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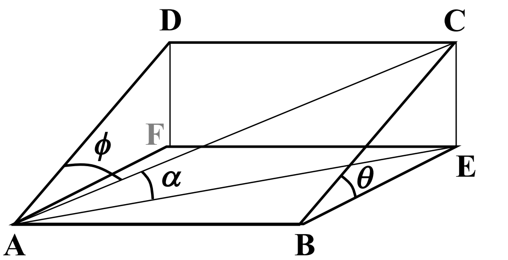
Pan Pearl River Delta Physics Olympiad 2018
2018 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 6 Problems, 45 Points) 简体版卷-1（共6题，45分）
(9:00 am – 11:45 am, 22 February, 2018)

Please fill in your final answers to all problems on the **answer sheet**.
请在答题纸上填上各题的最后答案。
At the end of the competition, please submit the **answer sheet only**. Question papers and working sheets will **not** be collected.
比赛结束时，请只交回答题纸，题目纸和草稿纸将不会收回。

1. Length of Daytime (7 points) 白昼长度（7 分）

(a) In the figure, ABCD is a rectangle lying on an inclined plane making an angle θ with the horizontal plane. ABEF is the projection of the rectangle on the horizontal plane. If the measure of the angle DAC is ϕ, derive an expression for the angle α. [2]
如图所示，矩形 ABCD 位于斜面上，斜面与水平面夹角为θ。ABEF 为该矩形于水平面的投影。设角 DAC 为ϕ，试推导角α的表达式。[2]



(b) The ecliptic is the plane on which the Earth revolves around the Sun. The axis of rotation of the Earth is inclined at an angle of 23.4° with the normal to the ecliptic. The day of the Winter Solstice (in the Northern Hemisphere) is 21 December. Using the result of (a) or otherwise, calculate the incident angle of sunlight relative to Earth's equatorial plane today (22 February). [2]
黄道面是指地球围绕太阳公转的平面。地球的自转轴相对于黄道面的法线倾斜，角度为 23.4°。在北半球，冬至的日期为 12 月 21 日。试用(a)部结果或其他方法，计算今天（2 月 22 日）阳光相对于赤道面的角度。[2]

(c) The latitude of Hong Kong is β = 22.25°. Calculate the length of daytime in Hong Kong today (22 February). Give your answer in hours to 3 significant figures. [3]
香港位于北纬β = 22.25°。试计算今天(2 月 22 日) 香港白昼的长度。答案请以小时表达，给三位有效数字。[3]

2. Rotating Ball (6 points) 滚动的球（9 分）

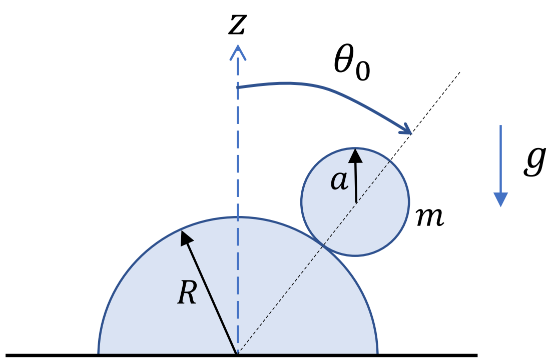
A ball of mass m and radius a is at rest on the surface of a sphere with radius R. The ball is initially at the angle θ_0 at $t = 0$. The bottom sphere is fixed and cannot move, but there is friction so the ball rolls without slipping until it leaves the surface of the sphere at angle θ_1. The moment of inertia of the ball about its axis is $I = \frac{2}{5}ma^2$. The gravity is directed as shown.
一个质量 m 和半径 a 的球放在一个半径 R 的球体表面上。球最初处于角度θ_0处。底部球体是固定的并且不能移动，但是存在摩擦力，因此球作纯滚动，直至它以角度θ_1离开球体表面。 球围绕其轴线的惯性矩是 $I = \frac{2}{5}ma^2$。引力方向如图所示。

(a) What is the frictional force acting on the ball at angle θ? [3]
在角度θ时作用在球上的摩擦力是多少？[3]

(b) What is the normal force acting on the ball at angle θ? [3]
在角度θ时作用在球上的支持力是多少？[3]

(c) Find θ_1 in terms of θ_0. [1]
试推导θ_1的表达式，用θ_0表示。[1]

(d) What is the speed of the ball at angle θ_1? [2]
在角度θ_1時球的速率是多少？[2]



3. Nearest Exoplanet Discovered (7 points) 发现最近的系外行星（7 分）

On 24th August 2016, astronomers discovered a planet orbiting the closest star to the Sun, Proxima Centauri, situated 4.22 light years away, which fulfils a long-standing dream of science-fiction writers: a world that is close enough for humans to send their first interstellar spacecraft.
2016 年 8 月 24 日，天文学家发现在距离太阳最近的恒星—比邻星（Proxima Centauri），有一颗行星围绕着它运行。比邻星距离太阳 4.22 光年。这发现实现了科幻小说作家的长期梦想：一个足够接近的世界，人类可以把第一艘星际航天器送达。

Astronomers have noted how the motion of Proxima Centauri changed in the first months of 2016, with the star moving towards and away from the Earth. In the figure below, the radial velocities of the star are measured and the direction of the radial velocities changed regularly. This regular pattern caused by an unseen planet, which they named Proxima Centauri B, repeats and results in tiny Doppler shifts in the star's light, making the light appear slightly redder, then bluer.
天文学家注意到比邻星的运动在 2016 年的头几个月的变化。恒星规律性地朝着和远离地球移动 。在下图中， 透过测量恒星的径向速度，發現径向速度的方向有规律地变化。径向速度的规律性变化是由一颗看不见的行星引起的，该行星称为比邻星 B，会重复导致恒星光线发生微小的多普勒频移，从而使光线稍微变红，然后变蓝。

It it is given that the star, Proxima Centauri, has a surface temperature of 3000 K and a radius of $R = 0.14R_{Sun}$ and the orbit of the unseen planet, Proxima Centauri B, around the star is circular. (Radius of the Sun $R_{Sun} = 6.96\times10^{8}\text{m}$, the gravitational constant $G = 6.674\times10^{-11}\text{m}^3/\text{kg}\cdot\text{s}^2$)
恒星比邻星的表面温度为3000 K，半径 $R = 0.14R_{Sun}$，行星比邻星 B 围绕恒星的轨道是圆形。（太阳半径$R_{Sun} = 6.96\times10^{8}\text{m}$，引力常数$G = 6.674\times10^{-11}\text{m}^3/\text{kg}\cdot\text{s}^2$）

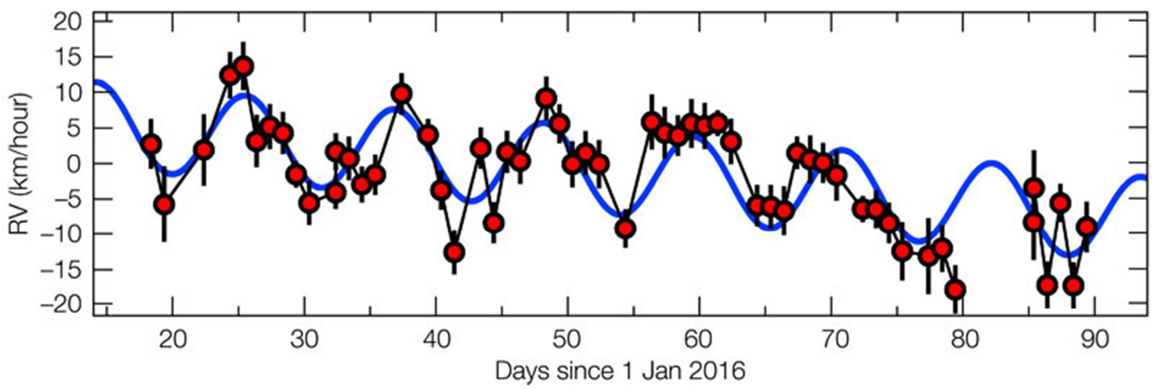


Figure: Measurements of the radial velocity of Proxima Centauri (denoted as RV in the figure) from 1 January 2016 onwards. The thick curve is the best fit curve of the data.
图：从 2016 年 1 月 1 日起，比邻星的径向速度（在图中以 RV 表示）测量结果。 粗线是数据的最佳拟合曲线。

(a) Proxima Centauri is a red dwarf star, unlike our Sun, with a mass of only 0.12 M_{Sun}. Estimate the radius of the planet's orbit using the given information. (The mass of sun, $M_{Sun} = 1.989\times10^{30}\text{kg}$) [2]
比邻星是一颗红矮星，与我们的太阳不同，质量仅为$0.12M_{Sun}$ 。试用所给资料估算行星轨道的半径。(太阳的质量 $M_{Sun} = 1.989\times10^{30}\text{kg}$) [2]

(b) Estimate the mass of the planet in terms of Earth mass. ($M_{Earth} = 5.972\times10^{24}kg$) [2]
试估算行星的质量，以地球质量为单位表示。（地球质量 $M_{Earth} = 5.972\times10^{24}\text{kg}$））[2]

(c) Estimate the equilibrium temperature of the planet by assuming that both the star and planet are black bodies. [3]
假设恒星和行星都是黑体，试估算行星的稳态温度。[3]

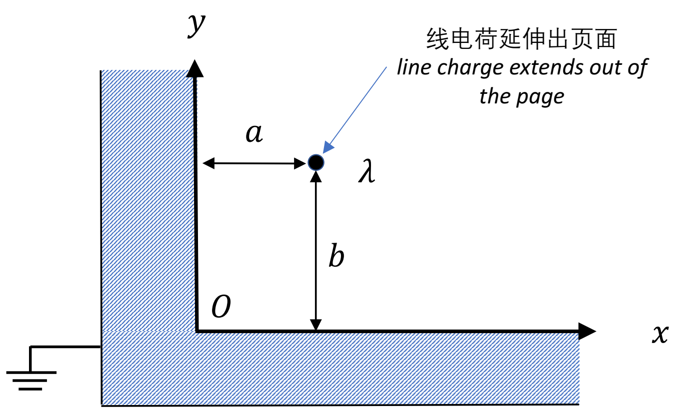
4. L-Shaped Conductor with a Wire (8 points) L 形导体和导线（8 分）

A L-shaped conductor consists of two semi-infinite conductors in the *xz* and *yz* planes where the cross section is shown in the figure. The L-shaped conductor is grounded and centered at the origin. A line of charge, with linear charge density λ runs parallel to the *z*-axis is located at (a, b) where $b > a > 0$.
L 形导体由 *xz* 和 *yz* 平面中的两个半无限平面组成，图中显示了导体的横截面。 L 形导体接地，中心点为原点。一条线性电荷密度为λ与 *z* 轴平行的电荷线位于（*a*,*b*)，其中 $b > a > 0$。

(a) Compute the electric potential $V(x, y, z)$ for $x > 0$ and $y > 0$. [3]
计算$x > 0$和$y > 0$时的电势 $V(x, y, z)$。[3]

(b) Compute the capacitance per unit length of a thin wire of radius r, placed at the point (a, b). Assume that the wire radius is much smaller than a and b (i.e. $r \ll a, b$) so that the solution of part (a) is approximately correct in the region exclusive of the conductors. [3]
计算放置在点(a, b)处、半径为 r 的细导线，其每单位长度的电容。 假定线半径 r 比 a 和 b 小得多（即$r \ll a, b$ ），使得在(a)部的解在除导体之外的区域中近似正确。[3]

(c) Compute the force per unit length on the wire (as a vector). [2]
计算导线上每单位长度的力（作为矢量）。[2]



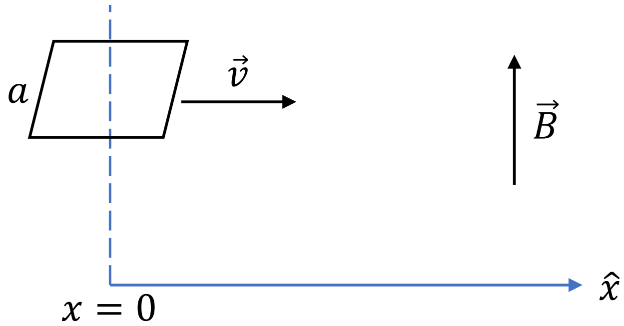
5. A Flying Square Loop (8 points) 一個飛行的方形环(8 分)

A square loop of side a and mass m is made of resistive material with a total resistance R. At $t = 0$, the loop is located at $x = 0$ and moves with velocity $v_0\hat{x}$. The loop lies in the *x*-*y* plane. There is a magnetic field $\vec{B} = B_0\left(\frac{x}{x_0}\right)\hat{z}$ where $B_0 > 0$ is a constant. In the problem, we neglect the effect of gravity.
一边长为 a、质量为 m 的方形环由电阻材料制成，总电阻为R。在$t = 0$时，环位于$x = 0$并以速度$v_0\hat{x}$ 移动。 环位于 *x*-*y* 平面中。 有一磁场 $\vec{B} = B_0\left(\frac{x}{x_0}\right)\hat{z}$，其中$B_0 > 0$是一常数。在这问题中，我们忽略重力的影响。

(a) What is the induced current on the loop when the center of the loop is at the point x with velocity $v\hat{x}$? What is the direction of the current? Is it clockwise/anticlockwise from above? [2]
当方形环中心位于x、速度为$v\hat{x}$ 时，环上的感应电流是多少？ 方向是什么？ 从上方观看是顺时针/逆时针？[2]

(b) What is the velocity of the square loop $v(t)$ at time t? [3]
方形环在时间t的速度$v(t)$ 是多少？[3]

(c) How far does the loop travel before stopping? [3]
方形环在停止前的行进距离是多少？[3]



6. The Phenomenon of the Halo (6 Points) 光晕現象（6 分）

Bright halos around the sun can be observed as in Figure 1. As shown in Fig. 2, this optical phenomenon is caused by the refraction of the sun's rays on ice crystals in the cirrostratus, a cloud genus that reaches a height of approximately 5.5 km.
我们有时候可以观察到太阳周围的明亮光晕圈，如图 1 所示。如图 2 所示，这种光学现象是由太阳光线在卷层云中的冰晶折射而产生的，该云层高度约 5.5 km。

To understand the phenomenon of the halo, we simplify the problem in two dimensions. In the following, we denote the angle of incidence on an ice crystal θ_i, the angle of refraction at the first interface θ_2, the angle of refraction at the exit of the crystal θ_o , and the angle of deflection between the ingoing and the outgoing sun ray θ_D.
为了理解光晕现象，我们将问题简化为两维。在下文中，我们以θ_i表示冰晶上的入射角，θ_2表示为经过第一个界面的折射角，θ_0表示为光线离开晶体的折射角，以及θ_D表示为入射和出射太阳光线之间的偏转角。

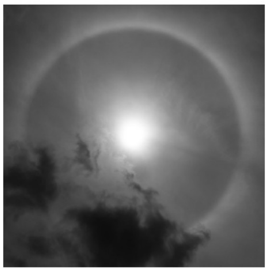


Fig. 1: A halo surrounding the sun
图1：太阳周围的光晕

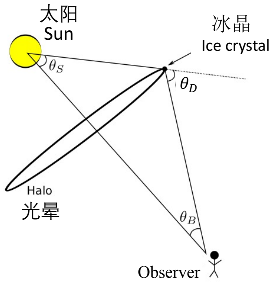


Fig. 2: Formation of a halo.
图2：晕圈的形成。

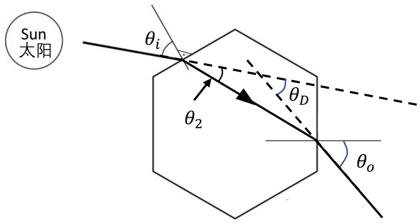


Fig. 3: Light refracted by an ice crystal
图3：由冰晶折射的光

(a) We consider an ice crystal in the form of a regular hexagon (Fig. 3). Derive an expression for θ_D as a function of θ_2, n_{Air} and n_{ice}, where n_i denotes the refractive index of the medium i. ($n_{Air} = 1,\ n_{ice} \approx 1.31$) [3]
我们考虑一个正六边形的冰晶(图 3)。试求出θ_D的表达式，并以θ_2, n_{Air}和n_{ice}作为其函数表示，其中n_i表示介质i的折射率。($n_{Air} = 1,\ n_{ice} \approx 1.31$) [3]

(b) Estimate the angular radius of the halo as measured by the observer on the ground. [2]
The identity may be useful:
试估算观察者在地面上看到光晕的角半径大小。 [2]
这个公式可能有用:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

(c) A closer look at a halo reveals the light spectrum along its entire circumference. Which of the colors, red or blue, is on the inner, which on the outer side of the halo? [1]
仔细观察一个光晕，可以在圆周上观察到不同颜色。问光晕的内侧是哪一种颜色，红色或蓝色，在光晕的外侧又是哪一种颜色？ [1]

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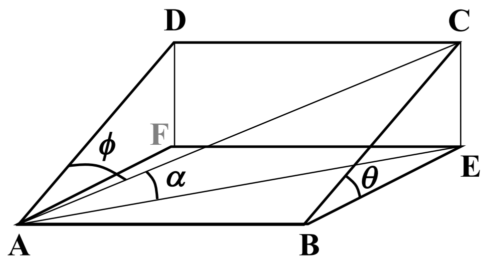
1. Length of Daytime (7 points) 白昼长度（7 分）

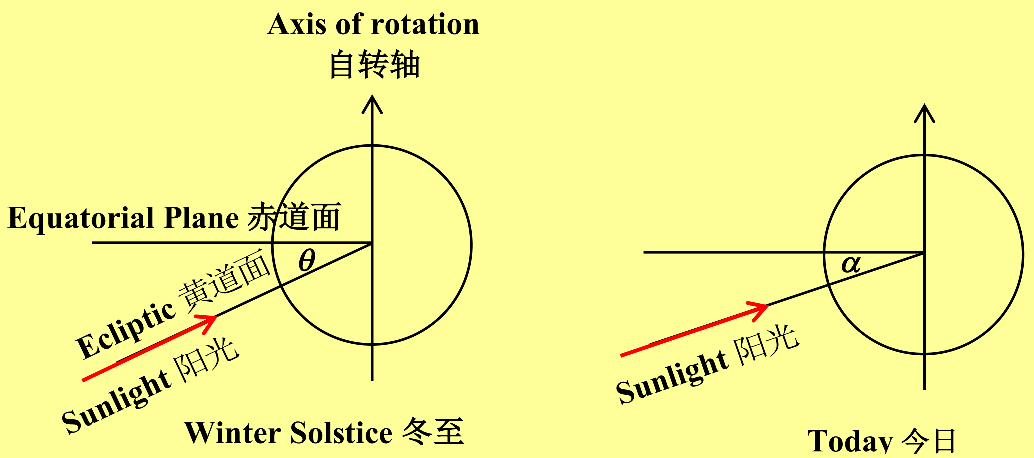
(a) In the figure, ABCD is a rectangle lying on an inclined plane making an angle θ with the horizontal plane. ABEF is the projection of the rectangle on the horizontal plane. If the measure of the angle DAC is ϕ, derive an expression for the angle α. [2]
如图所示，矩形 ABCD 位于斜面上，斜面与水平面夹角为θ。ABEF 为该矩形于水平面的投影。设角 DAC 为ϕ，试推导角α的表达式。[2]



Let h = AC. Then
$\mathrm{CE} = h\sin\alpha$.
$\mathrm{BC} = \mathrm{AD} = h\cos\phi$.
$\mathrm{CE} = \mathrm{BC}\sin\theta = h\cos\phi\sin\theta$.
Equating the expressions of CE, $h\sin\alpha = h\cos\phi\sin\theta \Rightarrow \alpha = \arcsin(\cos\phi\sin\theta)$.

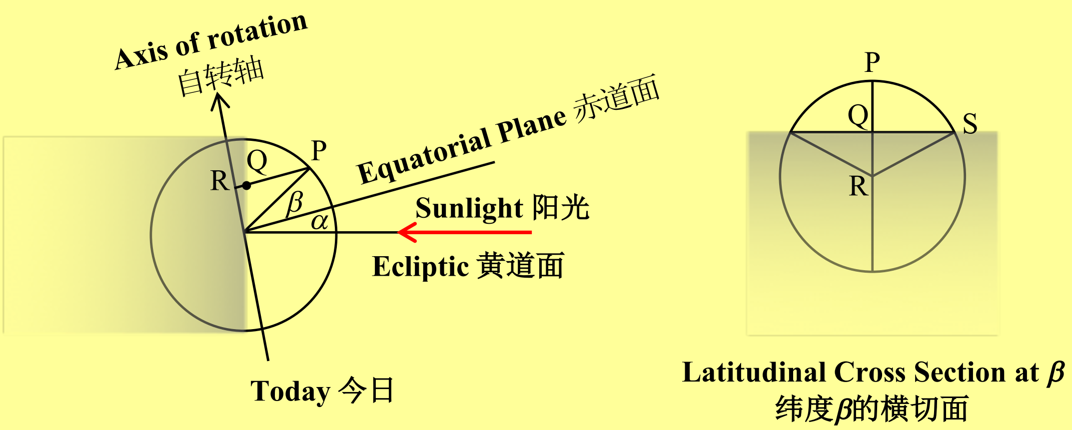
(b) The ecliptic is the plane on which the Earth revolves around the Sun. The axis of rotation of the Earth is inclined at an angle of 23.4° with the normal to the ecliptic. The day of the Winter Solstice (in the Northern Hemisphere) is 21 December. Using the result of (a) or otherwise, calculate the incident angle of sunlight relative to Earth's equatorial plane today (22 February). [2]
黄道面是指地球围绕太阳公转的平面。地球的自转轴相对于黄道面的法线倾斜，角度为 23.4°。在北半球，冬至的日期为 12 月 21 日。试用(a)部结果或其他方法，计算今天（2 月 22 日）阳光相对于赤道面的角度。[2]

In the figure above, consider ABEF to be the equatorial plane of the Earth, and ABCD the ecliptic. Then $\theta = 23.4^{\circ}$. When the Earth revolves around the Sun, sunlight is incident on the Earth from different directions lying on the plane ABCD. For example, on 21 December, sunlight is incident on the Earth in the direction AD, since this is the southernmost direction of sunlight. Similarly, during Spring Equinox and Autumn Equinox, sunlight is incident on the Earth in the direction AB or BA.



On 22 February, the time is 63 days after the Winter Solstice. Hence on the ecliptic, sunlight is incident from the angle $\phi = 360^0 \left(\frac{63}{365}\right) = 62.14^0$, and relative to the equatorial plane, it is incident from the angle
$\alpha = \arcsin(\cos\phi \sin\theta) = \arcsin[\cos(62.14^0)\sin(23.4^0)] = 10.70^0$.

(c) The latitude of Hong Kong is $\beta = 22.25°$. Calculate the length of daytime in Hong Kong today (22 February). Give your answer in hours to 3 significant figures. [3]
香港位于北纬$\beta = 22.25°$。试计算今天(2 月 22 日) 香港白昼的长度。答案请以小时表达，给三位有效数字。[3]



Consider the latitudinal cross section at β.
PR $= R\cos\beta$. (R is the radius of the Earth.)
QR $= R\sin\beta\tan\alpha$.
Hence the angle QRS is given by $\cos x = \frac{\text{QR}}{\text{SR}} = \frac{R\sin\beta\tan\alpha}{R\cos\beta} = \tan\beta\tan\alpha$.
$x = \arccos(\tan\beta\tan\alpha) = \arccos(\tan 22.25^0 \tan 62.14^0) = 85.57^0$.
Length of daytime in Hong Kong $= 24\left(\frac{85.57}{180}\right) = 11.4$ h

2. Rotating Ball (6 points) 滾動的球（9 分）

A ball of mass m and radius a is at rest on the surface of a sphere with radius R. The ball is initially at the angle θ_0 at $t = 0$. The bottom sphere is fixed and cannot move, but there is friction so the ball rolls without slipping until it leaves the surface of the sphere at angle θ_1.

The moment of inertia of the ball about its axis is $I = \frac{2}{5}ma^2$. The gravity is directed as shown.

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在角度θ时作用在球上的摩擦力是多少？[3]

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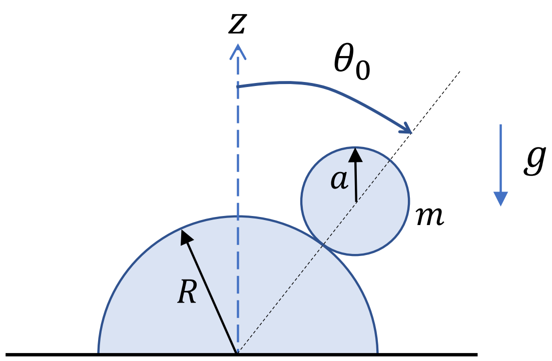
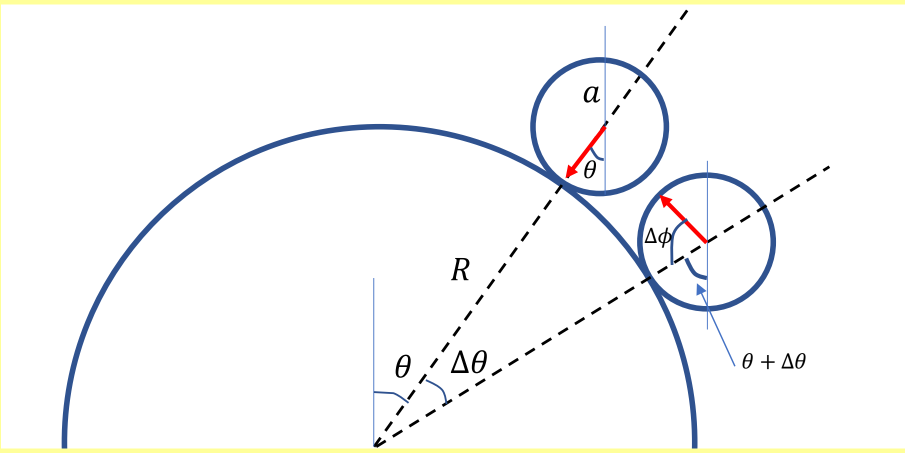
(c) Find θ_1 in terms of θ_0. [1]
试推导θ_1的表达式，用θ_0表示。[1]

(d) What is the speed of the ball at angle θ_1? [2]
在角度θ_1時球的速率是多少？[2]



Solution:
(a)



If the ball is rolling without slippering,

$$R\Delta\theta = a\Delta\phi$$

$$\omega \equiv \frac{(\theta + \Delta\theta + \Delta\phi) - \theta}{\Delta t} = \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{d\theta}{dt}\left(\frac{a+R}{a}\right)$$

$$(R + a)\Delta\theta = v\Delta t$$

$$\rightarrow v = (R+a)\frac{\Delta\theta}{\Delta t} = a\omega$$

By the conservation of energy and Newton's 2nd law, we have

$$mg(R+a)(1+\cos\theta_0) - mg(R+a)(1+\cos\theta) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$\rightarrow mg(R+a)(\cos\theta_0 - \cos\theta) = \frac{1}{2}mv^2 + \frac{1}{2}\frac{2}{5}ma^2\frac{v^2}{a^2} = \frac{7}{10}mv^2$$

$$\rightarrow v^2 = \frac{10}{7}g(R+a)(\cos\theta_0 - \cos\theta) \qquad (a)$$

From Newton's 2nd law on rotation

$$fa = I\frac{d\omega}{dt} \rightarrow f = \frac{2}{5}ma\frac{d\omega}{dt}$$

Newton's 2nd law along the tangential motion,

$$mg\sin\theta - f = m\frac{dv}{dt} = ma\frac{d\omega}{dt} = \frac{5}{2}f$$

$$\rightarrow f = \frac{2}{7}mg\sin\theta$$

Another way to get the result is from equation (a) without using the Newton's 2nd law.
Since $v = (R+a)\frac{d\theta}{dt}$, *Eq. (a) implies*

$$(R+a)^2\left(\frac{d\theta}{dt}\right)^2 = \frac{10}{7}g(R+a)(\cos\theta_0 - \cos\theta)$$

Differentiate on both sides,

$$2(R+a)^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = \frac{10}{7}g(R+a)\sin\theta\frac{d\theta}{dt} \rightarrow \frac{d^2\theta}{dt^2} = \frac{5}{7}\frac{g}{R+a}\sin\theta$$

And

$$f = \frac{2}{5}ma\frac{d\omega}{dt} = \frac{2}{5}m(a+R)\frac{d^2\theta}{dt^2} = \frac{2}{7}mg\sin\theta$$

(b) Newton's 2nd law gives

$$mg\cos\theta - N = \frac{mv^2}{R+a}$$

The normal force is

$$N = mg\cos\theta - \frac{mv^2}{R+a} = mg\cos\theta - \frac{m}{R+a}\frac{10}{7}g(R+a)(\cos\theta_0 - \cos\theta)$$

$$= mg(\frac{17}{7}\cos\theta - \frac{10}{7}\cos\theta_0)$$

(c) At θ_1, the normal force vanishes, we have

$$\cos\theta_1 = \frac{10}{17}\cos\theta$$

(d) the velocity of the ball at the angle θ is

$$v = \sqrt{g(R+a)(\cos\theta_0 - \cos\theta_1)} = \sqrt{\frac{7}{17}g(R+a)\cos\theta_0}$$

3. Nearest Exoplanet Discovered (7 points) 发现最近的系外行星（7 分）

On 24th August 2016, astronomers discovered a planet orbiting the closest star to the Sun, Proxima Centauri, situated 4.22 light years away, which fulfils a long-standing dream of science-fiction writers: a world that is close enough for humans to send their first interstellar spacecraft.
2016 年 8 月 24 日，天文学家发现在距离太阳最近的恒星一比邻星（Proxima Centauri），有一颗行星围绕着它运行。比邻星距离太阳 4.22 光年。这发现实现了科幻小说作家的长期梦想：一个足够接近的世界，人类可以把第一艘星际航天器送达。

Astronomers have noted how the motion of Proxima Centauri changed in the first months of 2016, with the star moving towards and away from the Earth. In the figure below, the radial velocities of the star are measured and the direction of the radial velocities changed regularly. This regular pattern caused by an unseen planet, which they named Proxima Centauri B, repeats and results in tiny Doppler shifts in the star’s light, making the light appear slightly redder, then bluer.
天文学家注意到比邻星的运动在 2016 年的头几个月的变化。恒星规律性地朝着和远离地球移动 。在下图中， 透过测量恒星的径向速度，發現径向速度的方向有规律地变化。径向速度的规律性变化是由一颗看不见的行星引起的，该行星称为比邻星 B，会重复导致恒星光线发生微小的多普勒频移，从而使光线稍微变红，然后变蓝。

It it is given that the star, Proxima Centauri, has a surface temperature of 3000 K and a radius of $R = 0.14R_{Sun}$ and the orbit of the unseen planet, Proxima Centauri B, around the star is circular. (Radius of the Sun $R_{Sun} = 6.96\times10^{8}\text{m}$, the gravitational constant $G = 6.674\times10^{-11}\text{m}^3/\text{kg}\cdot\text{s}^2$)
恒星比邻星的表面温度为3000 K，半径 $R = 0.14R_{Sun}$，行星比邻星 B 围绕恒星的轨道是圆形。（太阳半径$R_{Sun} = 6.96\times10^{8}\text{m}$，引力常数$G = 6.674\times10^{-11}\text{m}^3/\text{kg}\cdot\text{s}^2$）

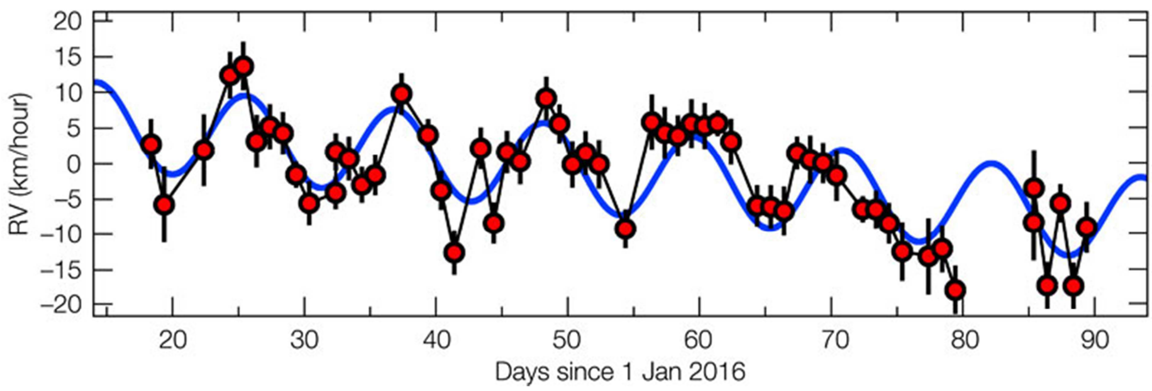


Figure: Measurements of the radial velocity of Proxima Centauri (denoted as RV in the figure) from 1 January 2016 onwards. The blue curve is the best fit curve of the data.
图：从 2016 年 1 月 1 日起，比邻星的径向速度（在图中以 RV 表示）测量结果。 蓝色曲线是数据的最佳拟合曲线。

(a) Proxima Centauri is a red dwarf star, unlike our Sun, with a mass of only 0.12 M_{Sun}. Estimate the radius of the planet’s orbit using the given information. (The mass of sun, $M_{Sun} = 1.989\times10^{30}\text{kg}$) [2]
比邻星是一颗红矮星， 与我们的太阳不同， 质量仅为$0.12M_{Sun}$ 。试用所给资料估算行星轨道的半径。(太阳的质量 $M_{Sun} = 1.989\times10^{30}\text{kg}$) [2]

(b) Estimate the mass of the planet in terms of Earth mass. ($M_{Earth} = 5.972\times10^{24} kg$) [2]
试估算行星的质量，以地球质量为单位表示。（地球质量 $M_{Earth} = 5.972\times10^{24}\text{kg}$））
[2]

(c) Estimate the equilibrium temperature of the planet by assuming that both the star and planet are black bodies. [3]
假设恒星和行星都是黑体，试估算行星的稳态温度。[3]

Solution:
(a) The period is the time interval between two consecutive peaks of the curve. From the radial velocity curve, the period is 11 days
(Full marks for the period within ±2 days)
Using Kepler's 3rd law: (Assuming $M_{star} \gg M_{planet}$)

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM_{star}} \to a = 7.14\times10^9\text{m}$$

(b) The orbital velocity for a circular orbit is:

$$v = \frac{2\pi a}{T} = 47200\text{m/s}$$

Here a is the distance between the star and the planet, that is, the sum of the orbital radii of the planet and the star, and we assume that the orbital radius of the star is negligible.
In the center of mass frame, $\vec{p}_{cm} = 0$.

$$\to Mv_s - mv_p = 0$$

From the figure, $v_s = 6.5\text{ km/h} = 1.806\text{ m/s}$.

$$\to m = \frac{v_s}{v_p}M = \left(\frac{1.806}{47200}\right)(0.12\times1.989\times10^{30}) = 9.1\times10^{24}kg \approx 1.5M_{Earth}$$

(c) By assuming thermal equilibrium, the heat absorbed by the planet (due to the radiation of star) is equal to the heat radiated by the planet (as a blackbody)

$$\left(\frac{4\pi R_s^2\sigma T_s^4}{4\pi a^2}\right)\pi R_p^2 = 4\pi R_p^2\sigma T_p^4 \to T_p^4 = \frac{R_s^2}{4a^2}T_s^4 \to T_p = T_s\sqrt{\frac{R_s}{2a}} \approx 248\text{K} = -25°C$$

4. L-Shaped Conductor with a Wire (8 points) L 形导体和导线（8 分）

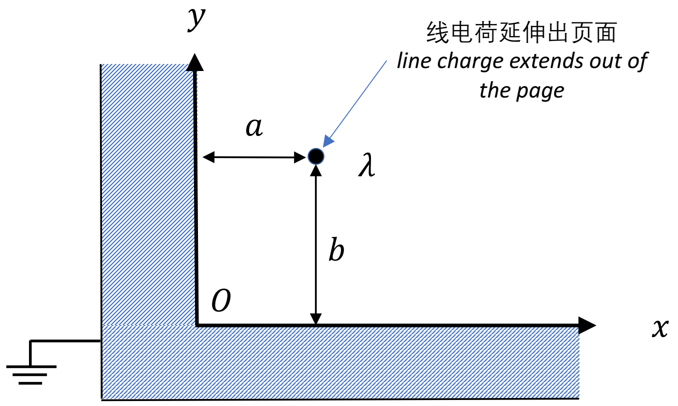
A L-shaped conductor consists of two semi-infinite conductors in the *xz* and *yz* planes where the cross section is shown in the figure. The L-shaped conductor is grounded and centered at the origin. A line of charge, with linear charge density λ runs parallel to the *z*-axis is located at (a, b) where $b > a > 0$.
L 形导体由 *xz* 和 *yz* 平面中的两个半无限平面组成，图中显示了导体的横截面。 L 形导体接地，中心点为原点。一条线性电荷密度为λ与 z 轴平行的电荷线位于（a,b)，其中 $b > a > 0$。

(a) Compute the electric potential $V(x, y, z)$ for $x > 0$ and $y > 0$. [3]
计算$x > 0$和$y > 0$时的电势 $V(x, y, z)$。[3]

(b) Compute the capacitance per unit length of a thin wire of radius r, placed at the point (a, b). Assume that the wire radius is much smaller than a and b (i.e. $r \ll a, b$) so that the solution of part (a) is approximately correct in the region exclusive of the conductors. [3]

计算放置在点(a,b)处、半径为 r 的细导线，其每单位长度的电容。 假定线半径 r 比 a 和 b 小得多（即$r \ll a,b$ ）， 使得在(a)部的解在除导体之外的区域中近似正确。[3]

(c) Compute the force per unit length on the wire (as a vector). [2]
计算导线上每单位长度的力（作为矢量）。[2]



Solution:
(a) We apply image method by adding 3 image line charge in the following ways:

1. $(-a,b)$: Charge density $-\lambda$
2. $(-a,-b)$: Charge density λ
3. $(a,-b)$: Charge density $-\lambda$

Hence the total electric field along the x-axis (at the point $(x,0)$) is

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{(x-a)^2+b^2}\big((x-a)\hat{\imath}-b\hat{\jmath}\big) - \frac{1}{(x+a)^2+b^2}\big((x+a)\hat{\imath}-b\hat{\jmath}\big) + \frac{1}{(x+a)^2+b^2}\big((x+a)\hat{\imath}+b\hat{\jmath}\big) - \frac{1}{(x-a)^2+b^2}\big((x-\mathrm{a})\hat{\imath}+b\hat{\jmath}\big)\right)$$
$$= \frac{\lambda}{2\pi\epsilon_0}\left(-\frac{2b}{(x-a)^2+b^2}+\frac{2b}{(x+a)^2+b^2}\right)\hat{\jmath}$$

which is along the y-direction and hence the electric potential which is constant along the x-axis.
Similarly, we can show the electric potential is also constant along the y-axis.
By setting the potential $V(0) = 0$ at the origin, the electric potential of an infinite line of charge at (a,b) is

$$V_0(x,y,z) = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{a^2+b^2}{(x-a)^2+(y-b)^2}\right)$$

Similarly, we can get the electric potential for other image wires

$$V_1(x,y,z) = -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{a^2+b^2}{(x+a)^2+(y-b)^2}\right)$$
$$V_2(x,y,z) = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{a^2+b^2}{(x+a)^2+(y+b)^2}\right)$$
$$V_3(x,y,z) = -\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{a^2+b^2}{(x-a)^2+(y+b)^2}\right)$$

The total electric potential becomes,

$$V(x,y,z) = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{(x+a)^2+(y-b)^2}{(x-a)^2+(y-b)^2}\right)\left(\frac{(x-a)^2+(y+b)^2}{(x+a)^2+(y+b)^2}\right)$$

(b)

$$C = \frac{Q}{\Delta V} = \frac{Q}{V(a-r,b,0) - V(0,0,0)} = \frac{\lambda L}{\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{(2a-r)^2}{(-r)^2}\right)\left(\frac{(r)^2+(2b)^2}{(2a-r)^2+(2b)^2}\right)}$$

$$= \frac{4\pi\epsilon_0 L}{\ln\left(\frac{(2ab)^2}{(a^2+b^2)r^2}\right)} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{2ab}{r\sqrt{a^2+b^2}}\right)}$$

$$\rightarrow \frac{C}{L} = \frac{2\pi\epsilon_0}{\ln\left(\frac{2ab}{r\sqrt{a^2+b^2}}\right)}$$

(c) The force on the wire is the electric force on the wire from three image wires.
From Gauss's law, the electric field created by three image wires at the point $(x, y, 0)$,

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\left(-\frac{1}{(x+a)^2+(y-b)^2}\left((x+a)\hat{\imath}+(y-b)\hat{\jmath}\right) + \frac{1}{(x+a)^2+(y+b)^2}\left((x+a)\hat{\imath}+(y+b)\hat{\jmath}\right) - \frac{1}{(x-a)^2+(y+b)^2}\left((x-a)\hat{\imath}+(y+b)\hat{\jmath}\right)\right)$$

Substitute $(x, y) = (a, b)$, we get

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{(a\hat{\imath}+b\hat{\jmath})}{a^2+b^2} - \frac{1}{a}\hat{\imath} - \frac{1}{b}\hat{\jmath}\right)$$

The force on the wire is

$$\vec{F} = q\vec{E} = \lambda L\frac{\lambda}{4\pi\epsilon_0}\left(\frac{(a\hat{\imath}+b\hat{\jmath})}{a^2+b^2} - \frac{1}{a}\hat{\imath} - \frac{1}{b}\hat{\jmath}\right) \rightarrow \frac{\vec{F}}{L} = \frac{\lambda^2}{4\pi\epsilon_0}\left(\frac{(a\hat{\imath}+b\hat{\jmath})}{a^2+b^2} - \frac{1}{a}\hat{\imath} - \frac{1}{b}\hat{\jmath}\right)$$

5. A Flying Square Loop (8 points) 一個飛行的方形環(8 分)

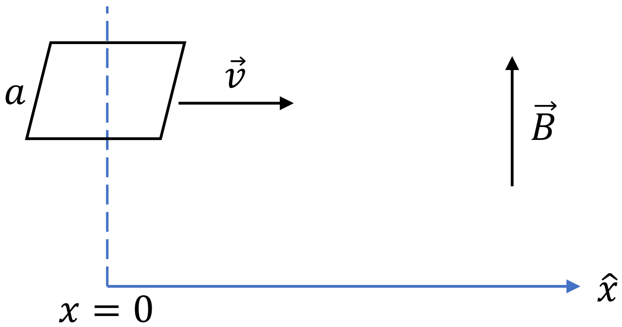
A square loop of side a and mass m is made of resistive material with a total resistance R. At $t = 0$, the loop is located at $x = 0$ and moves with velocity $v_0\hat{x}$. The loop lies in the x-y plane. There is a magnetic field $\vec{B} = B_0\left(\frac{x}{x_0}\right)\hat{z}$ where $B_0 > 0$ is a constant. In the problem, we neglect the effect of gravity.
一边长为 a、质量为 m 的方形环由电阻材料制成，总电阻为R。在$t = 0$时，环位于$x = 0$并以速度$v_0\hat{x}$ 移动。 环位于 x-y 平面中。 有一磁场 $\vec{B} = B_0\left(\frac{x}{x_0}\right)\hat{z}$，其中$B_0 > 0$是一常数。在这问题中，我们忽略重力的影响。

(a) What is the induced current on the loop when the center of the loop is at the point x with velocity $v\hat{x}$? What is the direction of the current? Is it clockwise/anticlockwise from above? [2]
当方形环中心位于x、速度为$v\hat{x}$ 时，环上的感应电流是多少？ 方向是什么？ 从上方观看是顺时针/逆时针？[2]

(b) What is the velocity of the square loop $v(t)$ at time t? [3]
方形环在时间t的速度$v(t)$ 是多少？[3]

(c) How far does the loop travel before stopping? [3]

方形环在停止前的行进距离是多少？[3]



Solution:

(a) When the center of the square loop is at the point x, the magnetic flux passes through it is,

$$\Phi = a\int_{x=x-\frac{a}{2}}^{x+\frac{a}{2}} B_0\left(\frac{x}{x_0}\right)dx = \frac{aB_0}{2x_0}\left(\left(x+\frac{a}{2}\right)^2 - \left(x-\frac{a}{2}\right)^2\right) = \frac{aB_0}{2x_0}(2ax) = \frac{a^2B_0x}{x_0}$$

The induced emf is

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{a^2B_0}{x_0}\left(\frac{dx}{dt}\right) = -\frac{a^2B_0}{x_0}v$$

The induced current is

$$I = \frac{\mathcal{E}}{R} = \frac{a^2B_0}{x_0R}v$$

and the current is clockwise from above.

