_p = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{\left(\sqrt{2}-1\right)^3 L^3} + \frac{2}{\left(L-\left(\sqrt{2}-1\right)L\right)^3}\right)\delta x^2 = \frac{(24+17\sqrt{2})q^2}{8L^3\pi\epsilon_0}\delta x^2. \text{ (1 point)}$$

$U_k = \frac{1}{2}m\delta\dot{x}^2$. (1 point)

Then, $a = \frac{(24+17\sqrt{2})q^2}{4\pi L^3\epsilon_0}$ and $b = m$. $\omega = \sqrt{\frac{a}{b}} = \left[\frac{(24+17\sqrt{2})q^2}{2L^3\pi\epsilon_0 m}\right]^{\frac{1}{2}} = 1.96\sqrt{\frac{q^2}{L^3 m\epsilon_0}}$. (1 point)

Alternative solution

Force $F = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{x^2} - \frac{2}{(L-x)^2}\right)$. (1 point)

Balance point at $F = 0$, $x_0 = (\sqrt{2}-1)L$. (1 point)

The elastic constant $K = -\frac{dF}{dx}|_{x=x_0} = \frac{q^2}{2\pi\epsilon_0 L^3}\left(\frac{1}{(\sqrt{2}-1)^3} + \frac{2}{(2-\sqrt{2})^2}\right) \approx 3.82\frac{q^2}{\epsilon_0 L^3}$. (1 point)

$\omega = \sqrt{\frac{K}{m}} = 1.95\sqrt{\frac{q^2}{\epsilon_0 m L^3}}$. (1 point)

Q2 (9 points)

Relativistic total energy is $E = \sqrt{p^2c^2 + m^2c^4}$ (1 point)

Here mc^2 is the rest energy. The kinetic energy is given by $K = \sqrt{p^2c^2 + m^2c^4} - mc^2$. (1 point)

So the momentum is $p = \frac{1}{c}\sqrt{(K+mc^2)^2 - m^2c^4} = \frac{1}{c}\sqrt{K^2 + 2Kmc^2}$. (1 point)

(i) $K << mc^2$, so

$p \approx \frac{1}{c}\sqrt{2mc^2K} = \frac{1}{c}\sqrt{2\times 0.511\times 10^{-6}} = 1.0\times 10^{-3} MeV/c$. (2 points)

(ii) $p = \frac{1}{c}\sqrt{K^2 + 2mcK} = \frac{1}{c}\sqrt{1 + 2\times 0.511} = 1.4 MeV/c$. (2 points)

(iii) $K >> mc^2$. $p \approx \frac{K}{c} = 1.0\times 10^6 MeV/c$. (2 points)

Q3 (8 points)

(a) Let one cable carry line charge density λ, and the other carry $-\lambda$. Choose the line joining the cable centers as the X-axis and choose $x = 0$ at the middle point between the cables, the total electric field on the X-axis is $E = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{d/2+x} + \frac{1}{d/2-x}\right)$. (1 point)

The voltage difference between the two cables is

$V = \int_{-(d/2-R)}^{d/2-R} E\cdot dx = \frac{\lambda}{2\pi\epsilon_0}\left[\ln\left(\frac{d}{2}+x\right) - \ln\left(\frac{d}{2}-x\right)\right]_{-(d/2-R)}^{d/2-R} = \frac{\lambda}{\pi\epsilon_0}\ln\left(\frac{d-R}{R}\right) \approx \frac{\lambda}{\pi\epsilon_0}\ln\left(\frac{d}{R}\right)$. (2 points)

$C = \frac{Q}{V} = \pi\epsilon_0 / \ln\left(\frac{d}{R}\right)$. (1 point)

(b) Employ the method of image charge, the electric field is $E = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{d+x} + \frac{1}{d-x}\right)$. The voltage difference is (1 point)

$V = \int_{-(d-R)}^{0} E\cdot dx = \frac{\lambda}{2\pi\epsilon_0}\left[\ln(d+x) - \ln(d-x)\right]_{-(d-R)}^{0} = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{2d-R}{R}\right) \approx \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{2d}{R}\right)$ (2 points)

$C = 2\pi\epsilon_0 / \ln\left(\frac{2d}{R}\right)$. (1 point)

Q4 (10 points)

(a)

(i) The force of plate-1 on plate-2 is determined by the field generated by plate-1 only. Apply Gauss law to the conducting plate, with a pillbox boundary for $\oint_S \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}$.

Since we are only considering the field of one plate, which is the same on both side of the plate, the field is then $\vec{E} = \frac{\sigma}{2\varepsilon_0}\vec{z}_0$. (1 point)

One can also use integration to calculate the field directly.

$$E=\frac{\sigma}{4\pi\varepsilon_0}\int_0^\infty \frac{2\pi rLdr}{(L^2+r^2)^{3/2}}=\frac{-\sigma}{2\varepsilon_0}\frac{L}{(L^2+r^2)^{1/2}}\Big|_0^\infty=\frac{\sigma}{2\varepsilon_0}.$$

The force per unit area is $\vec{F}=\dfrac{\sigma^2}{2\varepsilon_0}\vec{z}_0$. (1 point)

(ii) The total electric field is the superposition of the field generated by the two plates. The field between the plates is $\vec{E}=\dfrac{\sigma}{\varepsilon_0}\vec{z}_0$. Outside the plates the field is zero. (1 point)

(iii) The work done on an area A of the plate is

$\delta W=AF\cdot\delta D=\dfrac{\sigma^2 A}{2\varepsilon_0}\delta D$. (1 point)

(iv) The volume of the space where the field is non-zero is decreased by $A\delta D$. Let the energy density of the field by H. Then the energy stored in the field is decreased by $HA\delta D$. By energy conservation, $HA\delta D=\dfrac{\sigma^2 A}{2\varepsilon_0}\delta D$. So the energy density is given by $H=\dfrac{\sigma^2}{2\varepsilon_0}=\dfrac{1}{2}\varepsilon_0 E^2$, which is positive. (2 points)

(b)

(i) Gauss law for gravitational field $\oint\!\!\oint_S \vec{g}\cdot d\vec{S}=4\pi GM$

In the same way as part (a), we have $\vec{g}=2\pi G\sigma\vec{z}_0$ (1 point)

The force per unit area is $\vec{F}=2\pi G\sigma^2\vec{z}_0$. (1 point)

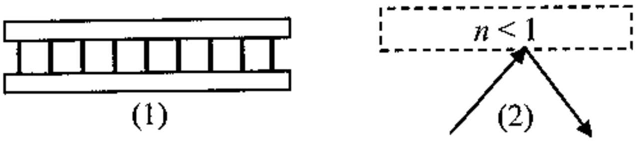
(ii) The field in the space between the plates is zero. Outside the field is $g=4\pi G\sigma$

(iii) $\delta W=AF\cdot\delta D=2\pi G\sigma^2 A\delta D$. (1 point)

(iv) The volume of space where the field is non-zero is ***increased*** by $A\delta D$. The energy stored in the field is decreased by $-HA\delta D$. So $H=-2\pi G\sigma^2=-2\pi G\dfrac{g^2}{(4\pi G)^2}=-\dfrac{1}{2}\dfrac{g^2}{4\pi G}$. (2 points)

Q5 (5 points)

A homogenous sheet with $n>1$ is incorrect. Possible solutions are given below. Any other ones that are correct in principle should also be given full mark.



(1) (2)

Q6 (10 points)

(a) $PV^\gamma=C\Rightarrow NKTV^{\gamma-1}=C$, (1 point)

Then $T_2/T_3=T_1/T_4=(V_1/V_2)^{\gamma-1}$. (1 point)

(b) Internal energy of ideal gas is $U=N\epsilon_0+Nc_VT$, (1 point)

Then $Q=\Delta U=Nc_V(T_2-T_1)$ and $c_V=\dfrac{k}{\gamma-1}$. (1 point)

Work is

$W=\int PdV=\int CV^{-\gamma}dV=\dfrac{C}{1-\gamma}V^{1-\gamma}$. (1 point)

For path A: $C=C_A=P_1V_2^\gamma=NkT_1V_2^{\gamma-1}$, and for path C: $C=C_C=P_2V_2^\gamma=NkT_2V_2^{\gamma-1}$. Then we have

$$W_A = \frac{C_A}{1-\gamma}\left(V_2^{1-\gamma} - V_1^{1-\gamma}\right) = \frac{Nk}{1-\gamma}\left(1-\left(\frac{V_1}{V_2}\right)^{1-\gamma}\right)T_1 = \frac{Nk}{1-\gamma}\left(T_1 - T_4\right) \text{ (1 point)}$$

$$W_C = -\frac{C_C}{1-\gamma}\left(V_2^{1-\gamma} - V_1^{1-\gamma}\right) = -\frac{Nk}{1-\gamma}\left(1-\left(\frac{V_1}{V_2}\right)^{1-\gamma}\right)T_2 = -\frac{Nk}{1-\gamma}\left(T_2 - T_3\right). \text{ (1 point)}$$

Then
$W_A / Q = -(T_1 - T_4)/(T_2 - T_1)$, and $W_C / Q = (T_2 - T_3)/(T_2 - T_1)$. (1 point)

(c)

$$\eta = \frac{W_A + W_C}{Q} = \frac{(T_2 + T_4) - (T_1 + T_3)}{T_2 - T_1}, \text{ (1 point)}$$

or $\eta = \dfrac{k}{c_V(1-\gamma)}\left(1-\left(\dfrac{V_1}{V_2}\right)^{1-\gamma}\right) = \left(1-\left(\dfrac{V_1}{V_2}\right)^{1-\gamma}\right)$. (1 point)

PART II

Question 1 (10 points)

Solution:

(a) The total energy is much larger than the rest energy. So the answer is '1', i. e., the neutrino is moving at speed very close to that of light. (3 points)

(b)

$E_1 - E_2 = \sqrt{p^2c^2 + m_1^2c^4} - \sqrt{p^2c^2 + m_2^2c^4}$ (1 point)

$\approx pc(1 - \dfrac{1}{2}\dfrac{m_1^2c^4}{pc}) - pc(1 - \dfrac{1}{2}\dfrac{m_2^2c^4}{pc}) = \dfrac{1}{2}\dfrac{(m_1^2 - m_2^2)c^4}{pc} = \dfrac{1}{2}\dfrac{\Delta m^2c^4}{pc}$ (4 points)

$T = \dfrac{h}{|E_1 - E_2|} = \dfrac{2Pch}{\Delta m^2 c^4}$. (2 points)

If the expansion is done in (c) instead in (b), full mark should still be given to (b).

(c) From expression of T in (b), we have $\Delta m^2 = \dfrac{2Pch}{Tc^4}$, (1 point)

Substitute $P = E/c$ (1 point) into above: $\Delta m^2 c^4 = \dfrac{2hcE}{Tc}$.

On the other hand, $cT/2 = 2R_{\text{earth}}$, (1 point) and $R_{\text{earth}} = 6\times10^6 m$. (1 point)

Then $\Delta m^2c^4 = \dfrac{2hcE}{4R_{\text{earth}}} = \dfrac{100\times10^6\times1.24\times10^{-6}}{2\times6\times10^6} = (1.0\pm0.5)\times10^{-5}\ \text{eV}^2$. (1 point)

Note: Any answer within the given range should get full mark.

Question 2 (15 points)

Solution:

(a)

(i) The impulse perpendicular to the path is $I_\perp = \int F_\perp(t)dt = \int F(t)\cos\theta(t)dt$. (1 point)

$\tan\theta = x/b$ and $x = vt$, we then have $dt = \dfrac{1}{v}dx = \dfrac{b}{v}d(\tan\theta) = \dfrac{b}{v}\dfrac{1}{\cos^2\theta}d\theta$. (1 point)

Then $I = \int_{-\pi/2}^{\pi/2}\dfrac{GMmb}{b^2v}\cos\theta d\theta = \dfrac{2GMm}{bv}$. (1 point)

The impulse along the path is

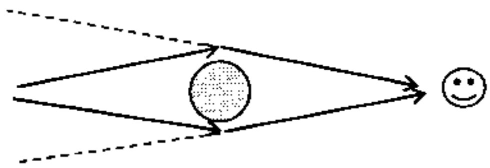
$I_{//} = \int F_{//}(t)dt = \int F(t)\sin\theta(t)dt = \dfrac{GMmb}{b^2v}\int_{-\pi/2}^{\pi/2}\sin\theta d\theta = 0$ (1 point)

(ii) $p = mv$, then $\theta \simeq \frac{I_\perp}{p} = \frac{2GM}{bv^2}$. (1 point)

(b)

(i) The angular diameter of the sun is $\theta_{sun} = R_{sun} / d = \frac{1.4\times10^6}{1.5\times10^8} = 9.3\times10^{-3}$. (1 point)

(ii) Correct drawing (2 points)



(iii)

$$\delta\theta = \frac{2GM}{Rc^2} = \frac{2\times6.67\times10^{-11}\times2\times10^{30}}{1.4\times10^6\times(3\times10^8)^2} = 2.1\times10^{-3},$$

(4 points)

The angular distance is then $\theta = 2\theta_{sum} + 2\delta\theta = 2.280\times10^{-2}$. (1 point)

(iv) No, because the sun is too bright. (1 point)

(v) No, because the angular diameter of the moon would only be half of the sun. (1 point)

Q3 (20 points)

(a)

(i) $\vec{E}_1 = \frac{x\vec{x}_0 + y\vec{y}_0 + (z-d)\vec{z}_0}{\left[x^2+y^2+(z-d)^2\right]^{3/2}}\frac{q}{\varepsilon_1} + \frac{x\vec{x}_0 + y\vec{y}_0 + (z+d)\vec{z}_0}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}q_2$, $z > 0$. (0.5 points)

$\vec{E}_2 = \frac{x\vec{x}_0 + y\vec{y}_0 + (z-d)\vec{z}_0}{\left[x^2+y^2+(z-d)^2\right]^{3/2}}(\frac{q}{\varepsilon_1} + q_1)$, $z < 0$. (0.5 points)

Full marks should be given if $\frac{1}{4\pi\varepsilon_0}$ appears in the above answers.

$\vec{D}_2 = \varepsilon_2\vec{E}_2$, $\vec{D}_1 = \varepsilon_1\vec{E}_1$

$\vec{D}_1^\perp = \vec{D}_2^\perp \Rightarrow \varepsilon_1 q_2 - q = -\varepsilon_2\left(\frac{q}{\varepsilon_1} + q_1\right)$ (1) (0.5 points)

$\vec{E}_1^{//} = \vec{E}_2^{//} \Rightarrow q_2 = q_1$ (2) (0.5 points)

Solving (1) and (2), $q_1 = q_2 = \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right)\left(\frac{q}{\varepsilon_1}\right)$. (1 point)

(ii) $\sigma = 4\pi(E_1^\perp - E_2^\perp) = 4\pi\left(\frac{d}{\left[x^2+y^2+d^2\right]^{3/2}}(q_2 - \frac{q}{\varepsilon_1} + \frac{q}{\varepsilon_1} + q_1)\right) = \frac{8\pi d q_1}{\left[x^2+y^2+d^2\right]^{3/2}}$. (1 point)

Full mark should be given if ε_0 instead of 4π is given.

(iii) $\varepsilon_1 = \varepsilon_2$, $q_2 = q_1 = 0$. No interface and no image charge. (1 point)

(iv) $q_1 = q_2 = -\frac{q}{\varepsilon_1}$. $\vec{E}_2 = 0$, no electric field in perfect metal. $\vec{E}_1$ is determined by an image charge $q_2 = -\frac{q}{\varepsilon_1}$ and a real charge $\frac{q}{\varepsilon_1}$. (1 point)

(b)

(i) $\vec{D}_2 = \varepsilon_2\vec{E}_2 - \beta\vec{B}_2$, $\vec{D}_1 = \varepsilon_1\vec{E}_1$, $\vec{H}_2 = \frac{\vec{B}_2}{\mu_2} + \beta\vec{E}_2$, $\vec{H}_1 = \frac{\vec{B}_1}{\mu_1}$.

$\vec{E}_1 = \frac{x\vec{x}_0 + y\vec{y}_0 + (z-d)\vec{z}_0}{\left[x^2+y^2+(z-d)^2\right]^{3/2}}\frac{q}{\varepsilon_1} + \frac{x\vec{x}_0 + y\vec{y}_0 + (z+d)\vec{z}_0}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}q_2$, $z > 0$. (1 point)

$\vec{E}_2 = \frac{x\vec{x}_0 + y\vec{y}_0 + (z-d)\vec{z}_0}{\left[x^2+y^2+(z-d)^2\right]^{3/2}}(\frac{q}{\varepsilon_1}+q_1)$, $z<0$. (1 point)

$\vec{B}_1 = \frac{x\vec{x}_0 + y\vec{y}_0 + (z+d)\vec{z}_0}{\left[x^2+y^2+(z+d)^2\right]^{3/2}} g_2$, $z>0$. (1 point)

$\vec{B}_2 = \frac{x\vec{x}_0 + y\vec{y}_0 + (z-d)\vec{z}_0}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} g_1$. $z<0$. (1 point)

At z = 0, apply the boundary conditions.

(i) $\vec{B}_1^{\perp} = \vec{B}_2^{\perp}$

$\vec{B}_1^{\perp} = \frac{g_2 d\vec{z}_0}{\left[x^2+y^2+d^2\right]^{3/2}}$, $\vec{B}_2^{\perp} = \frac{-g_1 d\vec{z}_0}{\left[x^2+y^2+d^2\right]^{3/2}}$. $\Rightarrow g_1 = -g_2$ (1). (1 point)

(ii) $\vec{D}_1^{\perp} = \vec{D}_2^{\perp}$

$$\vec{D}_1^{\perp} = \frac{-q d\vec{z}_0}{\left[x^2+y^2+d^2\right]^{3/2}} + \frac{\varepsilon_1 q_2 d\vec{z}_0}{\left[x^2+y^2+d^2\right]^{3/2}},$$

$$\vec{D}_2^{\perp} = \frac{-\left(\frac{q}{\varepsilon_1}+q_1\right)\varepsilon_2 d\vec{z}_0}{\left[x^2+y^2+d^2\right]^{3/2}} + \frac{\beta g_1 d\vec{z}_0}{\left[x^2+y^2+d^2\right]^{3/2}}.$$

So $\varepsilon_1 q_2 - q = \beta g_1 - \varepsilon_2\left(\frac{q}{\varepsilon_1}+q_1\right)$ (2). (1 point)

(iii) $\vec{E}_1^{//} = \vec{E}_2^{//}$

$$\vec{E}_1^{//} = \frac{x\vec{x}_0 + y\vec{y}_0}{\left[x^2+y^2+d^2\right]^{3/2}}\frac{q}{\varepsilon_1} + \frac{x\vec{x}_0 + y\vec{y}_0}{\left[x^2+y^2+d^2\right]^{3/2}} q_2,$$

$$\vec{E}_2^{//} = \frac{x\vec{x}_0 + y\vec{y}_0}{\left[x^2+y^2+d^2\right]^{3/2}}(\frac{q}{\varepsilon_1}+q_1).$$

So $q_2 = q_1$ (3) (1 point)

(iv) $\vec{H}_1^{//} = \vec{H}_2^{//}$

$$\vec{H}_1^{//} = \frac{\vec{B}_1^{//}}{\mu_1} = \frac{x\vec{x}_0 + y\vec{y}_0}{\left[x^2+y^2+d^2\right]^{3/2}}\frac{g_2}{\mu_1},$$

$$\vec{H}_2^{//} = \frac{\vec{B}_2^{//}}{\mu_2} + \beta\vec{E}_2^{//} = \frac{x\vec{x}_0 + y\vec{y}_0}{\left[x^2+y^2+d^2\right]^{3/2}}\frac{g_1}{\mu_2} + \frac{x\vec{x}_0 + y\vec{y}_0}{\left[x^2+y^2+d^2\right]^{3/2}}\beta(\frac{q}{\varepsilon_1}+q_1).$$

So $\frac{g_2}{\mu_1} = \frac{g_1}{\mu_2} + \beta(\frac{q}{\varepsilon_1}+q_1)$ (4). (1 point)

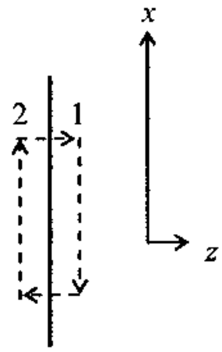
Putting (1) – (3) into (4), we finally get

$q_1 = q_2 = \left(\frac{(\varepsilon_1-\varepsilon_2)(1/\mu_1+1/\mu_2)-\beta^2}{(\varepsilon_1+\varepsilon_2)(1/\mu_1+1/\mu_2)+\beta^2}\right)\left(\frac{q}{\varepsilon_1}\right)$, (1 point)

$g_2 = -g_1 = \left(\frac{\beta}{(\varepsilon_1+\varepsilon_2)(1/\mu_1+1/\mu_2)+\beta^2}\right)\left(\frac{q}{\varepsilon_1}\right)$. (1 point)

In the unit system used here, the Gauss's Law become $\nabla \cdot \vec{E} = 4\pi\rho$, where ρ is the total charge density.
The electric charge sheet density is

$$\sigma = 4\pi(E_1^{\perp} - E_2^{\perp}) = 4\pi\left(\frac{d}{\left[x^2+y^2+d^2\right]^{3/2}}(q_2 - \frac{q}{\varepsilon_1} + \frac{q}{\varepsilon_1} + q_1)\right) = \frac{8\pi d q_1}{\left[x^2+y^2+d^2\right]^{3/2}}.\ \text{(1 point)}$$



Likewise, the Ampere's Law now is $\nabla \times \vec{B} = 4\pi\vec{J}$, where $\vec{J}$ is the electric current density. Do not deduct any points if μ_0 is used instead of 4π. Choose a point on the X-axis, and take a loop with length L and nearly zero height.

$\vec{B}_1^{//} = \frac{g_2}{x^2}\vec{x}_0 = -\frac{g_1}{x^2}\vec{x}_0$, $\vec{B}_2^{//} = \frac{g_1}{x^2}\vec{x}_0$. For a loop perpendicular to the X-axis and along the Y-axis (pointing out of the paper plane), the path integral

$\oint_l \vec{B}\cdot d\vec{l} = \vec{B}_1^{//}\cdot L\vec{y}_0 - \vec{B}_2^{//}\cdot L\vec{y}_0 = 0 - 0 = 0$. (1 point)

For a loop parallel to the X-axis as shown in the figure,

$\oint_l \vec{B}\cdot d\vec{l} = -\vec{B}_1^{//}\cdot L\vec{x}_0 + \vec{B}_2^{//}\cdot L\vec{x}_0 = \frac{Lg_1}{x^2} + \frac{Lg_1}{x^2} = \frac{2Lg_1}{x^2}$.

Le the surface current density be K, then $\frac{2Lg_1}{x^2} = \oint_l \vec{B}\cdot d\vec{l} = 4\pi KL$. (1 point)

Using the right hand rule, we have $\vec{K} = -\frac{g_1}{2\pi x^2}\vec{y}_0$, i. e., pointing into the paper plane. In general, we have $\vec{K} = \frac{g_1}{2\pi}\frac{(y\vec{x}_0 - x\vec{y}_0)}{(x^2+y^2)^{3/2}}$. (1 point)

Any other ways to reach the above expression for the current density are fine.

Pan Pearl River Delta Physics Olympiad 2012
2012 年泛珠三角及中华名校物理奥林匹克邀请赛
Part-1 (Total 6 Problems) 卷-1（共 6 题）
(9:00 am – 12:00 pm, 02-02-2012)

Q1 (5 points) 题 1 （5 分）

It is found that the speed v of the neutrinos with 10 MeV of total energy is $(1-v/c)<2\times10^{-9}$. Estimate the mass of the neutrinos in terms of eV, and determine whether the value you find is an upper or lower limit.
总能量为 10MeV 的微中子的速度 v 为 $(1-v/c)<2\times10^{-9}$。估算以电子伏特为单位的微中子质量，并决定你计算得到的值是上限还是下限。

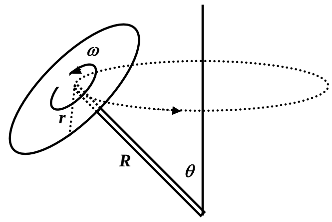
Q2 (10 points) 题 2 （10 分）

Usually we only consider the motion of a simple pendulum of length L in one dimension, while in fact the point mass can move in the horizontal plane, *i. e.*, with two degrees of freedom. Find an initial condition for the point mass such that its simple harmonic motion trajectory in the horizontal plane is (a) a straight line of length D; (b) a circle of radius R; and (c) an ellipse with long axis a and short axis b. All the length scales of the motion are much smaller than L.
通常我们只考虑长度为 L 的单摆一维的运动，但是实际上质点是可以在水平面上运动的，也就是说，有两个自由度。现在水平面上，所有运动的长度量级都远小于 L 的条件下，给出质点的初始速度和位置，使它的简谐运动轨迹是(a)长度为 D 的直线；(b)半径为 R 的圆形；(c)长轴为 a、短轴为 b 的椭圆。

Q3 (9 points) 题 3 （9 分）

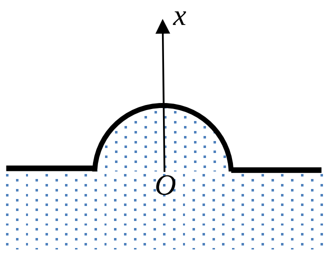
A satellite of mass m is revolving around Earth on a circular orbit of radius R with angular frequency ω_0. It is then hit by a meteoroid with a small impulse I in the inward radial direction. Determine the motion of the satellite afterwards in terms of the position as function of time.
质量为 m 的卫星原来围绕着地球运动，轨道半径为 R，角频率为 ω_0。一颗具有很小冲量 I 的流星沿向内的径向方向撞击了卫星。求此后卫星的位置与时间的关系。

Q4 (6 points) 题 4 （6 分）



As shown in the figure, a gyroscope consists of a uniform disk of radius r and an axle of length R through its center and along its axis. The other end of the axle is hinged on a table but is otherwise free to rotate in any direction. The mass of the disk is much larger than that of the axle. The gyroscope is spinning with angular velocity ω with the axle inclined to the vertical direction. Let g be the gravitational acceleration. Find its angular velocity of precession.
如图所示，一陀螺仪由半径为 r 的均匀盘和长度为 R 的转轴组成，转轴的质量远小于均匀盘。转轴的另外一端铰合在桌上，但是转轴可以在任何方向上自由转动。陀螺仪以角速度 ω 转动，转轴倾斜于垂直方向。令 g 为重力加速度。求陀螺仪进动的角速度。

Q5 (10 points) 题 5 （10 分）

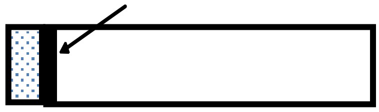


A point charge q is at $x_0 = 3R/2$ on the X-axis in front of a grounded conductor hemisphere of radius R on a large conductor plate perpendicular to the X-axis and in the Y-Z plane. The center of the hemisphere is at (0, 0, 0). Find the potential energy of the point charge. (Note: You must verify that the boundary conditions are preserved if you use image charge(s).)
垂直于 X 轴并在 Y-Z 平面的大导体平板上有一半径为 R 的接地导体半球。

半球的中心位于(0, 0, 0)。在半球前 X 轴上 $x_0 = 3R/2$ 处有电荷量为 q 的点电荷。求点电荷的势能。（注意：如果运用镜像电荷，则必须证明满足边界条件。）

Q6 (10 points) 题 6 （10 分）

As shown, a thermally insulated smooth container with a locked heavy piston contains n moles of single atom ideal gas at temperature T_0 in the left chamber and vacuum in the right chamber. The piston is then released and eventually sticks to the right wall of the container. A η portion of the kinetic energy of the piston is eventually absorbed as heat by the gas. The volume of the whole container is κ times the original volume of the gas.



(a) Find the kinetic energy of the piston right before it hits the wall, and verify your answer for $\kappa \to \infty$. (3 points)

(b) Find the change of entropy of the gas, and proof that the change is positive. (6 points)

(c) Verify your answer in (b) for the case $\eta = 1$. (1 point)

如图所示，一个绝热的光滑容器带有一个被锁定的重活塞。该容器左边的腔室内有 n 摩尔的单原子理想气体，气体温度为 T_0，右边腔室则为真空。然后，活塞被释放，撞击容器右壁后与之相连。最终，活塞动能中的某部分能量（假设该值为 η）被气体以热能的方式吸收。整个容器的体积是原来气体体积的 κ 倍。

(a) 求在活塞撞击右壁之前瞬间活塞的动能，并用 $\kappa \to \infty$ 验证你的答案。（3 分）

(b) 求气体熵的变化，并证明该变化为正值。（6 分）

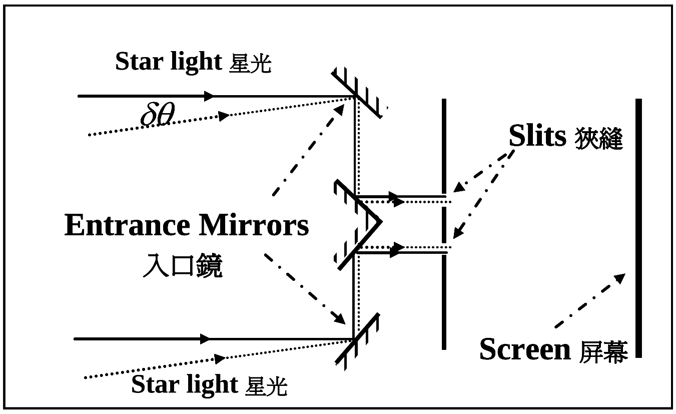
(c) 用 $\eta = 1$ 验证你(b)中的答案。（1 分）

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2012
2012 年泛珠三角及中华名校物理奥林匹克邀请赛
Part-2 (Total 3 Problems) 卷-2（共 3 题）
(2:30 pm – 5:30 pm, 02-02-2011)

Q1 Stella Interferometer (8 points) 题 1 恒星干涉仪（8 分）

A Stella Interferometer is used to accurately measure the angular separation between two close-by stars. As shown, the light from the stars can be treated as two broad parallel light waves of 5.0×10^{-7} meters in wavelength; one (wave-1) is at normal incidence and the other (wave-2) is off by a small angle $\delta\theta = 3.0\times10^{-6}$ degrees. Each wave is then split into two by the two entrance mirrors. The distance between the mirrors is D. There is no phase difference between the two waves split from wave-1 at the entrances. The waves at the entrances are then brought to the two narrow slits of a Young's interference experiment without introducing additional path difference. The entrance mirrors are moved slowly to increase the distance D until the fringes on the screen disappear.



Stella Interferometer 星光干涉儀

(a) Find the value of D. (6 points)

(b) If an optical telescope is used to observe the two stars, what should be the minimum diameter of the primary lens or mirror? (2 points)

恒星干涉仪可用来精确地测量双星之间微小的角距离。如图所示，从两颗恒星发射出来的光可当做两束宽大的平行光，其波长为 5.0×10^{-7} 米；光束-1 正入射，而光束-2 偏移一个微小的角度 $\delta\theta = 3.0\times10^{-6}$ 度入射。随后，每束光被两个入口镜分裂成两束光。镜子间的距离为 D。在入口处，光束-1 分裂出的两束光之间没有相位差。光束随后被引入到杨氏干涉实验的狭缝上，并且在此过程中没有引入任何另外的光程差。慢慢移动入口镜，增加彼此之间的距离 D 直到屏幕上的条纹消失。

(a) 求出 D 的值。（6 分）

(b) 如果用光学望远镜观测这两颗恒星，初级镜的最小直径应该是多少？（2 分）

Q2 Y-Meson (12 points) 题 2 Y 介子（12 分）

Y-mesons with rest mass $m_Y = 1.058\times10^{10}$ eV are produced by colliding electrons with positrons head-on in the reaction $e^+ + e^- \to Y$. Each Y-meson will decay immediately into a pair of B-mesons: $Y \to B^+ + B^-$. The rest mass of the B-mesons is $m_B = 5.28\times10^9$ eV and their lifetime is $\tau_0 = 1.5\times10^{-12}$ seconds. The rest mass of electrons and positrons is 5.11×10^5 eV. When the momentum of the electrons is the same as the positrons in the laboratory frame, the Y-mesons are at rest. ($c = 3.0\times10^8 m/s$)

(a) How large is the decay length (the distance it travels) of the B-mesons in the laboratory? (2 points)

(b) Assume that the B-mesons are moving along the electron-positron trajectory. To increase the decay length of half of the B-mesons, the Y-mesons need to be given momentum in the laboratory frame. This is done by colliding electrons with positrons with different energies. What momentum (in the unit of eV/c) should the B-mesons have if they will have decay length of 0.20 mm? (3 points)

(c) What is the total energy of the Y-mesons before they decay? (4 points)

(d) What should be the energies of the electrons and the positrons in order to produce the Y-mesons in (c)? (3 points)

通过电子和正电子对撞，即反应$e^+ + e^- \to Y$，可以产生静质量为1.058×10^{10} eV 的 Y 介子，而 Y 介子会立刻衰变成一对 B 介子，即反应$Y \to B^+ + B^-$。B 介子的静质量为5.28×10^9 eV，他们的寿命为$\tau_0 = 1.5\times10^{-12}$秒。电子和正电子的静质量为$5.11\times10^5$ eV。在实验室中，当电子的动量与正电子相一致时，Y 介子处于静止状态。（$c = 3.0\times10^8 m/s$）

(a) 在实验室中，B 介子的衰变长度（即走过的距离）是多少？（2 分）

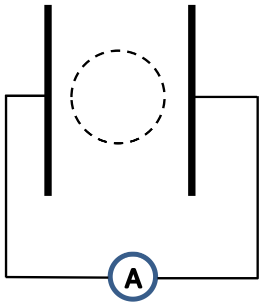
(b) 设 B 介子只沿着电子-正电子的轨迹运动，为了增加一半 B 介子的衰变长度，需要给 Y 介子动量。这可以通过用不同能量的正电子与电子碰撞来实现。如果希望 B 介子的衰变长度为 0.20 mm，那么 B 介子的动量应该为多少（以 eV/*c* 为单位）？（3 分）

(c) 在衰变前，Y 介子的总能量是多少？（4 分）

(d) 为了得到(c)中的 Y 介子，电子和正电子的能量分别应为多少？（3 分）

Q3 Penning Trap (30 points) 潘宁势井 （30 分）

(a) Consider an ion (mass *m* and charge *q*) with initial velocity *v* in the X-Y plane in a uniform magnetic field *B* along the Z-axis. Find its angular frequency ω_c (cyclotron frequency), and the kinetic energy in terms of ω_c and the orbital radius r_0. (1 point)



(b) Consider a point charge *q* between two grounded conductor plates perpendicular to the X-axis separated by a distance *D*. It can be shown that the induced charge on plate-1 and plate-2 are $Q_1 = q\left(\dfrac{x}{D} - 1\right)$ and $Q_2 = -q\dfrac{x}{D}$, respectively, where *x* is the distance between the charge and plate-1. Referring to the figure, find the electric current in the circuit if the center of the ion orbit in (a) is at $x = D/2$. To obtain larger current, should the orbital radius be bigger or smaller? (2 points)

(c) An AC electric field $E(t) = E_0\cos(\omega_c t)$ is applied across plate-1 and plate-2. Suppose the orbit of the ion remains nearly circular for each revolution and the energy gained from the AC electric field in each revolution is much smaller than the kinetic energy of the ion. The charges induced by the ion on the plates can be ignored. After time *T* which is much longer than the orbital period, find the radius of the orbit *R*. (3 points)

(d) The AC electric field in (c) is then turned off when 2*R* is still smaller than *D*. So far the ion is free to move along the z-direction, which means that it can escape from the uniform magnetic field region if it has initial velocity along the z-axis. To prevent that, another electric potential field in the form of $V(\vec{r}) = V_0(z^2 + \beta x^2 + \beta y^2)/z_0^2$, where $V_0 > 0$, is applied so that in the Z-direction the ion can only oscillate around $z = 0$. Find the oscillation frequency ω_z. (2 points)

(e) For the potential field in (d) to be valid in the region of empty space, find the constant β in (d). (1 point)

Such a combination of electric and magnetic field is called a Penning Trap. It is a device to trap an ion for a long time so its cyclotron frequency ω_c, and therefore its *q*/*m* ratio, can be measured with very high precision. Use ω_c and ω_z as known quantities for the remaining part of the question.

(f) The ion is now in a Penning Trap as above. Derive the differential equations for the position of the ion $x(t)$ and $y(t)$ in the X-Y plane. (4 points)

(g) Let $u(t) \equiv x(t) + iy(t)$, where $i \equiv \sqrt{-1}$. Find the differential equation for $u(t)$. (2 points)

(h) Try solution $u(t) = Ae^{-i\omega t}$, determine the two possible frequencies ω_+ and ω_-, with $\omega_+ > \omega_-$. (2 points)

(i) The general solution is then $u(t) = A_+e^{-i\omega_+ t} + A_-e^{-i\omega_- t}$. Suppose the electric trap potential is turned on after the magnetic field is on for a while. When the trap is turned on at $t = 0$ the ion is at $x = R$ on the X-axis, and the center of its orbit is at the origin of the X-Y plane. Determine A_+ and A_-, and take the approximation that $\omega_c >> \omega_z$. (5 points)

(j) To see what the trajectory of the ion looks like in (i), let us go to a rotating reference frame with angular frequency Ω. Using the definition of $u(t)$ in (g), find its expression $\tilde{u}(t)$ in the rotating frame. (3 points)

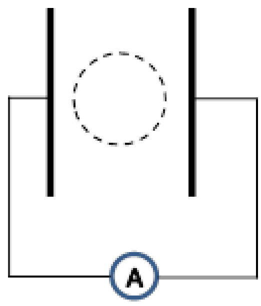
(k) Apply your answer in (j) to the answer in (i), and let $\Omega = \omega_-$. Draw a schematic diagram of the ion trajectory in the rotating frame on the X-Y plane. (1 point)

(l) Draw a schematic diagram of the ion trajectory on the X-Y plane in the laboratory frame. (1 point)

(m) Give three possible shapes of the ion trajectory on the X-Y plane in a rotating frame with angular frequency $\omega_c/2$ if the initial conditions are appropriate. (3 points)

(a) 考虑一个在 X-Y 平面上、初速度为 v 的离子（质量为 m，电量为 q），沿着 Z 方向有均匀的外磁场 B。求它的角频率 ω_c（回旋频率），并且用 ω_c 和轨道半径 r_0 表达它的动能。（1 分）

(b) 假设现有两个垂直于 X 轴的接地导体板，板之间的距离为 D，之间放置了一电量为 q 的点电荷。可以证明，板-1 和板-2 上的感应电荷分别为 $Q_1 = q(\frac{x}{D} - 1)$ 和 $Q_2 = -q\frac{x}{D}$，其中 x 是点电荷到板-1 的距离。如图所示，如果(a)中离子的轨道中心在 $x = D/2$ 处，求电路中的电流。为了获得更大的电流，轨道半径应该变小还是变大？（2 分）



(c) 现在板-1 和板-2 之间加上交流电场 $E(t) = E_0\cos(\omega_c t)$。假设在每次旋转中，离子的轨道还是保持接近于圆形，并且在每周期从交流电场获得的能量远小于离子的动能，同时忽略导体板上的感应电荷，求经过时间 T（远长于轨道周期）后轨道半径 R。（3 分）

(d) 在 $2R$ 大于 D 之前将(c)中的交流电场关掉。到目前为止，离子可以沿着 Z 轴方向自由运动，这就意味着，如果离子在 Z 方向上有初速度，那么它可以逃离均匀磁场区。为了防止此发生，需要再加一个电势场 $V(\vec{r}) = V_0(z^2 + \beta x^2 + \beta y^2)/z_0^2$，其中 $V_0 > 0$，使离子在 Z 方向上只能在 $z = 0$ 附近振动。求振动频率 ω_z。（2 分）

(e) 要使(d)中的电势场在真空中仍然有效，常数 β 应为何值？（1 分）

上述的电磁场组合称为潘宁势井。它可以长时间地捕获住离子，所以能够非常精确地测量出离子的回旋频率以及离子的 q/m 比值。
在解下面的问题时可把 ω_c 和 ω_z 当作是已知量。

(f) 现在，离子处于(e)中所描述的潘宁势井中。推导出离子在 X-Y 平面上位置 $x(t)$ 、$y(t)$ 所满足的微分方程。（4 分）

(g) 令 $u(t) \equiv x(t) + iy(t)$，其中 $i \equiv \sqrt{-1}$。导出 $u(t)$ 所满足的微分方程。（2 分）

(h) 尝试解 $u(t) = Ae^{-i\omega t}$，求出两个可能的频率 ω_+ 和 ω_-，其中 $\omega_+ > \omega_-$。（2 分）

(i) 令通解为 $u(t) = A_+ e^{-i\omega_+ t} + A_- e^{-i\omega_- t}$。假设在开启磁场后经过一段时间再加上电势场。在 $t = 0$ 时刻加上电势场时，离子处在 $x = R$ 处的 X 轴上，其轨道中心位置在 X-Y 平面的原点。计算 A_+ 和 A_-，并取近似 $\omega_c >> \omega_z$。（5 分）

(j) 为了方便确定(i)中离子运行轨迹的简图，假设我们处在一个角频率为 Ω 的旋转参考系中。利用(g)中 $u(t)$ 的定义，求在旋转参考系中它的表达式 $\tilde{u}(t)$。（3 分）

(k) 将你在(j)中得到的答案应用于(i)中，并令 $\Omega = \omega$。画出离子运动轨迹在旋转参考系中 X-Y 平面上的简图。（1 分）

(l) 画出离子在实验室参考系中 X-Y 平面上的运动轨迹简图。（1 分）

(m) 当角频率为 $\omega_c / 2$ 时，若初始条件合适，给出离子在旋转参考系中三种可能的 X-Y 平面上的运动轨迹。（3 分）

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2012
2012 年泛珠三角及中华名校物理奥林匹克邀请赛
Part-1 (Total 6 Problems) 卷-1（共 6 题）
(9:00 am – 12:00 pm, 02-02-2012)

Q1 (5 points)

$E=\gamma mc^2$, where $\gamma \equiv \sqrt{\frac{1}{(1-\beta)(1+\beta)}} \approx \sqrt{\frac{1}{2(1-\beta)}}$.

So $mc^2 = E\sqrt{2(1-\beta)} < 10\cdot\sqrt{2\cdot 2\cdot 10^{-9}} = 632\,\text{eV}$. (4 points)

Upper limit 上限. (1 point)

Q2 (10 points)

设 X-Y 平面为水平面，单摆平衡时质点在（0，0），质点现在位置是（x，y）。利用小幅振动近似的结果，我们知道细绳的张力等于质点的重力，张力在 X-Y 平面的投影的大小为 $mg\frac{\sqrt{x^2+y^2}}{L}$，方向指向（0，0）。因此，X 方向的分力为

$$F_x = -mg\frac{\sqrt{x^2+y^2}}{L}\frac{x}{\sqrt{x^2+y^2}} = -\frac{x}{L}mg \qquad \text{(i)。}$$

Y 方向的分力为 $F_y = -mg\frac{\sqrt{x^2+y^2}}{L}\frac{y}{\sqrt{x^2+y^2}} = -\frac{y}{L}mg$ (ii)。

动力学方程为 $-\frac{x}{L}g = \ddot{x}$，$-\frac{y}{L}g = \ddot{y}$。

通解为

$x(t) = A_x\cos(\omega t) + B_x\sin(\omega t)$，$y(t) = A_y\cos(\omega t) + B_y\sin(\omega t)$，

$\vec{r}(t) = x(t)\vec{x}_0 + y(t)\vec{y}_0$。 $\omega = \sqrt{g/L}$. (2 points)

设一般的初始条件 $\vec{r}(0) = X_0\vec{x}_0 + Y_0\vec{y}_0$，$\dot{\vec{r}}(0) = v_{x0}\vec{x}_0 + v_{y0}\vec{y}_0$，则

$x(t) = X_0\cos(\omega t) + \frac{v_{x0}}{\omega}\sin(\omega t)$，$y(t) = Y_0\cos(\omega t) + \frac{v_{y0}}{\omega}\sin(\omega t)$. (2 points)

As we only ask for examples, there can be many different ways.题目只要求给出例子，所以可以有多种答案。

(a) $X_0 = D/2$, others are 0 其余为 0. Then 则 $x(t) = \frac{D}{2}\cos(\omega t)$，$y(t) = 0$. (2 points)

(b) $\vec{r}(0) = R\vec{x}_0$, $\dot{\vec{r}}(0) = R\omega\vec{y}_0$. Then $x(t) = R\cos(\omega t)$, $y(t) = R\sin(\omega t)$. (2 points)

(c) $\vec{r}(0) = a\vec{x}_0$, $\dot{\vec{r}}(0) = b\omega\vec{y}_0$. Then $x(t) = a\cos(\omega t)$, $y(t) = b\sin(\omega t)$. (2 points)

Q3 (9 points)

In the rotating reference frame, the force along the radial direction is 在跟着卫星转的旋转参照系里，沿半径方向的力为：

$F_r(r) = m\omega^2 r - \dfrac{GMm}{r^2}$, and $F_r(R) = m\omega_0^2 R - \dfrac{GMm}{R^2} = 0$.

After the impact, the total angular momentum is still conserved 碰撞后，角动量仍然守恒:

$0 = d(\omega r^2)_{r=R} = R^2 d\omega + 2\omega_0 R dr$.

Let the orbit radius change by dr, then 令轨道的变化为 dr

$$dF_r(R) = m\omega_0^2 dr + 2\omega_0 R d\omega + m\frac{2GMm}{R^3} dr = -m\omega_0^2 dr.$$

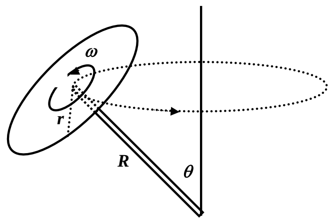
(Note that without taking into account the change of ω, the force is positive and the balance is unstable.若漏了考虑ω的变化，则力的变化是正的，原来的轨道运动变得不平衡了，那是不对的。)

This is a SHM with 上式结果显示卫星的径向运动是简谐振动，力常数为 $k = m\omega_0^2$, so 因此振动频率为 $\omega = \omega_0$.

The initial condition is 初始条件为 $v_0 = I/m$, and 和 $dr = 0$. So $dr(t) = \dfrac{-I}{m\omega_0}\sin(\omega_0 t)$.

$$x(t) = [R - \frac{I}{m\omega_0}\sin(\omega_0 t)]\cos(\omega_0 t),\ y(t) = [R - \frac{I}{m\omega_0}\sin(\omega_0 t)]\sin(\omega_0 t).$$

Q4 (6 points)



Angular momentum is 角动量为 $J = mr^2\omega/2$.

Torque 力矩 $\tau = mgR\sin\theta$, and pointing perpendicular to the paper plane 方向垂直于纸面.

The change of angular momentum is 角动量的变化为 $\Delta J = (J\sin\theta)\Delta\phi$.

$\tau = \dfrac{\Delta J}{\Delta t} = (J\sin\theta)\dfrac{\Delta\phi}{\Delta t} = (J\sin\theta)\Omega$. So $\Omega = \dfrac{2gR}{r^2\omega}$.

Q5 (10 points)

Let the image charge be at x on the X-axis 放个镜像电荷在 X-轴上 x 点处。

$$\Phi(R) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{R^2 + x^2 + 2Rx\cos\theta}} - \frac{q_1}{\sqrt{R^2 + b^2 + 2Rb\cos\theta}}\right) = 0$$

$\Rightarrow q^2(R^2 + b^2 + 2Rb\cos\theta) = q_1^2(R^2 + x^2 + 2Rx\cos\theta)$。

上式对任何角度θ都成立，所以有

$q^2(R^2 + b^2) = q_1^2(R^2 + x^2)$ (i)，

$q^2 b = q_1^2 x$ (ii)。

将 b 代掉，得

$x^2 q_1^4 - q^2(R^2 + x^2)q_1^2 + q^4 R^2 = 0$ (iii)。

解上式，得两个解。第一个是 $q_1 = -q$， $b = x$，即把球外的点电荷中和掉。因为这个镜像电荷没有放在球面内，不在所考虑的解的空间以外，所以不能用，舍去。第二个是 $q_1 = -Rq/x$， $b = R^2/x$。这

就是镜像电荷的值和位置。(It is OK if answers are given without derivations.若没有上述推导而只有答案也得全分。)

In order to make the potential on the plane zero, we need two more image charges, namely $q_3=-q_1$ at $-R^2/x$, and $q_3=-q$ at $-3R/2$. One can see that q_2 and q_3 combined will make the potential on the sphere surface zero. One can see that q_2 and q_3 combined will make the potential on the sphere surface zero. 为了使平面的电势为 0，我们需要另外两的镜像电荷，$q_3=-q_1$ 在 $-R^2/x$, and $q_3=-q$ 在 $-3R/2$. 而 q_2 、q_3 合起来也使球面的电势为 0。(2 points)

电荷受的力为：$F_1=\dfrac{qq_1}{4\pi\varepsilon_0}\dfrac{1}{\left(x-b\right)^2}=\dfrac{-q^2}{4\pi\varepsilon_0}\dfrac{R}{x\left(x-R^2/x\right)^2}=\dfrac{-q^2}{4\pi\varepsilon_0}\dfrac{xR}{\left(x^2-R^2\right)^2}$ 。

现在求 q 的电势，也就是把 q 从无穷远拉到现在位置所需的能量。用作用力做功的方法，

$$W_1=\int_{3R/2}^{\infty}F_1dx=\frac{-q^2R}{4\pi\varepsilon_0}\int_{3R/2}^{\infty}\frac{x}{\left(x^2-R^2\right)^2}dx=\frac{-q^2}{8\pi\varepsilon_0}\left(\frac{R}{d^2-R^2}\right)=\frac{-q^2}{8\pi\varepsilon_0R}\left(\frac{4}{9-4}\right)=\frac{-q^2}{10\pi\varepsilon_0R}$$ 。(4 points)

读者请注意，此势能和直接用电势所得的值是不同的。q 所在位置的电势由 q_1 产生，其值为

$U=\dfrac{q_1}{4\pi\varepsilon_0}\dfrac{1}{\left(d-b\right)}=\dfrac{-q}{4\pi\varepsilon_0}\dfrac{R}{\left(d^2-R^2\right)}=\dfrac{-q}{5\pi\varepsilon_0R}$. 所以 q 的势能为 $W=\dfrac{-q^2}{5\pi\varepsilon_0R}$.

正确答案是错误答案的一半。两者差别的主要原因，是因为镜像电荷的值随真电荷的位置而变。所以计算电荷势能最可靠的方法是用作用力的路径积分来做。

$$W_2=\int_{3R/2}^{\infty}F_2dx=\frac{q^2R}{4\pi\varepsilon_0}\int_{3R/2}^{\infty}\frac{x}{\left(x^2+R^2\right)^2}dx=\frac{q^2}{8\pi\varepsilon_0R}\left(\frac{4}{9+4}\right)=\frac{q^2}{26\pi\varepsilon_0R}.$$

$$W_3=\int_{3R/2}^{\infty}F_3dx=\int_{3R/2}^{\infty}\frac{q^2}{16\pi\varepsilon_0x^2}dx=-\frac{q^2}{24\pi\varepsilon_0R}$$ 。(2 points)

$$W=W_1+W_2+W_3=\frac{q^2}{2\pi\varepsilon_0R}\left(-\frac{1}{5}+\frac{1}{13}-\frac{1}{12}\right)=\frac{-161}{1560}\frac{q^2}{\pi\varepsilon_0R}.$$ (1 point)

Q6 (10 points)

(a) Let $c=\dfrac{3}{2}$, so that $c_v=cnR$. The expansion is an adiabatic process, 膨胀过程为绝热过程，所以有 $PV^{1+1/c}=con.$

The work done to the piston is 对活塞做的功为

$$W=\int_{V_0}^{\kappa V_0}PdV=P_0V_0^{1+1/c}\int_{V_0}^{\kappa V_0}V^{-1-1/c}dV=cP_0V_0(1-\kappa^{-1/c})=\frac{3}{2}nRT_0(1-\kappa^{-2/3}).$$ (2 points)

Maximum work is 最大功为 $W_{\max} = \frac{3}{2} nRT_0$, which is the total internal energy of the gas 等于气体的总内能，也就是气体可做的最大功. (1 point)

(b) $\frac{\Delta S}{nR} = \ln(\kappa) + c\ln(T/T_0) = \ln(\kappa) + c\ln\left(\frac{1}{T_0}(T_0 - \frac{\eta W}{cnR})\right)$

Put in the answer in (a) we get 代入(a) 里功的表达式，得 $\frac{\Delta S}{nR} = \ln(\kappa) + \frac{3}{2}\ln\left(1-\eta+\eta\kappa^{-2/3}\right)$. (5 points)

To see if ΔS is positive, let 令 $\Delta S = cnR\ln(m)$, so $m-1 = (1-\eta)(\kappa^{-2/3}-1) \geq 0$, and $\Delta S > 0$. (1 point)

(c) When $\eta = 1$, $\Delta S = 0$, which is consistent with the fact that there is no net loss in internal energy of the gas. 当 $\eta = 1$, $\Delta S = 0$, which is consistent with the fact that there is no net loss in internal energy of the gas 符合气体自由膨胀时熵不变这一结果。(1 point)

Pan Pearl River Delta Physics Olympiad 2012
2012 年泛珠三角及中华名校物理奥林匹克邀请赛
Part-2 (Total 3 Problems) 卷-2（共 3 题）

(2:30 pm – 5:30 pm, 02-02-2011)

Q1 Stella Interferometer (8 points)

(a) The difference between the path differences of the two pairs of split waves is $\Delta = D\cdot\delta\theta$. When $\Delta = \lambda/2$, the bright fringes of one star coincide with the dark fringes of the other. 两个平面波在两个入口镜的光程差的差为$\Delta = D\cdot\delta\theta$. 当$\Delta = \lambda/2$时, 一颗星的亮条纹与另一颗星的暗条纹刚好重叠，所以： $D = \frac{\lambda}{2\delta\theta} = \frac{180\cdot 5.0\cdot 10^{-7}}{2\cdot 3.14\cdot 3\cdot 10^{-6}} m = 4.8\ m$. (6 points)

(b) $\delta\theta = \frac{1.22\lambda}{D}$, so $D = \frac{1.22\lambda}{\delta\theta} = \frac{1.22\cdot 180\cdot 5.0\cdot 10^{-7}}{3.14\cdot 3\cdot 10^{-6}} m = 12\ m$. (2 points)

Q2 Y-particle (12 points)

(a) The two B-mesons should have the same momentum and energy because the original Y-meson is at rest. The kinetic energy of one B-meson is

$E_k = \frac{1}{2}(10.58 - 2\times 5.28) = 0.01$ GeV, which is much less than the rest energy of B-mesons.

So we can use $E_k = \frac{1}{2}m_B v_0^2$. Putting the numbers in we get $v_0 = \sqrt{\frac{0.01}{5.28}}\ c = 0.0615c$. As the B-mesons are moving at low speed their lifetime change can be ignored. So

$L_0 = v_0\tau_0 = 3\times 10^8 \times 0.0615 \times 1.5\times 10^{-12} = 0.028$ mm.

由于 Y 介子是静止的，所以两个 B 介子的动量相等，方向相反。B 介子的动能为

$E_k = \frac{1}{2}(10.58 - 2\times 5.28) = 0.01$ GeV, 比它的静止质量小亨多，所以可用经典力学

$E_k = \frac{1}{2}m_B v_0^2$ 来求它的速度。将数值代入后得 $v_0 = \sqrt{\frac{0.01}{5.28}}\ c = 0.0615c$. B 介子寿命因运动而延长的效应可忽略，因此

$L_0 = v_0\tau_0 = 3\times 10^8 \times 0.0615 \times 1.5\times 10^{-12} = 0.028$ mm. (2 points)

(b) Rough estimation: the speed of the B-mesons should be $v_0 L/L_0 = 0.44c$, so we need precise formula which takes into account the lifetime change. 粗略估计: B 介子的速度为$v_0 L/L_0 = 0.44c$, 所以必须考虑相对论效应.

$$L = \frac{v\tau_0}{\sqrt{1-(v/c)^2}}. \text{ So } v = \frac{c}{\sqrt{(c\tau_0/L)^2+1}} = 0.406c.$$

$$P = \gamma m_B v = 0.406\times 1.094\times 5.28(GeV/c) = 2.35\ \text{GeV/c (3 points)}$$

(c) The Y-mesons move with the center-of-mass (CoM) frame of the B-mesons. From (a) we get the speed of the B-mesons in that frame as v_0 = 0.0615c. From (b) we know that in the laboratory frame the speed of the B-mesons is v = 0.406c. Let the relative speed between the CoM frame and the laboratory frame be –*u*, which is also the speed of the Y-mesons in the laboratory frame, then $v=\dfrac{v_0+u}{1+v_0u/c^2}$. Solving it we get $u = 0.336c$. Y 介子的速度与 B 介子质心的速度一致，由 (a) 我们知道 B 介子在质心参照系的速度为 $v_0 = 0.0615c$. 由(b) 我们知道 B 介子在实验室参照系里的速度为 $v = 0.406c$. 令质心参照系相对于实验室的速度（也是 Y 介子在实验室的速度）为–*u*, 则 $v=\dfrac{v_0+u}{1+v_0u/c^2}$. 由此得 $u = 0.336c$. The total energy of the Y-mesons in the laboratory frame is Y 介子的总能量为 $E_Y=m_Yc^2/\sqrt{1-(u/c)^2}=1.062m_Yc^2$. (4 points)

(d) In the CoM an electrons has equal and opposite momentum as a positron, while together they must have the energy to create a Y-meson. The momentum 4-vector is then $\begin{pmatrix}0\\ m_Yc\end{pmatrix}$ in CoM. 在 Y 介子参照系里电子和正电子的动量相等，方向相反，总能量等于 Y 介子的静质量。因此 Y 介子的动量-能量 4 矢为 $\begin{pmatrix}0\\ m_Yc\end{pmatrix}$。In the laboratory frame the momentum 4-vector is 在实验室参照系，该 4 矢为

$$\begin{pmatrix}\gamma & \beta\gamma\\ -\beta\gamma & \gamma\end{pmatrix}\begin{pmatrix}0\\ m_Yc\end{pmatrix}=\begin{pmatrix}\beta\gamma m_Yc\\ \gamma m_Yc\end{pmatrix}.$$

Let the momenta of electron and positron be P_1 and P_2, respectively, and also note that as the electron (and positron) rest energy (0.511 MeV) is thousands of times less than that of the Y-meson, so the energy of an electron is simply cP_1. 令电子和正电子的动量分别为 P_1 、P_2，又由于现在电子的能量远大于它的静止质量 (0.511 MeV)，所以它的能量为 cP_1。We then have 由此我们得

$$\begin{pmatrix}P_1-P_2\\ P_1+P_2\end{pmatrix}=\begin{pmatrix}\beta\gamma m_Yc\\ \gamma m_Yc\end{pmatrix}$$, where 其中 $\beta\equiv u/c$.

The electron energy in the laboratory frame is then 电子在实验室的能量为

$$cP_1=\frac{1}{2}(1+\beta)\gamma m_Yc^2=\frac{1}{2}\times1.336\times1.062\times m_Yc^2=0.709m_Yc^2=7.51\ \text{GeV}.$$

正电子的能量为

$$cP_2=\frac{1}{2}(1-\beta)\gamma m_Yc^2=\frac{1}{2}\times0.664\times1.062\times m_Yc^2=0.353m_Yc^2=3.73\ \text{GeV}$$, or vise versa.

(3 points)

Q3 Penning Trap (30 points)

(a) $qBv=m\dfrac{v^2}{r_0}$, so $\omega_c=\dfrac{v}{r_0}=\dfrac{qB}{m}$. $E_k=\dfrac{1}{2}mv^2=\dfrac{1}{2}m\omega_c^2r_0^2$. (1 point)

(b) $x=\dfrac{D}{2}-r\cos(\omega_ct-\phi)$, $Q_1-Q_2=-2q\dfrac{r}{D}\cos(\omega_ct-\phi)$.

$I=\dfrac{dQ}{dt}=2q\dfrac{r_0\omega_c}{D}\sin(\omega_ct-\phi)$. Larger r_0 leads to larger current. (2 points)

(c) The energy gained in each cycle is 每周期得到的能量为

$$dE_k = \int_{T_c} qE_0 \cos(\omega_c t)\cdot r\omega_c \cos(\omega_c t)dt = \frac{1}{2}qE_0 r\omega_c T_c \text{, and } \frac{dE_k}{dt} = \frac{1}{2}qE_0 r\omega_c \quad \text{(1 point)}$$

On the other hand, from (a) we have 由(a) 我们得 $\frac{dE_k}{dt} = m\omega_c^2 r\frac{dr}{dt}$. (1 point)

So $m\omega_c^2 r\frac{dr}{dt} = \frac{1}{2}qE_0 r\omega_c$, and $R = r_0 + \frac{qE_0}{2m\omega_c}T$ (1 point)

(d) $E_z = -2V_0\frac{z}{z_0^2}$. $\omega_z^2 = \frac{2qV_0}{mz_0^2}$. (2 points)

(e) $\nabla^2 V(\vec{r}) = 0$ so $\beta = -1/2$. (1 point)

Use ω_c and ω_z as known for the remaining part of the question.

(f) The electric field in the x-y plane is 沿 X-Y 平面的电场为 $\vec{E} = -V_0\frac{x\vec{x}_0 + y\vec{y}_0}{z_0^2}$. (1 point)

The equation is 粒子的运动方程为

$$m(\ddot{x}\vec{x}_0 + \ddot{y}\vec{y}_0) = eV_0\frac{x\vec{x}_0 + y\vec{y}_0}{z_0^2} + eB(\dot{x}\vec{x}_0 + \dot{y}\vec{y}_0)\times\vec{z}_0 = eV_0\frac{x\vec{x}_0 + y\vec{y}_0}{z_0^2} - eB(\dot{x}\vec{y}_0 - \dot{y}\vec{x}_0)$$

(1 point)

$$\ddot{x} - \omega_c\dot{y} - \frac{1}{2}\omega_z^2 x = 0$$

$$\ddot{y} + \omega_c\dot{x} - \frac{1}{2}\omega_z^2 y = 0 \qquad \text{(2 points)}$$

(g) Multiply *i* to the first equation, and add to the second one, 将第一式乘 *i* 后与第二式相加，得 we get $\ddot{u} + ia\dot{u} + bu = 0$, where $a = \omega_c$ and $b = -\frac{1}{2}\omega_z^2$. (2 points)

(h) $\omega_\pm = \frac{\omega_c \pm \sqrt{\omega_c^2 - \omega_z^2}}{2}$ (2 points)

(i) $x(0) = R$, $y(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = -R\omega_c$ (2 points)

Then $R = A_+ + A_-$, $R\omega_c = \omega_+ A_+ + \omega_- A_-$ (1 point)

Solving the equations we get 解上述方程，得 $A_- = \frac{\omega_+ - \omega_c}{\omega_+ - \omega_-}R = -\frac{1}{2}\left(\frac{\omega_z}{\omega_c}\right)^2 R$,

$A_+ = \frac{\omega_c - \omega_-}{\omega_+ - \omega_-}R = R$ (2 points)

(j) $\tilde{x} = x\cos(\Omega t) + y\sin(\Omega t)$, $\tilde{y} = y\cos(\Omega t) - x\sin(\Omega t)$. So $\tilde{u} = ue^{-i\Omega t}$. (3 points)

(k) $\tilde{u} = ue^{i\omega_- t} = A_- + A_+ e^{-i(\omega_+ - \omega_-)t}$, which is a circle centered at $x = A_-$.这是一个中心在 $x = A_-$ 的圆。(1 point)

(l) (k)中的圆心绕原点作圆周运动。 (1 point)

(m) $\tilde{u} = A_+ e^{-i\omega_1 t} + A_- e^{i\omega_1 t}$ where $\omega_1 = \sqrt{\omega_c^2 - \omega_z^2}$. (1 point)

So $\tilde{x} = (A_+ + A_-)\cos(\omega_1 t)$, $\tilde{y} = (A_+ - A_-)\sin(\omega_1 t)$, which is an ellipse with $(A_+ + A_-)$ being one axis and $(A_+ - A_-)$ being the other. Under more special conditions, the ellipse can become a line or a circle. 这是个椭圆，在特定条件下可变成圆(A_- 或 A_+ 等于 0)，或直线($A_- = \pm A_+$)（(2 points)

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2013
2013 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Part-1 (Total 5 Problems)　卷-1（共 5 题）
(9:00 am – 12:00 pm, Feb. 15, 2013)

Q1 (9 points)

A bead of mass m and initial speed v_0 hits a uniform thin rod of mass m and length L perpendicularly at one end, which initially rests on a horizontal plane.

a) If the other end of the rod is fixed on a hinge which allows the rod to rotate freely in the horizontal plane, and the bead stays on the rod after collision, find the mechanical energy loss due to the collision.
b) If the rod is free to move on the plane and the bead stays on the rod after collision, find the mechanical energy loss due to the collision.
c) The rod is free to move on the plane. The collision is elastic. The velocity of the bead is perpendicular to the rod right after the collision. Find the angular speed of the rod, and the speeds of the bead and the center of mass of the rod.