(b) The dissipation power of the induced current compensated by the work done to decelerate the loop,

$$P = I\mathcal{E} = \frac{\mathcal{E}^2}{R} = \frac{1}{R}\left(\frac{a^2B_0}{x_0}\right)^2 v^2 = -m\dot{v}v$$

$$\rightarrow \frac{\dot{v}}{v} = -\frac{1}{mR}\left(\frac{a^2B_0}{x_0}\right)^2 = -\frac{\gamma}{m} \qquad where\ \gamma = \frac{1}{R}\left(\frac{a^2B_0}{x_0}\right)^2$$

The velocity of the loop is

$$v(t) = v_0e^{-\frac{\gamma}{m}t}$$

(c) Total distance travelled is

$$x = \int_{t=0}^{\infty} v(t)dt = \frac{v_0m}{\gamma} = Rmv_0\left(\frac{x_0}{a^2B_0}\right)^2$$

6. The Phenomenon of the Halo (6 Points) 光暈現象（6 分）

Bright halos around the sun can be observed as in Figure 1. As shown in Fig. 2, this optical phenomenon is caused by the refraction of the sun's rays on ice crystals in the cirrostratus, a cloud genus that reaches a height of approximately 5.5 km.

我们有时候可以观察到太阳周围的明亮光晕圈，如图 1 所示。如图 2 所示，这种光学现象是由太阳光线在卷层云中的冰晶折射而产生的，该云层高度约 5.5 km。

To understand the phenomenon of the halo, we simplify the problem in two dimensions. In the following, we denote the angle of incidence on an ice crystal θ_i, the angle of refraction at the first interface θ_2, the angle of refraction at the exit of the crystal θ_o, and the angle of deflection between the ingoing and the outgoing sun ray θ_D.

为了理解光晕现象，我们将问题简化为两维。在下文中，我们以θ_i表示冰晶上的入射角，θ_2表示为经过第一个界面的折射角，θ_0表示为光线离开晶体的折射角，以及θ_D表示为入射和出射太阳光线之间的偏转角。

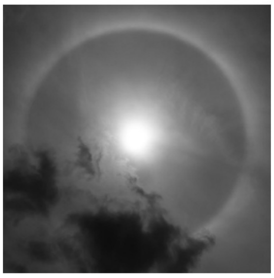


Fig. 1: A halo surrounding the sun
图1：太阳周围的光晕

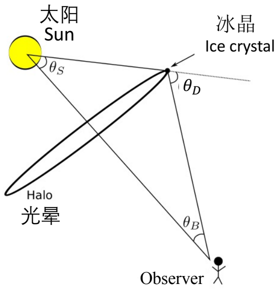


Fig. 2: Formation of a halo.
图2：晕圈的形成。

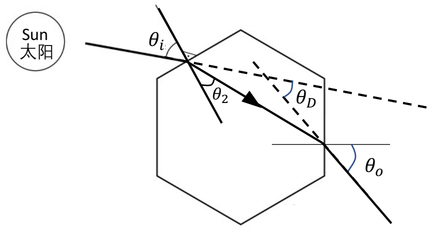


Fig. 3: Light refracted by an ice crystal
图3：由冰晶折射的光

(a) We consider an ice crystal in the form of a regular hexagon (Fig. 3). Derive an expression for θ_D as a function of θ_2, n_{Air} and n_{ice}, where n_i denotes the refractive index of the medium i. ($n_{Air} = 1,\ n_{ice} \approx 1.31$) [3]
(a) 我们考虑一个正六边形的冰晶(图 3)。试求出θ_D的表达式，并以θ_2, n_{Air}和n_{ice}作为其函数表示，其中n_i表示介质i的折射率。($n_{Air} = 1,\ n_{ice} \approx 1.31$) [3]

(b) Estimate the angular radius of the halo as measured by the observer on the ground. [2]
The identity may be useful:
试估算观察者在地面上看到光晕的角半径大小。[2]
这个公式可能有用:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

(c) A closer look at a halo reveals the light spectrum along its entire circumference. Which of the colors, red or blue, is on the inner, which on the outer side of the halo? [1]
仔细观察一个光晕，可以在圆周上观察到不同颜色。问光晕的内侧是哪一种颜色，红色或蓝色，在光晕的外侧又是哪一种颜色？[1]

Solution:
By Snell's law,

$$n_{air}\sin\theta_i = n_{ice}\sin\theta_2 \rightarrow \sin\theta_i = n_{ice}\sin\theta_2$$
$$n_{ice}\sin\theta_3 = \sin\theta_o$$
$$\theta_2 + \theta_3 = \frac{\pi}{3}$$
$$\theta_D = (\theta_i - \theta_2) + (\theta_o - \theta_3) = \theta_i - \frac{\pi}{3} + \theta_0$$
$$\theta_r = \arcsin\left(n_{ice}\sin\left(\frac{\pi}{3} - \arcsin\left(\frac{1}{n_{ice}}\sin\theta_i\right)\right)\right)$$
$$\theta_D = \arcsin(n_{ice}\sin\theta_2) - \frac{\pi}{3} + \arcsin\left(n_{ice}\sin\left(\frac{\pi}{3} - \theta_2\right)\right)$$

(b) The halo is observed at the stationary point of θ_D

$$\frac{d\theta_D}{d\theta_2} = \frac{1}{\sqrt{1 - n_{ice}^2 \sin^2 \theta_2}} n_{ice} \cos \theta_2 - \frac{1}{\sqrt{1 - n_{ice}^2 \sin^2 \left(\frac{\pi}{3} - \theta_2\right)}} n_{ice} \cos \left(\frac{\pi}{3} - \theta_2\right) = 0$$

$$\rightarrow \theta_2 = \frac{\pi}{3} - \theta_2 \quad i.e.\ \theta_2 = \frac{\pi}{6}$$

and the corresponding θ_D is:

$$\theta_{D,\min} = 21.8°$$

The angular radius of the halo is $\theta_B = \theta_{D,\min} - \theta_S \approx \theta_{D,\min} = 21.8°$

(c) The refractive index of red color is smaller than that of blue. The deflection angle corresponding to the blue color should be bigger and hence the outer of the halo should be blue, and inner of the halo is red.

2

Pan Pearl River Delta Physics Olympiad 2018
2018 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-2 (Total 2 Problems, 55 Points) 简体版卷-2（共2题，55分）
(1:45 pm – 5:00 pm, 22 February, 2018)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。
All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。
There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。
Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。
Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。
Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。
If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。
At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Oscillations of the Sun (22 points) 太阳的振荡（22 分）

The sun is made of compressible gas. It can oscillate in a variety of ways. Investigating these oscillations has provided rich information on the interior of the Sun. In this problem we study two kinds of waves: pressure waves and gravity waves.
太阳的成份是可压缩气体。它可以以各种方式振荡。研究这些振荡提供了有关太阳内部的丰富信息。在这问题中，我们研究两种波：压强波和重力波。

Part A. Pressure Waves (15 points) 压強波（15 分）

Most of us are familiar with sound waves propagating through Earth's atmosphere, which is a pressure wave. In the Sun, however, we need to consider the fact that gas density falls off with height because of gravity. In this problem, we will use the following notations:
我们大多数人都熟悉在地球大气层传播的声波，它是一种压強波。但是在太阳内，我们需要考虑由于重力而导致气体密度随高度下降。在这个问题中，我们将采用以下符号：

$\bar{m}$ = average mass of particles 粒子平均质量
g = gravitational acceleration 重力加速度
k_B= Boltzmann constant 波尔兹曼常数
T = absolute temperature 绝对温度
γ = ratio of the constant-pressure specific heat to the constant-volume specific heat 定压比热与定容比热之比

We model the Sun as an atmosphere whose density falls off with height because of gravity. For a thin layer of the atmosphere between heights x and $x + dx$, the equilibrium pressure at these locations are $P(x)$ and $P(x + dx)$ respectively. Assume that the gravitational acceleration and the temperature are constant.
我们将太阳模拟为一个大气层，其密度因重力而随高度下降。对于高度在x 和$x + dx$之间的薄层气体来说，这些位置的稳态压強分别为$P(x)$和$P(x + dx)$。假定重力加速度和温度是恒定的。

A1	Derive the differential equation for the atmospheric density $\rho(x)$. 试推导大气密度$\rho(x)$的微分方程。	**2 points 2 分**

A2	The scale height H of the atmosphere is the height through which the density becomes a factor of e^{-1} of the original density. Derive the expression of H. 大气的标度高度H是密度为起始密度e^{-1}倍的高度。求H的表达式。	**1 points 1 分**

When a pressure wave propagates vertically in the atmosphere, the particles will experience small vertical displacements. Let $u(x, t)$ denote the vertical displacement of the gas particles at time t whose undisturbed position is x.
当压強波在大气中垂直传播时，粒子将经历细小的垂直位移。设$u(x, t)$为气体粒子在時間t時的垂直位移，x为其不受干扰时的位置。

A3	As shown in the Fig. 1, there is a change in thickness of the thin layer. Express the change in thickness in terms containing the gradient $\partial u/\partial x$. (Remark: For u being a function of both x and t, $\partial u/\partial x$ is called the partial derivative of u with respect to x with t taken to be constant.) 如图 1 所示，薄层的厚度有变化。试以梯度$\partial u/\partial x$表示厚度变化。 （备注：u作为x和t二者的函数，$\partial u/\partial x$被称为u相对于x的偏导数，其中t在求导过程中视为常数。）	**1 points** **1 分**

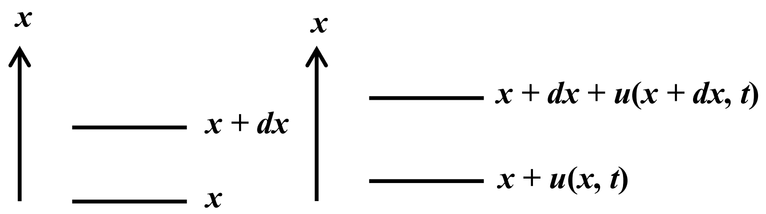


Fig. 1: The vertical displacements of a thin layer of gas particles caused by the propagation of a pressure wave. Note the change in the thickness of the layer.
图 1：由压強波传播引起薄层气体粒子的垂直位移。 请注意层厚度的变化。

A4	In turn, the vertical displacements produce small fluctuations in density and pressure, denoted as $\delta\rho(x,t)$ and $\delta P(x,t)$ respectively. Express the change in $\delta\rho(x,t)$ and $\delta P(x,t)$ in terms containing the gradient $\partial u/\partial x$. Assume that the heat transfer is negligible during the period of the pressure wave. 随之而来，垂直位移产生密度和压強的細小波动，分别表示为$\delta\rho(x,t)$和$\delta P(x,t)$ 。求$\delta\rho(x,t)$和$\delta P(x,t)$的表示式（以梯度$\partial u/\partial x$表示）。假设在压強波传播期间传热可以忽略不计。	**2 points** **2 分**

A5	Derive the differential equation of motion for $u(x,t)$. Simplify your expressions using the speed of sound $c_s = \sqrt{\gamma P/\rho}$. 试推导$u(x,t)$的微分运动方程。以音速$c_s = \sqrt{\gamma P/\rho}$简化你的答案。	**3 points** **3 分**

A6	Show that the solution of the equation of motion is equivalent to a pressure wave traveling through a uniform medium when the wavelength is shorter than a length scale. Derive this length scale. 试证明当波长短于某长度尺度时，运动方程的解等价于穿过均匀介质的压強波。求这个长度尺度。	**2 points** **2 分**

Next we seek a sinusoidal wave with angular frequency ω. The energy density of the wave, $\frac{1}{2}\rho\omega^2 u^2$, is expected to remain constant as the wave propagates upward with constant velocity in the direction of decreasing density $\rho(x)$. With this expectation in mind we let
接下来我们寻找一个角频率ω的正弦波。当波沿密度$\rho(x)$减小的方向以恒定速度向上传播时，我们预期波的能量密度$\frac{1}{2}\rho\omega^2 u^2$保持不变。因此我们可设

$$u(x,t) = \frac{f(x)}{\sqrt{\rho(x)}} e^{-i\omega t}.$$

A7	Derive the differential equation for $f(x)$. 试推导$f(x)$的微分方程。	**3 points** **3 分**

A8	When the frequency of the pressure wave is below a critical frequency ω_c below the Sun's surface, it becomes trapped inside the Sun. What is ω_c? 当太阳表面下的压强波频率低于临界频率ω_c时，它会被困于太阳内部。求 ω_c。	**1 points** **1 分**

Part B. Gravity Waves (7 points) 重力波（7 分）

In Part A we only included the restoring force due to the fluctuation in the pressure gradient for pressure waves traveling in the vertical direction of the Sun's atmosphere. However, for gravity waves propagating in a horizontal direction of the Sun's atmosphere, the buoyancy of the gas may also give rise to a restoring force which can sustain oscillations.
在 A 部，我们只考虑了由于压强波在太阳大气垂直方向传播的压强梯度的波动而产生的恢复力。然而，对于沿太阳大气的水平方向传播的重力波，气体的浮力也可成为维持振荡的恢复力。

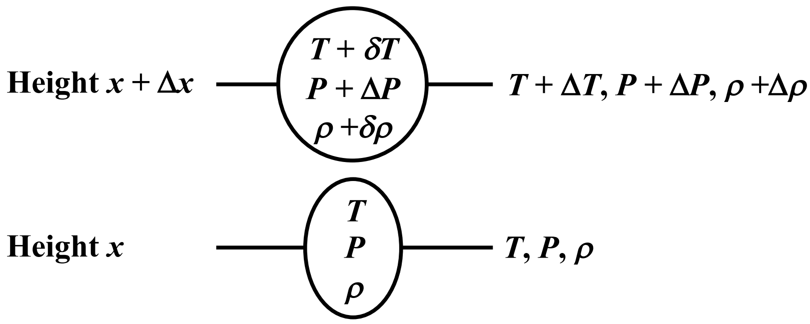


Fig. 2: Displacement of a pocket of gas from height x to height $x + \Delta x$.
图 2：一小口气体从高度x到高度$x + \Delta x$的位移。

To understand this, we consider a small vertical displacement of a pocket of gas in an environment of the same gas with both gradients in the temperature and pressure. As shown in Fig. 2, this pocket of gas has the same temperature, pressure and density as the surrounding gas. When its height is displaced by Δx, it enters an environment with temperature, pressure and density given by $T + \Delta T$, $P + \Delta P$ and $\rho + \Delta\rho$ respectively.
为了理解这一点，我们考虑一小口气体在相同气体的环境中，在温度梯度和压强梯度的影响下作垂直位移。如图 2 所示，这小口气体与周围气体具有相同的温度、压强和密度。当高度移动Δx时，它将进入温度为$T + \Delta T$、压强为$P + \Delta P$和密度为$\rho + \Delta\rho$ 的环境。

For the pocket of gas, the pressure inside the pocket responds rapidly to the environment so that its pressure also changes by ΔP. On the other hand, the change in temperature and density may be different. Suppose the temperature, pressure and density of the pocket of gas in the new environment are $T + \delta T$, $P + \Delta P$ and $\rho + \delta\rho$ respectively. Assume that there is insufficient time for heat conduction from the pocket of gas to the environment.
对这小口气体而言，内部压强迅速回应环境，使其压强也随之改变为$P + \Delta P$。另一方面，温度和密度的变化可能不同。假设新环境中这小口气体的温度为$T + \delta T$、压强为$P + \Delta P$和密度为$\rho + \delta\rho$。假设没有足够的时间从小口气体向环境传导热量。

B1	Express $\Delta\rho$ and $\delta\rho$ in terms of expressions containing ΔT and ΔP. 求$\Delta\rho$ 和$\delta\rho$的表达式(用ΔT和ΔP表示)。	**2 points** **2 分**

B2	Suppose the temperature and pressure gradients of the surrounding gas are dT/dx and dP/dx respectively. Derive the equation of motion of the pocket of gas. 假设周围气体的温度梯度和压强梯度分别为dT/dx和dP/dx。试推导这小口气体的运动方程。	**2 points** **2 分**

B3	Determine the range of temperature gradient dT/dx in which the pocket of gas can exhibit oscillations. Express the bound(s) of the temperature gradient in terms of T/H. 求这小口气体可出现振荡的温度梯度dT/dx范围(以T/H表示这范围的界限)。	**2 points** **2 分**

B4	How does the gas in the Sun behave when the temperature gradient is outside the range considered in B3? 当温度梯度超出 B3 考虑的范围时，太阳中的气体会有什么行为?	**1 points** **1 分**

END of Problem 1

问题 1 完

Problem 2: Plasmon Resonance and SERS 等离子共振和 SERS

Surface-enhanced Raman spectroscopy (SERS) is one of the most prominent optical phenomena in the last 40 years. SERS is based on **plasmon resonance**, referring to the significant increase in electric field intensity near the small metal granules under certain conditions. In order to determine these conditions, it is necessary to learn how to describe the properties of metals placed in oscillating electromagnetic fields.
表面增强拉曼光谱（SERS）是近 40 年来最重要的光学现象之一。 SERS 的基础是**等离子共振**，指在某些条件下小金属颗粒附近的电场强度显着增加。为了确定这些条件，我们有必要认识如何描述放置在振荡电磁场中的金属特性。

Properties of a medium in an electric field are described as follows:

$$\vec{D} = \epsilon\epsilon_0\vec{E} = \epsilon_0\vec{E} + \vec{P}$$

where $\vec{E}$ and $\vec{D}$ are the electric field intensity and the electric displacement respectively, ϵ is the permittivity of the medium, $\vec{P}$ is the electric polarization (electric dipole moments per unit volume), ϵ_0 is the vacuum permittivity. The boundary conditions in the absence of free charges are the continuity of electric field tangential to the boundary and the continuity of the electric displacement normal to the boundary.
介质在电场中的特性描述如下：

$$\vec{D} = \epsilon\epsilon_0\vec{E} = \epsilon_0\vec{E} + \vec{P}$$

其中$\vec{E}$ 和$\vec{D}$分别是电场强度和电位移，ϵ是介质的介电常数，$\vec{P}$是电极化强度（单位体积的电偶极矩），ϵ_0是真空介电常数。在没有自由电荷下的边界条件分别是与边界相切的电场的连续性和垂直于边界的电位移的连续性。

In an oscillating electromagnetic field, the permittivity of a media (including metals) is dependent on the electromagnetic field frequency, $\epsilon = \epsilon(\omega)$.
在振荡电磁场中，介质（包括金属）的介电常数取决于电磁场频率：$\epsilon = \epsilon(\omega)$。

Part A. Free Electron Gas (3 points) 自由电子气体（3 分）

Consider a metal occupying an infinite space. Positive ions form a crystal lattice. Free electrons move inside the lattice. The number density of positive ions and electrons are the same and equal to n.
考虑一占据无限空间的金属。正离子形成晶格。自由电子在晶格内移动。正离子和电子的数密度相同并等于n 。

A uniform oscillating electric field $\vec{E}_0\sin(\omega t)$ is applied in the metal. Assume that the ions are infinitely heavy and fixed. The effective mass and charge of an electron are denoted as m and $-e$ respectively. Within the simple framework of the free electron approximation one can assume that the field acting on an electron is equivalent to $\vec{E}_0\sin(\omega t)$. All other forces (including dissipative forces) are small and negligible.

在金属中施加均匀的振荡电场$\vec{E}_0 \sin(\omega t)$。假设离子无限重并且固定。电子的有效质量和电荷分别表示为m和$-e$。 在自由电子近似的简单框架内，可以假定作用于电子的场等价于$\vec{E}_0 \sin(\omega t)$。 所有其他力（包括耗散力）都很小并且可以忽略不计。

A1	The electric field drives the collective motion of the electrons $\vec{r}(t)$ along the electric field direction. Derive the expressions of $\vec{r}(t)$ and the polarization $\vec{P}(t)$ at the steady state. 电场驱动电子沿电场方向作集体运动$\vec{r}(t)$。试推导在稳定状态下的$\vec{r}(t)$和电极化强度$\vec{P}(t)$的表达式。	**2 points** **2 分**

A2	Determine the metal permittivity $\epsilon(\omega)$. 求金属的介电常数$\epsilon(\omega)$。	**1 points** **1 分**

Part B. Plasmon Resonance (16 points) 等离子共振（16 分）

In this part we consider a dielectric sphere of radius R and permittivity ϵ in a uniform electric field $\vec{E}_0$. Due to the polarization of the dielectric material, the electric field in the sphere and its neighborhood is modified. The polarization of the dielectric sphere is due to mobile charges being shifted in the uniform electric field. Here we model the dielectric effects by two oppositely charged spheres with radius R and charge density $\pm\rho$ being displaced along $\vec{E}_0$ by displacements $\pm\delta/2$ respectively (see Fig. 1).
在这部分我们考虑在均匀电场$\vec{E}_0$中半径为R和介电常数为ϵ的介电球。由于介质材料的极化作用，球体及其附近的电场被改变了。介质球的极化是由于电荷在均匀电场中产生移位。在这里，我们通过两个带相反电荷的球体来模拟介电效应。两个电荷球的半径为R、电荷密度分别为$\pm\rho$，沿$\vec{E}_0$方向的移位分别为$\pm\delta/2$（见图 1）。

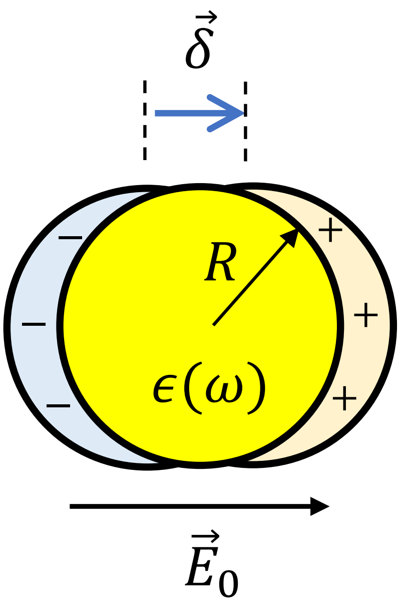


Fig. 1: Dielectric spheres in the uniform electric field.
图 1：在均匀电场中的介电球。

B1	The total electric field $\vec{E}_{in}$ inside the dielectric sphere is the superposition of the external field $\vec{E}_0$ and the electric fields due to the two charged spheres. Derive an expression for $\vec{E}_{in}$ in terms of $\vec{E}_0$ and the polarization $\vec{P}$ due to the two charged spheres. 介电球内部的总电场$\vec{E}_{in}$是由外加电场$\vec{E}_0$和两个带电球体引起的电场的叠加。试推导$\vec{E}_{in}$的表达式(以$\vec{E}_0$和两个带电球体引起的电极化强度$\vec{P}$表示)。	**2 points** **2 分**

B2	Due to the dielectric effects, surface charge is induced on the surface of the dielectric sphere. Consider a point on the surface of the dielectric sphere where the outward unit vector normal to the spherical surface is denoted as $\hat{n}$. In the limit of $\delta << R$, derive an expression for the induced surface charge density σ at this point in terms of the polarization $\vec{P}$ and $\hat{n}$. 由于介电效应，介电球表面上产生感应的表面电荷。考虑电介球表面上的一个点，它与球形表面垂直的向外单位矢量表示为$\hat{n}$。在$\delta << R$的极限下，试推导表面感应电荷密度的表达式(以电极化强度$\vec{P}$和$\hat{n}$表示)。	**2 points** **2 分**

B3	Following B2, derive the relation between the normal components of the electric fields $\vec{E}_{out} \cdot \hat{n}$ and $\vec{E}_{in} \cdot \hat{n}$ at the surface of the dielectric sphere. 根据 B2，试推导电场的法向分量$\vec{E}_{out} \cdot \hat{n}$和$\vec{E}_{in} \cdot \hat{n}$在介电球表面间的关系。	**1 point** **1 分**

B4	Express the induced electric dipole moment $\vec{d}_0$ of the dielectric sphere as a function of $\vec{E}_0$. 求介电球的感应电偶极矩$\vec{d}_0$(作为$\vec{E}_0$的函数表示)。	**3 points** **3 分**

Let us analyze the behavior of a metal sphere in an oscillating electric field of angular frequency ω and amplitude $\vec{E}_0$. The radius of the sphere is R. When the wavelength and field penetration depth are both much greater than the size of the sphere, one can consider the metal sphere as a dielectric in a uniform electric field, except that one has to use $\epsilon(\omega)$ (analogous to the one expressed in the previous part) in place of the permittivity. Hence the external electric field is $\vec{E} = \vec{E}_0 \cos \omega t$, and the dipole moment is $\vec{d} = \vec{d}_0 \cos \omega t$.
让我们分析在角频率为ω、振幅为$\vec{E}_0$的振荡电场下金属球的行为。 球体的半径是R。当波长和电场穿透深度都远大于球体的尺度时，可以将金属球看作均匀电场中的电介质，除了必须使用$\epsilon(\omega)$（类似于前面部分中所表达的那种） 代替介电常数。 因此，外部电场是$\vec{E} = \vec{E}_0 \cos \omega t$， 偶极矩是$\vec{d} = \vec{d}_0 \cos \omega t$ 。

B5	Sketch qualitatively the electric field lines (inside, near and far from the ball) in the system assuming $\epsilon(\omega) = -3$. 假设$\epsilon(\omega) = -3$，定性描绘系统中的电场线（球内、附近和远距）。	**2 points** **2 分**

B6	When $\omega = \omega_{res}$, resonance takes place and the internal electric intensity $\vert\vec{E}_{in}\vert$ increases to infinity. Determine ϵ_{res}, the value of $\epsilon(\omega)$ when $\omega = \omega_{res}$. 当$\omega = \omega_{res}$时，共振发生並且内部电场强度增加到无穷大。求ϵ_{res}，即$\epsilon(\omega)$在 $\omega = \omega_{res}$时的值。	**1 points** **1 分**

Significant increase in the electric field amplitude with frequency equaling ω_{res} is called the **plasmon resonance**. Assuming that there is no power dissipation, $|\vec{E}_{in}|$ approaches infinity. Taking into account dissipation, the major loss of power comes from dipole radiation.
电场振幅随着频率接近ω_{res}而显着增加, 这现象称为**等离子共振**。假设没有功耗，$|\vec{E}_{in}|$趋向无穷大。考虑到耗散，功率的主要损失来自偶极辐射。

B7	An oscillating dipole emits energy. Estimate the power I of this energy loss using dimensional analysis. A dipole radiation intensity depends on the dipole moment amplitude $\vert\vec{d}_0\vert$, its oscillation frequency ω_{res}, speed of light c and vacuum permittivity ϵ_0. 振荡偶极子发射能量。试用量纲分析来估算这种能量损失的功率 I 。偶极子辐射强度取决于偶极矩振幅$\vert\vec{d}_0\vert$、振荡频率ω_{res}、光速c和真空介电常数ϵ_0。	**3 points** **3 分**

B8	In practice, $\vert\vec{E}_{in}\vert$ is finite due to power dissipation at the plasmon resonance frequency ω_{res}. Suggest an approximate expression of the internal electric field intensity $\vert\vec{E}_{in}\vert$ using the condition that the power output is balanced by the mean power pumped into the system by the external field during plasmon resonance. 实际上，由于等离子共振频率下的功率损耗，$\vert\vec{E}_{in}\vert$是有限的。假設等离子共振时功率的输出与外场输入系统的平均功率平衡，試提出内部电场强度$\vert\vec{E}_{in}\vert$的近似表达式。	**2 points** **2 分**

Part C. Raman Spectroscopy (7 points) 拉曼光谱（7 分）

SERS is based on the phenomenon of **Raman scattering**, referring to the interaction of electromagnetic waves with mechanical vibrations of molecules. First we consider a molecule configuration. We assume that a molecule is made up of a number of atoms connected by chemical bonds that behave like springs. Hereafter we consider a diatomic molecule.
SERS 的基础是**拉曼散射现象**，指的是电磁波与分子机械振动的相互作用。 首先我们考虑一个分子的组态。我们假设一个分子是由许多通过化学键连接起来的原子组成的，化学键的作用如同弹簧。我们这里考虑双原子分子。

C1	Consider two masses m_1 and m_2 connected by a spring of spring constant k. Determine the frequency ω_0 of small-amplitude system oscillations. 考虑两个质量m_1和m_2，它们通过弹簧常数为k的弹簧连接。求系统在小振幅时的振荡频率ω_0。	**1 points** **1 分**

A polyatomic molecule is characterized by its spectrum of resonant frequencies. One can identify the molecule with the knowledge of its spectrum. This is the basic idea of SERS.
多原子分子的特征是其共振频率谱。人们可以通过其频谱的认识来识别分子。这是 SERS 的基本思想。

Let us analyze the behavior of a molecule in an external electric field $\vec{E}_0 \cos(\omega t)$. We assume that atoms have no charge, i.e. the molecule has no dipole moment in the absence of the external electric field. However, a molecule is polarized by the external electric field
让我们分析一个分子在外部电场$\vec{E}_0 \cos(\omega t)$中的行为。我们假设原子没有电荷，即在没有外部电场的情况下分子没有偶极矩。然而，分子能被外部电场极化

$$\vec{d} = \epsilon_0 \alpha \vec{E}_0 \cos(\omega t)$$

where α is the polarizability of the molecule. We assume that an induced dipole moment $\vec{d}$ is parallel to the electric field $\vec{E}$. Due to thermal agitation, mechanical oscillations of the molecules always exist at finite temperatures, and we assume that the thermally agitated angular frequency is ω_0.
其中α为分子的极化率。我们假设感应偶极矩$\vec{d}$与电场$\vec{E}$平行。由于热搅动，分子的机械振荡总是存在于有限的温度下，并且我们假定热搅动的角频率是ω_0 。

During molecular oscillations, the distance between atoms in a molecule deviates from its equilibrium value. Suppose the deviation x of the interatomic distance is given by $x = x_0 \cos(\omega_0 t)$. Furthermore, when the interatomic distance changes, the polarizability of the atoms changes accordingly, i.e., $\alpha = \alpha(x)$ (see Fig. 2).
在分子振荡过程中，分子中原子之间的距离偏离其平衡值。假设原子间距离的偏差x由$x = x_0 \cos(\omega_0 t)$ 给出。此外，当原子间距改变时，原子的极化率相应地改变，即$\alpha = \alpha(x)$（见图 2）。

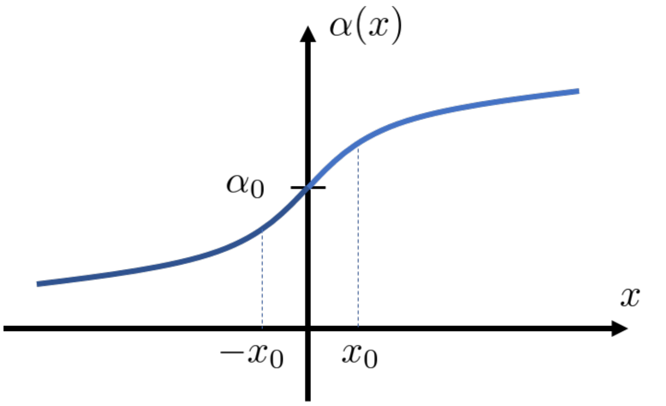


Fig. 2: The dependence of the polarizability of atoms in a molecule on the deviation of the interatomic distance during molecular oscillations.
图 2：分子中原子的极化率对分子振荡过程中原子间距的偏差的依赖性。

C2	Assuming that the amplitude of mechanical oscillations is small, express $\alpha(x)$ using linear approximation, given that $\alpha(0) = \alpha_0$ and $\left.\frac{d\alpha}{dx}\right\|_{x=0} = \beta_0$. 假设机械振荡的幅度很小，试以$\alpha(0) = \alpha_0$ 和 $\left.\frac{d\alpha}{dx}\right\|_{x=0} = \beta_0$ 将 $\alpha(x)$作线性近似表示。	**1 points** **1 分**

C3	Determine the diatomic molecule dipole moment in the external field $\vec{E}_0\cos(\omega t)$. Provide the answer in the form: 求外场$\vec{E}_0\cos(\omega t)$中的双原子分子偶极矩，答案以下列形式表示： $\vec{d} = \sum_i \vec{d}_i \cos(\omega_i t) \quad (1)$	**2 points** **2 分**

C4	A detector logs the dipole radiation of the molecule and detects several peaks with frequencies ω_i, corresponding to expression (1). The height of each peak is equal to the radiation intensity of the dipole $\vec{d}_i$. Determine frequency and height of each peak. Express the answer in terms of $\epsilon_0, \alpha_0, \beta_0, x_0, \omega, \omega_0$ and $\vec{E}_0$. 检测器记录分子的偶极辐射，并检测到数个具有频率ω_i，对应于表达式（1）的峰值。每个峰值的高度等于偶极子$\vec{d}_i$的辐射强度。求每个峰值的频率和高度，以$\epsilon_0, \alpha_0, \beta_0, x_0, \omega, \omega_0$ 和 $\vec{E}_0$表达答案。	**3 points** **3 分**

The presence of peaks with frequencies differing from ω in the spectrum is called **Raman scattering**. The stronger the external electric field the higher the signal detected from one molecule. A strong electric field can be obtained using the phenomenon of **plasmon resonance**. This is a difference between SERS and ordinary Raman spectroscopy.
频谱具有不同于ω的峰值的现象，被称为**拉曼散射**。 外部电场越强，从一个分子检测到的信号就越高。 利用**等离子共振**现象可以获得强大的电场。这是 SERS 和普通拉曼光谱之间的一个区别。

Part D. Surface-Enhanced Raman Spectroscopy (SERS) (7 points)
表面增强拉曼光谱（SERS）（7 分）

Consider a sphere of permittivity $\epsilon(\omega)$ in the uniform oscillating electric field of amplitude $|\vec{E}_0|$ in the case of plasmon resonance.
考虑在等离子共振情况下，在均匀振荡电场(振幅为 $|\vec{E}_0|$)中的介电球，其介电常数为$\epsilon(\omega)$。

D1	The molecule of the investigated material has to be placed into the points of maximal electric intensity. Locate these points in the figure on the answer sheet. 被检测材料的分子必须放置在最大电场强度的位置。 请在答题纸的图中标出这些位置。	**2 points** **2 分**

D2	Determine the enhancement factor $g(\omega)$ of the electric field at these points, where $g(\omega) = \max_{\vec{r}} \frac{\lvert\vec{E}(\vec{r})\rvert}{\lvert\vec{E}_0\rvert}$. Express the answer in terms of the metal permittivity $\epsilon(\omega)$. 求在这些位置电场的增强因子$g(\omega)$，其中$g(\omega) = \max_{\vec{r}} \frac{\lvert\vec{E}(\vec{r})\rvert}{\lvert\vec{E}_0\rvert}$。以金属介电常数$\epsilon(\omega)$表示答案。	**1 points** **1 分**

Metal beads enhance the external electric field radiation of amplitude $\vec{E}_0$ and dipole radiation of the molecule as well. The second process is characterized by the enhancement factor $g'(\omega, \omega_0)$. When $\omega \gg \omega_0$, one can assume $g' \approx g$. Then the signal intensity in SERS is g^4 times greater than that in ordinary Raman spectroscopy.
金属珠增强了外部电场辐射振幅$\vec{E}_0$和分子的偶极辐射。第二个过程以增强因子$g'(\omega, \omega_0)$为特征。 当$\omega \gg \omega_0$时，可假设$g' \approx g$。因此，SERS 中的信号强度比普通拉曼光谱中的信号强度大g^4倍。

Usually the signal comes from many molecules. Dipole radiations of the molecules are not coherent to each other. Thus a total radiation intensity formed by N molecules is equal to NI_0, where I_0 is the intensity of dipole radiation from a single molecule. An example of the experimental data is presented in Fig. 3.
信号通常来自许多分子。分子的偶极辐射彼此不相干。因此，由N个分子形成的总辐射强度为NI_0，其中I_0是来自单个分子的偶极辐射强度。图 3 给出了一组实验数据。

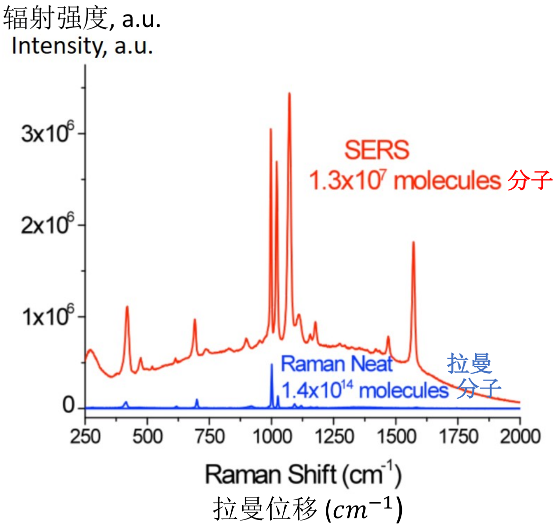


Fig. 3: [Credit: U.S. Naval Research Laboratory] Raman signal spectra. Red curve corresponds to SERS, blue to ordinary Raman spectroscopy. X-axis corresponds to Raman shift $k_0 = \omega_0/c$, and Y-axis corresponds to radiation intensity in arbitrary units. Note the different number of molecules in these experiments.
图 3：[美国海军研究实验室] 拉曼信号光谱。 红色曲线对应于 SERS，蓝色曲线对应于普通拉曼光谱。X轴对应于拉曼位移（$k_0 = \omega_0/c$），而 Y轴对应于任意单位下的辐射强度。 注意这些实验中分子的数量不同。

D3	By analyzing the experimental data presented in Fig. 3, estimate enhancement factor g due to the plasmon resonance at the peak of Raman shift $\omega_0/c = 1000\ \mathrm{cm}^{-1}$. Assume that $\omega_0 \ll \omega$. 通过分析图 3 中给出的实验数据，试估算在拉曼位移峰值$\omega_0/c = 1000\ \mathrm{cm}^{-1}$处由等离子共振引起的增强因子$g$。假设$\omega_0 \ll \omega$ 。	**2 points 2 分**

D4	Based on the results of Part B, estimate the radius R of the metal beads used in the experiment. Assume that the wavelength of the external radiation $\lambda = 785$ nm. 根据 B 部的结果，试估算实验中使用的金属珠的半径R。 假定外部辐射的波长为$\lambda = 785$ nm。	**2 points 2 分**

END of Problem 2
问题 2 完

Problem 1: Oscillations of the Sun (22 points) 太阳的振荡（22 分）

The sun is made of compressible gas. It can oscillate in a variety of ways. Investigating these oscillations has provided rich information on the interior of the Sun. In this problem we study two kinds of waves: pressure waves and gravity waves.
太阳的成份是可压缩气体。它可以以各种方式振荡。研究这些振荡提供了有关太阳内部的丰富信息。在这问题中，我们研究两种波：压力波和重力波。

Part A. Pressure Waves (15 points) 压力波（15 分）

Most of us are familiar with sound waves propagating through Earth's atmosphere, which is a pressure wave. In the Sun, however, we need to consider the fact that gas density falls off with height because of gravity. In this problem, we will use the following notations:
我们大多数人都熟悉在地球大气层传播的声波，它是一种压力波。但是在太阳内，我们需要考虑由于重力而导致气体密度随高度下降。在这个问题中，我们将采用以下符号：

$\bar{m}$ = average mass of particles 粒子平均质量
g = gravitational acceleration 重力加速度
k_B= Boltzmann constant 波尔兹曼常数
T = absolute temperature 绝对温度
γ = ratio of the constant-pressure specific heat to the constant-volume specific heat 定压比热与定容比热之比

We model the Sun as an atmosphere whose density falls off with height because of gravity. For a thin layer of the atmosphere between heights x and $x + dx$, the equilibrium pressure at these locations are $P(x)$ and $P(x + dx)$ respectively. Assume that the gravitational acceleration and the temperature are constant.
我们将太阳模拟为一个大气层，其密度因重力而随高度下降。对于高度在x 和$x + dx$之间的薄层气体来说，这些位置的稳态压力分别为$P(x)$和$P(x + dx)$。假定重力加速度和温度是恒定的。

A1	Derive the differential equation for the atmospheric density $\rho(x)$. 试推导大气密度$\rho(x)$的微分方程。	**2 points** **2 分**

Let A be the cross-section area of a column of the atmosphere.
External force acting on the gas = $[P(x) - P(x + dx)]A$.
Weight of the gas = $[\rho(x)Adx]g$.
Condition for equilibrium: $[P(x) - P(x + dx)]A = [\rho(x)Adx]g$.
In the limit $dx \to 0$, $P(x + dx) = P(x) + \frac{dP}{dx}dx$.
Hence $\frac{dP}{dx} = -\rho(x)g$.
Gas law: $P = \frac{\rho k_B T}{\bar{m}}$.
$\frac{d\rho}{dx} = -\frac{\bar{m}g}{k_B T}\rho$.

A2	The scale height H of the atmosphere is the height through which the density	**1 points**

	becomes a factor of e^{-1} of the original density. Derive the expression of H. 大气的标度高度H是密度为起始密度e^{-1}倍的高度。求H的表达式。	**1 分**

The solution of the differential equation is $\rho(x) = \rho(0)\exp\left(-\frac{\bar{m}g}{k_B T}x\right)$.
Hence $\frac{\bar{m}g}{k_B T}H = 1 \Rightarrow H = \frac{k_B T}{\bar{m}g}$.

When a pressure wave propagates vertically in the atmosphere, the particles will experience small vertical displacements. Let $u(x,t)$ denote the vertical displacement of the gas particles at time t whose undisturbed position is x.
当压力波在大气中垂直传播时，粒子将经历细小的垂直位移。设$u(x,t)$为气体粒子在時間t時的垂直位移，x为其不受干扰时的位置。

A3	As shown in the Fig. 1, there is a change in thickness of the thin layer. Express the change in thickness in terms containing the gradient $\partial u/\partial x$. (Remark: For u being a function of both x and t, $\partial u/\partial x$ is called the partial derivative of u with respect to x with t taken to be constant.) 如图 1 所示，薄层的厚度有变化。试以梯度$\partial u/\partial x$表示厚度变化。 （备注：u作为x和t二者的函数，$\partial u/\partial x$被称为u相对于x的偏导数，其中t在求导过程中视为常数。）	**1 points** **1 分**

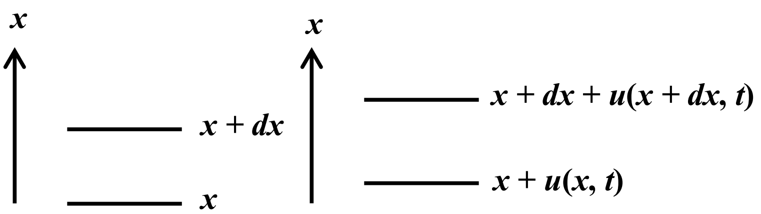


Fig. 1: The vertical displacements of a thin layer of gas particles caused by the propagation of a pressure wave. Note the change in the thickness of the layer.
图 1：由压力波传播引起薄层气体粒子的垂直位移。 请注意层厚度的变化。

Change in thickness $= u(x+dx,t) - u(x,t) = \frac{\partial u}{\partial x}dx$.

A4	In turn, the vertical displacements produce small fluctuations in density and pressure, denoted as $\delta\rho(x,t)$ and $\delta P(x,t)$ respectively. Express the change in $\delta\rho(x,t)$ and $\delta P(x,t)$ in terms containing the gradient $\partial u/\partial x$. Assume that the heat transfer is negligible during the period of the pressure wave. 随之而来，垂直位移产生密度和压力的細小波动，分别表示为$\delta\rho(x,t)$和$\delta P(x,t)$ 。求$\delta\rho(x,t)$和$\delta P(x,t)$的表示式（以梯度$\partial u/\partial x$表示）。假设在压力波传播期间传热可以忽略不计。	**2 points** **2 分**

Since the density is inversely proportional to the thickness of the layer,
$\frac{\delta\rho(x,t)}{\rho(x)} = -\frac{\partial u}{\partial x} \Rightarrow \delta\rho(x,t) = -\rho(x)\frac{\partial u}{\partial x}$.
Since for adiabatic processes, $P \propto \rho^\gamma$,
$\frac{\delta P(x,t)}{P(x)} = \gamma\frac{\delta\rho(x,t)}{\rho(x)} = -\gamma\frac{\partial u}{\partial x} \Rightarrow \delta P(x,t) = -\gamma P(x)\frac{\partial u}{\partial x}$.

A5	Derive the differential equation of motion for $u(x,t)$. Simplify your expressions using the speed of sound $c_s = \sqrt{\gamma P/\rho}$. 试推导$u(x,t)$的微分运动方程。以音速$c_s = \sqrt{\gamma P/\rho}$简化你的答案。	**3 points** **3 分**

Let A be the cross-section area of a column of the atmosphere.

$\rho(x)Adx\frac{\partial^2 u}{\partial t^2} = [\delta P(x,t) - \delta P(x+dx,t)]A.$

$\rho(x)\frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial x}\delta P(x,t) = \frac{\partial}{\partial x}\left[\gamma P(x)\frac{\partial u}{\partial x}\right].$

$\rho(x)\frac{\partial^2 u}{\partial t^2} = \gamma\frac{\partial P(x)}{\partial x}\frac{\partial u}{\partial x} + \gamma P(x)\frac{\partial^2 u}{\partial x^2}.$

From A1, $\frac{dP}{dx} = -\rho(x)g$. Hence

$\frac{\partial^2 u}{\partial t^2} = -\gamma g\frac{\partial u}{\partial x} + c_s^2\frac{\partial^2 u}{\partial x^2}.$

A6	Show that the solution of the equation of motion is equivalent to a pressure wave traveling through a uniform medium when the wavelength is shorter than a length scale. Derive this length scale. 试证明当波长短于某长度尺度时，运动方程的解等价于穿过均匀介质的压力波。求这个长度尺度。	**2 points** **2 分**

The solution of the differential equation is equivalent to a pressure wave traveling through a uniform medium if the first term is negligible compared with the second term.

First term $= \gamma g\frac{\partial u}{\partial x} \sim \frac{\gamma g}{\lambda}$

Second term $= c_s^2\frac{\partial^2 u}{\partial x^2} \sim \frac{c_s^2}{\lambda^2}$

$\frac{\gamma g}{\lambda} \ll \frac{c_s^2}{\lambda^2} \Rightarrow \lambda \ll \frac{c_s^2}{\gamma g} = \frac{P}{\rho g} = \frac{k_B T}{\bar{m} g} = H.$

Next we seek a sinusoidal wave with angular frequency ω. The energy density of the wave, $\frac{1}{2}\rho\omega^2 u^2$, is expected to remain constant as the wave propagates upward with constant velocity in the direction of decreasing density $\rho(x)$. With this expectation in mind we let
接下来我们寻找一个角频率ω的正弦波。当波沿密度$\rho(x)$减小的方向以恒定速度向上传播时，我们预期波的能量密度$\frac{1}{2}\rho\omega^2 u^2$保持不变。因此我们可设

$$u(x,t) = \frac{f(x)}{\sqrt{\rho(x)}}e^{-i\omega t}.$$

A7	Derive the differential equation for $f(x)$. 试推导$f(x)$的微分方程。	**3 points** **3 分**

$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u = -\omega^2\frac{f(x)}{\sqrt{\rho(x)}}e^{-i\omega t}.$

$\frac{\partial u}{\partial x} = \frac{f'(x)}{\sqrt{\rho(x)}}e^{-i\omega t} - \frac{f(x)\rho'(x)}{2\rho(x)^{\frac{3}{2}}}e^{-i\omega t}.$

From A1, $\rho'(x) = -\frac{\rho(x)}{H}$. Hence $\frac{\partial u}{\partial x} = \frac{f'(x)}{\sqrt{\rho(x)}}e^{-i\omega t} + \frac{f(x)}{2H\sqrt{\rho(x)}}e^{-i\omega t}.$

$\frac{\partial^2 u}{\partial x^2} = \frac{f''(x)}{\sqrt{\rho(x)}}e^{-i\omega t} + \frac{f'(x)}{H\sqrt{\rho(x)}}e^{-i\omega t} + \frac{f(x)}{4H^2\sqrt{\rho(x)}}e^{-i\omega t}.$

Substituting into A6,

$$-\omega^2 f(x) = -\gamma g f'(x) - \frac{\gamma g}{2H} f(x) + c_s^2\left[f''(x) + \frac{f'(x)}{H} + \frac{f(x)}{4H^2}\right].$$

$$f''(x) + \left(\frac{1}{H} - \frac{\gamma g}{c_s^2}\right) f'(x) + \left(\frac{1}{4H^2} - \frac{\gamma g}{2Hc_s^2} + \frac{\omega^2}{c_s^2}\right) f(x) = 0.$$

$$f''(x) + \left(\frac{\omega^2}{c_s^2} - \frac{1}{4H^2}\right) f(x) = 0.$$

A8	When the frequency of the pressure wave is below a critical frequency ω_c below the Sun's surface, it becomes trapped inside the Sun. What is ω_c? 当太阳表面下的压力波频率低于临界频率ω_c时，它会被困于太阳内部。求 ω_c。	**1 points** **1 分**

The pressure wave cannot propagate if $\frac{\omega^2}{c_s^2} - \frac{1}{4H^2} < 0 \Rightarrow \omega < \frac{c_s}{2H}$ Hence $\omega_c = \frac{c_s}{2H}$.

Part B. Gravity Waves (7 points) 重力波（7 分）

In Part A we only included the restoring force due to the fluctuation in the pressure gradient for pressure waves traveling in the vertical direction of the Sun's atmosphere. However, for gravity waves propagating in a horizontal direction of the Sun's atmosphere, the buoyancy of the gas may also give rise to a restoring force which can sustain oscillations.
在 A 部，我们只考虑了由于压力波在太阳大气垂直方向传播的压力梯度的波动而产生的恢复力。然而，对于沿太阳大气的水平方向传播的重力波，气体的浮力也可成为维持振荡的恢复力。

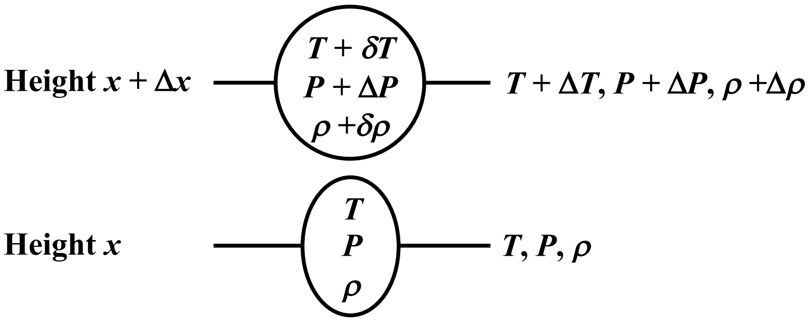


Fig. 2: Displacement of a pocket of gas from height x to height $x + \Delta x$.
图 2：一小口气体从高度x到高度$x + \Delta x$的位移。

To understand this, we consider a small vertical displacement of a pocket of gas in an environment of the same gas with both gradients in the temperature and pressure. As shown in Fig. 2, this pocket of gas has the same temperature, pressure and density as the surrounding gas. When its height is displaced by Δx, it enters an environment with temperature, pressure and density given by $T + \Delta T$, $P + \Delta P$ and $\rho + \Delta\rho$ respectively.
为了理解这一点，我们考虑一小口气体在相同气体的环境中，在温度梯度和压力梯度的影响下作垂直位移。如图 2 所示，这小口气体与周围气体具有相同的温度、压力和密度。当高度移动Δx时，它将进入温度为$T + \Delta T$、压力为$P + \Delta P$和密度为$\rho + \Delta\rho$ 的环境。

For the pocket of gas, the pressure inside the pocket responds rapidly to the environment so that its pressure also changes by ΔP. On the other hand, the change in temperature and

density may be different. Suppose the temperature, pressure and density of the pocket of gas in the new environment are $T+\delta T$, $P+\Delta P$ and $\rho+\delta\rho$ respectively. Assume that there is insufficient time for heat conduction from the pocket of gas to the environment.
对这小口气体而言，内部压力迅速回应环境，使其压力也随之改变为$P+\Delta P$。另一方面，温度和密度的变化可能不同。假设新环境中这小口气体的温度为$T+\delta T$、压力为$P+\Delta P$和密度为$\rho+\delta\rho$。假设没有足够的时间从小口气体向环境传导热量。

B1	Express $\Delta\rho$ and $\delta\rho$ in terms of expressions containing ΔT and ΔP. 求$\Delta\rho$ 和$\delta\rho$的表达式(用ΔT和ΔP表示)。	**2 points** **2 分**

Since the surrounding gas satisfies the ideal gas law $P=\frac{\rho k_B T}{\bar{m}}$, we have $\rho\propto P/T$ and
$\frac{\Delta\rho}{\rho}=\frac{\Delta P}{P}-\frac{\Delta T}{T}\Rightarrow\ \Delta\rho=\rho\left(\frac{\Delta P}{P}-\frac{\Delta T}{T}\right)$.
Since the pocket of gas undergoes an adiabatic process, $P\propto\rho^{\gamma}$,
$\frac{\delta\rho}{\rho}=\frac{1}{\gamma}\frac{\Delta P}{P}\Rightarrow\ \delta\rho=\frac{\rho}{\gamma}\frac{\Delta P}{P}$.

B2	Suppose the temperature and pressure gradients of the surrounding gas are dT/dx and dP/dx respectively. Derive the equation of motion of the pocket of gas. 假设周围气体的温度梯度和压力梯度分别为dT/dx和dP/dx。试推导这小口气体的运动方程。	**2 points** **2 分**

Let m be the mass of the pocket of gas.
Buoyancy of the pocket of gas $=\frac{m}{\rho+\delta\rho}(\rho+\Delta\rho)g-mg\approx\frac{mg}{\rho}(\Delta\rho-\delta\rho)$
$=mg\left(\frac{\gamma-1}{\gamma}\frac{\Delta P}{P}-\frac{\Delta T}{T}\right)$.
Applying Newton's law of motion,
$m\frac{d^2}{dt^2}\Delta x=mg\left(\frac{\gamma-1}{\gamma}\frac{\Delta P}{P}-\frac{\Delta T}{T}\right)$
$\frac{d^2}{dt^2}\Delta x=g\left(\frac{\gamma-1}{\gamma P}\frac{dP}{dx}-\frac{1}{T}\frac{dT}{dx}\right)\Delta x$

B3	Determine the range of temperature gradient dT/dx in which the pocket of gas can exhibit oscillations. Express the bound(s) of the temperature gradient in terms of T/H. 求这小口气体可出现振荡的温度梯度dT/dx范围(以T/H表示这范围的界限)。	**2 points** **2 分**

Oscillations are possible when $\frac{\gamma-1}{\gamma P}\frac{dP}{dx}-\frac{1}{T}\frac{dT}{dx}<0$.
$-\frac{dT}{dx}<-\frac{\gamma-1}{\gamma}\frac{T}{P}\frac{dP}{dx}$.
From A1, $\frac{dP}{dx}=-\rho g\Rightarrow-\frac{dT}{dx}<\frac{\gamma-1}{\gamma}\frac{\rho gT}{P}=\frac{\gamma-1}{\gamma}\frac{\bar{m}g}{k_B}=\frac{\gamma-1}{\gamma}\frac{T}{H}$.

B4	How does the gas in the Sun behave when the temperature gradient is outside the range considered in B3? 当温度梯度超出 B3 考虑的范围时，太阳中的气体会有什么行为?	**1 points** **1 分**

The gas will become **convective**.

Reference for this problem: A. C. Phillips, The Physics of Stars, 2nd edition (Wiley, 1994)

END of Problem 1
问题 1 完

Problem 2: Plasmon Resonance and SERS 等离子共振和 SERS

Surface-enhanced Raman spectroscopy (SERS) is one of the most prominent optical phenomena in the last 40 years. SERS is based on **plasmon resonance**, referring to the significant increase in electric field intensity near the small metal granules under certain conditions. In order to determine these conditions, it is necessary to learn how to describe the properties of metals placed in oscillating electromagnetic fields.
表面增强拉曼光谱（SERS）是近 40 年来最重要的光学现象之一。 SERS 的基础是**等离子共振**，指在某些条件下小金属颗粒附近的电场强度显着增加。为了确定这些条件，我们有必要认识如何描述放置在振荡电磁场中的金属特性。

Properties of a medium in an electric field are described as follows:

$$\vec{D} = \epsilon\epsilon_0\vec{E} = \epsilon_0\vec{E} + \vec{P}$$

where $\vec{E}$ and $\vec{D}$ are the electric field intensity and the electric displacement respectively, ϵ is the permittivity of the medium, $\vec{P}$ is the electric polarization (electric dipole moments per unit volume), ϵ_0 is the vacuum permittivity. The boundary conditions in the absence of free charges are the continuity of electric field tangential to the boundary and the continuity of the electric displacement normal to the boundary.
介质在电场中的特性描述如下：

$$\vec{D} = \epsilon\epsilon_0\vec{E} = \epsilon_0\vec{E} + \vec{P}$$

其中$\vec{E}$ 和$\vec{D}$分别是电场强度和电位移，ϵ是介质的介电常数，$\vec{P}$是电极化强度（单位体积的电偶极矩），ϵ_0是真空介电常数。在没有自由电荷下的边界条件分别是与边界相切的电场的连续性和垂直于边界的电位移的连续性。

In an oscillating electromagnetic field, the permittivity of a media (including metals) is dependent on the electromagnetic field frequency, $\epsilon = \epsilon(\omega)$.
在振荡电磁场中，介质（包括金属）的介电常数取决于电磁场频率：$\epsilon = \epsilon(\omega)$。

Part A. Free Electron Gas (3 points) 自由电子气体（3 分）

Consider a metal occupying an infinite space. Positive ions form a crystal lattice. Free electrons move inside the lattice. The number density of positive ions and electrons are the same and equal to n.
考虑一占据无限空间的金属。正离子形成晶格。自由电子在晶格内移动。正离子和电子的数密度相同并等于n 。

A uniform oscillating electric field $\vec{E}_0\sin(\omega t)$ is applied in the metal. Assume that the ions are infinitely heavy and fixed. The effective mass and charge of an electron are denoted as m and $-e$ respectively. Within the simple framework of the free electron approximation one can assume that the field acting on an electron is equivalent to $\vec{E}_0\sin(\omega t)$. All other forces (including dissipative forces) are small and negligible.
在金属中施加均匀的振荡电场$\vec{E}_0\sin(\omega t)$。假设离子无限重并且固定。电子的有效质量和电荷分别表示为m和$-e$。 在自由电子近似的简单框架内，可以假定作用于电子的场等价于$\vec{E}_0\sin(\omega t)$。 所有其他力（包括耗散力）都很小并且可以忽略不计。

A1	The electric field drives the collective motion of the electrons $\vec{r}(t)$ along the electric field direction. Derive the expressions of $\vec{r}(t)$ and the polarization $\vec{P}(t)$ at the steady state. 电场驱动电子沿电场方向作集体运动$\vec{r}(t)$。试推导在稳定状态下的$\vec{r}(t)$和电极化强度$\vec{P}(t)$的表达式。	**2 points** **2 分**

Newton's second law of motion for electron,

$$m\frac{d^2}{dt^2}\vec{r}(t) = -e\vec{E}_0\sin(\omega t)$$

One can get

$$\vec{r}(t) = \frac{e\vec{E}_0}{m\omega^2}\sin(\omega t)$$

Since the heavy (position) ion does not move, the electric dipole moment per atom is

$$\vec{p} = -e\vec{r}(t) \rightarrow \vec{P}(t) = n\vec{p} = -\frac{ne^2\vec{E}_0}{m\omega^2}\sin(\omega t)$$

A2	Determine the metal permittivity $\epsilon(\omega)$. 求金属的介电常数$\epsilon(\omega)$。	**1 points** **1 分**

$$\epsilon\epsilon_0\vec{E} = \epsilon_0\vec{E} + \vec{P} \rightarrow \epsilon\epsilon_0\vec{E} = \epsilon_0\vec{E} - \frac{ne^2}{m\omega^2}\vec{E} \rightarrow \epsilon = 1 - \frac{ne^2}{\epsilon_0 m\omega^2}$$

Part B. Plasmon Resonance (16 points) 等离子共振（16 分）

In this part we consider a dielectric sphere of radius R and permittivity ϵ in a uniform electric field $\vec{E}_0$. Due to the polarization of the dielectric material, the electric field in the sphere and its neighborhood is modified. The polarization of the dielectric sphere is due to mobile charges being shifted in the uniform electric field. Here we model the dielectric effects by two oppositely charged spheres with radius R and charge density $\pm\rho$ being displaced along $\vec{E}_0$ by displacements $\pm\delta/2$ respectively (see Fig. 1).

在这部分我们考虑在均匀电场$\vec{E}_0$中半径为R和介电常数为ϵ的介电球。由于介质材料的极化作用，球体及其附近的电场被改变了。介质球的极化是由于电荷在均匀电场中产生移位。在这里，我们通过两个带相反电荷的球体来模拟介电效应。两个电荷球的半径为R、电荷密度分别为$\pm\rho$，沿$\vec{E}_0$方向的移位分别为$\pm\delta/2$（见图 1）。

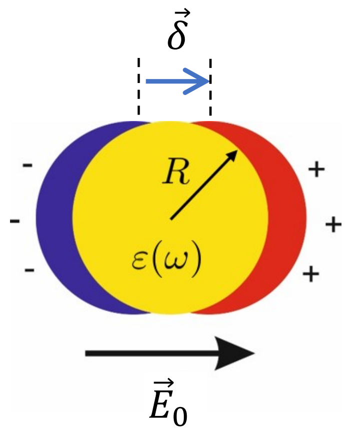


Fig. 1: Dielectric spheres in the uniform electric field.
图 1：在均匀电场中的介电球。

B1	The total electric field $\vec{E}_{in}$ inside the dielectric sphere is the superposition of the external field $\vec{E}_0$ and the electric fields due to the two charged spheres. Derive an expression for $\vec{E}_{in}$ in terms of $\vec{E}_0$ and the polarization $\vec{P}$ due to the two charged spheres. 介电球内部的总电场$\vec{E}_{in}$是由外加电场$\vec{E}_0$和两个带电球体引起的电场的叠加。试推导$\vec{E}_{in}$的表达式(以$\vec{E}_0$和两个带电球体引起的电极化强度$\vec{P}$表示)。	**2 points** **2 分**

Electric field at position $\vec{r}$ due to the positively charged sphere:

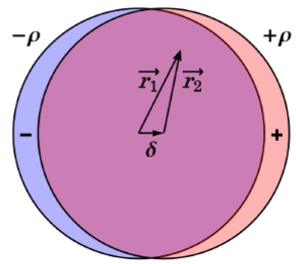
$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{4\pi r_2^3}{3}\rho\right)\frac{\vec{r}_2}{r_2^3} = \frac{\rho}{3\epsilon_0}\vec{r}_2 \quad \text{where } \vec{r}_2 = \vec{r} - \frac{\vec{\delta}}{2}$$

Similarly, electric field due to the negatively charged sphere:

$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{4\pi r_1^3}{3}\rho\right)\frac{\vec{r}_1}{r_1^3} = \frac{\rho}{3\epsilon_0}\vec{r}_1 \quad \text{where } \vec{r}_1 = \vec{r} + \frac{\vec{\delta}}{2}$$

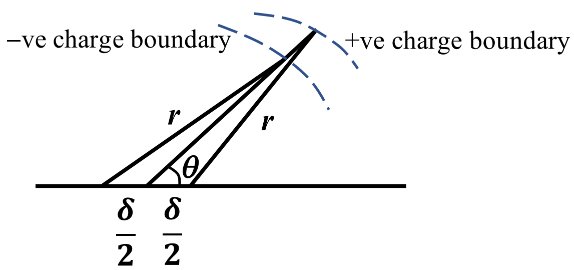
Hence

$$\vec{E}_{in} = \vec{E}_0 + \vec{E}_2 - \vec{E}_1 = \vec{E}_0 + \frac{\rho}{3\epsilon_0}(\vec{r}_2 - \vec{r}_1) = \vec{E}_0 - \frac{\rho\vec{\delta}}{3\epsilon_0} = \vec{E}_0 - \frac{\vec{P}}{3\epsilon_0}.$$



B2	Due to the dielectric effects, surface charge is induced on the surface of the dielectric sphere. Consider a point on the surface of the dielectric sphere where the outward unit vector normal to the spherical surface is denoted as $\hat{n}$. In the limit of $\delta << R$, derive an expression for the induced surface charge density σ at this point in terms of the polarization $\vec{P}$ and $\hat{n}$. 由于介电效应，介电球表面上产生感应的表面电荷。考虑电介球表面上的一个点，它与球形表面垂直的向外单位矢量表示为$\hat{n}$。在$\delta << R$ 的极限下，试推导表面感应电荷密度的表达式(以电极化强度$\vec{P}$和$\hat{n}$表示)。	**2 points** **2 分**

Consider a point making an angle θ at the center of the sphere with the electric field. As shown in the figure, distance of the boundary of the positively charged sphere from the center $\approx r + \frac{\delta}{2}\cos\theta$.



Similarly, distance of the boundary of the negatively charged sphere from the center $\approx r - \frac{\delta}{2}\cos\theta$.

Hence, thickness of the layer of the induced charge

$$\approx \left(r + \frac{\delta}{2}\cos\theta\right) - \left(r - \frac{\delta}{2}\cos\theta\right) = \delta\cos\theta.$$

Surface charge density: $\sigma \approx \rho\delta\cos\theta = P\cos\theta = \vec{P}\cdot\hat{n}$.

B3	Following B2, derive the relation between the normal components of the electric fields $\vec{E}_{out}\cdot\hat{n}$ and $\vec{E}_{in}\cdot\hat{n}$ at the surface of the dielectric sphere.	**1 point** **1 分**

	根据 B2，试推导电场的法向分量$\vec{E}_{out}\cdot\hat{n}$和$\vec{E}_{in}\cdot\hat{n}$在介电球表面间的关系。	

Using Gauss' law, $\vec{E}_{out}\cdot\hat{n} = \vec{E}_{in}\cdot\hat{n} + \frac{\sigma}{\epsilon_0} = \vec{E}_{in}\cdot\hat{n} + \frac{\vec{P}\cdot\hat{n}}{\epsilon_0}$.

B3	Express the induced electric dipole moment $\vec{d}_0$ of the dielectric sphere as a function of $\vec{E}_0$. 求介电球的感应电偶极矩$\vec{d}_0$(作为$\vec{E}_0$的函数表示)。	**3 points** **3 分**

From B.1, $\vec{E}_{in}\cdot\hat{n} = \vec{E}_0\cdot\hat{n} - \frac{\vec{P}\cdot\hat{n}}{3\epsilon_0}$.

From B.3, $\vec{E}_{out}\cdot\hat{n} = \vec{E}_{in}\cdot\hat{n} + \frac{\vec{P}\cdot\hat{n}}{\epsilon_0}$.

Eliminating the polarization, $\vec{E}_{out}\cdot\hat{n} = 3\vec{E}_0\cdot\hat{n} - 2\vec{E}_{in}\cdot\hat{n}$

Consider the boundary condition: $\vec{E}_{out}\cdot\hat{n} = \epsilon\vec{E}_{in}\cdot\hat{n}$.

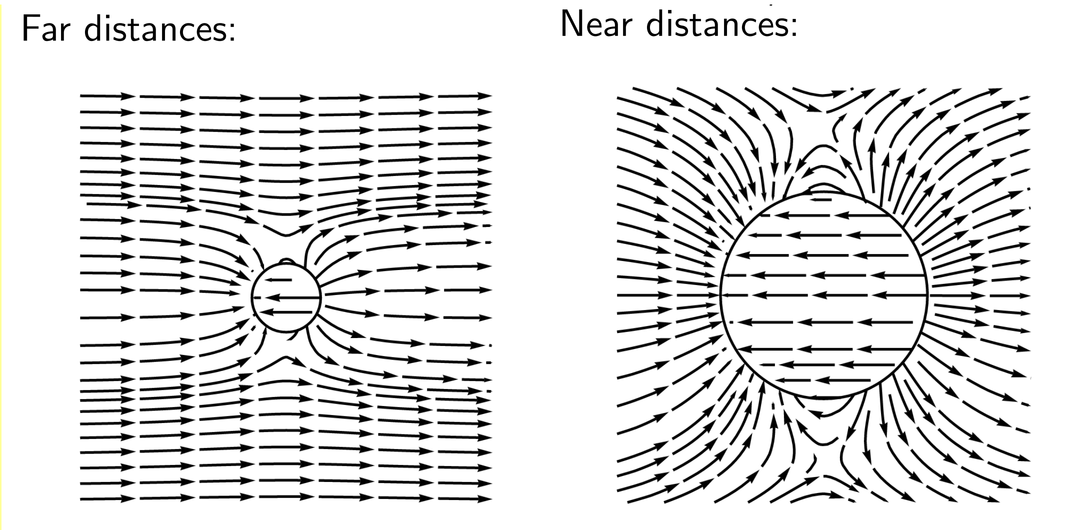
Eliminating $\vec{E}_{out}$: $\vec{E}_{in}\cdot\hat{n} = \frac{3}{\epsilon+2}\vec{E}_0\cdot\hat{n}$ and $\vec{P}\cdot\hat{n} = 3\epsilon_0\left(\vec{E}_0\cdot\hat{n} - \vec{E}_{in}\cdot\hat{n}\right) = 3\epsilon_0\left(\frac{\epsilon-1}{\epsilon+2}\right)\vec{E}_0\cdot\hat{n}$

Since $\vec{P}$ and $\vec{E}_0$ are parallel, $\vec{P} = 3\epsilon_0\left(\frac{\epsilon-1}{\epsilon+2}\right)\vec{E}_0$ and $\vec{d}_0 = \frac{4}{3}\pi R^3\vec{P} = 4\pi R^3\epsilon_0\left(\frac{\epsilon-1}{\epsilon+2}\right)\vec{E}_0$.

Let us analyze the behavior of a metal sphere in an oscillating electric field of angular frequency ω and amplitude $\vec{E}_0$. The radius of the sphere is R. When the wavelength and field penetration depth are both much greater than the size of the sphere, one can consider the metal sphere as a dielectric in a uniform electric field, except that one has to use $\epsilon(\omega)$ (analogous to the one expressed in the previous part) in place of the permittivity. Hence the external electric field is $\vec{E} = \vec{E}_0\cos\omega t$, and the dipole moment is $\vec{d} = \vec{d}_0\cos\omega t$.

让我们分析在角频率为ω、振幅为$\vec{E}_0$的振荡电场下金属球的行为。 球体的半径是R。当波长和电场穿透深度都远大于球体的尺度时，可以将金属球看作均匀电场中的电介质，除了必须使用$\epsilon(\omega)$（类似于前面部分中所表达的那种） 代替介电常数。 因此，外部电场是$\vec{E} = \vec{E}_0\cos\omega t$，偶极矩是$\vec{d} = \vec{d}_0\cos\omega t$ 。

B5	Sketch qualitatively the field lines (inside, near and far from the ball) in the system assuming $\epsilon(\omega) = -3$. 假设$\epsilon(\omega) = -3$，定性描绘系统中的场线（球内、附近和远距）。	**2 points** **2 分**



For $\epsilon(\omega) = -3$, $\vec{E}_{in} \cdot \hat{n} = \frac{3}{\epsilon+2}\vec{E}_0 \cdot \hat{n} = -3\vec{E}_0 \cdot \hat{n}$, and $\vec{E}_{out} \cdot \hat{n} = 9\vec{E}_0 \cdot \hat{n}$.
The sketch should show the following features:
In distant regions, the field lines are parallel to $\vec{E}_0$.
In the upper and lower regions the field lines are continuous from left to right.
In the neighborhood of the sphere the field lines terminate or originate at the sphere.
The direction at which the field lines terminate or originate at the sphere lies on the equatorial side of the normal.
Near the equatorial plane the field lines originate and terminate at the sphere in the direction opposite to $\vec{E}_0$.
There is a point of zero field above and below the sphere.
Inside the dielectric sphere the field lines are antiparallel to $\vec{E}_0$.

B6	When $\omega = \omega_{res}$, resonance takes place and the internal electric intensity $\lvert\vec{E}_{in}\rvert$ increases to infinity. Determine ϵ_{res}, the value of $\epsilon(\omega)$ when $\omega = \omega_{res}$. 当$\omega = \omega_{res}$时，共振发生並且内部电场强度增加到无穷大。求ϵ_{res}，即$\epsilon(\omega)$在 $\omega = \omega_{res}$时的值。	**1 points** **1 分**

$\vec{E}_{in} = \frac{3}{\epsilon+2}\vec{E}_0 \rightarrow \epsilon = -2$

Significant increase in the electric field amplitude with frequency equaling ω_{res} is called the **plasmon resonance**. Assuming that there is no power dissipation, $\lvert\vec{E}_{in}\rvert$ approaches infinity. Taking into account dissipation, the major loss of power comes from dipole radiation.
电场振幅随着频率接近ω_{res}而显着增加, 这现象称为**等离子共振**。假设没有功耗，$\lvert\vec{E}_{in}\rvert$趋向无穷大。考虑到耗散，功率的主要损失来自偶极辐射。

B7	An oscillating dipole emits energy. Estimate the power I of this energy loss using dimensional analysis. A dipole radiation intensity depends on the dipole moment amplitude $\lvert\vec{d}_0\rvert$, its oscillation frequency ω_{res}, speed of light c and vacuum permittivity ϵ_0. 振荡偶极子发射能量。试用*量纲*分析来估算这种能量损失的功率 I 。偶极子辐射强度取决于偶极矩振幅$\lvert\vec{d}_0\rvert$、振荡频率ω_{res}、光速c和真空介电常数ϵ_0。	**3 points** **3 分**

Using the dimensional analysis we obtain:

$I = \epsilon_0^{\alpha} \left|\vec{d}_0\right|^{\beta} \omega_{res}^{\gamma} c^{\eta}$

$$[I] = N \cdot m \cdot s^{-1}$$

$$[\epsilon_0] = C^2 \cdot N^{-1} \cdot m^{-2}, \quad \left[\left|\vec{d}_0\right|\right] = C \cdot m, \quad [\omega_{res}] = s^{-1}, \qquad [c] = m \cdot s^{-1}$$

Solving the equations, we get

$I = \frac{\left|\vec{d}_0\right|^2 \omega_{res}^4}{\epsilon_0 c^3}$ [2.0 : 0.5 for each exponents]

B8	In practice, $\left|\vec{E}_{in}\right|$ is finite due to power dissipation at the plasmon resonance frequency ω_{res}. Suggest an approximate expression of the internal electric field intensity $\left|\vec{E}_{in}\right|$ using the condition that the power output is balanced by the mean power pumped into the system by the external field during plasmon resonance. 实际上，由于等离子共振频率下的功率损耗，$\left|\vec{E}_{in}\right|$是有限的。假設等离子共振时功率的输出与外场输入系统的平均功率平衡，試提出内部电场强度$\left|\vec{E}_{in}\right|$的近似表达式。	**2 points** **2 分**

Work done in producing an electric dipole is $E_0\left(\frac{4}{3}\pi R^3 \rho\right)\delta \approx E_0 d_0$.

Power consumed in producing the oscillating electric dipole,

$\frac{d}{dt}\left(\vec{E}_0 \cdot \vec{d}\right) \approx \vec{E}_0 \cdot \frac{d}{dt}\vec{d} \approx E_0 d_0 \omega_{res}$

By energy balance and $\vec{d}_0 = \frac{4\pi}{3} R^3 \epsilon_0 (\epsilon - 1)\vec{E}_{in} \to d_0 \approx \epsilon_0 R^3 E_{in}$, we have

$E_0 d_0 \omega_{res} = \frac{d_0^2 \omega_{res}^4}{\epsilon_0 c^3} \to E_0 = \frac{d_0 \omega_{res}^3}{\epsilon_0 c^3} \to E_{in} = E_0 \left(\frac{c}{\omega_{res} R}\right)^3$

Part C. Raman Spectroscopy (7 points) 拉曼光谱（7 分）

SERS is based on the phenomenon of **Raman scattering**, referring to the interaction of electromagnetic waves with mechanical vibrations of molecules. First we consider a molecule configuration. We assume that a molecule is made up of a number of atoms connected by chemical bonds that behave like springs. Hereafter we consider a diatomic molecule.

SERS 的基础是**拉曼散射现象**，指的是电磁波与分子机械振动的相互作用。 首先我们考虑一个分子的组态。我们假设一个分子是由许多通过化学键连接起来的原子组成的，化学键的作用如同弹簧。我们这里考虑双原子分子。

C1	Consider two masses m_1 and m_2 connected by a spring of spring constant k. Determine the frequency ω_0 of small-amplitude system oscillations. 考虑两个质量m_1和m_2，它们通过弹簧常数为k的弹簧连接。求系统在小振幅时的振荡频率ω_0。	**1 points** **1 分**

Using the reduced mass,

$\omega_0 = \sqrt{\frac{k(m_1+m_2)}{m_1 m_2}}$

A polyatomic molecule is characterized by its spectrum of resonant frequencies. One can identify the molecule with the knowledge of its spectrum. This is the basic idea of SERS.
多原子分子的特征是其共振频率谱。人们可以通过其频谱的认识来识别分子。这是 SERS 的基本思想。

Let us analyze the behavior of a molecule in an external electric field $\vec{E}_0\cos(\omega t)$. We assume that atoms have no charge, i.e. the molecule has no dipole moment in the absence of the external electric field. However, a molecule is polarized by the external electric field
让我们分析一个分子在外部电场$\vec{E}_0\cos(\omega t)$中的行为。我们假设原子没有电荷，即在没有外部电场的情况下分子没有偶极矩。然而，分子能被外部电场极化

$$\vec{d} = \epsilon_0 \alpha \vec{E}_0 \cos(\omega t)$$

where α is the polarizability of the molecule. We assume that an induced dipole moment $\vec{d}$ is parallel to the electric field $\vec{E}$. Due to thermal agitation, mechanical oscillations of the molecules always exist at finite temperatures, and we assume that the thermally agitated angular frequency is ω_0.
其中α为分子的极化率。我们假设感应偶极矩$\vec{d}$与电场$\vec{E}$平行。由于热搅动，分子的机械振荡总是存在于有限的温度下，并且我们假定热搅动的角频率是ω_0 。

During molecular oscillations, the distance between atoms in a molecule deviates from its equilibrium value. Suppose the deviation x of the interatomic distance is given by $x = x_0\cos(\omega_0 t)$. Furthermore, when the interatomic distance changes, the polarizability of the atoms changes accordingly, i.e., $\alpha = \alpha(x)$ (see Fig. 2).
在分子振荡过程中，分子中原子之间的距离偏离其平衡值。假设原子间距离的偏差x由$x = x_0\cos(\omega_0 t)$ 给出。此外，当原子间距改变时，原子的极化率相应地改变，即$\alpha = \alpha(x)$（见图 2）。

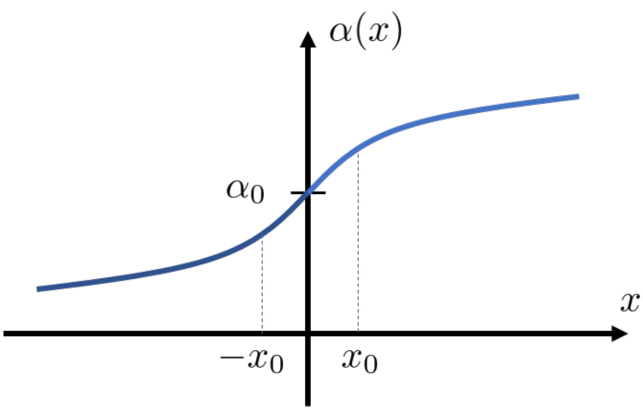


Fig. 2: The dependence of the polarizability of atoms in a molecule on the deviation of the interatomic distance during molecular oscillations.
图 2：分子中原子的极化率对分子振荡过程中原子间距的偏差的依赖性。

C2	Assuming that the amplitude of mechanical oscillations is small, express $\alpha(x)$ using linear approximation, given that $\alpha(0) = \alpha_0$ and $\left.\frac{d\alpha}{dx}\right\|_{x=0} = \beta_0$. 假设机械振荡的幅度很小，试以$\alpha(0) = \alpha_0$ 和 $\left.\frac{d\alpha}{dx}\right\|_{x=0} = \beta_0$ 将 $\alpha(x)$作线	**1 points** **1 分**

	性近似表示。	

$\alpha(x) = \alpha_0 + \beta_0 x$

C3	Determine the diatomic molecule dipole moment in the external field $\vec{E}_0 \cos(\omega t)$. Provide the answer in the form: 求外场$\vec{E}_0 \cos(\omega t)$中的双原子分子偶极矩，答案以下列形式表示： $\vec{d} = \sum_i \vec{d}_i \cos(\omega_i t)$ (1)	**2 points** **2 分**

$\vec{d} = \epsilon_0(\alpha_0 + \beta_0 x_0 \cos(\omega_0 t))\vec{E}_0 \cos(\omega t)$

$\Rightarrow \vec{d} = \epsilon_0 \alpha_0 \vec{E}_0 \cos(\omega t) + \frac{1}{2}\epsilon_0 \beta_0 x_0 \cos(\omega - \omega_0)t + \frac{1}{2}\epsilon_0 \beta_0 x_0 \cos(\omega + \omega_0)t$

C4	A detector logs the dipole radiation of the molecule and detects several peaks with frequencies ω_i, corresponding to expression (1). The height of each peak is equal to the radiation intensity of the dipole $\vec{d}_i$. Determine frequency and height of each peak. Express the answer in terms of $\epsilon_0, \alpha_0, \beta_0, x_0, \omega, \omega_0$ and $\vec{E}_0$. 检测器记录分子的偶极辐射，并检测到数个具有频率ω_i，对应于表达式（1）的峰值。每个峰值的高度等于偶极子$\vec{d}_i$的辐射强度。求每个峰值的频率和高度，以$\epsilon_0, \alpha_0, \beta_0, x_0, \omega, \omega_0$ 和 $\vec{E}_0$表达答案。	**3 points** **3 分**

The dipole radiation intensity reads

$$I = \frac{\left|\vec{d}_0\right|^2 \omega_{res}^4}{\epsilon_0 c^3}$$

where ω_{res} is the oscillation frequency. Hence we can observe 3 peaks with amplitudes:

Frequency	Amplitude
ω	$\frac{\epsilon_0(\alpha_0 E_0)^2\omega^4}{c^3}$
$\omega - \omega_0$	$\frac{\epsilon_0(\beta_0 x_0 E_0)^2(\omega - \omega_0)^4}{4c^3}$
$\omega + \omega_0$	$\frac{\epsilon_0(\beta_0 x_0 E_0)^2(\omega + \omega_0)^4}{4c^3}$

[0.5 for each of number in the table]

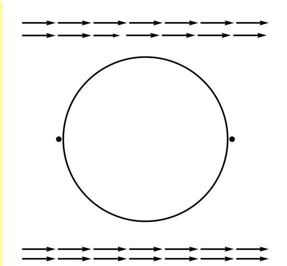
The presence of peaks with frequencies differing from ω in the spectrum is called **Raman scattering**. The stronger the external electric field the higher the signal detected from one molecule. A strong electric field can be obtained using the phenomenon of **plasmon resonance**. This is a difference between SERS and ordinary Raman spectroscopy.

频谱具有不同于ω的峰值的现象，被称为**拉曼散射**。 外部电场越强，从一个分子检测到的信号就越高。 利用**等离子共振**现象可以获得强大的电场。这是 SERS 和普通拉曼光谱之间的一个区别。

Part D. Surface-Enhanced Raman Spectroscopy (SERS) (7 points)
表面增强拉曼光谱（SERS）（7 分）

Consider a sphere of permittivity $\epsilon(\omega)$ in the uniform oscillating electric field of amplitude $|\vec{E}_0|$ in the case of plasmon resonance.
考虑在等离子共振情况下，在均匀振荡电场(振幅为 $|\vec{E}_0|$)中的介电球，其介电常数为 $\epsilon(\omega)$。

D1	The molecule of the investigated material has to be placed into the points of maximal electric intensity. Locate these points in the figure on the answer sheet. 被检测材料的分子必须放置在最大电场强度的位置。 请在答题纸的图中标出这些位置。	**2 points 2 分**



D2	Determine the enhancement factor $g(\omega)$ of the electric field at these points, where $g(\omega) = \max\limits_{\vec{r}} \frac{|\vec{E}(\vec{r})|}{|\vec{E}_0|}$. Express the answer in terms of the metal permittivity $\epsilon(\omega)$. 求在这些位置电场的增强因子$g(\omega)$，其中$g(\omega) = \max\limits_{\vec{r}} \frac{|\vec{E}(\vec{r})|}{|\vec{E}_0|}$。以金属介电常数$\epsilon(\omega)$表示答案。	**1 points 1 分**

$$g(\omega) = \max_{\vec{r}} \frac{|\vec{E}(\vec{r})|}{|\vec{E}_0|} = \left|\epsilon(\omega)\frac{E_{in}}{E_0}\right| = \left|\frac{3\epsilon(\omega)}{2+\epsilon(\omega)}\right|$$

Metal beads enhance the external electric field radiation of amplitude $\vec{E}_0$ and dipole radiation of the molecule as well. The second process is characterized by the enhancement factor $g'(\omega, \omega_0)$. When $\omega \gg \omega_0$, one can assume $g' \approx g$. Then the signal intensity in SERS is g^4 times greater than that in ordinary Raman spectroscopy.
金属珠增强了外部电场辐射振幅$\vec{E}_0$和分子的偶极辐射。第二个过程以增强因子$g'(\omega, \omega_0)$为特征。 当$\omega \gg \omega_0$时，可假设$g' \approx g$。因此，SERS 中的信号强度比普通拉曼光谱中的信号强度大g^4倍。

Usually the signal comes from many molecules. Dipole radiations of the molecules are not coherent to each other. Thus a total radiation intensity formed by N molecules is equal to NI_0, where I_0 is the intensity of dipole radiation from a single molecule. An example of the experimental data is presented in Fig. 3.
信号通常来自许多分子。分子的偶极辐射彼此不相干。因此，由N个分子形成的总辐射强度为NI_0，其中I_0是来自单个分子的偶极辐射强度。图 3 给出了一组实验数据。

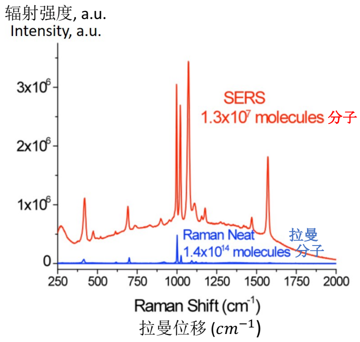


Fig. 3: [Credit: U.S. Naval Research Laboratory] Raman signal spectra. Red curve corresponds to SERS, blue to ordinary Raman spectroscopy. *X*-axis corresponds to Raman shift $k_0 = \omega_0/c$, and *Y*-axis corresponds to radiation intensity in arbitrary units. Note the different number of molecules in these experiments.
图 3：[美国海军研究实验室] 拉曼信号光谱。 红色曲线对应于 SERS，蓝色曲线对应于普通拉曼光谱。X轴对应于拉曼位移（$k_0 = \omega_0/c$），而 Y轴对应于任意单位下的辐射强度。 注意这些实验中分子的数量不同。

D3	By analyzing the experimental data presented in Fig. 3, estimate enhancement factor g due to the plasmon resonance at the peak of Raman shift $\omega_0/c = 1000\ \mathrm{cm}^{-1}$. Assume that $\omega_0 \ll \omega$. 通过分析图 3 中给出的实验数据，试估算在拉曼位移峰值$\omega_0/c = 1000\ \mathrm{cm}^{-1}$处由等离子共振引起的增强因子$g$。假设$\omega_0 \ll \omega$ 。	**2 points** **2 分**

$$\left(\frac{I_{SERS}}{N_{SERS}}\right) / \left(\frac{I_{RAM}}{N_{RAM}}\right) = g^4$$

$$\Longrightarrow g = \sqrt[4]{\left(\frac{I_{SERS}}{N_{SERS}}\right) / \left(\frac{I_{RAM}}{N_{RAM}}\right)} \approx 90$$

[2.0 pt: 80-100, 1.0 pt: 60-120, 0 pt: otherwise]

D4	Based on the results of Part B, estimate the radius R of the metal beads used in the experiment. Assume that the wavelength of the external radiation $\lambda = 785$ nm. 根据 B 部的结果，试估算实验中使用的金属珠的半径R。 假定外部辐射的波长为$\lambda = 785$ nm。	**2 points** **2 分**

$\omega_{res} = 2\pi \frac{c}{\lambda} \to \frac{c}{\omega_{res}} = \frac{\lambda}{2\pi}$

$\frac{3\epsilon(\omega_{res})}{2+\epsilon(\omega_{res})} \approx 90 \to \epsilon(\omega_{res}) \approx -2.069$

$E_{in} = E_0 \left(\frac{c}{\omega_{res} R}\right)^3.$

$g(\omega) = \max_{\vec{r}} \frac{|\vec{E}(\vec{r})|}{|\vec{E}_0|} = |\epsilon(\omega_{res})| \left(\frac{c}{\omega_{res} R}\right)^3 \approx 90$

$\Longrightarrow R \approx 36$ nm

Acknowledgment: We thank Dr. Vitaly Shevchenko for contributing this interesting problem.