第一题 (9 分)

一质量为 m 的小球以初速度 v_0 垂直撞击一个质量同为 m 长度为 L 的均匀细杆端点。细杆初始静止于水平面上。

a) 如果杆的另一端固定在一个光滑铰链上使它可以在水平面自由转动，碰撞后小球黏在杆上，计算碰撞所带来的动能损耗。
b) 如果杆可在平面自由运动，碰撞后小球黏在杆上，计算碰撞所带来的动能损耗。
c) 如果杆可在平面自由运动，而碰撞是弹性的。在碰撞后的瞬间，小球的速度垂直于杆。计算碰撞后杆的角速度、质心速度和小球的速度。

Q2 (10 points)

Consider a pair of stars, with masses m_1 and m_2, respectively, bound by their mutual gravity (binary system). Denote their positions as $\vec{r}_1(t)$ and $\vec{r}_2(t)$, respectively.

a) Write down the differential equations for $\vec{r}_1(t)$ and $\vec{r}_2(t)$.
b) Let $\vec{r}_c(t) = \dfrac{m_1\vec{r}_1(t) + m_2\vec{r}_2(t)}{m_1 + m_2}$, $\vec{r}(t) = \vec{r}_1(t) - \vec{r}_2(t)$, find the differential equations for $\vec{r}_c$ and $\vec{r}$.
c) Suppose initially $\vec{r}_c(t=0) = 0$, and $\dfrac{d\vec{r}_c}{dt}(t=0) = 0$, solve the equation in b) for $\vec{r}_c(t)$.
d) Assume that $|\vec{r}(t)| = a$ and a is a time-independent constant, find the solution in b) for $\vec{r}(t)$ with initial condition $\vec{r}(t=0) = a\vec{x}_0$.
e) The period of the orbital motion of a binary system, which is made of two neutron stars each having 1.0 solar-mass, is $T = 3.0\times10^4\,s$. Find the distance between the stars. ($G = 6.7 \times 10^{-11}$ N m^2/kg^2, Mass of the sun = 2.0×10^{30} kg,)

第二题 (10 分)

一对质量分别为m_1 和m_2的星体由万有引力束缚在一起 (双星系统)。用$\vec{r}_1(t)$和$\vec{r}_2(t)$分别表示它们的位置。

a) 写出$\vec{r}_1(t)$和$\vec{r}_2(t)$所遵循的微分方程。

b) 定义$\vec{r}_c(t)=\dfrac{m_1\vec{r}_1(t)+m_2\vec{r}_2(t)}{m_1+m_2}$，$\vec{r}(t)=\vec{r}_1(t)-\vec{r}_2(t)$，写出$\vec{r}_c(t)$和$\vec{r}(t)$遵循的微分方程。

c) 如果$\vec{r}_c(t)$的初始状态为$\vec{r}_c(t=0)=0$、$\dfrac{d\vec{r}_c}{dt}(t=0)=0$，求 b)中微分方程的解$\vec{r}_c(t)$。

d) 如果$|\vec{r}(t)|=a$ 而且a是与时间无关的常数，求 b)中微分方程的解$\vec{r}(t)$。其初始状态为$\vec{r}(t=0)=a\vec{x}_0$。

e) 一个由一对中子星(质量各为 1.0 太阳质量)构成的双星系统的轨道周期是$T=3.0\times10^4 s$。求两中子星之间的距离。(G = 6.7 × 10^{-11} N m^2/kg^2, 太阳质量 = 2.0 × 10^{30} kg)

Q3 (12 points)

A perfect conductor shell of radius R is charge neutral. Its center is at the origin of the X-Y coordinate. The answers for the following should be expressed in terms of L, R, Q, and a fundamental constant, where $L < R$.

a) Find the work done by electrostatics to move a point charge $q_1 = Q$ from the shell center to (L, 0).

b) Another point charge $q_2=\dfrac{L}{R}Q$ is fixed at ($\dfrac{R^2}{L}$, 0), find the work done by electrostatics to move the point charge q_1 from the shell center to (L, 0).

c) If charge q_1 is fixed at (L, 0), find the work done by electrostatics to move charge q_2 from (∞, 0) to ($\dfrac{R^2}{L}$, 0).

d) If charge q_1 is fixed at (L, 0), find the work done by electrostatics to move charge q_2 from $(0,\infty)$ to $(0,\dfrac{R^2}{L})$.

第三题(12 分)

一个中性的理想导体球壳半径为R，中心位于坐标原点。在下述题目中结果须用L、R、Q和相关的基本常数表示。这里$L<R$。

a) 计算把一个点电荷$q_1=Q$从球壳中心移到$(L,0)$的过程中静电力所做的功。

b) 如果另一个点电荷$q_2=\dfrac{L}{R}Q$固定在$(\dfrac{R^2}{L},0)$，计算把q_1从球壳中心移到$(L,0)$的过程中静电力所做的功。

c) 如果固定q_1在$(L,0)$，计算将点电荷q_2从$(\infty,0)$移到$(\dfrac{R^2}{L},0)$静电力所做的功。

d) 如果固定q_1在$(L,0)$，计算将点电荷q_2从$(0,\infty)$移到$(0,\dfrac{R^2}{L})$静电力所做的功。

Q4 (9 points)

Consider a faraway object with speed v and distance L from us. At one time, it is seen by us at certain position in the sky. After some time Δt, it is seen to have moved by a small angular displacement $\Delta\alpha$. Visually, the lateral distance covered by the object is $L\cdot\Delta\alpha$, and its lateral visual speed is $v_{visual}=\frac{L\cdot\Delta\alpha}{\Delta t}$. The actual velocity of the object $\vec{v}$ is at an angle θ to the line-of-sight.

a) If we observe the object by detecting the light it emits, find v_{visual}. Verify if v_{visual} exceeds the speed of light c for $v = 0.9c$ and $\theta = 45^\circ$.

b) If we observe the object by the sound (sound speed is v_s) it emits, like a bat tracking a moth, find v_{visual}, and the range of the object speed such that the object seems to move in the opposite direction as in a).

c) If we observe the object by detecting the solid beads emitted from it, and the relative speed between the beads and the object is v_b. All speeds are much smaller than that of light. Find v_{visual}. Can the object seems to move in the opposite direction as in a)?

Answers should be expressed only in terms of θ and the relevant speeds v, c, v_s, or v_b.

第四题(9 分)

一个遥远的物体速度 v，离我们的距离 L。在某一时刻，我们观察到该物体位于天空中的某一点。过了时间 Δt 之后，我们观察到它移动了一个小小的角位移 $\Delta\alpha$。我们测到的物体横向移动距离为 $L\cdot\Delta\alpha$，而其视觉横向速度为 $v_{視覺}=\frac{L\cdot\Delta\alpha}{\Delta t}$。物体运动的真实速度与观察方向的夹角为 θ。

a) 如果以物体发出的光作为观察信号，求物体的 $v_{視覺}$，并判断当 $v = 0.9c$ 、 $\theta = 45^\circ$ 时 $v_{視覺}$ 是否超过光速 c。

b) 如果以物体发出的声音(其速度为 v_s)作为观察信号，如同蝙蝠追踪飞蛾那样，求物体的视觉横向速度 $v_{視覺}$， 和视觉横向速度与 a) 的视觉运动方向相反时物体的真实速度范围。

c) 如果以物体发射出来的小球作为观察信号，小球与物体间的相对速度为 v_b，所有的速度都远小于光速，求物体的视觉横向速度 $v_{視覺}$。这时可以观测到物体与 a)中视觉运动方向相反的运动么？

答案须以 θ 和相应的速度 v 、 c 、 v_s 或者 v_b 表示。

Q5 (10 points)

Consider the process of pumping a flat tire. The atmospheric temperature is T_a. The atmospheric pressure is P_a. The initial pressure inside the tire is P_i, and the temperature inside is also T_a. After pumping, it reaches a maximum pressure $P_{\max}$ and temperature $T_{\max}$. The tire then cools slowly back to temperature T_a, and the pressure drops to the desired pressure P_f.

Note:

1. Air is ideal gas with heat capacity ratio γ.
2. The volume of the tire is a constant V_0.
3. The process of pumping is adiabatic.
4. Express your answer in terms of γ, V_0, P_i, T_a, P_a, P_f, R, and V_p for part-b.

(a) Pumping by a large compressor (4 points)

The pressure of the air inside the compressor is P_c. The temperature is T_a. Both remain the same during pumping. Find (i) the work done by the compressor, (ii) the internal energy gained by the air inside the tire (including the air pumped into the tire), and (iii) the minimum P_c required to obtain the final desired pressure P_f.

(b) Pumping by a small hand pump (6 points)

The volume of the pump is V_p. The air inside is initially at atmospheric pressure P_a. Because V_p is very small, a large number of strokes of the hand pump is required. Assume that in the j-th stroke, the pressure inside the tire is increased from P_{j-1} to P_j, where $P_0 = P_i$. Our model is that the air inside the pump is first compressed adiabatically so that its pressure increases from P_a to P_j. Then it is transferred into the tire under constant pressure P_j. (i) Find the number of strokes needed to fill up the tire. (ii) Find the internal energy gained by the gas after the j-th stroke in terms of P_j and the given parameters. (iii) By using integration as approximation of discrete sum, show that the pressure inside the tire just after pumping is

$$P_{max} = P_i\left[1+\left(\frac{P_a}{P_i}\right)^{\frac{1}{\gamma}}\left(\frac{P_f-P_i}{P_a}\right)\right]^{\gamma}$$

第五题(10 分)

已知环境温度为T_a，大气压力为P_a，一个轮胎中的初始压力为P_i，温度还是T_a。刚打完气时轮胎最大压强为$P_{\max}$，这时的温度为$T_{\max}$，之后慢慢冷却到T_a，压强则下降到最终的目标压强P_f。

条件：

1. 空气是理想气体，其热容比为γ。
2. 轮胎的体积为常数V_0。
3. 打气过程是绝热过程。
4. 答案用γ、V_0、P_i、T_a、P_a、P_c、P_f 、R 和 V_p (b-部分) 表示。
   a) 用一个大型空气压缩机打气。(4 分)
   设压缩机中的气压 P_c 在打气的过程中维持不变，其温度为T_a。求(i)完成打气压缩机须做的功，(ii)轮胎中空气(包括被打入轮胎的空气)内能的增加，(iii)为了得到最终压强 P_f 所需要的最小 P_c。
   b) 用小型手动打气筒打气。(6 分)
   打气筒的体积为V_p，其中气体的初始压强为P_a。因为V_p很小，因此需要很多次打气冲程。假设在第 j 次冲程，轮胎中的压强从P_{j-1}增加到P_j， $P_0 = P_i$。这里我们假设打气过程如下：空气在打气筒中先绝热压缩使得压强由P_a增加到P_j，而后在恒定压强P_j下输进轮胎。(i)求冲程总数。(ii)求第 j 次冲程后胎内空气内能的增加(用P_j和已知量表达)。(iii)用积分代替离散求和，证明刚打完气时轮胎的压强$P_{\max}$为

$$P_{max} = P_i\left[1+\left(\frac{P_a}{P_i}\right)^{\frac{1}{\gamma}}\left(\frac{P_f-P_i}{P_a}\right)\right]^{\gamma}$$

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2013
2013 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Part-2 (Total 2 Problems) 卷-2（共 2 题）
(2:00 pm – 5:00 pm, Feb. 15, 2013)

Q1 Inertia Inductance and Negative Refraction Medium 惯性电感与负折射介质

A conductor cylinder has certain electromagnetic inductance due to Faraday Effect. Its exact calculation is quite challenging in mathematics. Its order of magnitude, however, is the same as the model in a).
由于法拉第电磁感应效应，一个圆柱形导体是有一定电感的。该电感的严格求解在数学上相当困难。但电感的数量级可以用以下模型来计算。

a) Consider an infinitely long cylinder of radius *r*. The electric current is uniformly distributed in the cross section perpendicular to the cylinder axis. On the surface of the cylinder there is a thin shell of conduct insulated from the cylinder that carries the current in the opposite direction. Find the inductance per length. (1 point)
一个无限长的圆柱导体，其半径为 *r*。电流在导体内部均匀分布，并沿垂直于导体横截面的方向流动。在导体的表面有一层很薄的与圆柱导体绝缘的导体壳，里面的电流沿反方向流动。求导体单位长度的电感。（1 分）

For the rest of the problem there is no shell on the cylinder as in a).
In a real conductor electrons (mass *m*) driven by external electric fields are constantly colliding with defects and impurities. The average effect of such collisions is like a viscosity force $\vec{f}_c = -m\vec{v}/\tau$, where τ is a constant parameter called collision time.

<u>**解以下题目时不用考虑薄导体壳。**</u>

实际导体内的电子(质量 *m*)在外电场作用下运动，并不断与导体中的缺陷、杂质碰撞。碰撞的平均效果相当于粘滞阻力 $\vec{f}_c = -m\vec{v}/\tau$，其中 τ 为常数，称作碰撞时间。

b) Write down the differential equation for the velocity $\vec{v}(t)$ of an electron in the conductor in an external electric field $\vec{E}(t)$. Ignore electron-electron interactions. (1 point)
给出导体中电子在外电场 $\vec{E}(t)$ 作用下运动的微分方程。电子间的相互作用可忽略。（1 分）

c) Suppose $\vec{E}(t) = \vec{E}_0 e^{i\omega t}$, find the velocity of the electron. (2 points)
若外加电场为 $\vec{E}(t) = \vec{E}_0 e^{i\omega t}$，求电子运动的速度。（2 分）

d) Suppose the electron number density in the conductor is *n*, find the electric current density *J*. (2 points)
导体中电子的数量密度为 *n*，求电流密度 *J* 。（2 分）

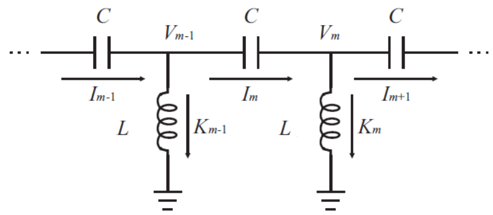
e) For a conductor cylinder (length *D* and radius *r*), find the relation between the voltage across the two ends and the electric current. The inductance in a) can be ignored for the time being. (2 points)
求一长为 *D*，半径为 *r* 的圆柱导体两端的电压与电流的关系。忽略 a)中所求的电感。（2 分）

f) From the answer in e), identify the term which is like a resistor and the one which is like an inductor L_I. L_I is called inertia inductance because it is not due to Faraday's law. (2 points)
在 e)中所得的答案中，找出相当于电阻、电感 L_I 的项。该电感 L_I 称为惯性电感，与法拉第定律无关。（2 分）

g) The resistance of a 1.0 μm long metal wire is 1.0 ohm and the collision time is $\tau = 2.0\times10^{-9}$ s at low temperature, find the value of L_I, and compare it with the inductance due to Faraday Effect. Which one is much bigger? ($\mu_0 = 4\pi\times10^{-7} N/A^2$) (2 points)

在低温下，长为 1.0 μm 的金属导线的电阻为 1.0 ohm，其电子碰撞时间为 $\tau = 2.0\times10^{-9}$ 秒。求惯性电感 L_I，并与电磁感应所形成的电感比较。哪个更大？($\mu_0 = 4\pi\times10^{-7} N/A^2$)。（2 分）

For a wave $A = A_0\cos(kx-\omega t)$, where $\omega = ck$, c is the wave speed. For conventional materials, c is almost a constant, so k increases with increasing ω. However, for some extraordinary artificial materials, c is a strong function of ω, so much so that k decreases with increasing ω. Such phenomenon is called Negative Refraction. It is like having a refraction index that is negative. Consider an infinitely long chain of inductor-capacitor (LC) as shown in the figure. The length of each LC segment is a.



考虑一波动 $A = A_0\cos(kx-\omega t)$，其中 $\omega = ck$，c 为波速。对于传统材料，c 为常数，故 k 随 ω 的增加而增加。但是，对于某些超常人工材料，c 可以是 ω 的函数，以至于在 ω 增加时，k 反而减小。该现象称作负折射。如图所示，考虑一个无限长的电感电容(LC)电路。每个 LC 单元的长度为 a。

h) Find the differential equation that relates $V_{m-1}(t)$, $V_m(t)$, and $V_{m+1}(t)$. (5 points)

给出 $V_{n-1}(t)$、$V_n(t)$、$V_{n+1}(t)$ 所满足的微分方程。（5 分）

i) Assume a wave-like solution $V_m(t) = V_0 e^{i(\omega t - mka)}$, find the relation between ω and k. (2 points)

设方程有波动形式的解 $V_m(t) = V_0 e^{i(\omega t - mka)}$，找出 ω 和 k 的关系。（2 分）

j) Does the answer in i) imply Negative Refraction? (1 point)

请问 i)中答案是否意味着负折射的存在？（1 分）

Q2 Gravitational Waves 引力波 (30 points)（30 分）

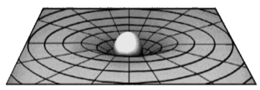
Gravitational waves (GW) are distortion, such as extension/compression, of space. The amplitude of a GW is given by $\varepsilon = \Delta L / L$, where L is the original length of space and ΔL is the change of length. So GW is like elastic waves in solids, except that here space is the medium for the waves.

The necessary constants for answering the following questions are given below.

引力波描述的是空间的扭曲变形(拉伸/压缩)。引力波的振幅为 $\varepsilon = \Delta L / L$，其中 L 是空间的原有长度，ΔL 是引力波所引起的长度变化。所以引力波类似于固体中的弹性波，只不过引力波是以空间为介质。求解以下题目时会用到以下物理常数：

Speed of light in vacuum 真空光速 $c = 3.0\times10^8 m/s$	Mass of the sun 太阳质量 $= 2.0\times10^{30}$ kg
Gravitational constant 引力常数, $G = 6.7\times10^{-11}$ N m^2/kg^2	Sun-Earth distance 太阳−地球距离$= 1.5\times10^{11}$ m
Boltzmann constant 波茨曼常数 $= 1.4\times10^{-23}$ J/K	Mass of Earth 地球质量 $= 6.0\times10^{24}$ kg

- Generation of gravitational waves by a binary star system 双星系统产生的引力波



a) Shown in the figure is the space distortion by a heavy mass. Consider two stars bound by their mutual gravity on a circular orbit. Their masses are M_1 and M_2, respectively. If viewed sideway (in the orbital plane) from Earth, when the (visual) separation between two stars is largest/smallest, is the space between the stars stretched or compressed? (1 point)

图中所示为一重物对空间产生的扭曲。考虑两个由自身引力束缚并作圆周运动的星体 。它们的质量分别为 M_1 和 M_2。从侧面观察（沿轨道所在平面），可见双星的距离随时间变化，并有最大和最小值。问空间何时被拉伸，何时被压缩？（1 分）

b) Such space disturbance in a) is propagating through space as gravitational waves. Suppose the angular frequency of the orbital motion in a) is ω, and the radius of the orbit is R, then the GW emission power L_G of the binary is given by the expression $L_G = A^a B^b M^2 R^4 \omega^6$, where M is called the reduced mass, $M \equiv M_1 M_2 / (M_1 + M_2)$. The constant A is a fundamental physics constant related to Relativity, and constant B is a fundamental physics constant related to Gravity. Determine the two constants and their exponent powers a and b by using dimension analysis. (2 points)
双星对空间的扰动会以引力波的形式向外传播。已知轨道运动的角频率为 ω，轨道半径为 R，则引力波的辐射功率为 $L_G = A^a B^b M^2 R^4 \omega^6$，其中 $M \equiv M_1 M_2 / (M_1 + M_2)$ 是约化质量。常数 A 是与相对论相关的物理常数，B 为与引力相关的物理常数。用量纲分析找出 A、B 和指数 a、b。（2 分）

c) Suppose the motion and the mechanical energy of the binary can be described by classical mechanics, find the total mechanical energy of the orbital motion of the binary system in terms of their orbital period. (2 points)
用经典力学求双星运动的总机械能。答案用公转周期表示。（2 分）

d) Estimate the numerical value of the orbital period decrease rate $\frac{dT}{dt}$ due to gravitational wave emission of a pair of neutron stars, each having 1.0 solar-mass, with orbital period $T = 3.0 \times 10^4 s$. (4 points)
考虑一个由两个相同中子星组成的双星系统，一个中子星的质量等于太阳质量。双星公转周期为 $T = 3.0 \times 10^4 s$。因引力波辐射，其周期会以 $\frac{dT}{dt}$ 逐渐减小。估算 $\frac{dT}{dt}$ 的数值。（4 分）

e) The Sun-Earth can also be considered as a binary system. Find the time needed, in terms of years, for Earth orbital period to reduce by 1 second. (2 points)
太阳-地球同样可被视为双星系统。估算地球公转周期减小 1 秒所需的时间，以年作单位。（2 分）

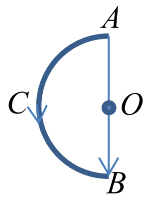
- Estimation of the emission power of a pair of black holes right before coalescence occurs
估算两个黑洞融合前的引力波辐射功率。

f) A black hole can be regarded as a point object of mass M. Find the distance R_s from the black hole where the escape velocity is equal to the speed of light c. (1 point)
黑洞可看作是质量为 M 的质点。距离该质点 R_s 的逃逸速度为光速 c。求 R_s。（1 分）

g) Estimate the numerical value of the emission power of a pair of identical black holes, each having 2 times the solar mass, right before coalescence occurs by using the following procedure: i) Start from the result in b), replace $R\omega$ by orbital speed v; ii) Take the distance between the two black holes as $2R_s$; iii) Replace M by R_s; iv) Take $v \approx c$. (3 points)
通过以下步骤，估算黑洞双星系统即将融合时的引力波辐射功率，一个黑洞的质量等于 2 倍太阳质量：i) 用 b)的结果，用轨道线速度 v 代替 $R\omega$；ii) 设两黑洞间距离为 $2R_s$； iii) 用 R_s 代替 M；iv)取 $v \approx c$。（3 分）

- Detection of gravitational waves 引力波的探测

h) Estimate the space elongation around a black hole by using the following procedure. As shown in the figure, consider a light beam from A to B. Without the black hole at



O the light will travel in a straight line AOB. With the black hole, the space is severely distorted and the light beam will travel along the semi-circle ACB. The distortion of the space ε is close to 0.1, 1, or 10? (2 points)
通过以下步骤，估算黑洞附近空间的拉伸。如图所示，考虑一束光从 A 传播至 B。若黑洞不存在，则光线沿直线 AOB 传播。若 O 点有一黑洞，则空间的扭曲使光线沿半圆 ACB 传播。空间的扭曲ε接近于 0.1, 1, 还是 10？（2 分）

i) Suppose the GW amplitude at the distance R_s from a black hole is the value you choose in h), given that the energy flux of GW is proportional to the square of amplitude, estimate the amplitude of G-wave at 10^4 light years away. (3 points)
假设引力波在距离黑洞 R_s 处的振幅为你在 h)中所选的值。已知引力波的能量通量与振幅的平方成正比，估算该引力波传播至 10^4 光年外的振幅。（3 分）

j) One type of GW detector is a straight uniform aluminum bar. As the bar is elongated by a GW by ΔL, such distortion remains in the bar for several thousand seconds as a standing sound wave, which can be detected by sensitive electronics. In the GW detection project AURIGA in Italy, the length of the bar is 3.0 meters and the speed of sound in aluminum is 6.4×10^3 m/s, determine the lowest resonant frequency of the bar. (2 points)
一根均匀铝棒可作为一种引力波探测器。铝棒被引力波拉伸变长ΔL，该扰动会以声波的驻波形式在铝棒内振荡数千秒钟。在意大利的 AURIGA 引力波探测项目中，铝棒的长度为 3.0 米，铝中声速为 6.4×10^3 m/s。求铝棒的最低共振频率。（2 分）

k) The length of the bar does not remain constant even without the GW because of the thermal fluctuation. The amplitude of the length fluctuation can be determined by treating the vibration of the bar as two half masses (1.1×10^3 kg) joint by a spring with the natural frequency equal to the resonant frequency of the bar in j). Find the average vibration amplitude at 4.2 K due to thermal fluctuation. (4 points)
由于热涨落，铝棒的长度即使在没有引力波的情况下仍然会改变。此时铝棒的运动可被看作两个质量相等的重物(1.1×10^3 kg)在一个弹簧连接下的振动，其频率等于 j)中所求得的共振频率。求温度在 4.2K 时，热涨落导致的铝棒长度涨落。（4 分）

l) Find the maximum distance within which AURIGA can detect a black hole coalescence event in g). (2 points)
用 AURIGA 探测 g)中两个黑洞的融合。求 AURIGA 所能探测的最大距离。（2 分）

m) At such distance in l), suppose one millionth (10^{-6}) of the energy released in the coalescence process in g) is in the form of electromagnetic (EM) waves instead of GW, compare the EM wave energy intensity Earth would be exposed to with that from the sun, which is 1.4×10^3 W/m^2. Are we in danger of global annihilation if this happens? (2 points)
根据 l)中所求得的距离，如果黑洞融合的能量中有百万分之一(10^{-6})是以电磁波的形式释放出来的，相比于地球上接受到太阳的电磁辐射（1.4×10^3 W/m^2），该电磁辐射是否会造成地球生物大灭绝？（2 分）

《THE END 完》

Part-I

Q1 (**9 points**)

Rotational inertia of the beam 杆的转动惯量的一般表达式: $I(l_1,l_2)=\int_{l_1}^{l_2} l^2 m/L dl=\frac{m}{3L}l^3\Big|_{l_1}^{l_2}$.

a) $I=I(0,L)=\frac{m}{3}L^2$

Conservation of angular momentum 角动量守恒:

$I\omega+mL^2\omega=mLv_0 \quad\Rightarrow\quad \omega=\frac{Lmv_0}{I+L^2m}=\frac{3v_0}{4L}$.

Energy lost 动能损失:

$\Delta E=\frac{1}{2}mv_0^2-\left(\frac{1}{2}I\omega^2+\frac{1}{2}mL^2\omega^2\right)=mv_0^2/8$. (3 points)

b) $I=I(-\frac{3}{4}L,\frac{1}{4}L)=\frac{7}{48}mL^2$, or $I=\frac{1}{12}mL^2+m\left(\frac{L}{4}\right)^2=\left(\frac{1}{12}+\frac{1}{16}\right)mL^2=\frac{7}{48}mL^2$

Conservation of momentum and angular momentum 角动量、动量守恒:

$$\begin{cases} mv_0=2mv', \\ I\omega+m\left(\frac{L}{4}\right)^2\omega=m\frac{L}{4}v_0 \end{cases} \Rightarrow \begin{cases} \omega=\frac{6v_0}{5L}, \\ v'=v_0/2 \end{cases}.$$

Energy lost 动能损失:

$\Delta E=\frac{1}{2}mv_0^2-\left(\frac{1}{2}I\omega^2+\frac{1}{2}m\left(\frac{L}{4}\right)^2\omega^2+mv'^2\right)=mv_0^2/10$. (3 points)

c) $I=I(-\frac{1}{2}L,\frac{1}{2}L)=\frac{1}{12}mL^2$

Conservation of momentum, angular momentum and energy 角动量、动量、能量守恒:

$$\begin{cases} mv_0=mv_1+mv_2 \\ I\omega+m\frac{L}{2}v_2=m\frac{L}{2}v_0 \\ \frac{1}{2}I\omega^2+\frac{1}{2}mv_2^2+\frac{1}{2}mv_1^2=\frac{1}{2}mv_0^2 \end{cases} \Rightarrow \begin{cases} \omega=\frac{12v_0}{5L} \\ v_1=\frac{2v_0}{5} \\ v_2=\frac{3v_0}{5} \end{cases} \quad \text{(3 points)}$$

Q2 (**10 points**)

a) $m_1\ddot{\vec{r}}_1=G\frac{m_1m_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_2-\vec{r}_1), \quad m_2\ddot{\vec{r}}_2=G\frac{m_1m_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2)$. (1 point)

b) $\vec{r}_1=\frac{m_2}{m_1+m_2}\vec{r}+\vec{r}_c, \quad \vec{r}_2=\frac{m_1}{m_1+m_2}\vec{r}+\vec{r}_c$.

$$\begin{cases} (m_1+m_2)\ddot{\vec{r}}_c = 0 \quad \Rightarrow \quad \ddot{\vec{r}}_c = 0 \\ \dfrac{m_1 m_2}{m_1+m_2}\ddot{\vec{r}} = -G\dfrac{m_1 m_2}{|\vec{r}|^3}\vec{r} \quad \Rightarrow \quad \ddot{\vec{r}} = -G\dfrac{m_1+m_2}{|\vec{r}|^3}\vec{r} \end{cases} \qquad \text{(2 points)}$$

c) $\vec{r}_c = 0$. (1 point)

d) The equation for $\vec{r}$ is the same as a uniform circular motion，$\vec{r}$ 满足的方程与匀速圆周运动满足的方程一致. Therefore 因此,

$$\frac{G(m_1+m_2)}{a^2} = \omega^2 a \quad \Rightarrow \quad \omega = \frac{\sqrt{G(m_1+m_2)}}{a^{3/2}}.$$

$\vec{r}(t) = a\cos(\omega t)\vec{x}_0 + a\sin(\omega t)\vec{y}_0$. (3 points)

e) $m_1 = m_2 = m$,

$$T = \frac{2\pi a^{3/2}}{\sqrt{2Gm}} \quad \Rightarrow \quad a = \left(\frac{2T^2 Gm}{4\pi^2}\right)^{1/3} = \left(\frac{9\times10^8\times6.7\times10^{-11}\times4\times10^{30}}{4\pi^2}\right)^{1/3} = \left(6.116\times10^{27}\right)^{1/3} = 1.8\times10^9 \text{ m}$$. (3 points)

Q3 (**12 points**)

a) $q' = -QR/x, \quad x' = R^2/x.$

$$f = -\frac{1}{4\pi\varepsilon_0}\frac{Qq'}{(x-x')^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2 Rx}{(x^2-R^2)^2}$$

$$W = \int_0^L f(x)dx = \frac{1}{8\pi\varepsilon_0}\frac{L^2Q^2}{R(R^2-L^2)}.$$ (3 points) 不用积分扣 2 分。

b) Since the outside charge does not create any field inside the conductor sphere, the work done here is no different from the one in a):外电荷对导体内部无影响，所以结果与 a)一样。

$$W = \frac{1}{8\pi\varepsilon_0}\frac{L^2e^2}{R(R^2-L^2)}.$$ (2 points)

c) The static electric field on the charge $q_2 = LQ/R$ outside the conducting sphere is equivalent to a field generated by three point charges: $q_1 = Q$ at $(0,0)$, image charge $q_2' = -q_2R/x$ at $(l' = R^2/l, 0)$, and $q_2'' = -q_2'$ at $(0,0)$. Therefore

球外 $q_2 = LQ/R$ 受的静电力来自以下三个点电荷: $q_1 = Q$ 在 $(0,0)$，镜像电荷 $q_2' = -q_2R/x$ 在 $(l' = R^2/l, 0)$，以及 $q_2'' = -q_2'$ 在 $(0,0)$。

$$f(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_2q_2'}{(x-x')^2} + \frac{q_2(q_1+q_2'')}{x^2}\right] = \frac{1}{4\pi\epsilon_0}\left[-\frac{q_2^2xR}{(x^2-R^2)^2} + \frac{q_2(xq_1+Rq_2)}{x^3}\right].$$ (2 points)

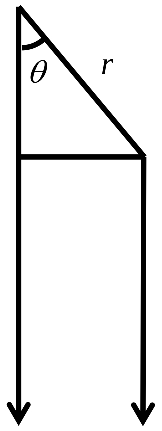
And the work done by electric field is then 电场做功

$$\int_{\infty}^{R^2/L} f(x)dx = \frac{1}{8\pi\varepsilon_0}\left[\frac{L^2 q_2^2}{R^3 - L^2 R} - \frac{Lq_2(Lq_2 + 2Rq_1)}{R^3}\right]$$

$$= \frac{1}{8\pi\varepsilon_0}\left[\frac{L^4 Q^2}{R^5 - R^3 L^2} + \frac{L^2(2R^2 - L^2)Q^2}{R^5}\right] = \underline{\frac{1}{8\pi\varepsilon_0}\frac{L^2(L^4 - 2L^2R^2 + 2R^4)Q^2}{R^5(R^2 - L^2)}} \quad \text{(3 points)}$$

d) $W = W_{\text{in c)}}$ (2 points)

Q4 (**9 points**)



a) $\Delta t = \dfrac{r}{v} - \dfrac{r\cos\theta}{c}$

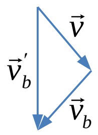
$\underline{\tilde{v} = \dfrac{r\sin\theta}{\Delta t} = \dfrac{cv\sin\theta}{c - v\cos\theta}}$ (2 points)

$\underline{\tilde{v} = \dfrac{0.9\cdot 0.707c}{1 - 0.9\cdot 0.707} = 1.75c}$. (1 point)

b) $\underline{\tilde{v} = \dfrac{v_s v\sin\theta}{v_s - v\cos\theta} < 0 \quad \Rightarrow \quad v_s - v\cos\theta < 0}$ (2 points)

Thus $\underline{v > v_s / \cos\theta}$. (1 point)

c) Let the emission angle of the beads be θ'. Then to ensure the net velocity of the beads is along the Y-direction, we have 令小球的发射角为θ'，为使球的合速度沿 Y-方向，须有



$$v\sin\theta = v_b \sin\theta', \quad v_b' = v\cos\theta + v_b\cos\theta' = v\cos\theta + v_b\sqrt{1 - (v/v_b)^2} > v\cos\theta$$

$$\Delta t = \frac{r}{v} - \frac{r\cos\theta}{v_b'}.$$

$$\underline{\tilde{v} = \frac{v\sin\theta(v\cos\theta + v_b\sqrt{1-(v/v_b)^2})}{v_b\sqrt{1-(v/v_b)^2}} > 0} \quad \text{(2 points)}$$

<u>So it will not happen 不会发生</u>. (1 point)

Q5

(a) The initial number of mole of air in the tire is 轮胎内原有气体摩尔量 $n_i = \dfrac{P_i V_0}{RT_a}$

The final number of mole of air in the tire is 打完气轮胎内有气体摩尔量 $n_f = \dfrac{P_f V_0}{RT_a}$

So number of mole of air pumped into the tire is 因此，打进轮胎的气体摩尔量为

$$n_f - n_i = \frac{\left(P_f - P_i\right)V_0}{RT_a}$$ 。

Inside the compressor, this amount of air has volume 在压缩机里，此摩尔量的气体的体积为

$$V' = \frac{(n_f - n_i)RT_a}{P_c} = \frac{P_f - P_i}{P_c}V_0$$

The work done by the compressor is hence 压缩机做功 $\underline{W_c = P_c V' = \left(P_f - P_i\right)V_0}$ (1 point)

The internal energy of the gas now in the tire increases by 轮胎内气体内能增加为

$$\underline{\Delta E = (n_f - n_i)C_V T_a + W_c = \frac{R}{\gamma - 1}(n_f - n_i)T_a + \left(P_f - P_i\right)V_0} \qquad \text{(1 point)}$$

The maximum temperature is then 因此这时的温度

$$T_{max} = T_a + \frac{W_c}{n_f C_V} = T_a + \frac{\left(P_f - P_i\right)V_0}{\frac{P_f V_0}{RT_a}\frac{R}{\gamma - 1}} = \left[\gamma - (\gamma - 1)\frac{P_i}{P_f}\right]T_a$$

The maximum pressure is 气压

$$P_{max} = \frac{n_f RT_{max}}{V_0} = \left[\gamma - (\gamma - 1)\frac{P_i}{P_f}\right]P_f = \gamma P_f - (\gamma - 1)P_i$$

The minimum P_c required is 最小 P_c 必须满足

$$\underline{P_c \geq P_{max} = \gamma P_f - \left(\gamma - 1\right)P_i} \qquad \text{(2 points)}$$

(b)

The total number of stokes is 打气总次数 $\underline{N = \frac{n_f - n_i}{P_a V_p / RT_a} = \frac{\left(P_f - P_i\right)V_0}{P_a V_p}}$. (1 point)

During the j-th stroke, in the adiabatic compression inside the pump from P_a to P_j 第 j 个斯托克循环绝热压缩使压强从 P_a 变为 P_j

$$P_a V_p{}^{\gamma} = P_j V'^{\gamma}$$

where V' is the volume of the air inside the pump after the adiabatic compression. V'：压缩后气体体积

The internal energy of the air at this moment is 此时气体的内能

$$\frac{1}{\gamma - 1}P_j V' = \frac{1}{\gamma - 1}P_j\left(\frac{P_a}{P_j}\right)^{\frac{1}{\gamma}}V_p$$

The amount of work done to inject this amount of air into the tire is 打气所做的总功

$$P_j V' = P_j\left(\frac{P_a}{P_j}\right)^{\frac{1}{\gamma}}V_p$$

Hence the change in internal energy of the air inside the tire during the j-th stroke is 内能的变化

$$\underline{\Delta U = \frac{1}{\gamma - 1}P_j V' + P_j V' = \frac{\gamma}{\gamma - 1}P_j V' = \frac{\gamma}{\gamma - 1}P_j\left(\frac{P_a}{P_j}\right)^{\frac{1}{\gamma}}V_p} \quad \text{(2 points)}$$

On the other hand, 另一方面

$$\Delta U=\Delta\left(\frac{PV}{\gamma-1}\right)=\frac{1}{\gamma-1}\left(P_j-P_{j-1}\right)V_0=\frac{1}{\gamma-1}V_0\Delta P=\frac{\gamma}{\gamma-1}V_pP_a^{\frac{1}{\gamma}}P_j^{\frac{\gamma-1}{\gamma}}$$

So 因此

$$P_j^{\frac{1-\gamma}{\gamma}}\Delta P=\gamma\frac{V_p}{V_0}P_a^{\frac{1}{\gamma}}$$

Replacing the finite-difference equation by differential equation and integrate 改用微分形式表示

$$P^{\frac{1-\gamma}{\gamma}}dP=\gamma\frac{V_p}{V_0}P_a^{\frac{1}{\gamma}}dj$$

$$\int_{P_i}^{P_{max}}P^{\frac{1-\gamma}{\gamma}}dP=\int_0^N\gamma\frac{V_p}{V_0}P_a^{\frac{1}{\gamma}}dj$$

We have 我们得到

$$\gamma\left(P_{max}^{\frac{1}{\gamma}}-P_i^{\frac{1}{\gamma}}\right)=\gamma\frac{V_p}{V_0}P_a^{\frac{1}{\gamma}}N$$

$$P_{max}^{\frac{1}{\gamma}}-P_i^{\frac{1}{\gamma}}=P_a^{\frac{1-\gamma}{\gamma}}\left(P_f-P_i\right)$$

$$\underline{P_{max}=P_i\left[1+\left(\frac{P_a}{P_i}\right)^{\frac{1}{\gamma}}\frac{P_f-P_i}{P_a}\right]^{\gamma}}\quad\text{(3 points)}$$

Part-II

Q1 **(20 points)**

a) $2\pi rB = \mu_0 J\pi r^2 \Rightarrow B = \mu_0 Jr/2$

Energy per length is 单位长度能量 $W = \dfrac{1}{\mu_0}\int_0^R B^2 2\pi r dr = \dfrac{\mu_0 I^2}{8\pi}$

Compare with 相比较 $\underline{W = \dfrac{1}{2}LI^2 \Rightarrow L = \dfrac{\mu_0}{4\pi}}$ (H/m) (1 point)

b) Newton's second law gives us the differential equation 牛顿第二定律引出的方程

$$m\ddot{\vec{x}} = -e\vec{E} - m\frac{\dot{\vec{x}}}{\tau} \quad \text{(1 point)}$$

c)

$$m\ddot{\vec{x}} = -e\vec{E}_0 e^{i\omega t} - m\frac{\dot{\vec{x}}}{\tau}$$

$$i\omega m\dot{\vec{x}} = -e\vec{E}_0 - m\frac{\dot{\vec{x}}}{\tau}$$

$$\underline{\dot{x} = \frac{-e\tau E_0}{(1+i\omega\tau)m}} \quad \text{(2 points)}$$

d) The current density is given by 电流密度 $\underline{J = -ne\dot{x} = \dfrac{e^2\tau nE_0}{(1+i\omega\tau)m}}$ (2 points)

e) $\dfrac{I}{\pi r^2} = \dfrac{e^2\tau n}{(1+i\omega\tau)m}\dfrac{V}{D}$, so $\underline{V = \dfrac{D}{\pi r^2}\dfrac{(1+i\omega\tau)m}{e^2\tau n}I}$ (2 points)

f) The real part of the impedance represents the resistance 阻抗实部 $\underline{R = \dfrac{D}{\pi r^2}\dfrac{m}{e^2\tau n}}$ (1 point)

The imaginary part represents 虚部 $\omega L_I = \dfrac{D}{\pi r^2}\dfrac{m\omega}{e^2 n}$, so $\underline{L_I = \dfrac{D}{\pi r^2}\dfrac{m}{e^2 n}}$. (1 point)

g) $\underline{L_I = R\tau = 2.0\times10^{-9}(H)}$. $\underline{L_{Faraday} = \dfrac{\mu_0}{4\pi}D = 1.0\cdot10^{-7}\cdot10^{-6} = 1.0\cdot10^{-13}(H)}$ (2 points)

h)

$$V_m - V_{m-1} = \frac{Q}{C} = \frac{I_m}{i\omega C}$$

$$V_m = i\omega LK_m,\ V_{m-1} = i\omega LK_{m-1}$$

$$\begin{cases} I_{m+1} + K_m = I_m \\ I_m + K_{m-1} = I_{m-1} \end{cases}$$

$$\underline{-i\omega C\left(V_{m+1}-2V_m-V_{m-1}\right)+\frac{V_m}{\omega^2 LC}=0}\quad \text{(5 points)}$$

i)

$$e^{-ika}+e^{ika}-2+\frac{1}{\omega^2 LC}=0$$

$$\underline{\omega=\sqrt{\frac{1}{4LC}}\left(\sin\frac{ka}{2}\right)^{-1}}\qquad \text{(2 points)}$$

j) Yes 是(1 point)

Q2 (**30 points**)

a) Largest 最大: compressed 压缩; Smallest 最小: stretched 伸展, or vice versa 反之亦然 (1 point)

b) L_G: Joule/s → N · m/s → kg · m^{2}s^{-3}

RHS: $A^a\mathrm{B}^b\mathrm{kg}^2\mathrm{m}^4\mathrm{s}^{-6}$

$\underline{\Rightarrow\ A^a=c^{-5}, B^b=G}$ (1 point)

$\underline{\Rightarrow\ L_G=c^{-5}GM^2R^4\omega^6}$ (1 point)

c)

$$G\frac{M_1M_2}{R^2}=M_1\omega^2R_1=M_2\omega^2R_2$$

with $R_1+R_2=R$

$$\Longrightarrow\ R^3=\frac{G(M_1+M_2)}{\omega^2}$$

Kinetic energy: $\bar{M}\equiv\frac{M_1M_2}{M_1+M_2}$, $M\equiv M_1+M_2$

$$\underline{KE=\frac{1}{2}M_1\omega^2R_1^2+\frac{1}{2}M_2\omega^2R_2^2=\frac{1}{2}\frac{\bar{M}GM}{R}=\frac{1}{2}(2\pi)^{2/3}\bar{M}(GM)^{2/3}T^{-2/3}}\qquad \text{(2 points)}$$

d) Neutron binary 双中子星系统

$\frac{dE_k}{dT}=-\frac{1}{3}(2\pi)^{2/3}\bar{M}(GM)^{2/3}T^{-5/3}$, (1 point)

$L_G=\frac{dE_k}{dt}=\frac{dE_k}{dT}\frac{dT}{dt}$ (1 point)

$-\frac{dT}{dt}=3G^{\frac{5}{3}}c^{-5}M_1M_2(M_1+M_2)^{-\frac{1}{3}}(2\pi)^{\frac{8}{3}}T^{-\frac{5}{3}}$ (1 point)

$$-\frac{dT}{dt}=1.4\times10^{-67}\cdot4\cdot10^{60}\cdot(2)^{-\frac{1}{3}}(3\cdot10^{4})^{-\frac{5}{3}}=1.4\times10^{-67}\cdot4\cdot10^{60}\cdot0.79\cdot3.4\cdot10^{-8}=1.5\times10^{-14}$$ (1 point)

e) For Sun-Earth system,日-地系统 with $\Delta T=1s$

$$-\frac{dT}{dt}=1.4\times10^{-67}\cdot2\cdot10^{30}\cdot6\cdot10^{24}(365.24\cdot86400)^{-\frac{5}{3}}=1.7\cdot10^{-12}\cdot3.2\cdot10^{-13}=5.4\cdot10^{-25}$$ (1 point)

$$t=\frac{dt}{dT}=1.8\cdot10^{25}s=\frac{1.8\cdot10^{25}}{365.24\cdot86400}=\frac{1.8\cdot10^{25}}{3.155\cdot10^{6}}y=5.7\cdot10^{18}y$$. (1 point)

f) $R_s=\frac{MG}{c^2}$ (1 point)

g) $R_s=GMc^{-2}$, or $R_sc^2G^{-1}=M$. (1 point)

Also $v=\omega R/2$. (1 point)

$$L_G=Gc^{-5}\bar{M}^2R^4\omega^6=16Gc^{-5}c^4G^{-2}R_s^2R_s^{-2}v^6=16\frac{c^5}{G}\left(\frac{v}{c}\right)^6\approx16\frac{c^5}{G}$$ (1 point) 数量级正确即可

$$=16\frac{3^5}{6.7}\cdot10^{40}\cdot10^{11}=5.9\cdot10^{53}W$$ (Order of magnitude only)

h) $\underline{L_0=2R,L'=\pi R\quad\Longrightarrow\quad\varepsilon\approx1}$ (2 points)

i) $(R'/R_s)^2=1/\varepsilon'^2$ with $R'=10^4$light year $=9.46\times10^{19}m,R_s=\frac{M_SG}{c^2}=1.49\times10^3m$

$\varepsilon'=\varepsilon R_s/R'=1.5\cdot1.5\cdot10^3\cdot10^{-29}/0.95=2.4\cdot10^{-22}$. (3 points)

j) $\lambda_1=2L=6\text{m},f_1=\frac{v}{\lambda_1}=1067\text{Hz}$ (2 points)

k) 1st resonance: 第一个共振态

$$\frac{1}{2}m\omega^2x^2=\frac{1}{2}k_BT\Rightarrow x=\left(\frac{k_BT}{m\omega^2}\right)^{1/2}=\left(\frac{42\cdot1.4\cdot10^{-24}}{1.1\cdot10^3}\right)^{1/2}\frac{1}{2\pi\cdot1.07\cdot10^3}=3.3\cdot10^{-17}m$$ (4 points)

l) The change in the length due to gravitational wave should be at least in the same order of magnitude as 10^{-15}m 重力波所引起的长度改变量至少为10^{-15}m量级

$$\frac{x}{L}=\frac{R_s}{D}\Rightarrow D=\frac{R_sL}{x}=\frac{9\cdot10^3}{3.3\cdot10^{-17}}=2.7\cdot10^{20}m=\frac{2.7\cdot10^{20}}{9.5\cdot10^{15}}=2.8\cdot10^4Ly$$ (2 points)

m) $$I=\frac{10^{-6}L_G}{4\pi D^2}=\frac{5.9\cdot10^{53}\cdot10^{-6}}{4\pi\cdot2.7^2\cdot10^{40}}=6.9\cdot10^5(W/m^2)$$. We would be toast. (2 points)

Pan Pearl River Delta Physics Olympiad 2014

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Part-1 (Total 6 Problems) 卷-1（共6 题）

(9:00 am – 12:00 pm, 6 February, 2014)

1. Sunset Twice a Day (6 points) 一天两观日落 （6 分）

Presently the tallest tower in the world is Burj Khalifa in Dubai. Its height is 828 m. An Internet news article reported that one can watch sunset twice in one day with this tower.
杜拜的哈利法塔是现时世界上最高的建筑，高度为 828 米。互联网上有新闻文章报导，可以利用这塔在一天内两次观看到日落。

(a) What is the time range of sunset between the bottom and the top of the tower? Give your answer in minutes. Parameters: Earth’s radius = 6400 km. Distance between Sun and Earth = 1.5×10^{11} m. (3 points)
试求塔底和塔顶之间日落时间的范围。答案请以分钟为单位。参数：地球半径 = 6400 公里。太阳和地球之间的距离 = 1.5×10^{11} 米。（3 分）

(b) Burj Khalifa also has the world’s third fastest elevator (lift) with a speed of v = 10 m/s. Immediately before the elevator starts moving upwards at the speed v from the bottom of the tower, a tourist in the elevator views the sunset. When he reaches the observatory at the height of 452 m, he found that the Sun has risen. Calculate the inclination angle of the Sun above the horizon. Give your answer in degrees. (3 points)
哈里发塔还拥有世界第三快的电梯，速度可达 v = 10 m/s。有电梯内的游客，在电梯从塔底开始上升前一瞬看到日落，其后电梯以速度 v 上升。当他到达在 452 米高度的观景台时，发现太阳上升了。试计算太阳在地平线以上的仰角。答案请以度为单位。（3 分）

2. Radiocarbon Dating (5 points) 放射性碳年龄测定法 （5 分）

Radiocarbon dating is a technique used in archeology to estimate the age of organic materials, such as wood and leather. It uses the fact that the density of ^{14}C atoms in the atmosphere is constantly around 1.3 atoms of ^{14}C in every 10^{12} atoms of all isotopes of carbon. However, when an organism dies, ^{14}C cannot be replenished and decreases due to β decay with a half-life of 5730 years. The radioactive decay can be written in the following form:
放射性碳年龄测定法是考古学上用来估计有机物料（如木材和皮革）年龄的技术。它的根据，在于 ^{14}C 原子在大气中，浓度恒常处于每 10^{12} 粒碳原子中（包括所有同位素）有 1.3 粒 ^{14}C 原子。但是，生物死亡后，^{14}C 不能得到补充，并因β衰变逐渐降低，半衰期为 5730 年。这放射性衰变可以写成以下形式：

$$^{14}_{6}C \rightarrow {}^{14}_{7}N + e^{-} + \overline{\nu}_e$$

(a) Suppose we obtain 50 grams of carbon from a piece of wood dated back to a prehistoric tomb. Using the carbon average atomic mass of 2×10^{-26} kg, calculate the number N_0 of ^{14}C atoms when the wood was still part of a living tree. (1 point)
假设我们从史前古墓的一块木头得到 50 克碳。已知碳的平均原子质量为 2×10^{-26} 千克，试计算木材仍是活树一部分时，^{14}C 原子的数目 N_0。（1 分）

(b) We can determine the age of the tomb if we know the number N of ^{14}C atoms from the 50 grams of carbon. There is no way to directly count the number of ^{14}C atoms, but we detect a total of 935 electrons emitted from the 50 grams of carbon in 10 minutes. How old is the tomb? (3 points)
要估算古墓的年代，我们需要知道该 50 克碳中 ^{14}C 原子的数目 N。我们无法直接数算 ^{14}C 原子的数目，但我们发现 50 克碳在 10 分钟内放射了共 935 粒电子。古墓的年龄是多少？（3 分）

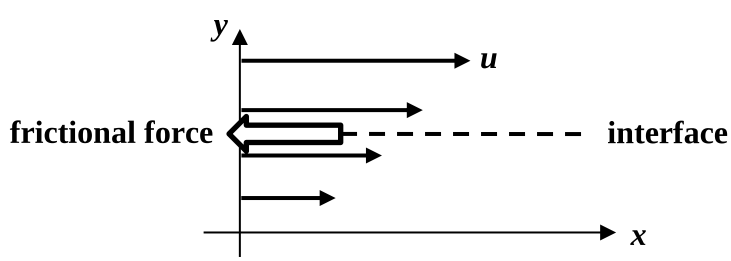
(c) An archaeologist claims that he/she discovered a fossil plant with an age of 2×10^8 years using the method of radiocarbon dating. A scientist says that this result is nonsense. Which side will you stand on? Please explain your reasons. (1 point)
某考古学家声称，他/她利用放射性碳年龄测定法，发现年代为 2×10^8 年的化石植物。某科学家说，这结果是无稽之谈。你认为哪方较合理？请解释你的理由。（1 分）

3. Viscosity (7 points) 粘度 （7 分）

When uneven forces are applied to a fluid, the flow velocities at different locations will be different. For viscous fluids, frictional forces will be present when two adjacent layers of fluids flow at different velocities. As shown in the figure, the viscosity μ of the fluid is defined by the equation:
当不均匀的力施加到流体中，流速在不同的位置将是不同的。对于粘性流体，当相邻的两层流体以不同速度流动时，摩擦力便会存在。如下图所示，流体的粘度μ由下式定义:

$$F = -\mu \frac{du}{dy} \Delta A ,$$

where F is the frictional force experienced by the fluid at an interface of area ΔA in the x direction, u is the x component of the velocity and du/dy is the velocity gradient. In this problem, we will analyze the viscosity using the kinetic theory of gases.
其中F是流体在x方向、面积为ΔA的界面上的摩擦力，u是方向的速度，du/dy是速度梯度。在这问题中，我们将以气体运动理论分析流体的粘度。



Let τ be the average time between successive collisions of a gas molecule with other molecules. Molecule i moves with velocity $\vec{v}_i$ in random directions, and the average velocity at height y is $\bar{u}(y)$.
设τ为气体分子与其他分子连续碰撞之间的平均时间。分子i以速度$\vec{v}_i$沿随机方向运动，而在高度y的平均速度为$\bar{u}(y)$。

(a) Suppose the interface is at a height y. What is the average x component of the momentum at height $y + \Delta y$? (1 point)
假设界面高度为y。在高度$y + \Delta y$的动量，其平均x分量是多少？（1分）

(b) An incident molecule arrives at height y. The y component of its velocity is v_y. What is Δy of the height where the molecule experiences the collision last time? (1 point)
一分子入射到高度y。其速度的y分量为v_y。分子上一次遇到碰撞的高度的Δy是什么？（1分）

(c) Compared with the average x component of the momentum of the gas molecules at the interface, what is the average extra x-momentum carried by the incident molecules of a given v_y when it arrives at height y? (1 point)
当给定v_y的入射分子到达高度y时，它的平均额外x-动量是什么（与界面上的气体分子动量的平均 x 分量相比）？ （1分）

(d) The gas contains n molecules per unit volume. What is the rate of x-momentum transfer through an area ΔA? Hence find an approximate expression for the viscosity of the fluid according to the kinetic theory of gases. How does the viscosity depend on temperature? (4 points)
气体单位体积含有n粒分子。通过面积ΔA的x-动量，传递率是什么？试根据气体运动理论，由此推导流体粘度的近似表达式。粘度与温度有何关系？（4分）

4. Age of the Universe (10 points) 宇宙的年龄 （10 分）

Hubble discovered that the velocities v of galaxies receding from Earth are proportional to their distance d from Earth,
哈勃发现星系远离地球的速度 v 与地球距离 d 成正比，

$$v = H_0 d \ ,$$

where H_0 is the Hubble constant at the present age of the universe. It was recently measured to be 68 km/s/Mpc.
其中 H_0 为宇宙目前的哈勃常数。最近测得为 68 km/s/Mpc。

(a) Assuming that the universe expanded from the beginning to the present at a uniform speed, estimate the age of the universe. Give your answer in billion years. Parameters: 1 Mpc = 3.26×10^6 light years, speed of light = 300,000 km/s. (2 points)
假设宇宙从太初到现在以均匀速率膨胀，试估计宇宙的年龄。答案请以 billion years（十亿年）为单位。参数：1 Mpc = 3.26×10^6 光年，光速 = 300,000 km/s。 （2 分）

(b) However, the universe does not expand at a speed uniform in time due to the gravitational attraction of matter. Friedmann modeled the universe as an expanding sphere of matter with uniform density $\rho(t)$ at time t. Consider a test mass m on the surface of the sphere of radius $r(t)$ at time t. The total energy of the test mass is mU. Find the relation between the expansion velocity $v(t)$ and radius $r(t)$ at time t based on Newtonian mechanics. You may use G to represent the universal gravitational constant. (1 point)
但是，由于物质的万有引力，宇宙膨胀的速率在时间上不是均匀的。弗里德曼模拟宇宙为一膨胀中的均匀密度球体，在时间 t 其密度是$\rho(t)$。考虑在时间 t 时，在半径为 $r(t)$的球体表面上有一测试质量 m。测试质量的总能量为 mU。根据牛顿力学，找出在时间 t 的膨胀速度 $v(t)$和半径 $r(t)$之间的关系。你可用 G 代表万有引力常数。（1 分）

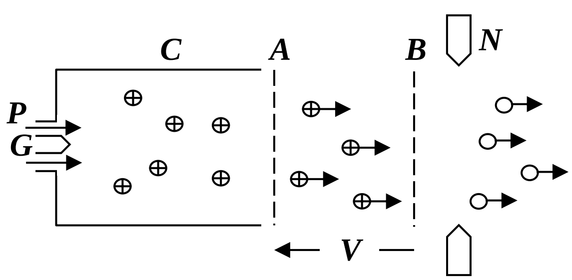
(c) Recent satellite data shows that U is negligible. In this case, the expansion of the universe is described by the power-law $\frac{r(t)}{r_0} = \left(\frac{t}{t_0}\right)^n$, where r_0 and t_0 are the present values of $r(t)$ and t respectively. Find n and t_0. Express your answer in terms of G and the density ρ_0 of the present universe. (4 points)

最近的卫星数据显示，U 可以忽略不计。在这情况下，宇宙的膨胀可用幂律 $\frac{r(t)}{r_0} = \left(\frac{t}{t_0}\right)^n$ 描述，其中 r_0 和 t_0 分别为 $r(t)$ 和 t 的现值。求 n 和 t_0。答案请以 G 和宇宙密度的现值 ρ_0 表达。（4 分）

(d) Express the present age of the universe in terms of the present value of the Hubble constant. Estimate the age of the universe in this Newtonian picture. Give your answer in billion years. Based on your understanding about current developments in physics research, how is this result different from the current estimate of the age of the universe? (3 points)
试以哈勃常数的现值，表达宇宙目前的年龄。试以此牛顿力学的角度，估计宇宙的年龄。答案请以 billion years（十亿年）为单位。根据你对物理学研究当代发展的理解，这结果与当前对宇宙年龄的估计有何不同？（3 分）

5. Electrostatic Ion Thrusters (12 points) 静电离子推进器 （12 分）

Electrostatic ion thrusters are used in spacecraft to control their trajectories in space. Its operating principle is shown in the following figure.
静电离子推进器用于控制航天器在太空的轨迹。它的工作原理如下图所示。



Streams of propellant atoms P are injected into the chamber C. The rate of injection is R, measured in the number of atoms per unit time. The atoms are ionized by bombarding with electrons shot from electron gun G. The positive ions are accelerated from grid electrode A to grid electrode B by the accelerating voltage V between them. The neutralizing electrode N emits electrons to neutralize the ion beam, preventing the spacecraft from gaining a net negative charge.
推进剂原子 P 被喷注入腔室 C。喷注的速率为 R，R 的单位为单位时间内的原子数目。原子被从电子枪 G 射出的电子碰撞而离子化。栅电极 A 到栅电极 B 之间的加速电压 V，使正离子加速。中和电极 N 发射电子，把离子束中和，以防止太空船带负电荷。

(a) Calculate the ratio of thrust F and the current I of the ion beam consisting of ions of mass m and charge ze, where z is a positive integer and e is the electronic charge. Express your answer in m, V, z and e. (4 points)
离子束由质量为 m、电荷为 ze 的离子组成，其中 z 是正整数，e 是电子电荷。试计算推力 F 与离子电流 I 之比。答案请以 m, V, z 和 e 表达。 （4 分）

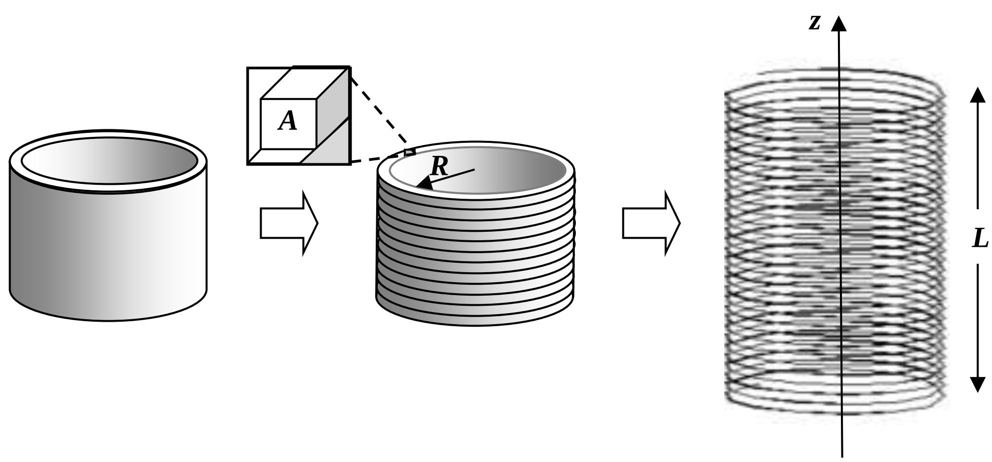
(b) Calculate the ratio of thrust F and the power W spent in accelerating the ion beam. Express your answer in m, V, z and e. (2 points)
试计算推力 F 与加速离子束所耗功率 W 之比。答案请以 m, V, z 和 e 表达。（2 分）

(c) To save power in space travel, should one prefer using light or heavy ions? Should one prefer using ions with single or multiple charges? Should one prefer using low or high accelerating voltages? (3 points)
为节省太空行程的功率，应该使用较轻抑较重的离子？应该使用单电荷离子抑多电荷离子？应该使用低加速电压抑高加速电压？（3 分）

(d) A 10 kW electrostatic ion thruster using xenon atoms as propellant is designed. The accelerating voltage is 10 kV. Calculate the exhaust speed of the ions. Give your answer in km/s. Parameters: ionized xenon carries a single charge, atomic mass of xenon = 131, proton mass = 1.67×10^{-27} kg, electronic charge $e = 1.6 \times 10^{-19}$ C. (1 point)
一个 10 kW 的静电离子推进器的设计，使用氙原子作为推进剂。加速电压为 10 kV。试计算离子排出的速率。答案请以 km/s 为单位。参数：氙离子带单电荷，氙的原子质量 = 131，质子质量 = 1.67×10^{-27} kg，电子电荷 $e = 1.6 \times 10^{-19}$ C。（1 分）

(e) If the neutralizing electrode N of the thruster described in (d) is switched off, calculate the time taken by the body of the spacecraft to gain a voltage equal to the accelerating voltage; at that moment the thruster ceases to operate because the ions follow the thruster. Assume that the spacecraft is spherical and has a radius of 1 m. Parameters: $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m. (2 points)
若在(d)中描述的推进器的中和电极 N 被关闭，试计算航天器身体上的电压变至与加速电压相等所需的时间；在那一刻因为离子不能离开推进器，将导致推进器停止操作。可假设航天器是球形的，半径为 1 m。参数：$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m。（2 分）

6. Slinky (10 points) 机灵鬼 （10 分）



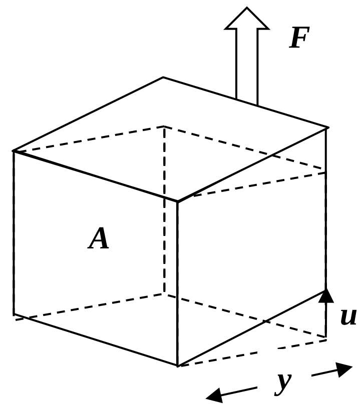
The slinky is a spring first put on sale in 1940's, and soon became a popular toy. As shown in the figure, a slinky can be manufactured from a hollow metal cylinder of radius R by cutting it into a helical thin strip. The helix consists of N turns and has a cross sectional area A. Let ρ be the density of the metal.

机灵鬼弹簧首次于 1940 年代发售，很快便成为一种流行的玩具。如图所示，一個机灵鬼由半径为 R 的空心金属圆筒切割成螺旋形的薄带。螺旋线有 N 匝，其横截面面积为 A。设ρ是金属的密度。

(a) In this problem we assume that the deformation of a stretched slinky is mainly due to shear deformation. Let G be the shear modulus of the metal. What is the tension T in the slinky when it is stretched to a length L that is much greater than its original length?
在这问题中，我们假设机灵鬼被拉伸时的形变，主要是剪切形变。设 G 是金属的剪切模量。當把机灵鬼拉伸到长度 L 时（L 比机灵鬼原本的长度大得多），机灵鬼中的张力 T 是什么？

The shear modulus G of a solid is defined as $G = \dfrac{F\ /\ A}{u\ /\ y}$ where, as shown in the figure, F is the force acting on the vertical side of the solid with area A, y is the width of the solid, and u is the shear distortion of the solid. (2 points)

如图所示，固体的剪切模量 G 被定义为$G = \dfrac{F\ /\ A}{u\ /\ y}$， 其中 F 是作用在固体侧面（面积为 A）的力，y 为固体的宽度，u 是固体的剪切形变。（2 分）



(b) To study how distortions propagate as a longitudinal wave in the slinky stretched to length L, we approximate the slinky by discrete particles separated by small distance ds connected by strings with tension T. Let $u_n(t)$ by the displacement of the n^{th} particle at time t. Derive the equation of motion of the particles. Neglect gravitational effects. (3 points)
为了研究形变如何以纵波在长度拉至 L 的机灵鬼上传播，我们将机灵鬼近似为一串离散的粒子，间距为 ds，由张力为 T 的繩子连接起来。设 $u_n(t)$ 是第 n 个粒子在时间 t 的位移。试推导粒子的运动方程。可忽略重力效应。（3 分）

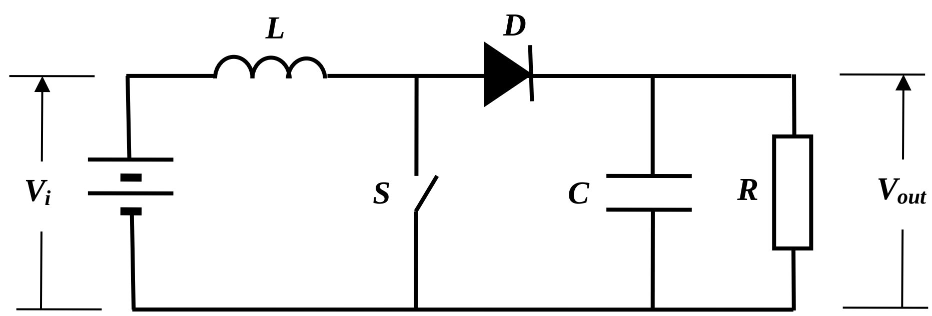
(c) Show that $u_n(\mathrm{t}) = C\sin(kz_n - \omega t)$ is a solution of the equation of motion, where z_n is the position of the n^{th} particle along the axis of the slinky. Find the relation between k and ω. Hence find the velocity of longitudinal wave propagation along the axis of the slinky. (5 points)
试证明 $u_n(\mathrm{t}) = C\sin(kz_n - \omega t)$是运动方程的解，其中 z_n 是沿机灵鬼轴线第 n 个粒子的位置，试找出 k 和ω之间的关系。由此推導沿机灵鬼轴线传播的纵波速度。（5 分）

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2014
2014 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Part-2 (Total 2 Problems) 卷-2（共2 题）
(2:00 pm – 5:00 pm, 6 February, 2014)

1. DC Step-up Converter (25 points) 增压转换器 （25 分）

Modern electric and gasoline hybrid cars require high voltages to drive their motors from batteries of lower voltages. Alternating current (AC) voltages can be stepped up easily by using transformers, but direct current (DC) voltages require more sophisticated designs. In this problem we analyze the step-up converter circuit as shown in the following figure.
现代电力和汽油混合动力汽车需要从低电压的电池产生的高电压驱动马达。交流电（AC）电压可以很容易地通过使用变压器增强，但直流（DC）电压需要更复杂的设计才能做到這一點。在这个问题中，我们分析如下图所示的增压转换电路。



The circuit consists of an input voltage V_i, an inductor of inductance L, a capacitor of capacitance C, and a load of resistance R. D is a diode whose resistance is effectively zero when the electric potential is higher on the left end, and effectively infinite when the electric potential is lower on the left end.
该电路包括一个输入电压 V_i，一個电感为 L 的电感器，一個电容为 C 的电容器，和一個电阻为 R 的负载。D 是一个二极管，當左端电勢高時，二极管的有效电阻是零，當左端电勢低時，二极管的有效电阻是無限大。

S is a switch operated by an electronic circuit not shown in the figure. It switches on and off periodically at a rather high frequency. Each period consists of an on-state and an off-state. During the on-state, it is switched on for a time t_1, and during the off-state, it is switched off for a time t_0.
S 是一個由图中未显示的电子电路所控制的开关。它以一个相当高的频率作周期性地开关。每个周期包括一个导通状态和一個关断状态。在导通状态时，它被接通的时间为 t_1，在关断状态时，它处于关闭状态的时间为 t_0。

(a) Consider the initial condition that the current in the circuit is 0 and the capacitor is uncharged. At $t = 0$, switch S is closed. Calculate the current through the inductor at $t = t_1$. (2 points)
考虑初始状态時电路中的电流为 0，电容器是不带电的。在 $t = 0$ 时，开关 S 闭合。试计算在 $t = t_1$ 時通过电感器的电流。（2 分）

(b) At $t = t_1$, switch S is open. Calculate the current through the inductor at time t during the off-state ($t_1 < t < t_1 + t_0$). You may assume that the load resistance R is so large that the current it draws is negligible. (6 points)
在 $t = t_1$ 时，开关 S 断开。试计算在关断状态中时间为 t（$t_1 < t < t_1 + t_0$）时，通过电感器的电流。你可以假设负载电阻 R 很大，通過它的电流可以忽略不計。（6 分）

(c) When the device continues to operate, we will consider the high frequency limit in which $t_1 + t_0 << \sqrt{LC}$ in the rest of the problem. In this regime, it is sufficient to keep terms up to first order of t_0 and t_1. Find the relation between the current through the inductor at the end of the $(n-1)^{\text{th}}$ off-state and that of the n^{th} on-state, denoted as $I_0(n-1)$ and $I_1(n)$ respectively. (2 points)
当电路持续工作時，我们将在下面的问题中，考虑高频极限 $t_1 + t_0 << \sqrt{LC}$。在此条件下，只需考虑 t_0 和 t_1 的一阶项。试找出通过电感器的电流在第 $n-1$ 次关断状态结束时（定义为 $I_0(n-1)$）与在第 n 次导通状态結束时（定义为 $I_1(n)$）之间的关系。（2 分）

(d) By including the load in the circuit during the n^{th} on-state, find the relation between the voltage across the capacitor at the end of the $(n-1)^{\text{th}}$ off-state and that of the n^{th} on-state, denoted as $V_0(n-1)$ and $V_1(n)$ respectively. (2 points)
试在第 n 次导通状态期间考虑把负载包括在电路中，从而找出电容器两端的电压在第 $n-1$ 次关断状态結束時（定義為 $V_0(n\text{-}1)$）与在第 n 次导通状态結束时（定義為 $V_1(n)$）之间的关系。（2 分）

(e) At the end of the n^{th} on-state, the current through the inductor is $I_1(n)$, and the voltage across the capacitor is $V_1(n)$. Calculate the current $I_0(n)$ through the inductor and the voltage $V_0(n)$ across the capacitor at the end of the immediately following off-state. (5 points)
在第 n 次导通状态結束时，通过电感器的电流为 $I_1(n)$, 在电容器两端的电压是 $V_1(n)$。试計算在紧随的关断状态結束时通过电感器的电流 $I_0(n)$以及电容器两端的电压 $V_0(n)$。（5 分）

(f) When the device reaches the steady state, calculate the step-up voltage ratio V_{out}/V_i to the lowest order. How should we set t_1 and t_0 to raise the ratio? (3 points)
当电路达到稳定状态时，试计算增压电压比率 V_{out}/V_i，以最低阶解答即可。为了提高這个比率，该如何设置 t_1 和 t_0？ （3 分）

(g) Calculate the on-state current through the inductor at the steady state. Explain the physical meaning of the result. (3 points)
试计算当电路达到稳定状态后，导通状态时通过电感器的电流。解释结果的物理意义。（3 分）

(h) Explain the importance of the diode in producing the step-up voltage. (1 point)
试解释二极管在产生增压电压中的重要性。（1 分）

(i) Estimate the time taken to reach the steady state. Use only the variables t_1, t_0, L, C, R to express your result. (1 point)
试估计达到稳定状态所需的时间。只可使用变量 t_1, t_0, L, C, R 表达你的结果。（1 分）

2. White Dwarf (25 points) 白矮星 （25 分）

At the end of lives of stars with comparable masses as the Sun, the gravitational force compresses the star inward to form white dwarfs, and is eventually balanced by the quantum

mechanical pressure of the electrons (known as the degeneracy pressure). This determines the size of the white dwarfs, which is comparable to that of the Earth. In this problem we analyze the size of white dwarfs.
当质量与太阳相近的恒星终结时，引力会使恒星向内坍塌形成白矮星。引力最终与电子气体的量子效应造成的压力（称为简并压）平衡。这决定了白矮星的大小与地球近似。本体旨在分析白矮星的大小。

(a) First consider an electron of mass m_e confined in a one-dimensional box of length L. Its kinetic energy is given by $E = \frac{p^2}{2m_e}$, where p is the momentum of the electron. In quantum theory, the electrons are described by waves whose wavelengths λ determine the momenta by the de Broglie relation $p = \frac{h}{\lambda}$. Only standing waves with nodal points at the wall of the box give rise to the allowed electronic states of the electrons. This enables us to calculate the energy of the n^{th} state as $E_n = E_1 n^2$. Derive the expression E_1. (3 points)
首先考虑质量为 m_e 的电子局限在长度为 L 的一维盒子中。其动能为 $E = \frac{p^2}{2m_e}$，p 是电子的动量。在量子理论中，电子可以用波来描述，其波长透过德布罗意关系决定动量 $p = \frac{h}{\lambda}$。只有电子波形成驻波的节点处于盒子两端时，才是允许的电子态。这使我们能够计算的第 n 个电子态的能量为 $E_n = E_1 n^2$。试导出 E_1 的表达式。 （3 分）

(b) To simplify the picture, we consider the white dwarf as a three-dimensional cubic box with volume V. The energy of an electronic state in the box is $E = E_1(n_x^2 + n_y^2 + n_z^2)$, where n_x, n_y, n_z are positive integers. Calculate the total number of electronic states with energy below the maximum energy $E_{\max}$. Assume that $E_{\max}$ is much greater than E_1. (2 points)
作为简化模型，我们把白矮星考虑成一个体积为 V 的三维立方盒子。电子态的能量为 $E = E_1(n_x^2 + n_y^2 + n_z^2)$，其中 n_x, n_y, n_z 是正整数。试计算低于最大能量 $E_{\max}$ 的电子态的总数。假定 $E_{\max}$ 远大于 E_1。（2 分）

(c) Suppose there are N protons and N electrons in the white dwarf. Due to the famous Pauli exclusion principle in quantum mechanics, each electronic state can only accommodate 2 electrons. The electrons will fill up the electronic states from low to high energy up to a maximum energy called the Fermi energy E_F. Calculate E_F. (2 points)
假设白矮星内有 N 个质子和 N 个电子。根据量子力学中著名的泡利不相容原理，每个电子态只能容纳 2 个电子。电子会按能量从低到高填满所有可能的电子态，直到能量达到最大能量 E_F，E_F 称为费米能。试计算 E_F。 （2 分）

(d) Calculation shows that the average energy per electron is $3E_F/5$. Considering the electrons as a gas, what is the pressure of the electron gas? Is the pressure inward or outward? (4 points)
计算显示，平均每个电子的能量为 $3E_F/5$。将电子作为气体，电子气的压强是多少？压力是向内还是向外？ （4 分）

(e) Compare the degeneracy pressure due to protons with that due to electrons. (1 point)
比较电子气的简并压与质子气的简并压。 （1 分）

(f) The gravitational potential energy is dominated by protons and neutrons. Let m_p be the mass of a proton or a neutron. Assume that the number of protons and neutrons are the same, and the mass density is approximately constant inside the star. Calculate the gravitational potential energy of the star of radius R. (4 points)
引力势能主要由质子和中子贡献。质子或中子的质量为 m_p。设质子和中子的数目相同，并且恒星内质量密度近似为常数。试计算半径为 R 的恒星的引力势能。（4 分）

(g) Derive the expression of the radius of the white dwarf. Does the radius increase or decrease with increasing mass of the white dwarf? (4 points)
试推导白矮星半径的表达式。若白矮星质量增加，半径是增加还是减少？ （4 分）

(h) Calculate the radius of the white dwarf with the same mass as the Sun. Give your answer in multiples of Earth's radius. You are given the following parameters: (2 points)
试计算质量与太阳相同的白矮星半径。答案请以地球半径为单位。可用以下参数：（2 分）

h = Planck's constant 普朗克常数 = 6.626×10^{-34} Js
G = gravitational constant 万有引力常数 = 6.67×10^{-11} $\mathrm{Nm^2/kg^2}$
m_p = mass of a proton or neutron 质子或中子质量 = 1.67×10^{-27} kg
m_e = mass of an electron 电子质量 = 9.11×10^{-31} kg
m_{Sun} = mass of Sun 太阳质量 = 1.99×10^{30} kg
R_{E} =radius of Earth 地球半径 = 6380 km

(i) Estimate the mass of the white dwarf when the velocity of electrons becomes comparable to the velocity of light $c = 3 \times 10^8$ m/s. Give your answer in multiples of solar mass. What will happen to the white dwarf? (3 points)
当电子速度接近光速 $c = 3 \times 10^8$ m/s 时，试估计白矮星的质量。请以太阳质量为单位。白矮星将有什么发生？（3 分）

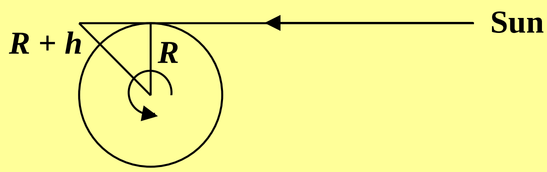
《THE END 完》

Part-1 (Total 6 Problems) 卷-1（共6 题）

1. Sunset Twice a Day (6 points) 一天两观日落 （6 分）

Presently the tallest tower in the world is Burj Khalifa in Dubai. Its height is 828 m. An Internet news article reported that one can watch sunset twice in one day with this tower.
杜拜的哈利法塔是现时世界上最高的建筑，高度为 828 米。互联网上有新闻文章报导，可以利用这塔在一天内两次观看到日落。

(a) What is the time range of sunset between the bottom and the top of the tower? Give your answer in minutes. Parameters: Earth's radius = 6400 km. Distance between Sun and Earth = 1.5×10^{11} m. (3 points)
试求塔底和塔顶之间日落时间的范围。答案请以分钟为单位。参数：地球半径 = 6400 公里。太阳和地球之间的距离 = 1.5×10^{11} 米。（3 分）



Neglecting the tilt of Earth's axis and the latitude of Dubai, the angular displacement of Earth between the two sunsets at the bottom and top of the tower
假设地球转轴的倾角和杜拜的纬度可略，则在塔底和塔顶两次日落之间地球的角位移为

$$\sin\theta = \frac{\sqrt{(R+h)^2 - R^2}}{R+h} \qquad [1]$$

$$\approx \frac{\sqrt{2Rh}}{R} = \sqrt{\frac{2h}{R}} << 1 \quad \Rightarrow \quad \theta \approx \sqrt{\frac{2h}{R}} = \sqrt{\frac{(2)(828)}{6400\times 10^3}} = 0.0161\,\text{radian} \qquad [1]$$

Time between the two sunsets 两次日落之间的时间

$$= \left(\frac{0.0161}{2\pi}\right)(24)(60)\text{ min} = 3.7\text{ min} \qquad [1]$$

(b) Burj Khalifa also has the world's third fastest elevator (lift) with a speed of v = 10 m/s. Immediately before the elevator starts moving upwards at the speed v from the bottom of the tower, a tourist in the elevator views the sunset. When he reaches the observatory at the height of 452 m, he found that the Sun has risen. Calculate the inclination angle of the Sun above the horizon. Give your answer in degrees. (3 points)
哈里发塔还拥有世界第三快的电梯，速度可达 v = 10 m/s。有电梯内的游客，在电梯从塔底开始上升前一瞬看到日落，其后电梯以速度 v 上升。当他到达在 452 米高度的观景台时，发现太阳上升了。试计算太阳在地平线以上的仰角。答案请以度为单位。（3 分）

Time to travel to the observatory 前往观景台的时间 $t = \dfrac{h}{v}$

Earth's angular speed 地球的角速度 $\omega = \dfrac{2\pi}{(24)(60)(60)}$

Earth's angular displacement 地球的角位移

$$\theta = \omega t = \frac{2\pi}{(24)(60)(60)}\left(\frac{452}{10}\right) = 0.00329 \text{ radian}$$

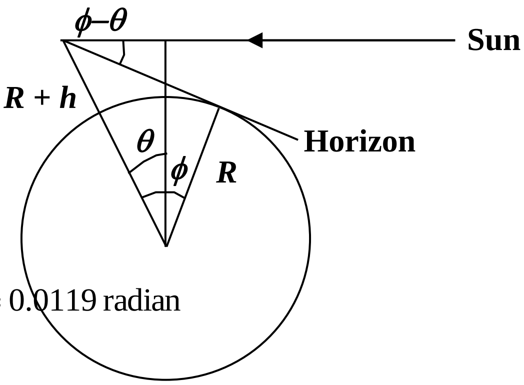
[1]

Change in the horizon 地平线的改变

$$\cos\phi = \frac{R}{R+h} \Rightarrow$$

$$\phi \approx \sin\phi = \sqrt{1-\left(\frac{R}{R+h}\right)^2} \approx \sqrt{\frac{2h}{R}} = \sqrt{\frac{(2)(452)}{6400\times 10^3}} = 0.0119 \text{ radian}$$

[1]



Elevation angle 仰角

$$\phi - \theta \approx 0.0119 - 0.0033 = 0.0086 \text{ radian} = 0.49^{\circ}$$ [1]

Remark: This is roughly the angular size of the Sun. So the tourist can view almost the entire Sun.

备注：这大致是太阳的角直径。因此，游客可以看到几乎整个太阳。

2. Radiocarbon Dating (5 points) 放射性碳年龄测定法 （5 分）

Radiocarbon dating is a technique used in archeology to estimate the age of organic materials, such as wood and leather. It uses the fact that the density of ^{14}C atoms in the atmosphere is constantly around 1.3 atoms of ^{14}C in every 10^{12} atoms of all isotopes of carbon. However, when an organism dies, ^{14}C cannot be replenished and decreases due to β decay with a half-life of 5730 years. The radioactive decay can be written in the following form:

放射性碳年龄测定法是考古学上用来估计有机物料（如木材和皮革）年龄的技术。它的根据，在于 ^{14}C 原子在大气中，浓度恒常处于每 10^{12} 粒碳原子中（包括所有同位素）有 1.3 粒 ^{14}C 原子。但是，生物死亡后，^{14}C 不能得到补充，并因β衰变逐渐降低，半衰期为 5730 年。这放射性衰变可以写成以下形式：

$$^{14}_{6}C \rightarrow ^{14}_{7}N + e^{-} + \bar{\nu}_e$$

(a) Suppose we obtain 50 grams of carbon from a piece of wood dated back to a prehistoric tomb. Using the carbon average atomic mass of 2×10^{-26} kg, calculate the number N_0 of ^{14}C atoms when the wood was still part of a living tree. (1 point)

假设我们从史前古墓的一块木头得到 50 克碳。已知碳的平均原子质量为 2×10^{-26} 千克，试计算木材仍是活树一部分时， ^{14}C 原子的数目 N_0。（1 分）

$$N_0 = \left(\frac{50\times 10^{-3}}{2\times 10^{-26}}\right)(1.3\times 10^{-12}) = 3.25\times 10^{12}$$ [1]

(b) We can determine the age of the tomb if we know the number N of ^{14}C atoms from the 50 grams of carbon. There is no way to directly count the number of ^{14}C atoms, but we detect a total of 935 electrons emitted from the 50 grams of carbon in 10 minutes. How old is the tomb? (3 points)

要估算古墓的年代，我们需要知道该 50 克碳中 ^{14}C 原子的数目 N。我们无法直接数算 ^{14}C 原子的数目，但我们发现 50 克碳在 10 分钟内放射了共 935 粒电子。古墓的年龄是多少？（3 分）

$N(t) = N_0 e^{-\frac{t}{\tau}}$

$\frac{1}{2} = e^{-t_{1/2}/\tau} \Rightarrow \tau = \frac{t_{1/2}}{\ln 2} = 8267$ years [1]

Decay rate 衰变率 $R(t) = -\frac{dN}{dt} = \frac{N_0}{\tau} e^{-\frac{t}{\tau}} \Rightarrow t = -\tau \ln\left(\frac{R\tau}{N_0}\right)$ [1]

$t = -8267 \ln\left(\frac{(93.5)(8267 \times 365 \times 24 \times 60)}{3.25 \times 10^{12}}\right) = 17190$ years [1]

(c) An archaeologist claims that he/she discovered a fossil plant with an age of 2×10^8 years using the method of radiocarbon dating. A scientist says that this result is nonsense. Which side will you stand on? Please explain your reasons. (1 point)
某考古学家声称，他/她利用放射性碳年龄测定法，发现年代为 2×10^8 年的化石植物。某科学家说，这结果是无稽之谈。你认为哪方较合理？请解释你的理由。（1 分）

The scientist is more reasonable. To see this, let us do a calculation basing on 50 grams of carbon:
科学家较合理。要了解这一点，让我们根据 50 克碳作一计算：

$$N(t) = N_0 e^{-\frac{t}{\tau}} = (3.25 \times 10^{12}) \exp\left(-\frac{2 \times 10^8}{8267}\right) = 2.27 \times 10^{-10495} << 1.$$

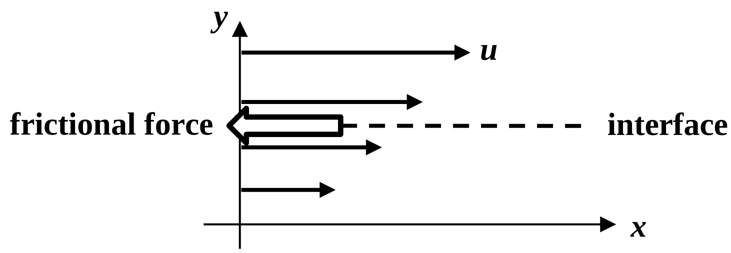
This is impossible to be detected. ($N \geq \sqrt{N} \Rightarrow N \geq 1$ for shot noise limited perfect detection.)
In order to have at least 1 ^{14}C atom left today, the archaeologist needs at least $50\,\text{g} \times 2 \times 10^{10483} = 10^{10482}\,\text{kg}$, which is impossible to be obtained (it is more than the mass of the Earth).
这是不可能被检测出来的。（$N \geq \sqrt{N} \Rightarrow N \geq 1$，是散粒噪声对準确检测的限制。）
若要至少有 1 粒 ^{14}C 原子到今天仍然存留，考古学家至少需要 $50\,\text{g} \times 2 \times 10^{10483} = 10^{10482}\,\text{kg}$（这质量比地球质量更大）。 [1]

3. Viscosity (7 points) 粘度 （7 分）

When uneven forces are applied to a fluid, the flow velocities at different locations will be different. For viscous fluids, frictional forces will be present when two adjacent layers of fluids flow at different velocities. As shown in the figure, the viscosity μ of the fluid is defined by the equation:
当不均匀的力施加到流体中，流速在不同的位置将是不同的。对于粘性流体，当相邻的两层流体以不同速度流动时，摩擦力便会存在。如下图所示，流体的粘度μ由下式定义:

$$F = -\mu \frac{du}{dy} \Delta A,$$

where F is the frictional force experienced by the fluid at an interface of area ΔA in the x direction, u is the x component of the velocity and du/dy is the velocity gradient. In this problem, we will analyze the viscosity using the kinetic theory of gases.
其中F是流体在x方向、面积为ΔA的界面上的摩擦力，u是方向的速度，du/dy是速度梯度。在这问题中，我们将以气体运动理论分析流体的粘度。



Let τ be the average time between successive collisions of a gas molecule with other molecules. Molecule i moves with velocity $\vec{v}_i$ in random directions, and the average velocity at height y is $\vec{u}(y)$.

设τ为气体分子与其他分子连续碰撞之间的平均时间。分子i以速度$\vec{v}_i$沿随机方向运动，而在高度y的平均速度为$\vec{u}(y)$。

(a) Suppose the interface is at a height y. What is the average x component of the momentum at height $y + \Delta y$? (1 point)
假设界面高度为y。在高度$y + \Delta y$的动量，其平均x分量是多少？（1分）Average x-momentum at height $y + \Delta y$ 在高度$y + \Delta y$的动量，其平均x分量是

$$= m\left(u + \frac{du}{dy}\Delta y\right).$$ [1]

(b) An incident molecule arrives at height y. The y component of its velocity is v_y. What is Δy of the height where the molecule experiences the collision last time? (1 point)
一分子入射到高度y。其速度的y分量为v_y。分子上一次遇到碰撞的高度的Δy是什么？（1分）

Height where the incident molecule experiences the collision last time $\Delta y = -v_y\tau$.
分子上一次遇到碰撞的高度 $\Delta y = -v_y\tau$。 [1]

(c) Compared with the average x component of the momentum of the gas molecules at the interface, what is the average extra x-momentum carried by the incident molecules of a given v_y when it arrives at height y? (1 point)
当给定v_y的入射分子到达高度y时，它的平均额外x-动量是什么（与界面上的气体分子动量的平均 x 分量相比）？（1分）

Average extra x-momentum平均额外x-动量 $m\left(u - \frac{du}{dy}v_y\tau\right) - mu = -\tau m v_y \frac{du}{dy}$ [1]

(d) The gas contains n molecules per unit volume. What is the rate of x-momentum transfer through an area ΔA? Hence find an approximate expression for the viscosity of the fluid according to the kinetic theory of gases. How does the viscosity depend on temperature? (4 points)
气体单位体积含有n粒分子。通过面积ΔA的x-动量，传递率是什么？试根据气体运动理论，由此推导流体粘度的近似表达式。粘度与温度有何关系？（4分）

Number of incident molecules passing through the area per unit time $= n v_y \Delta A$.
每单位时间入射分子通过面积的数目 $= n v_y \Delta A$

Each molecule transport an *x*-momentum equal to $-\tau m v_y \dfrac{du}{dy}$.

每个分子运输的*x*-动量等于 $-\tau m v_y \dfrac{du}{dy}$。

Hence the rate of *x*-momentum transfer in the upward direction is

因此，*x*-动量向上的传递率是

$$-\left(\tau m v_y \frac{du}{dy}\right)\left(n v_y \Delta A\right) = -\tau n m v_y^2 \frac{du}{dy}\Delta A \qquad [1]$$

Using Newton's second law of motion, frictional force experienced by the layer above the interface is

利用牛顿第二运动定律，界面上层受到的摩擦力为

$$F = -\tau n m\langle v_y^2\rangle \frac{du}{dy}\Delta A \quad \Rightarrow \quad \mu = \tau n m\langle v_y^2\rangle = \frac{1}{3}\tau n m\langle v^2\rangle \qquad [1]$$

According to the kinetic theory of gases, 根据气体运动理论，

$$\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT \quad \Rightarrow \quad \mu = \tau n m k T \qquad [1]$$

Viscosity is proportional to T. 粘度与T成正比。 [1]

Remark 1: The expression derived by using the Maxwell-Boltzmann distribution is different only in the coefficient.

备注1：通过使用麦克斯韦 - 玻尔兹曼分布得出的表达式，仅在系数有分别。

Remark 2: In practice, τ is also temperature dependent since $\tau = \lambda/v$, where λ is the mean free path that is mainly dependent on the density of gas, and v is proportional to $\sqrt{T}$. Combining, μ is proportional to $\sqrt{T}$.

备注2：在现实中，τ也随溫度改变，因为$\tau = \lambda/v$，其中λ是平均自由程，主要依赖於氣體的密度，v与$\sqrt{T}$ 成正比。結合后，μ与$\sqrt{T}$ 成正比。

4. Age of the Universe (10 points) 宇宙的年龄 （10 分）

Hubble discovered that the velocities v of galaxies receding from Earth are proportional to their distance d from Earth,

哈勃发现星系远离地球的速度 v 与地球距离 d 成正比，

$$v = H_0 d \, ,$$

where H_0 is the Hubble constant at the present age of the universe. It was recently measured to be 68 km/s/Mpc.

其中 H_0 为宇宙目前的哈勃常数。最近测得为 68 km/s/Mpc。

(a) Assuming that the universe expanded from the beginning to the present at a uniform speed, estimate the age of the universe. Give your answer in billion years. Parameters: 1 Mpc = 3.26×10^6 light years, speed of light = 300,000 km/s. (2 points)

假设宇宙从太初到现在以均匀速率膨胀，试估计宇宙的年龄。答案请以 billion years（十亿年）为单位。参数：1 Mpc = 3.26×10^6 光年，光速 = 300,000 km/s。 （2 分）

At uniform rate, 在均匀速率下， $v = \dfrac{d}{t}$.

Substituting into Hubble's law, 代入哈勃定律， $\dfrac{d}{t} = H_0 d \quad \Rightarrow \quad t = \dfrac{1}{H_0}$ [1]

$= \dfrac{3.26 \times 10^6 \times 300{,}000}{68} = 14.4 \text{ billion years}$ [1]

(b) However, the universe does not expand at a speed uniform in time due to the gravitational attraction of matter. Friedmann modeled the universe as an expanding sphere of matter with uniform density $\rho(t)$ at time t. Consider a test mass m on the surface of the sphere of radius $r(t)$ at time t. The total energy of the test mass is mU. Find the relation between the expansion velocity $v(t)$ and radius $r(t)$ at time t based on Newtonian mechanics. You may use G to represent the universal gravitational constant. (1 point)
但是，由于物质的万有引力，宇宙膨胀的速率在时间上不是均匀的。弗里德曼模拟宇宙为一膨胀中的均匀密度球体，在时间 t 其密度是$\rho(t)$。考虑在时间 t 时，在半径为 $r(t)$的球体表面上有一测试质量 m。测试质量的总能量为 mU。根据牛顿力学，找出在时间 t 的膨胀速度 $v(t)$和半径 $r(t)$之间的关系。你可用 G 代表万有引力常数。（1 分）

Since the attraction due to matter outside the sphere vanishes, the gravitational potential energy of the test mass is only due to matter inside the sphere. Using the conservation of energy,
在球体外的引力抵消，测试质量的引力势能只需考虑球体内的物质。利用能量守恒定律，

$$\frac{1}{2}mv^2 - \frac{Gm}{r}\left(\frac{4}{3}\pi r^3 \rho\right) = mU \quad \Rightarrow \quad \frac{1}{2}v^2 - \frac{G}{r}\left(\frac{4}{3}\pi r^3 \rho\right) = U \qquad [1]$$

(c) Recent satellite data shows that U is negligible. In this case, the expansion of the universe is described by the power-law $\dfrac{r(t)}{r_0} = \left(\dfrac{t}{t_0}\right)^n$, where r_0 and t_0 are the present values of $r(t)$ and t respectively. Find n and t_0. Express your answer in terms of G and the density ρ_0 of the present universe. (4 points)

最近的卫星数据显示，U 可以忽略不计。在这情况下，宇宙的膨胀可用幂律 $\dfrac{r(t)}{r_0} = \left(\dfrac{t}{t_0}\right)^n$ 描述，其中 r_0 和 t_0 分别为 $r(t)$ 和 t 的现值。求 n 和 t_0。答案请以 G 和宇宙密度的现值ρ_0 表达。（4 分）

When $U = 0$, 当 $U = 0$, $\dfrac{1}{2}v^2 = \dfrac{G}{r}\left(\dfrac{4}{3}\pi r^3 \rho\right)$.

Note that 注意 $\rho(t) = \rho_0 \left(\dfrac{r_0^3}{r(t)^3}\right)$.

Hence 所以 $v^2 = \dfrac{8\pi G r_0^3 \rho_0}{3r}$. [1]

$v = \dfrac{dr}{dt} = \dfrac{n r_0}{t_0}\left(\dfrac{t}{t_0}\right)^{n-1}$. [1]

Substituting, 代入上式， $\frac{n^2 r_0^2}{t_0^2}\left(\frac{t}{t_0}\right)^{2n-2} = \frac{8\pi G r_0^2 \rho_0}{3}\left(\frac{t}{t_0}\right)^{-n}$.

Comparing exponents and coefficients, 比较指数和系数，

$n = \frac{2}{3}$ [1]

$t_0 = \frac{1}{\sqrt{6\pi G\rho_0}}$ [1]

(d) Express the present age of the universe in terms of the present value of the Hubble constant. Estimate the age of the universe in this Newtonian picture. Give your answer in billion years. Based on your understanding about current developments in physics research, how is this result different from the current estimate of the age of the universe? (3 points)
试以哈勃常数的现值，表达宇宙目前的年龄。试以此牛顿力学的角度，估计宇宙的年龄。答案请以 billion years（十亿年）为单位。根据你对物理学研究当代发展的理解，这结果与当前对宇宙年龄的估计有何不同？（3 分）

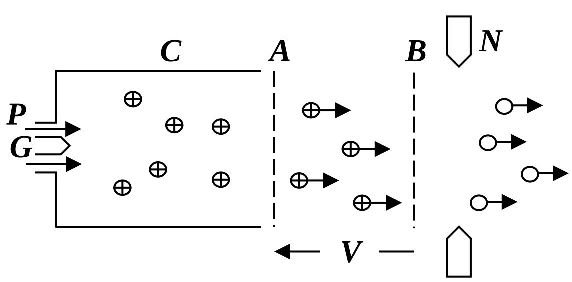
Since $\frac{r(t)}{r_0} = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$, $v = \frac{dr}{dt} = \frac{2r_0}{3t_0}\left(\frac{t}{t_0}\right)^{-\frac{1}{3}}$. At the present age, $t = t_0$. Hence $v_0 = \frac{2r_0}{3t_0}$. [1]

$H_0 = \frac{v_0}{r_0} = \frac{2}{3t_0} \quad \Rightarrow \quad t_0 = \frac{2}{3H_0} = \frac{(2)(3.26\times 10^6 \times 300{,}000)}{(3)(68)} = 9.6 \text{ billion years}$ [1]

This is less than the present estimate of the age of the universe (14 billion years). [1]

5. Electrostatic Ion Thrusters (12 points) 静电离子推进器 （12 分）

Electrostatic ion thrusters are used in spacecraft to control their trajectories in space. Its operating principle is shown in the following figure.
静电离子推进器用于控制航天器在太空的轨迹。它的工作原理如下图所示。



Streams of propellant atoms P are injected into the chamber C. The rate of injection is R, measured in the number of atoms per unit time. The atoms are ionized by bombarding with electrons shot from electron gun G. The positive ions are accelerated from grid electrode A to grid electrode B by the accelerating voltage V between them. The neutralizing electrode N emits electrons to neutralize the ion beam, preventing the spacecraft from gaining a net negative charge.
推进剂原子 P 被喷注入腔室 C。喷注的速率为 R，R 的单位为单位时间内的原子数目。原子被从电子枪 G 射出的电子碰撞而离子化。栅电极 A 到栅电极 B 之间的加速电压 V，使正离子加速。中和电极 N 发射电子，把离子束中和，以防止太空船带负电荷。

(a) Calculate the ratio of thrust F and the current I of the ion beam consisting of ions of mass m and charge ze, where z is a positive integer and e is the electronic charge. Express your answer in m, V, z and e. (4 points)
离子束由质量为 m、电荷为 ze 的离子组成，其中 z 是正整数，e 是电子电荷。试计算推力 F 与离子电流 I 之比。答案请以 m, V, z 和 e 表达。（4 分）

Current 电流: $I = Rze$ [1]

Using Newton's second law, 利用牛顿第二定律， $F = Rmv$ [1]

$$\frac{F}{I} = \frac{mv}{ze}$$

Using the conservation of energy, 利用能量守恒，

$$\frac{1}{2}mv^2 = zeV \quad [1]$$

$$\Rightarrow \quad v = \sqrt{\frac{2zeV}{m}} \quad \Rightarrow \quad \frac{F}{I} = \sqrt{\frac{2mV}{ze}} \quad [1]$$

(b) Calculate the ratio of thrust F and the power W spent in accelerating the ion beam. Express your answer in m, V, z and e. (2 points)
试计算推力 F 与加速离子束所耗功率 W 之比。答案请以 m, V, z 和 e 表达。（2 分）

Power 功率: $W = IV = RzeV$ [1]

$$\frac{F}{W} = \frac{mv}{zeV} = \sqrt{\frac{2m}{zeV}} \quad [1]$$

(c) To save power in space travel, should one prefer using light or heavy ions? Should one prefer using ions with single or multiple charges? Should one prefer using low or high accelerating voltages? (3 points)
为节省太空行程的功率，应该使用较轻抑较重的离子？应该使用单电荷离子抑多电荷离子？应该使用低加速电压抑高加速电压？（3 分）

Since F/W is proportional to $\sqrt{m}$, heavy ions are preferred.
由于 F/W 与 $\sqrt{m}$ 成正比，应该使用较重的离子。 [1]

Since F/W is proportional to $\sqrt{1/z}$, ions with single charge are preferred.
由于 F/W 与 $\sqrt{1/z}$ 成正比，应该使用单电荷离子。 [1]

Since F/W is proportional to $\sqrt{1/V}$, low accelerating voltages are preferred.
由于 F/W 与 $\sqrt{1/V}$ 成正比，应该使用低加速电压。 [1]

(d) A 10 kW electrostatic ion thruster using xenon atoms as propellant is designed. The accelerating voltage is 10 kV. Calculate the exhaust speed of the ions. Give your answer in km/s. Parameters: ionized xenon carries a single charge, atomic mass of xenon = 131, proton mass = 1.67×10^{-27} kg, electronic charge $e = 1.6 \times 10^{-19}$ C. (1 point)
一个 10 kW 的静电离子推进器的设计，使用氙原子作为推进剂。加速电压为 10 kV。试计算离子排出的速率。答案请以 km/s 为单位。参数：氙离子带单电荷，氙的原子质量 = 131，质子质量 = 1.67×10^{-27} kg，电子电荷 $e = 1.6 \times 10^{-19}$ C。（1 分）

$$v=\sqrt{\frac{2zeV}{m}}=\sqrt{\frac{(2)(1)(1.6\times10^{-19})(10\times10^{3})}{(131\times1.67\times10^{-27})}}=121\,\text{km/s}$$ [1]

(e) If the neutralizing electrode N of the thruster described in (d) is switched off, calculate the time taken by the body of the spacecraft to gain a voltage equal to the accelerating voltage; at that moment the thruster ceases to operate because the ions follow the thruster. Assume that the spacecraft is spherical and has a radius of 1 m. Parameters: ε_0 = 8.854 × 10^{-12} F/m. (2 points)
若在(d)中描述的推进器的中和电极 N 被关闭，试计算航天器身体上的电压变至与加速电压相等所需的时间；在那一刻因为离子不能离开推进器，将导致推进器停止操作。可假设航天器是球形的，半径为 1 m。参数：ε_0 = 8.854 × 10^{-12} F/m。（2 分）

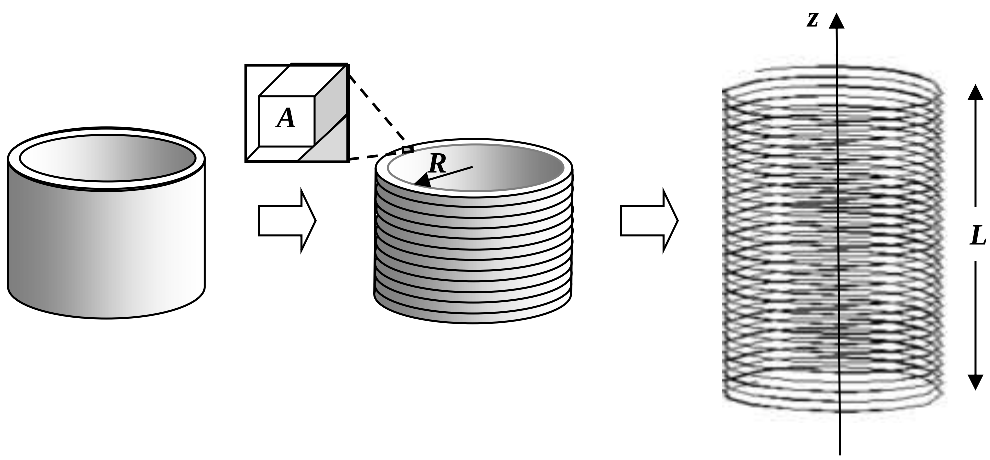
When the voltage has dropped to minus the accelerating voltage,
当电压降到加速电压的负值时，

$$V=\frac{Q}{4\pi\varepsilon_0 R}\quad\Rightarrow\quad Q=4\pi\varepsilon_0 RV=(4\pi)(8.854\times10^{-12})(1)(10\times10^{3})=1.11\times10^{-6}\text{ C}$$ [1]

$$I=\frac{W}{V}=\frac{10}{10}=1\text{ A}$$

Time for the voltage to drop 降低电压的时间: $t=\frac{Q}{I}=1.11\times10^{-6}\text{ s}=1.11\,\mu\text{s}$ [1]

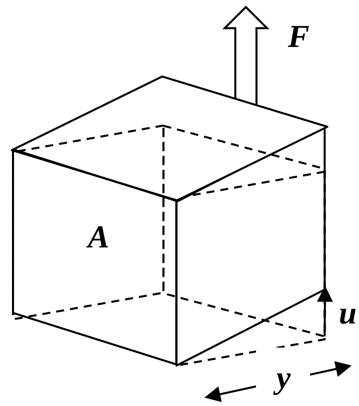
6. Slinky (10 points) 机灵鬼 （10 分）



The slinky is a spring first put on sale in 1940's, and soon became a popular toy. As shown in the figure, a slinky can be manufactured from a hollow metal cylinder of radius R by cutting it into a helical thin strip. The helix consists of N turns and has a cross sectional area A. Let ρ be the density of the metal.
机灵鬼弹簧首次于 1940 年代发售，很快便成为一种流行的玩具。如图所示，一個机灵鬼由半径为 R 的空心金属圆筒切割成螺旋形的薄带。螺旋线有 N 匝，其横截面面积为 A。设ρ是金属的密度。

(a) In this problem we assume that the deformation of a stretched slinky is mainly due to shear deformation. Let G be the shear modulus of the metal. What is the tension T in the slinky when it is stretched to a length L that is much greater than its original length?
在这问题中，我们假设机灵鬼被拉伸时的形变，主要是剪切形变。设 G 是金属的剪切模量。當把机灵鬼拉伸到长度 L 时（L 比机灵鬼原本的长度大得多），机灵鬼中的张力 T 是什么？

The shear modulus G of a solid is defined as $G=\dfrac{F/A}{u/y}$ where, as shown in the figure, F is the force acting on the vertical side of the solid with area A, y is the width of the solid, and u is the shear distortion of the solid. (2 points)
如图所示，固体的剪切模量 G 被定义为 $G=\dfrac{F/A}{u/y}$， 其中 F 是作用在固体侧面（面积为 A）的力，y 为固体的宽度，u 是固体的剪切形变。（2 分）



Considering the slinky as a sheared long strip, we have $y = 2\pi RN$ and $u = L$. Hence
把机灵鬼考虑成一条被剪切的长带，得 $y = 2\pi RN$ 和 $u = L$。所以

$$F=GA\frac{u}{y}=\frac{GAL}{2\pi RN}$$ [1]

Tension in the slinky 机灵鬼中的张力

$$T\sin\alpha=F$$

where α is the pitch angle given by $\sin\alpha=\dfrac{L}{2\pi RN}$.

其中 α 是斜角，由 $\sin\alpha=\dfrac{L}{2\pi RN}$ 给定。

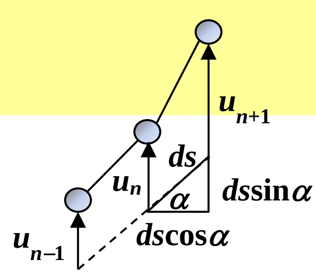
$$T=\frac{F}{\sin\alpha}=\left(\frac{GAL}{2\pi RN}\right)\left(\frac{2\pi RN}{L}\right)=GA$$ [1]

(b) To study how distortions propagate as a longitudinal wave in the slinky stretched to length L, we approximate the slinky by discrete particles separated by small distance ds connected by strings with tension T. Let $u_n(t)$ by the displacement of the n^{th} particle at time t. Derive the equation of motion of the particles. Neglect gravitational effects. (3 points)
为了研究形变如何以纵波在长度拉至 L 的机灵鬼上传播，我们将机灵鬼近似为一串离散的粒子，间距为 ds，由张力为 T 的繩子连接起来。设 $u_n(t)$ 是第 n 个粒子在时间 t 的位移。试推导粒子的运动方程。可忽略重力效应。（3 分）

$$\rho ds\ddot{u}_n=T\sin\theta_n-T\sin\theta_{n-1}$$

where θ_n is the inclination of the string from the n^{th} to $(n + 1)^{\text{th}}$ particle.
其中 θ_n 是从第 n 个粒子到第 $n + 1$ 个粒子的仰角。 [1]

$$\tan\theta_n=\frac{ds\sin\alpha+u_{n+1}-u_n}{ds\cos\alpha}\quad\Rightarrow$$



$$\sin\theta_n \approx \frac{ds\sin\alpha + u_{n+1} - u_n}{ds} = \sin\alpha + \frac{u_{n+1}-u_n}{ds} \qquad [1]$$

$$\sin\theta_{n-1} \approx \sin\alpha + \frac{u_n - u_{n-1}}{ds}$$

$$\rho ds\ddot{u}_n = T\sin\theta_n - T\sin\theta_{n-1} = \frac{GA}{ds}\left(u_{n+1} - 2u_n + u_{n-1}\right) \qquad [1]$$

(c) Show that $u_n(t) = C\sin(kz_n - \omega t)$ is a solution of the equation of motion, where z_n is the position of the n^{th} particle along the axis of the slinky. Find the relation between k and ω. Hence find the velocity of longitudinal wave propagation along the axis of the slinky. (5 points)

试证明 $u_n(t) = C\sin(kz_n - \omega t)$是运动方程的解，其中 z_n是沿机灵鬼轴线第 n 个粒子的位置，试找出 k 和ω之间的关系。由此推導沿机灵鬼轴线传播的纵波速度。（5 分）

Left hand side 左方: $\rho ds\ddot{u}_n = -\rho ds\omega^2 C\sin(kz_n - \omega t)$ [1]

Right hand side 右方: $\frac{GA}{ds}C\left[\sin(kz_{n+1} - \omega t) - 2\sin(kz_n - \omega t) + \sin(kz_{n-1} - \omega t)\right]$

$$= \frac{GA}{ds}C\left[2\sin\left(\frac{kz_{n+1}+kz_{n-1}}{2} - \omega t\right)\cos\left(\frac{kz_{n-1}-kz_{n+1}}{2}\right) - 2\sin(kz_n - \omega t)\right]$$

$$= -\frac{GA}{ds}2C\left[1 - \cos(kz_{n-1} - kz_{n+1})\right]\sin(kz_n - \omega t) \qquad [1]$$

Since 由于 $z_{n+1} - z_n = ds\sin\alpha$, $1 - \cos\left(\frac{kz_{n-1} - kz_{n+1}}{2}\right) = 1 - \cos(kds\sin\alpha) \approx \frac{1}{2}k^2 ds^2\sin^2\alpha$, [1]

Right hand side 右方 $= -GAdsk^2\sin^2\alpha C\sin(kz_n - \omega t)$.

Comparing both sides, 比较两方，

$$\rho\omega^2 = GA\sin^2\alpha k^2 \quad \Rightarrow \quad \omega^2 = \frac{GA\sin^2\alpha}{\rho}k^2 = \frac{GAL^2}{4\rho\pi^2R^2N^2}k^2 \qquad [1]$$

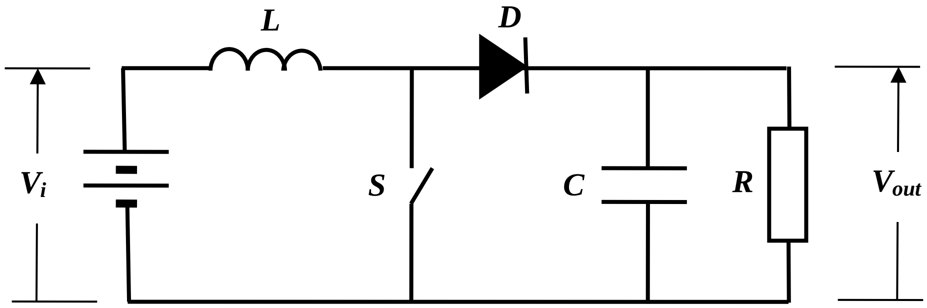
Wave velocity 波速: $c = \frac{\omega}{k} = \frac{L}{2\pi RN}\sqrt{\frac{GA}{\rho}}$ [1]

Part-2 (Total 2 Problems) 卷-2（共2 题）

1. DC Step-up Converter (25 points) 增压转换器 （25 分）

Modern electric and gasoline hybrid cars require high voltages to drive their motors from batteries of lower voltages. Alternating current (AC) voltages can be stepped up easily by using transformers, but direct current (DC) voltages require more sophisticated designs. In this problem we analyze the step-up converter circuit as shown in the following figure.

现代电力和汽油混合动力汽车需要从低电压的电池产生的高电压驱动马达。交流电（AC）电压可以很容易地通过使用变压器增强，但直流（DC）电压需要更复杂的设计才能做到這一點。在这个问题中，我们分析如下图所示的增压转换电路。



The circuit consists of an input voltage V_i, an inductor of inductance L, a capacitor of capacitance C, and a load of resistance R. D is a diode whose resistance is effectively zero when the electric potential is higher on the left end, and effectively infinite when the electric potential is lower on the left end.
该电路包括一个输入电压 V_i，一個电感为 L 的电感器，一個电容为 C 的电容器，和一個电阻为 R 的负载。D 是一个二极管，當左端电势高時，二极管的有效电阻是零，當左端电势低時，二极管的有效电阻是無限大。

S is a switch operated by an electronic circuit not shown in the figure. It switches on and off periodically at a rather high frequency. Each period consists of an on-state and an off-state. During the on-state, it is switched on for a time t_1, and during the off-state, it is switched off for a time t_0.
S 是一個由图中未显示的电子电路所控制的开关。它以一个相当高的频率作周期性地开关。每个周期包括一个导通状态和一個关断状态。在导通状态时，它被接通的时间为 t_1，在关断状态时，它处于关闭状态的时间为 t_0。

(a) Consider the initial condition that the current in the circuit is 0 and the capacitor is uncharged. At $t = 0$, switch S is closed. Calculate the current through the inductor at $t = t_1$. (2 points)
考虑初始状态時电路中的电流为 0，电容器是不带电的。在 $t = 0$ 时，开关 S 闭合。试计算在 $t = t_1$ 時通过电感器的电流。（2 分）

When switch S is closed, the diode, capacitor and resistor can be ignored. Hence
当开关 S 关闭时，二极管，电容器和负载可忽略不计。故

$$V_i - L\frac{dI}{dt} = 0 \qquad [1]$$

$$\Rightarrow \quad I = \int_0^{t_1} \frac{V_i}{L} dt = \frac{V_i}{L} t_1 \qquad [1]$$

(b) At $t = t_1$, switch S is open. Calculate the current through the inductor at time t during the off-state ($t_1 < t < t_1 + t_0$). You may assume that the load resistance R is so large that the current it draws is negligible. (6 points)
在 $t = t_1$ 时，开关 S 断开。试计算在关断状态中时间为 t（$t_1 < t < t_1 + t_0$）时，通过电感器的电流。你可以假设负载电阻 R 很大，通過它的电流可以忽略不計。（6 分）

When switch S is open, the circuit consists of the battery, inductor, diode and capacitor (the resistor is ignored). Assuming that the diode resistance is 0,
当开关 S 断开时，电路包括电池，电感器，二极管和电容器（负载可忽略）。假设二极管电阻为 0，

$V_i - L\frac{dI}{dt} - \frac{q}{C} = 0$ [1]

$I = \frac{dq}{dt} \quad \Rightarrow \quad \frac{d^2q}{dt^2} + \frac{q}{LC} = \frac{V_i}{L} \quad \Rightarrow \quad \frac{d^2q}{dt^2} + \frac{1}{LC}(q - CV_i) = 0$ [1]

The motion of $q - CV_i$ is a simple harmonic motion with angular frequency $\omega = \frac{1}{\sqrt{LC}}$.

$q - CV_i$ 的变化是简谐运动，其角频率为 $\omega = \frac{1}{\sqrt{LC}}$ [1]

Let 设 $q = CV_i + A\sin\omega(t - t_1) + B\cos\omega(t - t_1) \Rightarrow I = \omega A\cos\omega(t - t_1) - \omega B\sin\omega(t - t_1)$.

The initial condition is $q = 0$ and $I = V_i t_1/L$ at $t = t_1$.

初始条件为 $q = 0$ and $I = V_i t_1/L$ at $t = t_1$.

$B = -CV_i$ [1]

$A = \frac{V_i t_1}{\omega L}$ [1]

$I = \frac{V_i t_1}{L}\cos\omega(t - t_1) + \omega CV_i \sin\omega(t - t_1)$ [1]

(c) When the device continues to operate, we will consider the high frequency limit in which $t_1 + t_0 << \sqrt{LC}$ in the rest of the problem. In this regime, it is sufficient to keep terms up to first order of t_0 and t_1. Find the relation between the current through the inductor at the end of the $(n - 1)^{\text{th}}$ off-state and that of the n^{th} on-state, denoted as $I_0(n - 1)$ and $I_1(n)$ respectively. (2 points)

当电路持续工作時，我们将在下面的问题中，考虑高频极限 $t_1 + t_0 << \sqrt{LC}$。在此条件下，只需考虑 t_0 和 t_1 的一阶项。试找出通过电感器的电流在第 $n - 1$ 次关断状态结束时（定义为 $I_0(n - 1)$）与在第 n 次导通状态結束时（定义为 $I_1(n)$）之间的关系。（2 分）

During the next on-state, switch S is closed. Hence

在下一次导通状态，开关 S 闭合。故

$V_i - L\frac{dI}{dt} = 0$ [1]

$\Rightarrow \quad I_1(n) = I_0(n-1) + \int_0^{t_1}\frac{V_i}{L}dt = I_0(n-1) + \frac{V_i}{L}t_1$ [1]

(d) By including the load in the circuit during the n^{th} on-state, find the relation between the voltage across the capacitor at the end of the $(n - 1)^{\text{th}}$ off-state and that of the n^{th} on-state, denoted as $V_0(n - 1)$ and $V_1(n)$ respectively. (2 points)

试在第 n 次导通状态期间考虑把负载包括在电路中，从而找出电容器两端的电压在第 $n - 1$ 次关断状态結束時（定義為 $V_0(n\text{-}1)$）与在第 n 次导通状态結束时（定義為 $V_1(n)$）之间的关系。（2 分）

During the n^{th} on-state, the capacitor discharges and produces the current through the load.

在第 n 次导通状态中，电容器放电并产生通过负载的电流。

$\frac{q}{C} - IR = 0$ [1]

$$I=-\frac{dq}{dt}\quad\Rightarrow\quad\frac{dq}{dt}+\frac{q}{RC}=0\quad\Rightarrow\quad q=q_0(n-1)e^{-\frac{t}{RC}}$$

$$V_1(n)=V_0(n-1)e^{-\frac{t_1}{RC}}$$

$$\approx V_0(n-1)-\frac{V_0(n-1)}{RC}t_1$$ [1]

(e) At the end of the n^{th} on-state, the current through the inductor is $I_1(n)$, and the voltage across the capacitor is $V_1(n)$. Calculate the current $I_0(n)$ through the inductor and the voltage $V_0(n)$ across the capacitor at the end of the immediately following off-state. (5 points)
在第 n 次导通状态結束时，通过电感器的电流为 $I_1(n)$, 在电容器两端的电压是 $V_1(n)$。试計算在紧随的关断状态結束时通过电感器的电流 $I_0(n)$以及电容器两端的电压 $V_0(n)$。 （5 分）

During the immediately following off-state, switch S is open. Hence
在紧随的关断状态結束时，开关 S 断开。故

$$V_i-L\frac{dI}{dt}-\frac{q}{C}=0$$ [1]

The solution is the same as that of (b), but the initial condition is modified.
方程解與(b)相同，但初始条件不同。

Let $q=CV_i+A\sin(\omega t)+B\cos(\omega t)$, where the time is measured from the end of the n^{th} on-state.
设 $q=CV_i+A\sin(\omega t)+B\cos(\omega t)$, 其中时间从第 n 次导通状态结束时开始计算。

$I=\omega A\cos(\omega t)-\omega B\sin(\omega t)$.

The initial condition is 初始条件为 $q = CV_1(n)$ and $I = I_1(n)$ at $t = 0$.

$$B=CV_1(n)-CV_i$$ [1]

$$A=\frac{I_1(n)}{\omega}$$ [1]

$$I_0(n)=I_1(n)\cos(\omega t_0)-\omega\left[CV_1(n)-CV_i\right]\sin(\omega t_0)$$

$$\approx I_1(n)-\left[CV_1(n)-CV_i\right]\omega^2t_0=I_1(n)-\frac{V_1(n)-V_i}{L}t_0$$ [1]

$$V_0(n)=\frac{q(t_0)}{C}=V_i+\frac{I_1(n)}{\omega C}\sin(\omega t_0)+\left[V_1(n)-V_i\right]\cos(\omega t_0)$$

$$\approx V_1(n)+\frac{I_1(n)}{C}t_0$$ [1]

(f) When the device reaches the steady state, calculate the step-up voltage ratio V_{out}/V_i to the lowest order. How should we set t_1 and t_0 to raise the ratio? (3 points)
当电路达到稳定状态时，试计算增压电压比率 V_{out}/V_i，以最低阶解答即可。为了提高這个比率，该如何设置 t_1 和 t_0？ （3 分）

When the device reaches the steady state, 当電路达到稳定状态，
$V_1(n-1) = V_1(n)$ and $I_1(n-1) = I_1(n)$.

$$I_1=I_0+\frac{V_i}{L}t_1$$

$$I_0\approx I_1-\frac{V_1-V_i}{L}t_0$$ [1]

Eliminating I_1 and I_0, 消去 I_1 和 I_0， $V_1 \approx V_i \dfrac{t_1 + t_0}{t_0}$.

To the lowest order, 保留最低阶， $V_0 \approx V_1$.

Hence 所以 $\dfrac{V_{\text{out}}}{V_i} \approx \dfrac{t_1 + t_0}{t_0}$. [1]

To raise the step-up voltage ratio, t_1 should be set longer than t_0.
为了提高增压电压比率，t_1 应该比 t_0 長。 [1]

(g) Calculate the on-state current through the inductor at the steady state. Explain the physical meaning of the result. (3 points)
试计算当电路达到稳定状态后，导通状态时通过电感器的电流。解释结果的物理意义。（3 分）

$$V_1 \approx V_0\left(1 - \frac{t_1}{RC}\right)$$

$$V_0 \approx V_1 + \frac{I_1}{C} t_0 \quad [1]$$

Eliminating V_0, 消去 V_0, $I_1 \approx \dfrac{V_1}{R}\dfrac{t_1}{t_0}$ [1]

Physical meaning: $I_1 t_0$ is the charge stored in the capacitor during the off-state. V_1/R is the current flowing out of the capacitor during the on-state, and $V_1 t_1/R$ is the charge drained from the capacitor during the on-state. The two quantities are the same due to charge conservation.
物理意义：$I_1 t_0$ 是在关断状态存储在电容器中的电荷。V_1/R 是在导通状态下从电容器流出的电流，$V_1 t_1/R$ 是在导通状态下从电容流出的电荷。由于电荷守恒，这两个量是相同的。 [1]

(h) Explain the importance of the diode in producing the step-up voltage. (1 point)
试解释二极管在产生增压电压中的重要性。（1 分）

The diode prevents the capacitor to discharge through shorted circuit during the on-state, and through the battery and the inductor during the off-state, so that the voltage across the capacitor can build up to a high value.
在导通状态下，二极管可防止电容器通过短路放电。在关断状态下，二极管可防止电容器通过电池和电感器放电。这样电容器两端的电压可以升高。 [1]

(i) Estimate the time taken to reach the steady state. Use only the variables t_1, t_0, L, C, R to express your result. (1 point)
试估计达到稳定状态所需的时间。只可使用变量 t_1, t_0, L, C, R 表达你的结果。（1 分）

The current increases by $V_i t_1/L$ per period. The steady state current is $V_1 t_1/Rt_0$. Hence we estimate the number of periods to reach the steady state is $\left(\dfrac{V_1 t_1}{Rt_0}\right)\left(\dfrac{V_i t_1}{L}\right)^{-1} = \left(\dfrac{V_1}{V_i}\right)\left(\dfrac{L}{Rt_0}\right)$. The estimated time is $\left(\dfrac{V_1}{V_i}\right)\left(\dfrac{L}{Rt_0}\right)(t_1 + t_0) = \left(\dfrac{t_1 + t_0}{t_0}\right)^2\left(\dfrac{L}{R}\right)$.

电流在每個周期增加 $V_i t_1/L$。稳态电流是 $V_1 t_1/Rt_0$。因此我们估计达到稳定状态的周期数是 $\left(\dfrac{V_1 t_1}{Rt_0}\right)\left(\dfrac{V_i t_1}{L}\right)^{-1} = \left(\dfrac{V_1}{V_i}\right)\left(\dfrac{L}{Rt_0}\right)$，所需時間約為 $\left(\dfrac{V_1}{V_i}\right)\left(\dfrac{L}{Rt_0}\right)(t_1 + t_0) = \left(\dfrac{t_1 + t_0}{t_0}\right)^2\left(\dfrac{L}{R}\right)$。 [1]

2. White Dwarf (25 points) 白矮星 （25 分）

At the end of lives of stars with comparable masses as the Sun, the gravitational force compresses the star inward to form white dwarfs, and is eventually balanced by the quantum mechanical pressure of the electrons (known as the degeneracy pressure). This determines the size of the white dwarfs, which is comparable to that of the Earth. In this problem we analyze the size of white dwarfs.

当质量与太阳相近的恒星终结时，引力会使恒星向内坍塌形成白矮星。引力最终与电子气体的量子效应造成的压力（称为简并压）平衡。这决定了白矮星的大小与地球近似。本体旨在分析白矮星的大小。

(a) First consider an electron of mass m_e confined in a one-dimensional box of length L. Its kinetic energy is given by $E=\frac{p^2}{2m_e}$, where p is the momentum of the electron. In quantum theory, the electrons are described by waves whose wavelengths λ determine the momenta by the de Broglie relation $p=\frac{h}{\lambda}$. Only standing waves with nodal points at the wall of the box give rise to the allowed electronic states of the electrons. This enables us to calculate the energy of the n^{th} state as $E_n = E_1 n^2$. Derive the expression E_1. (3 points)

首先考虑质量为 m_e 的电子局限在长度为 L 的一维盒子中。其动能为 $E=\frac{p^2}{2m_e}$，p 是电子的动量。在量子理论中，电子可以用波来描述，其波长透过德布罗意关系决定动量 $p=\frac{h}{\lambda}$。只有电子波形成驻波的节点处于盒子两端时，才是允许的电子态。这使我们能够计算的第 n 个电子态的能量为 $E_n = E_1 n^2$。试导出 E_1 的表达式。 （3 分）

The n^{th} standing wave in the box is given by 盒中第 n 个驻波满足 $\frac{n\lambda}{2}=L \quad \Rightarrow \quad \lambda=\frac{2L}{n}$. [1]

The momentum of the n^{th} electronic state is 第 n 个电子态的动量为

$$p=\frac{h}{\lambda} \quad \Rightarrow \quad p=\frac{h}{2L}n. \qquad [1]$$

The energy of the n^{th} electronic state is 第 n 个电子态的能量为 $E=\frac{h^2}{8m_eL^2}n^2$.