END of Problem 2
问题 2 完

Pan Pearl River Delta Physics Olympiad 2019
2019 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 4 Problems, 40 Points) 简体版卷-1（共4题，40分）
(9:00 am – 11:30 am, 15 February, 2019)

Please fill in your final answers to all problems on the **answer sheet**.
请在答题纸上填上各题的最后答案。
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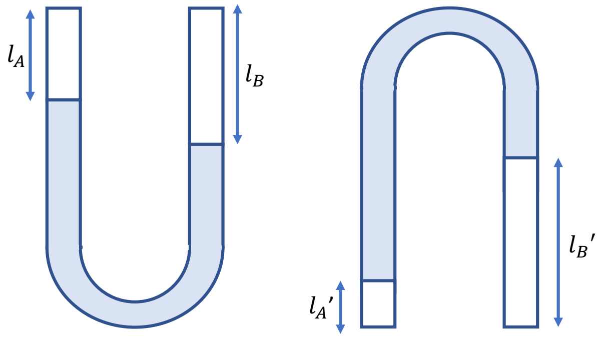
1. A U-shaped glass tube (9 points) U 形玻璃管（9 分）

A U-shaped glass tube with a constant cross-sectional area contains mercury (with density $\rho_{Hg} = 1.36 \times 10^4$ kg·m^{-3}). The two ends of the tubes are sealed; one contains gas A, the other contains gas B, both of which are ideal gases. In this problem, you can take the gravitational acceleration $g = 9.8$ ms^{-2}.

具有恒定横截面积的 U 形玻璃管含有汞（具有密度$\rho_{Hg} = 1.36 \times 10^4$ kg·m^{-3}）。管的两端是密封的;一端含有气体 A，另一端含有气体 B，两者都是理想气体。在这个问题上，你可以采取重力加速度 $g = 9.8$ ms^{-2}。



(a) (b)

First, we set the tubes vertically, with the two ends up (Fig. 2a). The parts filled with gases A and B have lengths $l_A = 12$ cm and $l_B = 18$ cm, respectively. Then we turn the tubes upside down (Fig. 2b), the length of the parts filled by gas A and gas B are $l_A' = 6$ cm and l_B' respectively. The ambient temperature is $T = 20°$C.

首先，我们将管子垂直放置，两端朝上（图 2a）。填充有气体 A 和 B 的部份分别具有长度$l_A = 12$ cm和长度$l_B = 18$ cm。然后我们将管子倒置（图 2b），由气体 A 和气体 B 填充的部分的长度分别为$l_A' = 6$ cm和l_B'。环境温度是$T = 20°$C。

(a) Calculate the numerical value of the length l_B' of the part filled with gas B when the tube is turned upside down. [1]

(a) 计算当玻璃管倒置时填充有气体 B 的部份的长度l_B'的数值。 [1]

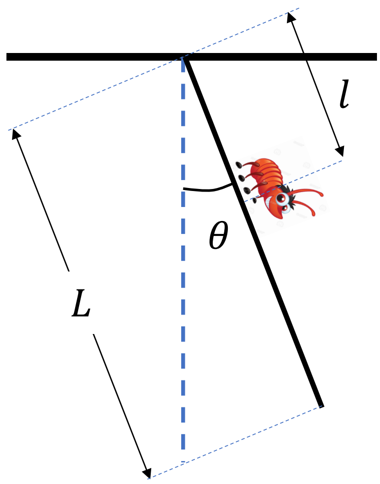
(b) Calculate the numerical values of the pressures p_A, p_B, p_A' and p_B' of the gases for the two orientations of the tubes. [4]

(b) 计算玻璃管内气体分别在两个方向的压强p_A，p_B，p'_A 和p'_B的数值。 [4]

(c) We look at the vertical tubes with two ends up again (Fig. 2a), but now we increase the ambient temperature by $\Delta T = 20°\mathrm{C}$ (i.e. $T \to T + \Delta T$). This leads to the changes in length $l_i \to l_i + \Delta l_i$ and pressure $p_i \to p_i + \Delta p_i$ $(i = A, B)$ for gases A and B. Determine the changes $\Delta l_A, \Delta l_B, \Delta p_A$ and Δp_B numerically. [4]
(c) 我们再看两端向上的垂直玻璃管（图 2a），但现在我们增加环境温度 $\Delta T = 20°\mathrm{C}$（即$T \to T + \Delta T$）。这导致气体 A 和 B 的长度$l_i \to l_i + \Delta l_i$和压强$p_i \to p_i + \Delta p_i(i = A, B)$发生变化。試計算長度变化 $\Delta l_A, \Delta l_B$ 和压强变化$\Delta p_A, \Delta p_B$的数值 。[4]

2. Bug on a rod (11 points) 杆上的虫子（11 分）

A pendulum consists of a uniform rigid rod of length L, mass M, a bug of mass $M/3$ which can crawl along the rod. The rod is pivoted at one end and swings in a vertical plane. Initially the bug is at the pivot-end of the rod, which is at rest at an angle θ_i ($\theta_i \ll 1$ rad) from the vertical as shown in the figure, is released. For $t > 0$ the bug crawls slowly with constant speed V along the rod towards the bottom end of the rod.
单摆由长度L、质量M的均匀刚性杆组成， 一只质量为$M/3$的虫子沿着杆爬行。杆在一端枢转并在垂直平面中摆动。虫子最初位于杆的枢轴处 ，此时杆与垂直方向成一定夾角θ_i ($\theta_i \ll 1$ rad) (如图所示)。當$t > 0$， 虫子沿着杆朝着杆的底端以恒定速度V缓慢地爬行。



(a) What is the moment of inertia I of the rod and bug about the pivot when the bug has reached a distance l along the rod. [1]
(a) 当虫子沿着杆爬至距离 l 时，杆和虫子相對於枢轴的转动慣量 I 是什么? [1]

(b) Find the angular frequency ω of the swing of the pendulum when the bug has reached a distance l along the rod. Express your answer in terms of L and l. [1]
(b) 当虫子沿着杆爬至距离 l 时，找出单摆摆动的角频率ω。答案用L和l表达 。 [1]

From now on, you can assume the speed of the bug is so small that l hardly changes in a period of oscillation and can be taken to be constant, and the motion of the rod can be effectively described by simple harmonic motion, that is,
从现在开始，您可以假设虫子的速率非常小，因此在振荡周期内 l 几乎不会发生变化，并可当为常数 ，而杆的运动可以用简谐运动描述，即是

$$\theta(t) = \theta_0(l) \sin \omega t$$

where ω is the angular frequency you obtained in part (b) and $\theta_0(l)$ is the amplitude of the oscillation which will vary as the bug crawled.
其中 ω 是你在（b）部份中得到的角频率，$\theta_0(l)$ 是振荡的幅度，并随着虫子爬行而变化。

After the bug has reached a distance l along the rod, calculate the following quantities when it further crawls a short distance Δl. In parts (c) to (f), express your answers in terms of Δl,

$\Delta\theta_0$ and other parameters in the problem, where $\Delta\theta_0$ is the change of the angular amplitude during the displacement Δl.
试计算在虫子沿着杆子爬至距离 l后，当虫子继续爬行短距离Δl时下列的物理量。在(c)至(f)部份中，答案以 Δl、$\Delta\theta_0$及本题中的其他参数表达，其中$\Delta\theta_0$是振荡幅度在位移Δl后的改变量。

(c) Calculate the time-averaged work done ΔW by the bug on the rod-bug system. [1]
(c) 试计算虫子作用于杆子-虫子系统的时间平均功ΔW. [1]

(d) Calculate the change $\Delta\omega^2$ in the term ω^2. [1]
(d) 试计算ω^2 的改变量$\Delta\omega^2$。[1]

(e) Calculate the time-averaged change ΔK in the kinetic energy of the whole system. [1]
(e) 试计算整个系统的时间平均动能改变量ΔK。[1]

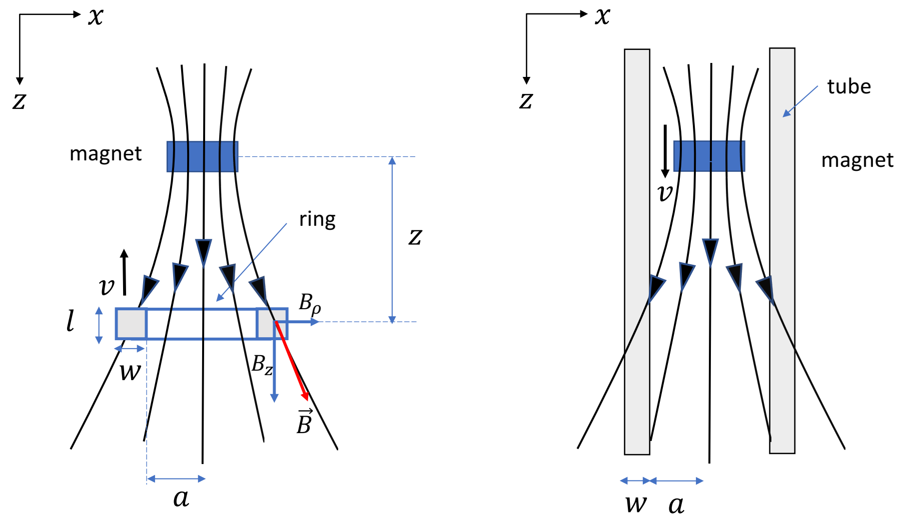
(f) Calculate the time-averaged change ΔU in the potential energy of the whole system. [1]
(f) 试计算整个系统的时间平均势能改变量ΔU。[1]

(g) Summarizing the above steps, calculate the relation between $\Delta\theta_0$ and Δl. [3]
(g) 总结上述步骤，计算$\Delta\theta_0$和Δl之间的关系。[3]

(h) Find the amplitude of the swing of the pendulum when the bug reaches the bottom end of the rod ($l = L$). Express your answer in terms of $\theta_0(l = 0) = \theta_i$. [2]
(h) 当虫子爬至杆的底端时($l = L$)，找出单摆摆动的幅度。答案用 $\theta_0(l = 0) = \theta_i$ 表达。[2]

3. Falling magnet inside a conductive pipe (10 points)
在导电管内下落的磁铁（10 分）

In this question, we consider the motion of a strong tiny magnet with mass M and magnetic dipole moment μ falling inside a vertical conducting non-magnetic tube.
在这个问题中，我们考虑具有质量M和磁偶极矩μ的强磁铁落入垂直的导电非磁性管内的运动。



(a) (b)

(a) We first consider a ring (with radius a, length l, thickness w ($w \ll a$) and conductivity σ) moving towards the magnet with speed v as shown in figure (a). The magnetic field at position $\vec{r}$ due to a magnetic dipole $\vec{\mu} = \mu\hat{z}$ (pointing downward as positive) at origin is given by
我们首先考虑一个环（半径a，长度l，厚度w（$w \ll a$）和导电率σ）以速度v向磁铁移动，如图（a）所示。由位於原點的磁偶极子$\vec{\mu} = \mu\hat{z}$（指向下为正）所產生的磁场，在位置$\vec{r}$处为

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi r^3}\left(\frac{3(\vec{\mu}\cdot\vec{r})\vec{r}}{r^2} - \vec{\mu}\right)$$

(i) Calculate the components of the magnetic field $B_\rho(\vec{r})$ and $B_z(\vec{r})$ at the point $\vec{r}$ with cylindrical coordinate $\vec{r} = (\rho = a, \phi = 0, z)$ [2]
(i) 计算磁场在位置$\vec{r}$处的分量$B_\rho(\vec{r})$和$B_z(\vec{r})$，$\vec{r}$的圆柱坐标为$\vec{r} = (\rho = a, \phi = 0, z)$。[2]

(ii) Calculate the induced emf and current on the ring. [3]
(ii) 计算环上的感生电动势和电流。 [3]

(iii) Calculate the magnetic force experienced by the coil. [1]
(iii) 计算环所感受的磁力。 [1]

(b) Next we consider a magnet falling inside along a vertical conducting non-magnetic tube of infinite length (with radius a, thickness w and conductivity σ) as shown in figure (b).
(b) 接下来我们考虑沿着一个无限长度（具有半径a，厚度w和导电率σ）的垂直导电非磁性管内落下的磁铁，如图（b）所示。

(i) When the magnet falls with the speed v, it experiences a damping force with the magnitude equal to γv. Calculate the damping constant γ. [3]
(i) 当磁铁以速度v下降时，它会感受一个量值为γv的阻尼力。试计算阻尼常数γ。 [3]

Hints: The following mathematical identity may be useful.
提示：下列数学公式可能有用。

$$\int_{-\infty}^{\infty}\frac{u^2du}{(1+u^2)^5}=\frac{5\pi}{128}$$

(ii) Determine the terminal velocity of the magnet when it falls inside the tube. [1]
(ii) 試計算磁铁在管内落下时的终端速度。 [1]

4. Heat flux between two plates (10 points) 两块板之间的热通量（10 点）

A system composed of two parallel plates at distance L from each other, which are at temperature T_1 and T_2 respectively.
一个系统由两块互相分隔、距离為 L 的平行板组成，分别处于温度 T_1 和 T_2。

(a) Calculate the heat flux density P (i.e. rate of heat energy flow per unit area) between two plates if the space between the plates is vacuum and each of the plates has emissivity ϵ. [4]
(a) 如果两块板之间的空间是真空并且每个板具有比輻射率 ϵ，计算两块板之间的热通量密度 P（即每单位面积的热能流速率）。[4]

(b) Now the space between the plates is filled with a monoatomic gas of molar density n and molar mass M. You need to estimate the heat flux density P between two plates according to the following approximations:

- The gas density is so low that the mean free path $\lambda \gg L$.
- $T_1 \gg T_2$
- When gas molecules bounce from the plates, they obtain the temperature of the respective plates (for instance, if they are absorbed/bounded for a short time by the molecules of the plate, and then released back into the space between the plates).
- You may neglect the black body radiation.
- “Estimate” means that the numeric prefactor of your expression does not need to be accurate.

(b) 现在两块板之间的空间充满了摩尔密度 n 和摩尔质量 M 的单原子气体。您需要跟据以下近似估算两块板之间的热通量密度 P：

- 气体密度低至平均自由程 $\lambda \gg L$.
- $T_1 \gg T_2$
- 当气体分子从板上反弹时，它们会获得相应板的温度（例如，如果它们被板的分子吸收/束缚很短的时间，然后释放回板之间的空间）。
- 您可忽略黑体辐射。
- “估算”表示答案的数字前因子不需要准确。

(i) Consider that there is an atom colliding with the hot plate and remains in thermal equilibrium with the hot plate when it is reflected by the plate. Calculate the average velocity square $\langle v_1^2\rangle$, and estimate the average horizontal velocity $\langle v_{1x}\rangle$ of the atom. [0.5]

(i) 考虑一粒原子与热板碰撞并且当它被板反射时与热板保持热平衡。计算平均速度平方$\langle v_1^2 \rangle$，并估算原子的平均水平速度$\langle v_{1x} \rangle$。 [0.5]

(ii) Consider that there is an atom colliding with the cold plate and remains in thermal equilibrium with the cold plate when it is reflected by the plate. Calculate the average velocity square $\langle v_2^2 \rangle$ and estimate the average horizontal velocity $\langle v_{2x} \rangle$ of the atom. [0.5]
(ii) 考虑一粒原子与冷板碰撞并且当它被板反射时与冷板保持热平衡。计算平均速度平方$\langle v_2^2 \rangle$，并估算原子的平均水平速度$\langle v_{2x} \rangle$。 [0.5]

(iii) Find the average energy transmitted by an atom when it moves from the hot to the cold plate. [1]
(iii) 找出一个原子从热板移动到冷板时传输的平均能量。 [1]

(iv) Estimate the heat flux density P between two plates. [4]
(iv) 估算两块板之间的热通量密度 P。 [4]

1

Solution

Pan Pearl River Delta Physics Olympiad 2019
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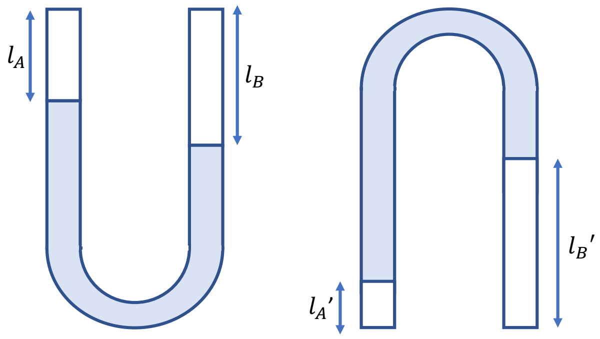


(a) (b)

First, we set the tubes vertically, with the two ends up (Fig. 2a). The parts filled with gases A and B have lengths $l_A = 12$ cm and $l_B = 18$ cm, respectively. Then we turn the tubes upside down (Fig. 2b), the length of the parts filled by gas A and gas B are $l_A' = 6$ cm and l_B' respectively. The ambient temperature is $T = 20°C$.

首先，我们将管子垂直放置，两端朝上（图 2a）。填充有气体 A 和 B 的部份分别具有长度$l_A = 12$ cm和长度$l_B = 18$ cm。然后我们将管子倒置（图 2b），由气体 A 和气体 B 填充的部分的长度分别为$l_A' = 6$ cm和l_B'。环境温度是$T = 20°C$。

(a) Calculate the numerical value of the length l_B' of the part filled with gas B when the tube is turned upside down. [1]
(a) 计算当玻璃管倒置时填充有气体 B 的部份的长度l_B'的数值。 [1]

Solution

(b) Calculate the numerical values of the pressures p_A, p_B, p'_A and p'_B of the gases for the two orientations of the tubes. [4]
(b) 计算玻璃管内气体分别在两个方向的压强p_A，p_B，p'_A 和p'_B的数值。 [4]

(c) We look at the vertical tubes with two ends up again (Fig. 2a), but now we increase the ambient temperature by $\Delta T = 20°\mathrm{C}$ (i.e. $T \to T + \Delta T$). This leads to the changes in length $l_i \to l_i + \Delta l_i$ and pressure $p_i \to p_i + \Delta p_i$ $(i = A, B)$ for gases A and B. Determine the changes $\Delta l_A, \Delta l_B, \Delta p_A$ and Δp_B numerically. [4]
(c) 我们再看两端向上的垂直玻璃管（图 2a），但现在我们增加环境温度 $\Delta T = 20°\mathrm{C}$（即$T \to T + \Delta T$）。这导致气体 A 和 B 的长度$l_i \to l_i + \Delta l_i$和压强$p_i \to p_i + \Delta p_i(i = A, B)$发生变化。試計算長度变化 $\Delta l_A, \Delta l_B$ 和压强变化$\Delta p_A, \Delta p_B$的数值 。[4]

Solution:
(a) By assuming the volume of the mercury doesn't change, the volume filled by the gases doesn't change. We have

$$Sl_A + Sl_B = Sl'_A + Sl'_B$$
$$\Rightarrow l'_B = l_A + l_B - l'_A = 24\ \mathrm{cm}$$

(b) By assuming the gases are ideal and the temperature doesn't change, $p_i V_i$ is conserved.

$$p_i V_i = p'_i V'_i$$
$$\Rightarrow p_A l_A = p'_A l_A{}' \ \text{ and } \ p_B l_B = p'_B l_B{}'$$

In the case of the vertical tube (Fig. 2a), the pressure difference between two parts is

$$p_A + \rho_{Hg} g (l_B - l_A) = p_B$$
$$\Rightarrow p_B - p_A = \rho_{Hg} g (l_B - l_A)$$

Similarly, in the configuration (Fig.2b),

$$p'_B + \rho_{Hg} g (l'_B - l_A{}') = p'_A$$
$$\Rightarrow p'_A - p'_B = \rho_{Hg} g (l'_B - l'_A)$$

Now we have 4 equations and 4 unknowns, we can solve them uniquely.

$$p_B = p_A + \rho_{Hg} g (l_B - l_A)$$
$$\Rightarrow p_A = \rho_{Hg} g \left(l'_B - l'_A + \frac{l_B}{l_B{}'}(l_B - l_A) \right) \left(\frac{l_A}{l_A{}'} - \frac{l_B}{l_B{}'} \right)^{-1}$$

We get

$$p_A = 24\ \mathrm{kPa}$$
$$p_B = 32\ \mathrm{kPa}$$
$$p_A{}' = 48\ \mathrm{kPa}$$
$$p_B{}' = 24\ \mathrm{kPa}$$

(c) (Method 1: Keep 1st order term) By increasing the temperature, all variables are changing accordingly,

$$T \to T + \Delta T$$
$$p_i \to p_i + \Delta p_i$$
$$l_i \to l_i + \Delta l_i$$

The ideal gas law becomes,

1

Solution

$$(p_i + \Delta p_i)(l_i + \Delta l_i)S = n_i R(T + \Delta T)$$

$$\Rightarrow p_i l_i S + (p_i \Delta l_i + \Delta p_i l_i)S + \cancel{\Delta p_i \Delta l_i S} = n_i RT + n_i R\Delta T$$

By dividing $n_i R = \frac{p_i l_i S}{T}$, we get

$$\frac{\Delta l_i}{l_i} + \frac{\Delta p_i}{p_i} = \frac{\Delta T}{T} \quad (i = A, B)$$

Next, we consider the length variation,

$$(p_B + \Delta p_B) - (p_A + \Delta p_A) = \rho_{Hg} g\big((l_B + \Delta l_B) - (l_A + \Delta l_A)\big)$$

$$\Rightarrow (p_B - p_A) + (\Delta p_B - \Delta p_A) = \rho_{Hg} g(l_B - l_A) + \rho_{Hg} g(\Delta l_B - \Delta l_A)$$

$$\Rightarrow (\Delta p_B - \Delta p_A) = \rho_{Hg} g(\Delta l_B - \Delta l_A)$$

Finally, we assume the volume of the mercury doesn't change,

$$\Delta l_A + \Delta l_B = 0$$

We get 4 equations and we can solve for 4 unknowns Δp_i and Δl_i.

$$\Rightarrow \Delta p_i = \left(\frac{\Delta T}{T} - \frac{\Delta l_i}{l_i}\right) p_i$$

$$\Rightarrow \Delta l_A = -\Delta l_B$$

$$\left(\frac{\Delta T}{T} - \frac{\Delta l_B}{l_B}\right) p_B - \left(\frac{\Delta T}{T} + \frac{\Delta l_B}{l_A}\right) p_A = \rho_{Hg} g(\Delta l_B + \Delta l_B)$$

$$\Rightarrow \frac{\Delta T}{T}(p_B - p_A) = \Delta l_B \left(2\rho_{Hg} g + \frac{p_B}{l_B} + \frac{p_A}{l_A}\right)$$

$$\Rightarrow \Delta l_B = \frac{\Delta T(p_B - p_A)}{T\left(2\rho_{Hg} g + \frac{p_B}{l_B} + \frac{p_A}{l_A}\right)}$$

The numerical values is

$$\Delta l_B = 0.085 \text{ cm}$$

$$\Delta l_A = -\Delta l_B = -0.085 \text{ cm}$$

$$\Delta p_B = 2033 \text{ Pa}$$

$$\Delta p_A = 1808 \text{ Pa}$$

(Method 2: Exact. Keep cubic term) We can retain the 2nd order of the equation,

$$(p_i \Delta l_i + \Delta p_i l_i)S + \Delta p_i \Delta l_i S = n_i R\Delta T$$

$$\Rightarrow \frac{\Delta l_i}{l_i} + \frac{\Delta p_i}{p_i} + \frac{\Delta p_i}{p_i}\frac{\Delta l_i}{l_i} = \frac{\Delta T}{T}$$

Together with the equations,

$$\Delta l_A + \Delta l_B = 0$$

$$(\Delta p_B - \Delta p_A) = \rho_{Hg} g(\Delta l_B - \Delta l_A) = 2\rho_{Hg} g\Delta l_B \qquad (A)$$

$$\frac{\Delta p_A}{p_A}\left(1 - \frac{\Delta l_B}{l_A}\right) = \frac{\Delta T}{T} + \frac{\Delta l_B}{l_A}$$

$$\frac{\Delta p_B}{p_B}\left(1 + \frac{\Delta l_B}{l_B}\right) = \frac{\Delta T}{T} - \frac{\Delta l_B}{l_B}$$

Sub. Into Eqtn. (A),

Solution

$$\Rightarrow \left(\frac{\frac{\Delta T}{T}-\frac{\Delta l_B}{l_B}}{1+\frac{\Delta l_B}{l_B}}\right)p_B-\left(\frac{\frac{\Delta T}{T}+\frac{\Delta l_B}{l_A}}{1-\frac{\Delta l_B}{l_A}}\right)p_A=2\rho_{Hg}g\Delta l_B$$

Numerically, we get $\Delta l_B = 0.000814$ m. (The other two roots are larger than l_B and hence we neglected.

$$\Delta l_A = -\Delta l_B = -0.000814\, m$$

Accordingly, we get

$$\Delta p_A = 1813 \text{ Pa}$$
$$\Delta p_B = 2030 \text{ Pa}$$

(Method 3: Keep 2nd order term) When the temperature increases from T to $T_1 = T + \Delta T$, Eq. (1) becomes

$$\frac{l_B}{l_B^1}p_B-\frac{l_A}{l_A^1}p_A=\rho g h_1\frac{T}{T_1}.$$

Let $l_A^1 = l_A + x$. Then $l_B^1 = l_B - x$ and $h_1 = l_B^1 - l_A^1 = (l_A - l_B) - 2x = h - 2x$. The equation becomes

$$\frac{p_B l_B}{l_B-x}-\frac{p_A l_A}{l_A+x}=\rho g(h-2x)\frac{T}{T_1}.$$

Expanding and keeping only up to quadratic terms of x,

$$p_B l_B(l_A+x)-p_A l_A(l_B-x)=\rho g\frac{T}{T_1}(h-2x)(l_B-x)(l_A+x).$$

$$(p_B-p_A)l_A l_B+(p_A l_A+p_B l_B)x$$
$$=\rho g\frac{T}{T_1}[h l_A l_B+(h l_B-h l_A-2 l_A l_B)x+(2l_A-2l_B-h)x^2].$$

$$6\rho g h^3+18\rho g h^2 x=\rho g\frac{T}{T_1}[6h^3-11h^2x-3hx^2].$$

$$\left(\frac{T_1}{T}-1\right)2h^2+\left(6\frac{T_1}{T}+\frac{11}{3}\right)hx+x^2=0.$$

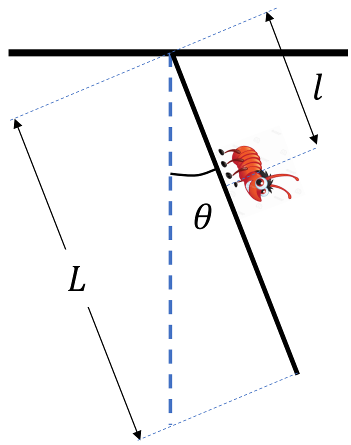
$$x=-0.0814 \text{ cm}.$$

which is the same as the exact solution.

Solution

1

2. Bug on a rod (11 points) 杆上的虫子（11 分）



A pendulum consists of a uniform rigid rod of length L, mass M, a bug of mass $M/3$ which can crawl along the rod. The rod is pivoted at one end and swings in a vertical plane. Initially the bug is at the pivot-end of the rod, which is at rest at an angle θ_i ($\theta_i \ll 1$ rad) from the vertical as shown in the figure, is released. For $t > 0$ the bug crawls slowly with constant speed V along the rod towards the bottom end of the rod.

单摆由长度L、质量M的均匀刚性杆组成， 一只质量为$M/3$的虫子沿着杆爬行。杆在一端枢转并在垂直平面中摆动。虫子最初位于杆的枢轴处 ，此时杆与垂直方向成一定夾角θ_i ($\theta_i \ll 1$ rad) (如图所示)。當$t > 0$， 虫子沿着杆朝着杆的底端以恒定速度V缓慢地爬行。

(a) What is the moment of inertia I of the rod and bug about the pivot when the bug has reached a distance l along the rod. [1]
(a) 当虫子沿着杆爬至距离 l 时，杆和虫子相對於枢轴的转动惯量 I 是什么? [1]

(b) Find the angular frequency ω of the swing of the pendulum when the bug has reached a distance l along the rod. Express your answer in terms of L and l. [1]
(b) 当虫子沿着杆爬至距离 l 时，找出单摆摆动的角频率ω。答案用L和l表达 。 [1]

From now on, you can assume the speed of the bug is so small that l hardly changes in a period of oscillation and can be taken to be constant, and the motion of the bug can be effectively described by simple harmonic motion, that is,
从现在开始，您可以假设虫子的速率非常小，因此在振荡周期内几乎不会发生变化，并可当为常数 ，而虫子的运动可以用简谐运动描述，即是

$$\theta(t) = \theta_0(l) \sin \omega t$$

where ω is the angular frequency you obtained in part (b) and $\theta_0(l)$ is the amplitude of the oscillation which will vary as the bug crawled.
其中 ω 是你在（b）部份中得到的角频率，$\theta_0(l)$ 是振荡的幅度，并随着虫子爬行而变化。

After the bug has reached a distance l along the rod, calculate the following quantities when it further crawls a short distance Δl. In parts (c) to (f), express your answers in terms of Δl, $\Delta\theta_0$ and other parameters in the problem, where $\Delta\theta_0$ is the change of the angular amplitude during the displacement Δl.
试计算在虫子沿着杆子爬至距离 l后，当虫子继续爬行短距离Δl时下列的物理量。在(c)至(f)部份中，答案以 Δl、$\Delta\theta_0$及本题中的其他参数表达，其中$\Delta\theta_0$是振荡幅度在位移Δl后的改变量。

(c) Calculate the time-averaged work done ΔW by the bug on the rod-bug system. [1]
(c) 试计算虫子作用于杆子-虫子系统的时间平均功ΔW. [1]

Solution

1

(d) Calculate the change $\Delta\omega^2$ in the term ω^2. [1]
(d) 试计算ω^2 的改变量$\Delta\omega^2$。[1]

(e) Calculate the time-averaged change ΔK in the kinetic energy of the whole system. [1]
(e) 试计算整个系统的时间平均动能改变量ΔK。[1]

(f) Calculate the time-averaged change ΔU in the potential energy of the whole system. [1]
(f) 试计算整个系统的时间平均势能改变量ΔU。[1]

(g) Summarizing the above steps, calculate the relation between $\Delta\theta_0$ and Δl. [3]
(g) 总结上述步骤，计算$\Delta\theta_0$和Δl之间的关系。

(h) Find the amplitude of the swing of the pendulum when the bug reaches the bottom end of the rod ($l = L$). Express your answer in terms of $\theta_0(l = 0) = \theta_i$. [2]
(h) 当虫子爬至杆的底端时($l = L$)，找出单摆摆动的幅度。答案用 $\theta_0(l = 0) = \theta_i$ 表达。[2]

Solution:
(a) When the bug has crawled at distance l, the moment of inertia of the rod and bug along the pivot is

$$I = \frac{1}{3}ML^2 + \frac{1}{3}Ml^2 = \frac{1}{3}M(L^2 + l^2)$$

(b) The equation of motion of the pendulum is

$$\frac{d}{dt}(I\dot{\theta}) = -Mg\frac{L}{2}\sin\theta - \frac{1}{3}Mgl\sin\theta$$
$$\Rightarrow \frac{1}{3}M(L^2 + l^2)\ddot{\theta} + \frac{2}{3}Ml\dot{l}\dot{\theta} = -Mg\sin\theta\left(\frac{L}{2} + \frac{l}{3}\right)$$

For small oscillations, it becomes:

$$\ddot{\theta} + \frac{2l\dot{l}\dot{\theta}}{L^2 + l^2} + \frac{g\left(l + \frac{3L}{2}\right)\theta}{L^2 + l^2} = 0$$

If the bug crawls so slowly that the change in l in a period of oscillation is negliglbe, i.e. $\dot{l} = v \ll l\omega$, we can ignore the second term and write,

$$\ddot{\theta} + \frac{g\left(l + \frac{3L}{2}\right)\theta}{L^2 + l^2} = 0$$

Hence the angular frequency of oscillation is

$$\omega = \sqrt{\frac{g(3L + 2l)}{2(L^2 + l^2)}}$$

(c) Consider the motion of the bug along the rod,

$$\frac{M}{3}(\ddot{l} - l\dot{\theta}^2) = \frac{Mg\cos\theta}{3} - f$$

when f is the force exerted on the bug by the rod. As the bug crawls with constant speed, $\ddot{l} = 0$. We get

$$f = \frac{Mg}{3}\cos\theta + \frac{Ml\dot{\theta}^2}{3}$$

Solution

The frictional force is pointing towards the pivot. This force is exerted by the bug on the rod-bug system to maintain its slow motion. Hence

$$\Delta W = -f\Delta l = -\frac{M}{3}\left(g\cos\theta + l\dot{\theta}^2\right)\Delta l \approx \left(-\frac{Mg}{3} + \frac{Mg}{6}\theta^2 - \frac{Ml}{3}\dot{\theta}^2\right)\Delta l.$$

Since the bug moves slowly along the rod, the motion can be approximated by simple harmonic motion. Averaging over time,

$$\langle\theta^2\rangle = \theta_0^2\langle\sin^2\omega t\rangle = \frac{1}{2}\theta_0^2,$$

$$\langle\dot{\theta}^2\rangle = \theta_0^2\omega^2\langle\cos^2\omega t\rangle = \frac{1}{2}\omega^2\theta_0^2.$$

Hence

$$\Delta W \approx \left(-\frac{Mg}{3} + \frac{Mg}{12}\theta_0^2 - \frac{Ml}{6}\omega^2\theta_0^2\right)\Delta l.$$

(d)

$$\frac{d\omega^2}{dl} = \frac{d}{dl}\left[\frac{g(3L+2l)}{2(L^2+l^2)}\right] = g\left[\frac{1}{L^2+l^2} - \frac{l(3L+2l)}{(L^2+l^2)^2}\right].$$

$$\Delta\omega^2 = g\left[\frac{1}{L^2+l^2} - \frac{l(3L+2l)}{(L^2+l^2)^2}\right]\Delta l.$$

(e) The kinetic energy is

$$K = \frac{1}{2}\frac{M}{3}(L^2+l^2)\dot{\theta}^2 + \frac{1}{2}\frac{M}{3}V^2.$$

Averaging over time,

$$K \approx \frac{M}{12}(L^2+l^2)\omega^2\theta_0^2 + \frac{M}{6}V^2.$$

Hence

$$\begin{aligned}\Delta K &\approx \frac{M}{6}(L^2+l^2)\omega^2\theta_0\Delta\theta_0 + \frac{M}{6}l\omega^2\theta_0^2\Delta l + \frac{M}{12}(L^2+l^2)\theta_0^2\Delta\omega^2 \\ &= \frac{M}{6}(L^2+l^2)\omega^2\theta_0\Delta\theta_0 + \frac{M}{6}l\omega^2\theta_0^2\Delta l + \frac{Mg}{12}\theta_0^2\left[1 - \frac{l(3L+2l)}{L^2+l^2}\right]\Delta l.\end{aligned}$$

Substituting the expression of ω^2,

$$\Delta K \approx \frac{Mg}{12}(3L+2l)\theta_0\Delta\theta_0 + \frac{Mg}{12}\theta_0^2\Delta l.$$

(f) The potential energy is

$$U = -\frac{MgL}{2}\cos\theta - \frac{Mgl}{3}\cos\theta \approx -\frac{Mg}{6}(3L+2l) + \frac{Mg}{12}(3L+2l)\theta^2.$$

Averaging over time,

$$U \approx -\frac{Mg}{6}(3L+2l) + \frac{Mg}{24}(3L+2l)\theta_0^2.$$

Hence

$$\Delta U \approx -\frac{Mg}{3}\Delta l + \frac{Mg}{12}(3L+2l)\theta_0\Delta\theta_0 + \frac{Mg}{12}\theta_0^2\Delta l.$$

(g) Using the work-energy theorem,

$$\Delta W = \Delta K + \Delta U.$$

$$\left(-\frac{Mg}{3} + \frac{Mg}{12}\theta_0^2 - \frac{Ml}{6}\omega^2\theta_0^2\right)\Delta l \approx \frac{Mg}{12}(3L+2l)\theta_0\Delta\theta_0 + \frac{Mg}{12}\theta_0^2\Delta l$$

$$-\frac{Mg}{3}\Delta l+\frac{Mg}{12}(3L+2l)\theta_0\Delta\theta_0+\frac{Mg}{12}\theta_0^2\Delta l.$$

Simplifying,

$$\frac{\Delta\theta_0}{\theta_0}+\frac{1}{2}\left(\frac{1}{3L+2l}+\frac{l}{L^2+l^2}\right)\Delta l\approx 0.$$

$$\Delta\theta_0\approx-\frac{\theta_0}{2}\left(\frac{1}{3L+2l}+\frac{l}{L^2+l^2}\right)\Delta l.$$

(h) Integrating,

$$\ln\theta_0+\frac{1}{4}\ln(3L+2l)+\frac{1}{4}\ln(L^2+l^2)\approx\text{constant}.$$

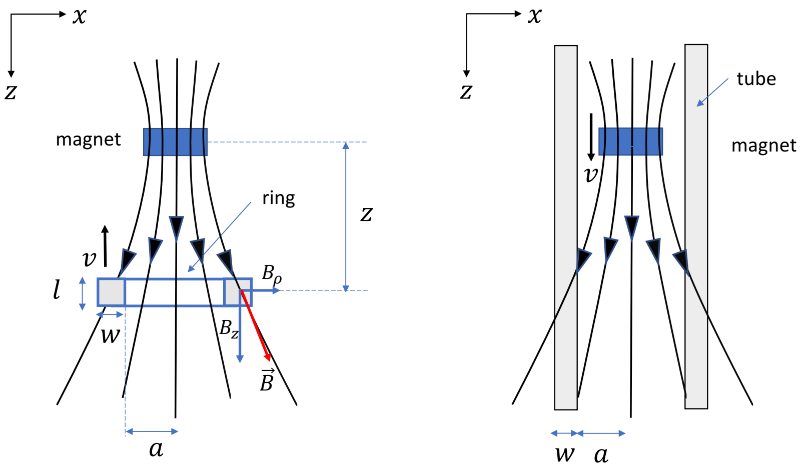
$$\theta_0(3L+2l)^{\frac{1}{4}}(L^2+l^2)^{1/4}\approx\text{constant}.$$

$$\theta_0(l)\approx\left(\frac{3}{10}\right)^{\frac{1}{4}}\theta_0(l=0)=0.74\,\theta_i.$$

3. Falling magnet inside a conductive pipe (10 points) 在导电管内下落的磁铁（10 分）

In this question, we consider the motion of a strong tiny magnet with mass M and magnetic dipole moment μ falling inside a vertical conducting non-magnetic tube.

在这个问题中，我们考虑具有质量M和磁偶极矩μ的强磁铁落入垂直的导电非磁性管内的运动。



(a) (b)

(a) We first consider a ring (with radius a, length l, thickness w ($w \ll a$) and conductivity σ) moving towards the magnet with speed v as shown in figure (a). The magnetic field at position $\vec{r}$ due to a magnetic dipole $\vec{\mu}=\mu\hat{z}$ (pointing downward as positive) at origin is given by

我们首先考虑一个环（半径a，长度l，厚度w（$w \ll a$）和导电率σ）以速度v向磁铁移动，如图（a）所示。由位於原點的磁偶极子$\vec{\mu}=\mu\hat{z}$（指向下为正）所產生的磁场，在位置$\vec{r}$处为

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi r^3}\left(\frac{3(\vec{\mu}\cdot\vec{r})\vec{r}}{r^2}-\vec{\mu}\right)$$

Solution

1

(i) Calculate the components of the magnetic field $B_\rho(\vec{r})$ and $B_z(\vec{r})$ at the point $\vec{r}$ with cylindrical coordinate $\vec{r} = (\rho = a, \phi = 0, z)$ [2]
(i) 计算磁场在位置$\vec{r}$处的分量$B_\rho(\vec{r})$和$B_z(\vec{r})$，$\vec{r}$的圆柱坐标为$\vec{r} = (\rho = a, \phi = 0, z)$。[2]

(ii) Calculate the induced emf and current on the ring. [3]
(ii) 计算环上的感生电动势和电流。 [3]

(iii) Calculate the magnetic force experienced by the coil. [1]
(iii) 计算环所感受的磁力。 [1]

(b) Next we consider a magnet falling inside along a vertical conducting non-magnetic tube of infinite length (with radius a, thickness w and conductivity σ) as shown in figure (b).
(b) 接下来我们考虑沿着一个无限长度（具有半径a，厚度w和导电率σ）的垂直导电非磁性管内落下的磁铁，如图（b）所示。

(i) When the magnet falls with the speed v, it experiences a damping force with the magnitude equal to γv. Calculate the damping constant γ. [3]
(i) 当磁铁以速度v下降时，它会感受一个量值为γv的阻尼力。试计算阻尼常数γ。 [3]

Hints: The following mathematical identity may be useful.
提示：下列数学公式可能有用。

$$\int_{-\infty}^{\infty} \frac{u^2 du}{(1+u^2)^5} = \frac{5\pi}{128}$$

(ii) Determine the terminal velocity of the magnet when it falls inside the tube. [1]
(ii) 試計算磁铁在管内落下时的终端速度。 [1]

Solution:
(ai) We can use the cylindrical coordinate where

$$\vec{r} = z\hat{z} + a\hat{\rho}$$

Hence

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{(z^2+a^2)^{3/2}}\left(\frac{3z\mu}{z^2+a^2}(z\hat{z}+a\hat{\rho}) - \mu\hat{z}\right)$$

$$\Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0\mu}{4\pi}\frac{1}{(z^2+a^2)^{5/2}}\left((2z^2-a^2)\hat{z} + 3az\hat{\rho}\right)$$

$$B_z = \frac{\mu_0\mu}{4\pi}\frac{(2z^2-a^2)}{(a^2+z^2)^{5/2}}$$

$$B_\rho = \frac{\mu_0\mu}{4\pi}\frac{3za}{(a^2+z^2)^{5/2}}$$

(aii) The induced emf is given by

$$\epsilon = \oint(\vec{v}\times\vec{B})\cdot d\vec{l} = 2\pi a v B_\rho = \frac{\mu_0\mu v}{2}\frac{3za^2}{(a^2+z^2)^{5/2}}$$

Alternatively, the flux through the ring is given by

Solution

$$\Phi = \int_0^a 2\pi\rho d\rho B_z(\rho,z) = \frac{\mu_0\mu}{4}\int_0^a 2\rho d\rho\left[\frac{3z^2}{(\rho^2+z^2)^{\frac{5}{2}}} - \frac{1}{(\rho^2+z^2)^{\frac{3}{2}}}\right]$$

$$= \frac{\mu_0\mu}{4}\left[-\frac{2z^2}{(\rho^2+z^2)^{\frac{3}{2}}} + \frac{2}{(\rho^2+z^2)^{\frac{1}{2}}}\right]_0^a = \frac{\mu_0\mu}{2}\frac{a^2}{(a^2+z^2)^{\frac{3}{2}}}.$$

The induced emf is given by

$$\epsilon = -\frac{d\Phi}{dt} = \frac{\mu_0\mu v}{2}\frac{3a^2z}{(a^2+z^2)^{\frac{5}{2}}}$$

and the resistance of the coil is

$$R = \frac{2\pi a}{\sigma wl}$$

Hence the induced current is

$$I = \frac{\epsilon}{R} = \frac{2\pi a v B_\rho}{2\pi a}\sigma wl = vB_\rho\sigma wl = \frac{\mu_0\mu v\sigma wl}{4\pi}\frac{3za}{(a^2+z^2)^{5/2}}$$

(aiii) The magnetic force experienced by the coil is

$$F = I(2\pi a)B_\rho = (2\pi alw)\sigma vB_\rho^2$$

(bi) The total magnetic force experienced by the magnet is

$$F = \int dF = (2\pi a^3 w\sigma v)\left(\frac{3\mu_0\mu}{4\pi}\right)^2\int_{-\infty}^{\infty}\frac{z^2dz}{(a^2+z^2)^5}$$

$$\Rightarrow F = \frac{9\mu_0^2\mu^2 w\sigma v}{8\pi a^4}\times\int_{\infty}^{\infty}\frac{u^2du}{(1+u^2)^5} = \frac{45}{1024}\left(\frac{\mu_0^2\mu^2 w\sigma}{a^4}\right)v = \gamma v$$

(bii) Newton's 2nd law gives

$$mg - \gamma\frac{dz}{dt} = m\frac{d^2z}{dt^2}$$

At the terminal velocity, $\frac{d^2z}{dt^2} = 0$ and

$$v_t = \frac{mg}{\gamma} = \frac{1024}{45}\left(\frac{a^4 g}{\mu_0\mu^2 w\sigma}\right)$$

4. Heat flux between two plates (10 points) 两块板之间的热通量（10 点）

A system composed of two parallel plates at distance L from each other, which are at temperature T_1 and T_2 respectively.

一个系统由两块互相分隔、距离為 L 的平行板组成，分别处于温度 T_1 和 T_2。

(a) Calculate the heat flux density P (i.e. rate of heat energy flow per unit area) between two plates if the space between the plates is vacuum and each of the plates has emissivity ϵ. [4]

(a) 如果两块板之间的空间是真空并且每个板具有比輻射率 ϵ，计算两块板之间的热通量密度 P（即每单位面积的热能流速率）。[4]

Solution

(b) Now the space between the plates is filled with a monoatomic gas of molar density n and molar mass M. You need to estimate the heat flux density P between two plates according to the following approximations:

- The gas density is so low that the mean free path $\lambda \gg L$.
- $T_1 \gg T_2$
- When gas molecules bounce from the plates, they obtain the temperature of the respective plates (for instance, if they are absorbed/bounded for a short time by the molecules of the plate, and then released back into the space between the plates).
- You may neglect the black body radiation.
- "Estimate" means that the numeric prefactor of your expression does not need to be accurate.

(b) 现在两块板之间的空间充满了摩尔密度 n 和摩尔质量 M 的单原子气体。您需要跟据以下近似估算两块板之间的热通量密度 P：

- 气体密度低至平均自由程 $\lambda \gg L$.
- $T_1 \gg T_2$
- 当气体分子从板上反弹时，它们会获得相应板的温度（例如，如果它们被板的分子吸收/束缚很短的时间，然后释放回板之间的空间）。
- 您可忽略黑体辐射。
- “估算”表示答案的数字前因子不需要准确。

(i) Consider that there is an atom colliding with the hot plate and remains in thermal equilibrium with the hot plate when it is reflected by the plate. Calculate the average velocity square $\langle v_1^2 \rangle$, and estimate the average horizontal velocity $\langle v_{1x} \rangle$ of the atom. [0.5]
(i) 考虑一粒原子与热板碰撞并且当它被板反射时与热板保持热平衡。计算平均速度平方$\langle v_1^2 \rangle$，并估算原子的平均水平速度$\langle v_{1x} \rangle$。 [0.5]

(ii) Consider that there is an atom colliding with the cold plate and remains in thermal equilibrium with the cold plate when it is reflected by the plate. Calculate the average velocity square $\langle v_2^2 \rangle$ and estimate the average horizontal velocity $\langle v_{2x} \rangle$ of the atom. [0.5]
(ii) 考虑一粒原子与冷板碰撞并且当它被板反射时与冷板保持热平衡。计算平均速度平方$\langle v_2^2 \rangle$，并估算原子的平均水平速度$\langle v_{2x} \rangle$。 [0.5]

(iii) Find the average energy transmitted by an atom when it moves from the hot to the cold plate. [1]
(iii) 找出一个原子从热板移动到冷板时传输的平均能量。 [1]

(iv) Estimate the heat flux density P between two plates. [4]
(iv) 估算两块板之间的热通量密度 P。 [4]

Solution:
(a) Without considering reflection of heat, it is reasonable to estimate the heat flux density

$$P \approx \sigma\epsilon(T_1^4 - T_2^4)$$

(N.B. You will receive 1 points for this answer)

To get the exact result, we consider the recursive reflection of heat flux between two walls.

1

Solution

Let $Q_2 = A\sigma\epsilon T_2^4$ be the initial inward flux and $Q_1 = A\sigma\epsilon T_1^4$ be the initial outward flux without any reflection of heat.

Surface 1 emits	Q_1
Surface 2 absorbs	ϵQ_1
Surface 2 reflects	$(1-\epsilon)Q_1$
Surface 1 absorbs	$(1-\epsilon)\epsilon Q_1$
Surface 1 reflects	$(1-\epsilon)^2 Q_1$
Surface 2 absorbs	$(1-\epsilon)^2\epsilon Q_1$
Surface 2 reflects	$(1-\epsilon)^3 Q_1$
Surface 1 absorbs	$(1-\epsilon)^3\epsilon Q_1$

The same is true for the heat emitted from surface 2.

Hence the total heat flux radiated from surface 1 and reabsorbed by surface 1 is

$$\tilde{Q}_1 = (1-\epsilon)\epsilon Q_1 + (1-\epsilon)^3\epsilon Q_1 + (1-\epsilon)^5\epsilon Q_1 + \cdots$$
$$= (1-\epsilon)\epsilon Q_1\left(1 + (1-\epsilon)^2 + (1-\epsilon)^4 + \cdots\right) = \frac{(1-\epsilon)\epsilon Q_1}{1-(1-\epsilon)^2}$$

Similarly, the total heat flux radiated from surface 2 and reabsorbed by surface 2 is

$$\tilde{Q}_2 = \frac{(1-\epsilon)\epsilon Q_2}{1-(1-\epsilon)^2}$$

And hence the total heat flux radiated from surface 2 and absorbed by surface 1 is

$$Q_2 - \tilde{Q}_2 = \frac{1-(1-\epsilon)^2-(1-\epsilon)\epsilon}{1-(1-\epsilon)^2}Q_2 = \frac{\epsilon}{1-(1-\epsilon)^2}Q_2$$

The net heat flux from surface 1 to surface 2 is

$$P = Q_1 - \frac{(1-\epsilon)\epsilon Q_1}{1-(1-\epsilon)^2} - \frac{\epsilon}{1-(1-\epsilon)^2}Q_2 = \frac{\sigma\epsilon(T_1^4 - T_2^4)}{2-\epsilon}$$

(N.B. When $\epsilon = 1$, we have ideal blackbody. The heat will be completely absorbed by the plates and two results are identical.)

(b)
(i) From the kinetic theory of ideal gas, we know

$$\frac{1}{2}m\overline{v_1^2} = \frac{3}{2}kT_1 \Rightarrow \overline{v_1^2} = \frac{3kT_1}{m}$$

As an approximation,

Solution

$$v_{1x} \approx \sqrt{\overline{v_{1x}^2}} = \sqrt{\frac{1}{3}\overline{v_1^2}} = \sqrt{\frac{kT_1}{m}}$$

Remark: To be precise, we can calculate $\bar{v}_{1x}$ using the Maxwell distribution. Since the particle can only move in one direction, the Maxwell distribution becomes $g(v_{1x})dv_{1x} = 2\sqrt{\frac{m}{2\pi kT_1}}\exp\left(-\frac{mv_{1x}^2}{2kT_1}\right)dv_{1x}$ and the mean value of v_{1x} is

$$\overline{v_{1x}} = \int_0^{\infty} 2\sqrt{\frac{m}{2\pi kT_1}}\exp\left(-\frac{mv_{1x}^2}{2kT_1}\right)v_{1x}dv_{1x} = \sqrt{\frac{2}{\pi}} \times \sqrt{\frac{kT_1}{m}}$$

which is different by a factor of order 1.

(ii) Similarly we have

$$\overline{v_2^2} = \frac{3kT_2}{m}$$

$$v_{2x} \approx \sqrt{\overline{v_{2x}^2}} = \sqrt{\frac{1}{3}\overline{v_2^2}} = \sqrt{\frac{kT_2}{m}}$$

(iii) Since $\lambda \gg L$, the probability of collision of two molecules is ignorably small, so we can imagine the molecules as independent ones bouncing back and forth between the plates. We are now considering a two-way journey of a molecule. Let the velocity of a molecule when leaving the hot plate be v_1, while that when leave the cold plate v_2; the component perpendicular to the plates is v_{1x} and v_{2x} respectively.

The net transmitted energy is $\Delta E = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = \frac{3k}{2}(T_1 - T_2) \approx \frac{3k}{2}T_1$

(iv) The time required to cover the normal distance L back and forth is

$$\Delta t = \frac{L}{v_{1x}} + \frac{L}{v_{2x}}.$$

Since $T_1 \gg T_2$, the velocities satisfy $v_1 \gg v_2$ and

$$\Delta E \approx \frac{1}{2}mv_1^2 \text{ and } \Delta t \approx \frac{L}{v_{2x}}$$

The power transmitted (per atom) during this journey is

$$P = \frac{\Delta E}{\Delta t} = \frac{m}{2L}v_1^2 v_{2x}$$

And the heat flux density due to all particles is

$$P_{tot} = \frac{m}{2L}v_1^2 v_{2x} \times (nN_A AL)$$

where A is the surface area of the plate. The heat flux density is

$$\Rightarrow P \approx \frac{nN_A}{2\sqrt{m}}k^{\frac{3}{2}}T_1 T_2^{1/2}$$

Since $N_A m = M$ and $kN_A = R$, we have

Solution

$$P = \frac{n}{2}\frac{R^{\frac{3}{2}}}{\sqrt{M}}T_1\sqrt{T_2}$$

N.B. Any answer of the form

$$P = C\frac{nR^{\frac{3}{2}}}{\sqrt{M}}T_1\sqrt{T_2}$$

for some dimensionless number C (of order one) will receive full credits.

2

Pan Pearl River Delta Physics Olympiad 2019
2019 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助

Simplified Chinese Part-2 (Total 2 Problems, 60 Points)
简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 15 February, 2019)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。
All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。
There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。
Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。
Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。
Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。
If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。
At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Gravitational Waves (26 points) 引力波 (26 分)

Gravitational waves (GW) are the "ripples of space" predicted by Einstein in 1916. GW are transverse waves travelling at the speed of light. They are sourced by the change of mass distribution in space. In 2015, GW were detected directly by the Laser Interferometer Gravitational-Wave Observatory (LIGO). The detection is not only a verification of Einstein's prediction after 100 years, but also provides a completely new probe of our universe and opens a new era of GW astronomy.

1916 年，爱因斯坦预言了空间的“涟漪”——引力波。引力波是横波，以光速传播，其来源是空间中物质质量分布的变化。2015 年，激光干涉引力波天文台(LIGO)发现引力波。引力波的发现不仅验证了爱因斯坦的百年预言，更是一种探测宇宙的全新手段。引力波的发现开启了引力波天文学的新时代。

In this problem, we will work in Newtonian mechanics and Newtonian gravity (instead of general relativity), and ignore the expansion of the universe, unless stated otherwise.

在本题中，除了有特殊说明之处外，我们将使用牛顿力学和牛顿引力（而不是广义相对论），且忽略宇宙膨胀。

You may find the following quantities useful here
本题中你可能用到以下数值：

Gravitational constant 牛顿引力常数	$G_N = 6.67\times10^{-11}\mathrm{m^3kg^{-1}s^{-2}}$
Speed of light 光速	$c = 3.00\times10^8\mathrm{m/s}$
Mass of the sun 太阳质量	$M_\odot = 1.99\times10^{30}\mathrm{kg}$
Mass of the earth 地球质量	$M_E = 5.97\times10^{24}\mathrm{kg}$
Radius of the earth 地球半径	$r_E = 6.37\times10^6\mathrm{m}$

Part A. Indirect Evidence of GW (9 points) 引力波的间接证据（9 分）

Before the direct discovery of GW, indirect evidence of GW has been found in a binary pulsar system in the 1970s. The binary consists of two pulsars rotating around each other in circular orbit with radius R. Let us assume each pulsar has mass M and radius r.

早在 20 世纪 70 年代，天文学家已经在双脉冲星系统中发现了引力波存在的间接证据。双脉冲星系统中，两颗脉冲星在半径为 R 的圆轨道上互相绕转。每颗脉冲星的质量为 M，半径为 r。

2

A1	Calculate the period of the binary system T_B. 求双星系统的运动周期 T_B。	**2 points** **2 分**

A2	Calculate the frequency of emitted gravitational wave f_{GW}. 求双星系统发射引力波的频率 f_{GW}。	**1 point** **1 分**

A3	The binary system emits GW with power $P = \frac{2}{5c^5} \times G_N^{\alpha} M^{\beta} R^{\gamma}$, where c is the speed of light. Calculate the values of α, β, γ. 双星系统辐射引力波的功率为 $P = \frac{2}{5c^5} \times G_N^{\alpha} M^{\beta} R^{\gamma}$，其中 c 是光速。求 α，β 和 γ 的值。	**2 points** **2 分**

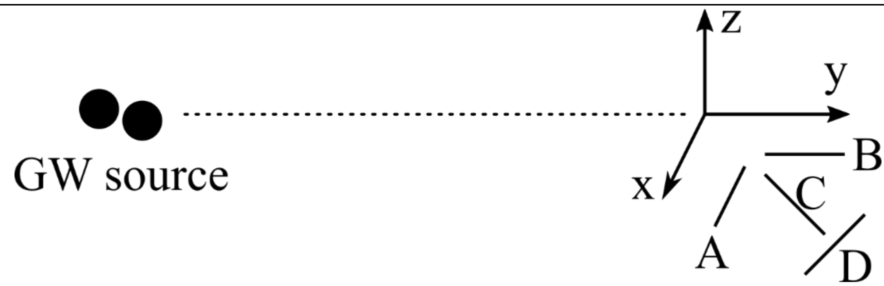
A4	After time T_C, the two pulsars collide due to GW emission. Calculate T_C. 因为引力波辐射，在 T_C 时间后，两颗脉冲星碰撞。求 T_C。	**3 points** **3 分**

A5	Apart from GW emission, in general relativity, there is also a correction to the gravitational potential energy $\Delta V_{GR} \propto -1/R^2$ (note the negative sign) for the pulsar system. For the same values of M, R, will this correction ΔV_{GR} increase or decrease T_B (no quantitative analysis required)? 除引力辐射外，广义相对论中，双星系统的引力势能有一个额外的修正项：$\Delta V_{GR} \propto -1/R^2$ (注意负号)。对于同样的 M 和 R，ΔV_{GR} 会增加还是减少双星系统的周期 T_B？(定性分析即可，不需要定量计算。)	**1 point** **1 分**

Part B. Direct Detection of GW (7 points) 引力波的发现 (7 分)

In 2016, GW were detected directly from distant merging black holes by the LIGO experiment.
2016 年，LIGO 实验组在遥远黑洞的并合事件中直接发现了引力波。

B1	Assume that we know the GW source is in the minus y direction, as shown in the below figure. 假如我们知道引力波源的方向是负 y 方向，如下图所示。	**1 point** **1 分**



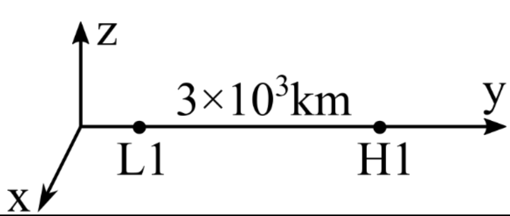
Now we measure the GW signal by measuring the change of distance of two free particles. Which of the following orientation of the particles can detect the biggest signal? Choose one from A-D below.
A. Along the x-axis;
B. Along the y-axis;
C. Along 45 degree in the x-y plane;
D. Along -45 degree in the x-y plane

我们通过测量两个自由粒子之间距离的变化来探测引力波。当粒子沿以下哪种方向放置时，测得的信号最大？从 A-D 中选择一个。
A. 沿 x 轴；
B. 沿 y 轴；
C. 沿 x-y 平面中 45 度角方向；
D. 沿 x-y 平面中-45 度角方向

B2

The LIGO experiment has two GW detectors L1 and H1. L1 and H1 are separated by 3×10^3km. The GW signal first arrived at L1, and arrived at H1 after 0.007s. Calculate the angle between the GW source and the L1-H1 line. Show the possible direction of the GW source in 3-dimensional space. (Draw the direction on the figure in the answer sheet.)

LIGO 实验有两个探测器 L1 和 H1。 L1 和 H1 相距 3×10^3km。引力波信号首先到达 L1，而后到达 H1。两者时间差为 0.007s。求引力波源与 L1-H1 连线的夹角，并画出引力波源在三维空间中的可能方位 (在答题纸上的图上画出方位)。



2 points
2 分

B3	For the 2016 LIGO event, the binary black hole system is about 10^{22}km away from the earth. The two initial black holes are $36M_{\odot}$ and $29M_{\odot}$. The final black hole is $62M_{\odot}$. The missing mass has all emitted away as GW energy. At 1000 km away from the center of the black hole merger event, the amplitude of GW reaches $A_0 \simeq 0.01$. The energy density of GW is proportional to its amplitude squared. What's the GW energy E_R that passes through the earth? 在 2016 年 LIGO 发现的引力波事件中，双黑洞系统距地球约10^{22}km。两个黑洞的初始质量为 $36M_{\odot}$ 和 $29M_{\odot}$，合并后的黑洞质量为 $62M_{\odot}$。系统损失的质量全部以引力波能量的形式辐射出去。在距离黑洞合并事件中心 1000 km 处，引力波的振幅为 $A_0 \simeq 0.01$。引力波能量密度正比于其振幅平方。求穿过地球的引力波能量 E_R。	**2 points** **2 分**

B4	For the same conditions as given in B3, what's the amplitude A of the GW when they pass by the earth? 在与 B3 相同的条件下，求引力波到达地球时的振幅 A。	**1 point** **1 分**

B5	GW can be used to study astronomy and cosmology. Consider the expansion of the universe as an example. In the Newtonian gravity concepts, the expansion of the universe can be considered as objects running away (recession) from us. GW astronomy can be a way of studying the relation between distance and the receding velocity, and thus studying the expansion history of the universe. 我们可以用引力波来研究天文学和宇宙学。例如，研究宇宙膨胀。在牛顿引力中，宇宙膨胀可以看成是天体远离我们而去 (退行)。通过引力波天文学，可以研究天体距离与退行速度之间的关系，进而研究宇宙的膨胀历史。 For GW events with optical counterpart (for example, neutron star mergers), the receding velocity is measured by the Doppler effect. The Doppler effect of which of the following can be used to directly measure the receding velocity? Choose one from A-D below: A. Synchrotron radiation from the neutron star B. Emission or absorption spectrum of elements C. Charged particles emitted from the neutron star system D. GW emitted from the neutron star system 对于存在光学对应体的引力波事件(例如双中子星并合)，可以通过多普勒效应测量天体的退行速度。以下哪个物理过程的多普勒效应可以直接用来测量退行速度？从 A-D 中选择一个。	**1 point** **1 分**

	A. 来自中子星的同步辐射 B. 元素的发射光谱或吸收光谱 C. 中子星系统辐射出的带电粒子 D. 中子星系统辐射出的引力波	

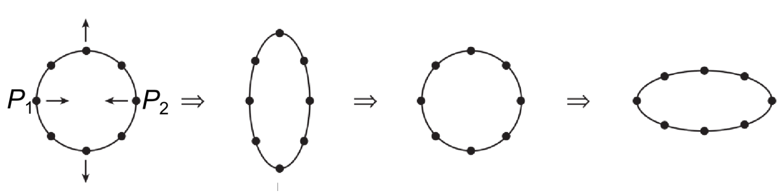
Part C. Interaction between GW and Matter (10 points) 引力波和物质的相互作用 (10 分)

The "ripples of space" is too rough for understanding the effect of GW on matter. More concretely, one can use Newtonian physics to understand GW when its amplitude A is small (the calculation can be reproduced in general relativity in a local Lorentz frame). In Newtonian physics, the spatial length is not fluctuating. Rather, the effect of GW on matter can be considered as a periodic force proportional to $\sin(\omega_{GW}t)$ acting on matter when GW (assuming GW is plane wave with constant amplitude) passes by.

要进一步理解引力波物理，“时空的涟漪”这种说法过于粗略。更具体地，当引力波的振幅 A 很小时，我们可以在牛顿力学的框架下计算引力波的效应 (在广义相对论中，通过取局域洛伦兹系，可以验证牛顿力学的计算)。在牛顿物理中，空间距离不会有涨落变化。引力波与物质的作用体现为，引力波给物质一个正比于 $\sin(\omega_{GW}t)$ 的作用力。这里假设引力波为平面波，振幅为常数。

The amplitude A (assuming $A \ll 1$) of GW has the following effect on matter: if two free test mass particles (each has mass m) are separated by r without GW. With GW passing by its perpendicular direction (throughout Part C, we assume the propagation direction of GW is perpendicular to the line of the two particles), their distance changes from $r(1-A)$ to $r(1+A)$ periodically. The oscillation of test particle is of the pattern below (we draw many test particles to show the effect of GW more clearly, but in the problem let us just consider the two particles P_1 and P_2).

振幅为 A (假设 $A \ll 1$) 的引力波会为物质带来如下效应：考虑两个自由的检验粒子，每个粒子的质量都为 m。当没有引力波通过时，两个粒子之间的距离为 r。当有引力波沿垂直于粒子连线的方向通过时 (整个 C 部分中，均考虑引力波传播方向与两粒子连线方向垂直)，两个粒子之间的距离在 $r(1-A)$ 与 $r(1+A)$ 之间周期性变化。检验粒子的振荡如下图所示。 (下图中画了多个检验粒子的运动，以更清楚地显示引力波的作用。在本题中，我们只考虑两个粒子P_1和P_2。)



C1	Now, connect the two particles with a spring with spring constant k and unstretched length r (i.e. the spring does not change the initial distance between particles without GW). Assume that the spring is light and the force that GW acts on the spring is negligible. For $k \gg m\,\omega_{GW}^2$, calculate the oscillation amplitude A' between these two particles when GW with amplitude A pass by, such that the distance between the two particles change between $r(1-A')$ and $r(1+A')$. Note: Here we assume the two particles have minimal kinetic energy as allowed in the above setup. (Otherwise additional kinetic energy can cause oscillations with larger amplitudes for the spring.) 现在，将两个粒子用弹簧连接起来。弹簧的弹性系数为 k，原长为 r（也就是说，没有引力波通过时，弹簧不改变粒子的初始距离）。假设弹簧很轻，引力波作用在弹簧上的力可以忽略。在 $k \gg m\,\omega_{GW}^2$ 的情况下，计算振幅为 A 的引力波通过时，粒子的振幅 A'。这里 A'的定义为，引力波通过时，两粒子距离在 $r(1-A')$ 和 $r(1+A')$ 之间变化。 注：假设两粒子的动能是可以满足题设条件的最小动能。（否则，额外的动能可以导致额外的振动，以及更大的振幅。）	**3 points** **3 分**

C2	For the same setup as Part C1, but $k \ll m\,\omega_{GW}^2$, calculate A'. Keep the linear terms in k, and the higher order terms in k can be neglected (i.e. if the exact result is $A+Bk+Ck^2+\cdots$, we require that you get $A+Bk$ and you can neglect higher terms Ck^2 and so on). 设 $k \ll m\,\omega_{GW}^2$，其他条件与 C1 中相同，计算 A'。这里，只需保留至 k 的线性阶，k 的高阶项可以忽略。（也就是说，如果精确结果是 $A+Bk+Ck^2+\cdots$，则你需要写出 $A+Bk$。你可以忽略 Ck^2 等高阶项。）	**3 points** **3 分**

C3	For the same setup as Part C1, but $k = m\,\omega_{GW}^2/2$, qualitatively describe how the distance between two particles changes with time. No explicit calculation is needed. 设 $k = m\,\omega_{GW}^2/2$，其他条件与 C1 中相同，定性描述两粒子间的距离将如何随时间变化。无需定量计算。	**1 point** **1 分**

C4	We'd like to estimate when GW pass through the earth, how much GW energy the earth can absorb. The earth is a system that the pressure of matter balances self-gravity. The real earth is too complicated but let's consider a toy model of the earth, as two particles at rest separated by $r = 6000$ km, each particle has mass $m = 3\times10^{24}$ kg. The self-gravity between these two particles are balanced by force provided by a light spring connecting these particles. And the unstretched length of the spring is 7000 km if no force acts on it. For the GW signal described by Part B3 and B4, with frequency $f =$	**2 points** **2 分**

	100Hz, estimate the order-of-magnitude of energy absorbed by the earth from one period of GW oscillation. 我们将估计当引力波穿过地球时，地球可以吸收多少能量。地球是一个自身压强与引力平衡的系统。真实的地球非常复杂。这里我们考虑一个地球的玩具模型：考虑两个静止粒子相距 $r = 6000\ \mathrm{km}$。每个粒子的质量为 $m = 3 \times 10^{24}\ \mathrm{kg}$。两个粒子之间由一根轻弹簧连接。弹簧的弹力与两粒子之间的引力平衡。假如不受力，弹簧的原长为$7000\ \mathrm{km}$。对于 B3、B4 中描述的，频率为 $f = 100\ \mathrm{Hz}$的引力波信号，求在引力波的一个振荡周期中，地球吸收的能量 (估计数量级即可)。	

C5	Black holes are so dense objects that even objects travelling at the speed of light (such as GW) cannot escape. For a black hole with mass the same as that of the earth (and at the same location as the earth), calculate the amount of GW energy that the black hole absorbs for the event described in Part B3 and B4. 黑洞是一种极端致密的天体：即使以光速运动的物体 (如引力波) 也不能逃出黑洞。考虑一个黑洞，与地球有同样的质量，并处于与地球同样的位置。对于 B3、B4 中描述的引力波信号，求此黑洞吸收的引力波能量。	**1 point** **1 分**

END of Problem 1
问题 1 完

Problem 2 Synchronization (34 marks) 同步现象（34 分）

Synchronization is a very common physical phenomenon. As early as in the 17th century, the famous Dutch scientist Christiaan Huygens observed that when two pendulum clocks are suspended from a common beam, they tend to oscillate in synchrony. In part A of this problem, we will consider a model of this phenomenon. In part B of this problem, we will consider a modern example of synchronization. Students can work on either part first before working on the other part.
同步是一种非常常见的物理现象。早在 17 世纪，著名的荷兰科学家克里斯蒂安•惠更斯就观察到，当两个单摆时钟悬挂在同一根梁上时，它们往往会同步振荡。在这个问题的 A 部份中，我们将考虑这种现象的一个模型。在这个问题的 B 部份中，我们将考虑一个现代的同步示例。同学可先完成任一部分，再完成另一部份。

A. The Pendulums (23 marks) 單擺 (23 分)

A single pendulum consists of a bob with mass m suspended vertically from a fixed point with a massless string of length L, subject to gravitational acceleration g. Let $q(t)$ be the angular displacement of the pendulum from the vertical at time t. When the bob moves, it encounters a constant frictional force of magnitude mLb in the opposite direction of motion.
单摆由一个质量m的小物块组成，物块从一固定点用绳子垂直悬挂，并受到重力加速度g的影响。绳子没有质量，长度为L。 在时间t，设单摆从垂直方向的角位移为$q(t)$。当物块移动时，有一数值为 mLb 的恒定摩擦力作用于它运动的反方向上。

A1	Write the dynamical equation of $q(t)$ for small oscillations. 写出小振动時$q(t)$的动力学方程式。	**2 points** **2 分**

Remark: To keep your equation simple, you may introduce the angular frequency
备注：为了使方程式更简洁，您可引入角频率

$$\omega = \sqrt{\frac{g}{L}},$$

and use the sign function defined by
并使用下面所定义的正負函数

$$\mathrm{sign} f = \begin{cases} 1 \text{ for } f > 0, \\ 0 \text{ for } f = 0, \\ -1 \text{ for } f < 0. \end{cases}$$

To compensate the loss of kinetic energy due to the friction in each cycle, the pendulum receives a kick every cycle. To simplify the calculations, we assume that the kick takes place when $q = -b/\omega^2$ and its angular velocity is positive.
为了补偿于每个周期中由摩擦力所引致的动能损失，单摆每个周期都会受到一次踢动。为了简化计算，我们假设踢动发生在$q = -b/\omega^2$并且当它的角速度为正时。

A2	Suppose that the angular velocity of the pendulum is u_n immediately after the n^{th} kick. Calculate $q(t)$ and $\dot{q}(t)$ in the cycle after the n^{th} kick. For convenience, we choose $t = 0$ at the n^{th} kick in this part and below. 假设单摆的角速度在第n次踢动后的瞬间为u_n。在第n次踢动后的周期中，计算$q(t)$和$\dot{q}(t)$。为方便起见，在此部份及以下部份中，我们设在第n次踢动的时间为$t = 0$。 For clarity, give your answer in three parts: 为清楚起见，请分三部份给出答案： (a) The first quarter of the cycle, (a) 第一个四分之一的周期， (b) the second and third quarters of the cycle, (b) 第二和第三个四分之一的周期， (c) the fourth quarter of the cycle. (c) 第四个四分之一的周期。	**2+2+2 points** **2+2+2 分**

A3	Suppose at each kick, a fixed amount of kinetic energy of the magnitude $mL^2h^2/2$ is injected to the pendulum, where h has the dimension of an angular velocity. Calculate the relation between u_{n+1} and u_n. 假设在每次踢动时，有数值为 $mL^2h^2/2$ 的动能被注入到单摆中，其中h具有角速度的量纲。计算u_{n+1}和之u_n间的关系。	**2 points** **2 分**

A4	What is the value of u_n after many kicks? 经过多次踢动后，u_n的数值是什么？	**2 points** **2 分**

A5	Suppose that at time t_0 during the first quarter of the cycle after the n^{th} kick, the pendulum receives an angular impulse equal to $mL^2\alpha$. Calculate the time at which: 假设在第n次踢动后的第一个四分之一周期内，單摆接受了数值为$mL^2\alpha$的角冲量。计算以下情况的时间： (a)the friction changes sign the first time, 摩擦力第一次改变方向时， (b) the friction changes sign the second time, 摩擦力第二次改变方向时， (c) the pendulum receives the $(n+1)^{\text{th}}$ kick. 單摆受到第$n+1$次踢動时。 Give your answer to the first order in α. 答案的表达式展开至α的第一阶。	**3 points** **3 分**

A6	Suppose that the time t_0 at which the pendulum receives an angular impulse equal to $mL^2\alpha$ is in the fourth quarter of the cycle after the n^{th} kick instead of the first quarter. Calculate the time at which the pendulum receives the $(n+1)^{\text{th}}$ kick. Give your answer to the first order in α. 假设单摆接受角冲量 $mL^2\alpha$ 的时间t_0是在第n次踢动后的第四个四分之一而不是第一个四分之一的周期内。计算单摆接受第$n+1$次踢動的时间。答案的表达式展开至α的第一阶。	**2 points 2 分**

Now consider two pendulum clocks. Let $q_1(t)$ and $q_2(t)$ be the angular displacements of the two clocks. The bob mass m, length L, friction parameter b and kick size h of the two pendulums are identical. Suppose that when $q_2 = -b/\omega^2$, pendulum 2 sends a small angular impulse equal to $mL^2\alpha$ on pendulum 1, and when $q_1 = -b/\omega^2$, pendulum 1 sends a small angular impulse equal to $mL^2\alpha$ on pendulum 2. (Here, $\alpha > 0$.)
现在考虑两个单摆。设两个单摆的角位移分别為$q_1(t)$和$q_2(t)$。两个单摆的小物塊质量m、长度L、摩擦参数b和踢動的大小h均是相同。假设当$q_2 = -b/\omega^2$时，单摆 2 发出一个数值為$mL^2\alpha$的小角冲量給单摆 1。当$q_1 = -b/\omega^2$时，单摆 1 发出一个数值為$mL^2\alpha$的小角冲量給单摆 2 。（这里 $\alpha > 0$。）

A7	Suppose the phase lag of pendulum 2 relative to pendulum 1 is ϕ_n at the beginning of n^{th} cycle of pendulum 1, and $0 \le \phi_n < \pi/2$. Calculate the relation between ϕ_{n+1} and ϕ_n. 假设在单摆 1 第n个周期的开始时，单摆 2 相对于单摆 1 的相位滞后為ϕ_n，其中$0 \le \phi_n < \pi/2$。 试计算ϕ_{n+1}和ϕ_n之间的关系。	**4 points 4 分**

A8	When ϕ_n is very small, calculate the number of cycles for ϕ_n to reduce by a factor of 10. 当ϕ_n非常少時，试计算ϕ_n減少 10 倍所需的周期数。	**2 points 2 分**

B. The Power Grid (11 marks) 电网 (11 分)

Synchronization is an important concept in the transmission of electricity in the power grid. The power grid is a network of nodes and links. Each node is an electric generator or other power consumption devices. The links are the transmission cables. Electric power is transmitted in the alternating current (AC) mode at 50 Hz or 60 Hz at a fixed voltage. However, the AC voltage of each node in the network has a slightly different phase.

同步是电网输电中的一个重要概念。电网是一个由节点和链路组成的网络。每个节点是一部发电机或其他功耗设备。链路是传输电缆。电力运用交流电（AC）模式以 50 Hz 或 60 Hz 频率经固定电压传输。但是，网络中每个节点的交流电压具有略微不同的相位。

B1	Consider a transmission cable connecting nodes 1 and 2. The inductance of the cable is L. The electric potentials of nodes 1 and 2 are $V_j(t) = V\cos(\omega t + \theta_j)$ for $j = 1, 2$. Calculate the time-averaged power transmitted from node 1 to 2. You may neglect the time dependence of θ_j. 考虑连接节点 1 和 2 的传输电缆。电缆的电感是L。节点 1 和 2 的电势为 $V_j(t) = V\cos(\omega t + \theta_j)$, 其中$j = 1, 2$。计算从节点 1 传输到节点 2 的时间平均功率。你可以忽略θ_j的时间依赖性。	**3 points** **3 分**

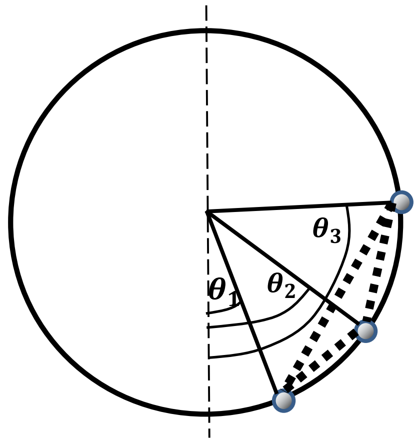
A network of electric generators and motors, labeled $j = 1, 2, \ldots, N$, are connected with each other. Their electric potentials are $V_j(t) = V\cos(\omega t + \theta_j)$ for $j = 1, 2, \ldots, N$, and the inductances of the connecting cables are L. The generator or motor at node j rotates with the phase angle $\omega t + \theta_j$ and its moment of inertia is I. The external power source or drain is P_j ($P_j > 0$ if j is a generator, and $P_j < 0$ if j is a motor). At the same time, the power dissipation due to friction is given by $\kappa(\omega + \dot{\theta}_j)^2/2$ at node j.

一个网络，由发电机和电动机彼此完全连接而成，发电机和电动机的标记為$j = 1, 2, \ldots, N$。它们的电势為$V_j(t) = V\cos(\omega t + \theta_j)$，其中$j = 1, 2, \ldots, N$。用于连接它们各点之间的电缆，其电感为$L$。节点处的发电机或电动机以相角$\omega t + \theta_j$旋转，其转动惯量为$I$。外部电能的供应或消耗功率為$P_j$（如果$j$是发电机，$P_j > 0$ 。如果j是电动机，$P_j < 0$）。同时，在节点j由摩擦引起的损耗功率是$\kappa\left(\omega + \dot{\theta}_j\right)^2/2$。

B2	Derive the dynamical equation for θ_j as a function of time. Assume that the rates of change of θ_j are much less than ω, such that the dynamical equation can be approximated by retaining only terms up to the first order of θ_j. 推导出θ_j以时间為函数的动力学方程式。可假设θ_j的变化率远小于ω，使得动力学方程式可通过仅保留至θ_j的第一阶项来近似。	**2 points** **2 分**

The dynamical equation of θ_j is similar to that of the coupled oscillator network shown in the figure. It consists of N particles free to slide on a smooth circular rail of radius R without collision. Each particle has mass m and is subject to a force F_j for particle j in the tangential direction of the circle. When a particle moves, it experiences a damping force that is equal to minus the velocity times the damping constant b. Each pair of particles is connected by a spring of very short equilibrium length and force constant k.

θ_j的动力学方程式类似于图中所示的耦合振荡器网络。它由N个可在半径R的平滑圆形轨道上自由滑动而没有碰撞的粒子组成。每个粒子的质量为m。粒子j 在圆的切线方向上受到力 F_j的作用。当粒子移动时，它受到的阻尼力等于速度乘以阻尼常数b的负值。每对粒子通过平衡长度非常短和劲度系数為k的弹簧连接。



B3	Derive the dynamical equation of the angular positions θ_j, and fill in the table on the answer sheet with the physical terms for the coupled oscillator network and the corresponding terms in the power grid. 导出角位置θ_j的动力学方程式，并在答题纸上的表格内填写耦合振荡器网络的物理项和电网中的相应物理项。	**3 points** **3 分**

B4	Consider a fully connected power grid with N_c consumer nodes and N_g generator nodes, and friction is negligible. Each consumer node consumes power P and the total consumed power is evenly provided by the generator nodes. Calculate the phase difference between the generators and the consumers at the steady state. 考虑一个由N_c个功耗设备节点和N_g个发电机节点組成的完全连接电网，其中摩擦可忽略不计。每个功耗设备节点消耗功率P，并且总消耗功率由发电机节点均匀提供。计算稳定状态下发电机和功耗设备之间的相位差。	**2 points** **2 分**

B5	Calculate the minimum number of generators to maintain effective electricity transmission in the power grid of part (B4). 计算在（B4）部份中的电网若要维持在有效传输状态，发电机数量最少需要多少部?	**1 points** **1 分**

END of Problem 2
问题 2 完

Pan Pearl River Delta Physics Olympiad 2019
2019 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助

Simplified Chinese Part-2 (Total 2 Problems, 60 Points)
简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 15 February, 2019)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。
All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。
There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。
Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。
Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。
Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。
If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。
At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Gravitational Waves (26 points) 引力波 (26 分)

Gravitational waves (GW) are the “ripples of space” predicted by Einstein in 1916. GW are transverse waves travelling at the speed of light. They are sourced by the change of mass distribution in space. In 2015, GW were detected directly by the Laser Interferometer Gravitational-Wave Observatory (LIGO). The detection is not only a verification of Einstein’s prediction after 100 years, but also provides a completely new probe of our universe and opens a new era of GW astronomy.

1916 年，爱因斯坦预言了空间的“涟漪”——引力波。引力波是横波，以光速传播，其来源是空间中物质质量分布的变化。2015 年，激光干涉引力波天文台(LIGO)发现引力波。引力波的发现不仅验证了爱因斯坦的百年预言，更是一种探测宇宙的全新手段。引力波的发现开启了引力波天文学的新时代。

In this problem, we will work in Newtonian mechanics and Newtonian gravity (instead of general relativity), and ignore the expansion of the universe, unless stated otherwise.

在本题中，除了有特殊说明之处外，我们将使用牛顿力学和牛顿引力（而不是广义相对论），且忽略宇宙膨胀。

You may find the following quantities useful here
本题中你可能用到以下数值：

Gravitational constant 牛顿引力常数	$G_N = 6.67\times10^{-11}\mathrm{m^3kg^{-1}s^{-2}}$
Speed of light 光速	$c = 3.00\times10^8\mathrm{m/s}$
Mass of the sun 太阳质量	$M_\odot = 1.99\times10^{30}\mathrm{kg}$
Mass of the earth 地球质量	$M_E = 5.97\times10^{24}\mathrm{kg}$
Radius of the earth 地球半径	$r_E = 6.37\times10^6\mathrm{m}$

Part A. Indirect Evidence of GW (10 points) 引力波的间接证据（9 分）

Before the direct discovery of GW, indirect evidence of GW has been found in a binary pulsar system in the 1970s. The binary consists of two pulsars rotating around each other in circular orbit with radius R. Let us assume each pulsar has mass M and radius r.

早在 20 世纪 70 年代，天文学家已经在双脉冲星系统中发现了引力波存在的间接证据。双脉冲星系统中，两颗脉冲星在半径为 R 的圆轨道上互相绕转。每颗脉冲星的质量为 M，半径为 r。

2

A1	Calculate the period of the binary system T_B. 求双星系统的运动周期 T_B。	**2 points** **2 分**

$\frac{Mv^2}{R} = \frac{G_N M^2}{4R^2}$. Thus $v = \sqrt{\frac{G_N M}{4R}}$ (1’).