Hence 所以 $E_1=\frac{h^2}{8m_eL^2}$ [1]

(b) To simplify the picture, we consider the white dwarf as a three-dimensional cubic box with volume V. The energy of an electronic state in the box is $E=E_1(n_x^2+n_y^2+n_z^2)$, where n_x, n_y, n_z are positive integers. Calculate the total number of electronic states with energy below the maximum energy $E_{\max}$. Assume that $E_{\max}$ is much greater than E_1. (2 points)

作为简化模型，我们把白矮星考虑成一个体积为 V 的三维立方盒子。电子态的能量为 $E=E_1(n_x^2+n_y^2+n_z^2)$，其中 n_x, n_y, n_z 是正整数。试计算低于最大能量 $E_{\max}$ 的电子态的总数。假定 $E_{\max}$ 远大于 E_1。（2 分）

We have to find the number of lattice points satisfying $n_x^2+n_y^2+n_z^2 \le E_{\max}/E_1$. In the three-dimensional space, these points are enclosed in the first octant of the sphere of radius $\sqrt{E_{\max}/E_1}$.
满足 $n_x^2+n_y^2+n_z^2 \le E_{\max}/E_1$ 的正整数组 (n_x, n_y, n_z) 的数目即三维空间第一卦限中，半径为 $\sqrt{E_{\max}/E_1}$ 的球体内的整格点数。 [1]

Since the volume enclosing a point is 1, the number of lattice points in one-eighth of a sphere is
由于每个格点体积为 1，八分之一个球体包含格点数为

$$N_{\text{state}} = \frac{1}{8}\frac{4}{3}\pi\left(\frac{E_{\max}}{E_1}\right)^{\frac{3}{2}} = \frac{\pi V}{6h^3}\left(8m_e E_{\max}\right)^{\frac{3}{2}}. \quad [1]$$

(c) Suppose there are N protons and N electrons in the white dwarf. Due to the famous Pauli exclusion principle in quantum mechanics, each electronic state can only accommodate 2 electrons. The electrons will fill up the electronic states from low to high energy up to a maximum energy called the Fermi energy E_F. Calculate E_F. (2 points)
假设白矮星内有 N 个质子和 N 个电子。根据量子力学中著名的泡利不相容原理，每个电子态只能容纳 2 个电子。电子会按能量从低到高填满所有可能的电子态，直到能量达到最大能量 E_F，E_F 称为费米能。试计算 E_F。（2 分）

$$N = 2N_{\text{state}} = \frac{\pi V}{3h^3}\left(8m_e E_F\right)^{\frac{3}{2}} \quad [1]$$

$$\Rightarrow \quad E_F = \frac{h^2}{8m_e}\left(\frac{3N}{\pi V}\right)^{\frac{2}{3}} \quad [1]$$

(d) Calculation shows that the average energy per electron is $3E_F/5$. Considering the electrons as a gas, what is the pressure of the electron gas? Is the pressure inward or outward? (4 points)
计算显示，平均每个电子的能量为 $3E_F/5$。将电子作为气体，电子气的压强是多少？压力是向内还是向外？（4 分）

Total kinetic energy of the electron gas 电子气的总动能 $E = \frac{3}{5}NE_F = \frac{3h^2}{40m_e}\left(\frac{3}{\pi V}\right)^{\frac{2}{3}} N^{\frac{5}{3}}$ [1]

Using the first law of thermodynamics, 利用热力学第一定律，$dE = -PdV$.

Therefore, 所以，$P = -\frac{dE}{dV}$ [1]

$$= \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{\frac{2}{3}}\left(\frac{N}{V}\right)^{\frac{5}{3}} \quad [1]$$

The pressure is outward. 压力向外。 [1]

(e) Compare the degeneracy pressure due to protons with that due to electrons. (1 point)
比较电子气的简并压与质子气的简并压。（1 分）

Since the pressure is inversely proportional to the mass of the particle, the degeneracy pressure of protons is much less than that of electrons.
由于压力与粒子的质量成反比，质子的简并压力比电子的少得多。 [1]

(f) The gravitational potential energy is dominated by protons and neutrons. Let m_p be the mass of a proton or a neutron. Assume that the number of protons and neutrons are the same, and the mass density is approximately constant inside the star. Calculate the gravitational potential energy of the star of radius R. (4 points)
引力势能主要由质子和中子贡献。质子或中子的质量为 m_p。设质子和中子的数目相同，并且恒星内质量密度近似为常数。试计算半径为 R 的恒星的引力势能。（4 分）

Mass of the star of radius r 半径为 r 的恒星质量为 $2N\left(\frac{r^3}{R^3}\right)m_p$。 [1]

Suppose its radius increases by a thin shell of thickness dr. Mass of the thin shell:
若增加一厚度为 dr 的薄球壳，球壳质量:

$$dm = 2N\left(\frac{4\pi r^2 dr}{4\pi R^3/3}\right)m_p = \frac{6Nm_p r^2}{R^3}dr \qquad [1]$$

Change in potential energy 引力势能变化

$$dU = -G2N\left(\frac{r^3}{R^3}\right)m_p\frac{dm}{r} = -\frac{12GN^2m_p^2r^4}{R^6}dr \qquad [1]$$

Gravitational potential energy 总引力势能

$$U = -\int_0^R \frac{12GN^2m_p^2r^4}{R^6}dr = -\frac{12GN^2m_p^2}{5R} \qquad [1]$$

(g) Derive the expression of the radius of the white dwarf. Does the radius increase or decrease with increasing mass of the white dwarf? (4 points)
试推导白矮星半径的表达式。若白矮星质量增加，半径是增加还是减少？ （4 分）

Total energy of the white dwarf 白矮星的总能量

$$E_{\text{tot}} = \frac{3h^2}{40m_e}\left(\frac{3}{\pi V}\right)^{\frac{2}{3}}N^{\frac{5}{3}} - \frac{12GN^2m_p^2}{5R} = \frac{3h^2}{40m_e}\left(\frac{9}{4\pi^2}\right)^{\frac{2}{3}}\frac{N^{\frac{5}{3}}}{R^2} - \frac{12GN^2m_p^2}{5R} \qquad [1]$$

Minimizing the total energy, 平衡时总能量最小，

$$\frac{dE_{\text{tot}}}{dR} = -\frac{3h^2}{20m_e}\left(\frac{9}{4\pi^2}\right)^{\frac{2}{3}}\frac{N^{\frac{5}{3}}}{R^3} + \frac{12GN^2m_p^2}{5R^2} = 0$$

[1]

$$R = \frac{h^2}{16Gm_p^2m_e}\left(\frac{9}{4\pi^2}\right)^{\frac{2}{3}}\left(\frac{1}{N}\right)^{\frac{1}{3}} \qquad [1]$$

Since R is proportional to $N^{-1/3}$, the radius decreases with increasing mass of the white dwarf.
由于 R 与 $N^{-1/3}$ 成正比，白矮星质量增加则半径减小。 [1]

(h) Calculate the radius of the white dwarf with the same mass as the Sun. Give your answer in multiples of Earth's radius. You are given the following parameters: (2 points)
试计算质量与太阳相同的白矮星半径。答案请以地球半径为单位。可用以下参数：（2 分）

h = Planck's constant 普朗克常数 = 6.626×10^{-34} Js
G = gravitational constant 万有引力常数 = 6.67×10^{-11} Nm^2/kg^2
m_p = mass of a proton or neutron 质子或中子质量 = 1.67×10^{-27} kg

m_e = mass of an electron 电子质量 = 9.11×10^{-31} kg
m_{Sun} = mass of Sun 太阳质量 = 1.99×10^{30} kg
R_E =radius of Earth 地球半径 = 6380 km

Number of protons in the star 白矮星内质子数目 $N = \dfrac{m_{\text{Sun}}}{2m_p}$ [1]

$$R = \frac{(6.626\times10^{-34})^2}{(16)(6.67\times10^{-11})(1.67\times10^{-27})^2(9.11\times10^{-31})}\left(\frac{9}{4\pi^2}\right)^{\frac{2}{3}}\left(\frac{(2)(1.67\times10^{-27})}{1.99\times10^{30}}\right)^{\frac{1}{3}}$$

=7181 km = $1.13R_E$ [1]

(i) Estimate the mass of the white dwarf when the velocity of electrons becomes comparable to the velocity of light $c = 3 \times 10^8$ m/s. Give your answer in multiples of solar mass. What will happen to the white dwarf? (3 points)
当电子速度接近光速 $c = 3 \times 10^8$ m/s 时，试估计白矮星的质量。请以太阳质量为单位。白矮星将有什么发生？（3 分）

We estimate the velocity of electrons in the white dwarf by 白矮星内电子的速度可估计为

$$v = \frac{p}{m_e} = \frac{1}{m_e}\sqrt{p_x^2 + p_y^2 + p_z^2} = \frac{h}{2m_e L}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

When 当 $v = c$, $\sqrt{n_x^2 + n_y^2 + n_z^2} = \dfrac{2m_e Lc}{h} = \dfrac{2m_e c}{h}V^{\frac{1}{3}}$. [1]

The number of states enclosed by a sphere with this radius 在此球体中的电子状态数

$$N_{\text{state}} = \frac{1}{8}\frac{4}{3}\pi\left(\frac{2m_e c}{h}\right)^3 V = \frac{4\pi m_e^3 c^3}{3h^3}V$$

The number of protons in the white dwarf 白矮星中的质子数

$$N = 2N_{\text{state}} = \frac{8\pi m_e^3 c^3}{3h^3}\left(\frac{4}{3}\pi R^3\right) = \frac{32\pi^2 m_e^3 c^3}{9h^3}R^3$$

Using the result from (g), 从（g）中的结果可得

$$N = \frac{9c^3h^3}{2048\pi^2 G^3 m_p^6}\left(\frac{1}{N}\right) \quad\Rightarrow\quad N = \sqrt{\frac{9c^3h^3}{2048\pi^2 G^3 m_p^6}} = \frac{3}{16\pi m_p^3}\left(\frac{hc}{2G}\right)^{\frac{3}{2}}$$

Mass of the white dwarf 白矮星的质量为 $2Nm_p = \dfrac{3}{8\pi m_p^2}\left(\dfrac{hc}{2G}\right)^{\frac{3}{2}}$

$$= \frac{3}{8\pi(1.67\times10^{-27})^2}\left(\frac{(6.626\times10^{-34})(3\times10^8)}{2(6.67\times10^{-11})}\right)^{\frac{3}{2}} = 2.46\times10^{30}\ \text{kg} = 1.2m_{\text{Sun}}$$ [1]

When the mass of the white dwarf reaches this limit, it will no longer exist. 当白矮星的质量达到此阈值时，将不再存在。
Remark: This answer is comparable to the Chandrasekhar limit of 1.4 m_{Sun}. Beyond this limit, it will collapse into a neutron star.
备注：此答案接近钱德拉塞卡极限 1.4 m_{Sun}。超过此限，星体会坍塌形成中子星。 [1]

Pan Pearl River Delta Physics Olympiad 2015

2015 年泛珠三角及中华名校物理奥林匹克邀请赛
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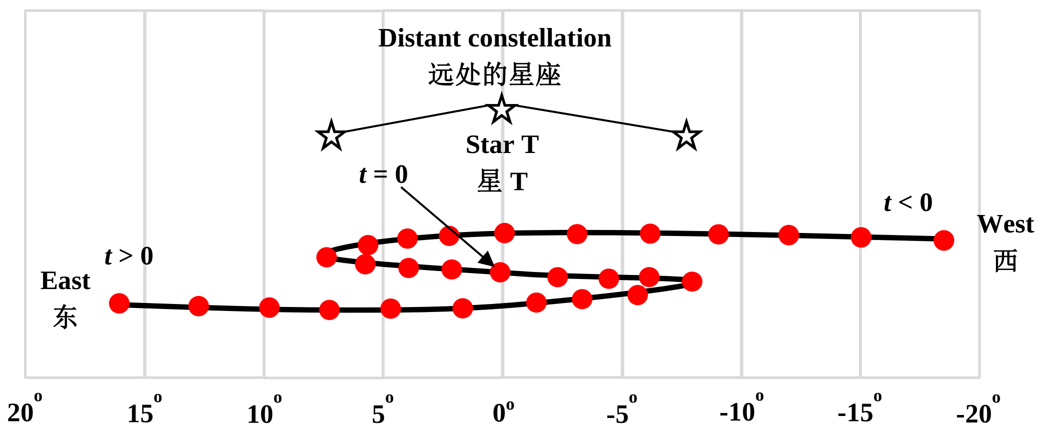
Part-1 (Total 5 Problems) 卷-1（共5题）

(9:00 am – 12:00 pm, 25 February, 2015)

Numerical answers should be given to 3 significant figures.数字答案请给三位有效数字。

1. Retrograde Motion of Mars (9 points) 火星的逆行运动 （9 分）

In the history of astronomy, the phenomenon of the retrograde motion played an important role. Suppose we observe the position of Mars at midnight every night for many nights. Using distant stars and constellations as the background, we will find that Mars moves from West to East most of the time. However, there are periods of time that Mars is observed to move in opposite direction, as shown in the figure. The orbital period of Mars is 1.88 y. Assume that the orbits of Earth and Mars are circular, and the tilting of Earth's axis can be ignored.
在天文史上，行星的逆行运动扮演了重要的角色。假设我们连续多个晚上在午夜观察火星的位置。若以远处的星体和星座为背景，我们会发现大部分时间火星是从西到东运动，但也有些时段是逆向运动，如图所示。火星的轨道周期是 1.88 年。假设地球和火星的轨道都是圆的，地轴的倾斜可略。



(a) What is the orbital radius R_M of Mars? Give your answer in AU (Astronomical Units, 1 AU is the average distance between Sun and Earth.) (1 points)
试求火星的轨道半径 R_M。答案请以 AU 为单位。（1 AU 是太阳与地球的平均距离。）（1 分）

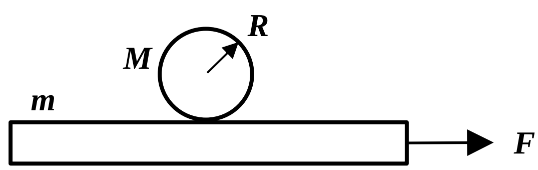
(b) At $t = 0$, Sun, Earth and Mars lie on a straight line. Sketch a figure indicating the positions of Sun, Earth, Mars, and star T when $t > 0$. Label them by letters S, E, M, and T respectively. Mark the angular displacements θ_E and θ_M of Earth and Mars respectively (starting from $t = 0$), and the angle θ that gives the angular position of Mars as observed from Earth using distant stars and constellations as the background. (2 points)
在 $t = 0$ 时，太阳、地球、火星成一直线。试作一草图，显示在 $t > 0$ 时，太阳、地球、火星和星 T 的位置，以 S，E，M 和 T 标示。在图上标示地球和火星的角位移分别为θ_E 和 θ_M（自 $t = 0$ 开始），和地球观察火星的角位置θ（以远处的星体和星座为背景）。（2 分）

(c) Derive an expression for the angular position θ of Mars at time t. Express your answer in terms R_E, R_M, ω_E, ω_M and t, where ω_E and ω_M are the orbital angular velocity of Earth and Mars respectively. (3 points)
试推导火星在时间 t 时的角位置θ。答案请以 R_E, R_M, ω_E, ω_M 和 t 表示，其中ω_E和ω_M分别为地球与火星的角速度。（3 分）

(d) Calculate the angular position θ of Mars at t = 0.1 y, 0.2 y and 0.3 y. Give your answer in degrees. (3 points)
试计算火星在 t = 0.1 年, 0.2 年和 0.3 年时的角位置θ。答案请以度数表示。（3 分）

2. Rolling Ball on a Racket (10 points) 球拍滚球（10 分）

As shown in the figure, a hollow spherical ball of mass M and radius R is placed on a racket of mass m. The racket has a flat surface with coefficient of static friction μ_s and coefficient of kinetic friction μ_k and is held horizontally.
如图所示，一个质量为 M，半径为 R 的空心园球被放置在质量为 m 的球拍上。球拍具有一个平坦的表面，其静摩擦系数为μ_s，动摩擦系数为μ_k，并且被保持在水平位置。



(a) The racket is driven horizontally by a periodic force $F(t) = F_0 \cos\omega_0 t$, with the ball remaining non-slipping. Calculate the maximum velocities of the oscillations of the racket and the ball, denoted as u_x and u_y respectively. (The moment of inertia of a hollow sphere of mass M and radius R is $I = 2MR^2/3$.) (5 points)
球拍被周期性的力 $F(t) = F_0 \cos\omega_0 t$ 沿水平方向驱动，园球维持在不滑动的状态。试计算球拍与球振动时的最大速度，分别表示为 u_x 和 u_y。（质量为 M，半径为 R 的空心球体的转动惯量为 $I = 2MR^2/3$。） (5 分)

(b) At the moment the racket is oscillating at its maximum velocity, its motion is brought to rest abruptly by an external force much stronger than the limiting frictional force between the racket and the ball in a very short duration of time. What is the final velocity of the ball? If the final velocity of the ball is 0, what is the displacement of the ball? (5 points)
在球拍振动至最大速度的一刻，其运动突然被外力煞停，这外力比球拍与球之间的极限摩擦力强得多，作用的时间也很短。问球的最终速度是多少？若球的最终速度为 0，其位移是多少？（5 分）

3. Balloon (10 points) 气球（10 分）

The work done in stretching a spring is converted to its spring energy. Likewise, the work done in stretching a membrane is converted to its surface energy, given by $E = \gamma S$, where γ is called the *surface tension* of the membrane, and S is its surface area.
拉伸弹簧所做的功被转换成弹簧的内能。同样，拉伸薄膜所做的功被转换成它的表面能$E = \gamma S$，其中γ称为薄膜的*表面张力*，而S是其表面面积。

(a) Consider a balloon of radius R. What is the change in surface energy when the radius changes by dR? Hence derive an expression for the pressure due to surface tension. (2 points)
考虑半径为R的气球。当半径改变为dR时，表面能的变化是多少？由此推导表面张力形成的压力的表达式。（2分）

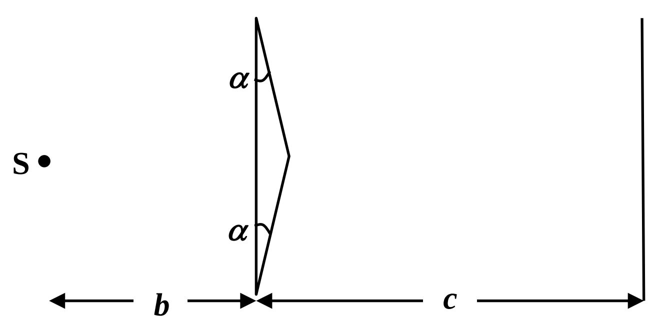
(b) The surface tension of balloon A is γ. When it is filled with a diatomic ideal gas, its radius becomes R_0. The surface tension of balloon B is 2γ. When it is filled with the same kind of ideal diatomic gas, its radius becomes R_0. The temperature of the environment is T. The two balloons are then connected so that the gases are free to exchange between them until a steady state is reached. The final temperature is the same as that of the environment. What are the final radii of the two balloons respectively? You may neglect the atmospheric pressure in the analysis. (4 points)
气球A的表面张力为γ。当它充满了一种双原子的理想气体，其半径是R_0。气球B的表面张力为2γ。当它被相同的双原子理想气体充满时，其半径是R_0。环境的温度为T。然后两个气球被连接，使得气体可以在它们之间自由交流，直至达到稳定状态。最终温度与环境相同。问两个气球最终的半径分别是什么？在分析中你可以忽略大气压力。（4分）

(c) What are the amounts of heat gain by the gases in balloons A and B respectively during the gas exchange process in (b)? (4 points)
在(b)部的气体交流过程中，气球A和B增加的热能分别是什么？（4分）

4. Fresnel Biprism (10 points) 菲涅耳双棱镜（10 分）

Fresnel biprism was devised shortly after the famous Young's double slit experiment to confirm the interference phenomenon. Nowadays, it is widely used in different applications. As shown in the figure, it consists of a single light source S and a pair of wedge-shaped prisms arranged back to back. We introduce the following notations:
在著名的杨氏双缝实验面世后不久，便产生了菲涅耳双棱镜的设计，用以确认干涉现象。如今，它被广泛用于不同的应用。如图所示，它由一个单一的光源 S 和一对背对背的楔形棱镜组成。我们引入以下符号：

n = refractive index of the biprism 双棱镜的折射率
α = apex angle of each prism 双棱镜的顶角
b = distance between light source and biprism 光源与双棱镜的距离
c = distance between biprism and screen 双棱镜与屏幕的距离
λ = wavelength of light 光的波长



(a) Derive an expression for the angular deviation after a light beam has passed through one of the two prisms. (3 points)
试推导光束经过其中一个棱镜后偏转角的表达式。（3 分）

(b) Derive an expression for the separation of the fringes on the screen. (4 points)
试推导屏幕上条纹距离的表达式。（4 分）

(c) In a modern application on electron microscopes, the single light source is replaced by a parallel beam of wave incident normally to the flat surface of the biprism. Derive an expression for the separation of the fringes on the screen. (3 points)
在现代，这原理已应用到电子显微镜中。在这应用中，单个光源被替换成入射的平行波束，垂直于双棱镜的平面。试推导屏幕上条纹距离的表达式。（3 分）

5. Ionic Crystals (11 points) 离子晶体（11 分）

An ionic crystal can be modeled by a chain of positively and negatively charged ions. The ionic separation is a. The positive ions with atomic mass M are located at the positions $x = na$ where n is even. The negative ions with atomic mass m ($m < M$) are located at the positions $x = na$ where n is odd. The ions are coupled to their neighbors by springs, which provide restoring forces to their transverse displacements. The returning force is proportional to the displacements of the ions relative to their neighbors, and the spring constant is k.
我们可以一串带正电和带负电的离子，作为离子晶体的模型。离子间的距离为 a。正离子的原子质量为 M，处于位置 $x = na$，其中 n 是偶数。负离子的原子质量为 m（$m < M$），处于在位置 $x = na$，其中 n 是奇数。相邻的离子有弹簧耦合，弹簧为离子的横向位移提供返回力。返回力正比于离子相对于相邻离子的位移，并且弹簧常数为 k。

(a) Let $u_n(t)$ be the transverse displacement of the ion at $x = na$ and time t. Derive the equations of motion for both types of ions. Show that the solution of the equation of motion can be written as
令 $u_n(t)$为处于 $x= na$ 的离子在时间 t 的横向位移。试推导两种类型离子的运动方程。表明运动方程的解可以写成

$$u_n(t) = \begin{cases} A_M \sin(qna - \omega t) & n \text{ even}, \\ A_m \sin(qna - \omega t) & n \text{ odd}. \end{cases}$$

Find the relation between q and ω. (3 points) 试找出 q 与ω的关系。（3 分）

(b) Find the solutions of ω in the limit $q = 0$, and the relation between A_M and A_m for each solution. (2 points)
在极限 $q = 0$，求ω的所有解，并且求在每个解中 A_M 与 A_m 间的关系。（2 分）

(c) In the limit $q = 0$, calculate the wave velocity of the low frequency mode. (1 point)
在极限 $q =0$，试计算低频模式的波速。（1 分）

(d) In the limit $q = \pi/2a$, find the solutions of ω, and the relation between A_M and A_m for each solution. (2 points)
在极限 $q = \pi/2a$，求ω的所有解，并且求在每个解中 A_M 与 A_m 间的关系。（2 分）

(e) Sketch the angular frequency ω as a function of the wavenumber q from $q = -\pi/2a$ to $q = \pi/2a$. (2 points)
试绘出角频率ω作为波数 q 的函数的草图，范围从 $q = -\pi/2a$ 到 $q = \pi/2a$。（2 分）

(f) An electromagnetic wave is incident on the crystal. Which frequency mode will be excited? (1 point) 有电磁波入射到晶体。哪种频率模式会被激发？（1 分）

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2015
2015 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Part-2 (Total 2 Problems) 卷-2（共2 题）
(2:00 pm – 5:00 pm, 25 February, 2015)

1. Exoplanet Microlensing (25 points) 系外行星的微透镜效应 （25 分）

With the discovery of planets orbiting around stars in recent years, the observation of exoplanets from astronomical distances became a challenge to scientists. Gravitational microlensing is one of the detection methods. It makes use of Einstein's discovery in general relativity that when a light ray passing near a spherically symmetric body of mass M, its direction will be deflected towards the body by a small angle given by
随着近年发现不少绕着恒星运行的行星，怎样观察相隔天文距离的系外行星便成为科学家的挑战。引力微透镜是其中一种检测方法。它利用爱因斯坦在广义相对论里发现的原理，就是当光线经过一个质量为 M 的球对称物体时，方向会朝向物体偏转，偏转的小角度为

$$\alpha = \frac{4GM}{rc^2},$$

where G is the gravitational constant, c is the speed of light, and r is the distance of closest approach of the light ray to the body. In this problem, we will study the principle of detecting exoplanet by microlensing.
其中 G 是万有引力常数，c 是光速，r 是光线和物体的最短距离。在这个问题中，我们将研究通过微透镜效应探测系外行星的原理。

(a) Consider a distant star S located at a distance D_s from Earth E, acting as the light source. Another star L of mass M and located at distance D_l ($< D_s$) from Earth acts as the lens. The lines EL and ES make a small angle β between them. Construct the following sketch in the answer book: (a1) the line EL, (a2) the line ES, (a3) the distances D_l and D_s, (a4) the angle β (remark: although this angle is small in practice, it should not be drawn too small for the purpose of clarity), (a5) a line perpendicular to EL through L, acting as the gravitational lens, (a6) the light ray from S to E, assuming that each of the segments between S and the lens and that between the lens and E are straight lines, (a7) the deflection angle α, (a8) the apparent angle θ of the star S as observed on Earth (relative to line EL). (3 marks)
考虑一个遥远的恒星 S，离地球 E 的距离为 D_s，作为光源。另一颗恒星 L，质量为 M，离地球的距离为 D_l（< D_s），作为透镜。线 EL 和 ES 间的小角度为β。试在答题簿上绘出以下草图：（a1）线 EL，（a2）线 ES，（a3）距离 D_l 和 D_s，（a4）角度β（注：虽然该角度实际上很小，但为清楚起见，不应把它绘得太小），（a5）一条垂直于 EL 而通过 L 的线，作为引力透镜，（a6）从 S 到 E 的光线，假定 S 和透镜之间的线段及透镜和 E 之间的线段各可视作直线，（a7）偏转角α，（a8）从地球观察星 S 的视角θ（相对于线 EL）。（3 分）

(b) Derive an equation for the angle θ in terms of the parameters D_s, D_l, G, M, c and β, assuming that all angles are small. (3 points)
试推导θ的方程式，以参数 D_s, D_l, G, M, c 和 β表达，可假设所有角度都很小。（3 分）

(c) Consider the case that the lens is exactly aligned with the source ($\beta = 0$). The image of S appears to be a ring known as an Einstein ring. Derive the expression for the angular radius θ_E of the Einstein ring. (2 points)
考虑透镜与光源对準的情况（$\beta = 0$）。 S 的影象呈环形，称为爱因斯坦环。试推导爱因斯坦环的角半径θ_E的表达式。 （2 分）

(d) Calculate the Einstein radius for the following typical values:
试以下列的典型值，计算爱因斯坦半径：
$M = 0.3$ solar mass, $D_s = 10$ kpc. $D_l = 3$ kpc.
Give your answer in milli-arc-seconds. You may use the following constants:
请以 milli-arc-seconds 表达你的答案。您可以使用以下参量：
$G = 6.67 \times 10^{-11}$ Nm^{2}kg^{-2}, 1 solar mass = 1.99×10^{30} kg, $c = 3 \times 10^8$ ms^{-1}, 1 kpc = 3.09×10^{19} m, 1 radian = 206265 arc seconds. (1 point) （1 分）

(e) When the lens and the source are not exactly aligned, there will be two images of S. It is convenient to express the angles β and θ in multiples of the Einstein radius θ_E. Hence we define $u \equiv \dfrac{\beta}{\theta_E}$ and $y \equiv \dfrac{\theta}{\theta_E}$. Derive the expressions for the angular positions y of the two images in terms of u. (2 points)
当透镜和光源不完全对齐时，S 将有两个影像。为方便起见，我们以爱因斯坦半径θ_E的倍数表达角β 和θ。因此我们定义$u \equiv \dfrac{\beta}{\theta_E}$和$y \equiv \dfrac{\theta}{\theta_E}$。试推导两个影像的角位置 y，以 u表示。（2 分）

(f) To study the effect of the finite size of star S, we introduce Cartesian coordinates on the plane normal to ES and through S, with the y axis lying in the plane containing E, L and S. Consider the corners $(0, u + \delta)$ and (δ, u) of a square on the surface of star S ($\delta << u$). Calculate the coordinates of the two corners of the two images when viewed from Earth. (2 points)
为研究星 S 有限大小的影响，我们在垂直于 ES 和通过 S 的平面上，引入一平面直角坐标，其中 y 轴位于包含 E，L 和 S 的平面中。考虑星 S 表面上一个正方形的角$(0, u + \delta)$ and (δ, u) ($\delta << u$)。试计算从地球观察时，这两个影像的两个角的坐标。（2 分）

(g) Calculate the areal magnifications of the two images of star S in terms of u. Following the practice in astronomical observations, give your answer in absolute values. (2 points)
试计算星 S 的两个影像的面积放大率，请以 u 表达。按照天文观测的习惯，请以绝对值为答案。（2 分）

(h) In practice, since the images cannot be resolved, astronomers measure the sum of the magnifications of the two images. Derive the expression for the total magnification. Describe its behavior when star S is remote (u approaches infinity) and when S approaches perfect alignment with L and E (u approaches 0). (3 points)
实际上，由于影象不易分辨，天文学家只测量两个影像的放大率的总和。试推导总放大率的表达式。试描述星 S 在远处时（u 趋近无穷大），及星 S 趋近对準 L 与 E 时(u 趋近 0)，总放大率的行为。（3 分）

(i) A planet P of star L has mass m and is located in the plane of E, L and S at the same distance D_l from Earth. EP and EL makes an angle θ_p. Derive an equation for the angle θ taking into account the gravitational lensing effects of both star L and planet P. Expressions in the

equation should be written in terms of the parameters D_s, D_l, G, M, c, β, m and θ_p, assuming that all angles are small. Simplify the equation by introducing the mass ratio $q \equiv \frac{m}{M}$ and the rescaled positions $u_p \equiv \frac{\theta_p}{\theta_E}$, $u \equiv \frac{\beta}{\theta_E}$, $y \equiv \frac{\theta}{\theta_E}$. (3 points)
星 L 旁有一行星 P 位于 E、L 和 S 的平面上，其质量为 m，与地球距离跟星 L 同为 D_l，EP 与 EL 间角度为θ_p 。考虑到星 L 和行星 P 两者的引力透镜作用，试推导角θ的方程式，式中的表达式应以 D_s、D_l、G、 M、 c、β、 m 和θ_p表达。可假设所有角度都很小。引入质量比$q \equiv \frac{m}{M}$和重整位置$u_p \equiv \frac{\theta_p}{\theta_E}$, $u \equiv \frac{\beta}{\theta_E}$, $y \equiv \frac{\theta}{\theta_E}$，以简化方程式。（3 分）

(j) In typical exoplanet detections, there is a motion of star S relative to star L. As star S approaches the closest distance to star L and moves away, u decreases with time to a minimum value u_0 and increases again. By plotting the magnification of the image of star S versus time, one observes a smooth and relatively broad peak in the magnification curve due to gravitational lensing by star L. In addition, one can observe a side peak due to the presence of the planet. For $q \ll 1$, estimate the width of this side peak, that is, the range of u in which the side peak is significant. (1 point)
在典型的系外行星检测中，星 S 对于星 L 有相对运动。星 S 趋近星 L 至最短距离，然后离开，过程中 u 随时间降到最小值 u_0 然后再增加。把星 S 影像的放大率与时间的关系绘成图表，放大率曲线上可以看到一个平滑和较宽的主峰，是由星 L 的引力透镜作用形成的。另外，我们可以观察到一个侧峰，是由行星形成的。对于 $q \ll 1$，试估计这个侧峰的宽度，也就是可以显著看到侧峰的 u 数值范围。（1 分）

(k) For $q \ll 1$, consider the situation that light rays pass very near to planet P, so that the gravitational lensing by star L becomes relatively insignificant. Calculate the position of star S where the total magnification of its image diverges, and the behavior of the total magnification in the neighborhood of this location. (3 points)
当 $q \ll 1$ 时，考虑光线非常靠近行星 P 的情况，在这情况下星 L 的引力透镜作用相对很弱。试计算当星 S 图像的总放大率发散时星 S 的位置，和这位置附近总放大率的行为。（3 分）

2. Cosmic Gravitational Waves (28 points) 宇宙引力波（28 分）

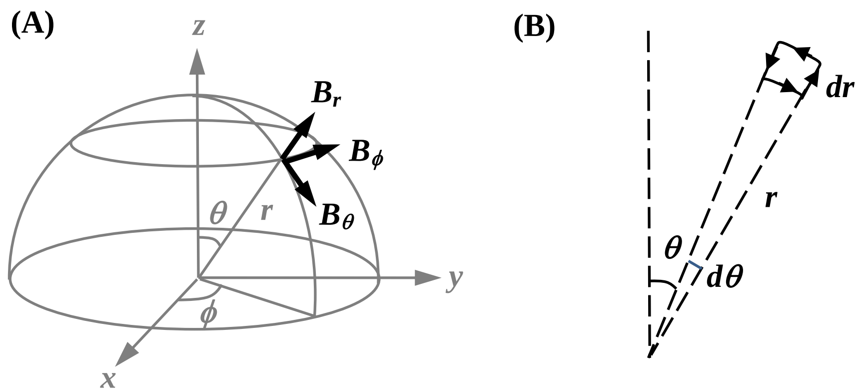
In March 2014, scientists operating gravitational wave detectors in the South Pole claimed that they found evidences of gravitational waves originated from the early universe in the cosmic microwave background radiation. While the evidence is still being debated, it is interesting to understand how gravitational waves interact with electromagnetic (EM) waves. To approach this issue, we start by considering how molecules scatter EM waves.
2014 年 3 月，操作南极引力波探测器的科学家，声称在宇宙微波背景辐射中，发现来自早期宇宙的引力波的证据。虽然证据还存在争议，但了解引力波如何作用于电磁（EM）波是一个有趣的课题。为了处理这个问题，我们首先考虑分子是如何散射电磁波。

(a) An oscillating electric dipole consists of charges oscillating at an angular frequency ω. Specifically, the charges are $Q(t) = \pm Q_0 \cos\omega t$, located at $(x, y, z) = (0, 0, \pm s)$ respectively. What is the current between them? (1 point)

一个振动的电偶极子，包含以角频率ω振动的电荷。具体来说，电荷分别为 $Q(t) = \pm Q_0\cos\omega t$, 位于$(x, y, z) = (0, 0, \pm s)$。它们之间的电流是什么？ （1 分）

(b) In spherical coordinates, we denote the components of the magnetic field as B_r, B_θ and B_ϕ, as shown in figure (A). Calculate $B_\phi(r, \theta, t)$ according to Biot-Savart's law at time t and distance r from the origin making an angle θ with the z axis. Note that due to the finite speed of light c, the magnetic field at a distant location is due to the time-changing current at an earlier instant. Hence the *retarded* magnetic field takes the form $\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r})\cos(\omega t - kr + \psi)$, where $k \equiv \dfrac{\omega}{c}$ is the *wavenumber*, and ψ is the phase shift. Express your answer in terms of the magnitude of the dipole moment $p \equiv 2Qs$ in the limit s approaches 0. Below, your answer to this part will be denoted as $B_{BS}(r, \theta, t)$. (3 points)

在球坐标中，我们以 B_r, B_θ 和 B_ϕ,表示磁场的分量，如图（A）所示。根据毕奥 - 萨伐尔定律，试计算磁场 $B_\phi(r, \theta, t)$，其中 r 为位置与原点的距离，θ为位置与 z 轴形成的角，t 为时间。注意，由于光以有限速率 c 传播，在远处的磁场是源于某一较早时刻的电流（电流随时间变化）。因此，*延迟*磁场的形式为$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r})\cos(\omega t - kr + \psi)$，其中 $k \equiv \dfrac{\omega}{c}$ 是*波数*，而ψ是相移。答案请以偶极矩 $p \equiv 2Qs$ 表达（取 s 趋于 0 的极限）。下面，你在这部的答案将被表示为 $B_{BS}(r, \theta, t)$。（3 分）



(c) However, Biot-Savart's law is only applicable to steady state currents. It is incomplete even after including the retarded nature of the oscillating current. By considering the wave nature of the magnetic field, the complete expression of the magnetic field is given by

但是，毕奥 - 萨伐尔定律只适用于稳态电流。甚至考虑了振动电流的滞后性质后，它还是不完整的。通过考虑磁场的波动性，完整的磁场表达式是

$$\mathbf{B}(r,\theta,t) = \mathbf{B}_{BS}(r,\theta,t) + \mathbf{B}_{wave}(r,\theta,t),$$

where 其中

$$\mathbf{B}_{wave}(r,\theta,t) = \frac{\mu_0}{4\pi c}\int\left[\frac{d}{dt}I\left(t-\frac{r}{c}\right)\right]\frac{d\bar{l}\times\hat{r}}{r}.$$

Derive an expression for the B_ϕ component of $\mathbf{B}_{wave}$ at (r, θ, t). (3 points)

试推导 $\mathbf{B}_{wave}$ 在 (r, θ, t)的 B_ϕ 分量的表达式。（3 分）

(d) Compare the amplitudes of B_{BS} and B_{wave} at large distance r. Derive the condition of r such that B_{BS} becomes negligible when compared with B_{wave}. (2 points)
比较 B_{BS} 和 B_{wave} 在距离 r 很大时的幅度。试推导 B_{BS} 相比 B_{wave} 变得微不足道时，关于 r 的条件。（2 分）

(e) At large distance r, the electric field at (r, θ, t) is mainly due to the electromagnetic induction by the magnetic field B_{wave}. By considering the electromotive force along the circuit shown in figure (B), derive the relation between $\frac{\partial E_\theta}{\partial r}$ and $\frac{\partial B_\phi}{\partial t}$. Here, $\frac{\partial E_\theta}{\partial r}$ is known as the partial derivative of E_θ with respect to r, meaning that other variables such as θ and t are considered fixed. Similarly, $\frac{\partial B_\phi}{\partial t}$ is the partial derivative of B_ϕ with respect to t, with other variables such as r and θ being fixed. You may assume that only the E_θ component of the electric field is significant at large distance r. (3 points)
在距离 r 很大时，在(r, θ, t)的电场主要是源于 B_{wave} 的电磁感应。通过考虑沿著图（B）中闭路的电动势，试推导$\frac{\partial E_\theta}{\partial r}$与$\frac{\partial B_\phi}{\partial t}$之间的关系。这里，$\frac{\partial E_\theta}{\partial r}$被称为 E_θ相对于 r 的偏导数，意味着其他变量如θ和 t 被假定为固定的。同样地，$\frac{\partial B_\phi}{\partial t}$是 B_ϕ相对于 t 的偏导数，当中假定其他变量如 r 和θ 为固定的。你可以假设在距离 r 很大时，电场仅有 E_θ分量是显著的。（3 分）

(f) At large distance r, the electric field is given by $E_\theta(r,\theta,t) = \frac{A(\theta)}{r}\cos(\omega t - kr)$. Find $A(\theta)$. (2 points)
在距离 r 很大时，电场为$E_\theta(r,\theta,t) = \frac{A(\theta)}{r}\cos(\omega t - kr)$。试找出 $A(\theta)$。（2 分）

(g) The magnitude and direction of the power per unit area of the EM wave are given by the Poynting vector. Calculate the time-averaged power per unit area at large distance r. This will be denoted as the radiation intensity $I(r)$. (3 points)
电磁波每单位面积传播功率的大小和方向，是由 Poynting 矢量给定的。试计算在距离 r 很大时，每单位面积按时间平均的传播功率。这将被表示为辐射强度 $I(r)$。（3 分）

(h) When an EM wave is incident on a molecule, its electric field $\mathbf{E}$ will drive the molecule into an oscillating dipole moment given by $\mathbf{p} = \alpha\mathbf{E}$, where α is the polarizability of the molecule. In turn, the oscillating dipole will radiate power. This is called a scattering process. Consider an EM wave incident from the x direction, given by $\mathbf{E}_i = \mathbf{E}_{x0}\cos(\omega t - kx)$. If $\mathbf{E}_{x0}$ is polarized at an angle θ_x with the z axis, calculate:
当电磁波射向一分子时，其电场 $\mathbf{E}$ 会使该分子产生振动偶极矩 $\mathbf{p} = \alpha\mathbf{E}$，其中$\alpha$是该分子的极化度。随之振动偶极子会辐射功率。这就是所谓的散射过程。考虑电磁波从 x 方向入射，由 $\mathbf{E}_i = \mathbf{E}_{x0}\cos(\omega t - kx)$给出。若 $\mathbf{E}_{x0}$ 的偏振方向与 z 轴成角度θ_x，试计算：
(h1) the intensity $I_x(r)$ of the radiation scattered to the z direction,
散射至 z 方向的辐射强度 $I_x(r)$，
(h2) the electric field polarization of the scattered wave along that direction,
沿该方向的散射波的电场偏振方向，

(h3) the intensity $\langle I_x(r)\rangle$ of the radiation scattered to the z direction for an unpolarized incident beam (that is, the polarization angle θ_x has a uniform distribution). (3 points)
非偏振入射光束（即偏振角θ_x均匀分布）散射至 z 方向的辐射强度$\langle I_x(r)\rangle$。（3 分）

(i) Next, consider an EM wave incident from the y direction, given by $\mathbf{E}_i = \mathbf{E}_{y0}\cos(\omega t - ky)$. If $\mathbf{E}_{y0}$ is polarized at an angle θ_y with the z axis, calculate:
接下来，考虑电磁波从 y 方向入射，由 $\mathbf{E}_i = \mathbf{E}_{y0}\cos(\omega t - ky)$给出。若 $\mathbf{E}_{y0}$ 的偏振方向与 z 轴成角度θ_y，试计算：
(i1) the electric field polarization of the scattered wave along the z direction,
沿 z 方向的散射波的电场偏振方向，
(i2) the intensity $\langle I_y(r)\rangle$ of the radiation scattered to the z direction for an unpolarized incident beam (that is, the polarization angle θ_y has a uniform distribution). (2 points)
非偏振入射光束（即偏振角θ_y均匀分布）散射至 z 方向的辐射强度$\langle I_y(r)\rangle$。（2 分）

(j) During the rapid expansion of the early universe, gravitational waves are formed. They consist of *quadrupolar* temperature oscillations，meaning that the directions of the maxima and minima of the oscillations are separated by an angle of $\pi/2$. Hence to analyze their effects on EM waves, we consider two incoherent incident beams of EM waves of the same frequency $\omega/2\pi$, one from the x direction and the other from the y direction. The amplitudes of their electric fields are E_{x0} and E_{y0} respectively. Suppose the EM radiations in the x and y directions correspond to temperatures $T + \Delta T$ and T respectively ($\Delta T << T$ and is positive). What is the ratio $\frac{\langle I_x(r)\rangle}{\langle I_y(r)\rangle}$? (1 point)
早期宇宙的迅速膨胀，形成引力波。它引起温度的振动，呈*四偶极*分布。这意味着振动的最大值和最小值的方向以$\pi/2$ 角度分开。因此，要分析它们对电磁波的影响，我们考虑两束频率同为$\omega/2\pi$的非相干入射光，一束来自 x 方向，另一束则来自 y 方向，其电场的幅度分别是 E_{x0} 和 E_{y0}。假设在 x 和 y 方向的电磁辐射分别对应于温度 $T + \Delta T$ 和 T（$\Delta T << T$，且是正的）。比例$\frac{\langle I_x(r)\rangle}{\langle I_y(r)\rangle}$是什么？（1 分）

(k) The degree of polarization of the scattered radiation is given by 下式是散射辐射的偏振度

$$\Pi = \frac{\left|\langle I_x(r)\rangle - \langle I_y(r)\rangle\right|}{\langle I_x(r)\rangle + \langle I_y(r)\rangle}.$$

Calculate Π. What is the direction of the electric field polarization in the scattered wave? (2 points)
试计算Π。散射辐射中电场的偏振方向是什么？（2 分）

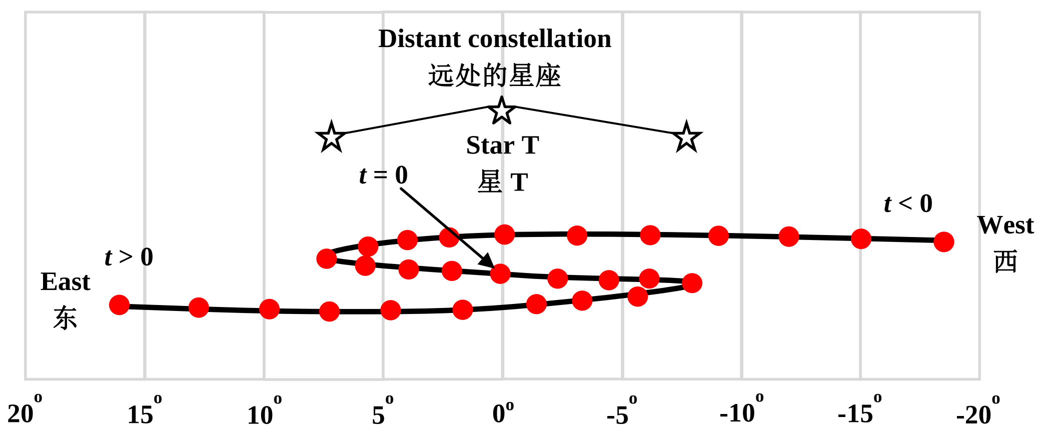
《THE END 完》

Pan Pearl River Delta Physics Olympiad 2015
2015 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Part-1 (Total 5 Problems) 卷-1（共5题）
(9:00 am – 12:00 pm, 25 February, 2015)

Numerical answers should be given to 3 significant figures.数字答案请给三位有效数字。

1. Retrograde Motion of Mars (9 points) 火星的逆行运动 （9 分）

In the history of astronomy, the phenomenon of the retrograde motion played an important role. Suppose we observe the position of Mars at midnight every night for many nights. Using distant stars and constellations as the background, we will find that Mars moves from West to East most of the time. However, there are periods of time that Mars is observed to move in opposite direction, as shown in the figure. The orbital period of Mars is 1.88 y. Assume that the orbits of Earth and Mars are circular, and the tilting of Earth's axis can be ignored.
在天文史上，行星的逆行运动扮演了重要的角色。假设我们连续多个晚上在午夜观察火星的位置。若以远处的星体和星座为背景，我们会发现大部分时间火星是从西到东运动，但也有些时段是逆向运动，如图所示。火星的轨道周期是 1.88 年。假设地球和火星的轨道都是圆的，地轴的倾斜可略。

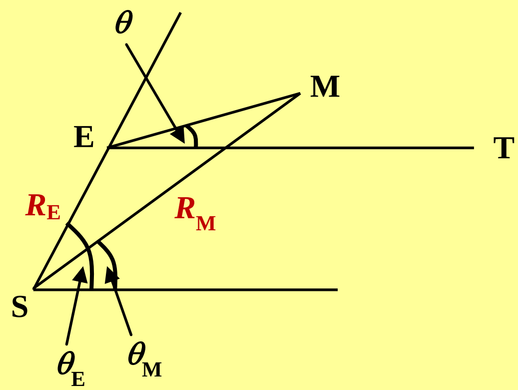


(a) What is the orbital radius R_M of Mars? Give your answer in AU (Astronomical Units, 1 AU is the average distance between Sun and Earth.) (1 points)
试求火星的轨道半径 R_M。答案请以 AU 为单位。(1 AU 是太阳与地球的平均距离。)（1 分）

Using Kepler's Law, 按开普勒定律， $R_M = R_E\left(\frac{T_M}{T_E}\right)^{\frac{2}{3}} = 1\left(\frac{1.88}{1}\right)^{\frac{2}{3}} = 1.5233\text{ AU} \approx 1.52\text{ AU}$

(b) At $t = 0$, Sun, Earth and Mars lie on a straight line. Sketch a figure indicating the positions of Sun, Earth, Mars, and star T when $t > 0$. Label them by letters S, E, M, and T respectively. Mark the angular displacements θ_E and θ_M of Earth and Mars respectively (starting from t =

0), and the angle θ that gives the angular position of Mars as observed from Earth using distant stars and constellations as the background. (2 points)
在 t = 0 时，太阳、地球、火星成一直线。试作一草图，显示在 t > 0 时，太阳、地球、火星和星 T 的位置，以 S、E、M 和 T 标示。在图上标示地球和火星的角位移分别为 θ_E 和 θ_M（自 t = 0 开始），和地球观察火星的角位置θ（以远处的星体和星座为背景）。（2 分）

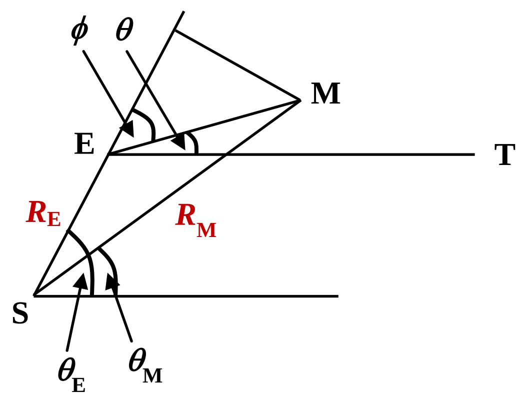


(c) Derive an expression for the angular position θ of Mars at time t. Express your answer in terms R_E, R_M, ω_E, ω_M and t, where ω_E and ω_M are the orbital angular velocity of Earth and Mars respectively. (4 points)
试推导火星在时间 t 时的角位置θ。答案请以 R_E, R_M, ω_E, ω_M 和 t 表示，其中ω_E 和ω_M 分别为地球与火星的角速度。（3 分）

$\theta_E = \omega_E t$, $\theta_M = \omega_M t$
In triangle SEM, we need to find the exterior angle at E. By constructing a perpendicular line from M to SE, this angle is
在三角形 SEM 里，需找出角 E 的外角。从点 M 作一线垂直于 SE，可见这角为

$$\phi = \tan^{-1}\left(\frac{R_M \sin(\theta_E - \theta_M)}{R_M \cos(\theta_E - \theta_M) - R_E}\right).$$

$$\theta = \omega_E t - \tan^{-1}\left(\frac{R_M \sin(\omega_E t - \omega_M t)}{R_M \cos(\omega_E t - \omega_M t) - R_E}\right)$$



(d) Calculate the angular position θ of Mars at t = 0.1 y, 0.2 y and 0.3 y. Give your answer in degrees. (3 points)
试计算火星在 t = 0.1 年, 0.2 年和 0.3 年时的角位置θ。答案请以度数表示。（3 分）

$$\text{At } t = 0.1,\ \theta = 2\pi(0.1) - \tan^{-1}\left(\frac{1.5233\sin[2\pi(0.1) - 2\pi(0.1)/1.88]}{1.5233\cos[2\pi(0.1) - 2\pi(0.1)/1.88] - 1}\right) = -7.963^\circ \approx -8.00^\circ$$

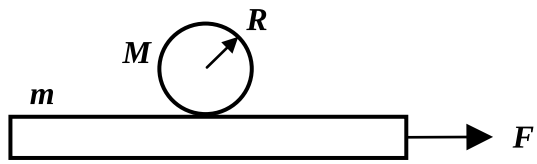
$$\text{At } t = 0.2,\ \theta = 2\pi(0.2) - \tan^{-1}\left(\frac{1.5233\sin[2\pi(0.2) - 2\pi(0.2)/1.88]}{1.5233\cos[2\pi(0.2) - 2\pi(0.2)/1.88] - 1}\right) = -0.4538^\circ \approx -0.454^\circ$$

At $t = 0.3$, $\theta = 2\pi(0.3) - \tan^{-1}\left(\dfrac{1.5233\sin[2\pi(0.3) - 2\pi(0.3)/1.88]}{1.5233\cos[2\pi(0.3) - 2\pi(0.3)/1.88] - 1}\right) = -16.43^{\circ} \approx 16.4^{\circ}$

2. Rolling Ball on a Racket (10 points) 球拍滚球（10 分）

As shown in the figure, a hollow spherical ball of mass M and radius R is placed on a racket of mass m. The racket has a flat surface with coefficient of static friction μ_s and coefficient of kinetic friction μ_k and is held horizontally.

如图所示，一个质量为 M，半径为 R 的空心园球被放置在质量为 m 的球拍上。球拍具有一个平坦的表面，其静摩擦系数为μ_s，动摩擦系数为μ_k，并且被保持在水平位置。



(a) The racket is driven horizontally by a periodic force $F(t) = F_0 \cos\omega_0 t$, with the ball remaining non-slipping. Calculate the maximum velocities of the oscillations of the racket and the ball, denoted as u_x and u_y respectively. (The moment of inertia of a hollow sphere of mass M and radius R is $I = 2MR^2/3$.) (5 points)

球拍被周期性的力 $F(t) = F_0 \cos\omega_0 t$ 沿水平方向驱动，园球维持在不滑动的状态。试计算球拍与球振动时的最大速度，分别表示为 u_x 和 u_y。（质量为 M，半径为 R 的空心球体的转动惯量为 $I = 2MR^2/3$。） (5 分)

Let x and y be the displacements of the racket and the ball respectively. Let θ be the angular displacement of the ball (counted in the direction of x at the contact point with the racket). Let f be the frictional force between the racket and the ball. Applying Newton's law,

令 x 和 y 分别为球拍与球的位移。令θ为球的角位移（方向按照在与球拍的接触点沿位移 x 方向计算）。设 f 是球拍与球之间的摩擦力。运用牛顿定律，

$$m\ddot{x} = F - f \qquad (1)$$

$$M\ddot{y} = f \qquad (2)$$

$$\frac{2}{3}MR^2\ddot{\theta} = fR \qquad (3)$$

Condition for no slipping: 不滑动的条件： $\theta = \dfrac{x - y}{R}$ (4)

From Eqs. (3) and (4), 从方程式（3）和（4）， $\dfrac{2}{3}M(\ddot{x} - \ddot{y}) = f$ (5)

Combining with Eq. (2), 结合方程式（2）： $\ddot{y} = \dfrac{2}{5}\ddot{x}$ and $f = \dfrac{2}{5}M\ddot{x}$ (6)

Substituting into Eq. (1), 代入方程式（1）， $\left(m + \dfrac{2}{5}M\right)\ddot{x} = F_0 \cos\omega_0 t$

Solution: 解： $x = -\dfrac{5F_0 \cos\omega_0 t}{(2M + 5m)\omega_0^2}$, $y = -\dfrac{2F_0 \cos\omega_0 t}{(2M + 5m)\omega_0^2}$, $\theta = -\dfrac{3F_0 \cos\omega_0 t}{(2M + 5m)\omega_0^2 R}$,

$\dot{x} = \frac{5F_0 \sin \omega_0 t}{(2M+5m)\omega_0}$, $\dot{y} = \frac{2F_0 \sin \omega_0 t}{(2M+5m)\omega_0}$, $\dot{\theta} = \frac{3F_0 \sin \omega_0 t}{(2M+5m)\omega_0 R}$.

Hence 因此 $u_x = \frac{5F_0}{(2M+5m)\omega_0}$, $u_y = \frac{2F_0}{(2M+5m)\omega_0}$.

(b) At the moment the racket is oscillating at its maximum velocity, its motion is brought to rest abruptly by an external force much stronger than the limiting frictional force between the racket and the ball in a very short duration of time. What is the final velocity of the ball? If the final velocity of the ball is 0, what is the displacement of the ball? (5 points)
在球拍振动至最大速度的一刻，其运动突然被外力煞停，这外力比球拍与球之间的极限摩擦力强得多，作用的时间也很短。问球的最终速度是多少？若球的最终速度为 0，其位移是多少？（5 分）

The impulse acting on the racket is given by the external force multiplied by the time duration of the force, whereas the impulse acting on the ball is given by the limiting frictional force multiplied by the duration. Hence the impulse acting on the ball is negligible. Hence when the racket stops moving, the ball continues to move with the velocity u_y and angular velocity $\omega \equiv \frac{3F_0}{(2M+5m)\omega_0 R} = \frac{3u_y}{2R}$. Since $u_y \neq -R\omega$, the ball will slide until it finally rolls. Applying Newton's law,
作用在球拍的冲量是外力乘以力作用的时间，而作用在球上的冲量是极限摩擦力乘以力作用的时间。因此，作用在球上的冲量可以忽略不计。因此，当球拍停止移动时，球继续以速度 u_y 和角速度 $\omega \equiv \frac{3F_0}{(2M+5m)\omega_0 R} = \frac{3u_y}{2R}$ 移动。因 $u_y \neq -R\omega$，球会滑动，直到它最终滚动。运用牛顿定律，

$$M\ddot{y} = -\mu_k Mg \quad \Rightarrow \quad \dot{y} = u_y - \mu_k gt$$

$$\frac{2}{3}MR^2\ddot{\theta} = -fR \quad \Rightarrow \quad R\dot{\theta} = \frac{3}{2}u_y - \frac{3}{2}\mu_k gt$$

When the ball stops sliding, 当球停止滑动时， $R\dot{\theta} = -\dot{y} \quad \Rightarrow \quad u_y = \mu_k gt \quad \Rightarrow \quad \dot{y} = 0$

$$y = u_y t - \frac{1}{2}\mu_k g t^2 = \frac{u_y^2}{2\mu_k g}$$

Hence the final velocity of the ball is 0, and its displacement is 因此球的最终速度为 0，其位移为 $y = \frac{u_y^2}{2\mu_k g}$.

3. Balloon (10 points) 气球（10 分）

The work done in stretching a spring is converted to its spring energy. Likewise, the work done in stretching a surface of a membrane is converted to its surface energy, given by $E = \gamma S$, where γ is called the *surface tension* of the membrane, and S is its surface area.
拉伸弹簧所做的功被转换成弹簧的内能。同样，拉伸一个薄膜表面所做的功被转换成它的表面能 $E = \gamma S$，其中 γ 称为薄膜的*表面张力*，而 S 是其表面面积。

(a) Consider a balloon of radius R. What is the change in surface energy when the radius changes by dR? Hence derive an expression for the pressure due to surface tension. (2 points)
考虑半径为R的气球。当半径改变为dR时，表面能的变化是多少？由此推导表面张力形成的压力的表达式。（2分）

The surface energy of the balloon is $E = \gamma 8\pi R^2$ (the balloon has both inner and outer surfaces).气球的表面能是 $E = \gamma 8\pi R^2$（气球有里外两面）。
Hence $dE = \gamma 16\pi R dR$. Equating this to the work done by pressure 把这等同压强做的功

$$dW = pdV = p4\pi R^2 dR,\ \gamma 16\pi R dR = p4\pi R^2 dR \quad \Rightarrow \quad p = \frac{\gamma 16\pi R dR}{4\pi R^2 dR} = \frac{4\gamma}{R}.$$

(b) The surface tension of balloon A is γ. When it is filled with a diatomic ideal gas, its radius becomes R_0. The surface tension of balloon B is 2γ. When it is filled with the same kind of ideal diatomic gas, its radius becomes R_0. The temperature of the environment is T. The two balloons are then connected so that the gases are free to exchange between them until a steady state is reached. The final temperature is the same as that of the environment. What are the final radii of the two balloons respectively? You may neglect the atmospheric pressure in the analysis. (4 points)
气球A的表面张力为γ。当它充满了一种双原子的理想气体，其半径是R_0。气球B的表面张力为2γ。当它被相同的双原子理想气体充满时，其半径是R_0。环境的温度为T。然后两个气球被连接，使得气体可以在它们之间自由交流，直至达到稳定状态。最终温度与环境相同。问两个气球最终的半径分别是什么？在分析中你可以忽略大气压力。（4分）

Since the initial pressure in balloon B is higher, the gas will flow from balloon B to A. The radius of balloon B decreases and that of balloon A increases. Hence the pressure in balloon A and B increases and decreases respectively. The pressure difference increases, driving the system further away from equilibrium. This continues until all gases flow into balloon A. Hence $R_B = 0$.
因为气球B的初始压强较高，引致气体从气球B流向A。气球B的半径减少，气球A的半径增加。因此，气球A和B的压强分别增大和减小，使系统进一步远离平衡。这情况持续，直到所有的气体流入气球A。因此，$R_B = 0$。
To find R_A, we consider the initial number of moles of gas in balloon A:
要找出R_A，我们考虑起初时气球A中气体的摩尔数

$$n_A = \frac{p_A V_A}{RT} = \frac{1}{RT}\left(\frac{4\gamma}{R_0}\right)\left(\frac{4\pi R_0^3}{3}\right) = \frac{16\pi\gamma R_0^2}{3RT}.$$

Similarly, the initial number of moles of gas in balloon B:
同样，起初时气球B中气体的摩尔数：

$$n_B = \frac{p_B V_B}{RT} = \frac{1}{RT}\left(\frac{8\gamma}{R_0}\right)\left(\frac{4\pi R_0^3}{3}\right) = \frac{32\pi\gamma R_0^2}{3RT}.$$

Since the number of moles of gas is conserved, 因为气体的摩尔数守恒，

$$\frac{16\pi\gamma R_A^2}{3RT} = \frac{16\pi\gamma R_0^2}{3RT} + \frac{32\pi\gamma R_0^2}{3RT} \quad \Rightarrow \quad R_A = \sqrt{3}R_0.$$

(c) What are the amounts of heat gain by the gases in balloons A and B respectively during the gas exchange process in (b)? (4 points)
在(b)部的气体交流过程中，气球A和B增加的热能分别是什么？（4分）

Using the first law of thermodynamics, 应用热力学第一定律，
Heat gain = internal energy change + work done by the gas
热能增加 = 内能改变 + 气体做的功
The internal energy of an ideal gas is independent of its volume.
理想气体的内能与体积无关。
For balloon A, 对气球A来说，

Internal energy change: 内能改变： $\Delta U_A = \frac{16\pi\gamma}{3RT}\left(\sqrt{3}R_0\right)^2 c_V T - \frac{16\pi\gamma R_0^2}{3RT} c_V T = \frac{32\pi c_V \gamma R_0^2}{3R}$.

Work done by the gas is equal to the change in surface energy of the balloon:
气体做的功等于气球表面能的改变： $W_A = \gamma 8\pi\left(\sqrt{3}R_0\right)^2 - \gamma 8\pi R_0^2 = 16\pi\gamma R_0^2$.

For diatomic ideal gases, 在双原子的理想气体中， $c_V = \frac{5}{2}R$.

Hence heat gain: 所以热能增加是： $Q_A = \Delta U_A + W_A = \frac{32\pi\gamma R_0^2}{3R}\left(\frac{5}{2}R\right) + 16\pi\gamma R_0^2 = \frac{128\pi\gamma R_0^2}{3}$.

Similarly, for balloon B, 同样，对气球B来说，

$\Delta U_B = -\frac{32\pi\gamma R_0^2}{3RT} c_V T = -\frac{32\pi c_V \gamma R_0^2}{3R}$.

$W_B = -\gamma 16\pi R_0^2$.

$Q_B = \Delta U_B + W_B = -\frac{32\pi\gamma R_0^2}{3R}\left(\frac{5}{2}R\right) - 16\pi\gamma R_0^2 = -\frac{128\pi\gamma R_0^2}{3}$.

Remark: A common mistake is to assume that the pressures in both balloons are the same when the gas exchange process has reached steady state. This implies
注：一个常见的错误，是假设当气体交流过程达到稳定状态时，两气球的压强相同。这显示

$\frac{4\gamma}{R_A} = \frac{8\gamma}{R_B} \Rightarrow R_B = 2R_A$.

Since the number of moles of gas is conserved, 因为气体的摩尔数守恒，

$R_A^2 + 2(2R_A)^2 = 3R_0^2 \Rightarrow R_A = \frac{R_0}{\sqrt{3}}$ and $R_B = \frac{2R_0}{\sqrt{3}}$ and $Q_A = -\frac{128\pi\gamma R_0^2}{9}$, $Q_B = \frac{128\pi\gamma R_0^2}{9}$.

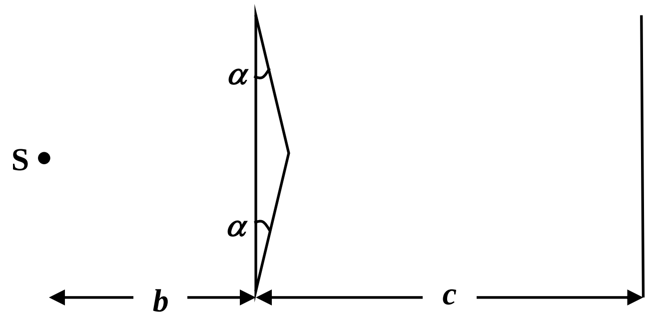
However, this equilibrium state is unstable. 可是，这平衡态是不稳定的。

4. Fresnel Biprism (10 points) 菲涅耳双棱镜（10 分）

Fresnel biprism was devised shortly after the famous Young's double slit experiment to confirm the interference phenomenon. Nowadays, it is widely used in different applications. As shown in the figure, it consists of a single light source S and a pair of wedge-shaped prisms arranged back to back. We introduce the following notations:

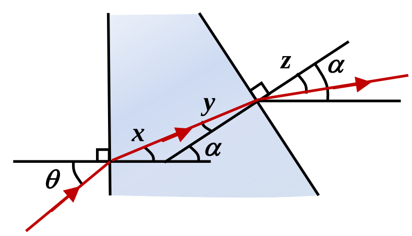
在著名的杨氏双缝实验面世后不久，便产生了菲涅耳双棱镜的设计，用以确认干涉现象。如今，它被广泛用于不同的应用。如图所示，它由一个单一的光源 S 和一对背对背的楔形棱镜组成。我们引入以下符号：

n = refractive index of the biprism 双棱镜的折射率
α = apex angle of each prism 双棱镜的顶角
b = distance between light source and biprism 光源与双棱镜的距离
c = distance between biprism and screen 双棱镜与屏幕的距离
λ = wavelength of light 光的波长



(a) Derive an expression for the angular deviation after a light beam has passed through one of the two prisms. (3 points)
试推导光束经过其中一个棱镜后偏转角的表达式。（3 分）

Consider a light beam incident on the upper prism. Let θ be the incident angle. For the angles shown in the figure,
考虑入射上棱镜的光束。设θ为入射角。对图中显示的角度来说，



$$x = \frac{\theta}{n}, \quad y = \alpha - x = \alpha - \frac{\theta}{n}, \quad z = ny = n\alpha - \theta.$$

The angle between the deflected beam and the horizontal direction 偏转光束与平行方向的角 $= \alpha - z = \theta - (n-1)\alpha$.

Hence the angular deviation of the beam is 所以光束的偏转角是$(n-1)\alpha$.

(b) Derive an expression for the separation of the fringes on the screen. (4 points)
试推导屏幕上条纹距离的表达式。（4 分）

When $\theta = (n-1)\alpha$, the beam emerges parallel to the horizontal direction. Hence the image of S in the upper prism is located at a distance $b\theta = (n-1)b\alpha$ above S.
当$\theta = (n-1)\alpha$，光束在偏转后在以水平方向出现。因此 S 在上棱镜中的影像位于 S 之上，距离为 $b\theta = (n-1)b\alpha$。
Similarly, the image of S in the lower prism is located at $(n-1)b\alpha$ below S.
同样地，S 在下棱镜中的影像位于 S 之下，距离为 $b\theta = (n-1)b\alpha$。
This is equivalent to the Young's double slit experiment with slit separation $d = 2(n-1)b\alpha$.
这布局等同杨氏双缝实验，双缝距离为 $d = 2(n-1)b\alpha$。
Condition for constructive interference: 相长干涉的条件：$d\sin\theta = m\lambda$.

Positions of the bright fringes:光条纹的位置： $y=(b+c)\tan\theta\approx(b+c)\sin\theta=(b+c)\dfrac{m\lambda}{d}$.

Hence the fringe separation:因此条纹间的距离： $\Delta y=(b+c)\dfrac{\lambda}{d}=\dfrac{(b+c)\lambda}{2(n-1)b\alpha}$.

(c) In a modern application on electron microscopes, the single light source is replaced by a parallel beam of wave incident normally to the flat surface of the biprism. Derive an expression for the separation of the fringes on the screen. (3 points)
在现代，这原理已应用到电子显微镜中。在这应用中，单个光源被替换成入射的平行波束，垂直于双棱镜的平面。试推导屏幕上条纹距离的表达式。（3 分）

When the waves are incident on the screen at an angle ϕ, it sets up a traveling wave in the transverse direction with a wavelength $\lambda/\sin\phi$.
当波以角度ϕ入射到屏幕上，它产生了一个波长为$\lambda/\sin\phi$的横向行波。
In the biprism setup, we have $\phi=(n-1)\alpha$ and two waves with the same frequency and wavelength traveling in opposite directions. Hence a standing wave is formed.
在双棱镜设置中，我们有$\phi=(n-1)\alpha$，和两个具有相同频率和波长、但沿相反方向行进的行波。因此，驻波便形成了。
Fringes in standing waves are separated by half wavelengths.
驻波中条纹的距离为波长的一半。
Hence the fringe separation is: 因此，条纹间距为: $\Delta y=\dfrac{\lambda}{2\sin\phi}\approx\dfrac{\lambda}{2(n-1)\alpha}$.

5. Ionic Crystals (11 points) 离子晶体（11 分）

An ionic crystal can be modeled by a chain of positively and negatively charged ions. The ionic separation is a. The positive ions with atomic mass M are located at the positions $x = na$ where n is even. The negative ions with atomic mass m ($m < M$) are located at the positions $x = na$ where n is odd. The ions are coupled to their neighbors by springs, which provide restoring forces to their transverse displacements. The returning force is proportional to the displacements of the ions relative to their neighbors, and the spring constant is k.
我们可以一串带正电和带负电的离子，作为离子晶体的模型。离子间的距离为 a。正离子的原子质量为 M，处于位置 $x = na$，其中 n 是偶数。负离子的原子质量为 m（$m < M$），处于在位置 $x = na$，其中 n 是奇数。相邻的离子有弹簧耦合，弹簧为离子的横向位移提供返回力。返回力正比于离子相对于相邻离子的位移，并且弹簧常数为 k。

(a) Let $u_n(t)$ be the transverse displacement of the ion at $x = na$ and time t. Derive the equations of motion for both types of ions. Show that the solution of the equation of motion can be written as
令 $u_n(t)$为处于 $x= na$ 的离子在时间 t 的横向位移。试推导两种类型离子的运动方程。表明运动方程的解可以写成

$$u_n(t)=\begin{cases}A_M\sin(qna-\omega t) & n\text{ even,}\\ A_m\sin(qna-\omega t) & n\text{ odd.}\end{cases}$$

Find the relation between q and ω. (3 points) 试找出 q 与ω的关系。（3 分）

Using Newton’s law,利用牛顿定律，

$M\ddot{u}_n = k(u_{n+1}-u_n)-k(u_n-u_{n-1}) = ku_{n-1}-2ku_n+ku_{n+1}$ for n even, n 是偶数。

$m\ddot{u}_n = k(u_{n+1}-u_n)-k(u_n-u_{n-1}) = ku_{n-1}-2ku_n+ku_{n+1}$ for n odd. n 是奇数。

For even n, n 是偶数时，

$$-M\omega^2 A_M \sin(qna-\omega t)$$
$$= kA_m \sin[q(n-1)a-\omega t]-2kA_M\sin(qna-\omega t)+kA_m\sin[q(n+1)a-\omega t]$$
$$= 2kA_m\cos qa\sin(qna-\omega t)-2kA_M\sin(qna-\omega t)$$
$$\Rightarrow\quad -M\omega^2A_M = 2kA_m\cos qa-2kA_M$$
$$\Rightarrow\quad (M\omega^2-2k)A_M+2k\cos qaA_m = 0. \qquad (1)$$

For odd n, n 是奇数时，

$$-m\omega^2 A_m \sin(qna-\omega t)$$
$$= kA_M \sin[q(n-1)a-\omega t]-2kA_m\sin(qna-\omega t)+kA_M\sin[q(n+1)a-\omega t]$$
$$= 2kA_M\cos qa\sin(qna-\omega t)-2kA_m\sin(qna-\omega t)$$
$$\Rightarrow\quad -m\omega^2A_m = 2kA_M\cos qa-2kA_m$$
$$\Rightarrow\quad (m\omega^2-2k)A_m+2k\cos qaA_M = 0. \qquad (2)$$

Eqs. (1) and (2) have non-trivial solutions if 方程（1）和（2）有非零解的条件是

$$\frac{A_M}{A_m} = -\frac{2k\cos qa}{M\omega^2-2k} = -\frac{m\omega^2-2k}{2k\cos qa}$$
$$\Rightarrow\quad (M\omega^2-2k)(m\omega^2-2k) = 4k^2\cos^2 qa$$
$$\Rightarrow\quad Mm\omega^4-2k(M+m)\omega^2+4k^2\sin^2 qa) = 0$$
$$\Rightarrow\quad \omega = \sqrt{\frac{k(M+m)\pm k\sqrt{(M+m)^2-4Mm\sin^2 qa}}{Mm}}.$$

(b) Find the solutions of ω in the limit q = 0, and the relation between A_M and A_m for each solution. (2 points)

在极限 $q = 0$，求ω的所有解，并且求在每个解中 A_M 与 A_m 间的关系。（2 分）

In the limit $q = 0$, the high frequency solution is 在极限 $q = 0$，高频解是

$$\omega^2 = \frac{k(M+m)+k|M+m|}{Mm} = \frac{2k(M+m)}{Mm} \quad\Rightarrow\quad \omega = \sqrt{\frac{2k(M+m)}{Mm}}.$$
$$\frac{A_M}{A_m} = -\frac{2k\cos qa}{M\omega^2-2k} = -\frac{2k}{2k(M+m)/m-2k} = -\frac{m}{M}.$$

The motions of the two ions are out of phase. 两种离子的运动是反相的。

The low frequency solution is 低频解是 $\omega^2 = \dfrac{1}{Mm}\left[k(M+m)-k(M+m)\sqrt{1-\dfrac{4Mm(qa)^2}{(M+m)^2}}\right]$

$$\approx \frac{k(M+m)}{Mm}\left[1-\left(1-\frac{2Mm(qa)^2}{(M+m)^2}\right)\right] = \frac{2kq^2a^2}{M+m} \quad\Rightarrow\quad \omega = \sqrt{\frac{2k}{M+m}}qa.$$
$$\frac{A_M}{A_m} = -\frac{2k\cos qa}{M\omega^2-2k} = -\frac{2k}{-2k} = 1.$$

The motions of the two ions are in phase. 两种离子的运动是同相的。

(c) In the limit q = 0, calculate the wave velocity of the low frequency mode. (1 point)
在极限 q =0，试计算低频模式的波速。（1 分）

Wave velocity: 波速： $v=\frac{\omega}{q}=a\sqrt{\frac{2k}{M+m}}$.

(d) In the limit $q = \pi/2a$, find the solutions of ω, and the relation between A_M and A_m for each solution. (2 points)
在极限 $q = \pi/2a$，求ω的所有解，并且求在每个解中 A_M 与 A_m 间的关系。（2 分）

In the limit $q = \pi/2a$, the high frequency solution is 在极限 $q = \pi/2a$，高频解是

$$\omega^2=\frac{k(M+m)+k|M-m|}{Mm}=\frac{2k}{m} \quad \Rightarrow \quad \omega=\sqrt{\frac{2k}{m}}.$$

$$\frac{A_M}{A_m}=-\frac{2k\cos qa}{M\omega^2-2k}=0.$$

Only the lighter ion moves.只有较轻的离子在运动。

The low frequency solution is 低频解是 $\omega^2=\frac{k(M+m)-k|M-m|}{Mm}=\frac{2k}{M} \quad \Rightarrow \quad \omega=\sqrt{\frac{2k}{M}}$.

$$\frac{A_M}{A_m}=-\frac{2k\cos qa}{M\omega^2-2k}.$$

Since both the denominator and numerator vanish, we have to consider higher order terms.
因为分母和分子同时消失，我们必须考虑高阶项

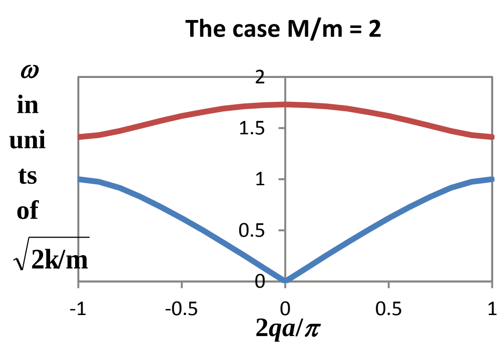
$$\omega^2=\frac{k(M+m)-k\sqrt{(M-m)^2+4Mm\cos^2 qa}}{Mm}=\frac{k}{Mm}\left[M+m-(M-m)\sqrt{1+\frac{4Mm\cos^2 qa}{(M-m)^2}}\right]$$

$$\approx\frac{k}{Mm}\left[M+m-(M-m)-\frac{2Mm\cos^2 qa}{M-m}\right]=\frac{2k}{M}\left[1-\frac{M\cos^2 qa}{M-m}\right]$$

$$\Rightarrow \quad \frac{A_M}{A_m}\approx-\frac{2k\cos qa}{2k-2kM\cos^2 qa/(M-m)-2k}=\frac{M-m}{M\cos qa}\to\infty.$$

Only the more massive ion moves. 只有较重的离子在运动。

(e) Sketch the angular frequency ω as a function of the wavenumber q from $q = -\pi/2a$ to $q = \pi/2a$. (2 points)
试绘出角频率ω作为波数 q 的函数的草图，范围从 $q = -\pi/2a$ 到 $q = \pi/2a$。（2 分）



(f) An electromagnetic wave is incident on the crystal. Which frequency mode will be excited? (1 point) 有电磁波入射到晶体。哪种频率模式会被激发？（1 分）

Since in the high frequency mode, positive and negative ions oscillate out of phase, oscillating electric dipole moments will be formed. Hence the high frequency mode will be excited. 由于在高频率的模式中，正负离子以反相振动，振动电偶极矩将形成。因此，高频模式将被激发。

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2015

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Part-2 (Total 2 Problems) 卷-2（共2 题）

(2:00 pm – 5:00 pm, 25 February, 2015)

1. Exoplanet Microlensing (25 points) 系外行星的微透镜效应 （25 分）

Reference: 参考文献：B. S. Gaudi, Exoplanet Microlensing, EXOPLANETS, edited by S. Seager, Space Science Series of the University of Arizona Press (Tucson, AZ, 2010).

With the discovery of planets orbiting around stars in recent years, the observation of exoplanets from astronomical distances became a challenge to scientists. Gravitational microlensing is one of the detection methods. It makes use of Einstein's discovery in general relativity that when a light ray passing near a spherically symmetric body of mass M, its direction will be deflected towards the body by a small angle given by

随着近年发现不少绕着恒星运行的行星，怎样观察相隔天文距离的系外行星便成为科学家的挑战。引力微透镜是其中一种检测方法。它利用爱因斯坦在广义相对论里发现的原理，就是当光线经过一个质量为 M 的球对称物体时，方向会朝向物体偏转，偏转的小角度为

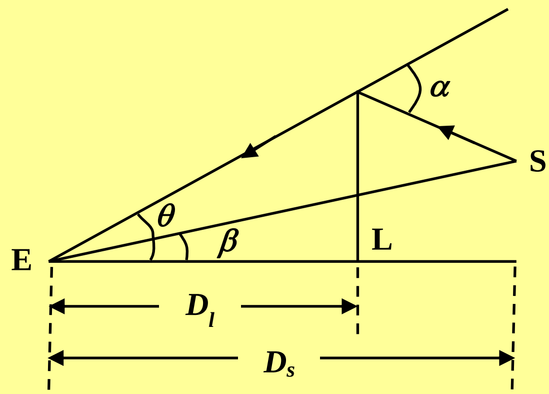
$$\alpha = \frac{4GM}{rc^2},$$

where G is the gravitational constant, c is the speed of light, and r is the distance of closest approach of the light ray to the body. In this problem, we will study the principle of detecting exoplanet by microlensing.

其中 G 是万有引力常数，c 是光速，r 是光线和物体的最短距离。在这个问题中，我们将研究通过微透镜效应探测系外行星的原理。

(a) Consider a distant star S located at a distance D_s from Earth E, acting as the light source. Another star L of mass M and located at distance D_l ($< D_s$) from Earth acts as the lens. The lines EL and ES make a small angle β between them. Construct the following sketch in the answer book: (a1) the line EL, (a2) the line ES, (a3) the distances D_l and D_s, (a4) the angle β (remark: although this angle is small in practice, it should not be drawn too small for the purpose of clarity), (a5) a line perpendicular to EL through L, acting as the gravitational lens, (a6) the light ray from S to E, assuming that each of the segments between S and the lens and that between the lens and E are straight lines, (a7) the deflection angle α, (a8) the apparent angle θ of the star S as observed on Earth (relative to line EL). (3 marks)

考虑一个遥远的恒星 S，离地球 E 的距离为 D_s，作为光源。另一颗恒星 L，质量为 M，离地球的距离为 D_l（$< D_s$），作为透镜。线 EL 和 ES 间的小角度为β。试在答题簿上绘出以下草图：（a1）线 EL，（a2）线 ES，（a3）距离 D_l 和 D_s，（a4）角度β（注：虽然该角度实际上很小，但为清楚起见，不应把它绘得太小），（a5）一条垂直于 EL 而通过 L 的线，作为引力透镜，（a6）从 S 到 E 的光线，假定 S 和透镜之间的线段及透镜和 E 之间的线段各可视作直线，（a7）偏转角α，（a8）从地球观察星 S 的视角θ（相对于线 EL）。（3 分）



(b) Derive an equation for the angle θ in terms of the parameters D_s, D_l, G, M, c and β, assuming that all angles are small. (3 points)
试推导θ的方程式，以参数 D_s, D_l, G, M, c 和 β表达，可假设所有角度都很小。（3 分）

Constructing the vertical line XY through S, we have
作一垂直线 XY 通过 S，我们得

XY = XS + SY,

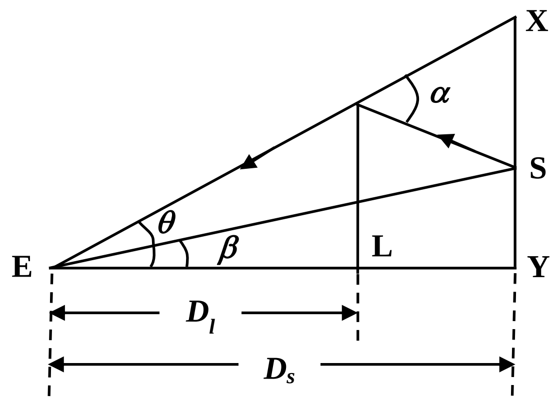
$$D_s\theta = D_s\beta + (D_s - D_l)\alpha$$

Substituting $\alpha = \dfrac{4GM}{rc^2}$ and $r = D_l\theta$,

代入$\alpha = \dfrac{4GM}{rc^2}$ 和 $r = D_l\theta$，

$$\theta = \beta + \left(\frac{D_s - D_l}{D_s D_l}\right)\frac{4GM}{c^2\theta}.$$

$$\beta = \theta - \frac{4GM}{c^2\theta}\left(\frac{D_s - D_l}{D_s D_l}\right).$$



(c) Consider the case that the lens is exactly aligned with the source (β = 0). The image of S appears to be a ring known as an Einstein ring. Derive the expression for the angular radius θ_E of the Einstein ring. (2 points)
考虑透镜与光源对準的情况（β = 0）。 S 的影象呈环形，称为爱因斯坦环。试推导爱因斯坦环的角半径θ_E的表达式。 （2 分）

$$0 = \theta - \frac{4GM}{c^2\theta}\left(\frac{D_s - D_l}{D_s D_l}\right) \Rightarrow \quad \theta_E = \sqrt{\frac{4GM}{c^2}\left(\frac{D_s - D_l}{D_s D_l}\right)}.$$

(d) Calculate the Einstein radius for the following typical values:
试以下列的典型值，计算爱因斯坦半径：
M = 0.3 solar mass, D_s = 10 kpc. D_l = 3 kpc.
Give your answer in milli-arc-seconds. You may use the following constants:
请以 milli-arc-seconds 表达你的答案。您可以使用以下参量：
$G = 6.67 \times 10^{-11}$ Nm^{2}kg^{-2}, 1 solar mass = 1.99×10^{30} kg, $c = 3 \times 10^8$ ms^{-1}, 1 kpc = 3.09×10^{19} m, 1 radian = 206265 arc seconds. (1 point) （1 分）

$$\theta_E = \sqrt{\frac{4(6.67\times10^{-11})(0.3)(1.99\times10^{30})}{(3\times10^8)^2}\left[\frac{10-3}{(10)(3)(3.09\times10^{19})}\right]} = 3.66\times10^{-9}\text{ rad}$$
= 0.754 milli-arc-seconds.

(e) When the lens and the source are not exactly aligned, there will be two images of S. It is convenient to express the angles β and θ in multiples of the Einstein radius θ_E. Hence we define $u \equiv \frac{\beta}{\theta_E}$ and $y \equiv \frac{\theta}{\theta_E}$. Derive the expressions for the angular positions y of the two images in terms of u. (2 points)

当透镜和光源不完全对齐时，S 将有两个影像。为方便起见，我们以爱因斯坦半径θ_E的倍数表达角β 和θ。因此我们定义$u \equiv \frac{\beta}{\theta_E}$ 和 $y \equiv \frac{\theta}{\theta_E}$。试推导两个影像的角位置 y，以 u 表示。（2 分）

$$\beta = \theta - \frac{4GM}{c^2\theta}\left(\frac{D_s - D_l}{D_s D_l}\right) \Rightarrow \beta = \theta - \frac{\theta_E^2}{\theta} \Rightarrow u = y - \frac{1}{y} \Rightarrow y^2 - uy - 1 = 0 \Rightarrow$$
$$y = \frac{u \pm \sqrt{u^2+4}}{2}.$$

(f) To study the effect of the finite size of star S, we introduce Cartesian coordinates on the plane normal to ES and through S, with the y axis lying in the plane containing E, L and S. Consider the corners $(0, u+\delta)$ and (δ, u) of a square on the surface of star S ($\delta << u$). Calculate the coordinates of the two corners of the two images when viewed from Earth. (2 points)

为研究星 S 有限大小的影响，我们在垂直于 ES 和通过 S 的平面上，引入一平面直角坐标，其中 y 轴位于包含 E，L 和 S 的平面中。考虑星 S 表面上一个正方形的角$(0, u+\delta)$ 和 (δ, u) ($\delta << u$)。试计算从地球观察时，这两个影像的两个角的坐标。（2 分）

The image of (δ, u) is $\left(\frac{y}{u}\delta, y\right)$. (δ, u)的影像是$\left(\frac{y}{u}\delta, y\right)$。

$$y = \frac{u \pm \sqrt{u^2+4}}{2} \Rightarrow \delta y = \frac{1}{2}\left(1 \pm \frac{u}{\sqrt{u^2+4}}\right)\delta u = \frac{\sqrt{u^2+4} \pm u}{2\sqrt{u^2+4}}\delta u = \pm\frac{y\delta u}{\sqrt{u^2+4}}.$$

Hence the image of $(0, u+\delta)$ is 所以$(0, u+\delta)$的影像是$\left(0, \frac{u \pm \sqrt{u^2+4}}{2}\left[1 \pm \frac{\delta}{\sqrt{u^2+4}}\right]\right)$.

In summary, for the image at $y = \frac{u+\sqrt{u^2+4}}{2}$, the images of the corners $(0, u+\delta)$ and (δ, u) are respectively

总结一下，对于在 $y = \frac{u+\sqrt{u^2+4}}{2}$ 的影像，角$(0, u+\delta)$和(δ, u)的影像分别是

$$\left(0, \frac{u+\sqrt{u^2+4}}{2}\left[1+\frac{\delta}{\sqrt{u^2+4}}\right]\right) \text{ and } \left(\frac{u+\sqrt{u^2+4}}{2u}\delta, \frac{u+\sqrt{u^2+4}}{2}\right).$$

For the image at $y=\dfrac{u-\sqrt{u^2+4}}{2}$, the images of the corners (0, u + δ) and (δ, u) are respectively 对于在 $y=\dfrac{u-\sqrt{u^2+4}}{2}$,的影像， 角(0, u + δ)和(δ, u)的影像分别是 $\left(0,\dfrac{u-\sqrt{u^2+4}}{2}\left[1-\dfrac{\delta}{\sqrt{u^2+4}}\right]\right)$和$\left(\dfrac{u-\sqrt{u^2+4}}{2u}\delta,\dfrac{u-\sqrt{u^2+4}}{2}\right)$.

(g) Calculate the areal magnifications of the two images of star S in terms of u. Following the practice in astronomical observations, give your answer in absolute values. (2 points)
试计算星 S 的两个影像的面积放大率，请以 u 表达。按照天文观测的习惯，请以绝对值为答案。（2 分）

The length scales are magnified by $\dfrac{u\pm\sqrt{u^2+4}}{2u}$ and $\dfrac{u\pm\sqrt{u^2+4}}{\pm 2\sqrt{u^2+4}}$ in the x and y directions respectively. Hence the areal magnifications are:
长度分别在 x 和 y 方向放大了 by $\dfrac{u\pm\sqrt{u^2+4}}{2u}$ 和 $\dfrac{u\pm\sqrt{u^2+4}}{\pm 2\sqrt{u^2+4}}$。因此，面积放大率为：

$$\left|\left(\frac{u\pm\sqrt{u^2+4}}{2u}\right)\left(\frac{u\pm\sqrt{u^2+4}}{\pm 2\sqrt{u^2+4}}\right)\right|=\frac{\left(u\pm\sqrt{u^2+4}\right)^2}{4|u|\sqrt{u^2+4}}=\frac{1}{2}\left(\frac{u^2+2}{|u|\sqrt{u^2+4}}\pm 1\right).$$

(h) In practice, since the images cannot be resolved, astronomers measure the sum of the magnifications of the two images. Derive the expression for the total magnification. Describe its behavior when star S is remote (u approaches infinity) and when S approaches perfect alignment with L and E (u approaches 0). (3 points)
实际上，由于影象不易分辨，天文学家只测量两个影像的放大率的总和。试推导总放大率的表达式。试描述星 S 在远处时（u 趋近无穷大），及星 S 趋近对準 L 与 E 时(u 趋近 0)，总放大率的行为。（3 分）

Total magnification: 总放大率：

$$A=\frac{1}{2}\left(\frac{u^2+2}{|u|\sqrt{u^2+4}}+1\right)+\frac{1}{2}\left(\frac{u^2+2}{|u|\sqrt{u^2+4}}-1\right)=\frac{u^2+2}{|u|\sqrt{u^2+4}}.$$

When u approaches infinity, 当 u 趋近无穷大： $A\to 1$.

When u approaches 0,当 u 趋近 0： $A\to\dfrac{1}{|u|}$.

(i) A planet P of star L has mass m and is located in the plane of E, L and S at the same distance D_l from Earth. EP and EL makes an angle θ_p. Derive an equation for the angle θ taking into account the gravitational lensing effects of both star L and planet P. Expressions in the equation should be written in terms of the parameters D_s, D_l, G, M, c, β, m and θ_p, assuming that all angles are small. Simplify the equation by introducing the mass ratio $q\equiv\dfrac{m}{M}$ and the rescaled positions $u_p\equiv\dfrac{\theta_p}{\theta_E}$, $u\equiv\dfrac{\beta}{\theta_E}$, $y\equiv\dfrac{\theta}{\theta_E}$. (3 points)

星 L 旁有一行星 P 位于 E、L 和 S 的平面上，其质量为 m，与地球距离跟星 L 同为 D_l，EP 与 EL 间角度为θ_p。考虑到星 L 和行星 P 两者的引力透镜作用，试推导角θ的方程式，式中的表达式应以 D_s、D_l、G、 M、 c、β、 m 和θ_p表达。可假设所有角度都很小。引入质量比$q\equiv\dfrac{m}{M}$和重整位置$u_p\equiv\dfrac{\theta_p}{\theta_E}$，$u\equiv\dfrac{\beta}{\theta_E}$，$y\equiv\dfrac{\theta}{\theta_E}$，以简化方程式。（3 分）

As shown in the figure,如图所示，

XZ = XY + YS + SZ.

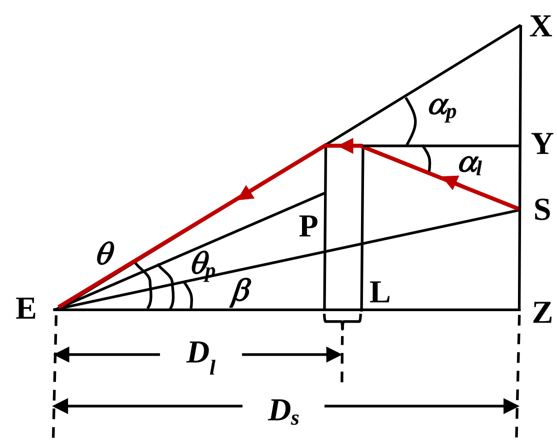
$$D_s\theta=(D_s-D_l)\alpha_p+(D_s-D_l)\alpha_l+D_s\beta.$$

Substituting 代入 $\alpha_l=\dfrac{4GM}{r_lc^2}$,

$$\alpha_p=\frac{4Gm}{r_pc^2},\ r_l=D_l\theta,\ r_p=D_l(\theta-\theta_p),$$

$$\theta=\beta+\left(\frac{D_s-D_l}{D_sD_l}\right)\frac{4G}{c^2}\left(\frac{M}{\theta}+\frac{m}{\theta-\theta_p}\right).$$

$$\beta=\theta-\frac{4G}{c^2}\left(\frac{D_s-D_l}{D_sD_l}\right)\left(\frac{M}{\theta}+\frac{m}{\theta-\theta_p}\right).$$



The equation can be simplified to 方程式可简化为

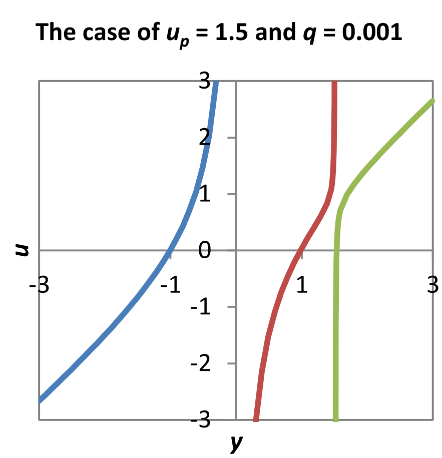
$$u=y-\frac{1}{y}-\frac{q}{y-u_p}.$$

(j) In typical exoplanet detections, there is a motion of star S relative to star L. As star S approaches the closest distance to star L and moves away, u decreases with time to a minimum value u_0 and increases again. By plotting the magnification of the image of star S versus time, one observes a smooth and relatively broad peak in the magnification curve due to gravitational lensing by star L. In addition, one can observe a side peak due to the presence of the planet. For $q \ll 1$, estimate the width of this side peak, that is, the range of u in which the side peak is significant. (1 point)

在典型的系外行星检测中，星 S 对于星 L 有相对运动。星 S 趋近星 L 至最短距离，然后离开，过程中 u 随时间降到最小值 u_0 然后再增加。把星 S 影像的放大率与时间的关系绘成图表，放大率曲线上可以看到一个平滑和较宽的主峰，是由星 L 的引力透镜作用形成的。另外，我们可以观察到一个侧峰，是由行星形成的。对于 $q \ll 1$，试估计这个侧峰的宽度，也就是可以显著看到侧峰的 u 数值范围。（1 分）

Plotting u as a function of y, we see in the figure shown that there are three solutions for each given u. The side peak is significant when the light rays fall within the Einstein radius of the planet. Following part (c), the Einstein radius is proportional to $\sqrt{M}$. Hence the width of the side peak is of the order $\sqrt{q}$.

把 u 作为 y 的函数绘成图表，我们看到对每个给定的 u 有三个解。当光线落在行星的爱因斯坦



半径范围内，侧峰便变为显著。从(c)部得知，爱因斯坦半径与$\sqrt{M}$成正比。因此，侧峰的宽度的量级为$\sqrt{q}$。

(k) For $q << 1$, consider the situation that light rays pass very near to planet P, so that the gravitational lensing by star L becomes relatively insignificant. Calculate the position of star S where the total magnification of its image diverges, and the behavior of the total magnification in the neighborhood of this location. (3 points)
当 $q << 1$ 时，考虑光线非常靠近行星 P 的情况，在这情况下星 L 的引力透镜作用相对很弱。试计算当星 S 图像的总放大率发散时星 S 的位置，和这位置附近总放大率的行为。（3 分）

When light rays pass very near to planet P, $y \approx u_p$. The equation for y becomes
当光线非常靠近行星 P 时，$y \approx u_p$。y 的方程式变成

$$u \approx y - \frac{q}{y - u_p}.$$

Let $y' = y - u_p$ and $u' = u - u_p$. Then the equation becomes $u' \approx y' - \dfrac{q}{y'}$.

设 $y' = y - u_p$ 和 $u' = u - u_p$。则方程式变成 $u' \approx y' - \dfrac{q}{y'}$.

The solutions are 方程式的解是 $y' \approx \dfrac{u' \pm \sqrt{u'^2 + 4q}}{2}$.

When $u' = 0$, $y' \approx \pm\sqrt{q}$, confirming that the Einstein radius is $\sqrt{q}$.
$u' = 0$ 时，$y' \approx \pm\sqrt{q}$，确认爱因斯坦半径为 $\sqrt{q}$.

Following parts (f) and (g), the areal magnifications are 跟随(f)和(g)部，面积放大率为

$$\left|\frac{y'}{u'}\left(\frac{dy'}{du'}\right)\right| \approx \left|\frac{u' \pm \sqrt{u'^2+4q}}{2u'}\frac{1}{2}\left(1 \pm \frac{u'}{\sqrt{u'^2+4q}}\right)\right| = \frac{\left(u' \pm \sqrt{u'^2+4q}\right)^2}{4|u'|\sqrt{u'^2+4q}} = \frac{1}{2}\left(\frac{u'^2+2q}{|u'|\sqrt{u'^2+4q}} \pm 1\right)$$

Total magnification: 总放大率：

$$A = \frac{1}{2}\left(\frac{u'^2+2q}{|u'|\sqrt{u'^2+4q}} + 1\right) + \frac{1}{2}\left(\frac{u'^2+2q}{|u'|\sqrt{u'^2+4q}} - 1\right) = \frac{u'^2+2q}{|u'|\sqrt{u'^2+4q}}.$$

Hence the total magnification diverges when star S is located at $u' = 0$, or $u = u_p$.
因此当 $u' = 0$ 或 $u = u_p$ 时，总放大率发散。

When u' approaches 0, 当 u' 趋近 0： $A \to \dfrac{\sqrt{q}}{|u'|} = \dfrac{\sqrt{q}}{|u - u_p|}$.

2. Cosmic Gravitational Waves (25 points) 宇宙引力波（25 分）

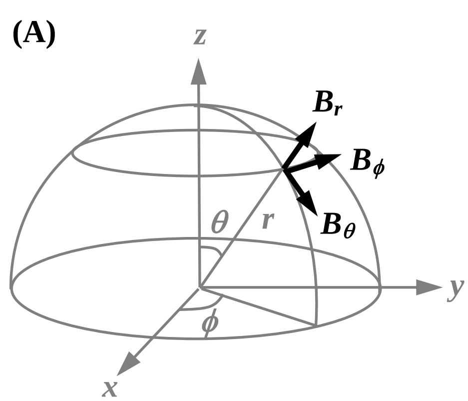
In March 2014, scientists operating gravitational wave detectors in the South Pole claimed that they found evidences of gravitational waves originated from the early universe in the cosmic microwave background radiation. While the evidence is still being debated, it is interesting to understand how gravitational waves interact with electromagnetic (EM) waves. To approach this issue, we start by considering how molecules scatter EM waves.

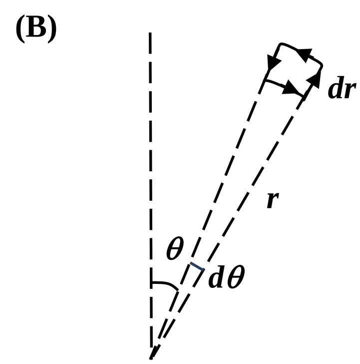
2014 年 3 月，操作南极引力波探测器的科学家，声称在宇宙微波背景辐射中，发现来自早期宇宙的引力波的证据。虽然证据还存在争议，但了解引力波如何作用于电磁（EM）波是一个有趣的课题。为了处理这个问题，我们首先考虑分子是如何散射电磁波。

(a) An oscillating electric dipole consists of charges oscillating at an angular frequency ω. Specifically, the charges are $Q(t) = \pm Q_0\cos\omega t$, located at $(x, y, z) = (0, 0, \pm s)$ respectively. What is the current between them? (1 point)
一个振动的电偶极子，包含以角频率ω振动的电荷。具体来说，电荷分别为 $Q(t) = \pm Q_0\cos\omega t$, 位于$(x, y, z) = (0, 0, \pm s)$。它们之间的电流是什么？（1 分）

$$I(t) = \frac{d}{dt}\left(Q_0\cos\omega t\right) = -\omega Q_0\sin\omega t.$$

(b) In spherical coordinates, we denote the components of the magnetic field as B_r, B_θ and B_ϕ, as shown in figure (A). Calculate $B_\phi(r, \theta, t)$ according to Biot-Savart's law at time t and distance r from the origin making an angle θ with the z axis. Note that due to the finite speed of light c, the magnetic field at a distant location is due to the time-changing current at an earlier instant. Hence the *retarded* magnetic field takes the form $\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r})\cos(\omega t - kr + \psi)$, where $k \equiv \frac{\omega}{c}$ is the *wavenumber*, and ψ is the phase shift. Express your answer in terms of the magnitude of the dipole moment $p \equiv 2Qs$ in the limit s approaches 0. Below, your answer to this part will be denoted as $B_{BS}(r, \theta, t)$. (3 points)
在球坐标中，我们以 B_r, B_θ 和 B_ϕ,表示磁场的分量，如图（A）所示。根据毕奥 - 萨伐尔定律，试计算磁场 $B_\phi(r, \theta, t)$，其中 r 为位置与原点的距离，θ为位置与 z 轴形成的角，t 为时间。注意，由于光以有限速率 c 传播，在远处的磁场是源于某一较早时刻的电流（电流随时间变化）。因此，*延迟*磁场的形式为$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r})\cos(\omega t - kr + \psi)$，其中$k \equiv \frac{\omega}{c}$是*波数*，而$\psi$是相移。答案请以偶极矩 $p \equiv 2Qs$ 表达（取 s 趋于 0 的极限）。下面，你在这部的答案将被表示为 $B_{BS}(r, \theta, t)$。（3 分）





Using Biot-Savart's law,利用根据毕奥 - 萨伐尔定律，

$$\mathbf{B}(r,\theta,t) = \frac{\mu_0}{4\pi}\int\frac{I(t - r/c)d\vec{l} \times \hat{r}}{r^2}.$$

$$B_\phi(r,\theta,t)=\left(\frac{\mu_0 I(t-r/c)\sin\theta}{4\pi r^2}\right)2s=-\frac{\mu_0\omega Q_0\sin(\omega t-kr)2s\sin\theta}{4\pi r^2}=-\frac{\mu_0\omega p_0\sin(\omega t-kr)\sin\theta}{4\pi r^2}$$

where $p_0 = 2Q_0s$.

(c) However, Biot-Savart's law is only applicable to steady state currents. It is incomplete even after including the retarded nature of the oscillating current. By considering the wave nature of the magnetic field, the complete expression of the magnetic field is given by
但是，毕奥 - 萨伐尔定律只适用于稳态电流。甚至考虑了振动电流的滞后性质后，它还是不完整的。通过考虑磁场的波动性，完整的磁场表达式是

$$\mathbf{B}(r,\theta,t)=\mathbf{B}_{\mathrm{BS}}(r,\theta,t)+\mathbf{B}_{\mathrm{wave}}(r,\theta,t),$$

where 其中

$$\mathbf{B}_{\mathrm{wave}}(r,\theta,t)=\frac{\mu_0}{4\pi c}\int\left[\frac{d}{dt}I\left(t-\frac{r}{c}\right)\right]\frac{d\bar{l}\times\hat{r}}{r}.$$

Derive an expression for the B_ϕ component of $\mathbf{B}_{\mathrm{wave}}$ at (r, θ, t). (3 points)
试推导 $\mathbf{B}_{\mathrm{wave}}$ 在 (r, θ, t)的 B_ϕ 分量的表达式。（3 分）

$$B_{\mathrm{wave}}(r,\theta,t)=\frac{\mu_0}{4\pi c}\int\frac{d}{dt}\left(-\omega Q_0\sin(\omega t-kr)\right)\frac{\sin\theta}{r}dl=-\frac{\mu_0}{4\pi c}\left(\omega^2 Q_0\cos(\omega t-kr)\frac{\sin\theta}{r}\right)2s$$

$$=-\frac{\mu_0\omega^2 p_0}{4\pi cr}\cos(\omega t-kr)\sin\theta.$$

(d) Compare the amplitudes of B_{BS} and B_{wave} at large distance r. Derive the condition of r such that B_{BS} becomes negligible when compared with B_{wave}. (2 points)
比较 B_{BS} 和 B_{wave} 在距离 r 很大时的幅度。试推导 B_{BS} 相比 B_{wave} 变得微不足道时，关于 r 的条件。（2 分）

$$\frac{|B_{\mathrm{BS}}(r,\theta,t)|}{|B_{\mathrm{wave}}(r,\theta,t)|}=\frac{\mu_0\omega p_0\sin\theta/4\pi r^2}{\mu_0\omega^2 p_0\sin\theta/4\pi cr}=\frac{c}{\omega r}<<1$$

$$\Rightarrow\quad r>>\frac{c}{\omega}=\frac{\lambda}{2\pi}.$$

(e) At large distance r, the electric field at (r, θ, t) is mainly due to the electromagnetic induction by the magnetic field B_{wave}. By considering the electromotive force along the circuit shown in figure (B), derive the relation between $\frac{\partial E_\theta}{\partial r}$ and $\frac{\partial B_\phi}{\partial t}$. Here, $\frac{\partial E_\theta}{\partial r}$ is known as the partial derivative of E_θ with respect to r, meaning that other variables such as θ and t are considered fixed. Similarly, $\frac{\partial B_\phi}{\partial t}$ is the partial derivative of B_ϕ with respect to t, with other variables such as r and θ being fixed. You may assume that only the E_θ component of the electric field is significant at large distance r. (3 points)
在距离 r 很大时，在(r, θ, t)的电场主要是源于 B_{wave} 的电磁感应。通过考虑沿著图（B）中闭路的电动势，试推导 $\frac{\partial E_\theta}{\partial r}$ 与 $\frac{\partial B_\phi}{\partial t}$ 之间的关系。这里，$\frac{\partial E_\theta}{\partial r}$ 被称为 E_θ相对于 r 的偏

导数，意味着其他变量如θ和 t 被假定为固定的。同样地，$\frac{\partial B_\phi}{\partial t}$ 是 B_ϕ相对于 t 的偏导数，当中假定其他变量如 r 和θ 为固定的。你可以假设在距离 r 很大时，电场仅有 E_θ分量是显著的。（3 分）

Consider the electromotive force along the circuit. Total electromotive force:
考虑沿著闭路的电动势。总电动势：

$$\text{emf} = -E_\theta(r+dr)(r+dr)d\theta + E_\theta(r)rd\theta.$$

Magnetic flux enclosed by the circuit: 闭路的磁通量：$\Phi = (-B_\phi)(rd\theta dr)$.

Using Faraday's law, 利用法拉第定律， $\text{emf} = -\frac{d\Phi}{dt}$.

$$-E_\theta(r+dr)(r+dr)d\theta + E_\theta(r)rd\theta = \frac{\partial B_\phi}{\partial t} rd\theta dr.$$

$$-E_\theta(r+dr) - \frac{E_\theta(r)}{r}dr + E_\theta(r) = \frac{\partial B_\phi}{\partial t}dr.$$

In the limit dr approaches 0, 在 dr 趋近 0 时， $\frac{\partial E_\theta}{\partial r} + \frac{E_\theta}{r} = -\frac{\partial B_\phi}{\partial t}$.

(f) At large distance r, the electric field is given by $E_\theta(r,\theta,t) = \frac{A(\theta)}{r}\cos(\omega t - kr)$. Find $A(\theta)$. (2 points)

在距离 r 很大时，电场为$E_\theta(r,\theta,t) = \frac{A(\theta)}{r}\cos(\omega t - kr)$。试找出 $A(\theta)$。（2 分）

$$\frac{\partial B_\phi}{\partial t} = \frac{\mu_0\omega^3 p_0}{4\pi cr}\sin(\omega t - kr)\sin\theta.$$

$$\frac{\partial E_\theta}{\partial r} = -\frac{A(\theta)}{r^2}\cos(\omega t - kr) + \frac{kA(\theta)}{r}\sin(\omega t - kr) \approx \frac{kA(\theta)}{r}\sin(\omega t - kr).$$

$$\frac{E_\theta}{r} = \frac{A(\theta)}{r^2}\cos(\omega t - kr) << \frac{\partial E_\theta}{\partial r} \quad \Rightarrow \quad A(\theta) = -\frac{\mu_0\omega^2 p_0}{4\pi}\sin\theta.$$

(g) The magnitude and direction of the power per unit area of the EM wave are given by the Poynting vector. Calculate the time-averaged power per unit area at large distance r. This will be denoted as the radiation intensity $I(r)$. (3 points)

电磁波每单位面积传播功率的大小和方向，是由 Poynting 矢量给定的。试计算在距离 r 很大时，每单位面积按时间平均的传播功率。这将被表示为辐射强度 $I(r)$。（3 分）

$$S(r) = \frac{1}{\mu_0}E_\theta B_\phi = \frac{1}{\mu_0}\left[-\frac{\mu_0\omega^2 p_0}{4\pi r}\cos(\omega t - kr)\sin\theta\right]\left[-\frac{\mu_0\omega^2 p_0}{4\pi cr}\cos(\omega t - kr)\sin\theta\right]$$

$$= \frac{\mu_0\omega^4 p_0^2}{16\pi^2 r^2 c}\cos^2(\omega t - kr)\sin^2\theta.$$

$$I(r) = \langle S(r)\rangle = \frac{\mu_0\omega^4 p_0^2}{16\pi^2 r^2 c}\langle\cos^2(\omega t - kr)\rangle\sin^2\theta = \frac{\mu_0\omega^4 p_0^2}{32\pi^2 r^2 c}\sin^2\theta.$$

(h) When an EM wave is incident on a molecule, its electric field **E** will drive the molecule into an oscillating dipole moment given by **p** = α**E**, where α is the polarizability of the molecule.

In turn, the oscillating dipole will radiate power. This is called a scattering process. Consider an EM wave incident from the x direction, given by $\mathbf{E}_i = \mathbf{E}_{x0}\cos(\omega t - kx)$. If $\mathbf{E}_{x0}$ is polarized at an angle θ_x with the z axis, calculate:
当电磁波射向一分子时，其电场 $\mathbf{E}$ 会使该分子产生振动偶极矩 $\mathbf{p} = \alpha\mathbf{E}$，其中$\alpha$是该分子的极化度。随之振动偶极子会辐射功率。这就是所谓的散射过程。考虑电磁波从 x 方向入射，由 $\mathbf{E}_i = \mathbf{E}_{x0}\cos(\omega t - kx)$给出。若 $\mathbf{E}_{x0}$ 的偏振方向与 z 轴成角度θ_x，试计算：
(h1) the intensity $I_x(r)$ of the radiation scattered to the z direction,
散射至 z 方向的辐射强度 $I_x(r)$，
(h2) the electric field polarization of the scattered wave along that direction,
沿该方向的散射波的电场偏振方向，
(h3) the intensity $\langle I_x(r)\rangle$ of the radiation scattered to the z direction for an unpolarized incident beam (that is, the polarization angle θ_x has a uniform distribution). (3 points)
非偏振入射光束（即偏振角θ_x均匀分布）散射至 z 方向的辐射强度$\langle I_x(r)\rangle$。（3 分）

(h1) If $\mathbf{E}_{x0}$ is polarized at an angle θ_x with the z axis, then the dipole moment lies in the yz plane making an angle θ_x with the z axis. Its intensity is
若 $\mathbf{E}_{x0}$ 的偏振与 z 轴成角度θ_x，则偶极矩位于 yz 平面与 z 轴成角度θ_x。辐射强度为

$$I_x(r) = \frac{\mu_0\omega^4 p_0^2}{32\pi^2 r^2 c}\sin^2\theta_x = \frac{\mu_0\omega^4\alpha^2}{32\pi^2 r^2 c}E_{x0}^2\sin^2\theta_x .$$

(h2) The electric field of the scattered waves becomes polarized in the y direction.
散射波的电场偏振方向是 y 方向。

(h3) $$\langle I_x(r)\rangle = \frac{\mu_0\omega^4\alpha^2}{32\pi^2 r^2 c}E_{x0}^2\langle\sin^2\theta_x\rangle = \frac{\mu_0\omega^4\alpha^2}{64\pi^2 r^2 c}E_{x0}^2 .$$

(i) Next, consider an EM wave incident from the y direction, given by $\mathbf{E}_i = \mathbf{E}_{y0}\cos(\omega t - ky)$. If $\mathbf{E}_{y0}$ is polarized at an angle θ_y with the z axis, calculate:
接下来，考虑电磁波从 y 方向入射，由 $\mathbf{E}_i = \mathbf{E}_{y0}\cos(\omega t - ky)$给出。若 $\mathbf{E}_{y0}$ 的偏振与 z 轴成角度θ_y，试计算：
(i1) the electric field polarization of the scattered wave along the z direction,
沿 z 方向的散射波的电场偏振方向，
(i2) the intensity $\langle I_y(r)\rangle$ of the radiation scattered to the z direction for an unpolarized incident beam (that is, the polarization angle θ_y has a uniform distribution). (2 points)
非偏振入射光束（即偏振角θ_y均匀分布）散射至 z 方向的辐射强度$\langle I_y(r)\rangle$。（2 分）

(i1) The electric field of the scattered waves becomes polarized in the x direction.
散射波的电场偏振方向是 x 方向。
(i2) If $\mathbf{E}_{y0}$ is polarized at an angle θ_y with the z axis, then the dipole moment lies in the xz plane making an angle θ_y. Its intensity is
若 $\mathbf{E}_{y0}$ 的偏振与 z 轴成角度θ_y，则偶极矩位于 xz 平面与 z 轴成角度θ_y。辐射强度为

$$I_y(r) = \frac{\mu_0\omega^4 p_0^2}{32\pi^2 r^2 c}\sin^2\theta_y = \frac{\mu_0\omega^4\alpha^2}{32\pi^2 r^2 c}E_{y0}^2\sin^2\theta_y .$$

$$\langle I_y(r)\rangle = \frac{\mu_0\omega^4\alpha^2}{32\pi^2 r^2 c}E_{y0}^2\langle\sin^2\theta_y\rangle = \frac{\mu_0\omega^4\alpha^2}{64\pi^2 r^2 c}E_{y0}^2 .$$

(j) During the rapid expansion of the early universe, gravitational waves are formed. They consist of *quadrupolar* temperature oscillations, meaning that the directions of the maxima and minima of the oscillations are separated by an angle of $\pi/2$. Hence to analyze their effects on EM waves, we consider two incoherent incident beams of EM waves of the same frequency $\omega/2\pi$, one from the x direction and the other from the y direction. The amplitudes of their electric fields are E_{x0} and E_{y0} respectively. Suppose the EM radiations in the x and y directions correspond to temperatures $T + \Delta T$ and T respectively ($\Delta T << T$ and is positive). What is the ratio $\dfrac{\langle I_x(r)\rangle}{\langle I_y(r)\rangle}$? (1 point)

早期宇宙的迅速膨胀，形成引力波。它引起温度的振动，呈*四偶极*分布。这意味着振动的最大值和最小值的方向以$\pi/2$ 角度分开。因此，要分析它们对电磁波的影响，我们考虑两束频率同为$\omega/2\pi$的非相干入射光，一束来自 x 方向，另一束则来自 y 方向，其电场的幅度分别是 E_{x0} 和 E_{y0}。假设在 x 和 y 方向的电磁辐射分别对应于温度 $T + \Delta T$ 和 T（$\Delta T << T$，且是正的）。比例 $\dfrac{\langle I_x(r)\rangle}{\langle I_y(r)\rangle}$ 是什么？（1 分）

$$\frac{\langle I_x(r)\rangle}{\langle I_y(r)\rangle} = \frac{T+\Delta T}{T}.$$

(k) The degree of polarization of the scattered radiation is given by 下式是散射辐射的偏振度

$$\Pi = \frac{\left|\langle I_x(r)\rangle - \langle I_y(r)\rangle\right|}{\langle I_x(r)\rangle + \langle I_y(r)\rangle}.$$

Calculate Π. What is the direction of the electric field polarization in the scattered wave? (2 points)

试计算Π。散射辐射中电场的偏振方向是什么？（2 分）

$$\Pi = \frac{\left|\langle I_x(r)\rangle - \langle I_y(r)\rangle\right|}{\langle I_x(r)\rangle + \langle I_y(r)\rangle} = . \ \Pi = \frac{\left|\langle I_x(r)\rangle - \langle I_y(r)\rangle\right|}{\langle I_x(r)\rangle + \langle I_y(r)\rangle} = \frac{T+\Delta T - T}{T+\Delta T + T} = \frac{\Delta T}{2T+\Delta T} \approx \frac{\Delta T}{2T}.$$

Since $\langle I_x(r)\rangle$ is stronger, the electric field polarization in the scattered radiation is the y direction.

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2016
2016 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 5 Problems) 简体版卷-1（共5题）
(9:00 am – 12:00 pm, 18 February, 2016)

Please fill in your final answers to all problems on the **summary sheet**.
请在**总答案纸**上填上各题的最后答案。
There are 5 problems. Please answer each problem using a **new sheet**.
卷一 5 题，每答 1 题，须采用**新一张纸**。
Please answer on each sheet using a **single column**. Do not use two columns on a single sheet.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。
Please answer on **only one side** of each sheet. Do not use both sides of the sheet.
每张纸**单面**作答。不可以双面作答。
At the end of the competition, please arrange the **summary sheet as the first page**, followed by the answers to each problem in sequential order of the problems. If your answer to a problem requires more than one sheet, please arrange the sheets of the same problem in sequential order of the parts.
比赛结束时，请将**总答案纸放在首页**，随后把答题纸按题目次序排好。若答一题超过一页，请按分部次序排好答题纸。

1. Electrostatic Force (4 marks) 静电力 （4 分）

Consider a 2017-side regular polygon. There are 2016 point charges, each with charge q and located at a vertex of the polygon. Another point charge Q is located at the center of the polygon. The distance from the center of the regular polygon to its vertices is a. Find the force experienced by Q.
考虑一 2017-边正多边形。其中 2016 个角上各有一点电荷 q。 另有一个点电荷 Q 位于多边形的中心。 中心到每一个角的距离为 a。 求 Q 所受的力。

2. Capacitors (13 marks)

(a-c) Consider two clusters of electric charges. Cluster A consists of N charges $q_1, q_2, \ldots, q_N$, located at positions $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ respectively. Cluster B consists of M charges $q'_1, q'_2, \ldots, q'_M$, located at positions $\vec{r}'_1, \vec{r}'_2, \ldots, \vec{r}'_M$ respectively.
(a-c) 考虑两组电荷。组 A 由 N 个电荷 $q_1, q_2, \ldots, q_N$ 组成, 并分别位于位置$\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$。组 B 由 M 个电荷$q'_1, q'_2, \ldots, q'_M$组成, 并分别位于位置$\vec{r}'_1, \vec{r}'_2, \ldots, \vec{r}'_M$。

(a) Write the electric potential $\phi_A(\vec{r})$ at position $\vec{r}$ due to the charges in cluster A. (1 mark)
写下于位置$\vec{r}$ 由组 A 电荷形成的电势$\phi_A(\vec{r})$。（1 分）

(b) Write the electric potential energy $E_{B|A}$ of cluster B due to the electric potential ϕ_A. (1 mark)
写下组 B 电荷因电势ϕ_A产生的电势能 $E_{B|A}$。 (1 分)

(c) What is the relation between $E_{B|A}$ and $E_{A|B}$? (1 mark)

$E_{B|A}$ 和 $E_{A|B}$有何关系?（1 分）

(d) Consider two large conducting plates as shown in Fig. 1a. The upper plate carries a uniform surface charge density σ' and the lower plate is grounded. Find the surface charge density of the lower plate and the potential $\phi'(z)$, where z is the height of an arbitrary location from the lower plate. (5 marks)
考虑如图 1a 所示两块很大的电导板。上板带有均匀面电荷密度σ'，而下板则接地。求下板的面电荷密度和电势$\phi'(z)$，其中z为任意一点距离下板的高度。（5 分）

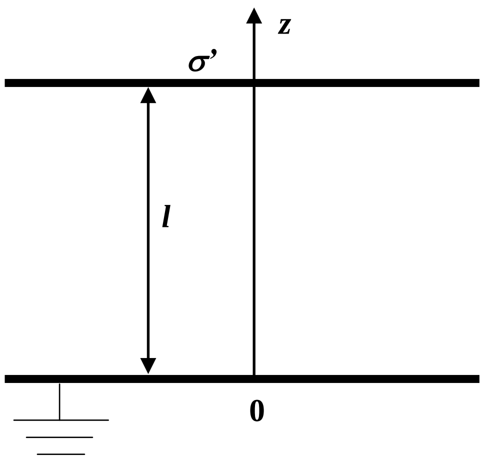


Figure 1a 图 1 a

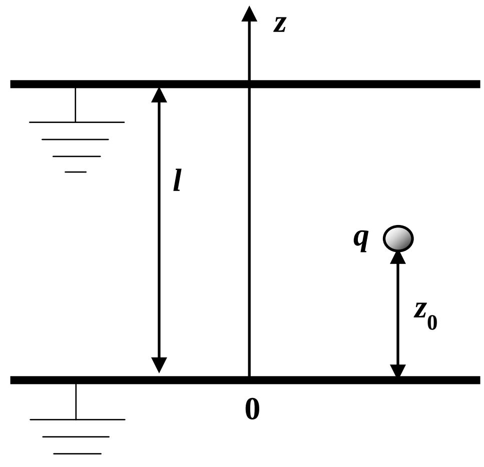


Figure 1b 图 1b

(e) A point charge q is placed between two very large grounded parallel conducting plates. If z_0 is the distance between q and the lower plate, find the total charge induced on the upper plate in terms of q, z_0, and l, where l is the distance between the plates, as shown in Fig. 1b. (5 marks)
如图 1b 所示，在相距为 l 的两块平行大电导板间放置电荷 q，其到下板的距离为 z_0。求上板的总感应电荷。以 q、z_0 和 l 表示你的答案。（5 分）

3. Cannonballs and Bombs (10 marks) 砲彈和炸彈 （10 分）

(a) Envelope of safety: A ground based cannon can fire a cannonball at a fixed speed of u in any direction. The envelope of safety is the curve inside which a target can be hit by the cannonball, and outside which there is no possibility of a target getting hit by the cannonball. Find the equation of the envelope of safety in space. (3 marks)
安全区域边界: 一门位于地面的大炮能以固定速率 u 向任何方向发射炮弹。 若目标在安全区域边界内, 则有可能被炮弹打中。 若在其外, 则不可能被炮弹打中。 求在空中的安全区域边界的方程式。 （3 分）

(b) A bomb explodes at a height of H into many small fragments. It is given that after the explosion the fragments have the same speed u and a uniform angular distribution in all directions. After some time all fragments hit the ground and the collisions with the ground are perfectly inelastic. Find the radius R of the distribution of the debris. (2 marks)

一个炸弹在高度 H 处爆炸成很多小碎片。 已知刚爆炸后各碎片以同样的速率 u 和均匀的角分布向各个方向散开。 其后各碎片都坠到地面上。 假设所有碎片与地面的碰撞皆为完全非弹性碰撞。 求炸弹残骸的分布半径 R。 (2 分)

(c) A bomb explodes on the ground. Its fragments are projected at the same speed u, and the angular distribution is uniform within a narrow angle α with the upward vertical direction. After some time all fragments hit the ground. Let the mass of the bomb be M. Find the radius R of the distribution of the debris up to order α. Calculate the radial density distribution $\rho(r)$ within radius R up to order r^2, where $\rho(r)2\pi r dr$ is the mass of the debris located at a distance r to $r + dr$ from the centre of the distribution. (5 marks)

[Remark: $\tan\varepsilon \approx \varepsilon\left(1+\frac{\varepsilon^2}{3}\right)$ and $\sin\varepsilon \approx \varepsilon\left(1-\frac{\varepsilon^2}{6}\right)$ for $\varepsilon \ll 1$.]

一个炸弹在地面爆炸。爆炸后各碎片以同样速率 u 射出，角度分布则限在与垂直向上方向的狭小夹角α内，而在这范围内角度分布均匀。其后各碎片都坠到地面上。设炸弹的质量为 M。求炸弹残骸的分布半径 R, 准确至α 的第一阶。定义径密度分布$\rho(r)$, 使得$\rho(r)2\pi r dr$为距离残骸中心 r 至 $r+dr$ 范围内的残骸质量。求半径 R 内的$\rho(r)$, 准确至 r 的第二阶。（5 分）

[注: 当 ε ≪1 时, $\tan\varepsilon \approx \varepsilon\left(1+\frac{\varepsilon^2}{3}\right)$及$\sin\varepsilon \approx \varepsilon\left(1-\frac{\varepsilon^2}{6}\right)$。]

4. Collisions (14 marks) 碰撞 (14 分)

A thin rod with length L, mass m and uniform density lies on the y-axis with its midpoint at the origin. A point object A with mass m travels with velocity u in the positive x direction hits the rod with impact parameter h, where $-L/2 \le h < L/2$, as shown in Fig. 2a. The collision is perfectly inelastic.

如图 2a 所示, 一根长度为 L、质量为 m、密度均匀的幼棒处在 y 轴上。 棒的中心点位于原点。 一质量为 m 的质点 A 以速度 u 向正 x 方向运动, 并以碰撞参数 h 与棒碰撞, 其中$-L/2 \le h < L/2$,。碰撞为完全非弹性碰撞。

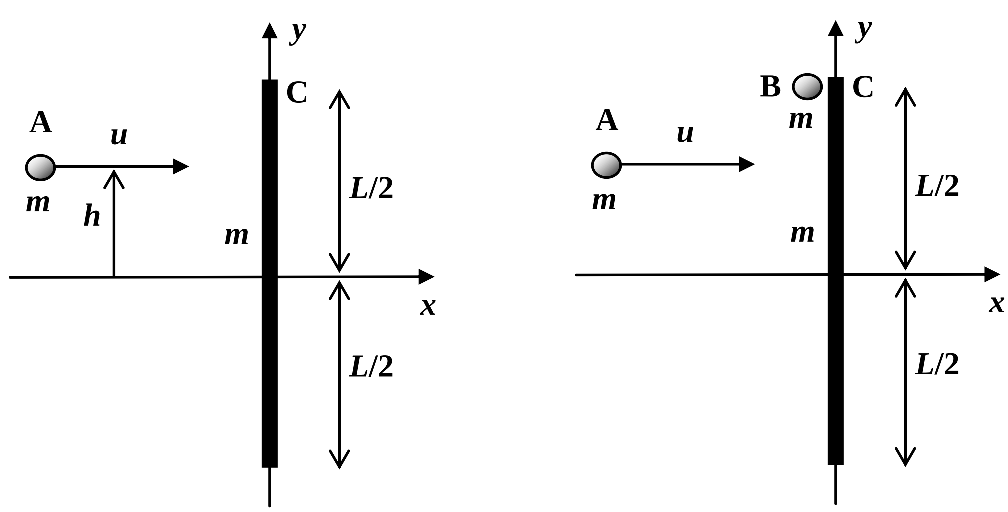


Figure 2a 图 2a　　Figure 2b 图 2b

(a) Find the total kinetic energy just after the collision between A and the rod. (6 marks)
求 A 刚与棒碰撞后系统的总动能。（6 分）

(b) Find the velocity v of point C at the top end of the rod as a function of h. (2 marks)
求棒上端点 C 的速度 v 与 h 的函数关系。（2 分）

(c) Find H such that $v(H) = 0$. (1 mark)
求 H 使得 $v(H) = 0$。（1 分）

(d) Suppose another point object B of mass m is located very close to point C, at the left hand side, as shown in Fig. 2b. Further suppose the point object A hits the rod at the lower end. Find the velocity of the point object B just after the rod collides elastically with it. (5 marks)
假设另一质量为 m 的质点 B 的位置与棒顶端 C 的左边非常接近，如图 2b 所示。再设点 A 撞到棒的下端。求棒与质点 B 产生完全弹性碰撞后，质点 B 的速度。（5 分）

5. Thermodynamic Cycle (9 marks) 热力学循环 (9 分)

Consider the thermodynamic cycle of an ideal monatomic gas shown in the pV diagram in Fig. 3. The cycle consists of four processes:

A → B: Isobaric expansion at pressure rp, where $r > 1$
B → C: Isothermal expansion at temperature T_2
C → D: Isobaric compression at pressure p
D → A: Isothermal compression at temperature T_1

考虑图 3 中所示一种单原子理想气体的热力学循环的 pV 图。 该循环包括四个过程:

A → B: 压强 rp 下的等压膨胀, 其中 $r > 1$
B → C: 温度 T_2 下的等温膨胀
C → D: 压强 p 下的等压压缩
D → A: 温度 T_1 下的等温压缩

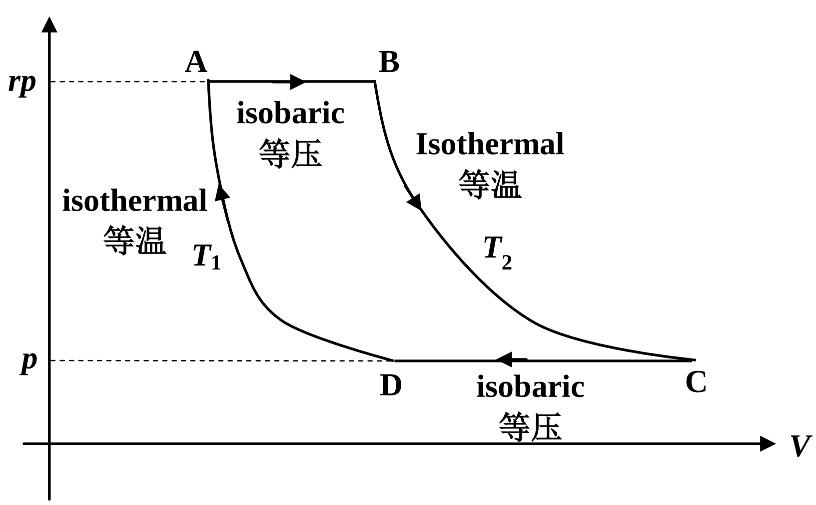


Figure 3 图 3

(a) Write the highest temperature T_H and lowest temperature T_C in the cycle. No proof is required. (1 mark)
无须证明, 写下循环中的最高温度 T_H和最低温度 T_C。（1 分）

(b) Write the efficiency e_C of a Carnot engine operating with a hot reservoir at temperature T_H and a cold reservoir at temperature T_C. (1 mark)
一卡诺热机在温度为 T_H的高温热库和温度为 T_C的低温热库间运行。 写下其热效率 e_C。(1 分)

(c) Given that the gas is in thermal contact with a hot reservoir with temperature T_H whenever heat is added to the gas, and in thermal contact with a cold reservoir with temperature T_C whenever heat is removed from the gas, find the efficiency e of an engine running the cycle in the pV diagram. Express your answer in terms of T_C, T_H, p, and r. (5 marks)
已知一热机在循环运行中，气体吸热时永远与温度为 T_H 的热库处于热接触, 气体放热时永远与温度为 T_C的热库处于热接触。求其热效率 e。 以 T_C、T_H、p 和 r 表示你的答案。（5 分）

(d) Find the ratio $\frac{e}{e_C}$. Hence suggest a parameter regime in which the efficiency approaches that of the ideal engine. (2 marks)
求比例 $\frac{e}{e_C}$。根据答案, 提出能使热效率趋近理想热机热效率的参数范围。 (2 分)

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2016
2016 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 5 Problems) 简体版卷-1（共5题）
(9:00 am – 12:00 pm, 18 February, 2016)

1. Electrostatic Force (4 marks) 静电力（4 分）

Consider a 2017-side regular polygon. There are 2016 point charges, each with charge q and located at a vertex of the polygon. Another point charge Q is located at the center of the polygon. The distance from the center of the regular polygon to its vertices is a. Find the force experienced by Q.
考虑一 2017-边正多边形。其中 2016 个角上各有一点电荷 q。 另有一个点电荷 Q 位于多边形的中心。 中心到每一个角的距离为 a。 求 Q 所受的力。

Consider the polygon with a charge q at each vertex. In other words, there are 2017 charges. The system has a discrete rotational symmetry and hence the force acting on Q must be zero. Now our system is equivalent to the above system but with a charge $-q$ added to one vertex. Hence the force is

$$\mathbf{F} = \frac{Qq}{4\pi\varepsilon_0 a^2}\hat{\mathbf{a}}$$

where $\hat{\mathbf{a}}$ is a unit vector pointing from the center to the empty vertex.