$T_B = \frac{2\pi R}{v} = 4\pi R\sqrt{\frac{R}{G_N M}}$ (1’).

A2	Calculate the frequency of emitted gravitational wave f_{GW}. 求双星系统发射引力波的频率 f_{GW}。	**1 point** **1 分**

GW are sourced by mass distribution. When the two pulsars interchange their position (i.e. after half a period), the mass distribution returns to the same state. Thus, the period of GW is $T_B/2$ and the frequency is $f_{GW} = 2/T_B$. ($1/T_B$ for 0.5’; factor of 2 for 0.5’)

A3	The binary system emits GW with power $P = \frac{2}{5c^5}\times G_N^\alpha M^\beta R^\gamma$, where c is the speed of light. Calculate the values of α, β, γ. 双星系统辐射引力波的功率为 $P = \frac{2}{5c^5}\times G_N^\alpha M^\beta R^\gamma$，其中 c 是光速。求 α，β 和 γ 的值。	**2 points** **2 分**

Dimensional analysis.

[P] = $\mathrm{m^2\ kg\ s^{-3}}$, $[c^{-5}] = \mathrm{m^{-5}s^5}$, $[G_N] = \mathrm{m^3kg^{-1}s^{-2}}$, [M] = kg, [R] = m.

Thus,

$2 = -5 + 3\alpha + \gamma$

$1 = -\alpha + \beta$

$-3 = 5 - 2\alpha$ (1’)

Thus,

$\alpha = 4$, $\beta = 5$, $\gamma = -5$ (1’)

A4	After time T_C, the two pulsars collide due to GW emission. Calculate T_C. 因为引力波辐射，在 T_C 时间后，两颗脉冲星碰撞。求 T_C。	**3 points** **3 分**

The kinetic energy of the binary is $E_k = 2\times\frac{1}{2}Mv^2 = \frac{G_N M^2}{4R}$. (0.5’)

The potential energy of the binary is $E_p = -\frac{G_N M^2}{2R}$. (0.5’)

The total energy of the binary is $E = E_k + E_p = -\frac{G_N M^2}{4R}$. (0.5’)

Thus, the change of energy over time is $\frac{dE}{dt} = \frac{G_N M^2}{4R^2}\frac{dR}{dt} = -P = -\frac{2}{5c^5}G_N^\alpha M^\beta R^\gamma = -\frac{2}{5c^5}G_N^4 M^5 R^{-5}$. (0.5’) (Minus sign in the last two terms because emission of GW means energy

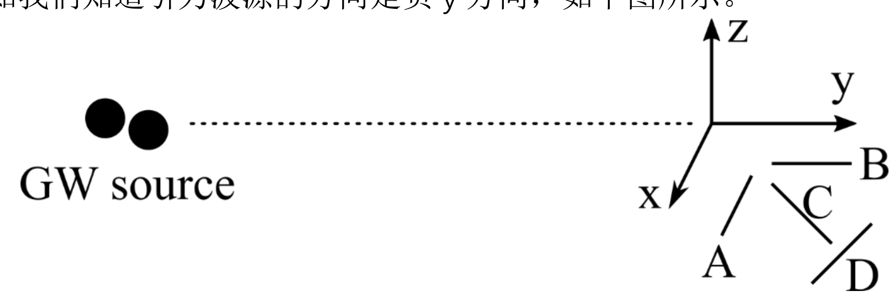
loss of the binary. We give result with both symbols α, β, γ, and their explicit values obtained in A3, to avoid double penalty. If the student is wrong in A3, he/she may still get full marks in A4.)
Integrate the above equation from R to r (corresponding time duration T_C), we get
$T_C = \frac{5c^5}{8(-\gamma-1)} G_N^{1-\alpha} M^{2-\beta} (R^{-\gamma-1} - r^{-\gamma-1}) = \frac{5c^5}{32} G_N^{-3} M^{-3} (R^4 - r^4)$. (0.5' for the integration result, and 0.5' for the correct upper/lower limits, i.e. $R^4 - r^4$ instead of R^4 only or $r^4 - R^4$)

A5	Apart from GW emission, in general relativity, there is also a correction to the gravitational potential energy $\Delta V_{GR} \propto -1/R^2$ (note the negative sign) for the pulsar system. For the same values of M, R, will this correction ΔV_{GR} increase or decrease T_B (no quantitative analysis required)? 除引力辐射外，广义相对论中，双星系统的引力势能有一个额外的修正项：$\Delta V_{GR} \propto -1/R^2$ (注意负号)。对于同样的 M 和 R，ΔV_{GR} 会增加还是减少双星系统的周期 T_B？(定性分析即可，不需要定量计算。)	**1 point** **1 分**

Decrease. (1')
For fixed M, R, more negative potential → need larger velocity to provide centrifugal force → smaller period.

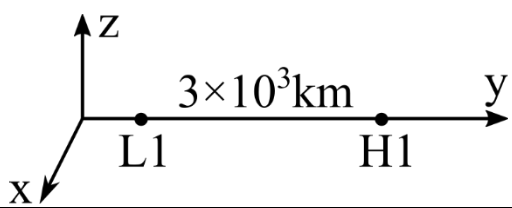
Part B. Direct Detection of GW (7 points) 引力波的发现 (7 分)

In 2016, GW were detected directly from distant merging black holes by the LIGO experiment.
2016 年，LIGO 实验组在遥远黑洞的并合事件中直接发现了引力波。

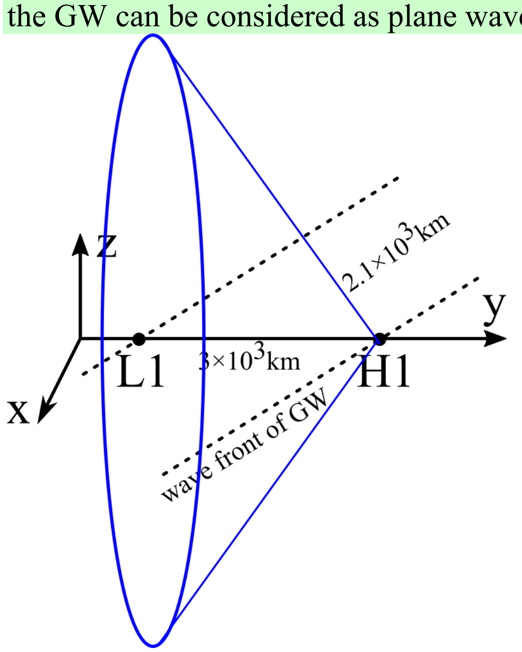
B1	Assume that we know the GW source is in the minus y direction, as shown in the below figure. 假如我们知道引力波源的方向是负 y 方向，如下图所示。  Now we measure the GW signal by measuring the change of distance of two free particles. Which of the following orientation of the particles can detect the biggest signal? Choose one from A-D below. A. Along the x-axis; B. Along the y-axis; C. Along 45 degree in the x-y plane; D. Along -45 degree in the x-y plane 我们通过测量两个自由粒子之间距离的变化来探测引力波。当粒子沿以	**1 point** **1 分**

	下哪种方向放置时，测得的信号最大？从 A-D 中选择一个。 A. 沿 x 轴； B. 沿 y 轴； C. 沿 x-y 平面中 45 度角方向； D. 沿 x-y 平面中-45 度角方向	

A (Transverse wave → oscillation of spatial length is along x and z directions.) (1’)

B2	The LIGO experiment has two GW detectors L1 and H1. L1 and H1 are separated by 3×10^3km. The GW signal first arrived at L1, and arrived at H1 after 0.007s. Calculate the angle between the GW source and the L1-H1 line. Show the possible direction of the GW source in 3-dimensional space. (Draw the direction on the figure in the answer sheet.) LIGO 实验有两个探测器 L1 和 H1。 L1 和 H1 相距 3×10^3km。引力波信号首先到达 L1，而后到达 H1。两者时间差为 0.007s。求引力波源与 L1-H1 连线的夹角，并画出引力波源在三维空间中的可能方位 (在答题纸上的图上画出方位)。 	**2 points** **2 分**

The distance between GW source and earth is much greater than the distance of detectors. Thus, the GW can be considered as plane waves, and a wave front can be approximated as planar.



$0.007\text{s}\times c = 2.1\times10^3\text{km}$ (0.5’)

The angle between the L1-H1 line and the GW source is thus

$\arccos 2.1/3 = 0.795\ \text{rad} = 45.6°$ (any of these is fine) (0.5’)

Since the situation is spherically symmetric, the possible sources form a cone. Note that there is essentially no difference whether to draw the cone starting from L1, H1, or the middle point since the source is far away.

(0.5’ for a line and another 0.5’ for the whole cone)

Note: If exact spherical waves are used, and the correct result is obtained, that’s also fine. But the computation is more complicated.

B3	For the 2016 LIGO event, the binary black hole system is about 10^{22}km away from the earth. The two initial black holes are $36M_{\odot}$ and $29M_{\odot}$. The final black hole is $62M_{\odot}$. The missing mass has all emitted away as GW energy. At 1000 km away from the center of the black hole merger event, the amplitude of GW reaches $A_0 \simeq 0.01$. The energy density of GW is proportional to its amplitude squared. What's the GW energy E_R that passes through the earth? 在 2016 年 LIGO 发现的引力波事件中，双黑洞系统距地球约10^{22}km。两个黑洞的初始质量为 $36M_{\odot}$ 和 $29M_{\odot}$，合并后的黑洞质量为 $62M_{\odot}$。系统损失的质量全部以引力波能量的形式辐射出去。在距离黑洞合并事件中心 1000 km 处，引力波的振幅为 $A_0 \simeq 0.01$。引力波能量密度正比于其振幅平方。求穿过地球的引力波能量 E_R。	**2 points** **2 分**

The energy emitted: $E = \Delta M\, c^2 = (36 + 29 - 62)\times 1.99\times 10^{30}\times(3\times 10^8)^2\ \mathrm{J} = 5.37\times 10^{47}\ \mathrm{J}$ (1')

The energy fraction that comes to the earth: $\frac{\pi(6370)^2}{4\pi\times(10^{22})^2} = 1.01\times 10^{-37}$. (0.5')

Thus, the energy passes by the earth is 5.45×10^{10} J. (0.5')

B4	For the same conditions as given in B3, what's the amplitude A of the GW when they pass by the earth? 在与 B3 相同的条件下，求引力波到达地球时的振幅 A。	**1 point** **1 分**

Consider a shell of GW. From energy conservation, energy density of GW ~ 1/(distance)^2. (1')
Thus, the amplitude of the GW decreases inversely proportional to distance:
$A = 0.01\times\frac{1000}{10^{22}} = 10^{-21}$. (1')

B5	GW can be used to study astronomy and cosmology. Consider the expansion of the universe as an example. In the Newtonian gravity concepts, the expansion of the universe can be considered as objects running away (recession) from us. GW astronomy can be a way of studying the relation between distance and the receding velocity, and thus studying the expansion history of the universe. 我们可以用引力波来研究天文学和宇宙学。例如，研究宇宙膨胀。在牛顿引力中，宇宙膨胀可以看成是天体远离我们而去 (退行)。通过引力波天文学，可以研究天体距离与退行速度之间的关系，进而研究宇宙的膨胀历史。 For GW events with optical counterpart (for example, neutron star mergers), the receding velocity is measured by the Doppler effect. The Doppler effect of which of the following can be used to directly measure the receding velocity? Choose one from A-D below: A. Synchrotron radiation from the neutron star	**1 point** **1 分**

	B. Emission or absorption spectrum of elements C. Charged particles emitted from the neutron star system D. GW emitted from the neutron star system 对于存在光学对应体的引力波事件(例如双中子星并合)，可以通过多普勒效应测量天体的退行速度。以下哪个物理过程的多普勒效应可以直接用来测量退行速度？从 A-D 中选择一个。 A. 来自中子星的同步辐射 B. 元素的发射光谱或吸收光谱 C. 中子星系统辐射出的带电粒子 D. 中子星系统辐射出的引力波	

B. (1’) Because only the element spectrum has known frequency at emission. By comparing the observed frequency and the known emission frequency, we know the receding velocity.

Part C. Interaction between GW and Matter (10 points) 引力波和物质的相互作用 (10 分)

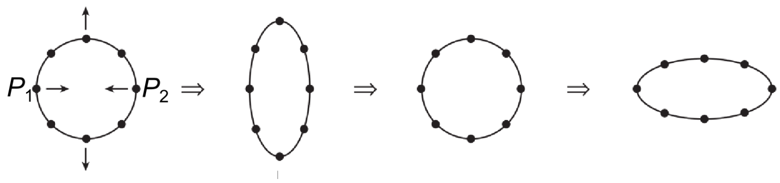
The “ripples of space” is too rough for understanding the effect of GW on matter. More concretely, one can use Newtonian physics to understand GW when its amplitude A is small (the calculation can be reproduced in general relativity in a local Lorentz frame). In Newtonian physics, the spatial length is not fluctuating. Rather, the effect of GW on matter can be considered as a periodic force proportional to $\sin(\omega_{GW} t)$ acting on matter when GW (assuming GW is plane wave with constant amplitude) passes by.

要进一步理解引力波物理，“时空的涟漪”这种说法过于粗略。更具体地，当引力波的振幅 A 很小时，我们可以在牛顿力学的框架下计算引力波的效应 (在广义相对论中，通过取局域洛伦兹系，可以验证牛顿力学的计算)。在牛顿物理中，空间距离不会有涨落变化。引力波与物质的作用体现为，引力波给物质一个正比于 $\sin(\omega_{GW} t)$ 的作用力。这里假设引力波为平面波，振幅为常数。

The amplitude A (assuming $A \ll 1$) of GW has the following effect on matter: if two free test mass particles (each has mass m) are separated by r without GW. With GW passing by its perpendicular direction (throughout Part C, we assume the propagation direction of GW is perpendicular to the line of the two particles), their distance changes from $r(1-A)$ to $r(1+A)$ periodically. The oscillation of test particle is of the pattern below (we draw many test particles to show the effect of GW more clearly, but in the problem let us just consider the two particles P_1 and P_2).

振幅为 A (假设 $A \ll 1$) 的引力波会为物质带来如下效应：考虑两个自由的检验粒子，每个粒子的质量都为 m。当没有引力波通过时，两个粒子之间的距离为 r。当有引力波沿垂直于粒子连线的方向通过时 (整个 C 部分中，均考虑引力波传播方向与两粒子连线方向垂直)，两个粒子之间的距离在 $r(1-A)$ 与 $r(1+A)$ 之间周期性变化。检验粒子的振荡如下图所

示。（下图中画了多个检验粒子的运动，以更清楚地显示引力波的作用。在本题中，我们只考虑两个粒子P_1和P_2。）



C1	Now, connect the two particles with a spring with spring constant k and unstretched length r (i.e. the spring does not change the initial distance between particles without GW). Assume that the spring is light and the force that GW acts on the spring is negligible. For $k \gg m\,\omega_{GW}^2$, calculate the oscillation amplitude A' between these two particles when GW with amplitude A pass by, such that the distance between the two particles change between $r(1-A')$ and $r(1+A')$. Note: Here we assume the two particles have minimal kinetic energy as allowed in the above setup. (Otherwise additional kinetic energy can cause oscillations with larger amplitudes for the spring.) 现在，将两个粒子用弹簧连接起来。弹簧的弹性系数为 k，原长为 r（也就是说，没有引力波通过时，弹簧不改变粒子的初始距离）。假设弹簧很轻，引力波作用在弹簧上的力可以忽略。在 $k \gg m\,\omega_{GW}^2$ 的情况下，计算振幅为 A 的引力波通过时，粒子的振幅 A'。这里 A'的定义为，引力波通过时，两粒子距离在$r(1-A')$ 和 $r(1+A')$ 之间变化。 注：假设两粒子的动能是可以满足题设条件的最小动能。（否则，额外的动能可以导致额外的振动，以及更大的振幅。）	**3 points** **3 分**

For free test particles separated by r, let the position of each particle is $x = \pm r/2$, and let's study the particle with $x = r/2$. Let the periodic force be $F = F_0 \sin(\omega_{GW} t)$ acting on this particle.

Integrate $F = m\ddot{x}$ (0.5') twice (or you first guess the form of x, and determine coefficients by taking derivative), we get $x = \frac{r}{2} - \frac{F_0 \sin(\omega_{GW} t)}{m\,\omega_{GW}^2}$. (0.5')

Compare it with the definition of A, we get $F_0 = \frac{Ar}{2} m\omega_{GW}^2$. (0.5')

An opposite force will act on the particle located at $x = -r/2$.

Now, for the two particles connected by a spring:

The condition $k \gg m\omega_{GW}^2$ indicates that the frequency of GW is very slow. Slowly acting a force on the spring, the length of the spring just follows the force (0.5'). The maximal length that the spring stretches is $r + \frac{F_0}{k} = r + \frac{Ar}{2k} m\omega_{GW}^2$ (0.5'), i.e. $A' = \frac{A}{2k} m\omega_{GW}^2$ (0.5')

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Alternative solution of C1 - C2: Instead of considering half spring, consider the full spring:

Let $x_1 = \frac{r}{2} + x_{10} \sin \omega_{GW} t$ and $x_2 = -\frac{r}{2} + x_{20} \sin \omega_{GW} t$.

$$\begin{pmatrix} -m\omega_{GW}^2 & 0 \\ 0 & -m\omega_{GW}^2 \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} -k & k \\ k & -k \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} + \begin{pmatrix} F_0 \\ -F_0 \end{pmatrix}$$

$$\begin{pmatrix} k - m\omega_{GW}^2 & -k \\ -k & k - m\omega_{GW}^2 \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} F_0 \\ -F_0 \end{pmatrix}$$

$$\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} k - m\omega_{GW}^2 & -k \\ -k & k - m\omega_{GW}^2 \end{pmatrix}^{-1} \begin{pmatrix} F_0 \\ -F_0 \end{pmatrix}$$

$$= \frac{1}{(k - m\omega_{GW}^2)^2 - k^2} \begin{pmatrix} k - m\omega_{GW}^2 & k \\ k & k - m\omega_{GW}^2 \end{pmatrix} \begin{pmatrix} F_0 \\ -F_0 \end{pmatrix}$$

Distance between the two particles:

$$d = x_1 - x_2 = r + (x_{10} - x_{20}) \sin \omega_{GW} t = r - \frac{2m\omega_{GW}^2 F_0}{(k - m\omega^2)^2 - k^2} \sin \omega_{GW} t$$

$$d = r \left[1 - \frac{(m\omega_{GW}^2)^2 A}{(k - m\omega_{GW}^2)^2 - k^2} \right] \sin \omega t$$

C1: When $k \gg m\omega^2$, $A' \approx \frac{\left(m\omega_{GW}^2\right)^2 A}{2km\omega_{GW}} = \frac{m\omega_{GW}^2}{2k} A$.

C2: When $k \ll m\omega^2$, $A' \approx \frac{\left(m\omega_{GW}^2\right)^2 A}{\left(m\omega_{GW}^2\right)^2 - 2km\omega_{GW}^2} \approx A + \frac{2k}{m\omega_{GW}^2} A$.

C2	For the same setup as Part C1, but $k \ll m\,\omega_{GW}^2$, calculate A'. Keep the linear terms in k, and the higher order terms in k can be neglected (i.e. if the exact result is $A + Bk + Ck^2 + \cdots$, we require that you get $A + Bk$ and you can neglect higher terms Ck^2 and so on). 设 $k \ll m\,\omega_{GW}^2$，其他条件与 C1 中相同，计算 A'。这里，只需保留至 k 的线性阶，k 的高阶项可以忽略。（也就是说，如果精确结果是 $A + Bk + Ck^2 + \cdots$，则你需要写出 $A + Bk$。你可以忽略 Ck^2 等高阶项。）	**3 points** **3 分**

From symmetry, the center of the spring will not move. Thus, let's consider half spring from the center to the particle to the right. The spring constant of half spring is $2k$ (0.5').

The Newtonian 2nd law: $m\ddot{x} = F_0 \sin(\omega_{GW} t) - 2k\left(x - \frac{r}{2}\right)$. (*) (0.5')

In principle, this equation (*) can be solved exactly without assuming the condition between k and $m\omega_{GW}^2$. But as we do not require to solve complicated differential equations, here we just work out the solution assuming small k.

Let $x = x^{(0)} + x^{(1)} + x^{(2)} + \cdots$. Where the superscript indicates the order in k. (0.5') We will ignore $x^{(2)}$ and higher terms.

We have already solved $x^{(0)} = \frac{r}{2} - \frac{Ar\,sin(\omega_{GW}t)}{2}$ (free particle with $k = 0$). Insert it into (*), we get

$\ddot{x}^{(1)} = \frac{kAr\sin(\omega_{GW}t)}{m}$. (0.5')

Similar to the case in C1, integrate this equation twice, we get

$x^{(1)} = -\frac{kAr\sin(\omega_{GW}t)}{m\omega_{GW}^2}$. (0.5')

Thus, up to linear order in k, the maximal distance is the maximal of $2(x^{(0)} + x^{(1)})$. The factor of two is because we have two particles at $x = \pm r/2$. Inserting the above results, we have

$A' = A + \frac{2kA}{m\omega_{GW}^2}$. (0.5')

Note: If you directly get the exact solution of (*), it is also correct. The form is

$A' = \frac{A}{1-\frac{2k}{m\omega_{GW}^2}}$.

C3	For the same setup as Part C1, but $k = m\,\omega_{GW}^2/2$, qualitatively describe how the distance between two particles changes with time. No explicit calculation is needed. 设 $k = m\,\omega_{GW}^2/2$，其他条件与 C1 中相同，定性描述两粒子间的距离将如何随时间变化。无需定量计算。	**1 point** **1 分**

Resonance happens. The distance oscillates (0.5') and the amplitude of oscillation grows with time (0.5').

C4	We'd like to estimate when GW pass through the earth, how much GW energy the earth can absorb. The earth is a system that the pressure of matter balances self-gravity. The real earth is too complicated but let's consider a toy model of the earth, as two particles at rest separated by $r = 6000$ km, each particle has mass $m = 3\times10^{24}$ kg. The self-gravity between these two particles are balanced by force provided by a light spring connecting these particles. And the unstretched length of the spring is 7000 km if no force acts on it. For the GW signal described by Part B3 and B4, with frequency $f = 100$Hz, estimate the order-of-magnitude of energy absorbed by the earth from one period of GW oscillation. 我们将估计当引力波穿过地球时，地球可以吸收多少能量。地球是一个自身压强与引力平衡的系统。真实的地球非常复杂。这里我们考虑一个地球的玩具模型：考虑两个静止粒子相距 $r = 6000$ km。每个粒子的质量为 $m = 3\times10^{24}$ kg。两个粒子之间由一根轻弹簧连接。弹簧的弹力与两粒子之间的引力平衡。假如不受力，弹簧的原长为7000 km。对于 B3、B4 中描述的，频率为 $f = 100$ Hz的引力波信号，求在引力波的一个振荡周期中，地球吸收的能量 (估计数量级即可)。	**2 points** **2 分**

First estimate the spring constant of the earth: $F_{\text{grav}} = k\Delta r = \frac{G_N m^2}{r^2}$. Thus, $k = 1.67\times10^{19}\text{kg/s}^2$. (0.5')
Now determine the spring constant is for which case (C1-C3):
$m\omega_{GW}^2 = 3\times10^{24}\text{kg}\times(100\times2\pi\ \text{Hz})^2 \gg k$. Thus, we can use the case of C2. (0.5')
For order-of-magnitude estimate, the size of the earth simply change by an amount $\delta r = rA = 6\times10^{-15}\text{m}$.
For each period, GW first slowly stretch the earth to do work, and then the work is dissipated to the earth (for example, the heat in the spring in this case) when the spring returns.
The force is about $F_0 = \frac{1}{2}m\omega_{GW}^2 \text{Ar} = 3.5\times10^{15}N$ (0.5')
The work GW does in each period is $W = F_0\delta r \sim 21$ J. (0.5') (To be accurate, there is an average of $\sin(\omega_{GW})\cos(\omega_{GW})$ over two of 1/4 periods. But here we are only interested in the order-of-magnitude.)

C5	Black holes are so dense objects that even objects travelling at the speed of light (such as GW) cannot escape. For a black hole with mass the same as that of the earth (and at the same location as the earth), calculate the amount of GW energy that the black hole absorbs for the event described in Part B3 and B4. 黑洞是一种极端致密的天体：即使以光速运动的物体 (如引力波) 也不能逃出黑洞。考虑一个黑洞，与地球有同样的质量，并处于与地球同样的位置。对于 B3、B4 中描述的引力波信号，求此黑洞吸收的引力波能量。	**1 point** **1 分**

Escape velocity: $v = \sqrt{\frac{2G_N m}{r}} = c$. (0.5') Thus, $r = \frac{2G_N m}{c^2} = 0.0088\text{m}$. (0.5')
Note: using Newtonian mechanics instead of relativity, we are actually making two mistakes: (1) light & GW are actually massless particles instead of massive ones, and their momentum-energy relations are relativistic; and (2) the Newtonian gravitational potential is not enough to describe gravity. Coincidentally, these two mistakes cancel each other's effect, and $r = \frac{2G_N m}{c^2}$ actually holds even in general relativity. But in general relativity, even if GW did not touch the horizon $r = \frac{2G_N m}{c^2}$, but instead reaches $r = \frac{3G_N m}{c^2}$ (photon sphere), GW will eventually fall into the black hole. Thus $r = \frac{3G_N m}{c^2}$ is also considered correct, though it does not follow from Newtonian mechanics.
Compared to problem B3, the GW energy come to the black hole is $5\times10^{10}\ \text{J}\times\frac{0.0088^2}{(6\times10^6)^2} = 1.04\times10^{-7}$ J. (0.5') Since GW cannot escape the black hole, this is the GW energy that the black hole absorbs. (0.5')

If you use $E = h\nu$ for the graviton energy, or use angular momentum conservation to calculate the condition for the GW to be absorbed, they can also be considered correct. There is an unique answer in general relativity, but you can use different ways to model it in Newtonian mechanics.

END of Problem 1
问题 1 完

Problem 2 Synchronization (34 marks) 同步（34 分）

Synchronization is a very common physical phenomenon. As early as in the 17th century, the famous Dutch scientist Christiaan Huygens observed that when two pendulum clocks are suspended from a common beam, they tend to oscillate in synchrony. In part A of this problem, we will consider a model of this phenomenon. In part B of this problem, we will consider a modern example of synchronization. Students can work on either part first before working on the other part.
同步是一种非常常见的物理现象。早在 17 世纪，著名的荷兰科学家克里斯蒂安•惠更斯就观察到，当两个单摆时钟悬挂在同一根梁上时，它们往往会同步振荡。在这个问题的 A 部份中，我们将考虑这种现象的一个模型。在这个问题的 B 部份中，我们将考虑一个现代的同步示例。同学可先完成任一部分，再完成另一部份。

A. The Pendulums (23 marks) 單擺 (23 分)

A single pendulum consists of a bob with mass m suspended vertically from a fixed point with a massless string of length L, subject to gravitational acceleration g. Let $q(t)$ be the angular displacement of the pendulum from the vertical at time t. When the bob moves, it encounters a constant frictional force of magnitude mLb in the opposite direction of motion.
单摆由一个质量m的小物块组成，物块从一固定点用绳子垂直悬挂，并受到重力加速度g的影响。绳子没有质量，长度为L。 在时间t，设单摆从垂直方向的角位移为$q(t)$。当物块移动时，有一数值为 mLb 的恒定摩擦力作用于它运动的反方向上。

A1	Write the dynamical equation of $q(t)$ for small oscillations. 写出小振动時$q(t)$的动力学方程式。	**2 points 2 分**

Remark: To keep your equation simple, you may introduce the angular frequency
备注：为了使方程式更简洁，您可引入角频率

$$\omega = \sqrt{\frac{g}{L}},$$

and use the sign function defined by
并使用下面所定义的正負函数

$$\mathrm{sign} f = \begin{cases} 1 \text{ for } f > 0, \\ 0 \text{ for } f = 0, \\ -1 \text{ for } f < 0. \end{cases}$$

Using Newton's law of motion,

$$mL\frac{d^2q}{dt^2} = -mLb\,\mathrm{sign}\left(\frac{dq}{dt}\right) - mg\sin\, q,$$

For small oscillations, $\sin q \approx q$. Hence

$$\ddot{q} = -b\ \mathrm{sign}\dot{q} - \omega^2 q,$$

where $\ddot{q} = d^2q/dt^2$ and $\dot{q} = dq/dt$.

To compensate the loss of kinetic energy due to the friction in each cycle, the pendulum receives a kick every cycle. To simplify the calculations, we assume that the kick takes place when $q = -b/\omega^2$ and its angular velocity is positive.

为了补偿于每个周期中由摩擦力所引致的动能损失，单摆每个周期都会受到一次踢动。为了简化计算，我们假设踢动发生在$q = -b/\omega^2$并且当它的角速度为正时。

A2	Suppose that the angular velocity of the pendulum is u_n immediately after the n^{th} kick. Calculate $q(t)$ and $\dot{q}(t)$ in the cycle after the n^{th} kick. For convenience, we choose $t = 0$ at the n^{th} kick in this part and below. 假设单摆的角速度在第n次踢动后的瞬间为u_n。在第n次踢动后的周期中，计算$q(t)$和$\dot{q}(t)$。为方便起见，在此部份及以下部份中，我们设在第n次踢动的时间为$t = 0$。 For clarity, give your answer in three parts: 为清楚起见，请分三部份给出答案： (a) The first quarter of the cycle, (a) 第一个四分之一的周期， (b) the second and third quarters of the cycle, (b) 第二和第三个四分之一的周期， (c) the fourth quarter of the cycle. (c) 第四个四分之一的周期。	**2+2+2 points** **2+2+2 分**

(A2a) In the first quarter of the cycle after the n^{th} kick, the angular velocity is positive. Hence

$$\ddot{q} = -\omega^2\left(q + \frac{b}{\omega^2}\right).$$

The solution is a simple harmonic motion centered at $q = -b/\omega^2$. Hence the solution takes the form

$$q(t) = -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin\omega t,$$

and

$$\dot{q}(t) = u_n\cos\omega t.$$

(A2b) In the second and third quarters of the cycle, the angular velocity is negative. Hence

$$\ddot{q} = -\omega^2\left(q - \frac{b}{\omega^2}\right).$$

The solution is a simple harmonic motion centered at $q = b/\omega^2$. The initial condition at $t = \pi/2\omega$ is $q\left(\frac{\pi}{2\omega}\right) = -\frac{b}{\omega^2} + \frac{u_n}{\omega^2}$ and $\dot{q}\left(\frac{\pi}{2\omega}\right) = 0$. Hence the solution takes the form

$$q(t) = \frac{b}{\omega^2} + \left[q\left(\frac{\pi}{2\omega}\right) - \frac{b}{\omega^2}\right]\cos\left(\omega t - \frac{\pi}{2}\right),$$
$$q(t) = \frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{2b}{\omega^2}\right)\sin\omega t,$$

and

$$\dot{q}(t) = \left(u_n - \frac{2b}{\omega}\right)\cos\omega t.$$

(A2c) In the fourth quarter of the cycle, the angular velocity is positive. Hence

$$\ddot{q} = -\omega^2\left(q + \frac{b}{\omega^2}\right).$$

The solution is a simple harmonic motion centered at $q = -b/\omega^2$. The initial condition at $t = 3\pi/2\omega$ is $q\left(\frac{3\pi}{2\omega}\right) = -\frac{u_n}{\omega} + \frac{3b}{\omega^2}$ and $\dot{q}\left(\frac{\pi}{2\omega}\right) = 0$. Hence the solution takes the form

$$q(t) = -\frac{b}{\omega^2} + \left[q\left(\frac{3\pi}{2\omega}\right) + \frac{b}{\omega^2}\right]\cos\left(\omega t - \frac{3\pi}{2}\right),$$
$$q(t) = -\frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{4b}{\omega^2}\right)\sin\omega t,$$

and

$$\dot{q}(t) = \left(u_n - \frac{4b}{\omega}\right)\cos\omega t.$$

A3	Suppose at each kick, a fixed amount of kinetic energy of the magnitude $mL^2h^2/2$ is injected to the pendulum, where h has the dimension of an angular velocity. Calculate the relation between u_{n+1} and u_n. 假设在每次踢动时，有数值为 $mL^2h^2/2$ 的动能被注入到单摆中，其中h具有角速度的量纲。计算u_{n+1}和之u_n间的关系。	**2 points** **2 分**

At the end of the n^{th} cycle,

$$\dot{q}(t) = u_n - \frac{4b}{\omega}.$$

At the beginning of the $(n+1)^{\text{th}}$ cycle,

$$\frac{1}{2}mL^2u_{n+1}^2 = \frac{1}{2}mL^2\left(u_n - \frac{4b}{\omega}\right)^2 + \frac{1}{2}mL^2h^2.$$

Therefore

$$u_{n+1} = \sqrt{\left(u_n - \frac{4b}{\omega}\right)^2 + h^2}.$$

A4	What is the value of u_n after many kicks? 经过多次踢动后，u_n的数值是什么？	**2 points** **2 分**

After many kicks, $u_{n+1} = u_n$. Hence

$$u_n^2 = \left(u_n - \frac{4b}{\omega}\right)^2 + h^2.$$
$$u_n = \frac{h^2\omega}{8b} + \frac{2b}{\omega}.$$

A5	Suppose that at time t_0 during the first quarter of the cycle after the n^{th} kick, the pendulum receives an angular impulse equal to $mL^2\alpha$. Calculate the time at which: 假设在第n次踢动后的第一个四分之一周期内，單摆接受了数值为 $mL^2\alpha$ 的角冲量。计算以下情况的时间： (a)the friction changes sign the first time, (a) 摩擦力第一次改变方向时， (b) the friction changes sign the second time, (b) 摩擦力第二次改变方向时， (c) the pendulum receives the $(n+1)^{\text{th}}$ kick. (c) 單摆受到第$n+1$次踢動时。 Give your answer to the first order in α. 答案的表达式展开至α的第一阶。	**3 points** **3 分**

(A5a) After the pendulum has received the angular impulse at time t_0, the pendulum motion takes the form

$$q(t) = -\frac{b}{\omega^2} + \left(q(t_0) + \frac{b}{\omega^2}\right)\cos(\omega t - \omega t_0) + \frac{\dot{q}(t_0)}{\omega}\sin(\omega t - \omega t_0).$$

Since $q(t_0) = -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin\omega t_0$ and $\dot{q}(t_0) = u_n\cos\omega t_0 + \alpha$, we have

$$\begin{gathered} q(t) = -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin\omega t_0\cos(\omega t - \omega t_0) + \frac{u_n}{\omega}\cos\omega t_0\sin(\omega t - \omega t_0) + \frac{\alpha}{\omega}\sin(\omega t - \omega t_0) \\ = -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin\omega t + \frac{\alpha}{\omega}\sin(\omega t - \omega t_0). \\ \dot{q}(t) = u_n\cos\omega t + \alpha\cos(\omega t - \omega t_0). \end{gathered}$$

The friction changes sign when $\dot{q}(t) = 0$. Suppose this takes place when $\omega t = \frac{\pi}{2} + \varepsilon$. Then

$$\begin{gathered} 0 = u_n\cos\left(\frac{\pi}{2} + \varepsilon\right) + \alpha\cos\left(\frac{\pi}{2} + \varepsilon - \omega t_0\right). \\ \varepsilon \approx \frac{\alpha\sin\omega t_0}{u_n}. \\ t = \frac{\pi}{2\omega} + \frac{\alpha\sin\omega t_0}{\omega u_n}. \end{gathered}$$

(A5b) After the friction has changed sign the first time, the pendulum motion takes the form

$$q(t) = \frac{b}{\omega^2} + \left[q\left(\frac{\pi}{2\omega} + \frac{\varepsilon}{\omega}\right) - \frac{b}{\omega^2}\right]\cos\left(\omega t - \frac{\pi}{2} - \varepsilon\right).$$

Note that

$$\begin{aligned} q\left(\frac{\pi}{2\omega} + \frac{\varepsilon}{\omega}\right) &= -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin\left(\frac{\pi}{2} + \varepsilon\right) + \frac{\alpha}{\omega}\sin\left(\frac{\pi}{2} + \varepsilon - \omega t_0\right) \\ &\approx -\frac{b}{\omega^2} + \frac{u_n}{\omega}\cos\varepsilon + \frac{\alpha}{\omega}\cos(\omega t_0 - \varepsilon) \approx \frac{u_n}{\omega} - \frac{b}{\omega^2} + \frac{\alpha}{\omega}\cos\omega t_0. \end{aligned}$$

Hence

$$q(t) = \frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{2b}{\omega^2} + \frac{\alpha}{\omega}\cos\omega t_0\right)\cos\left(\omega t - \frac{\pi}{2} - \varepsilon\right).$$

$$\dot{q}(t) = -\left(u_n - \frac{2b}{\omega} + \alpha\cos\omega t_0\right)\sin\left(\omega t - \frac{\pi}{2} - \varepsilon\right).$$

The friction changes sign the second time when $\dot{q}(t) = 0$. This takes place when $\omega t - \frac{\pi}{2} - \varepsilon = \pi$. Hence

$$t = \frac{3\pi}{2\omega} + \frac{\alpha\sin\omega t_0}{\omega u_n}.$$

(A5c) After the friction has changed sign the second time, the pendulum motion takes the form

$$q(t) = -\frac{b}{\omega^2} + \left[q\left(\frac{3\pi}{2\omega} + \frac{\varepsilon}{\omega}\right) + \frac{b}{\omega^2}\right]\cos\left(\omega t - \frac{3\pi}{2} - \varepsilon\right).$$

Note that

$$q\left(\frac{3\pi}{2\omega} + \frac{\varepsilon}{\omega}\right) = \frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{2b}{\omega^2} + \frac{\alpha}{\omega}\cos\omega t_0\right)\cos\left(\frac{3\pi}{2} + \varepsilon - \frac{\pi}{2} - \varepsilon\right) = -\frac{u_n}{\omega} + \frac{3b}{\omega^2} - \frac{\alpha}{\omega}\cos\omega t_0.$$

Hence

$$q(t) = -\frac{b}{\omega^2} - \left(\frac{u_n}{\omega} - \frac{4b}{\omega^2} + \frac{\alpha}{\omega}\cos\omega t_0\right)\cos\left(\omega t - \frac{3\pi}{2} - \varepsilon\right).$$

The pendulum receives the $(n+1)^{\text{th}}$ kick when $q(t) = -b/\omega^2$. This takes place when $\omega t - \frac{3\pi}{2} - \varepsilon = \frac{\pi}{2}$. Hence

$$t = \frac{2\pi}{\omega} + \frac{\alpha\sin\omega t_0}{\omega u_n}.$$

A6	Suppose that the time t_0 at which the pendulum receives an angular impulse equal to $mL^2\alpha$ is in the fourth quarter of the cycle after the n^{th} kick instead of the first quarter. Calculate the time at which the pendulum receives the $(n+1)^{\text{th}}$ kick. Give your answer to the first order in α. 假设单摆接受角冲量 $mL^2\alpha$ 的时间t_0是在第n次踢动后的第四个四分之一而不是第一个四分之一的周期内。计算单摆接受第$n+1$次踢動的时间。答案的表达式展开至α的第一阶。	**2 points** **2 分**

After the pendulum has received the angular impulse at time t_0, the pendulum motion takes the form

$$q(t) = -\frac{b}{\omega^2} + \left(q(t_0) + \frac{b}{\omega^2}\right)\cos(\omega t - \omega t_0) + \frac{\dot{q}(t_0)}{\omega}\sin(\omega t - \omega t_0).$$

Since $q(t_0) = -\frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{4b}{\omega^2}\right)\sin\omega t_0$ and $\dot{q}(t_0) = \left(u_n - \frac{4b}{\omega}\right)\cos\omega t_0 + \alpha$, we have

$$q(t) = -\frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{4b}{\omega^2}\right)\sin\omega t_0 \cos(\omega t - \omega t_0) + \left(\frac{u_n}{\omega} - \frac{4b}{\omega^2}\right)\cos\omega t_0 \sin(\omega t - \omega t_0) + \frac{\alpha}{\omega}\sin(\omega t - \omega t_0)$$
$$= -\frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{4b}{\omega^2}\right)\sin\omega t + \frac{\alpha}{\omega}\sin(\omega t - \omega t_0).$$
$$\dot{q}(t) = \left(u_n - \frac{4b}{\omega}\right)\cos\omega t + \alpha\cos(\omega t - \omega t_0).$$

The pendulum receives the $(n+1)^{\text{th}}$ kick when $q(t) = -b/\omega^2$. Suppose this takes place when $\omega t = 2\pi + \delta$. Then

$$\left(\frac{u_n}{\omega} - \frac{4b}{\omega^2}\right)\sin(2\pi + \delta) + \frac{\alpha}{\omega}\sin(2\pi + \delta - \omega t_0) = 0.$$
$$\delta \approx \frac{\alpha \sin\omega t_0}{u_n - \frac{4b}{\omega}}.$$
$$t = \frac{2\pi}{\omega} + \frac{\alpha\sin\omega t_0}{\omega\left(u_n - \frac{4b}{\omega}\right)}.$$

Now consider two pendulum clocks. Let $q_1(t)$ and $q_2(t)$ be the angular displacements of the two clocks. The bob mass m, length L, friction parameter b and kick size h of the two pendulums are identical. Suppose that when $q_2 = -b/\omega^2$, pendulum 2 sends a small angular impulse equal to $mL^2\alpha$ on pendulum 1, and when $q_1 = -b/\omega^2$, pendulum 1 sends a small angular impulse equal to $mL^2\alpha$ on pendulum 2. (Here, $\alpha > 0$.)
现在考虑两个单摆。设两个单摆的角位移分别為$q_1(t)$和$q_2(t)$。两个单摆的小物塊质量m、长度L、摩擦参数b和踢動的大小h均是相同。假设当$q_2 = -b/\omega^2$时，单摆 2 发出一个数值為$mL^2\alpha$的小角冲量給单摆 1。当$q_1 = -b/\omega^2$时，单摆 1 发出一个数值為$mL^2\alpha$的小角冲量給单摆 2 。（这里 $\alpha > 0$。）

A7	Suppose the phase lag of pendulum 2 relative to pendulum 1 is ϕ_n at the beginning of n^{th} cycle of pendulum 1, and $0 \le \phi_n < \pi/2$. Calculate the relation between ϕ_{n+1} and ϕ_n. 假设在单摆 1 第n个周期的开始时，单摆 2 相对于单摆 1 的相位滞后為ϕ_n，其中$0 \le \phi_n < \pi/2$。 试计算ϕ_{n+1}和ϕ_n之间的关系。	**4 points** **4 分**

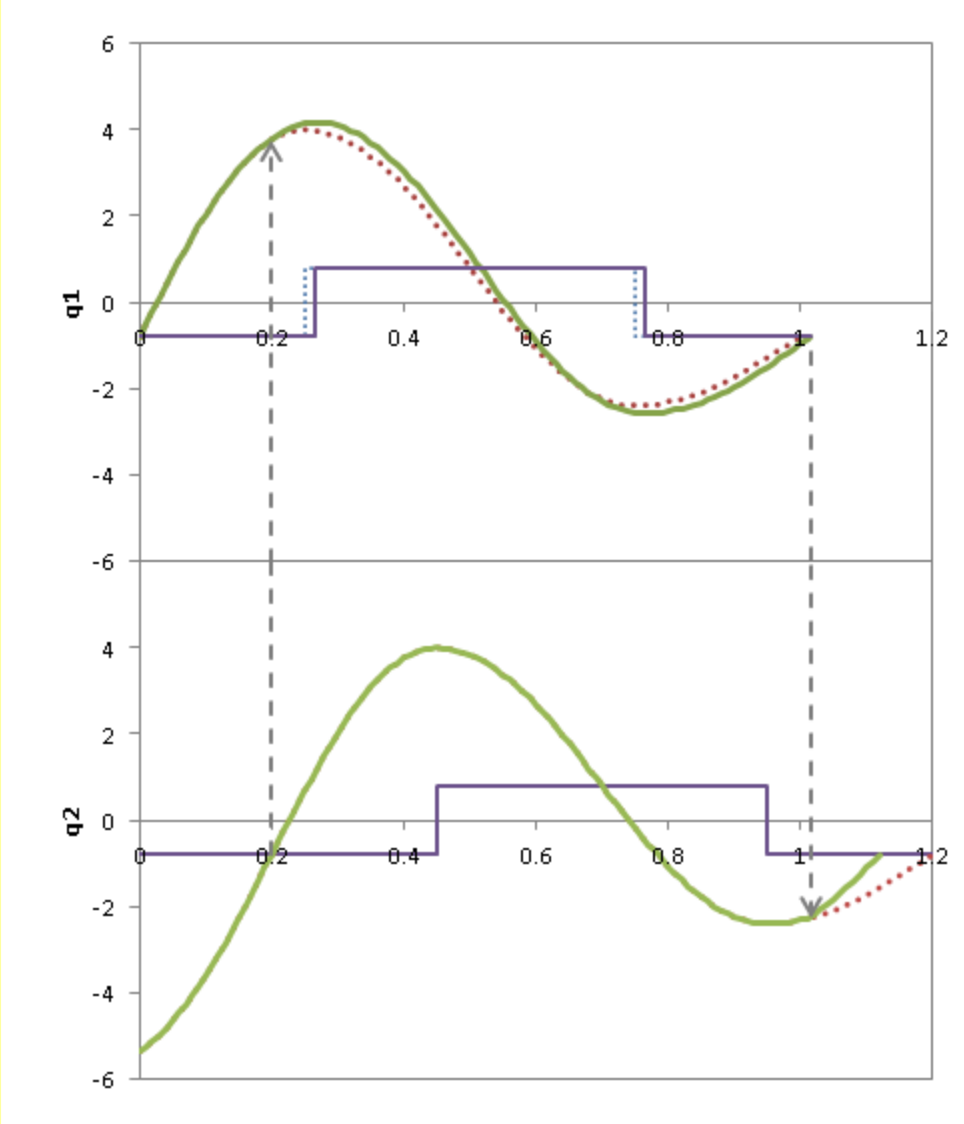


Figure: An example of how the phase lag of pendulum 2 relative to pendulum 1 is reduced in a cycle. Initially, the phase lag is 0.2 cycle. At $\omega t = 0.2$, pendulum 2 sends an angular impulse to pendulum 1. The angular velocity of pendulum 1 increases, causing the instant of the next kick to postpone from $\omega t = 1$ to $\omega t = 1.02$. At $\omega t = 1.02$, pendulum 1 sends an angular impulse to pendulum 2. The angular velocity of pendulum 2 increases, causing the instant of the next kick to move forward from $\omega t = 1.2$ to $\omega t = 1.12$. Hence the phase lag is reduced to $1.12 - 1.02 = 0.1$ cycle.

$$q_1 = -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin\omega t,$$
$$q_2 = -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin(\omega t - \phi_n).$$

At time $t_0 = \phi_n/\omega$, pendulum 2 sends an angular impulse on pendulum 1. At this instant, the phase of pendulum 1 is in the first quarter of the cycle. Using the result of (A5), pendulum 1 receives the next kick at

$$t_1 = \frac{2\pi}{\omega} + \frac{\alpha \sin \phi_n}{\omega u_n}.$$

At this instant, pendulum 1 sends an angular impulse on pendulum 2, and the phase of pendulum 2 is in the fourth quarter of the cycle. Using the result of (A6), pendulum 2 receives the next kick at

$$t_2 - \frac{\phi_n}{\omega} = \frac{2\pi}{\omega} + \frac{\alpha \sin(\omega t_1 - \phi_n)}{\omega \left(u_n - \frac{4b}{\omega}\right)} \approx \frac{2\pi}{\omega} - \frac{\alpha \sin \phi_n}{\omega \left(u_n - \frac{4b}{\omega}\right)}.$$

Phase difference:

$$\omega t_2 - \omega t_1 = 2\pi + \phi_n - \frac{\alpha \sin \phi_n}{u_n - \frac{4b}{\omega}} - 2\pi - \frac{\alpha \sin \phi_n}{u_n}.$$

$$\phi_{n+1} = \phi_n - \frac{\alpha \sin \phi_n}{u_n - \frac{4b}{\omega}} - \frac{\alpha \sin \phi_n}{u_n}.$$

A8	When ϕ_n is very small, calculate the number of cycles for ϕ_n to reduce by a factor of 10. 当ϕ_n非常少時，试计算ϕ_n减少 10 倍所需的周期数。	**2 points** **2 分**

When ϕ_n is very small,

$$\phi_n = \phi_{n-1} - \left(u_n - \frac{4b}{\omega}\right)^{-1} \alpha \sin \phi_{n-1} - u_n^{-1} \alpha \sin \phi_{n-1} \approx \left[1 - \alpha \left(u_n - \frac{4b}{\omega}\right)^{-1} - \alpha u_n^{-1}\right] \phi_{n-1}.$$

Note that for the system to sustain many kicks, $u_n - \frac{4b}{\omega} > 0$. For ϕ_n to reduce by a factor of 10,

$$\frac{1}{10} = \left[1 - \alpha \left(u_n - \frac{4b}{\omega}\right)^{-1} - \alpha u_n^{-1}\right]^N \approx \exp\left\{-\left[\left(u_n - \frac{4b}{\omega}\right)^{-1} + u_n^{-1}\right] \alpha N\right\},$$

$$N \approx \frac{1}{\alpha}\left[\left(u_n - \frac{4b}{\omega}\right)^{-1} + u_n^{-1}\right]^{-1} \ln 10.$$

Remarks: This part of the problem is adopted from [1]. In that reference, the angular impulses are negative, leading to the two clocks synchronizing oppositely. This agrees with Huygens' observation. On the other hand, there are experiments such as metronomes placed on flexible platforms that show congruent synchronization.

[1] H. M. Oliveira and L. V. Melo, Huygens synchronization of two clocks, Sci. Rep. 5: 11548 (2015).

B. The Power Grid (11 marks) 电网 (11 分)

Synchronization is an important concept in the transmission of electricity in the power grid. The power grid is a network of nodes and links. Each node is an electric generator or other power consumption devices. The links are the transmission cables. Electric power is transmitted in the alternating current (AC) mode at 50 Hz or 60 Hz at a fixed voltage. However, the AC voltage of each node in the network has a slightly different phase.

同步是电网输电中的一个重要概念。电网是一个由节点和链路组成的网络。每个节点是一部发电机或其他功耗设备。链路是传输电缆。电力运用交流电（AC）模式以 50 Hz 或 60 Hz 频率经固定电压传输。但是，网络中每个节点的交流电压具有略微不同的相位。

B1	Consider a transmission cable connecting nodes 1 and 2. The inductance of the cable is L. The electric potentials of nodes 1 and 2 are $V_j(t) = V\cos(\omega t + \theta_j)$ for $j = 1, 2$. Calculate the time-averaged power transmitted from node 1 to 2. You may neglect the time dependence of θ_j. 考虑连接节点 1 和 2 的传输电缆。电缆的电感是L。节点 1 和 2 的电势为 $V_j(t) = V\cos(\omega t + \theta_j)$，其中$j = 1, 2$。计算从节点 1 传输到节点 2 的时间平均功率。你可以忽略θ_j的时间依赖性。	**3 points** **3 分**

Potential difference of node 1 relative to 2:

$$V(t) = V\cos(\omega t + \theta_1) - V\cos(\omega t + \theta_2) = -2V\sin\left(\omega t + \frac{\theta_1 + \theta_2}{2}\right)\sin\left(\frac{\theta_1 - \theta_2}{2}\right).$$

Current from node 1 to 2: $V(t) = L\frac{dI(t)}{dt}$ implies

$$I(t) = -\int 2V\sin\left(\omega t + \frac{\theta_1 + \theta_2}{2}\right)\sin\left(\frac{\theta_1 - \theta_2}{2}\right)dt = \frac{2V}{\omega L}\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\omega t + \frac{\theta_1 + \theta_2}{2}\right).$$

Power transmitted from node 1 to 2:

$$P(t) = I(t)V\cos(\omega t + \theta_1) = \frac{2V^2}{\omega L}\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\omega t + \frac{\theta_1 + \theta_2}{2}\right)\cos(\omega t + \theta_1) =$$
$$= \frac{V^2}{\omega L}\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\left[\cos\left(\frac{\theta_1 - \theta_2}{2}\right) + \cos\left(2\omega t + \frac{3\theta_1 + \theta_2}{2}\right)\right].$$

Average power from 1 to 2:

$$\langle P\rangle = \frac{V^2}{\omega L}\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\left[\cos\left(\frac{\theta_1 - \theta_2}{2}\right) + \left\langle\cos\left(2\omega t + \frac{3\theta_1 + \theta_2}{2}\right)\right\rangle\right]$$
$$= \frac{V^2}{\omega L}\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\frac{\theta_1 - \theta_2}{2}\right) = \frac{V^2}{2\omega L}\sin(\theta_1 - \theta_2).$$

A network of electric generators and motors, labeled $j = 1, 2, \ldots, N$, are connected with each other. Their electric potentials are $V_j(t) = V\cos(\omega t + \theta_j)$ for $j = 1, 2, \ldots, N$, and the inductances of the connecting cables are L. The generator or motor at node j rotates with the phase angle $\omega t+\theta_j$ and its moment of inertia is I. The external power source or drain is P_j ($P_j>0$ if j is a generator, and $P_j<0$ if j is a motor). At the same time, the power dissipation due to friction is given by $\kappa(\omega+\dot{\theta}_j)^2/2$ at node j.

一个网络，由发电机和电动机彼此完全连接而成，发电机和电动机的标记為$j=1,2,\ldots,N$。它们的电势為$V_j(t)=V\cos(\omega t+\theta_j)$，其中$j=1,2,\ldots,N$。用于连接它们各点之间的电缆，其电感为$L$。节点处的发电机或电动机以相角$\omega t+\theta_j$旋转，其转动惯量为$I$。外部电能的供应或消耗為$P_j$（如果$j$是发电机，$P_j>0$ 。如果j是电动机，$P_j<0$）。同时，在节点j由摩擦引起的功耗是$\kappa\left(\omega+\dot{\theta}_j\right)^2/2$。

B2	Derive the dynamical equation for θ_j as a function of time. Assume that the rates of change of θ_j are much less than ω, such that the dynamical equation can be approximated by retaining only terms up to the first order of θ_j. 推导出θ_j以时间為函数的动力学方程式。可假设θ_j的变化率远小于ω，使得动力学方程式可通过仅保留至θ_j的第一阶项来近似。	**2 points** **2 分**

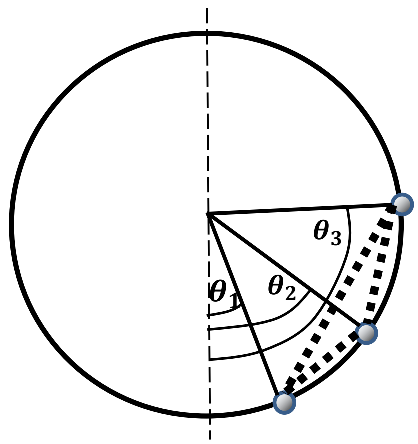
Using the conservation of power,

$$P_j=\frac{d}{dt}\left[\frac{I}{2}\left(\omega+\dot{\theta}_j\right)^2\right]+\sum_{k\neq j}\frac{V^2}{2\omega L}\sin\left(\theta_j-\theta_k\right)+\frac{\kappa}{2}(\omega+\dot{\theta}_j)^2$$
$$=I\left(\omega+\dot{\theta}_j\right)\ddot{\theta}_j+\sum_{k\neq j}\frac{V^2}{2\omega L}\sin\left(\theta_j-\theta_k\right)+\frac{\kappa}{2}(\omega+\dot{\theta}_j)^2$$
$$\approx I\omega\ddot{\theta}_j+\sum_{k\neq j}\frac{V^2}{2\omega L}\sin\left(\theta_j-\theta_k\right)+\frac{\kappa}{2}\omega^2+\kappa\omega\dot{\theta}.$$
$$I\ddot{\theta}_j+\kappa\dot{\theta}_j=\frac{P_j}{\omega}-\frac{\kappa}{2}\omega+\sum_{k\neq j}\frac{V^2}{2\omega L}\sin\left(\theta_k-\theta_j\right).$$

B3	The dynamical equation of θ_j is similar to that of the coupled oscillator network shown in the figure. It consists of N particles free to slide on a smooth circular rail of radius R without collision. Each particle has mass m and is subject to a force F_j for particle j in the tangential direction of the circle. When a particle moves, it experiences a damping force that is equal to minus the velocity times the damping constant b. Each pair of particles is connected by a spring of very short equilibrium length and force constant k. Derive the dynamical equation of the angular positions θ_j, and fill in the table on the answer sheet with the physical terms for the coupled oscillator network and the corresponding terms in the power grid. θ_j的动力学方程式类似于图中所示的耦合振荡器网络。它由N个可在半	**3 points** **3 分**

2

	径R的平滑圆形轨道上自由滑动而没有碰撞的粒子组成。每个粒子的质量为m。粒子j 在圆的切线方向上受到力 F_j的作用。当粒子移动时，它受到的阻尼力等于速度乘以阻尼常数b的负值。每对粒子通过平衡长度非常短和劲度系数為k的弹簧连接。导出角位置θ_j的动力学方程式，并在答题纸上的表格内填写耦合振荡器网络的物理项和电网中的相应物理项。	



Force of particle 2 on particle 1:

$$F_{12} = k2R\sin\left(\frac{\theta_2 - \theta_1}{2}\right).$$

Its tangential component is

$$F_{12t} = F_{12}\cos\left(\frac{\theta_2 - \theta_1}{2}\right) = k2R\sin\left(\frac{\theta_2 - \theta_1}{2}\right)\cos\left(\frac{\theta_2 - \theta_1}{2}\right) = kR\sin(\theta_2 - \theta_1).$$

Using Newton's law for circular motion,

$$I\frac{d^2\theta_j}{dt^2} = RF_j - Rb\frac{d\theta_j}{dt} + \sum_{k\neq j} kR\sin(\theta_k - \theta_j).$$

	Moment of inertia 转动惯量	Damping torque 阻尼力矩	Torque due to external force 由外力引起的扭矩	Torque on particle j due to particle k 粒子k作用於粒子j的扭矩

Coupled oscillator network 耦合振荡器网络	$I\dfrac{d^2\theta_j}{dt^2}$			
Corresponding term in the power grid 电网中的相应物理项	$I\dfrac{d^2\theta_j}{dt^2}$			

	Moment of inertia	Damping torque	Torque due to external force	Torque on particle j due to particle k
Coupled oscillator network	$I\dfrac{d^2\theta_j}{dt^2}$	$R^2 b\dfrac{d\theta_j}{dt}$	RF_j	$kR^2\sin(\theta_k-\theta_j)$
Corresponding term in the power grid	$I\dfrac{d^2\theta_j}{dt^2}$	$\kappa\dfrac{d\theta_j}{dt}$	$\dfrac{P_j}{\omega}-\dfrac{\kappa}{2}\omega$	$\dfrac{V^2}{2\omega^2 L}\sin(\theta_k-\theta_j)$

B4	Consider a fully connected power grid with N_c consumer nodes and N_g generator nodes, and friction is negligible. Each consumer node consumes power P and the total consumed power is evenly provided by the generator nodes. Calculate the phase difference between the generators and the consumers at the steady state. 考虑一个由N_c个功耗设备节点和N_g个发电机节点組成的完全连接电网，其中摩擦可忽略不计。每个功耗设备节点消耗功率P，并且总消耗功率由发电机节点均匀提供。计算稳定状态下发电机和功耗设备之间的相位差。	**2 points** **2 分**

At the steady state,

$$\frac{P_j}{\omega}+\sum_{k\neq j}\frac{V^2}{2\omega^2 L}\sin(\theta_k-\theta_j)=0.$$

For the generators,

$$\frac{N_c P}{N_g\omega}+\frac{N_c V^2}{2\omega^2 L}\sin(\theta_c-\theta_g)=0.$$

For the consumers,

$$-\frac{P}{\omega}+\frac{N_g V^2}{2\omega^2 L}\sin(\theta_g-\theta_c)=0.$$

The two equations are dependent. Solution:

$$\theta_g-\theta_c=\arcsin\left(\frac{2\omega LP}{N_g V^2}\right).$$

B5	Calculate the minimum number of generators to keep the power grid in part (B4) synchronized. 计算在（B4）部份中的电网若要维持在同步状态，发电机数量最小需要多少部？	**1 points** **1 分**

$$\frac{2\omega LP}{N_g V^2}\leq 1,$$

Hence the minimum number of generators is

$$N_{g,\min}=\frac{2\omega LP}{V^2},$$

(or more precisely, $N_{g,\min}=$ ceiling function of $\frac{2\omega LP}{V^2}$).

Reference: F. Dörfler, M. Chertkov, and F. Bullo, Synchronization in complex networks and smart grids, Proc. Natl. Acad. Sci. USA **110**, 2005-2010 (2013).

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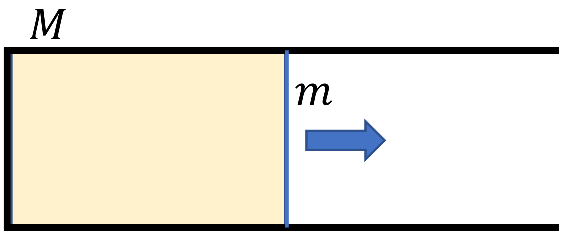
1. [10 points]
A static insulating container of mass M and cylindrical shape is placed in vacuum. One of its ends is closed. Initially, an insulating piston of mass m and negligible width separates the volume of the container into two equal parts. The closed part contains n moles of monoatomic ideal gas with molar mass M_0 and temperature T. Assume that the container wall is smooth.
1. [10 分] 将质量为 M 的圆柱形绝缘容器静止置于真空中。其一端是封闭的。最初，质量 m 和宽度可忽略的绝缘活塞将容器分成两个相等的部分。封闭部分包含 n 摩尔的单原子理想气体，其温度为 T，摩尔质量为 M_0。假設容器為光滑容器。



(a) [2 points] Assume that the state of gas during its expansion can be approximated by thermal equilibrium condition, what is the temperature of the gas T_f when the piston left the container?
(a) [2 分] 假设气体在膨胀过程中的状态可以通过熱平衡条件来近似，求活塞离开容器时气体的温度 T_f 。

(b) [4 points] At the moment when the piston leaves the container, the gas and the container will move with speed v and the piston will move with speed u. Find v and u.
(b) [4 分] 在活塞离开容器的那一刻，气体和容器将以速率 v 移动，而活塞以速率 u 移动。求 v 和 u。
(c) [4 points] When all the gas has left the container, the final speed of the container further changes from v to $v + v'$. Estimate v' using the kinetic theory of gases. Assume that the final speed of the container is much less than the thermal speed of the molecules.
(c) [4 分] 当所有气体都离开容器后，容器的最终速度进一步从 v 变为 $v + v'$。使用气体动力学理论估算 v'。假设容器的最终速度远小于分子的热速度。

The gas constant is R. There is no heat exchange between the gas, container and the piston. The change of the temperature of the gas, when it leaves the container, can be neglected. The gravitation of the Earth can be neglected.
气体常数为 R。气体、容器和活塞之间没有热交换。气体离开容器后的温度变化可以忽略不计。可以忽略地球的引力。

2. **[10 points]** A spherical dust particle falls from rest through a water mist cloud of uniform density. The initial mass and radius of the spherical dust is M_0 and R_0 respectively. The rate of accretion onto the droplet is equal to the volume of the mist cloud swept out by the droplet per unit time. Let ρ be the density of water mist and g be the gravitational acceleration.
We assume the density of water mist ρ does not change after accretion and ignore air friction other than that from accretion.
2. [10 分] 球形尘埃粒子从静止的地方通过均匀密度的水雾云落下。球形尘埃的初始质量和半径分别为M_0 和R_0。液滴上的吸积率等于每单位时间被液滴扫出的雾状云的体积。设 ρ 为水雾的密度，g 为重力加速度。
我们假设水雾的密度 ρ 在吸积后没有变化，也忽略除吸积过程以外的空气摩擦。

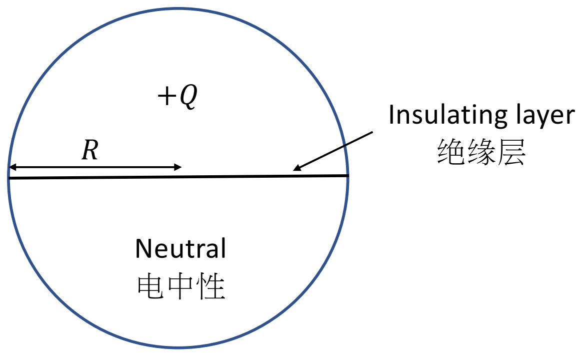
(a) [2 points] Let $M(t)$ and $R(t)$ be the mass and radius of the droplet at time t respectively. Find a relationship between $\frac{dM}{dt}$ and $\frac{dR}{dt}$.
(a) [2 分] 设 $M(t)$ 和 $R(t)$ 分别为液滴在时间 t 的质量和半径。求 $\frac{dM}{dt}$ 和$\frac{dR}{dt}$ 之间的关系。

(b) [2 points] Find the speed of the droplet at time t. Express the answer in term of ρ, R and $\dot{R}$.
(b) [2 分] 求在时间 t 的液滴速率。用 ρ, R 和 $\dot{R}$ 表示答案。

(c) [3 points] After a long time, the radius of the droplet increases with time as $R(t) = bt^n$. Find b and n. Express the answers in terms of ρ and g.
(c) [3 分] 长时间后，液滴的半径随时间增加，满足关系式 $R(t) = bt^n$。求 b 和 n。用 ρ 和 g 表示答案。

(d) [3 points] Find the value of the acceleration of the droplet after a long time. Express the answer in terms of ρ and g.
(d) [3 分] 找出长时间后液滴的加速度值。用 ρ 和 g 表示答案。

3. **[10 points]** There is a solid metallic sphere of radius R, which is cut into two identical hemispheres. The cut surface is coated with a thin insulating layer of thickness d, and the two parts are put together so that the original shape of the sphere is restored. Initially, the sphere is electrically neutral. Then one of the hemispheres is given a positive charge $+Q$ while the other one remains neutral. You can assume $d \ll R$ in this problem.



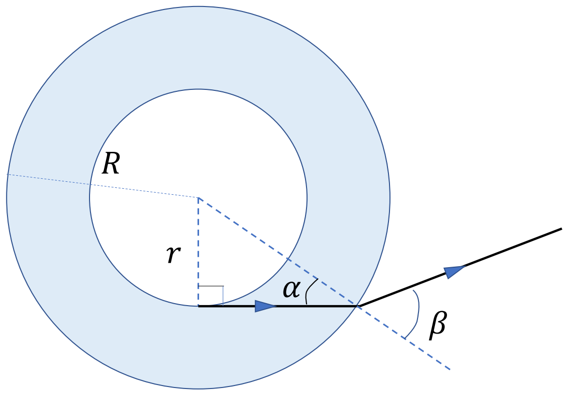
3. **[10 分]** 有一个半径为 R 的固体金属球，被切成两个相同的半球。切割表面涂上一层厚度为 d 的薄绝缘层，并将两部分放回一起，以恢复球形。最初，金属球是电中性的。然后，一个半球被赋予正电荷 $+Q$，而另一个则保持中性。在此问题中可假设 $d \ll R$。

(a) [4 points] Find the charge on each surfaces of the sphere.
(a) [4 分] 找出球体每个表面上的电荷。

(b) [3 points] Find the electrostatic interaction force between two hemispheres.
(b) [3 分] 找出两个半球之间的静电相互作用力。

(c) [3 points] Find the electrostatic energy of the sphere.
(c) [3 分] 求球体的静电能。

4. **[10 points]** Figure 4 shows a hollow glass tube with outer radius R and inner radius r respectively. The refractive index of the glass is n. From the outside air, the apparent inner radius of the tube is r'(i.e. the radius of the hollow portion observed from outside appears to be equal to r').
4. [10 分] 圖 4 表示一支中空玻璃管，其外半徑為 R，內半徑為 r，玻璃折射率為 n。由外面空氣中看來，該管之視內半徑為 r' (即空心部分的半徑從外面看起來等於 r')。



(a) [2 points] Find the ratio of the actual inner radius to the outer radius $\frac{r}{R}$. Express your answer in terms of α, β and n.
(a) [2 分] 求真內半徑與外半徑的比值 $\frac{r}{R}$，以α, β 和n来表示。

(b) [3 points] Find the ratio of the apparent inner radius to the outer radius $\frac{r'}{R}$. Express your answer in terms of α, β and n.
(b) [3 分] 求視內半徑與外半徑的比值 $\frac{r'}{R}$，以α, β 和n来表示。

(c) [5 points] If $R = 4.0$ mm, $r' = 0.50$ mm and $n = 1.6$, calculate the actual inner radius r of the glass tube up to 2 significant figures.
(c) [5 分] 若$R = 4.0$ mm， $r' = 0.50$ mm，$n = 1.6$，計算玻璃管的真內半徑r，答案準確至兩位有效數字。

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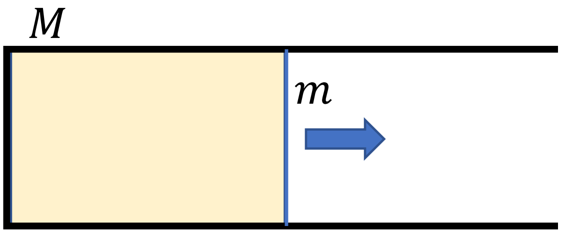

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Solution:
(a) When the gas expands, there is no heat exchange and the process is adiabatic. We have

$$TV^{\gamma-1} = T_f V_f^{\gamma-1}$$

where $\gamma = \frac{c_p}{c_v} = \frac{\frac{3}{2}+1}{\frac{3}{2}} = \frac{5}{3}$ and $V_f = 2V$. The final temperature is

$$T_f = 2^{-\frac{2}{3}} T$$

Reference: The validity range of the steady-state adiabatic law

Using the differential form of the first law of thermodynamics,

$$dU = \frac{3}{2} nRdT,$$

$$dQ = 0,$$

$$dW = pdV + Mvdv + \frac{1}{4} nM_0 (v-u)(dv-du) + mudu.$$

This yields

$$\frac{3}{2} nRdT = -pdV - \left[\left(M + \frac{1}{4} nM_0\right) v - \frac{1}{4} nM_0 u\right] dv - \left[\left(m + \frac{1}{4} nM_0\right) u - \frac{1}{4} nM_0 v\right] du.$$

Let A be the cross-section area of the cylinder. Then in time dt,

$$dV = A(u+v)dt.$$

Here we have defined v in the forward direction, and u in the backward direction. Using conservation of linear momentum,

$$\left(M + \frac{1}{2} nM_0\right) v = \left(m + \frac{1}{2} nM_0\right) u.$$

Expressing u and v in terms of V,

$$u = \left(\frac{M + \frac{1}{2} nM_0}{M + m + nM_0}\right) \frac{1}{A} \frac{dV}{dt},$$

$$v = \left(\frac{m + \frac{1}{2} nM_0}{M + m + nM_0}\right) \frac{1}{A} \frac{dV}{dt}.$$

Eliminating u and v, and dividing by dt, we obtain

$$\frac{3}{2}nR\frac{dT}{dt} = -p\frac{dV}{dt}$$
$$-\left[\left(M+\frac{1}{4}nM_0\right)\left(m+\frac{1}{2}nM_0\right)^2 - \frac{1}{2}nM_0\left(M+\frac{1}{2}nM_0\right)\left(m+\frac{1}{2}nM_0\right) + \left(m+\frac{1}{4}nM_0\right)\left(M+\frac{1}{2}nM_0\right)^2\right]\frac{1}{(M+m+nM_0)^2}\frac{1}{A^2}\frac{dV}{dt}\frac{d^2V}{dt^2}.$$

Using ideal gas law,

$$\frac{3}{2}\frac{dT}{dt} + \frac{T}{V}\frac{dV}{dt} = -\frac{\mu}{nRA^2}\frac{dV}{dt}\frac{d^2V}{dt^2}.$$

Here we have introduced the reduced mass:

$$\mu = \left[\left(M+\frac{1}{4}nM_0\right)\left(m+\frac{1}{2}nM_0\right)^2 - \frac{1}{2}nM_0\left(M+\frac{1}{2}nM_0\right)\left(m+\frac{1}{2}nM_0\right) + \left(m+\frac{1}{4}nM_0\right)\left(M+\frac{1}{2}nM_0\right)^2\right]\frac{1}{(M+m+nM_0)^2}.$$

Let $x = TV^{\frac{2}{3}}$. Then we have

$$\frac{d(x^2)}{dt} = -\frac{4\mu}{3nRA^2}V^{\frac{2}{3}}\frac{dV}{dt}\frac{d^2V}{dt^2}.$$

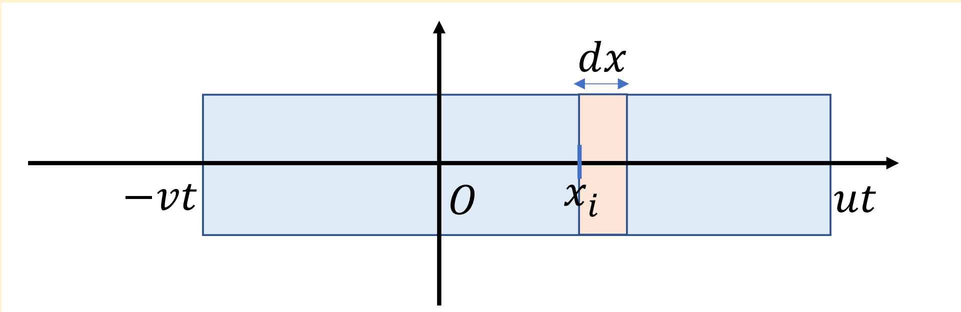
Although this equation is difficult to solve analytically, it shows that $TV^{\frac{2}{3}}$ is not a constant, that is, the steady-state adiabatic law is not applicable.

However, the steady-state adiabatic law is approximately correct if the motion during expansion is sufficiently low, since it is proportional to $\frac{1}{A}\frac{dV}{dt}$. In other words, the approximation is valid when the momentum of the system is negligible.

(b) By the conservation of momentum and energy,

$$(M+nM_0)v - mu = 0$$
$$\frac{1}{2}(M+nM_0)v^2 + \frac{1}{2}mu^2 = -\Delta U = \frac{3}{2}nR(T-T_f)$$
$$\Rightarrow v = \sqrt{3\left(1-2^{-\frac{2}{3}}\right)\frac{mnRT}{(nM_0+M)(m+nM_0+M)}}$$
$$\Rightarrow u = \sqrt{3\left(1-2^{-\frac{2}{3}}\right)\frac{(nM_0+M)nRT}{m(m+nM_0+M)}}$$

In fact, we can consider a more realistic situation. , If the thermal motion of the gas is much larger than v and u, we can assume that the density of gas ρ is uniform during the expansion.



Imagine that we have divided the gas into N slices, the mass of each slice is

$$\Delta m = \frac{nM_0}{N}$$

At time t, the location of the i-th slice is

$$x_i = -vt + \frac{(u+v)t}{N} i$$

for $i = 0,1,2,\dots,N-1$. Hence the velocity of the i-th slice is

$$\Rightarrow \dot{x}_i = -v + \frac{(u+v)}{N} i$$

The total momentum of the gas is

$$P_{gas} = \sum_i \Delta m \dot{x}_i = \frac{nM_0}{N} \sum_i \left(-v + \frac{u+v}{N} i\right) = nM_0 \left(-v + \frac{u+v}{N^2} \sum i\right)$$
$$= nM_0 \left(-v + \frac{u+v}{N^2} \frac{N(N+1)}{2}\right) \approx nM_0 \left(-v + \frac{u+v}{2}\right) = nM_0 \left(\frac{u-v}{2}\right)$$

The total kinetic energy of the gas is,

$$E_{gas} = \sum_i \frac{1}{2} \Delta m \dot{x}_i^2 = \frac{nM_0}{2N} \sum_i \left(v^2 - \frac{2v(u+v)}{N} i + \frac{(u+v)^2}{N^2} i^2\right)$$
$$= \frac{nM_0}{2} \left(v^2 - \frac{2v(u+v)}{N^2} \frac{N(N+1)}{2} + \frac{(u+v)^2}{N^3} \frac{N(N+1)(2N+1)}{6}\right)$$
$$\approx \frac{nM_0}{2} \left(v^2 - v(u+v) + \frac{(u+v)^2}{3}\right) = \frac{nM_0}{6} (u^2 - uv + v^2)$$

The conservation of energy and momentum give, (we define $\alpha = \frac{1}{2} nM_0$)

$$Mv + \alpha(v-u) - mu = 0$$
$$\frac{1}{2} Mv^2 + \frac{\alpha}{3}(u^2 - uv + v^2) + \frac{1}{2} mu^2 = -\Delta U = \frac{3}{2} nR(T - T_f)$$
$$\Rightarrow v^2 = \frac{3}{2} nRT \left(1 - 2^{-\frac{2}{3}}\right) \times \frac{(m+\alpha)^2}{(3M+\alpha)(m+\alpha)^2 + (3m+2\alpha)(M+\alpha)^2 - 2\alpha(M+\alpha)(m+\alpha)}$$
$$\Rightarrow u^2 = \frac{3}{2} nRT \left(1 - 2^{-\frac{2}{3}}\right) \times \frac{(M+\alpha)^2}{(3M+\alpha)(m+\alpha)^2 + (3m+2\alpha)(M+\alpha)^2 - 2\alpha(M+\alpha)(m+\alpha)}$$

(c) When the piston leaves the container, half of the gas particle will leave the container without any effect. And the other half will hit the bottom of the container before leaving the container. Each atom gives the container a momentum

$$\Delta p = 2m\overline{v_x}$$

where $m = M_0/N_A$ is the mass of each atom. By the kinetic theory,

$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}kT_f$$

and

$$\overline{v^2} = 3\overline{v_x^2}$$

We get

$$\overline{v_x} \approx \sqrt{\frac{1}{3}\overline{v^2}} = \sqrt{\frac{kT_f}{m}}$$

$$\Rightarrow \Delta p = 2\sqrt{mkT_f}$$

The total momentum received by the container (due to the half of the gas) is

$$p = \left(\frac{nN_A}{2}\right)\Delta p = nN_A\sqrt{mkT_f} = nN_A\sqrt{\frac{M_0}{N_A}\frac{RT_f}{N_A}} = n\sqrt{M_0RT_f} = 2^{-1/3}n\sqrt{M_0RT}$$

Notice that $nRT = NkT \Rightarrow k = \frac{n}{N}R = R/N_A$. And the gain of the velocity is

$$v' = \frac{p}{M} = 2^{-1/3}\frac{n\sqrt{M_0RT}}{M}$$

Alternative solution:

We first assume p(t) is decaying exponentially with the relaxation time as the time scale. This assumption is valid if the gas is allowed to diffuse in space freely, but due to the particular geometry of the cylinder, the gas can only diffuse out the cylinder at the end. This significantly modifies the exponential decay of the pressure.

Without solving the hydrodynamic equations, we adopt the following simplified picture:

At $t = 0$, the gas near the end ($x = L$) starts to diffuse out, whereas gases inside the cylinder remains at the initial pressure.

At $0 < t < \frac{L}{v_{rms}}$, gases within a distance $L - v_{rms}t$ starts to diffuse out, whereas gases deeper inside the cylinder remains at the initial pressure. The pressure is approximately

$$p(x,t) = p_f \quad \text{for } x < L - v_{rms}t,$$

$$p(x,t) = p_f \, exp\left[-\frac{x - L + v_{rms}t}{l}\right] \text{ for } x > L - v_{rms}t,$$

where l is mean free path of the gas.

At $t > \frac{L}{v_{rms}}$, even gases near the end wall starts to diffuse out. The pressure is approximately

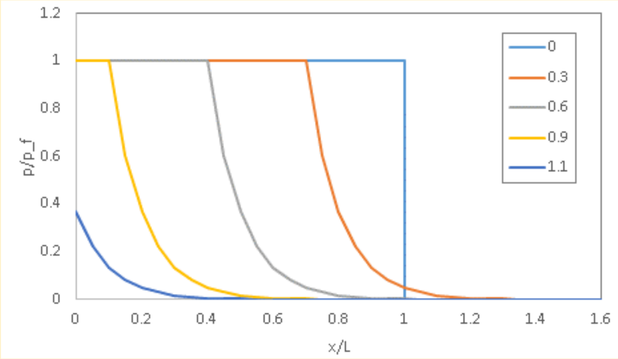
$$p(x,t) = p_f exp\left[-\frac{x - L + v_{rms}t}{l}\right].$$

Summarizing, we can write

$$p(x,t) = p_f \min\left\{1, exp\left[-\frac{x - L + v_{rms}t}{l}\right]\right\}.$$

The result is summarized in the following figure, where time is scaled in units of L/v_{rms}. For illustration, we have used $\frac{L}{l} = 10$, which is exaggerated.

1



Now we can focus on the pressure on the wall. The pressure is effectively unchanged before the time $\frac{L}{v_{rms}}$, and exponentially decay afterwards. At $x = 0$,

$$p(0,t) = p_f \quad \text{for } t < \frac{L}{v_{rms}},$$

$$p(0,t) = p_f exp\left[\frac{L - v_{rms}t}{l}\right] \quad \text{for } t > \frac{L}{v_{rms}}.$$

Hence the total momentum transfer to the wall is

$$\Delta p = A\, p_f\left(\frac{L}{v_{rms}}\right) + Ap_f \int_0^\infty exp\left(-\frac{t}{\tau}\right) dt,$$

where $\tau = \frac{l}{v_{rms}}$ is the relaxation time. The result is

$$\Delta p = Ap_f \frac{L}{v_{rms}} + Ap_f\tau. \qquad [1]$$

The first term is the result of kinetic theory, whereas the second term is due to the period of unchanged pressure. For $L >> l$, the first term is most important.

Interpretation of the second term

In eqtn [1], the physics of the first was originated from the kinetic theory of the molecules, which is valid at low density. The second term can be interpreted in the following way.

At higher density, a hydrodynamic model of the molecules is required. The mean free path l of the molecules is given by the condition

$$\rho\pi D^2 l = 1.$$

Here $\rho = nN_A/V_f$ is the number density of the molecules. N_A is the Avogadro's number. Hence

$$l = \frac{V_f}{\pi D^2 N_A n}.$$

The average speed of the molecules is given by

$$\sqrt{\langle v^2\rangle} = \sqrt{\frac{3k_BT_f}{m_0}} = \sqrt{\frac{3RT_f}{M_0}}.$$

Hence the relaxation time of gas density is given by

$$\tau = \frac{l}{\sqrt{\langle v^2 \rangle}} = \frac{V_f}{\pi D^2 N_A n}\sqrt{\frac{M_0}{3RT_f}}.$$

The momentum change of the container is given by the impulse, which is the integration of the force and the time. Hence the momentum change Δp is given by

$$\Delta p = Ap_f\tau = \frac{Ap_fV_f}{\pi D^2 N_A n}\sqrt{\frac{M_0}{3RT_f}}.$$

Using ideal gas law, $p_fV_f = nRT_f$,

$$\Delta p = Ap_f\tau = \frac{A}{\pi D^2 N_A}\frac{\sqrt{M_0RT}}{3^{\frac{1}{2}}2^{\frac{1}{3}}}.$$

And the change of the velocity is

$$v' = \frac{A}{\pi D^2 N_A}\frac{\sqrt{M_0RT}}{3^{1/2}2^{1/3}M}$$

2. **[10 points]** A spherical dust particle falls from rest through a water mist cloud of uniform density. The initial mass and radius of the spherical dust is M_0 and R_0 respectively. The rate of accretion onto the droplet is equal to the volume of the mist cloud swept out by the droplet per unit time. Let ρ be the density of water mist and g be the gravitational acceleration.
We assume the density of water mist ρ does not change after accretion and ignore air friction other than that from accretion.
2. [10 分] 球形尘埃粒子从静止的地方通过均匀密度的水雾云落下。球形尘埃的初始质量和半径分别为M_0 和R_0。液滴上的吸积率等于每单位时间被液滴扫出的雾状云的体积。设 ρ 为水雾的密度，g 为重力加速度。
我们假设水雾的密度 ρ 在吸积后没有变化，也忽略除吸积过程以外的空气摩擦。

(a) [2 points] Let $M(t)$ and $R(t)$ be the mass and radius of the droplet at time t respectively. Find a relationship between $\frac{dM}{dt}$ and $\frac{dR}{dt}$.
(a) [2 分] 设 $M(t)$ 和 $R(t)$ 分别为液滴在时间 t 的质量和半径。求 $\frac{dM}{dt}$ 和$\frac{dR}{dt}$ 之间的关系。

(b) [2 points] Find the speed of the droplet at time t. Express the answer in term of ρ, R and $\dot{R}$.
(b) [2 分] 求在时间 t 的液滴速度。用 ρ, R 和 $\dot{R}$ 表示答案。

(c) [3 points] After a long time, the radius of the droplet increases with time as $R(t) = bt^n$. Find b and n. Express the answers in terms of ρ and g.
(c) [3 分] 长时间后，液滴的半径随时间增加，满足关系式 $R(t) = bt^n$。求 b 和 n。用 ρ 和 g 表示答案。

(d) [3 points] Find the value of the acceleration of the droplet after a long time. Express the answer in terms of ρ and g.
(d) [3 分] 找出长时间后液滴的加速度值。用 ρ 和 g 表示答案。

Solution:
(a) Take the initial position of the dust particle as the origin and the x-axis along the downward vertical. Let $M(t)$ and $R(t)$ be the mass and radius of the droplet at time t respectively. Then

$$M(t) = M_0 + \frac{4}{3}\pi(R^3 - R_0^3)\rho$$

Where ρ is the density of the water mist.

$$\frac{dM}{dt} = 4\pi\rho R^2 \frac{dR}{dt}$$

(b) The droplet has a cross section πR^2 and sweeps out a cylinder of volume $\pi R^2 \dot{x}$ in unit time. As the rate of accretion is proportional to this volume, we have

$$\frac{dM}{dt} = \rho\pi R^2 \dot{x}$$

Hence, the speed of the droplet is,

$$\dot{x} = 4\dot{R}$$

(c) The momentum conservation gives

$$M(t+dt)\dot{x}(t+dt) - M(t)\dot{x}(t) = Mgdt$$

$$\Rightarrow \frac{d}{dt}(M\dot{x}) = Mg$$

$$\Rightarrow \dot{x}\frac{dM}{dt} + M\ddot{x} = Mg$$

For large t, $M(t) \approx \frac{4}{3}\pi R^3 \rho,\ \dot{x} = 4\rho\dot{R}$ and $\frac{dM}{dt} \approx \frac{3M\dot{R}}{R}$, we have

$$\ddot{R} + \frac{3\dot{R}^2}{R} = \frac{g}{4}$$

A particular solution for large t has the form

$$R(t) = bt^n$$

Substituting into the DE,

$$bn(n-1)t^{n-2} + \frac{3b^2n^2t^{2(n-1)}}{bt^n} = (bn(n-1) + 3bn^2)t^{n-2} = \frac{g}{4}$$

$$\Rightarrow n = 2$$

and

$$\Rightarrow b = \frac{g}{56}$$

$$\Rightarrow R(t) = \frac{g}{56}t^2$$

(d)

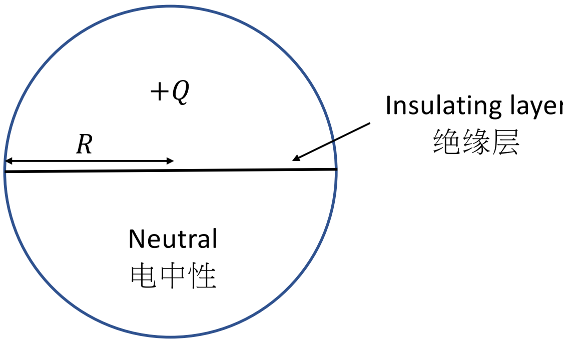
$$\dot{x} = 4\dot{R} = 4\left(\frac{g}{28}\right)t = \frac{g}{7}t$$

$$\Rightarrow \ddot{x} = \frac{g}{7}$$

The acceleration of the droplet is $g/7$ after a long time.