2. Capacitors (13 marks) 电容器（13 分）

(a-c) Consider two clusters of electric charges. Cluster A consists of N charges $q_1, q_2, \ldots, q_N$, located at positions $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ respectively. Cluster B consists of M charges $q'_1, q'_2, \ldots, q'_M$, located at positions $\vec{r}'_1, \vec{r}'_2, \ldots, \vec{r}'_M$ respectively.
(a-c) 考虑两组电荷。组 A 由 N 个电荷 $q_1, q_2, \ldots, q_N$ 组成, 并分别位于位置$\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$。组 B 由 M 个电荷$q'_1, q'_2, \ldots, q'_M$组成, 并分别位于位置$\vec{r}'_1, \vec{r}'_2, \ldots, \vec{r}'_M$。

(a) Write the electric potential $\phi_A(\vec{r})$ at position $\vec{r}$ due to the charges in cluster A. (1 mark)
写下于位置$\vec{r}$ 由组 A 电荷形成的电势$\phi_A(\vec{r})$。（1 分）

$$\phi_A(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}-\vec{r}_i|}.$$

(b) Write the electric potential energy $E_{B|A}$ of cluster B due to the electric potential ϕ_A. (1 mark)
写下组 B 电荷因电势ϕ_A产生的电势能 $E_{B|A}$。 (1 分)

$$E_{B|A} = \sum_{i=1}^{M} q'_i\phi_A(\vec{r}'_i) = \sum_{i=1}^{M} q'_i\left(\frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{N}\frac{q_j}{|\vec{r}'_i-\vec{r}_j|}\right) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{M}\sum_{j=1}^{N}\frac{q'_i q_j}{|\vec{r}'_i-\vec{r}_j|}.$$

(c) What is the relation between $E_{B|A}$ and $E_{A|B}$? (1 mark)
$E_{B|A}$ 和 $E_{A|B}$有何关系? (1 分)

$$E_{A|B} = \sum_{i=1}^{N} q_i \phi_B(\vec{r}_i) = \sum_{i=1}^{N} q_i \left(\frac{1}{4\pi\varepsilon_0} \sum_{j=1}^{M} \frac{q'_j}{|\vec{r}_i - \vec{r}'_j|} \right) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{q_i q'_j}{|\vec{r}_i - \vec{r}'_j|}.$$

Interchanging the indices i and j, and the order of summation, we have $E_{B|A} = E_{A|B}$.

(d) Consider two large conducting plates as shown in Fig. 1a. The upper plate carries a uniform surface charge density σ' and the lower plate is grounded. Find the surface charge density of the lower plate and the potential $\phi'(z)$, where z is the height of an arbitrary location from the lower plate. (5 marks)

考虑如图 1a 所示两块很大的电导板。上板带有均匀面电荷密度σ'，而下板则接地。求下板的面电荷密度和电势$\phi'(z)$，其中z为任意一点距离下板的高度。（5 分）

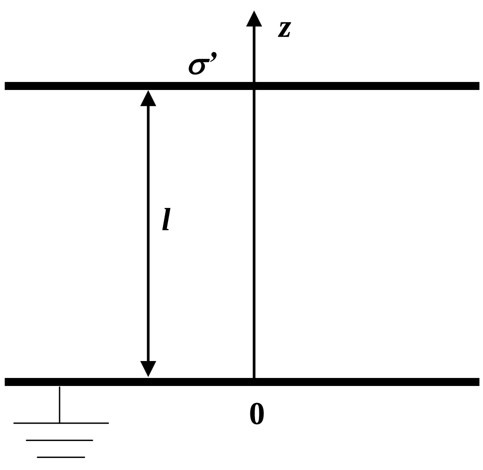


Figure 1a 图 1 a

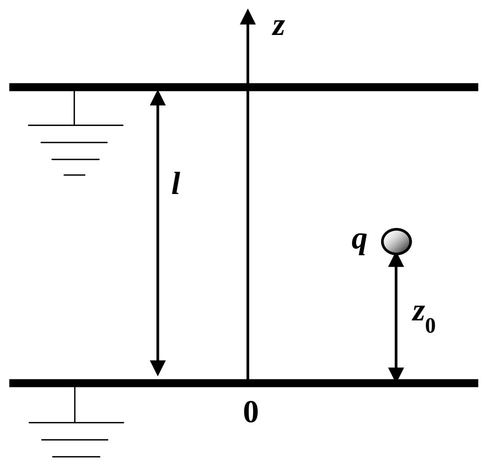


Figure 1b 图 1b

The surface charge density at the lower plate is $-\sigma'$.

Using Gauss' law, the electric field between the plates is $E = \frac{\sigma'}{\varepsilon_0}$.

The potential is $\phi'(z) = \begin{cases} 0 & z \leq 0, \\ \frac{\sigma'}{\varepsilon_0} z & 0 \leq z \leq l, \\ \frac{\sigma'}{\varepsilon_0} l & z > l. \end{cases}$

(e) A point charge q is placed between two very large grounded parallel conducting plates. If z_0 is the distance between q and the lower plate, find the total charge induced on the upper plate in terms of q, z_0, and l, where l is the distance between the plates, as shown in Fig. 1b. (5 marks)

如图 1b 所示，在相距为 l 的两块平行大电导板间放置电荷 q，其到下板的距离为 z_0。求上板的总感应电荷。以 q、z_0 和 l 表示你的答案。（5 分）

Consider the charge distribution in Fig. 1a to be cluster A, and that in Fig. 1b to be cluster B.

To calculate $E_{A|B}$, we note that there are electric charges in cluster A located at the upper plate only, but for cluster B, the electric potential at the upper plate is 0. Hence $E_{A|B} = 0$.

To calculate $E_{B|A}$, we note that there are electric charges in cluster B located at:

- the lower plate, but $\phi_A = 0$;
- the point charge q, where $\phi_A = \frac{\sigma'}{\varepsilon_0} z_0$;

- the upper plate with charge Q_u to be determined, where $\phi_A = \frac{\sigma'}{\varepsilon_0} l$.

Hence applying the result in part (c), $q\frac{\sigma'}{\varepsilon_0}z_0 + Q_u\frac{\sigma'}{\varepsilon_0}l = 0 \Rightarrow Q_u = -q\frac{z_0}{l}$

3. Cannonballs and Bombs (10 marks) 砲彈和炸彈 （10 分）

(a) Envelope of safety: A ground based cannon can fire a cannonball at a fixed speed of u in any direction. The envelope of safety is the curve inside which a target can be hit by the cannonball, and outside which there is no possibility of a target getting hit by the cannonball. Find the equation of the envelope of safety in space. (3 marks)

安全区域边界：一门位于地面的大炮能以固定速率 u 向任何方向发射炮弹。 若目标在安全区域边界内，则有可能被炮弹打中。 若在其外，则不可能被炮弹打中。 求在空中的安全区域边界的方程式。 （3 分）

Consider a target at (x, y). Let the cannonball fired at angle θ hit this point. Then

$$\begin{cases} x = ut\cos\theta, \\ y = ut\sin\theta - \frac{1}{2}gt^2. \end{cases}$$

Eliminating t, $y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$.

Note that $\frac{1}{\cos^2\theta} = \sec^2\theta = 1 + \tan^2\theta$, we arrive at the quadratic equation in $\tan\theta$.

$$\frac{gx^2}{2u^2}\tan^2\theta - x\tan\theta + \frac{gx^2}{2u^2} + y = 0.$$

Outside the envelope, there are no solutions.

Inside the envelope, there are two solutions.

Right on the envelope, there is only one solution.

Setting the discriminant to zero, $\Delta = (-x)^2 - 4\frac{gx^2}{2u^2}\left(\frac{gx^2}{2u^2} + y\right) = 0 \Rightarrow y = -\frac{g}{2u^2}x^2 + \frac{u^2}{2g}$ which is a parabola.

(b) A bomb explodes at a height of H into many small fragments. It is given that after the explosion the fragments have the same speed u and a uniform angular distribution in all directions. After some time all fragments hit the ground and the collisions with the ground are perfectly inelastic. Find the radius R of the distribution of the debris. (2 marks)

一个炸弹在高度 H 处爆炸成很多小碎片。 已知刚爆炸后各碎片以同样的速率 u 和均匀的角分布向各个方向散开。 其后各碎片都坠到地面上。 假设所有碎片与地面的碰撞皆为完全非弹性碰撞。 求炸弹残骸的分布半径 R。 (2 分)

Let the bomb be located at the origin, x be the horizontal axis and y be the vertical axis, with upward as positive. Using the result of (a), set $y = -H$, we have $-H = -\frac{g}{2u^2}R^2 + \frac{u^2}{2g} \Rightarrow$

$$R = \frac{u}{g}\sqrt{u^2 + 2gH}.$$

(c) A bomb explodes on the ground. Its fragments are projected at the same speed u, and the angular distribution is uniform within a narrow angle α with the upward vertical direction. After some time all fragments hit the ground. Let the mass of the bomb be M. Find the radius R of the distribution of the debris up to order α. Calculate the radial density distribution $\rho(r)$

within radius R up to order r^2, where $\rho(r)2\pi r dr$ is the mass of the debris located at a distance r to $r+dr$ from the centre of the distribution. (5 marks)

[Remark: $\tan\varepsilon \approx \varepsilon\left(1+\frac{\varepsilon^2}{3}\right)$ and $\sin\varepsilon \approx \varepsilon\left(1-\frac{\varepsilon^2}{6}\right)$ for $\varepsilon \ll 1$.]

一个炸弹在地面爆炸。爆炸后各碎片以同样速率 u 射出，角度分布则限在与垂直向上方向的狭小夹角α内，而在这范围内角度分布均匀。其后各碎片都坠到地面上。设炸弹的质量为 M。求炸弹残骸的分布半径 R, 准确至α 的第一阶。定义径密度分布$\rho(r)$，使得$\rho(r)2\pi r dr$为距离残骸中心 r 至 $r+dr$ 范围内的残骸质量。求半径 R 内的$\rho(r)$, 准确至 r 的第二阶。（5 分）

[注: 当 $\varepsilon \ll 1$ 时, $\tan\varepsilon \approx \varepsilon\left(1+\frac{\varepsilon^2}{3}\right)$及$\sin\varepsilon \approx \varepsilon\left(1-\frac{\varepsilon^2}{6}\right)$。]

$$\begin{cases} x = ut\cos\theta, \\ y = ut\sin\theta - \frac{1}{2}gt^2. \end{cases}$$

At $y=0$, $ut\sin\theta - \frac{1}{2}gt^2 = 0 \Rightarrow t = \frac{2u\sin\theta}{g} \Rightarrow x = u\cos\theta\left(\frac{2u\sin\theta}{g}\right) = \frac{u^2\sin 2\theta}{g}$

Substituting $\theta = \frac{\pi}{2} - \varepsilon$ and $x = r$,

$$r = \frac{u^2}{g}\sin(\pi - 2\varepsilon) = \frac{u^2}{g}\sin(2\varepsilon) \approx \frac{2u^2}{g}\varepsilon\left(1 - \frac{2}{3}\varepsilon^2\right).$$

Hence $R \approx \frac{2u^2}{g}\alpha$.

The angular distribution after the explosion is $\rho(\theta)d\theta = M\frac{2\pi\cos\theta d\theta}{2\pi(1-\cos\alpha)} \Rightarrow \rho(\theta) \approx \frac{2M}{\alpha^2}\cos\theta$

$\Rightarrow \rho(\varepsilon) = \frac{2M}{\alpha^2}\cos\left(\frac{\pi}{2}-\varepsilon\right) = \frac{2M}{\alpha^2}\sin\varepsilon \approx \frac{2M}{\alpha^2}\varepsilon\left(1-\frac{\varepsilon^2}{6}\right).$

$\rho(r) = \frac{\rho(\varepsilon)}{2\pi r}\frac{d\varepsilon}{dr}.$

$\frac{dr}{d\varepsilon} \approx \frac{2u^2}{g}(1-2\varepsilon^2) \Rightarrow \rho(r) \approx \frac{Mg}{2\pi r u^2\alpha^2}\varepsilon\left(1+\frac{11\varepsilon^2}{6}\right) \approx \frac{Mg^2}{4\pi u^4\alpha^2}\left(1+\frac{11\varepsilon^2}{6}\right)\left(1+\frac{2\varepsilon^2}{3}\right)$

$\approx \frac{Mg^2}{4\pi u^4\alpha^2}\left(1+\frac{5\varepsilon^2}{2}\right) \approx \frac{Mg^2}{4\pi u^4\alpha^2}\left(1+\frac{5g^2r^2}{8u^4}\right).$

4. Collisions (14 marks) 碰撞 (14 分)

A thin rod with length L, mass m and uniform density lies on the y-axis with its midpoint at the origin. A point object A with mass m travels with velocity u in the positive x direction hits the rod with impact parameter h, where $-L/2 \le h < L/2$, as shown in Fig. 2a. The collision is perfectly inelastic.

如图 2a 所示, 一根长度为 L、质量为 m、密度均匀的幼棒处在 y 轴上。 棒的中心点位于原点。 一质量为 m 的质点 A 以速度 u 向正 x 方向运动, 并以碰撞参数 h 与棒碰撞, 其中$-L/2 \le h < L/2$,。碰撞为完全非弹性碰撞。

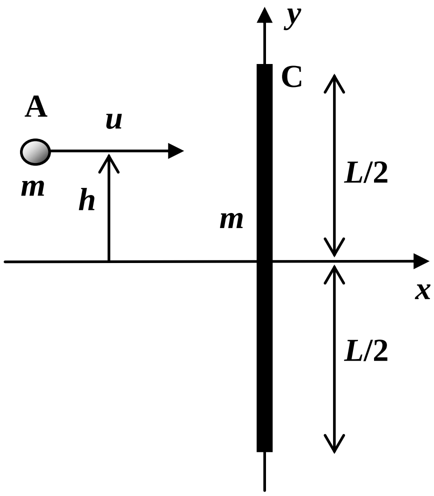


Figure 2a 图 2a

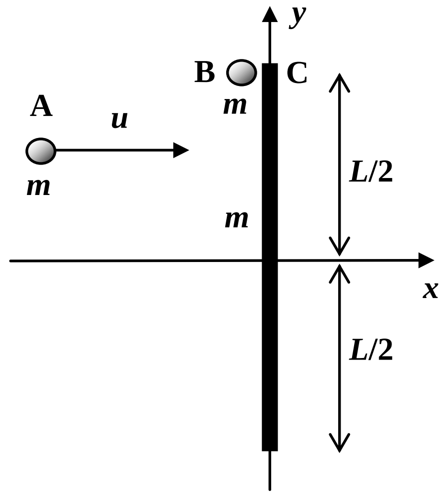


Figure 2b 图 2b

(a) Find the total kinetic energy just after the collision between A and the rod. (6 marks)
求 A 刚与棒碰撞后系统的总动能。（6 分）

Initial momentum: $p_i = mu$

Initial angular momentum about origin (clockwise positive): $L_i = muh$

Initial kinetic energy: $K_i = \frac{1}{2}mu^2$

Final momentum: $p_f = 2mv_{CM}$

Final angular momentum about origin: $L_f = 2mv_{CM}y_{CM} + I_{CM}\omega$ where $y_{CM} = \frac{m}{m+m}h = \frac{h}{2}$.

$$I_{CM} = \frac{1}{12}mL^2 + m\left(\frac{h}{2}\right)^2 + m\left(h - \frac{h}{2}\right)^2 = \frac{1}{12}mL^2 + \frac{1}{2}mh^2$$

Using the conservation of momentum, $mu = 2mv_{CM} \Rightarrow v_{CM} = \frac{u}{2}$.

Using the conservation of angular momentum,

$muh = (2m)\left(\frac{u}{2}\right)\frac{h}{2} + \left(\frac{1}{12}mL^2 + \frac{1}{2}mh^2\right)\omega \Rightarrow \omega = \frac{6uh}{L^2+6h^2}$.

Final kinetic energy:

$$K_f = \frac{1}{2}(2m)\left(\frac{u}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2 + \frac{1}{2}mh^2\right)\left(\frac{6uh}{L^2+6h^2}\right)^2 = \frac{mu^2}{4}\left(1 + \frac{6h^2}{L^2+6h^2}\right)$$

(b) Find the velocity v of point C at the top end of the rod as a function of h. (2 marks)
求棒上端点 C 的速度 v 与 h 的函数关系。（2 分）

$$v(h) = \frac{u}{2} + \frac{6uh}{L^2+6h^2}\left(\frac{L}{2} - \frac{h}{2}\right) = \frac{uL(L+6h)}{2(L^2+6h^2)}.$$

(c) Find H such that $v(H) = 0$. (1 mark)
求 H 使得 $v(H) = 0$。（1 分）

$v(H) = \frac{uL(L+6H)}{2(L^2+6H^2)} = 0 \Rightarrow H = -\frac{L}{6}$.

(d) Suppose another point object B of mass m is located very close to point C, at the left hand side, as shown in Fig. 2b. Further suppose the point object A hits the rod at the lower end. Find the velocity of the point object B just after the rod collides elastically with it. (5 marks)
假设另一质量为 m 的质点 B 的位置与棒顶端 C 的左边非常接近，如图 2b 所示。再设点 A 撞到棒的下端。求棒与质点 B 产生完全弹性碰撞后，质点 B 的速度。（5 分）

Let w_1 be the forward velocity of the center of mass of the rod after collision with B.
Let w_2 be the backward velocity of object B after the collision.
Let ω_1 be the clockwise angular velocity of the rod after collision with B.
Conservation of linear momentum: $mu = -mw_2 + 2mw_1 \Rightarrow w_2 = 2w_1 - u$.
Conservation of angular momentum about the origin: $muh = -mw_2\frac{L}{2} + I_{CM}\omega_1 + 2mw_1\left(\frac{h}{2}\right)$.
Since $I_{CM} = \frac{1}{12}mL^2 + \frac{1}{2}mh^2 = \frac{5}{24}mL^2$, this implies $-\frac{1}{2}u = -\frac{1}{2}w_2 + \frac{5}{24}L\omega_1 - \frac{1}{2}w_1$.
Eliminating w_2, $\omega_1 = \frac{12}{5L}(3w_1 - 2u)$.
Conservation of energy: $\frac{mu^2}{4}\left(1 + \frac{6h^2}{L^2+6h^2}\right) = \frac{1}{2}mw_2^2 + \frac{1}{2}I_{CM}\omega_1^2 + \frac{1}{2}(2m)w_1^2 \Rightarrow$
$\frac{2}{5}u^2 = \frac{1}{2}w_2^2 + \frac{5}{48}L^2\omega_1^2 + w_1^2$.
Substituting w_2 and ω_1, $\frac{2}{5}u^2 = \frac{1}{2}(2w_1 - u)^2 + \frac{3}{5}(3w_1 - 2u)^2 + w_1^2$.
This reduces to the quadratic equation $84w_1^2 - 92uw_1 + 25u^2 = 0 \Rightarrow w_1 = \frac{25}{42}u$ or $\frac{u}{2}$. The second solution is the velocity before collision. Hence $w_1 = \frac{25}{42}u \Rightarrow w_2 = \frac{4}{21}u$.

5. Thermodynamic Cycle (9 marks) 热力学循环 (9 分)

Consider the thermodynamic cycle of an ideal monatomic gas shown in the pV diagram in Fig. 3. The cycle consists of four processes:

A → B: Isobaric expansion at pressure rp, where $r > 1$
B → C: Isothermal expansion at temperature T_2
C → D: Isobaric compression at pressure p
D → A: Isothermal compression at temperature T_1

考虑图 3 中所示一种单原子理想气体的热力学循环的 pV 图。 该循环包括四个过程:

A → B: 压强 rp 下的等压膨胀, 其中 $r > 1$
B → C: 温度 T_2 下的等温膨胀
C → D: 压强 p 下的等压压缩
D → A: 温度 T_1 下的等温压缩

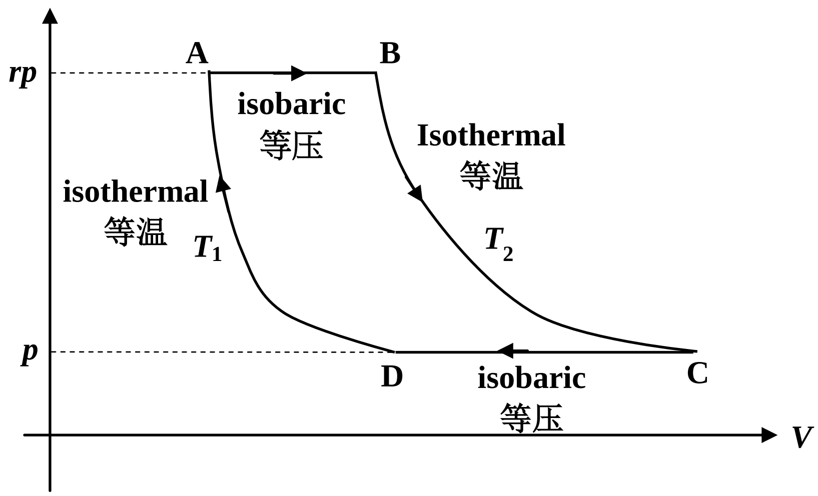


Figure 3 图 3

(a) Write the highest temperature T_H and lowest temperature T_C in the cycle. No proof is required. (1 mark)
无须证明, 写下循环中的最高温度 T_H和最低温度 T_C。（1 分）

$T_H = T_2$, $T_C = T_1$.

(b) Write the efficiency e_C of a Carnot engine operating with a hot reservoir at temperature T_H and a cold reservoir at temperature T_C. (1 mark)
一卡诺热机在温度为 T_H的高温热库和温度为 T_C的低温热库间运行。 写下其热效率 e_C。(1 分)

$e_C = 1 - \frac{T_C}{T_H}$.

(c) Given that the gas is in thermal contact with a hot reservoir with temperature T_H whenever heat is added to the gas, and in thermal contact with a cold reservoir with temperature T_C whenever heat is removed from the gas, find the efficiency e of an engine running the cycle in the pV diagram. Express your answer in terms of T_C, T_H, p, and r. (5 marks)
已知一热机在循环运行中，气体吸热时永远与温度为 T_H 的热库处于热接触, 气体放热时永远与温度为 T_C的热库处于热接触。求其热效率 e。 以 T_C、T_H、p 和 r 表示你的答案。（5 分）

In A → B

$$Q = \Delta U - W = \frac{3}{2}nR(T_H - T_C) + rp(V_B - V_A) = \frac{3}{2}nR(T_H - T_C) + nR(T_H - T_C)$$
$$= \frac{5}{2}nR(T_H - T_C)$$

In B → C

$$Q = -W = \int_{V_B}^{V_C} pdV = \int_{V_B}^{V_C} \frac{nRT_H}{V} dV = nRT_H \ln\frac{V_C}{V_B} = nRT_H \ln\frac{\frac{nRT_H}{p}}{\frac{nRT_H}{rp}} = nRT_H \ln r$$

In C → D

$$Q = \Delta U - W = \frac{3}{2}nR(T_C - T_H) - p(V_C - V_D) = \frac{3}{2}nR(T_C - T_H) - nR(T_H - T_C) = $$
$$= -\frac{5}{2}nR(T_H - T_C)$$

In D → A

$$Q = -W = \int_{V_D}^{V_A} pdV = \int_{V_D}^{V_A} \frac{nRT_C}{V}dV = -nRT_C \ln\frac{V_D}{V_A} = -nRT_C \ln\frac{\frac{nRT_D}{p}}{\frac{nRT_A}{rp}} = -nRT_C \ln r$$

Since $\Delta U = 0$ in a cycle, work done in a cycle = heat absorbed in a cycle
$= Q_{AB} + Q_{BC} + Q_{CD} + Q_{DA} = nR(T_H - T_C)\ln r$.

Heat is input during A → B and B → C, $Q_{AB} + Q_{BC} = \frac{5}{2}nR(T_H - T_C) + nRT_H \ln r$.

The efficiency is

$$e = \frac{nR(T_H - T_C)\ln r}{\frac{5}{2}nR(T_H - T_C) + nRT_H \ln r} = \frac{(T_H - T_C)\ln r}{\frac{5}{2}(T_H - T_C) + T_H \ln r}$$

(d) Find the ratio $\frac{e}{e_C}$. Hence suggest a parameter regime in which the efficiency approaches that of the ideal engine. (2 marks)

求比例 $\frac{e}{e_C}$。根据答案，提出能使热效率趋近理想热机热效率的参数范围。 (2 分)

$$\frac{e}{e_C} = \left[\frac{(T_H - T_C)\ln r}{\frac{5}{2}(T_H - T_C) + T_H \ln r}\right]\left(\frac{T_H}{T_H - T_C}\right) = \frac{T_H \ln r}{\frac{5}{2}(T_H - T_C) + T_H \ln r} < 1.$$

To make the ratio approaches 1, we can make $r \gg 1$ or make $T_H - T_C \ll T_H$.

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2016
2016 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-2 (Total 3 Problems) 简体版卷-2（共3题）
2:00 pm – 5:00 pm, 18 February, 2016

Please fill in your final answers to all problems on the **summary sheet**.
请在总答案纸上填上各题的最后答案。
There are 3 problems. Please answer each problem using a **new sheet**.
卷二 3 题，每答 1 题，须采用新一张纸。
Please answer on each sheet using a **single column**. Do not use two columns on a single sheet.
每页纸请用单一直列的方式答题。不可以在一页纸上以双直列方式答题。
Please answer on **only one side** of each sheet. Do not use both sides of the sheet.
每张纸单面作答。不可以双面作答。
At the end of the competition, please arrange the **summary sheet as the first page**, followed by the answers to each problem in sequential order of the problems. If your answer to a problem requires more than one sheet, please arrange the sheets of the same problem in sequential order of the parts.
比赛结束时，请将总答案纸放在首页，随后把答题纸按题目次序排好。若答一题超过一页，请按分部次序排好答题纸。

1. Gravitational Lens Simulator (9 marks) 引力透镜仿真镜 (9 分)

The gravitational field of a massive body exercises a lenslike condensing action upon radiation passing through it. A simulated gravitational lens was constructed of Plexiglas for use in demonstrating the lens phenomenon.
大质量物体附近的引力场会对经过的光线产生类似透镜的作用, 称为引力透镜效应。 这里我们考虑以有机玻璃制造一个光学透镜以仿真引力透镜现象。
According to general relativity, a light ray passing at a distance of closest approach r to the center of a spherically symmetric body of mass M, will be deflected toward the body through an angle which, for small deflections, is given by
根据广义相对论，当一束光线经过一个拥有球对称、质量为M的物体时,会产生屈折。当屈折角度很小，而光线与物体中心最靠近距离为 r 时,该角度可由下面的公式得出

$$\varepsilon = \frac{4GM}{rc^2}$$

where G is the gravitational constant and c is the speed of light. We, therefore, require of our simulator that it deflects transmitted light through an angle
在上式中G为引力常数， c为光速。 因此我们要求仿真镜以下式中的角度屈折光线

$$\varepsilon = \frac{R}{r}$$

where R is a constant. 在上式中R是某个常数。

The lens, illustrated in cross section in Fig. 1, is designed to be hand held within the range between roughly one foot and arm's length from the observer. The object for viewing is assumed to be at a distance much larger than one meter on the left hand side of the lens. This implies that one can assume the incident light ray to be normal to the plane front surface of the lens and refraction is thus assumed to take place entirely at the back surface. The angle of incidence of the light ray at the back surface is designated by θ, the angle of refraction by θ', and the angle of deflection by ε, which are all assumed to be small angles. The refractive index of the lens is n.
图一所示为仿真镜的横截面。仿真镜为观察者手举而设计，设计距离为大约一呎到手臂长度。观察对象位于图中仿真镜左边远大于一米处。因此我们可以假设入射光是垂直于仿真镜的前平面，而折射仅发生于仿真镜的后表面。 以θ 表示入射光与后表面法线的夹角，θ' 表示折射角度，ε表示屈折角度 。以上角度皆假设为小角度。仿真镜的折射系数是 n。

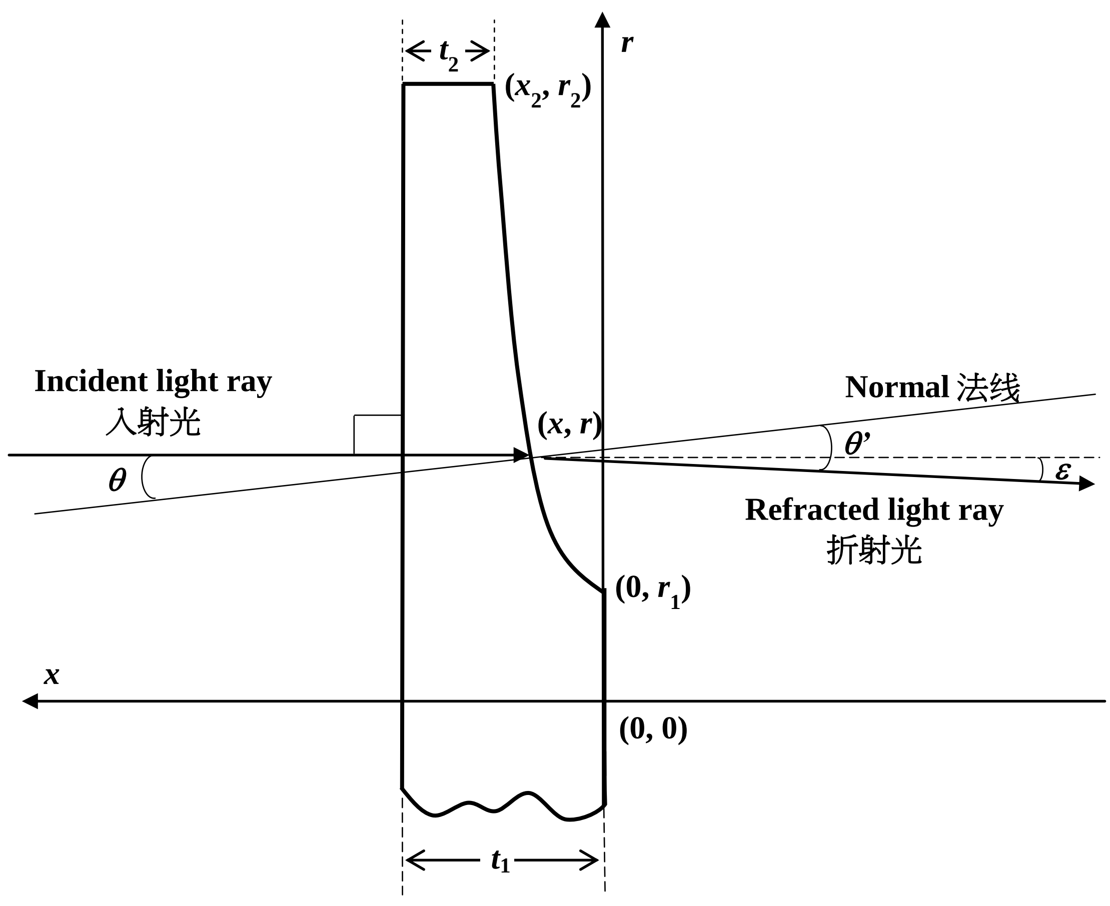


Figure 1 图一

(a) Derive the expression dx/dr for the slope of the back refracting surface of the lens in terms of ε and n. (4 marks)
试推导透镜后折射面的斜率 dx/dr，答案以ε和 n 表达。(4 分)

(b) Derive the expression $x(r)$ of the back refracting surface of the lens. Express your answer in terms of n, R and r_1. (3 marks)
试推导透镜后折射面的表达式 $x(r)$。答案以 n、R 和 r_1 表达。(3 分)

(c) A light ray is incident at impact parameter r equal to 2 cm. If we require the ray to cross the lens axis at a horizontal distance of 30 cm from the point at which it departs from the lens, find R. (1 mark)
考虑一束入射参数 r 为 2 cm 的光线。 如果要求该光线在离开透镜后于水平距离 30 cm 处与透镜中轴相交, 则 R 应取何值? (1 分)

(d) Find the effective gravitational mass of the lens. (1 mark)
计算透镜的有效引力质量。(1 分)

2. A String and a Mass (19 marks) 系着质量的弦 (19 分)

In this question, all oscillatory motions are assumed to be small.
在本题中，假设所有震荡皆为微小震荡。

As shown in Fig. 2, a string of mass m and length l with tension τ has a mass M attached to the end. The mass M can slide in a vertical direction on a frictionless rod at $x = l$. The shape of the string is described by a function $y(x, t)$. The string is fixed at the origin $y(0, t) = 0$.
如图二所示, 一条质量为 m、长度为 l、张力为 τ的弦一端系着质量 M。 该质量 M 可在一根位于 $x = l$ 处的平滑棒上自由垂直滑动。 弦的型态由函数 $y(x, t)$给出。 弦的另一端固定于原点处, 即 $y(0, t) = 0$。

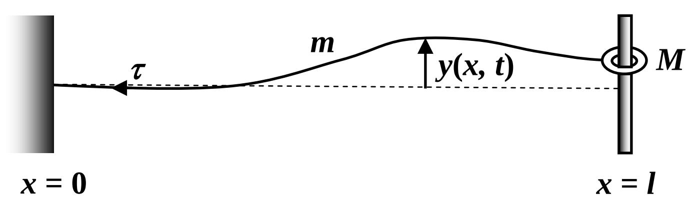


Figure 2 图二

(a) First assume that mass M is held fixed at $y = 0$. Write down the general solution $y(x, t)$ for the standing waves on the string. Express your answer in terms of the given parameters and arbitrary constants. (4 marks)
首先假设质量 M 被固定在 $y = 0$ 处。写下弦上驻波的一般解 $y(x, t)$。答案以给定的参数和任意常数表达。(4 分)

(b) Now assume that mass M can slide up and down on a frictionless rod at $x = l$. What is the boundary condition on $y(x, t)$ at $x = l$? You can assume that the oscillations are small. (1 mark)
现在假设质量 M 能沿着 $x = l$ 处的平滑棒上下移动，则 $y(x, t)$在 $x = l$ 处的边界条件该是甚么? 你可以假设震荡都是小震荡。(1 分)

(c) Write down an equation for the frequencies of the standing waves on the string when the mass M is free to slide. You do not need to solve the equation. (2 marks)
当质量 M 可自由滑动时, 写下满足弦驻波频率的方程式。你不需要解该方程式。(2 分)

(d) If $m << M$, find the two lowest normal mode frequencies. (2 marks)
如果 $m << M$, 找出最低频的两个正则模频率。(2 分)

(e) For $m << M$, calculate the ratio of the kinetic energy of the string to that of the mass M at the lowest normal mode frequency. (2 marks)
如果 $m << M$，试计算在最低频的正则模中，弦的动能与质量 M 的动能的比例。(2 分)

(f) If $m >> M$, find the two lowest normal mode frequency. (2 marks)
如果 $m >> M$, 找出最低頻的兩個正則模頻率。(2 分)

(g) For $m >> M$, calculate the ratio of the kinetic energy of the string to that of the mass M at the lowest normal mode frequency. (2 marks)
如果 $m >> M$，试计算在最低频的正则模中，弦的动能与质量 M 的动能的比例。(2 分)

(h) A traveling wave of angular frequency ω is generated near the end $x = l$. It propagates towards the mass M and is reflected with a phase shift of $\pi/2$. What is the value of ω in terms of τ, m, M and l? (4 marks)
在靠近 $x = l$ 处生成一角频率为ω的行波。该行波朝质量 M 方向传播，并以$\pi/2$ 的相移被反射。 求ω，答案以τ、m、M 和 l 表达。(4 分)

3. Maximum Mass of a Star (22 marks) 星體的最大質量 (22 分)

Consider a star of mass M and radius R. Assume that its density is uniform.
考虑一质量为 M、半径为 R 的星体。 假设其质量密度均匀。

(a) Its gravitational potential energy U can be calculated by considering the work done in bringing a thin layer of materials and depositing on the surface of a spherical protostar of radius r when r gradually grows from 0 to R. Calculate U. Express your answer in terms of G, M and R, where G is the gravitational constant. (4 marks)
要计算星体的引力势能 U，可考虑逐层将星体物质加至半径为 r 的球状准星体表面所作的功，并让 r 由 0 逐渐增加到 R。计算 U，答案以引力常数 G、M 和 R 表达。(4 分)

(b) Assume that the star is made up of protons and electrons, both behaving as ideal gases. It is known that during the formation of the star, half of the loss in gravitational potential energy is converted to thermal energy, while the other half is radiated away. Derive the temperature T of the star in terms of G, M, R, $\overline{m}$ and k_B, where $\overline{m}$ is the average mass of protons and electrons, and k_B is the Boltzmann constant. (2 marks)
假设星体由质子和电子组成, 其行为皆为理想气体。已知当星体形成时, 其引力势能的耗损一半会转化为热能, 另一半会被辐射掉。试推导星体温度 T，答案以 G、M、R、$\overline{m}$和 k_B表达。 这里$\overline{m}$为质子与电子的平均质量，k_B 为玻耳兹曼常数。(2 分)

(c) Derive the gas pressure P_g of the star in terms of G, M and R. (2 marks)
试推导星体的气体压强 P_g，答案以 G、M 和 R 表达。(2 分)

(d) The virial theorem states that the total pressure in a star is related to the gravitational potential energy by $P = -b\dfrac{U}{V}$. What is the value of b? (1 mark)
根据维里定理, 星体内的总压强与引力势能的关系为 $P = -b\dfrac{U}{V}$。求 b 的值。(1 分)

(e) At high temperature, photons in the star also contribute to the pressure. Derive the radiation pressure P_r by applying the kinetic theory of gases in a cubic box of volume L^3, in which the momenta of the photons are described by the de Broglie relation. Express your result in terms of the photon energy density u. (5 marks)
在高温下，星体中的光子也会施加压强。试应用气体运动论，考虑一体积为 L^3 的立方盒子中的光子，其中光子的动量满足德布罗意关系式，从而得出辐射压强 P_r。答案以光子能量密度 u 表达。(5 分)

(f) It is known that the photon energy density is given by $u = aT^4$, where a is determined from fundamental constants. Show that $\frac{P_r}{P_g} \propto M^c$. What is the value of c? (2 marks)
已知光子的能量密度为 $u = aT^4$, 其中 a 由基本常数决定。证明 $\frac{P_r}{P_g} \propto M^c$, 并求 c 的值。(2 分)

(g) Calculate the ratio $\frac{P_r}{P_g}$ for the Sun. You may use the following parameters: (1 mark)
计算太阳的 $\frac{P_r}{P_g}$。你可以使用以下参数: (1 分)
$a = 7.565 \times 10^{-16}$ JK^{-4}m^{-3}, $G = 6.673 \times 10^{-11}$ Nm^{2}kg^{-2}, $\bar{m} = 8.368 \times 10^{-28}$ kg, $M_{Sun} = 1.989 \times 10^{30}$ kg, $k_B = 1.381 \times 10^{-23}$ JK^{-1}.

(h) For stars more massive than the Sun, the radiation pressure becomes increasingly significant and the star becomes unstable. This implies that there is an upper limit on the mass of stable stars. Suppose the radiation pressure becomes equal to 1/3 the gas pressure at this limit. Calculate the temperature in terms of a, k_B, $\bar{m}$, M and R. (2 marks)
对于比太阳重的星体, 辐射压强越变得重要, 星体也变得越不稳定。这意味着稳定的星体有一个质量上限。假设在这个上限时辐射压强等于 1/3 气体压强。试计算此上限的温度，答案以 a、k_B,、$\bar{m}$、M 和 R.表达。 (2 分)

(i) Using the virial theorem in part (d), find this upper limit of stellar mass. Express your answer in units of solar mass. (3 marks)
应用(d)部中的维里定理, 求星体质量的上限。答案以太阳质量为单位。(3 分)

《THE END 完》

Pan Pearl River Delta Physics Olympiad 2016
2016 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-2 (Total 3 Problems) 简体版卷-2（共3题）
2:00 pm – 5:00 pm, 18 February, 2016

1. Gravitational Lens Simulator (9 marks) 引力透镜仿真镜 (9 分)

The gravitational field of a massive body exercises a lenslike condensing action upon radiation passing through it. A simulated gravitational lens was constructed of Plexiglas for use in demonstrating the lens phenomenon.
大质量物体附近的引力场会对经过的光线产生类似透镜的作用, 称为引力透镜效应。 这里我们考虑以有机玻璃制造一个光学透镜以仿真引力透镜现象。
According to general relativity, a light ray passing at a distance of closest approach r to the center of a spherically symmetric body of mass M, will be deflected toward the body through an angle which, for small deflections, is given by
根据广义相对论, 当一束光线经过一个拥有球对称、质量为M的物体时,会产生屈折。当屈折角度很小，而光线与物体中心最靠近距离为 r 时,该角度可由下面的公式得出

$$\varepsilon = \frac{4GM}{rc^2}$$

where G is the gravitational constant and c is the speed of light. We, therefore, require of our simulator that it deflects transmitted light through an angle
在上式中G为引力常数， c为光速。 因此我们要求仿真镜以下式中的角度屈折光线

$$\varepsilon = \frac{R}{r}$$

where R is a constant. 在上式中R是某个常数。
The lens, illustrated in cross section in Fig. 1, is designed to be hand held within the range between roughly one foot and arm's length from the observer. The object for viewing is assumed to be at a distance much larger than one meter on the left hand side of the lens. This implies that one can assume the incident light ray to be normal to the plane front surface of the lens and refraction is thus assumed to take place entirely at the back surface. The angle of incidence of the light ray at the back surface is designated by θ, the angle of refraction by θ', and the angle of deflection by ε, which are all assumed to be small angles. The refractive index of the lens is n.
图一所示为仿真镜的横截面。仿真镜为观察者手举而设计，设计距离为大约一呎到手臂长度。观察对象位于图中仿真镜左边远大于一米处。因此我们可以假设入射光是垂直于仿真镜的前平面, 而折射仅发生于仿真镜的后表面。 以θ 表示入射光与后表面法线的夹角, θ' 表示折射角度，ε表示屈折角度 。以上角度皆假设为小角度。仿真镜的折射系数是 n。

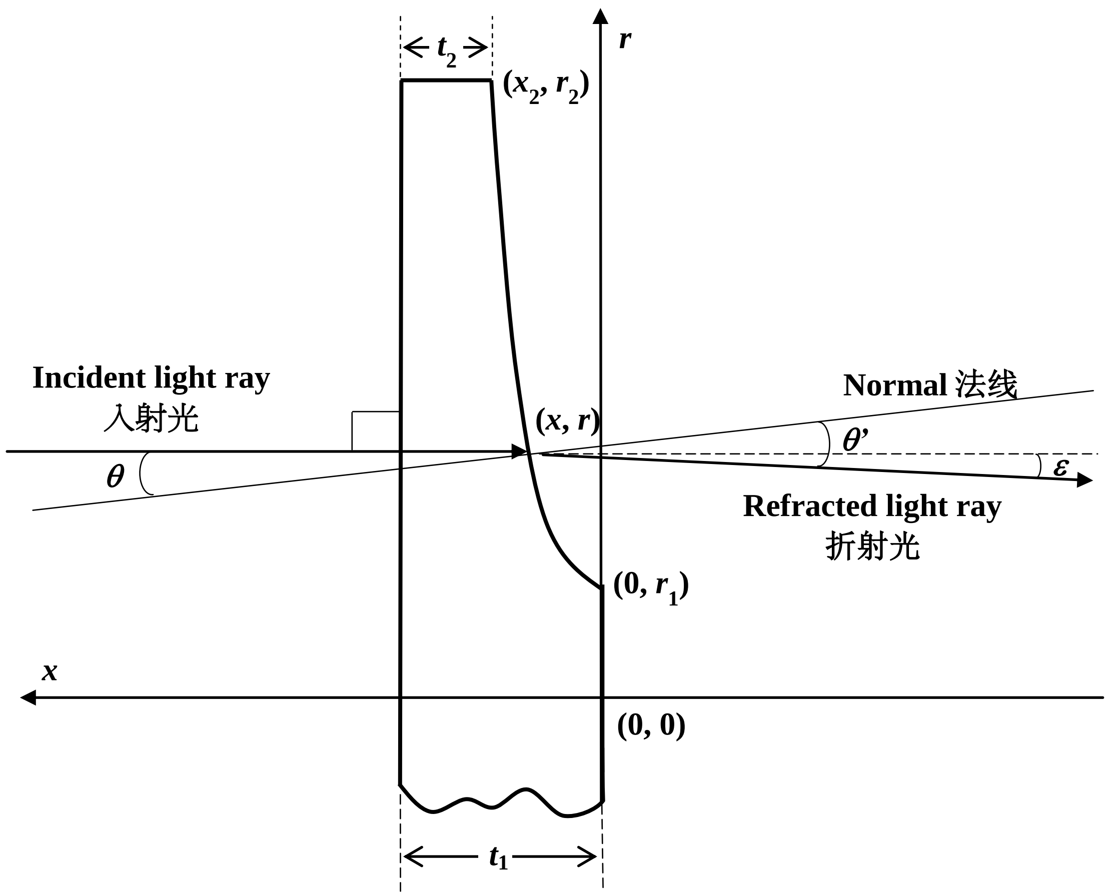


Figure 1 图一

(a) Derive the expression dx/dr for the slope of the back refracting surface of the lens in terms of ε and n. (4 marks)
试推导透镜后折射面的斜率 dx/dr，答案以ε和 n 表达。(4 分)

Using Snell's law, $n\sin\theta = \sin\theta'$
By small angle assumption, $n\theta = \theta'$
The angle of deflection is thus $\varepsilon = \theta' - \theta = (n-1)\theta$
We also have $\tan(-\theta) = -\frac{1}{dr/dx} \Rightarrow \frac{dx}{dr} = \tan\theta$
By small angle assumption $\frac{dx}{dr} = \theta = \frac{\varepsilon}{n-1}$

(b) Derive the expression $x(r)$ of the back refracting surface of the lens. Express your answer in terms of n, R and r_1. (3 marks)
试推导透镜后折射面的表达式 $x(r)$。答案以 n、R 和 r_1 表达。(3 分)

Using $\varepsilon = \frac{R}{r}$, we have $\frac{dx}{dr} = \frac{\varepsilon}{n-1} = \frac{R}{(n-1)r} \Rightarrow \ dx = \frac{Rdr}{(n-1)r}$.
Integrating, $x = \int \frac{R}{(n-1)r}dr = \frac{R}{n-1}\ln r + C$
Boundary condition: $0 = \frac{R}{n-1}\ln r_1 + C \Rightarrow C = -\frac{R}{n-1}\ln r_1$

$\Rightarrow x = \frac{R}{n-1}\ln r - \frac{R}{n-1}\ln r_1 \Rightarrow x = \frac{R}{n-1}\ln\frac{r}{r_1}$

(c) A light ray is incident at impact parameter r equal to 2 cm. If we require the ray to cross the lens axis at a horizontal distance of 30 cm from the point at which it departs from the lens, find R. (1 mark)

考虑一束入射参数 r 为 2 cm 的光线。 如果要求该光线在离开透镜后于水平距离 30 cm 处与透镜中轴相交, 则 R 应取何值? (1 分)

$\tan\varepsilon = \frac{2}{30} \Rightarrow \varepsilon = \tan^{-1}\frac{2}{30} = \frac{R}{2} \Rightarrow R = 2\tan^{-1}\frac{2}{30} = 0.1331 \approx 0.133$ cm

Remark: Also accept answer using small angle approximation: $\varepsilon \approx \tan\varepsilon = \frac{2}{30} = \frac{R}{2} \Rightarrow R = \frac{2^2}{30} = 0.1333 \approx 0.133$ cm.

(d) Find the effective gravitational mass of the lens. (1 mark)

计算透镜的有效引力质量。(1 分)

$R = \frac{4GM}{c^2} \Rightarrow M = \frac{Rc^2}{4G} = \frac{(0.001331)(3\times10^8)^2}{(4)(6.67\times10^{-11})} = 4.489\times10^{23} \approx 4.49\times10^{23}$kg

which is about 6 times the mass of the Moon.

Reference: S. Liebes Jr., Am. J. Phys. **37**, 103 (1969).

2. A String and a Mass (19 marks) 系着质量的弦 (19 分)

In this question, all oscillatory motions are assumed to be small.

在本题中，假设所有震荡皆为微小震荡。

As shown in Fig. 2, a string of mass m and length l with tension τ has a mass M attached to the end. The mass M can slide in a vertical direction on a frictionless rod at $x = l$. The shape of the string is described by a function $y(x, t)$. The string is fixed at the origin $y(0, t) = 0$.

如图二所示, 一条质量为 m、长度为 l、张力为τ的弦一端系着质量 M。 该质量 M 可在一根位于 $x = l$ 处的平滑棒上自由垂直滑动。 弦的型态由函数 $y(x, t)$给出。 弦的另一端固定于原点处, 即 $y(0, t) = 0$。

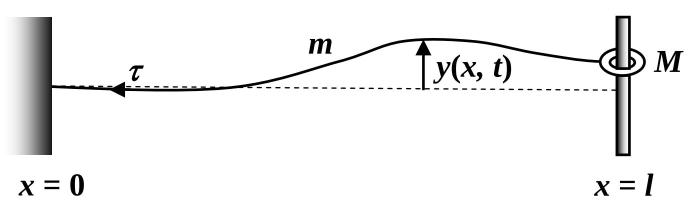


Figure 2 图二

(a) First assume that mass M is held fixed at $y = 0$. Write down the general solution $y(x, t)$ for the standing waves on the string. Express your answer in terms of the given parameters and arbitrary constants. (4 marks)

首先假设质量 M 被固定在 $y = 0$ 处。写下弦上驻波的一般解 $y(x, t)$。答案以给定的参数和任意常数表达。(4 分)

A standing wave has a general form $y(x,t) = (A\sin kx + B\cos kx)\sin(\omega t + \delta)$.

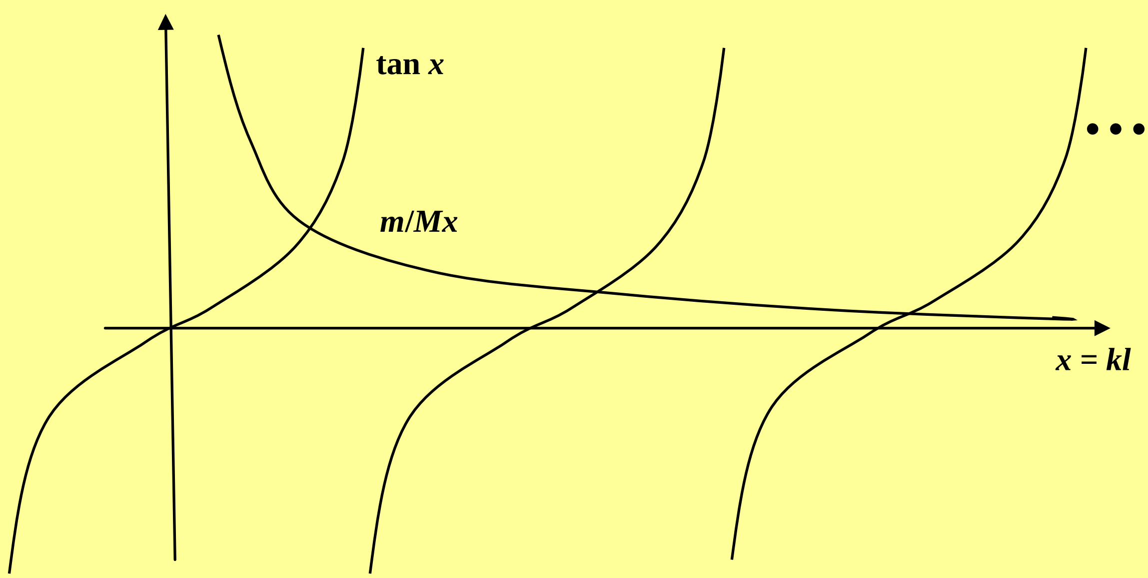
We can always choose the zero point of t so that $\delta = 0$.
Hence $y(x,t) = (A\sin kx + B\cos kx)\sin\omega t$.
The boundary conditions $y(0, t) = 0$ and $y(l, t) = 0$ impose conditions $B = 0$ and $kl = n\pi$ for $n = 1, 2, 3, \ldots$
In general for a wave $\omega = kv$ and for a string $v = \sqrt{\tau/\rho} = \sqrt{l\tau/m}$. General solution:
$y(x,t) = \sum_{n=1}^{\infty} A_n \sin k_n x \sin \omega_n t$, where $k_n = \frac{n\pi}{l}$ and $\omega_n = \sqrt{\frac{l\tau}{m}} k_n$.

(b) Now assume that mass M can slide up and down on a frictionless rod at $x = l$. What is the boundary condition on $y(x, t)$ at $x = l$? You can assume that the oscillations are small. (1 mark)
现在假设质量 M 能沿着 $x = l$ 处的平滑棒上下移动，则 $y(x, t)$在 $x = l$ 处的边界条件该是甚么? 你可以假设震荡都是小震荡。(1 分)

The force needed to accelerate the mass is given by the tension in the string. Using Newton's law,
$$M\frac{\partial^2 y(l,t)}{\partial t^2} = -\tau \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=l}$$

(c) Write down an equation for the frequencies of the standing waves on the string when the mass M is free to slide. You do not need to solve the equation. (2 marks)
当质量 M 可自由滑动时, 写下满足弦驻波频率的方程式。你不需要解该方程式。(2 分)

$$-M\omega^2 A\sin kl\sin\omega t = -\tau kA\cos kl\sin\omega t \Rightarrow Mk^2\frac{l\tau}{m}\sin kl = \tau k\cos kl \Rightarrow M\frac{kl}{m}\sin kl = \cos kl$$
$$kl\tan kl = \frac{m}{M}$$
There are infinitely many normal modes.



(d) If $m << M$, find the two lowest normal mode frequencies. (2 marks)
如果 $m << M$, 找出最低频的两个正则模频率。(2 分)

First normal mode:
Since kl <<1, $\tan kl \approx kl$ and we have

$k \approx \frac{1}{l}\sqrt{\frac{m}{M}} \Rightarrow \omega = vk \approx \sqrt{\frac{\tau l}{m}}\frac{1}{l}\sqrt{\frac{m}{M}} = \sqrt{\frac{\tau}{Ml}}.$

Second normal mode:

$k \approx \frac{\pi}{l} \Rightarrow \omega = vk \approx \sqrt{\frac{\tau l}{m}}\frac{\pi}{l} = \pi\sqrt{\frac{\tau}{ml}}.$

(e) For $m << M$, calculate the ratio of the kinetic energy of the string to that of the mass M at the lowest normal mode frequency. (2 marks)

如果 $m << M$，试计算在最低频的正则模中，弦的动能与质量 M 的动能的比例。(2 分)

Let A be the amplitude of oscillation of the mass M.

At the first normal mode, the amplitude of oscillation at position x of the string is Ax/l.

Hence the ratio of the kinetic energies is $R = \frac{\int\left(\frac{Ax}{l}\right)^2 dm}{MA^2} = \frac{\int x^2 dm}{Ml^2}.$

Note that the numerator is the moment of inertia of a rod about one end.

Hence $R = \frac{ml^2/3}{Ml^2} = \frac{m}{3M}.$

(f) If $m >> M$, find the two lowest normal mode frequency. (2 marks)

如果 $m >> M$, 找出最低頻的兩個正則模頻率。(2 分)

First normal mode:

$kl\tan kl = \frac{m}{M} \gg 1 \Rightarrow kl \approx \frac{\pi}{2} \Rightarrow \omega = vk \approx \sqrt{\frac{\tau l}{m}}\frac{\pi}{2l} = \frac{\pi}{2}\sqrt{\frac{\tau}{ml}}.$

Second normal mode: $kl \approx \frac{3\pi}{2} \Rightarrow \omega = vk \approx \sqrt{\frac{\tau l}{m}}\frac{3\pi}{2l} = \frac{3\pi}{2}\sqrt{\frac{\tau}{ml}}.$

(g) For $m >> M$, calculate the ratio of the kinetic energy of the string to that of the mass M at the lowest normal mode frequency. (2 marks)

如果 $m >> M$，试计算在最低频的正则模中，弦的动能与质量 M 的动能的比例。(2 分)

Let A be the amplitude of oscillation of the mass M.

At the first normal mode, the amplitude of oscillation at position x of the string is $A\sin(\pi x/2l)$.

Hence the ratio of the kinetic energies is $R = \frac{\int\left(A\sin\frac{\pi x}{2l}\right)^2 dm}{MA^2} = \frac{\langle\sin^2\frac{\pi x}{2l}\rangle m}{M} = \frac{m}{2M}.$

(h) A traveling wave of angular frequency ω is generated near the end $x = l$. It propagates towards the mass M and is reflected with a phase shift of $\pi/2$. What is the value of ω in terms of τ, m, M and l? (4 marks)

在靠近 $x = l$ 处生成一角频率为ω的行波。该行波朝质量 M 方向传播，并以$\pi/2$ 的相移被反射。 求ω，答案以τ、m、M 和 l 表达。(4 分)

The wave can be written as $y(x,t) = A\sin(k(x-l)-\omega t) + rA\sin\left(k(x-l)+\omega t+\frac{\pi}{2}\right)$

$= A\sin(k(x-l)-\omega t) + rA\cos(k(x-l)+\omega t).$

Substituting into the boundary condition,

$$M\left.\frac{\partial^2 y}{\partial t^2}\right|_{x=l} = M\omega^2 A[\sin(\omega t) - r\cos(\omega t)],$$

$$-\tau\left.\frac{\partial y}{\partial x}\right|_{x=l} = \tau k A[-\cos(\omega t) + r\sin(\omega t)].$$

For the boundary condition to satisfy for all t, we should have $M\omega^2 = \tau kr$ and $M\omega^2 r = \tau k \Rightarrow r = \frac{M\omega^2}{\tau k} = \frac{\tau k}{M\omega^2} \Rightarrow M\omega^2 = \tau k$ and $r = 1$. Hence

$M\omega^2 = \frac{\tau\omega}{v} \Rightarrow \omega = \frac{\tau}{Mv} = \frac{1}{M}\sqrt{\frac{m\tau}{l}}$.

3. Maximum Mass of a Star (22 marks) 星体的最大质量 (22 分)

Consider a star of mass M and radius R. Assume that its density is uniform.
考虑一质量为 M、半径为 R 的星体。 假设其质量密度均匀。

(a) Its gravitational potential energy U can be calculated by considering the work done in bringing a thin layer of materials and depositing on the surface of a spherical protostar of radius r when r gradually grows from 0 to R. Calculate U. Express your answer in terms of G, M and R, where G is the gravitational constant. (4 marks)
要计算星体的引力势能 U，可考虑逐层将星体物质加至半径为 r 的球状准星体表面所作的功，并让 r 由 0 逐渐增加到 R。计算 U，答案以引力常数 G、M 和 R 表达。(4 分)

Work done in depositing a layer of thickness dr on the surface of a protostar is the gravitational potential energy change of a layer of volume $4\pi r^2 dr$ brought in from infinity to distance r

$dU = -\frac{Gm(r)}{r}\rho 4\pi r^2 dr$, where $\rho = \frac{3M}{4\pi R^3}$ is the density and $m(r) = M\left(\frac{r}{R}\right)^3$ is the mass of the protostar of radius r. Hence

$$U = -\int_0^R \frac{GM}{r}\left(\frac{r}{R}\right)^3 \rho 4\pi r^2 dr = -\frac{4\pi\rho GM}{R^3}\int_0^R r^4 dr = -\frac{4\pi\rho GMR^2}{5} = -\frac{3GM^2}{5R}.$$

(b) Assume that the star is made up of protons and electrons, both behaving as ideal gases. It is known that during the formation of the star, half of the loss in gravitational potential energy is converted to thermal energy, while the other half is radiated away. Derive the temperature T of the star in terms of G, M, R, $\overline{m}$ and k_B, where $\overline{m}$ is the average mass of protons and electrons, and k_B is the Boltzmann constant. (2 marks)
假设星体由质子和电子组成，其行为皆为理想气体。已知当星体形成时，其引力势能的耗损一半会转化为热能，另一半会被辐射掉。试推导星体温度 T，答案以 G、M、R、$\overline{m}$和 k_B 表达。 这里$\overline{m}$为质子与电子的平均质量，k_B 为玻耳兹曼常数。(2 分)

Using the conservation of energy, $\frac{3}{2}Nk_BT = \frac{1}{2}\left(\frac{3GM^2}{5R}\right)$ where $N = \frac{M}{\overline{m}}$ is the number of protons and electrons. $\Rightarrow T = \frac{GM\overline{m}}{5k_BR}$.

(c) Derive the gas pressure P_g of the star in terms of G, M and R. (2 marks)
试推导星体的气体压力 P_g，答案以 G、M 和 R 表达。(2 分)

$$P_g = \frac{Nk_BT}{V} = \frac{GM^2}{5R}\frac{3}{4\pi R^3} = \frac{3GM^2}{20\pi R^4}.$$

(d) The virial theorem states that the total pressure in a star is related to the gravitational potential energy by $P = -b\dfrac{U}{V}$. What is the value of b? (1 mark)

根据维里定理, 星体内的总压强与引力势能的关系为 $P = -b\dfrac{U}{V}$。求 b 的值。(1 分)

Since $\dfrac{3}{2}Nk_B T = \dfrac{1}{2}\left(\dfrac{3GM^2}{5R}\right)$, we have $\dfrac{3}{2}PV = \dfrac{1}{2}(-U) \Rightarrow b = -\dfrac{PV}{U} = \dfrac{1}{3}$.

(e) At high temperature, photons in the star also contribute to the pressure. Derive the radiation pressure P_r by applying the kinetic theory of gases in a cubic box of volume L^3, in which the momenta of the photons are described by the de Broglie relation. Express your result in terms of the photon energy density u. (5 marks)

在高温下，星体中的光子也会施加压强。试应用气体运动论，考虑一体积为 L^3 的立方盒子中的光子，其中光子的动量满足德布罗意关系式，从而得出辐射压强 P_r。答案以光子能量密度 u 表达。(5 分)

Consider a box of width L. Let p_x be the momentum of a photon along the x direction.
Change in momentum of the photon when it hits the wall = $-2p_x$.
Time interval between two successive hits on the wall $= \dfrac{2L}{c\cos\theta_x}$, where θ_x is the angle between the photon momentum and the x axis.
Using Newton's second law, the force on the wall is the rate of change of momenta of the photons when they hit the wall $F = N\left\langle 2p_x \dfrac{c\cos\theta_x}{2L}\right\rangle = N\left\langle \dfrac{pc\cos^2\theta_x}{L}\right\rangle$, where N is the number of photons, and $\langle\ \rangle$ represents the average over the direction and magnitude of the photon momenta.
Note that for isotropic distributions, $\langle \cos^2\theta_x \rangle = \dfrac{1}{3}$. Pressure: $P_r = \dfrac{F}{L^2} = \dfrac{N}{3L^3}\langle pc \rangle$.
Using de Broglie relation, $pc = \dfrac{hc}{\lambda} = hf = \varepsilon$, where ε is the photon energy. Hence $N\langle pc\rangle$ is the total photon energy. Hence $P_r = \dfrac{E}{3V} = \dfrac{u}{3}$.

(f) It is known that the photon energy density is given by $u = aT^4$, where a is determined from fundamental constants. Show that $\dfrac{P_r}{P_g} \propto M^c$. What is the value of c? (2 marks)

已知光子的能量密度为 $u = aT^4$, 其中 a 由基本常数决定。证明 $\dfrac{P_r}{P_g} \propto M^c$, 并求 c 的值。(2 分)

$$P_r = \frac{a}{3}T^4 = \frac{a}{3}\left(\frac{GM\bar{m}}{5Rk_B}\right)^4 = \frac{aG^4M^4\bar{m}^4}{1875R^4k_B^4}.$$

$$\frac{P_r}{P_g}=\left(\frac{aG^4M^4\overline{m}^4}{1875R^4k_B^4}\right)\left(\frac{20\pi R^4}{3GM^2}\right)=\frac{4\pi aG^3m_H^4}{1125k_B^4}M^2.\text{ Hence }\frac{P_r}{P_g}\propto M^2\text{ and }c=2.$$

(g) Calculate the ratio $\frac{P_r}{P_g}$ for the Sun. You may use the following parameters: (1 mark)

计算太阳的$\frac{P_r}{P_g}$。你可以使用以下参数: (1 分)

a = 7.565 × 10^{-16} $JK^{-4}m^{-3}$, G = 6.673 × 10^{-11} Nm^2kg^{-2}, $\overline{m}$ = 8.368 × 10^{-28} kg, M_{Sun} = 1.989 × 10^{30} kg, k_B = 1.381 × 10^{-23} JK^{-1}.

$$\frac{P_r}{P_g}=\frac{4\pi(7.565\times10^{-16})(6.673\times10^{-11})^3(8.368\times10^{-28})^4}{1125(1.381\times10^{-23})^4}(1.989\times10^{30})^2=1.34\times10^{-4}$$

(h) For stars more massive than the Sun, the radiation pressure becomes increasingly significant and the star becomes unstable. This implies that there is an upper limit on the mass of stable stars. Suppose the radiation pressure becomes equal to 1/3 the gas pressure at this limit. Calculate the temperature in terms of a, k_B, $\overline{m}$, M and R. (2 marks)

对于比太阳重的星体，辐射压强越变得重要，星体也变得越不稳定。这意味着稳定的星体有一个质量上限。假设在这个上限时辐射压强等于 1/3 气体压强。试计算此上限的温度，答案以 a、k_B,、$\overline{m}$、M 和 R.表达。 (2 分)

When $P_r=\frac{1}{3}P_g$, $\frac{a}{3}T^4=\frac{Nk_BT}{3V}\Rightarrow T=\left(\frac{Nk_B}{aV}\right)^{\frac{1}{3}}=\left(\frac{3Mk_B}{4\pi a\overline{m}R^3}\right)^{\frac{1}{3}}$

(i) Using the virial theorem in part (d), find this upper limit of stellar mass. Express your answer in units of solar mass. (3 marks)

应用(d)部中的维里定理, 求星体质量的上限。答案以太阳质量为单位。(3 分)

$$P_g+P_r=-\frac{U}{3V}=-\frac{1}{3V}\left(-\frac{3GM^2}{5R}\right)=\frac{GM^2}{5RV}$$

Since $P_r=\frac{1}{3}P_g$, $\frac{4}{3}P_g=\frac{4Nk_BT}{3V}=\frac{GM^2}{5RV}\Rightarrow T=\frac{3GM^2}{20Nk_BR}=\frac{3GM\overline{m}}{20k_BR}$.

Combining with the result of part (g),

$$\frac{3GM\overline{m}}{20k_BR}=\left(\frac{3Mk_B}{4\pi a\overline{m}R^3}\right)^{\frac{1}{3}}\Rightarrow$$

$$M=\left(\frac{2000k_B^4}{9\pi aG^3\overline{m}^4}\right)^{\frac{1}{2}}=\left[\frac{(2000)(1.381\times10^{-23})^4}{9\pi(7.565\times10^{-16})(6.673\times10^{-11})^3(8.368\times10^{-28})^4}\right]^{\frac{1}{2}}=1.528\times10^{32}\text{ kg}$$

= 76.8 M_{Sun}

Remark: Although this estimate is based on the assumption of uniform stellar density, it agrees with the observation that stars with masses greater than 50 solar masses are rare.

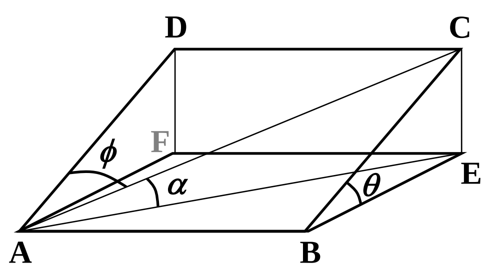
《THE END 完》

Pan Pearl River Delta Physics Olympiad 2017
2017 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 7 Problems, 45 Points) 简体版卷-1（共7题，45分）
(9:00 am – 12:00 pm, 3 February, 2017)

Please fill in your final answers to all problems on the **answer sheet**. 请在答题纸上填上各题的最后答案。 At the end of the competition, please submit the **answer sheet only**. Question papers and working sheets will **not** be collected. 比赛结束时，请只交回答题纸，题目纸和草稿纸将不会收回。

1. No-Shadow Day (6 points) 立竿无影（5 分）

(a) In the figure, ABCD is a rectangle lying on an inclined plane making an angle θ with the horizontal plane. ABEF is the projection of the rectangle on the horizontal plane. If the measure of the angle DAC is ϕ, derive an expression for the angle α. [1]
如图所示，矩形 ABCD 位于斜面上，斜面与水平面夹角为θ。ABEF 为该矩形于水平面的投影。设角 DAC 为ϕ，试推导角α的表达式。[1]



(b) The ecliptic is the plane on which the Earth revolves around the Sun. The axis of rotation of the Earth is inclined at an angle of 23.4^o with the normal to the ecliptic. The day of the Summer Solstice (in the Northern Hemisphere) is 21 June. The latitude of Hong Kong is 22.25^o, and the no-shadow days are those days on which the Sun does not cast a shadow of a vertical pole at noon in Hong Kong. Using the result of (a) or otherwise, derive the angular displacement of the Earth's revolution between the Summer Solstice and the no-shadow days in Hong Kong. Give your answer to 3 significant figures. [2]
黄道面是指地球围绕太阳公转的平面。地球的自转轴相对于黄道面的法线倾斜，角度为 23.4^o。在北半球，夏至的日期为 6 月 21 日。香港位于北纬 22.25^o，而当某日正午的太阳照在一支立于香港的垂直竿子时是没有影子的，那日就是香港的无影日了。试用(a)部结果或其他方法，推导在夏至和香港的无影日之间，地球公转的角位移。答案请给三位有效数字。[2]

(c) Write the dates of the no-shadow days in Hong Kong. [2]
试写下香港无影日的日期。[2]

2. Six Missiles (5 points) 六枚飞弹 （5 分）

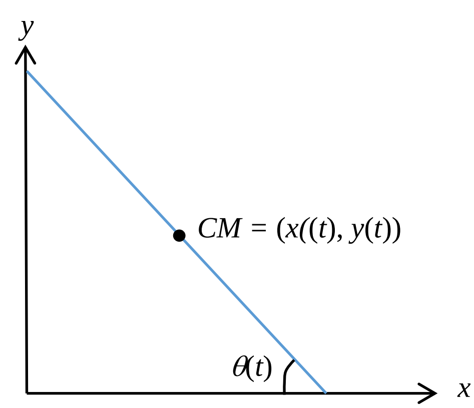
Six missiles are initially located at the six vertex of a regular hexagon with side length a. The speed of the missiles in the plane is v. Each missile is equipped with an automatic navigation system. The automatic navigation system of each missile guides itself to aim at the current position of its counterclockwise neighbor.
今有飞弹六枚, 分别位于一边长为 a 的正六边形的六个角上。 每枚飞弹都装置有自动导航系统。 该系统会指示飞弹永远以速率 v 飞向其逆时针方向之近邻。

(a) Find the radial component of the missile velocities relative to the center of the hexagon. [2]
找出飞弹指向六角形中心的径向速率。[2]

(b) Find the time taken for a missile to hit another. [3]
找出一枚飞弹击中另一枚所需的时间。[3]

3. Falling ladder (15 points) 下跌中的梯子（10 分）

A ladder of length $2l$ and mass m is standing up against a vertical wall with initial angle α relative to the horizontal. There is no friction between the ladder and the wall or the floor. The ladder begins to slide down with zero initial velocity. Denote $\theta(t)$ as the angle the ladder makes with the horizontal after it starts to slide and $(x(t), y(t))$ be the coordinate of the center of mass of the ladder. In this problem, you should take the gravitational potential energy to be zero at $y = 0$.
一个长度为$2l$、质量为 m 的梯子靠着一道垂直的墙，并与水平形成初始夹角 α。梯子与墙身和地面并没有摩擦力。梯子从零初始速度开始下滑。$\theta(t)$表示为梯子下滑期间与水平形成的夹角, $(x(t), y(t))$ 表示为梯子质心的坐标。在本题中，你应将在 $y = 0$ 处的引力势能取值为零。



(a) What is the initial total mechanical energy of the ladder in terms of α? [1]
梯子的初始总机械能是甚么？答案以α表示。[1]

(b) Write the potential energy of the ladder in terms of $\theta(t)$ when it is sliding. [1]
请用$\theta(t)$写下梯子下滑时的势能。[1]

(c) Write the total kinetic energy of the ladder in terms of $\dot{x}(t), \dot{y}(t), \dot{\theta}(t)$ when it is sliding. (Hint: The moment of inertia of a rod of length $2l$ and mass m about an axis through the center of mass and perpendicular to its length is $I = ml^2/3$.) [1]

请用$\dot{x}(t), \dot{y}(t), \dot{\theta}(t)$写下梯子下滑时的总动能。 (提示: 一条长度为$2l$、质量为 m 的杆子，相对於通过杆子质心并垂直于杆子的转动轴，其轉动惯量为$I = ml^2/3$。) [1]

(d) As long as the ladder is in contact with the wall, find the relation between $x(t)$ and $\theta(t)$, and similarly the relation between $y(t)$ and $\theta(t)$. [2]
当梯子靠着墙的时候，请找出$x(t)$和$\theta(t)$的关系式，而同样地，找出$y(t)$和$\theta(t)$的关系式。[2]

(e) Write the total mechanical energy of the ladder in terms of $\theta(t)$ and $\dot{\theta}(t)$ only by eliminating any dependence on $x(t)$ and $y(t)$. [2]
在消去$x(t)$和 $y(t)$后，只用$\theta(t)$ 和 $\dot{\theta}(t)$写下梯子的总机械能。[2]

(f) Derive the relation between $\theta(t)$ and $\ddot{\theta}(t)$. [1]
试推导$\theta(t)$和$\ddot{\theta}(t)$的关系式。[1]

(g) Find the angle θ_c when the ladder loses contact with the vertical wall. [2]
找出当梯子和墙身失去接触时的角度 θ_c。[2]

4. Photon Gas (5 points) 光子气体（5 分）

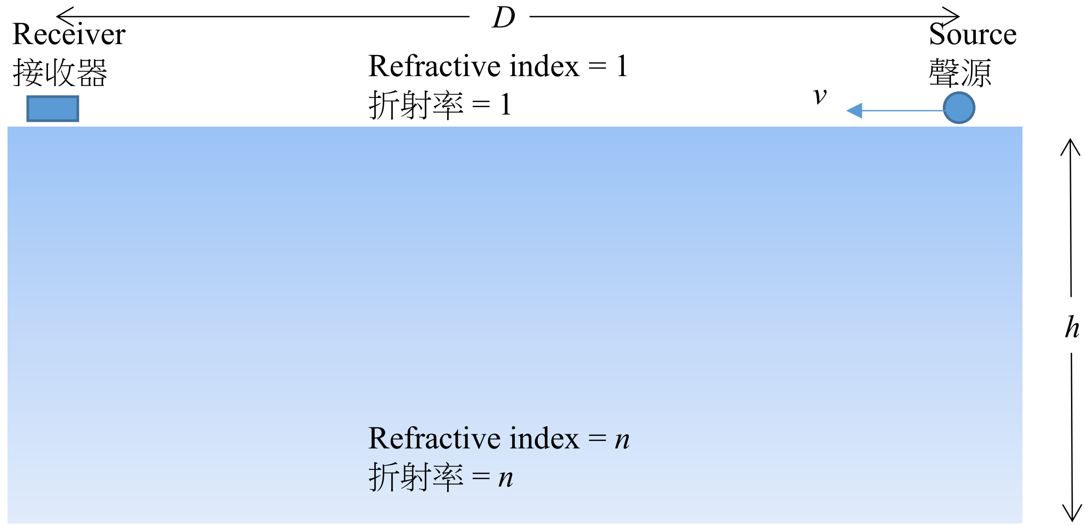
The kinetic theory is very useful in understanding the properties of gases. In this problem we apply the theory to a gas of N photons inside a cubic box with side length L. The energy-momentum relation of a photon is given by $E = |\mathbf{p}|c$.
分子运动理论于理解气体性质时非常有用。 在本题中我们把这理论应用於一长度为 L 的立方盒子中的 N 粒光子。 光子的能量-动量关系为 $E = |\mathbf{p}|c$。

(a) Express the time taken between two consecutive collisions of the same wall of the box normal to the x direction in terms of L, p, p_x and c. [2]
找出一光子连续两次撞击同一面向 x 方向的盒壁之间的时间间距。答案以 L, p, p_x 和 c 表达。[2]

(b) Express the internal energy U of the photon gas in terms of its pressure P and volume V. [3]
找出光子气体的内能 U。 答案以气体的压力 P 和体积 V 表达。[3]

5. Sea Surface Sound Transmission (5 points) 海面传音（5 分）

In a region where the ocean has a constant depth of h, a sound source emits sound wave with frequency f. Suppose the frequency is so high that sound waves can be treated as rays, and the refractive index of water for sound wave is n. Let the sound speed in air be c. The source is moving with constant horizontal speed $v < c$ towards a stationary receiver at distance D, both located just above the ocean surface, as shown in the figure. It is also assumed that the speed is low so that $\frac{D}{\tau} \gg v$, where $\tau \gg \frac{1}{f}$ is the observation time. Assume we can ignore reflection by the ocean surface and consider only reflection by the ocean floor.
在一处海床深度为常数 h 的海面上, 一声源发出频率为 f 的声波。假设该频率足够高使得声波可以被视为声线束, 而海水对声线束的折射率为 n。设空气中的声速为 c。如图中所示, 声源与一静止接收器皆位于海面上, 两者距离为 D。声源以均匀速率 $v < c$ 向接收器移动。假设该速率足够慢, 使得$\frac{D}{\tau} \gg v$，其中$\tau \gg \frac{1}{f}$为观察时间。假设海面的反射可以忽略, 只需考虑海床的反射。



Express your answers from (a) to (c) in terms of c, f, n and v.
在(a)至(c)中, 答案以 c, f, n, 和 v 表达。

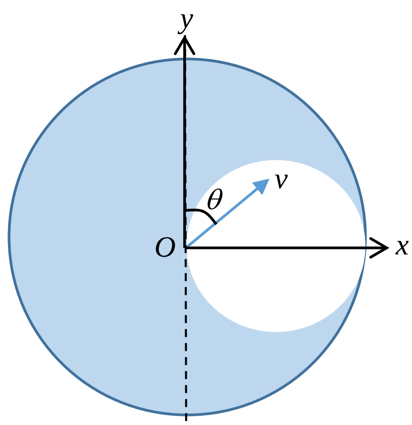
(a) Find the frequency of the sound arriving at the receiver through air. [2]
找出声音经过空气到达接收器的频率。 [2]

(b) Find the frequency of the sound that will arrive at the receiver via the ocean when $D = 2h$. [2]
找出当$D = 2h$时发出的声音经过海洋到达接收器的频率。[2]

(c) Find the instantaneous beat frequency when the sound waves in (b) reach the receiver. [1]
找出(b)中的声波到达接收器时的瞬时拍频频率。[1]

6. Electron Trajectory in a Cavity (10 points) 空腔中的电子轨迹（10 分）

As shown in the figure below, a spherical cavity is carved out from a uniformly positively charged sphere with radius R. The radius of the cavity is $R/2$, located at a distance $R/2$ from the center of the large sphere, O. The total positive charge in the system is Q.
如下图所示，在一个带均匀正电、半径为 R 的球体内挖出一半径为 $R/2$ 的球形空腔。空腔中心与大球中心 O 距离为 $R/2$。系统的总正电荷为 Q。



(a) Consider a point in the cavity at distance r and polar angle θ from the origin. Calculate the x and y components of the electric field at the point. [5]
考虑空腔内一点, 其与原点距离为 r, 极角为θ。计算该点电场的 x 分量和 y 分量。[5]

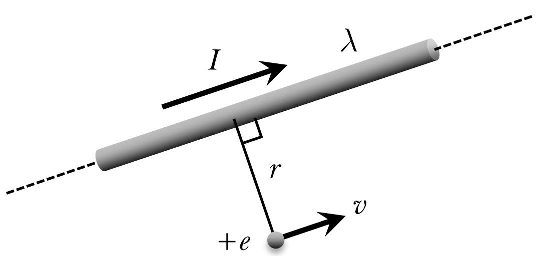
(b) As shown in the figure, electrons are emitted from O in all directions with speed v and direction θ ranging between 0 and π, but none of them can reach the opposite end of the diameter of the cavity. Gravitational forces are negligible. Find the equation of the envelope of all trajectories of the electrons. [3]
如图所示，现考虑许多电子由 O 以同样速率 v 向 0 至π的各方向θ射出。但其动能不足以到达空腔的直径对点。忽略万有引力。求所有可能的电子轨迹的包络线方程。[3]

(c) Find the maximum x coordinate where the electrons hit the inner surface of the cavity. Express your answer in terms of Q, the absolute value of the electronic charge e, electron mass m, R, and v. [2]
找出电子打中空腔内表面处最大可能的 x 坐标。答案以 Q、电子电荷绝对值 e、电子质量 m、R 和 v 表达。[2]

7. Proton Motion Near a Charged Current-Carrying Wire (5 points)
带电荷电流线邻近的质子运动（5 分）

As shown in the figure below, a proton of charge $+e$ moves with velocity v parallel to an infinitely-long, thin wire, at a distance r from the axis of the wire. The wire carries a current I, and its charge per unit length is λ (assumed positive and uniform). Both the proton and the wire are in vacuum.
如下图所示，一个带电荷+e 的质子在一条无限长的幼导线附近运动，速度为 v，方向与导线平行。质子与导线轴的距离为 r。导线载着电流 I，且每单位长度电荷为λ(假定为正及均匀)。质子与导线都处于真空中。

Express the answers from (a) to (c) in terms of r, I, λ, the permittivity of vacuum ε_0, the permeability of vacuum μ_0, the speed of light c, and the unit vectors in cylindrical coordinates $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$.
在(a)至(c)中，答案以 r、I、λ、真空电容率 ε_0、真空磁导率 μ_0、光速 c 和柱坐标的单位向量 $\mathbf{e}_r$、$\mathbf{e}_\theta$及 $\mathbf{e}_z$ 表示。



(a) Find the electric field experienced by the proton. [2]
求质子所感受到的电场 **E**。[2]

(b) Find the magnetic field experienced by the proton. [2]
求质子所感受到的磁场 **B**。[2]

(c) Find the speed of the proton such that it moves in a straight line parallel to the wire. [1]
求质子的速率，使其沿直线平行于导线运动。[1]

Pan Pearl River Delta Physics Olympiad 2017
2017 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
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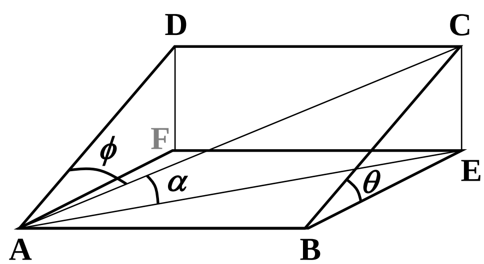
(9:00 am – 12:00 pm, 3 February, 2017)

1. No-Shadow Day (5 points) 立竿无影（5 分）

(a) In the figure, ABCD is a rectangle lying on an inclined plane making an angle θ with the horizontal plane. ABEF is the projection of the rectangle on the horizontal plane. If the measure of the angle DAC is ϕ, derive an expression for the angle α. [1]
如图所示，矩形 ABCD 位于斜面上，斜面与水平面夹角为θ。ABEF 为该矩形于水平面的投影。设角 DAC 为ϕ，试推导角α的表达式。[1]

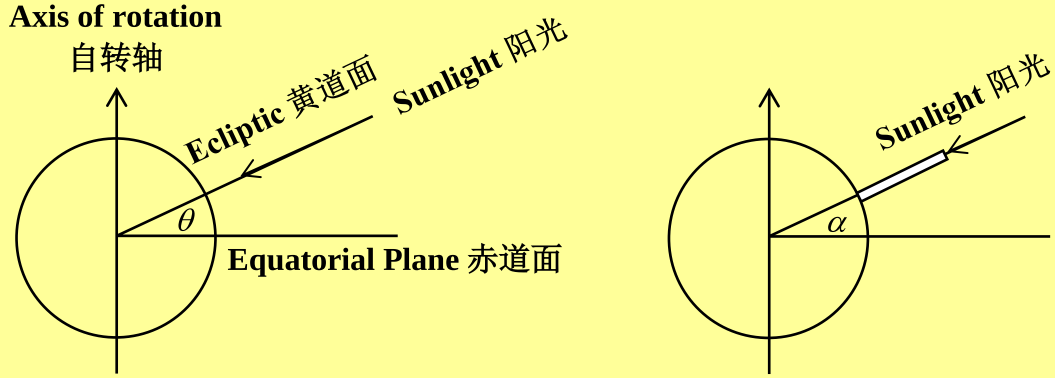


Let h = AC. Then
$\mathrm{CE} = h \sin \alpha$.
$\mathrm{BC} = \mathrm{AD} = h \cos \phi$.
$\mathrm{CE} = \mathrm{BC} \sin \theta \ = h \cos \phi \sin \theta$.
Equating the expressions of CE, $h \sin \alpha = h \cos \phi \sin \theta \ \Rightarrow \alpha = \arcsin(\cos \phi \sin \theta)$.

(b) The ecliptic is the plane on which the Earth revolves around the Sun. The axis of rotation of the Earth is inclined at an angle of 23.4° with the normal to the ecliptic. The day of the Summer Solstice (in the Northern Hemisphere) is 21 June. The latitude of Hong Kong is 22.25°, and the no-shadow days are those days on which the Sun does not cast a shadow of a vertical pole at noon in Hong Kong. Using the result of (a) or otherwise, derive the angular displacement of the Earth's revolution between the Summer Solstice and the no-shadow days in Hong Kong. Give your answer to 3 significant figures. [2]
黄道面是指地球围绕太阳公转的平面。地球的自转轴相对于黄道面的法线倾斜，角度为 23.4°。在北半球，夏至的日期为 6 月 21 日。香港位于北纬 22.25°，而当某日正午的太阳照在一支立于香港的垂直竿子时是没有影子的，那日就是香港的无影日了。试用(a)部结果或其他方法，推导在夏至和香港的无影日之间，地球公转的角位移。答案请给三位有效数字。[2]

In the figure above, consider ABEF to be the equatorial plane of the Earth, and ABCD the ecliptic. Then $\theta = 23.4^{\circ}$. When the Earth revolves around the Sun, sunlight is incident on the Earth from different directions lying on the plane ABCD. For example, on 21 June, sunlight is incident on the Earth in the direction DA, since this is the northernmost direction of sunlight.

Similarly, during Spring Equinox and Autumn Equinox, sunlight is incident on the Earth in the direction AB or BA.



Summer Solstice 夏至

No Shadow Day in Hong Kong 香港无影日

Identifying $\theta = 23.4^0$ and when the Sun does not cast a shadow of the vertical pole at noon in Hong Kong, $\alpha = 22.25^o$.

Hence the angle ϕ is given by

$\cos\phi = \frac{\sin\alpha}{\sin\theta} = \frac{\sin 22.25^0}{\sin 23.4^0} = 0.9534 \Rightarrow \phi = 17.56^0$

(c) Write the dates of the no-shadow days in Hong Kong. [2]
试写下香港无影日的日期。[2]

The number of days for the Earth to revolve around the Sun through this angle

$= 365\left(\frac{17.56}{360}\right) = 17.8$

Hence the days with no shadow in Hong Kong are 18 days before and after the Summer Solstice, that is, 3 June and 9 July.

2. Six Missiles (5 points) 六枚飞弹 （5 分）

Six missiles are initially located at the six vertex of a regular hexagon with side length a. The speed of the missiles in the plane is v. Each missile is equipped with an automatic navigation system. The automatic navigation system of each missile guides itself to aim at the current position of its counterclockwise neighbor.

今有飞弹六枚, 分别位于一边长为 a 的正六边形的六个角上。 每枚飞弹都装置有自动导航系统。 该系统会指示飞弹永远以速率 v 飞向其逆时针方向之近邻。

(a) Find the radial component of the missile velocities relative to the center of the hexagon. [2]
找出飞弹指向六角形中心的径向速率。[2]

By symmetry, all the six missiles hit at the same time.
By symmetry, they must hit at the center of the hexagon.
By symmetry, the missiles are always at the vertex of a rotating hexagon.
The radial speed is $v\cos(\pi/3) = v/2$.

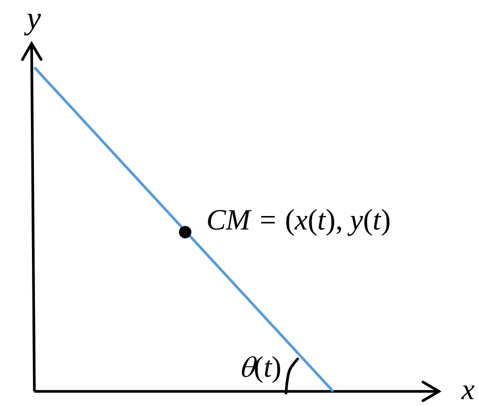
(b) Find the time taken for a missile to hit another. [3]
找出一枚飞弹击中另一枚所需的时间。[3]

The time taken is $a/(v/2) = 2a/v$.

3. Falling ladder (10 points) 下跌中的梯子（10 分）

A ladder of length $2l$ and mass m is standing up against a vertical wall with initial angle α relative to the horizontal. There is no friction between the ladder and the wall or the floor. The ladder begins to slide down with zero initial velocity. Denote $\theta(t)$ as the angle the ladder makes with the horizontal after it starts to slide and $(x(t), y(t))$ be the coordinate of the center of mass of the ladder. In this problem, you should take the gravitational potential energy to be zero at $y = 0$.

一个长度为$2l$、质量为 m 的梯子靠着一道垂直的墙，并与水平形成初始夹角 α。梯子与墙身和地面并没有摩擦力。梯子从零初始速度开始下滑。$\theta(t)$表示为梯子下滑期间与水平形成的夹角，$(x(t), y(t))$ 表示为梯子质心的坐标。在本题中，你应将在 $y = 0$ 处的引力势能取值为零。



(a) What is the initial total mechanical energy of the ladder in terms of α? [1]
梯子的初始总机械能是甚么？答案以α表示。[1]

$E_i = mgy = mgl\sin\alpha$

(b) Write the potential energy of the ladder in terms of $\theta(t)$ when it is sliding. [1]
请用$\theta(t)$写下梯子下滑时的势能。[1]

$U = mgy = mgl\sin\theta$

(c) Write the total kinetic energy of the ladder in terms of $\dot{x}(t), \dot{y}(t), \dot{\theta}(t)$ when it is sliding. (Hint: The moment of inertia of a rod of length $2l$ and mass m about an axis through the center of mass and perpendicular to its length is $I = ml^2/3$.) [1]
请用$\dot{x}(t), \dot{y}(t), \dot{\theta}(t)$写下梯子下滑时的总动能。（提示: 一条长度为$2l$、质量为 m 的杆子，相对於通过杆子质心并垂直于杆子的转动轴，其轉动惯量为$I = ml^2/3$。）[1]

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{6}ml^2\dot{\theta}^2$$

(d) As long as the ladder is in contact with the wall, find the relation between $x(t)$ and $\theta(t)$, and similarly the relation between $y(t)$ and $\theta(t)$. [2]
当梯子靠着墙的时候，请找出$x(t)$和$\theta(t)$的关系式，而同样地，找出$y(t)$和$\theta(t)$的关系式。[2]

$x = l\cos\theta$ and $y = l\sin\theta$

(e) Write the total mechanical energy of the ladder in terms of $\theta(t)$ and $\dot\theta(t)$ only by eliminating any dependence on $x(t)$ and $y(t)$. [2]
在消去$x(t)$和 $y(t)$后，只用$\theta(t)$ 和 $\dot\theta(t)$写下梯子的总机械能。[2]

$$\dot x = -(l\sin\theta)\dot\theta \quad \text{and} \quad \dot y = (l\cos\theta)\dot\theta$$

$$E = T + U = \frac{1}{2}ml^2\dot\theta^2(\sin^2\theta + \cos^2\theta) + \frac{1}{6}ml^2\dot\theta^2 + mgl\sin\theta = \frac{2}{3}ml^2\dot\theta^2 + mgl\sin\theta$$

(f) Derive the relation between $\theta(t)$ and $\ddot\theta(t)$. [1]
试推导$\theta(t)$和$\ddot\theta(t)$的关系式。[1]

$$\frac{2}{3}ml^2\dot\theta^2 + mgl\sin\theta = mgl\sin\alpha \Rightarrow \frac{4}{3}ml^2\dot\theta\ddot\theta + mgl\cos\theta\dot\theta = 0 \Rightarrow \ddot\theta = -\frac{3g}{4l}\cos\theta$$

(g) Find the angle θ_c when the ladder loses contact with the vertical wall. [2]
找出当梯子和墙身失去接触时的角度 θ_c。[2]

$$m\ddot x = -ml\sin\theta\ddot\theta - ml\cos\theta\dot\theta^2 = 0 \text{ and } \dot\theta^2 = \frac{3g}{2l}(\sin\alpha - \sin\theta)$$

$$\Rightarrow -ml\sin\theta\left(-\frac{3g}{4l}\cos\theta\right) - ml\cos\theta\frac{3g}{2l}(\sin\alpha - \sin\theta) = 0$$

$$\Rightarrow \frac{3}{4}\sin\theta - \frac{3}{2}\sin\alpha + \frac{3}{2}\sin\theta = 0 \Rightarrow \sin\theta_c = \frac{2}{3}\sin\alpha$$

4. Photon Gas (5 points) 光子气体（5 分）

The kinetic theory is very useful in understanding the properties of gases. In this problem we apply the theory to a gas of N photons inside a cubic box with side length L. The energy-momentum relation of a photon is given by $E = |\mathbf{p}|c$.
分子运动理论于理解气体性质时非常有用。 在本题中我们把这理论应用於一长度为 L 的立方盒子中的 N 粒光子。 光子的能量-动量关系为 $E = |\mathbf{p}|c$。

(a) Express the time taken between two consecutive collisions of the same wall of the box normal to the x direction in terms of L, p, p_x and c. [2]
找出一光子连续两次撞击同一面向 x 方向的盒壁之间的时间间距。答案以 L, p, p_x和 c 表达。[2]

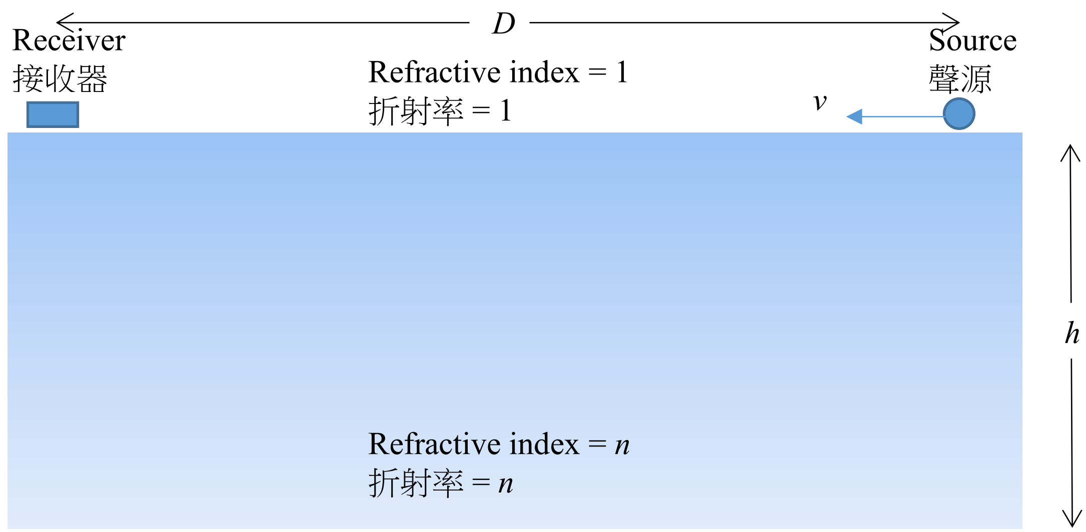
During every collision with the wall, the change in momentum is $2p_x$.
The time taken between two collisions is $\Delta t = 2L/c(p_x/p) = 2Lp/cp_x$.

(b) Express the internal energy U of the photon gas in terms of its pressure P and volume V. [3]
找出光子气体的内能 U。 答案以气体的压力 P 和体积 V 表达。[3]

The force is $2p_x/(2Lp/cp_x) = cp_x^2/pL$
The pressure is cp_x^2/pV.
The total pressure is P
$= c\,\langle p_x^2/p\rangle\, N/V = 1/3\, N/V\, c\langle(p_x^2 + p_y^2 + p_z^2)/p\rangle = 1/3\, N/V\, c\langle p\rangle = 1/3\, U/V$
Hence $U = 3PV$.

5. Sea Surface Sound Transmission (5 points) 海面传音（5 分）

In a region where the ocean has a constant depth of h, a sound source emits sound wave with frequency f. Suppose the frequency is so high that sound waves can be treated as rays, and the refractive index of water for sound wave is n. Let the sound speed in air be c. The source is moving with constant horizontal speed $v < c$ towards a stationary receiver at distance D, both located just above the ocean surface, as shown in the figure. It is also assumed that the speed is low so that $\frac{D}{\tau} \gg v$, where $\tau \gg \frac{1}{f}$ is the observation time. Assume we can ignore reflection by the ocean surface and consider only reflection by the ocean floor.
在一处海床深度为常数 h 的海面上，一声源发出频率为 f 的声波。 假设该频率足够高使得声波可以被视为声线束，而海水对声线束的折射率为 n。设空气中的声速为 c。如图中所示，声源与一静止接收器皆位于海面上，两者距离为 D。 声源以均匀速率 $v < c$ 向接收器移动。假设该速率足够慢，使得$\frac{D}{\tau} \gg v$，其中$\tau \gg \frac{1}{f}$为观察时间。 假设海面的反射可以忽略，只需考虑海床的反射。



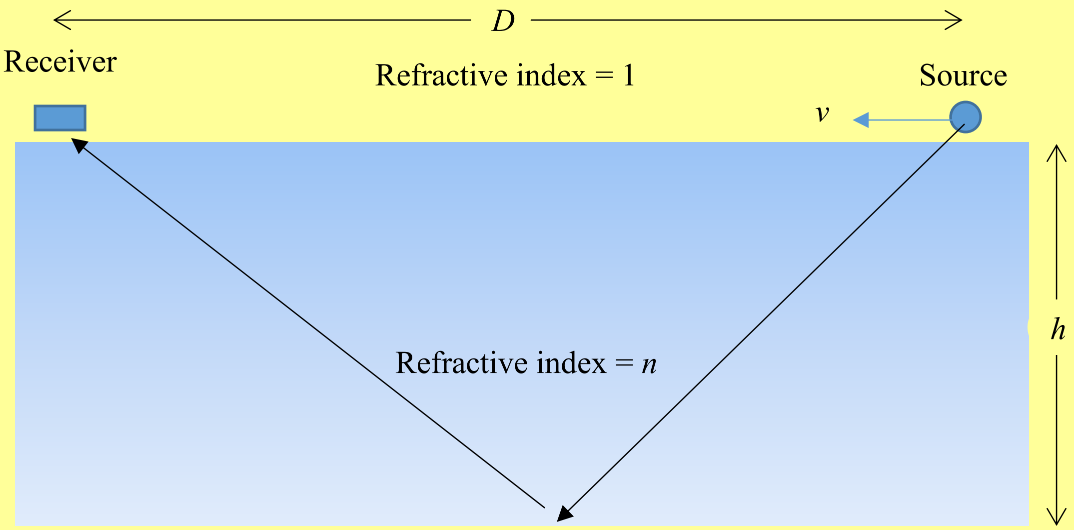
在(a)至(c)中，答案以 c, f, n, 和 v 表达。

(a) Find the frequency of the sound arriving at the receiver through air. [2]
找出声音经过空气到达接收器的频率。[2]

The Doppler shifted frequency is $f_1 = \frac{c}{c-v} f$.

(b) Find the frequency of the sound that will arrive at the receiver via the ocean when $D = 2h$. [2]
找出当$D = 2h$时发出的声音经过海洋到达接收器的频率。[2]

The shortest path via the ocean is reflected at the midpoint:



Component of the source velocity longitudinal to the sound wave in the ocean

$$v' = \frac{(D/2)v}{\sqrt{(\frac{D}{2})^2 + h^2}} = \frac{Dv}{\sqrt{D^2 + 4h^2}}$$

The Doppler shifted frequency is $f_2 = \frac{c/n}{c/n - v'} f = \frac{cf}{c - \frac{nDv}{\sqrt{D^2+4h^2}}}$

When $D = 2h$, we have

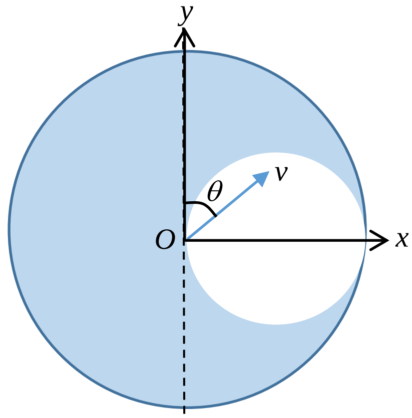
$$f_2 = \frac{cf}{c - \frac{nv}{\sqrt{2}}}$$

(c) Find the instantaneous beat frequency when the sound waves in (b) reach the receiver. [1]
找出(b)中的声波到达接收器时的瞬时拍频频率。[1]

The beat frequency is $f_b = |f_1 - f_2| = \left|\frac{f}{1-\frac{v}{c}} - \frac{f}{1-\frac{nv}{\sqrt{2}c}}\right| = \frac{\left|1-\frac{n}{\sqrt{2}}\right|}{\left(1-\frac{v}{c}\right)\left(1-\frac{nv}{\sqrt{2}c}\right)} \frac{v}{c} f$

6. Electron Trajectory in a Cavity (10 points) 空腔中的电子轨迹（10 分）

A spherical cavity is carved out from a uniformly positively charged sphere with radius *R*. The radius of the cavity is *R*/2, located at a distance *R*/2 from the center of the large sphere, *O*. The total positive charge in the system is *Q*.
在一个带均匀正电、半径为 *R* 的球体内挖出一半径为 *R*/2 的球形空腔。空腔中心与大球中心 *O* 距离为 *R*/2。 系统的总正电荷为 *Q*。



(a) Consider a point in the cavity at distance r and polar angle θ from the origin. Calculate the x and y components of the electric field at the point. [5]
考虑空腔内一点, 其与原点距离为 r, 极角为θ。计算该点电场的 x 分量和 y 分量。[5]

The configuration is equivalent to a fully filled large sphere with charge $8Q/7$ and a small sphere with charge $-Q/7$.
Electric field due to the large sphere

$$E_x = \frac{8Q}{7}\left(\frac{r}{R}\right)^3 \frac{1}{4\pi\varepsilon_0 r^2}\left(\frac{x}{r}\right) = \frac{2Qx}{7\pi\varepsilon_0 R^3},$$

$$E_y = \frac{8Q}{7}\left(\frac{r}{R}\right)^3 \frac{1}{4\pi\varepsilon_0 r^2}\left(\frac{y}{r}\right) = \frac{2Qy}{7\pi\varepsilon_0 R^3},$$

Electric field due to the small sphere

$$E_x = -\frac{Q}{7}\left(\frac{\sqrt{\left(x-\frac{R}{2}\right)^2+y^2}}{\frac{R}{2}}\right)^3 \frac{1}{4\pi\varepsilon_0\left[\left(x-\frac{R}{2}\right)^2+y^2\right]}\left(\frac{x-\frac{R}{2}}{\sqrt{\left(x-\frac{R}{2}\right)^2+y^2}}\right) = -\frac{2Q\left(x-\frac{R}{2}\right)}{7\pi\varepsilon_0 R^3},$$

$$E_y = -\frac{Q}{7}\left(\frac{\sqrt{\left(x-\frac{R}{2}\right)^2+y^2}}{\frac{R}{2}}\right)^3 \frac{1}{4\pi\varepsilon_0\left[\left(x-\frac{R}{2}\right)^2+y^2\right]}\left(\frac{y}{\sqrt{\left(x-\frac{R}{2}\right)^2+y^2}}\right) = -\frac{2Qy}{7\pi\varepsilon_0 R^3},$$

Total electric field

$$E_x = \frac{2Qx}{7\pi\varepsilon_0 R^3} - \frac{2Q\left(x-\frac{R}{2}\right)}{7\pi\varepsilon_0 R^3} = \frac{Q}{7\pi\varepsilon_0 R^2},$$

$$E_y = 0.$$

(b) As shown in the figure, electrons are emitted from O in all directions with speed v and direction θ ranging between 0 and π, but none of them can reach the opposite end of the diameter of the cavity. Gravitational forces are negligible. Find the equation of the envelope of all trajectories of the electrons. [3]
如图所示，现考虑许多电子由 O 以同样速率 v 向 0 至π的各方向θ射出。但其动能不足以到达空腔的直径对点。忽略万有引力。求所有可能的电子轨迹的包络线方程。[3]

The electron is subject to an effective acceleration g to the left with

$$g = \frac{eQ}{7\pi\varepsilon_0 m R^2}$$

The equation of motion of an electron is $x = vt\sin\theta - \frac{1}{2}gt^2,\ \ y = vt\cos\theta$

Eliminating t, the equation of the trajectory is $x = y\tan\theta - \frac{g\sec^2\theta}{2v^2}y^2$

For a given position, the angle θ required to reach the position is given by

$\frac{gy^2}{2v^2}\tan^2\theta - y\tan\theta + x + \frac{gy^2}{2v^2} = 0$

Solution exists if $y^2 \geq 4\left(\frac{gy^2}{2v^2}\right)\left(x + \frac{gy^2}{2v^2}\right)$

Hence the equation of the envelope of safety is $x = \frac{v^2}{2g} - \frac{g}{2v^2}y^2$

(c) Find the maximum x coordinate where the electrons hit the inner surface of the cavity. Express your answer in terms of Q, the absolute value of the electronic charge e, electron mass m, R, and v. [2]

找出电子打中空腔内表面处最大可能的 x 坐标。答案以 Q、电子电荷绝对值 e、电子质量 m、R 和 v 表达。[2]

The surface of the cavity is given by $(x - \frac{R}{2})^2 + y^2 = R^2/4$

The intersection with the envelope is given by $x = \frac{R}{2} + \frac{v^2}{g} \pm \sqrt{\frac{R^2}{4} + \frac{v^2R}{g}}$

Since $x = \frac{R}{2} + \frac{v^2}{g} + \sqrt{\frac{R^2}{4} + \frac{v^2R}{g}} > R$, it is rejected.

So $x = \frac{R}{2} + \frac{v^2}{g} - \sqrt{\frac{R^2}{4} + \frac{v^2R}{g}}$

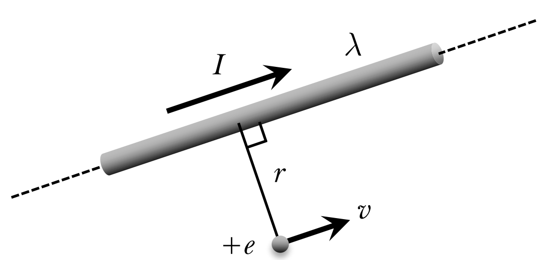
7. Proton Motion Near a Charged Current-Carrying Wire (5 points)
带电荷电流线邻近的质子运动（5 分）

As shown in the figure below, a proton of charge $+e$ moves with velocity v parallel to an infinitely-long, thin wire, at a distance r from the axis of the wire. The wire carries a current I, and its charge per unit length is λ (assumed positive and uniform). Both the proton and the wire are in vacuum.

如下图所示，一个带电荷$+e$ 的质子在一条无限长的幼导线附近运动，速度为 v，方向与导线平行。质子与导线轴的距离为 r。导线载着电流 I，且每单位长度电荷为λ(假定为正及均匀)。质子与导线都处于真空中。

Express the answers from (a) to (c) in terms of r, I, λ, the permittivity of vacuum ε_0, the permeability of vacuum μ_0, the speed of light c, and the unit vectors in cylindrical coordinates $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$.

在(a)至(c)中，答案以 r、I、λ、真空电容率 ε_0、真空磁导率 μ_0、光速 c 和柱坐标的单位向量 $\mathbf{e}_r$、$\mathbf{e}_\theta$及 $\mathbf{e}_z$ 表示。



(a) Find the electric field experienced by the proton. [2]
求质子所感受到的电场 **E**。[2]

By Gauss's law: $\oint \mathbf{E}\cdot d\mathbf{A} = \frac{\lambda}{\varepsilon_0} \Rightarrow \mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\mathbf{e}_r$.

(b) Find the magnetic field experienced by the proton. [2]
求质子所感受到的磁场 **B**。[2]

By Ampere's law: $\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I \Rightarrow \mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{e}_\theta$.

(c) Find the speed of the proton such that it moves in a straight line parallel to the wire. [1]
求质子的速率，使其沿直线平行于导线运动。[1]

Resultant force on the proton is $\mathbf{F}_\mathrm{e} + \mathbf{F}_\mathrm{m} = e\mathbf{E} + e\mathbf{v}\times\mathbf{B}$

$$= \frac{e\lambda}{2\pi\varepsilon_0 r}\mathbf{e}_r + (ev\mathbf{e}_z)\times\left(\frac{\mu_0 I}{2\pi r}\mathbf{e}_\theta\right) = \frac{e\lambda}{2\pi\varepsilon_0 r}\mathbf{e}_r + \frac{ev\mu_0 I}{2\pi r}(-\mathbf{e}_r) = \left(\frac{e\lambda}{2\pi\varepsilon_0 r} - \frac{ev\mu_0 I}{2\pi r}\right)\mathbf{e}_r$$

For the proton to move in a straight line parallel to the wire:

$$\frac{e\lambda}{2\pi\varepsilon_0 r} = \frac{ev\mu_0 I}{2\pi r} \Rightarrow v = \frac{\lambda}{\varepsilon_0\mu_0 I} = \frac{c^2\lambda}{I}$$

Remark: Some students include relativistic effects in calculating the electric and magnetic fields experienced by the proton. In that case, the correct answer can be obtained by either using (1) $\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ and $\mathbf{B}' = \gamma(\mathbf{B} - \mathbf{v}\times\mathbf{E}/c^2)$, or (2) $\lambda' = \gamma(\lambda - vI/c^2)$ and $\mathbf{I}' = \gamma(\mathbf{I} - \mathbf{v}\lambda)$, since ($\rho c$, **J**) is a 4-vector. The answers become $E' = \frac{\gamma}{2\pi\varepsilon_0 r}\left(\lambda - \frac{v}{c^2}I\right)$ and $B' = \frac{\mu_0\gamma}{2\pi r}(I - v\lambda)$.

Pan Pearl River Delta Physics Olympiad 2017
2017 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-2 (Total 2 Problems, 55 Points) 简体版卷-2（共2题，55分）
(2:00 pm – 5:00 pm, 3 February, 2017)

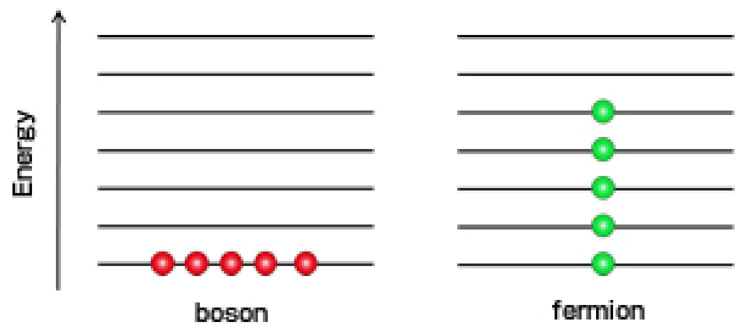
All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。
All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。
There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。
Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。
Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。
Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。
If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。
At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Bose Einstein Condensation (22 points) 玻色-爱因斯坦凝聚（22 分）

Planck’s constant 普朗克常数 $h = 6.626 \times 10^{-34}$ Js
Boltzmann constant 波尔兹曼常数 $k_B = 1.381 \times 10^{-23}$ JK^{-1}

In nature, particles are classified into two different kinds: bosons and fermions. Bosons (e.g. photons) are particles that like to be together in the same state. In contrast, fermions (e.g. electrons, protons and neutrons) are unlikely to go into an already occupied state according to the Pauli exclusion principle. Statistical mechanics tells us that when a system of bosons reaches a critical density in a trap it undergoes a transition that a large number of bosons will have a tendency to occupy the same lowest-energy state. This phenomenon is called Bose-Einstein condensation. The following figure shows how bosons and fermions occupy energy states when the temperature approaches 0 K.
在自然界中，粒子可以分为两种不同的类型：玻色子和费米子。玻色子（例如光子）是喜欢一起处于相同状态的粒子。相反，根据泡利不相容原理，费米子（例如电子、质子和中

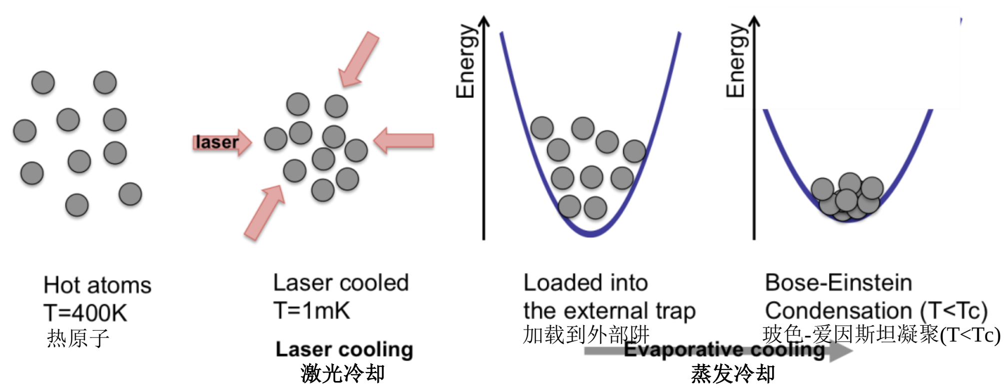
子）不可能进入已经被占据的状态。统计力学告诉我们，当一个玻色子系统在阱中达到临界密度时，它会经历相变，令大量的玻色子倾向占据相同的最低能阶。这种现象称为玻色-爱因斯坦凝聚。下图显示当温度接近 0 K 时，玻色子和费米子如何占据能阶。



Recent development of trapping and cooling ultracold atoms (e.g. Sodium, Rubidium and Lithium atoms) paved the way for the observation of Bose-Einstein condensation of atomic gases in ultracold temperature (Nobel prize in physics 2001), which had been theoretically predicted by Bose and Einstein in 1924. Several different cooling techniques have been employed to achieve ultracold temperature around 10-100 nK (note $1\ \text{nK} = 10^{-9}\text{K}$). For example, the hot Rubidium atoms prepared at 400 K are cooled down to ~1mK through the Laser cooling techniques (Nobel prize in physics in 1997). Such cold atoms prepared by laser cooling technique are typically loaded into the external trap (produced by either magnetic or optical fields) for further cooling as shown below.

在捕获和冷却超冷原子（例如钠、铷和锂原子）的技术上，近年的进展为观察超冷温度下原子气体的玻色-爱因斯坦凝聚（2001 年诺贝尔物理学奖）提供了有利条件，印证了 1924 年玻色和爱因斯坦的预测。几种不同的冷却技术已被采用以实现约 10-100 nK 的超冷温度（注意 $1\ \text{nK} = 10^{-9}\text{K}$）。例如，通过激光冷却技术（1997 年诺贝尔物理学奖），在 400K 下制备的热铷原子可以冷却至～1 mK。这种冷原子通常被加载到外部阱（由磁场或光场产生）中，用於进一步冷却，如下所示。

Laser cooling and trapping Rubidium atoms 激光冷却和捕获铷原子



A. Maxwell- Boltzmann distribution and the thermal de Broglie wavelength of the atoms 麦克斯韦-玻尔兹曼分布和原子的热德布罗意波长

Consider a dilute gas of atoms. The inter-particle interactions are very weak. In this case, the gas can be described by the ideal gas model in which the particles move freely inside a stationary trap without interacting with one another except for very brief elastic collisions to reach thermal equilibrium.
考虑稀释的原子气体。粒子间相互作用非常弱。在这种情况下，气体可以通过理想气体模型描述，其中粒子在固定阱内自由移动，除了在趋向热平衡的过程中会有非常短暂的弹性碰撞，彼此没有相互作用。

In this atomic gas system, the probability distribution of the particle speed v is given by Maxwell-Boltzmann distribution,
在这种原子气体系统中，粒子速度v 的概率分布由麦克斯韦 - 玻尔兹曼分布给出，

$$f(v) = \sqrt{\left(\frac{m}{2\pi k_B T}\right)^3} 4\pi v^2 e^{-\frac{mv^2}{2k_B T}},$$

where m is the mass of the atom, k_B is the Boltzmann constant, and T is the temperature of the gas in the unit of Kelvin [K].
其中 m 是原子的质量，k_B是玻尔兹曼常数，T 是气体温度，单位为[K]。

A1	Derive the most probable velocity v_{mp} of a particle at temperature T. 试推导温度为 T 时粒子最可能的速度v_{mp}。	**2 points** **2 分**

A2	Based on the most probable velocity v_{mp} obtained in A1, write down the characteristic de Broglie wavelength λ_{dB} of the particle in an atomic gas at temperature T. 根据在 A1 中求得的最可能速度v_{mp}，试写下温度为 T 时原子气体中粒子的特征德布罗意波长λ_{dB}。	**2 points** **2 分**

Since particles in a gas of atoms have different speed following Maxwell-Boltzmann distribution, it is useful to consider the *thermal de Broglie wavelength* (λ_T) defined as $\lambda_T = \lambda_{dB} \times \pi^{-\frac{1}{2}}$. Here we derive the Bose-Einstein temperature T_{C} for a gas of N non-interacting (bosonic) atoms of mass m in a three-dimensional box with volume V. We will consider the simple physical picture that Bose-Einstein condensation occurs when the characteristic inter-particle distance between bosonic atoms becomes comparable to the thermal de Broglie wavelength λ_T. (Planck's constant h = 6.626 × 10^{-34} Js, Boltzmann constant k_B = 1.381 × 10^{-23} JK^{-1})
由于原子气体中的粒子按著麦克斯韦-波尔兹曼分布，各有不同的速率，我们引入热德布罗意波长(λ_T)，定义为$\lambda_T = \lambda_{dB} \times \pi^{-\frac{1}{2}}$。在这里，我们会考虑在体积为 V 的三维盒子中的原子气体，其中有 N 个质量为 m 的非相互作用（玻色子）原子，我们会推导其玻色-爱

因斯坦温度 T_C 。我们将採用一幅简单的物理图画，就是当玻色子原子间的特征距离和热德布罗意波长λ_T相若时，玻色-爱因斯坦凝聚便会发生。

A3	What is the expected T_C of the $N = 10^5$ atoms of mass $m = 1.445 \times 10^{-25}$ kg trapped in the trap with a volume of $V = 10^5 \mu m^3$? ($1\ \mu m^3 = 10^{-18}\ m^3$) 在体积为 $V = 10^5 \mu m^3$的阱中，捕获 $N = 10^5$ 个质量为 $m = 1.445 \times 10^{-25}$ kg的原子，求 T_C 的预期值。($1\ \mu m^3 = 10^{-18}\ m^3$)	**3 points** **3 分**

B. Evaporative cooling in an external trap 在外部阱中的蒸发冷却

The temperatures reached by laser cooling are extremely low (< 1 mK), but they are not cold enough to realize Bose-Einstein condensation. To date, Bose-Einstein condensation of alkali atoms has been achieved by using evaporative cooling after atoms are loaded into the external trap. During evaporative cooling, when atoms escaping from a trap have a kinetic energy higher than the average energy of atoms in the trap, the remaining atoms become cooled.
激光冷却达到的温度极低（< 1 mK），但还是不够冷去实现玻色-爱因斯坦凝聚。到目前为止，碱金属原子的玻色-爱因斯坦凝聚可以通过把原子加载到外部阱之后，使用蒸发冷却实现。在蒸发冷却期间，当从阱中逸出的原子具有高于阱中原子平均能量的动能时，剩余的原子就会冷却。

In the following problems in part B, we will estimate the effect of evaporative cooling. For atoms trapped in a box of fixed volume and having no heat exchange with the surroundings, we assume that an average energy of trapped atoms is ϵ and a small number of atoms $|\Delta N|$ are evaporated within a short time $\Delta\tau$ with an average energy of $(1+\beta)\epsilon$ where $\beta > 0$. During the process, the small change in the number of atoms $\Delta N < 0$ leads to the change $\Delta\epsilon < 0$ in the average energy of the remaining atoms. We also assume that $\left|\frac{\Delta\epsilon}{\epsilon}\right| \ll 1$ and $\left|\frac{\Delta N}{N}\right| \ll 1$.
在下面 B 部的问题中，我们将估计蒸发冷却的影响。对于被捕获在固定体积的盒子中，并且没有与周围环境进行热交换的原子，我们假设被捕获原子的平均能量是ϵ。假设有小数目的原子$|\Delta N|$ 短时间$\Delta\tau$内蒸发，其平均能量为$(1+\beta)\epsilon$，其中$\beta > 0$。在这过程中，原子数目$\Delta N < 0$ 的小变化，导致剩余原子的平均能量的变化$\Delta\epsilon < 0$。 我们还假设$\left|\frac{\Delta\epsilon}{\epsilon}\right| \ll 1$和$\left|\frac{\Delta N}{N}\right| \ll 1$。
[Remark: In the derived relation, you may ignore the term $\frac{\Delta\epsilon}{\epsilon}\frac{\Delta N}{N}$ since $\frac{\Delta\epsilon}{\epsilon} \ll 1$ and $\left|\frac{\Delta N}{N}\right| \ll 1$.]
［备注：在推导的关系中，由于$\frac{\Delta\epsilon}{\epsilon} \ll 1$和$\left|\frac{\Delta N}{N}\right| \ll 1$，可以忽略 $\frac{\Delta\epsilon}{\epsilon}\frac{\Delta N}{N}$一项 。］

B1	Derive the relation between $\Delta\epsilon$ and ΔN with β, ϵ and N. 试用 β、 ϵ和 N，推导 $\Delta\epsilon$和 ΔN之间的关系。	**3 points** **3 分**

Now we consider cold atoms at the initial temperature of $T_i = 200\mu K$ in a trap. Assume that we remove 1% of atoms (i.e. $\left|\frac{\Delta N}{N}\right| = 0.01$) during each time period $\Delta\tau$ and $\beta = 2$.
现在我们考虑阱中的冷原子，初始温度为 $T_i = 200\mu K$。假设我们在每段时间 $\Delta\tau$期间去除1%的原子（即$\left|\frac{\Delta N}{N}\right| = 0.01$），并且$\beta = 2$。

B2	Then estimate the final temperature T_f of atoms after the evaporative cooling over the total time period of $350\Delta\tau$. 试估计在$350\Delta\tau$的总时间段内，经蒸发冷却后的原子的最终温度T_f。	**3 points** **3 分**

C. Bose-Einstein temperature T_C in a harmonic potential
谐波势中的玻色-爱因斯坦凝聚温度 T_C

In a real experiment with ultracold atomic gases, a gas of bosonic atoms is trapped in a three-dimensional harmonic trap generated by the laser beam or the magnetic field. Here we consider a three-dimensional trap characterized by the harmonic potential:
在超冷原子气体的真实实验中，玻色原子气体被捕获在由激光束或磁场产生的三维谐波阱中。这里我们考虑一个三维阱，可用谐波势描述：

$$U_{\text{trap}} = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2).$$

C1	Consider the fact that ultracold atoms are oscillating around the bottom of the trap with the characteristic trapping frequency $\omega_i/2\pi$ along the i-direction. Derive the characteristic volume confining the atoms in terms of T and $\omega_{x,y,z}$. 考虑超冷原子在阱底振荡，沿着 i 方向的特征捕获频率为 $\omega_i/2\pi$ 。试推导原子被限定的特征体积，答案以 T 和$\omega_{x,y,z}$表达。	**3 points** **3 分**

C2	Derive the Bose-Einstein condensation temperature T_C of the atoms trapped in a harmonic trap considered in Part C1 in terms of ω_i and N. 试推导 C1 部的谐波阱中捕获的原子的玻色-爱因斯坦凝聚温度 T_C，答案以 ω_i 和 N表达。	**2 points** **2 分**

C3	What is the Bose-Einstein condensation temperature T_c of the $N = 10^4$ atoms of mass $m = 1.445 \times 10^{-25}$ kg in the harmonic trap with trapping frequencies $\omega_x/2\pi = \omega_y/2\pi = \omega_z/2\pi = 100$ Hz? 谐波阱中有 $N = 10^4$个原子，每个原子的质量为$m = 1.445 \times 10^{-25}$ kg，谐波频率为 $\omega_x/2\pi = \omega_y/2\pi = \omega_z/2\pi = 100$ Hz。求玻色−爱因斯坦凝聚温度 T_c。	**1 point** **1 分**

Note that the evaporative cooling is efficient enough to achieve the Bose-Einstein condensation.
注意，蒸发冷却的效率足以实现玻色-爱因斯坦凝聚。

D. Adiabatic cooling by slowly expanding the trap 通过缓慢膨胀阱进行绝热冷却

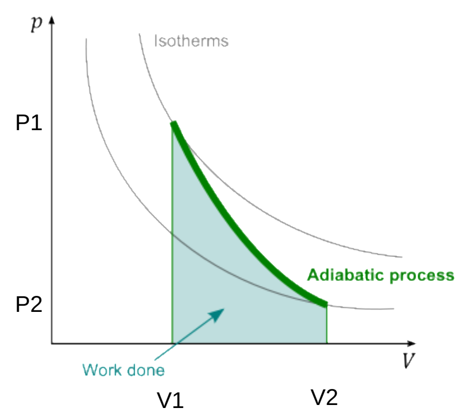
Cooling atomic gases to lower temperature has been motivated by the quest to observe new forms of matter such as superfluid. However the evaporative cooling we discussed in part B is not always preferable since a number of atoms leave the trap during the process. In this part we consider a different cooling technique (so-called adiabatic cooling) by slowly expanding the trap without losing atoms.

把原子气体冷却至更低温度的动机，是寻求物质的新状态（例如超流体）。然而，我们在 B 部中讨论的蒸发冷却，不一定是首选的方法，因为在该过程中有许多原子离开了阱。在这部中，我们会考虑另一冷却技术（所谓的绝热冷却），是通过缓慢地膨胀阱而不损失原子而达成的。

D1	Calculate the fraction of atoms remaining in the trap after the evaporative cooling described in part B2. 试计算在 B2 部中描述的蒸发冷却之后，留在阱中的原子的分数。	**1 point** **1 分**

Consider N atoms in an external harmonic trap with trapping frequencies of $\omega_x = \omega_y = \omega_z = 2\pi f_0$ at the temperature $T_1 = 105$ nK $= 1.05 \times 10^{-7}$ K. From now on, we assume that the whole atomic gas can be regarded as a monoatomic ideal gas. At this stage, the atomic gas has the pressure P_1 and the volume V_1 as described in the figure below.

考虑在外部谐波阱中有N个原子，捕获频率为$\omega_x = \omega_y = \omega_z = 2\pi f_0$，其温度为$T_1 = 105$ nK $= 1.05 \times 10^{-7}$ K。从现在开始，我们假设整个原子气体可视为单原子理想气体。这时，原子气体具有如下图所示的压强P_1和体积V_1。



Now consider the adiabatic decompression process of N atoms trapped in a harmonic trap. For this we adiabatically change the trapping frequencies of the harmonic potential trap from $\omega_{x,1} = \omega_{y,1} = \omega_{z,1} = 2\pi f_0$ to $\omega_{x,2} = 2\pi f_0$ and $\omega_{y,2} = \omega_{z,2} = \frac{2\pi f_0}{10}$ following the adiabatic process in the P-V diagram. Note that there is no heat exchange between the atomic gas and the environment (actually vacuum) and no atoms leave the trap during the process.

现在考虑捕获 N 个原子的谐波阱的绝热减压过程。为此，我们绝热地将谐波势阱的捕获频率从$\omega_{x,1} = \omega_{y,1} = \omega_{z,1} = 2\pi f_0$ 改变至 $\omega_{x,2} = 2\pi f_0$ 和 $\omega_{y,2} = \omega_{z,2} = \frac{2\pi f_0}{10}$。注意，原子气体和环境（实际上是真空）之间没有热交换，并且在该过程中没有原子离开阱。

D2	Calculate the final temperature of the atomic gas after adiabatic decompression of the trap. 试计算阱绝热减压后原子气体的最终温度。	**2 points** **2 分**

Problem 2: Swimming Microorganisms (33 points) 游泳微生物（33 分）

Although objects in water tend to sink in a gravitational field, microorganisms such as paramecium can control their swimming directions not necessarily subject to gravitational field. Recently, physicists proposed that their swimming patterns are related to their asymmetric shape. When they swim in a viscous fluid, they experience asymmetric resistance forces that may cause them to rotate.

虽然水中的物体倾向于在重力场中下沉，但是诸如草履虫的微生物可以控制它们的游泳方向，不一定受到重力场的影响。最近，物理学家提出他们的游泳模式和他们不对称的形状有关。当它们在粘性流体中游泳时，它们经历可能导致它们旋转的不对称抗阻力。

A. Resistive Forces and Torques in a Viscous Fluid 粘性流体中的抗阻力和力矩

For a rod having a translational motion in a viscous fluid, there are two kinds of resistive forces. In this question, we will refer to the resistive force acting in the normal direction of the rod as the *drag*, and the resistive force along the direction of the rod as the *friction*, as shown in Fig. 1(a). The drag per unit length is approximated as $\mu v_\perp$, and the friction per unit length as $\frac{\mu v_\parallel}{2}$, where $v_\perp$ and $v_\parallel$ are the velocity components normal and parallel to the axis of the rod respectively, and μ is a constant proportional to the viscosity of the fluid.

在粘性流体中有平移运动的杆子，存在两种抗阻力。在本题中，我们将作用在杆子的法向方向上的抗阻力，称为阻力，而将作用在沿杆方向的抗阻力，称为摩擦力，如图 1(a)所示。每单位长度的阻力近似为$\mu v_\perp$，而每单位长度的摩擦力为$\frac{\mu v_\parallel}{2}$，其中 $v_\perp$ 和 $v_\parallel$分别为垂直和平行于杆轴线的速度分量，μ是与流体粘度成比例的常数。

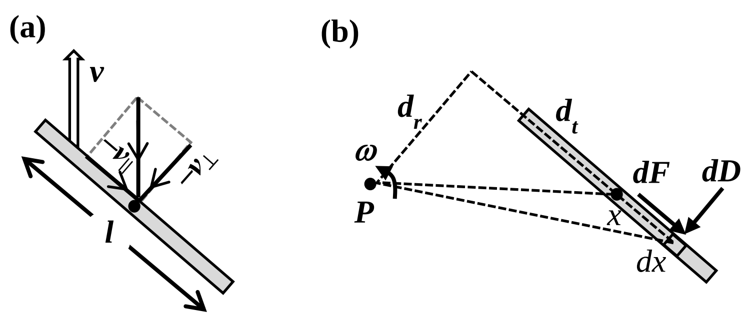


Fig. 1: (a) Directions of the resistive forces acting on a rod moving in a viscous fluid with velocity $\mathbf{v}$ indicated as the white arrow. The drag is directed along $-\mathbf{v}_\perp$, and the friction along $-\mathbf{v}_\parallel$. (b) The resistive forces acting on an element of the rod rotating about point P in the same plane at radial distance d_r and tangential distance d_t from its center.

图 1：（a）杆子在粘性流体中运动时，作用在杆子上的抗阻力的方向，速度 $\mathbf{v}$ 以白色箭头表示。阻力方向沿著$-\mathbf{v}_\perp$，摩擦力方向则沿著$-\mathbf{v}_\parallel$。（b）杆子围绕 P 点在同一平面旋转，杆子中心与 P 点的径向距离为 d_r，切向距离为 d_t。图示作用在杆子的一小段上的抗阻力。

As shown in Fig. 1(b), consider a reference point P whose radial and tangential distances from the center of the rod are d_r and d_t respectively. If the rod has a fixed position and orientation with respect to P, and P has a translational motion, then the resistive forces acting on the rod can be calculated using Fig. 1(a). However, if the rod also rotates in the same plane about P at an

angular velocity ω, there will be extra forces and torques acting on the rod due to drag and friction.
如图 1（b）所示，考虑一个参考点 P，与杆子中心的径向和切向距离分别为 d_r 和 d_t。如果杆子相对于 P 点的位置和取向固定，并且 P 在作平移运动，则可使用图 1(a)计算作用在杆子上的抗阻力。然而，如果杆子也在同一平面以角速度ω围绕 P 转动，阻力和摩擦力将产生额外的力和力矩作用在杆子上。

A1	Derive the friction F due to the rotational motion. 试推导由于旋转运动引起的摩擦力 F。	**1 point** **1 分**

A2	Derive the drag D due to the rotational motion. 试推导由于旋转运动引起的阻力 D。	**1 point** **1 分**

A3	Derive the torque τ_f about the axis of rotation due to the friction. 试推导摩擦力围绕旋转轴心的力矩τ_f。	**1 point** **1 分**

A4	Derive the torque τ_d about the axis of rotation due to the drag. 试推导阻力围绕旋转轴心的力矩τ_d。	**2 points** **2 分**

B. A Passive Microswimmer with No Rotation 无动力又不旋转的游泳微生物

An asymmetric microswimmer is L-shaped with the dimensions shown in Fig. 2(a). The mass of the microswimmer is m and the density is uniform. The lengths of the long and short arms are $4b$ and $2b$ respectively. The width and thickness of its two arms are negligible.
有不对称的游泳微生物具有 L 形的形状，尺寸如图 2（a）所示。游泳微生物的质量为 m，密度均匀。长臂和短臂的长度分别为 $4b$ 和 $2b$。两臂的宽度和厚度可忽略。

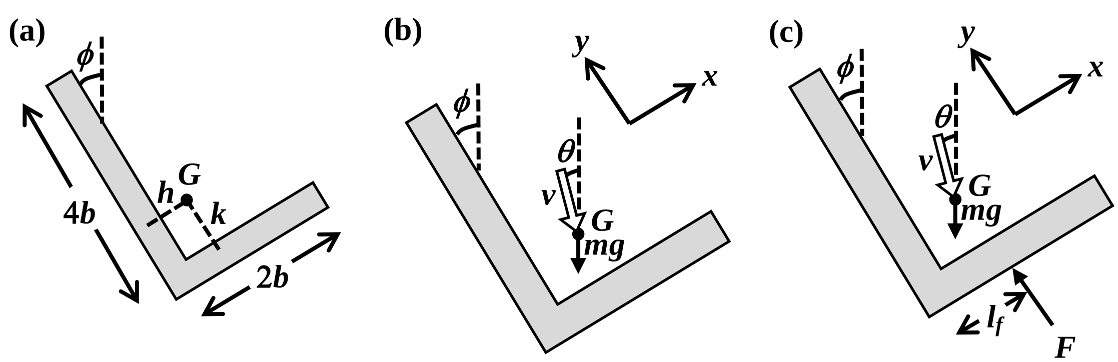


Fig. 2: (a) Dimensions of the microswimmer. (b) The weight and the velocity of a passive microswimmer. (c) An active microswimmer.
图 2：（a）游泳微生物的尺寸。（b）无动力游泳微生物的重量和速度。（c）具动力的游泳微生物。

A passive microswimmer does not have any self-propulsion. The center of mass G of the microswimmer is at a distance h and k from the midlines of the long and short arms respectively.
无动力的游泳微生物不具有任何自推进力。游泳微生物的质心 G 与长臂和短臂的中线的距离分别为 h 和 k。

B1	Write the expressions of h and k. 试写下 h 和 k 的表达式。	**2 points** **2 分**

The L-shaped microswimmer is tilted by an angle ϕ as shown in Fig. 2(b) and is sinking with velocity v in the direction inclined at an angle θ with the vertical in the presence of gravitational acceleration g. The microswimmer does not rotate. Assume that the upthrust of the fluid is negligible compared with the weight of the microswimmer.
L 形游泳微生物的倾斜角度为ϕ，如图 2（b）所示，并且在重力加速度 g 的影响下，以速度 v 下沉，速度相对垂直方向的倾斜角度为θ。游泳微生物不旋转。假设流体的浮力与游泳微生物的重量相比是可忽略的。

B2	Write the equation consisting of the components of all forces along the y axis (the direction of the long arm). 试写下沿 y 轴（长臂的方向）的所有分力组成的方程。	**2 points** **2 分**

B3	Write the equation consisting of the components of all forces along the x axis (direction of the short arm). 试写下沿 x 轴（短臂的方向）的所有分力组成的方程。	**2 points** **2 分**

B4	Write the equation consisting of the moments of all forces about the center of mass. 试写下所有围绕质心的力的力矩组成的方程。	**2 points** **2 分**

B5	Calculate the tilt angle ϕ of the microswimmer at the steady state. Give your answer in degrees. 试计算游泳微生物在稳态下的倾斜角ϕ。答案以度数表达。	**1 point** **1 分**

B6	Calculate the motion direction θ of the microswimmer at the steady state. Give your answer in degrees. 试计算游泳微生物在稳态时的运动方向θ。答案以度数表达。	**1 point**

C. An Active Microswimmer with Rotation 具动力又会旋转的游泳微生物

To model an active microswimmer, physicists implemented a laser-induced chemical reaction at a point on the shorter arm of the object so that it provides a self-propulsion force F normal to the short arm. The dynamical properties of the microswimmer are rather sensitive to the point of application of F. For convenience we consider the case that this point is located at a distance

$l_f = \frac{13}{24}b$ from the corner (see Fig. 2(c)). The force can be adjusted by tuning the laser intensity incident on the microswimmer. Note that it is possible that the microswimmer can rotate so that forces and torques due to rotation have to be included. The velocity v, direction θ and the tilt angle ϕ becomes time dependent, and you will need to include the angular velocity $\dot{\phi}$ as one of the variables.

为了模拟具动力的游泳微生物，物理学家在物体短臂上的一点加进可由激光诱导的化学反应，为它提供垂直于短臂的自推进力 F。游泳微生物的动力学性质对于 F 的作用点是相当敏感的。为了方便起见，我们考虑这一点位于距离角落$l_f = \frac{13}{24}b$的情况（参见图 2（c））。调节射在游泳微生物上的激光强度，可以调节推进力。注意，因为游泳微生物可以旋转，我们必须加入考虑由于旋转引起的力和力矩。速度 v、方向θ和倾斜角ϕ变得与时间相关，你需要将角速度$\dot{\phi}$ 包括为其中一个变量。

C1	Write the equation consisting of the components of all forces along the y axis (the direction of the long arm). 试写下沿 y 轴（长臂的方向）的所有分力组成的方程。	**2 points** **2 分**

C2	Write the equation consisting of the components of all forces along the x axis (the direction of the short arm). 试写下沿 x 轴（短臂的方向）的所有分力组成的方程。	**2 points** **2 分**

C3	Write the equation consisting of the moments of all forces about the center of mass. 试写下所有围绕质心的力的力矩组成的方程。	**4 points** **4 分**

The above three equations can be solved for the three variables $v\cos(\phi-\theta)$, $v\sin(\phi-\theta)$ and $\dot{\phi}$.
上述三个方程可以对$v\cos(\phi-\theta)$、$v\sin(\phi-\theta)$和$\dot{\phi}$三个变量求解。

C4	Eliminate $v\cos(\phi-\theta)$ and $v\sin(\phi-\theta)$ from the above equations to obtain an equation involving ϕ and $\dot{\phi}$ only. 从上述方程中消去$v\cos(\phi-\theta)$和$v\sin(\phi-\theta)$，以获得一个单涉及ϕ和$\dot{\phi}$ 的方程。	**2 points** **2 分**

C5	Derive the tilt angle ϕ when the microswimmer reaches the steady state of constant tilt. 试推导游泳微生物在固定倾斜稳态下的倾斜角ϕ。	**2 points** **2 分**

C6	Consider a microswimmer initially at the steady state with $F = 0$. At $t = 0$ the laser is switched on so that F becomes nonzero. Calculate $\phi(t)$ for $F << mg$. 考虑游泳微生物的初始状态处於 $F = 0$ 的稳态。在 $t = 0$ 时，激光亮了，使得 F 变为非零。在 $F << mg$ 的情况下，试计算$\phi(t)$。	**2 points** **2 分**

When F gradually increases from 0, the direction of linear motion gradually changes. When F exceeds a critical value, the tilt angle is no longer constant and the microswimmer takes a wheel-like trajectory.
当 F 从 0 渐渐增加时，游泳微生物线性运动的方向渐渐改变。当 F 超过临界值时，倾斜角不再恒定，游泳微生物的轨迹变成轮状。

C7	Write the maximum value of F for linear motion. 写下线性运动的最大 F 值。	**1 point** **1 分**

C8	To verify that the microswimmer can move in a wide range of directions, calculate the force(s) required for linear motion in the horizontal direction $\theta = \pi/2$. Give your answer in multiples of mg to 3 significant figures. 为了验证游泳微生物能够在广阔范围内的方向移动，试计算游泳微生物在水平方向 $\theta = \pi/2$ 作线性运动时，所需的(诸)力是多少。答案以 mg 的倍数表达，至 3 位有效数字。	**3 points** **3 分**

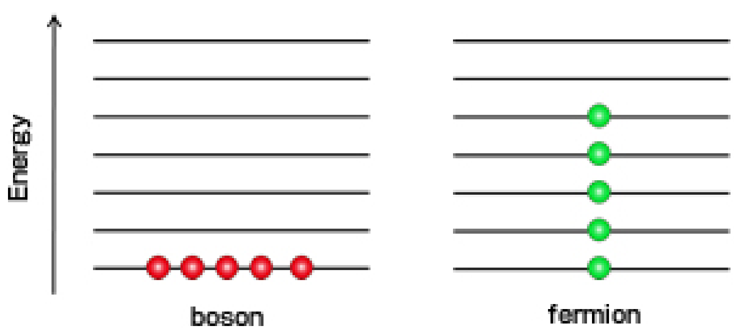
《THE END 完》

Pan Pearl River Delta Physics Olympiad 2017
2017 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-2 (Total 2 Problems, 55 Points) 简体版卷-2（共2题，55分）
(2:00 pm – 5:00 pm, 3 February, 2017)

Problem 1: Bose Einstein Condensation (22 points) 玻色-爱因斯坦凝聚（22 分）

Planck's constant 普朗克常数 $h = 6.626 \times 10^{-34}$ Js
Boltzmann constant 波尔兹曼常数 $k_B = 1.381 \times 10^{-23}$ JK^{-1}

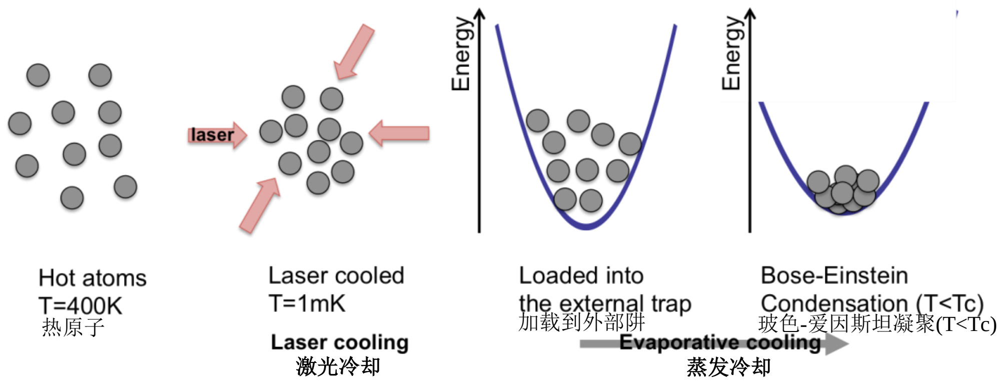
In nature, particles are classified into two different kinds: bosons and fermions. Bosons (e.g. photons) are particles that like to be together in the same state. In contrast, fermions (e.g. electrons, protons and neutrons) are unlikely to go into an already occupied state according to the Pauli exclusion principle. Statistical mechanics tells us that when a system of bosons reaches a critical density in a trap it undergoes a transition that a large number of bosons will have a tendency to occupy the same lowest-energy state. This phenomenon is called Bose-Einstein condensation. The following figure shows how bosons and fermions occupy energy states when the temperature approaches 0 K.
在自然界中，粒子可以分为两种不同的类型：玻色子和费米子。玻色子（例如光子）是喜欢一起处于相同状态的粒子。相反，根据泡利不相容原理，费米子（例如电子、质子和中子）不可能进入已经被占据的状态。统计力学告诉我们，当一个玻色子系统在阱中达到临界密度时，它会经历相变，令大量的玻色子倾向占据相同的最低能阶。这种现象称为玻色-爱因斯坦凝聚。下图显示当温度接近 0 K 时，玻色子和费米子如何占据能阶。



Recent development of trapping and cooling ultracold atoms (e.g. Sodium, Rubidium and Lithium atoms) paved the way for the observation of Bose-Einstein condensation of atomic gases in ultracold temperature (Nobel prize in physics 2001), which had been theoretically predicted by Bose and Einstein in 1924. Several different cooling techniques have been employed to achieve ultracold temperature around 10-100 nK (note $1\ \text{nK} = 10^{-9}\text{K}$). For example, the hot Rubidium atoms prepared at 400 K are cooled down to ~1mK through the Laser cooling techniques (Nobel prize in physics in 1997). Such cold atoms prepared by laser cooling technique are typically loaded into the external trap (produced by either magnetic or optical fields) for further cooling as shown below.

在捕获和冷却超冷原子（例如钠、铷和锂原子）的技术上，近年的进展为观察超冷温度下原子气体的玻色-爱因斯坦凝聚（2001 年诺贝尔物理学奖）提供了有利条件，印证了 1924 年玻色和爱因斯坦的预测。几种不同的冷却技术已被采用以实现约 10-100 nK 的超冷温度（注意 1 nK = 10^{-9}K）。例如，通过激光冷却技术（1997 年诺贝尔物理学奖），在 400K 下制备的热铷原子可以冷却至～1 mK。这种冷原子通常被加载到外部阱（由磁场或光场产生）中，用於进一步冷却，如下所示。

Laser cooling and trapping Rubidium atoms 激光冷却和捕获铷原子



A. Maxwell- Boltzmann distribution and the thermal de Broglie wavelength of the atoms
麦克斯韦-玻尔兹曼分布和原子的热德布罗意波长

Consider a dilute gas of atoms. The inter-particle interactions are very weak. In this case, the gas can be described by the ideal gas model in which the particles move freely inside a stationary trap without interacting with one another except for very brief elastic collisions to reach thermal equilibrium.
考虑稀释的原子气体。粒子间相互作用非常弱。在这种情况下，气体可以通过理想气体模型描述，其中粒子在固定阱内自由移动，除了在趋向热平衡的过程中会有非常短暂的弹性碰撞，彼此没有相互作用。

In this atomic gas system, the probability distribution of the particle speed v is given by Maxwell-Boltzmann distribution,
在这种原子气体系统中，粒子速度v 的概率分布由麦克斯韦 - 玻尔兹曼分布给出，

$$f(v) = \sqrt{\left(\frac{m}{2\pi k_B T}\right)^3}\, 4\pi v^2 e^{-\frac{mv^2}{2k_BT}},$$

where m is the mass of the atom, k_B is the Boltzmann constant, and T is the temperature of the gas in the unit of Kelvin [K].
其中 m 是原子的质量，k_B是玻尔兹曼常数，T是气体温度，单位为[K]。

A1	Derive the most probable velocity v_{mp} of a particle at temperature T.	**2 points**

	试推导温度为 T 时粒子最可能的速度v_{mp}。	**2 分**

$f'(v) = 0 \Rightarrow v_{mp} = \sqrt{\dfrac{2k_B T}{m}}$

(Remarks: 1 point if only $f'(v) = 0$ is given)

A2	Based on the most probable velocity v_{mp} obtained in A1, write down the characteristic de Broglie wavelength λ_{dB} of the particle in an atomic gas at temperature T. 根据在 A1 中求得的最可能速度v_{mp}，试写下温度为 T 时原子气体中粒子的特征德布罗意波长λ_{dB}。	**2 points** **2 分**

The de Broglie wavelength of the particle is

$$\lambda_{\mathrm{dB}} = \frac{h}{mv} \text{ (1 point)}$$

The characteristic de Broglie wavelength is estimated by replacing the velocity by the characteristic velocity v_{mp} as

$$\lambda_{\mathrm{dB}} = \sqrt{\frac{h^2}{2mk_B T}} \text{ (1 point)}$$

Since particles in a gas of atoms have different speed following Maxwell-Boltzmann distribution, it is useful to consider the <u>*thermal de Broglie wavelength* (λ_T)</u> defined as $\lambda_T = \lambda_{dB} \times \pi^{-\frac{1}{2}}$. Here we derive the Bose-Einstein temperature T_C for a gas of N non-interacting (bosonic) atoms of mass m in a three-dimensional box with volume V. We will consider the simple physical picture that Bose-Einstein condensation occurs when the characteristic inter-particle distance between bosonic atoms becomes comparable to the thermal de Broglie wavelength λ_T. (Planck's constant $h = 6.626 \times 10^{-34}$ Js, Boltzmann constant $k_B = 1.381 \times 10^{-23}$ JK^{-1})
由于原子气体中的粒子按著麦克斯韦-波尔兹曼分布，各有不同的速率，我们引入<u>热德布罗意波长(λ_T)</u>，定义为$\lambda_T = \lambda_{dB} \times \pi^{-\frac{1}{2}}$。在这里，我们会考虑在体积为 V 的三维盒子中的原子气体，其中有 N 个质量为 m 的非相互作用（玻色子）原子，我们会推导其玻色-爱因斯坦温度 T_C 。我们将採用一幅简单的物理图画，就是当玻色子原子间的特征距离和热德布罗意波长λ_T相若时，玻色-爱因斯坦凝聚便会发生。

A3	What is the expected T_C of the $N = 10^5$ atoms of mass $m = 1.445 \times 10^{-25}$ kg trapped in the trap with a volume of $V = 10^5 \mu\mathrm{m}^3$? ($1\ \mu\mathrm{m}^3 = 10^{-18}$ m^3) 在体积为 $V = 10^5 \mu\mathrm{m}^3$的阱中，捕获 $N = 10^5$ 个质量为 $m = 1.445 \times$	**3 points** **3 分**

	10^{-25} kg的原子，求 T_C 的预期值。(1 μm^3 = 10^{-18} m^3)	

At the Bose-Einstein temperature T_C, the inter-particle separation is equal to the de Broglie wavelength as

$\lambda_{dB} = \sqrt{\frac{h^2}{2\pi m k_B T}} \approx \left(\frac{V}{N}\right)^{\frac{1}{3}}$ (1 point for identifying the expression of the inter-particle separation)

Therefore, the T_C is given by

$$T_c = \frac{h^2}{2\pi m k_B}\left(\frac{N}{V}\right)^{\frac{2}{3}} \quad \text{(1 point)}$$

(Remarks: Note that this is an estimate, hence any final numerical result from an estimation containing $\sim \frac{h^2}{k_B m}\left(\frac{N}{V}\right)^{\frac{2}{3}}$ can be regarded as correct.)

For the given parameters, $N = 10^5$ atoms and trap volume $V = 10^5$ μm^3, $T_c \sim 35$ nK. (1 point)

(Note: The exact result is $T_c \approx 0.527 \frac{h^2}{2\pi m k_B}\left(\frac{N}{V}\right)^{\frac{2}{3}}$.

B. Evaporative cooling in an external trap 在外部阱中的蒸发冷却

The temperatures reached by laser cooling are extremely low (< 1 mK), but they are not cold enough to realize Bose-Einstein condensation. To date, Bose-Einstein condensation of alkali atoms has been achieved by using evaporative cooling after atoms are loaded into the external trap. During evaporative cooling, when atoms escaping from a trap have a kinetic energy higher than the average energy of atoms in the trap, the remaining atoms become cooled.

激光冷却达到的温度极低（< 1 mK），但还是不够冷去实现玻色-爱因斯坦凝聚。到目前为止，碱金属原子的玻色-爱因斯坦凝聚可以通过把原子加载到外部阱之后，使用蒸发冷却实现。在蒸发冷却期间，当从阱中逸出的原子具有高于阱中原子平均能量的动能时，剩余的原子就会冷却。

In the following problems in part B, we will estimate the effect of evaporative cooling. For atoms trapped in a box of fixed volume and having no heat exchange with the surroundings, we assume that an average energy of trapped atoms is ϵ and a small number of atoms $|\Delta N|$ are evaporated within a short time $\Delta\tau$ with an average energy of $(1+\beta)\epsilon$ where $\beta > 0$. During the process, the small change in the number of atoms $\Delta N < 0$ leads to the change $\Delta\epsilon < 0$ in the average energy of the remaining atoms. We also assume that $\left|\frac{\Delta\epsilon}{\epsilon}\right| \ll 1$ and $\left|\frac{\Delta N}{N}\right| \ll 1$.

在下面 B 部的问题中，我们将估计蒸发冷却的影响。对于被捕获在固定体积的盒子中，并且没有与周围环境进行热交换的原子，我们假设被捕获原子的平均能量是ϵ。假设有小数目的原子$|\Delta N|$ 短时间$\Delta\tau$内蒸发，其平均能量为$(1+\beta)\epsilon$，其中$\beta > 0$。在这过程中，原子数目$\Delta N < 0$ 的小变化，导致剩余原子的平均能量的变化$\Delta\epsilon < 0$。我们还假设$\left|\frac{\Delta\epsilon}{\epsilon}\right| \ll 1$和$\left|\frac{\Delta N}{N}\right| \ll 1$。

[Remark: In the derived relation, you may ignore the term $\frac{\Delta\epsilon}{\epsilon}\frac{\Delta N}{N}$ since $\frac{\Delta\epsilon}{\epsilon} \ll 1$ and $\left|\frac{\Delta N}{N}\right| \ll 1$.]

［备注：在推导的关系中，由于$\frac{\Delta\epsilon}{\epsilon} \ll 1$和$\left|\frac{\Delta N}{N}\right| \ll 1$，可以忽略 $\frac{\Delta\epsilon}{\epsilon}\frac{\Delta N}{N}$一项 。］

B1	Derive the relation between $\Delta\epsilon$ and ΔN with β, ϵ and N. 试用 β、 ϵ和 N，推导 $\Delta\epsilon$和 ΔN之间的关系。	**3 points** **3 分**

The average energy per atom after atom loss ΔN must be $\epsilon + \Delta\epsilon$.
Total (mechanical) energy of $(N + \Delta N)$ atoms after atom loss = $(N\epsilon + (1 + \beta)\epsilon\Delta N)$
Therefore, we have a relation:

$$\epsilon + \Delta\epsilon = \frac{N\epsilon + (1+\beta)\epsilon\Delta N}{N + \Delta N} \qquad \text{(1 point)}$$

Then

$$\beta\frac{\Delta N}{N} = \frac{\Delta\epsilon}{\epsilon} + \frac{\Delta\epsilon}{\epsilon}\frac{\Delta N}{N} \qquad \text{(1 point for steps)}$$

and

$$\beta\frac{\Delta N}{N} = \frac{\Delta\epsilon}{\epsilon} \qquad \text{(1 point)}$$

by ignoring the second order term as $\left|\frac{\Delta\epsilon}{\epsilon}\right|\left|\frac{\Delta N}{N}\right| \ll 1$.

Now we consider cold atoms at the initial temperature of $T_i = 200\mu\text{K}$ in a trap. Assume that we remove 1% of atoms (i.e. $\left|\frac{\Delta N}{N}\right| = 0.01$) during each time period $\Delta\tau$ and $\beta = 2$.
现在我们考虑阱中的冷原子，初始温度为 $T_i = 200\mu\text{K}$。假设我们在每段时间 $\Delta\tau$期间去除1%的原子（即$\left|\frac{\Delta N}{N}\right| = 0.01$），并且$\beta = 2$。

B2	Then estimate the final temperature T_f of atoms after the evaporative cooling over the total time period of $350\Delta\tau$. 试估计在$350\Delta\tau$的总时间段内，经蒸发冷却后的原子的最终温度T_f 。	**3 points** **3 分**

From the result of B1, we know $\beta\frac{\Delta N}{N} = \frac{\Delta T}{T}$. (1 point)
For each time period $\Delta\tau$, the temperature changes:

$$T_{\text{after}} = T_{\text{before}} + \Delta T = T_{\text{before}}\left(1 + \frac{\Delta T}{T_{\text{before}}}\right) = T_{\text{before}}\left(1 + \frac{\beta\Delta N}{N}\right) \qquad \text{(1 point)}$$

Here note that $\Delta T < 0$ and $\Delta N < 0$.

Therefore, after 350 time period $\Delta\tau$,

$$T_{\text{final}} = \left(1 - \frac{\beta\Delta N}{N}\right)^{350} T_{\text{initial}} = (1 - 2 \times 0.01)^{350} \times (2 \times 10^5)\text{nK} \qquad \text{(1 point)}$$

$$= 169.8 \text{ nK} \qquad \text{(1 point)}$$

[Remarks : Alternative approximation may give slightly different temperature. The final temperature T_{final} between 160 nK and 180 nK can be regarded as correct.]

C. Bose-Einstein temperature T_C in a harmonic potential
谐波势中的玻色-爱因斯坦凝聚温度 T_C

In a real experiment with ultracold atomic gases, a gas of bosonic atoms is trapped in a three-dimensional harmonic trap generated by the laser beam or the magnetic field. Here we consider a three-dimensional trap characterized by the harmonic potential:
在超冷原子气体的真实实验中，玻色原子气体被捕获在由激光束或磁场产生的三维谐波阱中。这里我们考虑一个三维阱，可用谐波势描述：

$$U_{\text{trap}} = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2).$$

C1	Consider the fact that ultracold atoms are oscillating around the bottom of the trap with the characteristic trapping frequency $\omega_i/2\pi$ along the i-direction. Derive the characteristic volume confining the atoms in terms of T and $\omega_{x,y,z}$. 考虑超冷原子在阱底振荡，沿着 i 方向的特征捕获频率为 $\omega_i/2\pi$ 。试推导原子被限定的特征体积，答案以 T 和$\omega_{x,y,z}$表达。	**3 points** **3 分**

Along the x-axis, the characteristic kinetic energy of the atom is given by $\frac{1}{2}k_B T$ from the kinetic theory of the ideal gas. Then the maximum displacement R_X of the atom trapped in the harmonic potential $U_{\text{trap}}^x = \frac{1}{2}m\omega_x^2 R_x^2$ is given by

$$\frac{1}{2}k_B T = \frac{1}{2}m\,\omega_x^2 R_x^2 \quad \text{(2 points)}$$

and

$$R_x = \sqrt{\frac{k_B T}{m\omega_x^2}}$$

In a similar way, one can derive

$$R_{y,z} = \sqrt{\frac{k_B T}{m\omega_{y,z}^2}}$$

Therefore, the characteristic volume V is given

$$V \sim R_x R_y R_z = \frac{\left(\frac{k_B T}{m}\right)^{\frac{3}{2}}}{\omega_x \omega_y \omega_z} \quad \text{(1 point)}$$

[Remarks: Note that this is an estimate, hence any final numerical result from an estimation containing $\sim \frac{\left(\frac{k_B T}{m}\right)^{\frac{3}{2}}}{\omega_x \omega_y \omega_z}$can be regarded as correct.]

C2	Derive the Bose-Einstein condensation temperature T_C of the atoms trapped in a harmonic trap considered in Part C1 in terms of ω_i and N. 试推导 C1 部的谐波阱中捕获的原子的玻色-爱因斯坦凝聚温度 T_C，答案以 ω_i 和 N表达。	**2 points** **2 分**

The characteristic inter-particle separation is given by

$$\left(\frac{V}{N}\right)^{\frac{1}{3}} \sim N^{-\frac{1}{3}}\left(\frac{k_B T}{m}\right)^{\frac{1}{2}}\left(\omega_x\omega_y\omega_z\right)^{\frac{1}{3}} \text{ (1 point)}$$

Considering $\lambda_{\mathrm{T}} \sim \left(\frac{V}{N}\right)^{\frac{1}{3}} = \sqrt{\frac{h^2}{2\pi m k_B T}}$,

$$T_C = \frac{h}{k_B\sqrt{2\pi}}\left(\omega_x\omega_y\omega_z\right)^{\frac{1}{3}} N^{\frac{1}{3}} \text{ (1 point)}$$

Note: The exact result is $T_{\mathrm{C}} \simeq 0.15\frac{h}{k_{\mathrm{B}}}\left(\omega_x\omega_y\omega_z\right)^{\frac{1}{3}} N^{\frac{1}{3}}$.

[Remark: Note that this is an estimate, hence any final numerical result from an estimation containing $\sim\frac{1}{k_B}\, h\left(\omega_x\omega_y\omega_z\right)^{\frac{1}{3}} N^{\frac{1}{3}}$ can be regarded as correct.]

C3	What is the Bose-Einstein condensation temperature T_c of the $N = 10^4$ atoms of mass $m = 1.445 \times 10^{-25}$ kg in the harmonic trap with trapping frequencies $\omega_x/2\pi = \omega_y/2\pi = \omega_z/2\pi = 100$ Hz? 谐波阱中有 $N = 10^4$个原子，每个原子的质量为$m = 1.445 \times 10^{-25}$ kg，谐波频率为 $\omega_x/2\pi = \omega_y/2\pi = \omega_z/2\pi = 100$ Hz。求玻色−爱因斯坦凝聚温度 T_c。	**1 point** **1 分**

From the result of part C2, $T_C = \frac{h}{k_B\sqrt{2\pi}}\left(\omega_x\omega_y\omega_z\right)^{\frac{1}{3}} N^{\frac{1}{3}}$.

Using $\omega_x = \omega_y = \omega_z = 2\pi \times 100$ rad/s, $N = 10^4$, $T_c = 259.1$ nK. (1 point)

Note that the evaporative cooling is efficient enough to achieve the Bose-Einstein condensation.
注意，蒸发冷却的效率足以实现玻色-爱因斯坦凝聚。

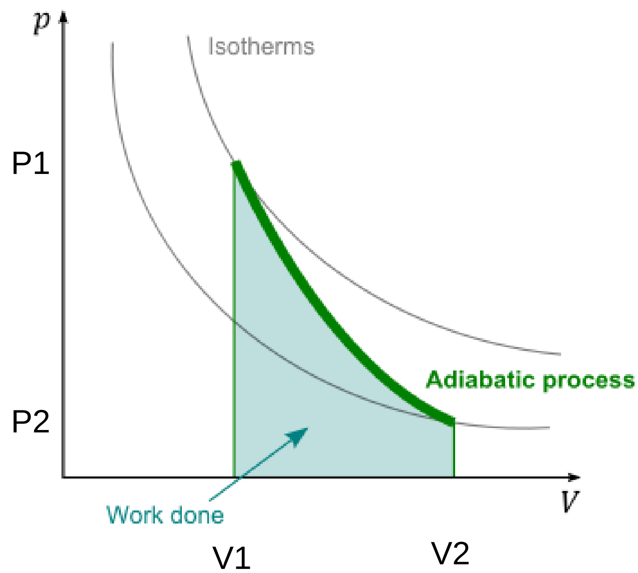
D. Adiabatic cooling by slowly expanding the trap 通过缓慢膨胀阱进行绝热冷却

Cooling atomic gases to lower temperature has been motivated by the quest to observe new forms of matter such as superfluid. However the evaporative cooling we discussed in part B is not always preferable since a number of atoms leave the trap during the process. In this part we consider a different cooling technique (so-called adiabatic cooling) by slowly expanding the trap without losing atoms.
把原子气体冷却至更低温度的动机，是寻求物质的新状态（例如超流体）。然而，我们在 B 部中讨论的蒸发冷却，不一定是首选的方法，因为在该过程中有许多原子离开了阱。在这部中，我们会考虑另一冷却技术（所谓的绝热冷却），是通过缓慢地膨胀阱而不损失原子而达成的。

D1	Calculate the fraction of atoms remaining in the trap after the evaporative cooling described in part B2. 试计算在 B2 部中描述的蒸发冷却之后，留在阱中的原子的分数。	**1 point** **1 分**

Fraction of remaining atoms $= (1 - 0.01)^{350} = 0.03$ (1 point)

Consider N atoms in an external harmonic trap with trapping frequencies of $\omega_x = \omega_y = \omega_z = 2\pi f_0$ at the temperature $T_1 = 105$ nK $= 1.05 \times 10^{-7}$ K. From now on, we assume that the whole atomic gas can be regarded as a monoatomic ideal gas. At this stage, the atomic gas has the pressure P_1 and the volume V_1 as described in the figure below.
考虑在外部谐波阱中有N个原子，捕获频率为$\omega_x = \omega_y = \omega_z = 2\pi f_0$，其温度为$T_1 = 105$ nK $= 1.05 \times 10^{-7}$ K。从现在开始，我们假设整个原子气体可视为单原子理想气体。这时，原子气体具有如下图所示的压强P_1和体积V_1。



Now consider the adiabatic decompression process of N atoms trapped in a harmonic trap. For this we adiabatically change the trapping frequencies of the harmonic potential trap from $\omega_{x,1} = \omega_{y,1} = \omega_{z,1} = 2\pi f_0$ to $\omega_{x,2} = 2\pi f_0$ and $\omega_{y,2} = \omega_{z,2} = \frac{2\pi f_0}{10}$ following the adiabatic process in the P-V diagram. Note that there is no heat exchange between the atomic gas and the environment (actually vacuum) and no atoms leave the trap during the process.
现在考虑捕获 N 个原子的谐波阱的绝热减压过程。为此，我们绝热地将谐波势阱的捕获频率从$\omega_{x,1} = \omega_{y,1} = \omega_{z,1} = 2\pi f_0$ 改变至 $\omega_{x,2} = 2\pi f_0$ 和 $\omega_{y,2} = \omega_{z,2} = \frac{2\pi f_0}{10}$。注意，原子气体和环境（实际上是真空）之间没有热交换，并且在该过程中没有原子离开阱。

D2	Calculate the final temperature of the atomic gas after adiabatic decompression of the trap. 试计算阱绝热减压后原子气体的最终温度。	**2 points** **2 分**

$PV^{\gamma} =$ constant and $PV = nRT$ implies $T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$. (1 point)
For the monoatomic gas with $\gamma = 5/3$. From the result of (C2), the volume changes as

$$V_2 = 100\, V_1$$

and thus

$$T_2 = \frac{T_1}{100^{\gamma-1}} = \frac{105}{100^{\frac{5}{3}-1}} = 5 \text{ nK (1 point)}$$

Remark: Some students first substituted the adiabatic relation into the result of part C2 to find the relation between the volume and the trapping frequencies. They found that V only expands 10 times and the final temperature is 23 nK. This answer has sound physics reasoning and is also considered correct. On the other hand, the question setter realized that in actual experiments, the volume does not change that dramatically on cooling due to the repulsive interaction. In retrospect, the problem might be less ambiguous by directly pointing out that the volume increases 100 times.

Remark: Using this adiabatic cooling via adiabatic decompression of the trap, researchers at MIT had achieved the coldest matter in universe around 500 pico-Kelvin in 2003 (research work reported in Science 301, 1513-1515 (2003)).

Acknowledgement: We thank Prof. Gyu-Boong Jo for contributing this interesting question.

Problem 2: Swimming Microorganisms (33 points) 游泳微生物（33 分）

Although objects in water tend to sink in a gravitational field, microorganisms such as paramecium can control their swimming directions not necessarily subject to gravitational field. Recently, physicists proposed that their swimming patterns are related to their asymmetric shape. When they swim in a viscous fluid, they experience asymmetric resistance forces that may cause them to rotate.
虽然水中的物体倾向于在重力场中下沉，但是诸如草履虫的微生物可以控制它们的游泳方向，不一定受到重力场的影响。最近，物理学家提出他们的游泳模式和他们不对称的形状有关。当它们在粘性流体中游泳时，它们经历可能导致它们旋转的不对称抗阻力。

A. Resistive Forces and Torques in a Viscous Fluid 粘性流体中的抗阻力和力矩

For a rod having a translational motion in a viscous fluid, there are two kinds of resistive forces. In this question, we will refer to the resistive force acting in the normal direction of the rod as the *drag*, and the resistive force along the direction of the rod as the *friction*, as shown in Fig. 1(a). The drag per unit length is approximated as $\mu v_\perp$, and the friction per unit length as $\frac{\mu v_\parallel}{2}$, where $v_\perp$ and $v_\parallel$ are the velocity components normal and parallel to the axis of the rod respectively, and μ is a constant proportional to the viscosity of the fluid.
在粘性流体中有平移运动的杆子，存在两种抗阻力。在本题中，我们将作用在杆子的法向方向上的抗阻力，称为阻力，而将作用在沿杆方向的阻力，称为摩擦力，如图 1(a)所示。每单位长度的阻力近似为$\mu v_\perp$，而每单位长度的摩擦力为$\frac{\mu v_\parallel}{2}$，其中 $v_\perp$ 和 $v_\parallel$分别为垂直和平行于杆轴线的速度分量，μ是与流体粘度成比例的常数。

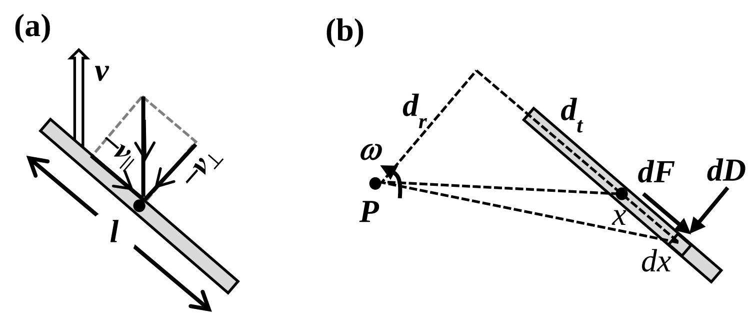


Fig. 1: (a) Directions of the resistive forces acting on a rod moving in a viscous fluid with velocity **v** indicated as the white arrow. The drag is directed along $-\mathbf{v}_\perp$, and the friction along $-\mathbf{v}_\parallel$. (b) The resistive forces acting on an element of the rod rotating about point P in the same plane at radial distance d_r and tangential distance d_t from its center.
图 1：（a）杆子在粘性流体中运动时，作用在杆子上的抗阻力的方向，速度 **v** 以白色箭头表示。阻力方向沿著$-\mathbf{v}_\perp$，摩擦力方向则沿著$-\mathbf{v}_\parallel$。（b）杆子围绕 P 点在同一平面旋转，杆子中心与 P 点的径向距离为 d_r，切向距离为 d_t。图示作用在杆子的一小段上的抗阻力。

As shown in Fig. 1(b), consider a reference point P whose radial and tangential distances from the center of the rod are d_r and d_t respectively. If the rod has a fixed position and orientation with respect to P, and P has a translational motion, then the resistive forces acting on the rod can be calculated using Fig. 1(a). However, if the rod also rotates in the same plane about P at an angular velocity ω, there will be extra forces and torques acting on the rod due to drag and friction.

如图 1（b）所示，考虑一个参考点 P，与杆子中心的径向和切向距离分别为 d_r 和 d_t。如果杆子相对于 P 点的位置和取向固定，并且 P 在作平移运动，则可使用图 1(a)计算作用在杆子上的抗阻力。然而，如果杆子也在同一平面以角速度ω围绕 P 转动，阻力和摩擦力将产生额外的力和力矩作用在杆子上。

A1	Derive the friction F due to the rotational motion. 试推导由于旋转运动引起的摩擦力 F。	**1 point** **1 分**

As shown in Fig. 1(c), $dF = \frac{\mu}{2}\left(\omega\sqrt{(x+d_t)^2+d_r^2}\,dx\right)\frac{d_r}{\sqrt{(x+d_t)^2+d_r^2}} = \frac{1}{2}\mu d_r\omega dx$

$F = \frac{1}{2}\mu l d_r \omega$ [1]

A2	Derive the drag D due to the rotational motion. 试推导由于旋转运动引起的阻力 D。	**1 point** **1 分**

As shown in Fig. 1(c), $dD = \mu\left(\omega\sqrt{(x+d_t)^2+d_r^2}\,dx\right)\frac{x+d_t}{\sqrt{(x+d_t)^2+d_r^2}} = \mu\omega(x+d_t)dx$

$D = \mu\omega\int_{-l/2}^{l/2}(x+d_t)dx = \mu l d_t \omega$ [1]

A3	Derive the torque τ_f about the axis of rotation due to the friction. 试推导摩擦力围绕旋转轴心的力矩 τ_f 。	**1 point** **1 分**

As shown in Fig. 1(c), $d\tau_f = \frac{\mu}{2}\left(\omega\sqrt{(x+d_t)^2+d_r^2}\,dx\right)\left(\frac{d_r}{\sqrt{(x+d_t)^2+d_r^2}}\right)d_r = \frac{1}{2}\mu\omega d_r^2 dx$

Torque $\tau_f = \frac{1}{2}\mu l d_r^2\omega$ [1]

A4	Derive the torque τ_d about the axis of rotation due to the drag. 试推导阻力围绕旋转轴心的力矩 τ_d 。	**2 points** **2 分**

As shown in Fig. 1(c), $d\tau_d = \mu\left(\omega\sqrt{(x+d_t)^2+d_r^2}\,dx\right)\frac{x+d_t}{\sqrt{(x+d_t)^2+d_r^2}}(x+d_t) = \mu\omega(x+d_t)^2 dx$

$\tau_d = \mu\omega\int_{-l/2}^{l/2}(x+d_t)^2dx = \mu\omega\int_{-l/2}^{l/2}(x^2+2d_tx+d_t^2)dx = \frac{1}{12}\mu l^3\omega + \mu l d_t^2\omega$ [1,1]

B. A Passive Microswimmer with No Rotation 无动力又不旋转的游泳微生物

An asymmetric microswimmer is L-shaped with the dimensions shown in Fig. 2(a). The mass of the microswimmer is m and the density is uniform. The lengths of the long and short arms are $4b$ and $2b$ respectively. The width and thickness of its two arms are negligible.
有不对称的游泳微生物具有 L 形的形状，尺寸如图 2（a）所示。游泳微生物的质量为 m，密度均匀。长臂和短臂的长度分别为 $4b$ 和 $2b$。两臂的宽度和厚度可忽略。

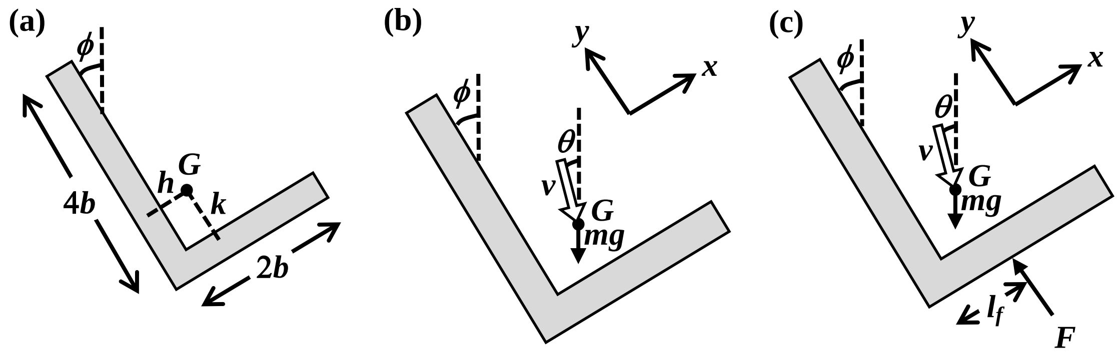


Fig. 2: (a) Dimensions of the microswimmer. (b) The weight and the velocity of a passive microswimmer. (c) An active microswimmer.
图 2:（a）游泳微生物的尺寸。（b）无动力游泳微生物的重量和速度。（c）具动力的游泳微生物。

A passive microswimmer does not have any self-propulsion. The center of mass G of the microswimmer is at a distance h and k from the midlines of the long and short arms respectively.
无动力的游泳微生物不具有任何自推进力。游泳微生物的质心 G 与长臂和短臂的中线的距离分别为 h 和 k。

B1	Write the expressions of h and k. 试写下 h 和 k 的表达式。	**2 points 2 分**

$h = \frac{b}{3}$, $k = \frac{4b}{3}$.   [1,1]

The L-shaped microswimmer is tilted by an angle ϕ as shown in Fig. 2(b) and is sinking with velocity v in the direction inclined at an angle θ with the vertical in the presence of gravitational acceleration g. The microswimmer does not rotate. Assume that the upthrust of the fluid is negligible compared with the weight of the microswimmer.
L 形游泳微生物的倾斜角度为ϕ，如图 2（b）所示，并且在重力加速度 g 的影响下，以速度 v 下沉，速度相对垂直方向的倾斜角度为θ。游泳微生物不旋转。假设流体的浮力与游泳微生物的重量相比是可忽略的。

B2	Write the equation consisting of the components of all forces along the y axis (the direction of the long arm). 试写下沿 y 轴（长臂的方向）的所有分力组成的方程。	**2 points 2 分**

Friction on the long arm: $F_a = \mu 4bv\cos(\phi - \theta)/2$ [0.5]

Drag on the short arm: $D_b = \mu 2bv\cos(\phi - \theta)$ [0.5]

Hence $F_a + D_b = mg\cos\phi \Rightarrow 4\mu bv\cos(\phi - \theta) = mg\cos\phi$ [1]

B3	Write the equation consisting of the components of all forces along the x axis (direction of the short arm). 试写下沿 x 轴（短臂的方向）的所有分力组成的方程。	**2 points** **2 分**

Drag on the long arm: $D_a = \mu 4bv\sin(\phi - \theta)$ [0.5]

Friction on the short arm: $F_b = \mu 2bv\sin(\phi - \theta)/2$ [0.5]

Hence $D_a + F_b = mg\sin\phi \Rightarrow 5\mu bv\sin(\phi - \theta) = mg\sin\phi$ [1]

B4	Write the equation consisting of the moments of all forces about the center of mass. 试写下所有围绕质心的力的力矩组成的方程。	**2 points** **2 分**

Clockwise moments of the long arm in Fig. 2(b): [0.5]

Drag on the long arm: $D_a(2b - k) = 8\mu b^2 v\sin(\phi - \theta)/3$

Friction on the long arm: $F_a(a - h) = 2\mu b^2 v\cos(\phi - \theta)/3$

Anticlockwise moments of the short arm in Fig. 2(b): [0.5]

Drag on the short arm: $D_b(b - h) = 4\mu b^2 v\cos(\phi - \theta)/3$

Friction on the short arm: $F_b(2b - k) = 4\mu b^2 v\sin(\phi - \theta)/3$

Hence $D_a(2b - k) + F_a(a - h) = D_b(b - h) + F_b(2b - k)$

$\Rightarrow 8\mu b^2 v\sin(\phi - \theta)/3 + 2\mu b^2 v\cos(\phi - \theta)/3 = 4\mu bv\cos(\phi - \theta)/3 + 4\mu bv\sin(\phi - \theta)/3$ [1]

$$\tan(\phi - \theta) = \frac{1}{2}$$

B5	Calculate the tilt angle ϕ of the microswimmer at the steady state. Give your answer in degrees. 试计算游泳微生物在稳态下的倾斜角ϕ。答案以度数表达。	**1 point** **1 分**

From B2 and B3,

$$mg\cos\phi = 4\mu bv\cos(\phi - \theta)$$

$$mg\sin\phi = 5\mu bv\cos(\phi - \theta)$$

Dividing, $\tan\phi = \frac{5}{4}\tan(\phi - \theta) = \left(\frac{5}{4}\right)\left(\frac{1}{2}\right) = \frac{5}{8} \Rightarrow \phi = \arctan\frac{5}{8} = 0.56\text{ rad} = 32^0$ [1]

B6	Calculate the motion direction θ of the microswimmer at the steady state. Give your answer in degrees.	**1 point**

	试计算游泳微生物在稳态时的运动方向θ。答案以度数表达。	

$$\tan\theta = \frac{\tan\phi - \tan(\phi-\theta)}{1+\tan\phi\tan(\phi-\theta)} = \frac{5/8-1/2}{1+(5/8)(1/2)} = \frac{2}{21} \Rightarrow \theta = \arctan\frac{2}{21} = 0.095 \text{ rad} = 5.4^0$$ [1]

C. An Active Microswimmer with Rotation 具动力又会旋转的游泳微生物

To model an active microswimmer, physicists implemented a laser-induced chemical reaction at a point on the shorter arm of the object so that it provides a self-propulsion force F normal to the short arm. The dynamical properties of the microswimmer are rather sensitive to the point of application of F. For convenience we consider the case that this point is located at a distance $l_f = \frac{13}{24}b$ from the corner (see Fig. 2(c)). The force can be adjusted by tuning the laser intensity incident on the microswimmer. Note that it is possible that the microswimmer can rotate so that forces and torques due to rotation have to be included. The velocity v, direction θ and the tilt angle ϕ becomes time dependent, and you will need to include the angular velocity $\dot{\phi}$ as one of the variables.
为了模拟具动力的游泳微生物，物理学家在物体短臂上的一点加进可由激光诱导的化学反应，为它提供垂直于短臂的自推进力 F。游泳微生物的动力学性质对于 F 的作用点是相当敏感的。为了方便起见，我们考虑这一点位于距离角落$l_f = \frac{13}{24}b$的情况（参见图 2（c））。调节射在游泳微生物上的激光强度，可以调节推进力。注意，因为游泳微生物可以旋转，我们必须加入考虑由于旋转引起的力和力矩。速度 v、方向θ和倾斜角ϕ变得与时间相关，你需要将角速度$\dot{\phi}$包括为其中一个变量。

C1	Write the equation consisting of the components of all forces along the y axis (the direction of the long arm). 试写下沿 y 轴（长臂的方向）的所有分力组成的方程。	**2 points 2 分**

Friction on the long arm due to translation: $F_{ta} = 2\mu bv\cos(\phi-\theta)$

Drag on the short arm due to translation: $D_{tb} = 2\mu bv\cos(\phi-\theta)$

Friction on the long arm due to rotation: $F_{ra} = \mu 4bh\dot{\phi}/2 = 2\mu b^2\dot{\phi}/3$

Drag on the short arm due to rotation: $D_{rb} = -\mu 2b(b-h)\dot{\phi} = -4\mu b^2\dot{\phi}/3$

Hence $F_{ta} + F_{ra} + D_{tb} + D_{rb} + F = mg\cos\phi \Rightarrow F + 4\mu bv\cos(\phi-\theta) - \frac{2}{3}\mu b^2\dot{\phi} = mg\cos\phi$

[1 for terms dependent on $\dot{\phi}$, 1 for the equation]

C2	Write the equation consisting of the components of all forces along the x axis (the direction of the short arm). 试写下沿 x 轴（短臂的方向）的所有分力组成的方程。	**2 points 2 分**

Drag on the long arm due to translation: $D_{ta} = 4\mu bv\sin(\phi-\theta)$

Friction on the short arm due to translation: $F_{tb} = \mu bv \sin(\phi - \theta)$

Drag on the long arm due to rotation: $D_{ra} = \mu 4b(2b-k)\dot{\phi} = 8\mu b^2\dot{\phi}/3$

Friction on the short arm due to rotation: $F_{rb} = -\mu 2bk\dot{\phi}/2 = -4\mu b^2\dot{\phi}/3$

Hence $D_{ta} + D_{ra} + F_{tb} + F_{rb} = mg \sin\phi \Rightarrow 5\mu bv \sin(\phi-\theta) + \frac{4}{3}\mu b^2\dot{\phi} = mg \sin\phi$

[1 for terms dependent on $\dot{\phi}$, 1 for the equation]

C3	Write the equation consisting of the moments of all forces about the center of mass. 试写下所有围绕质心的力的力矩组成的方程。	**4 points** **4 分**

Clockwise moments of the long arm in Fig. 2(c):

Drag on the long arm due to translation: $D_{ta}(2b-k) = 8\mu b^2 v\sin(\phi-\theta)/3$

Friction on the long arm due to translation: $F_{ta}(a-h) = 2\mu b^2 v\cos(\phi-\theta)/3$

Drag on the long arm due to rotation: $64\mu b^3\dot{\phi}/12 + D_{ra}(2b-k) = 64\mu b^3\dot{\phi}/9$ [0.5]

Friction on the long arm due to rotation: $F_{ra}h = 2\mu b^3\dot{\phi}/9$ [0.5]

Anticlockwise moments of the short arm in Fig. 2(c):

Drag on the short arm due to translation: $D_{tb}(b-h) = 4\mu b^2 v\cos(\phi-\theta)/3$

Friction on the short arm due to translation: $F_{tb}k = 4\mu b^2 v\sin(\phi-\theta)/3$

Drag on the short arm due to rotation: $-8\mu b^3/12 + D_{rb}(b-h) = -14\mu b^3\dot{\phi}/9$ [0.5]

Friction on the short arm due to rotation: $-F_{rb}k = -16\mu b^3\dot{\phi}/9$ [0.5]

Self-propulsion force: $F(13b/24 - h) = 5Fb/24$ [1]

$$8\mu b^2 v\sin(\phi-\theta)/3 + 2\mu b^2 v\cos(\phi-\theta)/3 + 64\mu b^3\dot{\phi}/9 + 2\mu b^3\dot{\phi}/9$$
$$= 4\mu b^2 v\cos(\phi-\theta)/3 + 4\mu b^2 v\sin(\phi-\theta)/3 - 14\mu b^3\dot{\phi}/9 - 16\mu b^3\dot{\phi}/9 + 5Fb/24$$

$$\frac{32}{3}\mu b^2\dot{\phi} = \frac{5}{24}F + \frac{2}{3}\mu bv\left[\cos(\phi-\theta) - 2\sin(\phi-\theta)\right]$$ [1]

The above three equations can be solved for the three variables $v\cos(\phi-\theta)$, $v\sin(\phi-\theta)$ and $\dot{\phi}$.
上述三个方程可以对$v\cos(\phi-\theta)$、$v\sin(\phi-\theta)$和$\dot{\phi}$三个变量求解。

C4	Eliminate $v\cos(\phi-\theta)$ and $v\sin(\phi-\theta)$ from the above equations to obtain an equation involving ϕ and $\dot{\phi}$ only. 从上述方程中消去$v\cos(\phi-\theta)$和$v\sin(\phi-\theta)$，以获得一个单涉及ϕ和$\dot{\phi}$的方程。	**2 points** **2 分**

From C1 to C3,

$$5\mu bv \sin(\phi-\theta) = mg\sin\phi - \frac{4}{3}\mu b^2\dot{\phi} \qquad (1)$$

$$4\mu bv\cos(\phi-\theta) = mg\cos\phi + \frac{2}{3}\mu b^2\dot{\phi} - F \qquad (2)$$

$$\frac{32}{3}\mu b^2\dot{\phi} = \frac{5}{24}F + \frac{2}{3}\mu bv\left[\cos(\phi-\theta) - 2\sin(\phi-\theta)\right] \qquad (3)$$

Substituting (1) and (2) into (3),

$$\frac{153}{15}\mu b^2\dot{\phi} = \frac{1}{24}F - \frac{mg}{30}(8\sin\phi - 5\cos\phi)$$

$$\dot{\phi} = \frac{1}{306\,\mu b^2}\left[\frac{5F}{4} - mg\,(8\sin\phi - 5\cos\phi)\right] = \frac{1}{306\,\mu b^2}\left[\frac{5F}{4} - \sqrt{89}\,mg\,\sin(\phi-\phi_0)\right] \text{where } \sin\phi_0 = \frac{5}{\sqrt{89}}$$

[1 point for steps, 1 point for the result]

C5	Derive the tilt angle ϕ when the microswimmer reaches the steady state of constant tilt. 试推导游泳微生物在固定倾斜稳态下的倾斜角ϕ。	**2 points** **2 分**

$$\frac{1}{306\,\mu b^2}\left[\frac{5F}{4} - \sqrt{89}\,mg\,\sin(\phi-\phi_0)\right] = 0 \;\Rightarrow\; \phi = \arcsin\frac{5}{\sqrt{89}} + \arcsin\frac{5F}{4\sqrt{89}\,mg}$$

C6	Consider a microswimmer initially at the steady state with $F = 0$. At $t = 0$ the laser is switched on so that F becomes nonzero. Calculate $\phi(t)$ for $F << mg$. 考虑游泳微生物的初始状态处於 $F = 0$ 的稳态。在 $t = 0$ 时，激光亮了，使得 F 变为非零。在 $F << mg$ 的情况下，试计算$\phi(t)$。	**2 points** **2 分**

$$\frac{d\phi}{dt} = \frac{1}{306\,\mu b^2}\left[\frac{5F}{4} - \sqrt{89}\,mg\,\sin(\phi-\phi_0)\right] \approx -\frac{\sqrt{89}\,mg}{306\,\mu b^2}\left(\phi - \phi_0 - \frac{5F}{4\sqrt{89}\,mg}\right) \qquad [1]$$

Solution: $\phi = \phi_0 + \frac{5F}{4\sqrt{89}\,mg}\left[1 - \exp\left(-\frac{\sqrt{89}\,mg}{306\,\mu b^2}t\right)\right]$ [1]

When F gradually increases from 0, the direction of linear motion gradually changes. When F exceeds a critical value, the tilt angle is no longer constant and the microswimmer takes a wheel-like trajectory.

当 F 从 0 渐渐增加时，游泳微生物线性运动的方向渐渐改变。当 F 超过临界值时，倾斜角不再恒定，游泳微生物的轨迹变成轮状。

C7	Write the maximum value of F for linear motion. 写下线性运动的最大 F 值。	**1 point** **1 分**

The equation $\frac{1}{306\mu b^2}\left[\frac{5F}{4} - \sqrt{89}\,mg\sin(\phi-\phi_0)\right] = 0$ has no solution when $F > \frac{4\sqrt{89}}{5}mg$. Hence the answer is $F = \frac{4\sqrt{89}}{5}mg$. [1]

C8	To verify that the microswimmer can move in a wide range of directions, calculate the force(s) required for linear motion in the horizontal direction $\theta = \pi/2$. Give your answer in multiples of mg to 3 significant figures. 为了验证游泳微生物能够在广阔范围内的方向移动，试计算游泳微生物在水平方向$\theta = \pi/2$ 作线性运动时，所需的(诸)力是多少。答案以 mg 的倍数表达，至 3 位有效数字。	**3 points** **3 分**

From C4, $\dot{\phi} = 0 \Rightarrow F = \frac{4}{5}mg(8\sin\phi - 5\cos\phi)$

From C1 and C2,

$$5\mu bv\sin(\phi-\theta) = mg\sin\phi$$
$$4\mu bv\cos(\phi-\theta) = mg\cos\phi - F$$ [1]

Dividing, $\tan(\phi-\theta) = \frac{4mg\sin\phi}{5(mg\cos\phi - F)} = \frac{4\sin\phi}{25\cos\phi - 32\sin\phi}$

When $\theta = \frac{\pi}{2}$, $\tan(\phi-\theta) = -\cot\phi = -\frac{\cos\phi}{\sin\phi} \Rightarrow -\frac{\cos\phi}{\sin\phi} = \frac{4\sin\phi}{25\cos\phi - 32\sin\phi}$

$$25\cos^2\phi - 32\sin\phi\cos\phi + 4\sin^2\phi = 0$$ [1]

Since $\cos^2\phi = \frac{1}{2} + \frac{1}{2}\cos 2\phi$, $\sin^2\phi = \frac{1}{2} - \frac{1}{2}\cos 2\phi$, $\sin\phi\cos\phi = \frac{1}{2}\sin 2\phi$,

$$32\sin 2\phi - 21\cos 2\phi = 29$$

$\sqrt{1465}\sin(2\phi - 2\phi_1) = 29$ where $\sin 2\phi_1 = \frac{21}{\sqrt{1465}}$

$$\phi = \frac{1}{2}\left(\arcsin\frac{29}{\sqrt{1465}} + \arcsin\frac{21}{\sqrt{1465}}\right) = 0.7202 \Rightarrow F = \frac{4}{5}mg(8\sin\phi - 5\cos\phi) = 1.21\,mg$$

$$\phi = \frac{1}{2}\left(\pi - \arcsin\frac{29}{\sqrt{1465}} + \arcsin\frac{21}{\sqrt{1465}}\right) = 1.4313 \Rightarrow F = \frac{4}{5}mg(8\sin\phi - 5\cos\phi) = 5.78\,mg$$

[1 for either result]

The background material of this problem can be found in the article: B. ten Hagen, F. Kümmel, R. Wittkowski, D. Takagi, H. Löwen, and C. Bechinger, “Gravitaxis of asymmetric self-propelled colloidal particles”, Nature Communications **5**, 4829 (2014).