3. **[10 points]** There is a solid metallic sphere of radius R, which is cut into two identical hemispheres. The cut surface is coated with a thin insulating layer of thickness d, and the two parts are put together so that the original shape of the sphere is restored. Initially, the sphere is electrically neutral. Then one of the hemispheres is given a positive charge $+Q$ while the other one remains neutral. You can assume $d \ll R$ in this problem.

3. **[10 分]** 有一个半径为 R 的固体金属球，被切成两个相同的半球。切割表面涂上一层厚度为 d 的薄绝缘层，并将两部分放回一起，以恢复球形。最初，金属球是电中性的。然后，一个半球被赋予正电荷 $+Q$ ，而另一个则保持中性。在此问题中可假设 $d \ll R$ 。



(a) [3 points] Find the charge on each surfaces of the sphere.
(a) [3 分] 找出球体每个表面上的电荷。

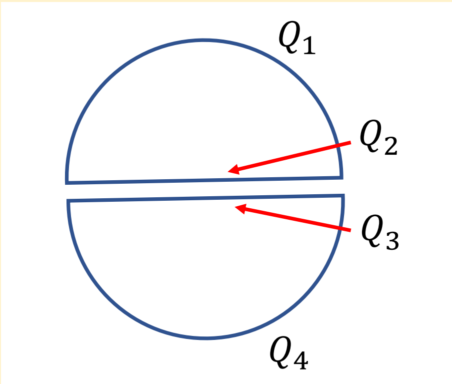
(b) [4 points] Find the electrostatic interaction force between two hemispheres.
(b) [4 分] 找出两个半球之间的静电相互作用力。

(c) [3 points] Find the electrostatic energy of the sphere.
(c) [3 分] 求球体的静电能。

Solution:

(a) Inside the metallic hemispheres, the electric field should be zero. Next, consider a Gauss surface surrounding the insulating layer. The electric flux is zero and hence the net charge inside the close surface is zero. This means the two sides of the insulating layer have the same amount but opposite charge. And the insulating layer is negligibly thin, that means the charge doesn't contribute any electric field except for the small space between the charge, just like two infinite-large parallel conductors.

The electric field is zero inside the conducting sphere. This can only be done if the charge uniformly distributes on the outer surface.



We have the conditions:

$$Q_1 = Q_4$$
$$Q_1 + Q_2 = Q$$
$$Q_2 = -Q_3$$
$$Q_3 + Q_4 = 0$$

We get

$$Q_1 = Q_2 = Q_4 = \frac{Q}{2}$$
$$Q_3 = -\frac{Q}{2}$$

And the charge distribution are

$$\sigma_1 = \sigma_4 = \frac{Q}{4\pi R^2}$$

$$\sigma_2 = \frac{Q}{2\pi R^2}$$
$$\sigma_3 = -\frac{Q}{2\pi R^2}$$

(b) The electric pressure (electric force per unit area) on the surface of a conductor is given by

$$P = \sigma E_{ext} = \frac{\sigma^2}{2\epsilon_0}$$

The net force acting on the upper hemisphere is

$$F_{upper} = (P_1 - P_2)\pi R^2 = \frac{\pi R^2}{2\epsilon_0}(\sigma_1^2 - \sigma_2^2) = \frac{\pi R^2}{2\epsilon_0}\left(\frac{Q^2}{16\pi^2 R^4} - \frac{Q^2}{4\pi^2 R^4}\right) = \frac{Q^2}{2\epsilon_0 \pi R^2}\left(\frac{1}{16} - \frac{1}{4}\right)$$
$$= -\frac{3Q^2}{32\epsilon_0 \pi R^2}$$

The force is acting downward.

For the lower hemisphere is, we have

$$F_{lower} = (P_3 - P_4)\pi R^2 = \frac{3Q^2}{32\epsilon_0 \pi R^2}$$

which is acting upward. Hence the electrostatic (attractive) force between two hemispheres is $F = \frac{3Q^2}{32\epsilon_0 \pi R^2}$.

(c) We can calculate the electrostatic energy of the sphere using the electric field energy density $u = \frac{1}{2}\epsilon_0 E^2$.

Inside the insulating layer, the energy is

$$U_1 = \frac{1}{2}\epsilon_0 \left(\frac{\sigma_2}{\epsilon_0}\right)^2 \times \pi R^2 d = \frac{Q^2}{8\epsilon_0 \pi R}\left(\frac{d}{R}\right)$$

where $d \ll R$ is the thickness of the layer.
Inside the conductor, the electric field is zero and there is no energy associated with the field.

$$U_2 = 0$$

Outside the sphere, the electric field is

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}$$

And the total energy is

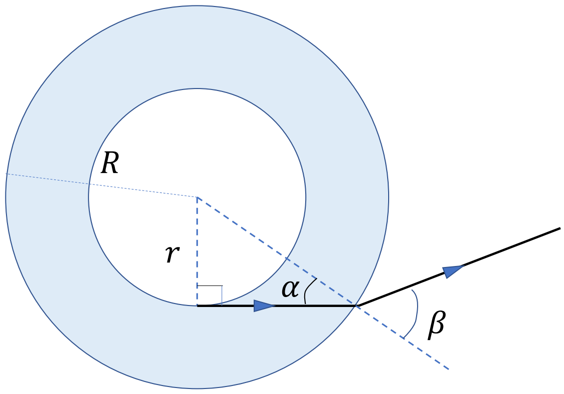
$$U_3 = \frac{1}{2}\epsilon_0 \int \frac{Q^2}{16\pi^2 \epsilon_0^2 r^4} 4\pi r^2 dr = \frac{Q^2}{8\pi\epsilon_0}\int_R^\infty \frac{1}{r^2} dr = \frac{Q^2}{8\epsilon_0 R}$$

The total electrostatic energy of the sphere is

$$E = E_1 + E_2 + E_3 \approx \frac{Q^2}{8\pi\epsilon_0 R}$$

4. **[10 points]** Figure 4 shows a hollow glass tube with outer radius R and inner radius r respectively. The refractive index of the glass is n. From the outside air, the apparent inner radius of the tube is r'(i.e. the radius of the hollow portion observed from outside appears to be equal to r').

4. [10 分] 圖 4 表示一支中空玻璃管，其外半徑為 R，內半徑為 r，玻璃折射率為 n。由外面空氣中看來，該管之視內半徑為 r' (即空心部分的半徑從外面看起來等於 r')。



(a) [2 points] Find the ratio of the actual inner radius to the outer radius $\frac{r}{R}$. Express your answer in terms of α, β and n.
(a) [2 分] 求真內半徑與外半徑的比值 $\frac{r}{R}$，以α, β 和n来表示。

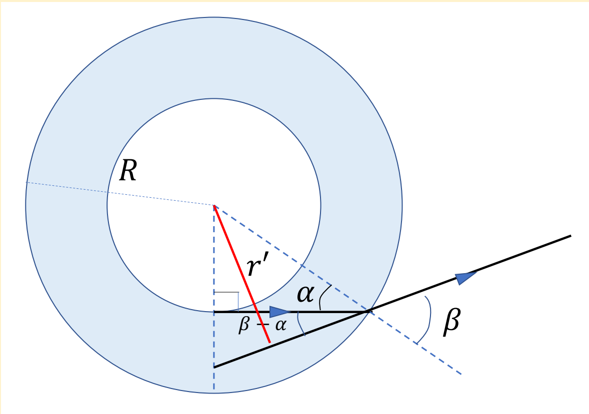
(b) [3 points] Find the ratio of the apparent inner radius to the outer radius $\frac{r'}{R}$. Express your answer in terms of α, β and n.
(b) [3 分] 求視內半徑與外半徑的比值 $\frac{r'}{R}$，以α, β 和n来表示。

(c) [5 points] If $R = 4.0$ mm, $r' = 0.50$ mm and $n = 1.6$, calculate the actual inner radius r of the glass tube up to 2 significant figures.
(c) [5 分] 若$R = 4.0$ mm， $r' = 0.50$ mm，$n = 1.6$，計算玻璃管的真內半徑r，答案準確至兩位有效數字。

Solution:
(a)

$$\frac{r}{R} = \sin\alpha$$

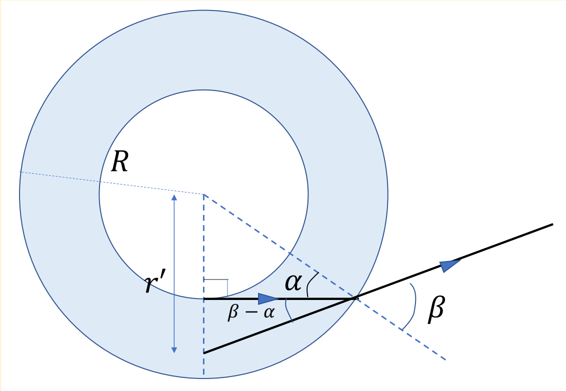
(b)



$$r' = R\sin\beta$$

$$\Rightarrow \frac{r'}{R} = \sin\beta = n\sin\alpha$$

Due to a different interpretation of the apparent radius, we also accept the following solution.



From the figure, we have

$$n\sin\alpha = \sin\beta$$

$$\frac{r'}{R} = \sin\alpha + \cos\alpha\tan(\beta-\alpha) = \frac{\sin\alpha\cos(\beta-\alpha)+\cos\alpha\sin(\beta-\alpha)}{\cos(\beta-\alpha)} = \frac{\sin\beta}{\cos(\beta-\alpha)} = \frac{n\sin\alpha}{\cos(\beta-\alpha)}$$

(c) It is given that

$$\frac{r'}{R} = 0.125 = 1.6\sin\alpha \Rightarrow \frac{r}{R} = \sin\alpha = \frac{0.125}{1.6} = 0.078125$$
$$\Rightarrow r = 0.31\text{ mm}$$

According to the other answer in part (b), we have

$$\frac{r'}{R} = 0.125 = 1.6\frac{\sin\alpha}{\cos(\beta-\alpha)}$$

$$\Rightarrow \frac{\sin\alpha}{\cos(\beta-\alpha)} = \frac{0.125}{1.6} = 0.078125 = x \ll 1 \qquad [4.1]$$

Since $\sin\alpha = \frac{r}{R} < \frac{r'}{R} = 0.125$ and $\sin\beta = n\sin\alpha < 1.6\times 0.125 = 0.2$ are small, we can approximate

$$\frac{\sin\alpha}{\cos(\beta-\alpha)} \approx \frac{\alpha}{1-\frac{1}{2}(\beta-\alpha)^2} \approx \frac{\alpha}{1-\frac{1}{2}(n-1)\alpha^2} = x$$

$$\Rightarrow \frac{1}{2}(n-1)x\alpha^2 + \alpha - x = 0$$

$$\alpha = 0.07798$$

$$\Rightarrow \frac{r}{R} = 0.07798$$

$$r = 0.31\text{ mm}$$

If you solve the equation [4.1] exactly, you will get $\alpha = 0.07812$ and $r = 0.312$ mm

Pan Pearl River Delta Physics Olympiad 2020
2020 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
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Simplified Chinese Part-2 (Total 2 Problems, 60 Points)
简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 8 August 2020)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。

All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。

There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。

Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。

Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。

Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。

If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。

At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Precision measurement of the gravitational constant *G* (27 points)
问题 1：重力常数 G 的精确测量（27 分）

The precision measurement of the gravitational constant G is important because it is a fundamental constant. Besides, it can play a role in verifying (or disproving) some recent proposed versions of string theory or the existence of the fifth fundamental force.
引力常数 G 是一个基本物理常数，所以对其精确测量非常重要。此外，引力常数的精确测量也有助于验证（或排除）一些弦理论的版本或第五种基本力的存在。

Part A. Estimation of the Gravitational Field Change During the Experiment (15 points)
对实验中引力场变化的估计 （15 分）

A1	Let R and M be the radius and the mass of the Earth, respectively, express the gravitational field g in terms of R, M, and G, while ignoring the spinning of the Earth. 设 R 和 M 分别是地球的半径和质量。忽略地球自转，求引力场的强度 g，用 R，M 和 G 表示。	1 point 1 分

In this problem we present a simplified version of the latest method, which can lower the relative error of G down to 10^{-5}. As shown in Fig. 1(a), laser interferometer 1 measures the spacing between the two pendulum bobs with respect to the reference spacing between the suspension points of the pendulum, which is measured by laser interferometer 2. When the four source masses are moved from the outer position (shown in Fig. 1(a)) to the inner position (shown in Fig. 1(b)), the pendulum bob separation changes. Not pictured is the vacuum chamber that encloses the pendulums but not the source masses.

本题简化地讨论一个测量引力常数的最新实验，可以将 G 的相对误差降低到 10^{-5}。如图 1(a) 所示，激光干涉仪 1 测量两个单摆摆锤之间的距离。激光干涉仪 2 测量两个单摆悬挂点之间的距离作为参照。当四个质量源从外侧位置（如图 1(a)所示）移动到内侧位置（如图 1(b)所示）时，两个摆锤之间的距离会改变。除了质量源外，整个实验装置置于真空环境内（没有画在图中）。

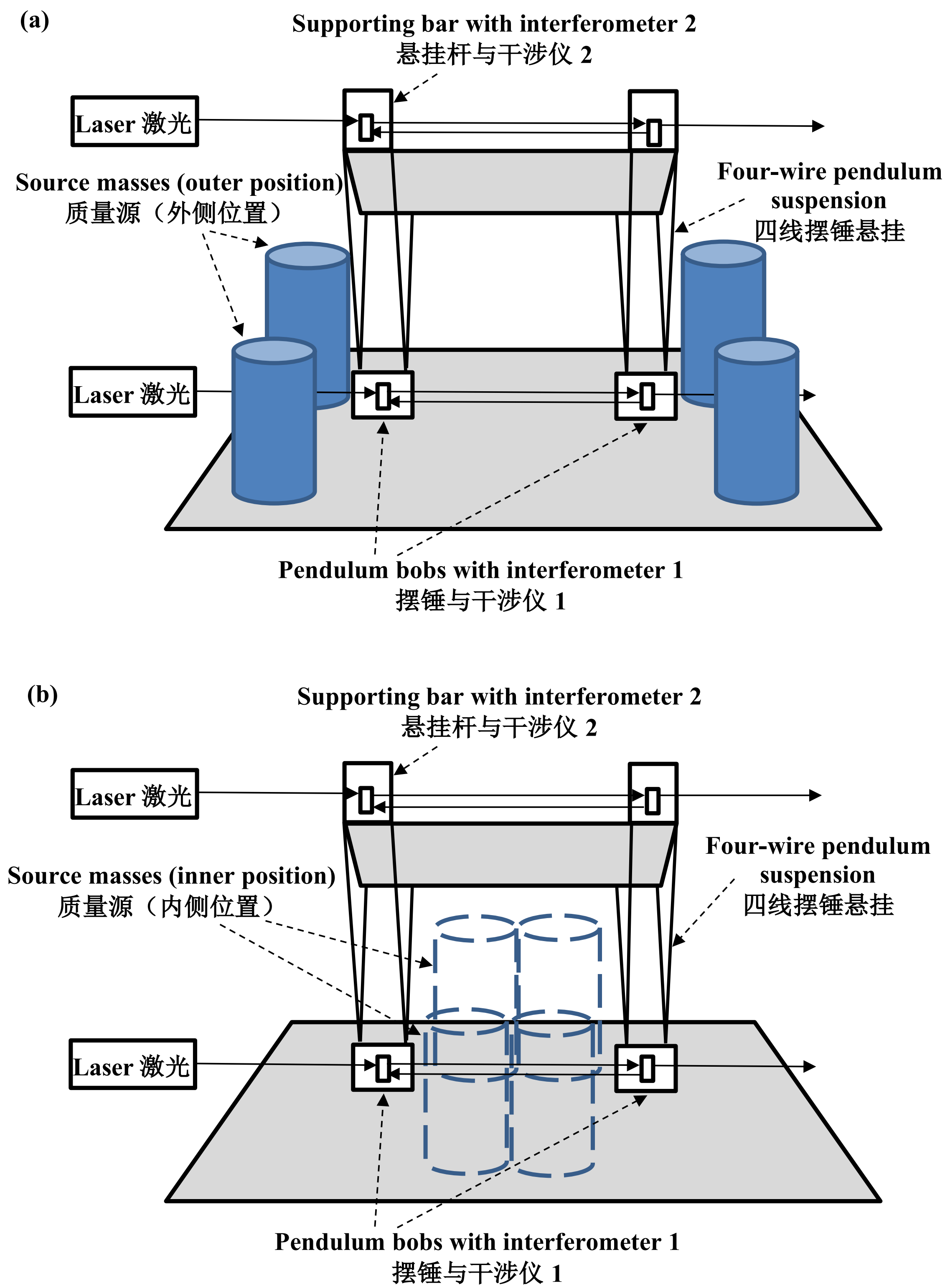


Figure 1: The setup of apparatus for measuring the gravitational constant precisely, with the source masses placed at (a) outer positions, (b) inner positions. 引力常数精确测量的实验装置，质量源置于 (a) 外侧位置，(b) 内侧位置。

Figures 2 and 3 show the top and side views of the apparatus. The outer and inner positions of the source masses, and the pendulum bobs (at the ends of interferometer 1) are located symmetrically with respect to the center of the vacuum system. The length scales a_1, a_2, b, d, h and R are shown in the figures.
图 2 和图 3 是实验装置的顶视图和侧视图。质量源的外侧和内侧位置，和摆锤（在干涉仪 1 的末端）的位置相对于真空系统的中心是对称的。图中显示距离 a_1, a_2, b, d, h 和 R。

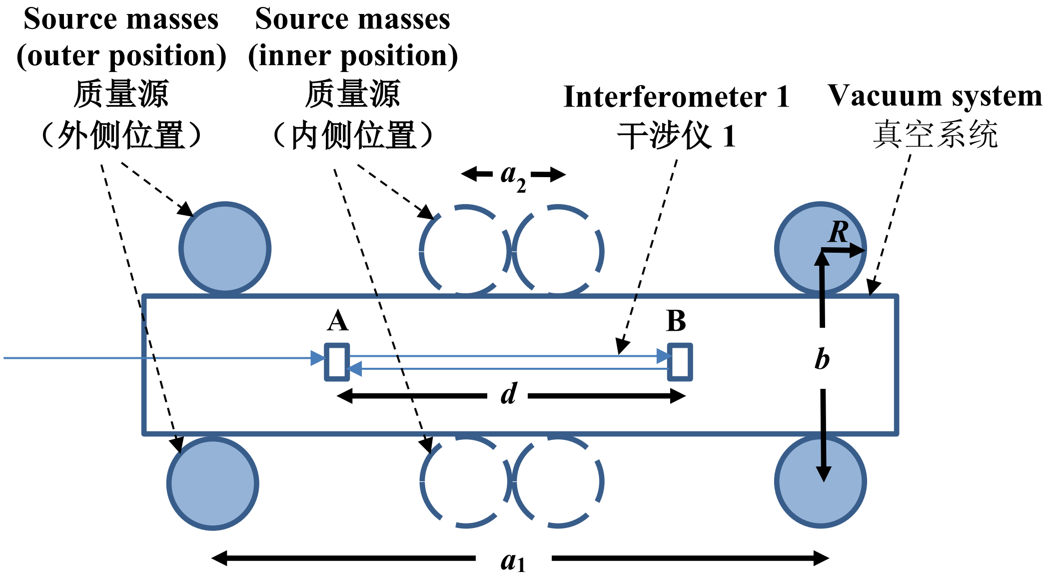


Figure 2: Top view of the apparatus. 仪器的顶视图。

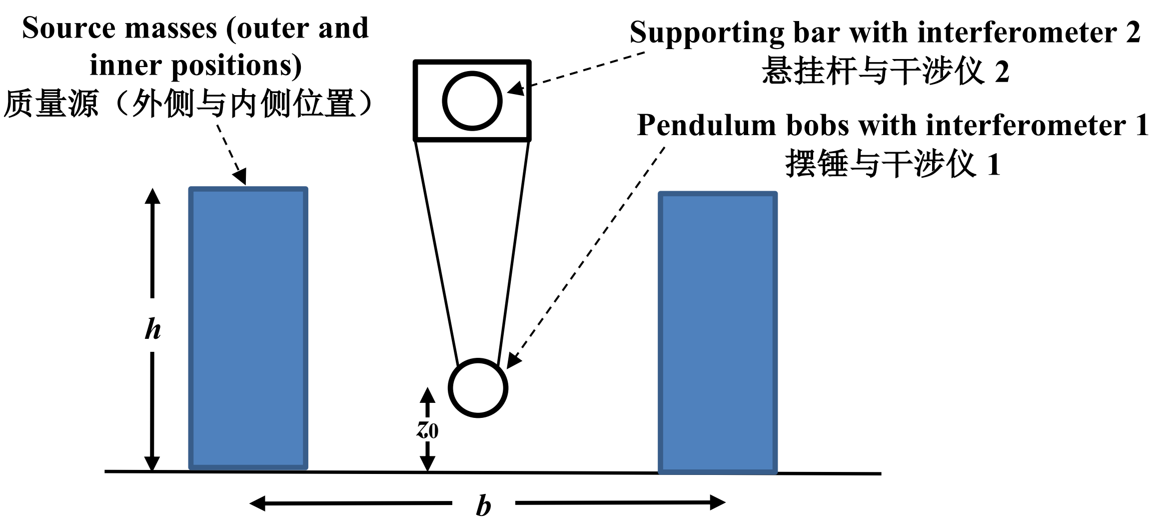


Figure 3: The side view of the apparatus. 仪器的侧视图。

It is very complicated to calculate the gravitational force of the 4 cylindrical source masses acting on pendulum bob A. Here we approximate each cylinder with uniform density, mass M, radius R and height h to be a thin wire with uniform density, mass M and height h passing through the axis of the cylinder.

计算四个圆柱形质量源对摆锤 A 的引力非常复杂。这里，我们将每个密度均匀、质量为M、半径为 R、高度为 h 的圆柱近似为沿圆柱轴向放置的细线，密度均匀、质量为 M、高度为 h 。

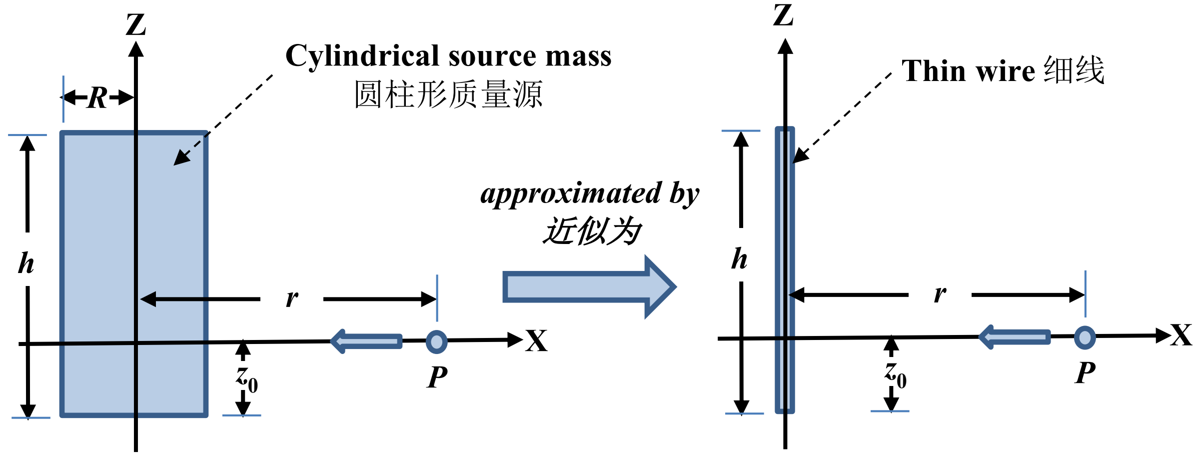


Figure 4: The X-component of the gravitational field at point P due to a cylindrical source mass is now approximated by that due to a thin wire. 计算在 P 点引力场的 X 分量时，将圆柱形质量源近似为细线。

You are provided the integral formula:
你可以使用以下积分公式：

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C,$$

A2	Derive an expression of the X-component of the gravitational field g_x at point P due to the thin wire. Express your answer in terms of G, M, h, z_0, and r. 推导由细线产生的引力场在 P 点的 X 分量 g_x。用 G, M, h, z_0 和 r 表示你的结果。	3 points 3 分

A3	Calculate the gravitational field g_x at point P when the point is very near the thin wire. 当点 P 非常接近细线时，计算引力场在 P 点的 X 分量 g_x。	1 point 1 分

A4	Applying Gauss' law, verify the result in part A3. Write your steps in the answer sheet. 应用高斯定律，验证 A3 中的结论。在答题纸上写下过程。	1 point 1 分

You are provided the following parameters:
你可以使用以下参数：

$G = 6.67 \times 10^{-11}$ Nkg^{-2}m^2	$M = 119.1$ kg	$a_1 = 0.568$ m
$a_2 = 0.166$ m	$b = 0.262$ m	$d = 0.34$ m
$h = 0.312$ m	$z_0 = 0.002$ m	$R = 0.083$ m

A5	Using the given parameters, and the thin wire approximation for the 4 cylindrical source masses, calculate the horizontal component of the gravitational field due to the 4 source masses at the position of pendulum bob A, when the source masses are located at the inner position. 使用上面给出的参数，以及用细线近似四个质量源。当质量源处于内侧位置时，计算摆锤 A 处四个质量源产生的引力场的水平分量。	3 points 3 分

A6	Similar to part A5, calculate the horizontal component of the total gravitational field at the position of pendulum bob A, when the source masses are located at the outer position. 与 A5 部分类似，当质量源处于外侧位置时，计算摆锤 A 处总引力场的水平分量。	3 points 3 分

A7	Calculate the change Δg_x of the horizontal component of the gravitational field at the position of pendulum bob A, when the source masses are moved from the outer position to the inner position. 当质量源从外侧位置移到内侧位置，计算摆锤 A 处引力场水平分量的变化Δg_x。	1 point 1 分

In this experiment, care has to be taken to monitor the uncertainties of measurements. Since the calculation is complicated, we will simply focus on the expression derived in Part A3. Consider the case that all mass measurements have an uncertainty of 0.6 parts in 10^5, and all dimension measurements have an uncertainty of 1.4 parts in 10^5.

本实验需要小心考虑实验误差。完整计算比较复杂，为了简化，我们使用 A3 部分中推导的表达式。假设所有质量测量的误差均为 0.6×10^{-5}，所有长度测量的误差均为 1.4×10^{-5}。

Remark: The uncertainty of a physical quantity $f(x_1, x_2, \ldots)$ is given by the standard deviation of f divided by f, that is, $\sqrt{\sigma_f^2/f^2}$, where $x_1, x_2, \ldots$ are independent measurements and σ_f^2 is calculated from $\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_2^2 + \cdots$.

注：物理量 $f(x_1, x_2, \ldots)$的误差为 f 的标准差除以 f 本身，即 $\sqrt{\sigma_f^2/f^2}$，其中 $x_1, x_2, \ldots$为独立测量的物理量，σ_f^2 可以用 $\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_2^2 + \cdots$ 计算。

A8	Calculate the uncertainty of g_x in Part A3. 计算 A3 部分中 g_x 的误差。	2 points 2 分

Part B. Measurement of the Gravitational Field Change During the Experiment (15 points)
测量实验中的引力场改变（15 分）

In Part A we estimated the gravitational field change when the source masses are moved from the outer to the inner position, assuming a certain value of the gravitational constant G. In this part we investigate how the gravitational field change can be measured experimentally, such that the gravitational constant G can be calculated. We let:

Δx = the horizontal displacement of the pendulum bob A when the source masses are moved from the outer to the inner position,
ω = the angular frequency of the natural oscillations of the pendulum bob,
m = mass of the pendulum bob.

在 Part A 中，我们假设给定引力常数 G，估算了质量源从外侧移到内侧时引力场的改变。本部分中，我们将研究如何在实验中测量引力场的改变，以及如何因此测量引力常数 G。我们定义三个量：

Δx = 当质量源从外侧移到内侧时，摆锤 A 在水平方向的偏移量，
ω = 单摆自然振动的角频率，
m = 摆锤的质量。

B1	Derive the expression of the change Δg_x in the horizontal component of the gravitational field at the position of pendulum bob A, when the source masses are moved from the outer position to the inner position. Express the result in terms of the 3 variables above. 当质量源从外侧移到内侧，求摆锤 A 处引力场水平分量的改变 Δg_x。用以上三个量表示结果。	1 point 1 分

The pendulum bobs are hung from the supporting bar at a distance $l = 0.738$ m vertically below the bar. The gravitational acceleration is $g = 9.8$ ms^{-2}.

摆锤从顶部悬挂下来，与悬挂杆的垂直距离为 $l = 0.738$ m。引力加速度是 g = 9.8 ms^{-2}。

B2	Using your result in part A7, calculate the change in the separation of the pendulum bobs. 利用 A7 部分的结果，计算摆锤位置的变化量。	1 point 1 分

There are corrections to the answer in B2 because besides the pendulum motion, there are other contributions to ω^2 such as the flexing of the wires. The frequencies were found to be (0.589 8171 ± 0.000 0023) Hz for one bob and (0.589 7069 ± 0.000 0013) Hz for the other, where the number following the plus-minus sign is the standard derivation of the quantity.

B2 中的答案还没有加入校正，因为除了单摆运动之外，还有其它因素影响ω^2，例如摆线的弹性。对一个摆锤，频率的测量值为 (0.589 8171 ± 0.000 0023) Hz，对另一个摆锤，频率的测量值为 (0.589 7069 ± 0.000 0013) Hz，其中正负号后面的数字为该量的标准差。

B3	Calculate the mean and the uncertainty of the average value of the pendulum frequency. 计算单摆频率的平均值，以及单摆频率平均值的误差。	2 points 2 分

B4	If the error of time measurement is 10^{-5} s, determine the number of periods to be measured such that the uncertainty of the measured period is 10^{-7}. 假设时间测量的绝对误差为10^{-5}秒，为了达到误差为 10^{-7} 的周期测量精度，求需要测量的周期数量。	1 point 1 分

The horizontal displacement of the pendulum bobs can be measured with high precision using the laser interferometer (commonly known as Fabry-Pérot interferometer). The cavity of the interferometer is formed by two highly reflective mirrors separated at a distance d = 0.34 m so that the laser team traveling between them forms a standing wave. The laser wavelength is 633 nm. When the spectrum is closely examined, one finds that it consists of a sequence of peaks as shown in Fig. 5.

摆锤的水平位移可以通过激光干涉仪（法布里-珀罗干涉仪）精确测量。干涉仪腔体的两端分别放置了一块反射系数很高的镜片。两镜片相距d = 0.34 m。激光在两镜片之间形成驻波。激光的波长为633 nm。如图 5 所示，干涉仪的光谱由一系列峰组成。

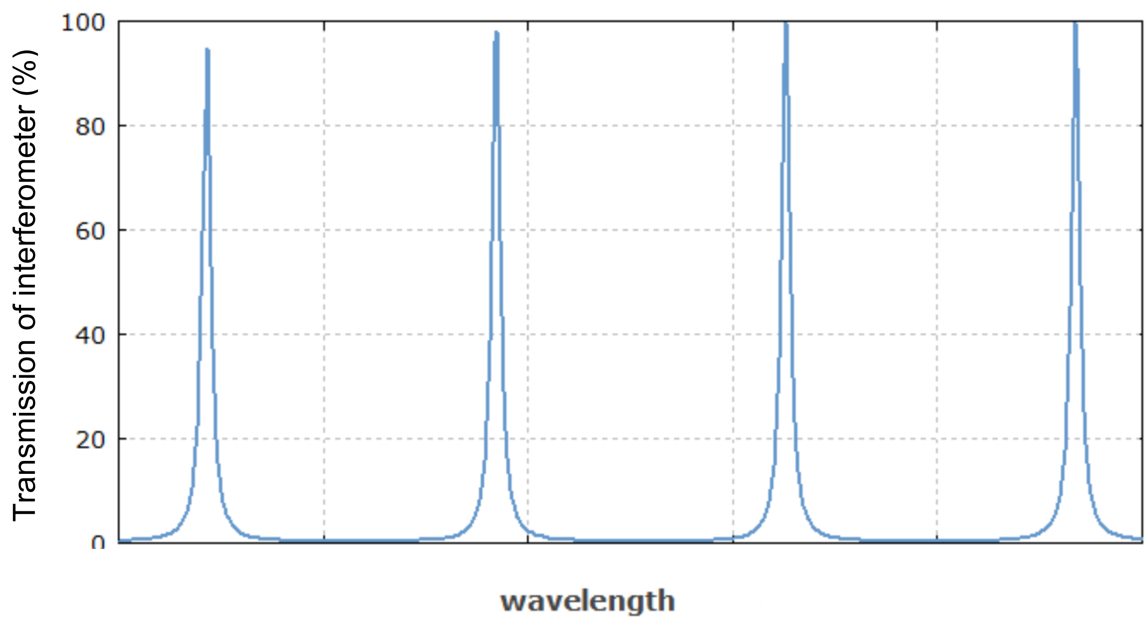


Figure 5: A sketch of the close-up of the spectrum of the interferometer. 干涉仪中光谱的细致结构示意图。

B5	Calculate the frequency separation of the neighboring peaks in the spectrum. Then calculate the wavelength separation of the neighboring peaks in the spectrum. 计算光谱中相邻两峰之间的频率差，并计算光谱中相邻两峰之间的波长差。	2 points 2 分

The width of the fringes in Fig. 5 is about 100 kHz.
图 5 中，条纹的宽度约为 100kHZ。

B6	Assuming that the major mechanism of power loss of the standing wave is the transmission through the mirrors, estimate the fraction of power loss from the interferometer per transmission. 假设驻波中主要的能量损失来自穿过镜面的透射。估算每次透射中能量损失占总能量的比例。	2 points 2 分

B7	Suppose the length of the interferometer changes by 1 nm. Calculate the corresponding change in the cavity frequency. 设干涉仪长度改变 1 nm，求腔体中频率的相应改变。	2 points 2 分

When the source masses are moved from the outer to the inner position, the beat frequency of the cavity frequencies of the two interferometers changes by 125 MHz.
当质量源从外侧移到内侧时，两个干涉仪腔体中频率的拍频改变了 125 MHz。

B8	Calculate the change in the separation of the pendulum bobs. (Remark: This result will be different from that in B2 due to the approximations made in Part A.) 计算两个摆锤间距离的改变。（注：由于 Part A 中做的近似，这个结果将与 B2 中的不同。）	1 point 1 分

Problem 2: Physics in Various Dimensions (33 points) 不同维度的物理 (33 分)

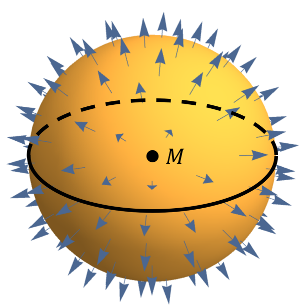
"What had she experienced? She had seen how a cruel attacker could lower the dimensions of space by one and destroy a solar system. What are dimensions?" -- Cixin LIU, *Death's End* (Translation: Ken LIU)

"她经历过什么？她刚刚看到，为了毁灭一个恒星系。残忍的攻击者把那里的空间维度降低了一维。空间维度，空间维度是什么？"—— 刘慈欣《三体 • 死神永生》

In this problem, we will explore 4-dimensional space, attack from 4-dimensional to 3-dimensional spaces, and the motion of celestial bodies in two dimensions. (Note: In this problem, whenever we mention the number of dimensions, we mean spatial dimensions and did not count in the time dimension. For example, when we mention 3-dimensional space, we mean the spacetime with 3 space dimensions and 1 time dimension). The setup of the problem is as follows:

本题中，我们将讨论四维空间、从四维到三维的维数攻击，以及二维空间中的天体运动问题。（注意：本题提到空间维度的数量时，并没有把时间维计算在内。例如，当我们提到三维空间时，指的是三维空间加上一维时间组成的时空。）问题设定如下：

- Let the mass of the star be M， the mass of the planet be $m \ll M$， So in the star-planet problems, the star can be considered at rest. The change of star position due to planet motion can be neglected.
- In n-dimensional space, Newtonian gravity can still be derived from Gauss flux law: For any (n-1)-dimensional closed surface enclosing the point particle M, the gravitational acceleration flux through the surface is $-A_{n-1}GM$, where A_{n-1} is the area of a $(n-1)$-dimensional unit sphere (understood as generalized area, for example, for $n=2$, $A_1 = 2\pi$ is length, for $n=3$, $A_2 = 4\pi$ is area, and for $n=4$, $A_3 = 2\pi^2$ is volume). The meaning of flux is: for a small area element, the flux is the dot product of the gravitational acceleration vector and the area vector. For the case when the gravitational acceleration is normal to the surface, flux is the magnitude of gravitational acceleration times this area. For example, the figure below illustrates how to derive Newtonian gravity from Gauss flux law for the case of 3 spatial dimensions.



Newtonian gravity from
Gauss flux law in 3 dimensions:

(flux through spherical surface)
$= (F/m) \times 4\pi r^2 = -A_2 GM$

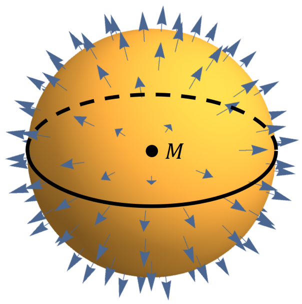
$\Rightarrow F = -GMm/r^2$

(Note: the Gauss flux law does not apply for general relativity.)

- Newton's three laws of motion still holds. Momentum conservation and angular momentum conservation laws still holds. The relativity of motion still holds.
- We approximate the stars and planets as particles, with negligible radius.

- 设恒星质量为 M，行星质量 $m \ll M$，故在行星围绕恒星运动问题中，恒星可以视为静止，行星运动对恒星位置的影响可以忽略。

- 在 n 维空间，牛顿引力可以由高斯引力通量定律推导出来，即对于包括了质点 M 的任何 $n-1$ 维闭合曲面，穿出此闭合曲面的引力加速度通量等于 $-A_{n-1}GM$，其中 A_{n-1} 为 $(n-1)$ 维单位球面的面积（理解为广义的面积，例如在 $n=2$ 情况下 $A_1=2\pi$ 为长度，$n=3$ 情况下 $A_2=4\pi$ 为面积，$n=4$ 情况下$A_3=2\pi^2$为体积）。通量的意思是：对一个小面积元，通量是引力加速度矢量点乘有向面积元。对引力加速度垂直于面积元的情况，通量为引力加速度的大小乘以面积元的面积。例如，下图中，对于三维空间，我们用高斯引理通量定律推导了牛顿引力定律。



三维空间中用高斯引力通量定律推导牛顿引力定律

(穿过球面的引力加速度通量) $=(F/m)\times 4\pi r^2=-A_2GM$

$\Rightarrow F=-GMm/r^2$

（注意：高斯通量定律并不适用于广义相对论。）

- 牛顿三定律成立。动量守恒、角动量守恒定律成立。运动的相对性成立。
- 所有星球视为质点，其半径足够小，可以忽略。

Part A. High-Dimensional World (13 points) 高维的世界（13 分）

In the whole part A, we consider Newtonian gravity in n-dimensional $(n\geq 4)$ space. Considering that the gravitational acceleration vector is parallel to the position vector, the orbit of the planet is within the 2-dimensional plane determined by the position and velocity vectors of the planet.

在整个 Part A 中，我们考虑 n $(n\geq 4)$ 维空间中的牛顿引力。由于引力加速度与位置矢量共线，行星围绕恒星的运动轨迹处于行星的位置矢量与速度矢量所决定的二维平面内。

A1	Applying the Gauss flux law to a $(n-1)$-dimensional sphere, one can derive the Newtonian law of gravity in n space dimensions. Due to attraction from the star, the gravity force exerted on the planet is $F=-GMmr^{\alpha}$ (the direction of the force points to the star); the gravitational potential energy is $V(r)=GMm\,r^{\beta}/\beta$, where r is the distance between the star and the planet. Express α,β in terms of n. 将高斯引力通量定律应用于 $(n-1)$ 维球面，可以推导 n 维空间中的牛顿引力定律。由于受到恒星吸引，行星受到的引力为 $F=-GMmr^{\alpha}$ （力的方向指向恒星），引力势能为 $V(r)=GMm\,r^{\beta}/\beta$，其中 r 为行星与恒星的距离，求 α,β（用 n 表示）。	**2 point** **2 分**

A2	The Kepler's second law in n dimensions is: $L = mr^{\lambda}\dot{\varphi}$ is a constant (where φ is the angle between the planet-star plane and the x-axis of the motion plane, $\dot{\varphi} \equiv d\varphi/dt$). Find λ. n 维的开普勒第二定律为： $L = mr^{\lambda}\dot{\varphi}$ 为常数（φ 为行星-恒星连线与其运动平面的 x 轴之间的夹角， $\dot{\varphi} \equiv d\varphi/dt$）。求 λ。	**1 point** **1 分**

A3	Calculate the speed of the planet along the radius direction $\dot{r} \equiv dr/dt$. Express your result using n, r, G, M, m, E, L, where E is the energy of the planet (including kinetic energy and gravitational potential energy). 求行星的径向速度 $\dot{r} \equiv dr/dt$，用 n, r, G, M, m, E, L 表示，其中 E 为行星的能量（包括动能和引力势能）。	**3 points** **3 分**

A4	Give the conditions for the planet to form a circular orbit (give algebraic equations using n, r, G, M, m, E, L, no need to solve these equations). 给出行星沿圆轨道运动的所有条件（给出关于 n, r, G, M, m, E, L 的代数方程组即可，不必解方程）。	**2 points** **2 分**

A5	For $n = 4$, when the values of r, G, M, m, E, L are such that a circular orbit is possible, and r is varied while other parameters are fixed, can the planet (1) form elliptical orbits? (2) move from finite r to $r \to \infty$? (3) move from finite r to $r \to 0$? 当 $n = 4$，并且 r, G, M, m, E, L 的取值使得行星有可能形成圆轨道时，固定其它参数而 r 取不同的值时，行星是否有可能 (1) 形成椭圆轨道 (2) 从有限的 r 运动到 $r \to \infty$ (3) 从有限的 r 运动到 $r \to 0$	**1.5 points** **1.5 分**

A6	For $n > 4$, when the values of n, r, G, M, m, E, L are such that circular orbit is possible, and r is varied while other parameters are fixed, can the planet (1) form elliptical orbits? (2) move from finite r to $r \to \infty$? (3) move from finite r to $r \to 0$? For the possible cases in the above questions, please state the possible range of r given n, r, G, M, m, E, L are fixed. (If the limit of the range is one of the roots of an algebraic equation, please specify which root without the need to solve the equation explicitly.) 当 $n > 4$，并且 n, r, G, M, m, E, L 的取值使得行星有可能形成圆轨道时，固定其它参数而 r 取不同的值时，行星是否有可能 (1) 形成椭圆轨道 (2) 从有限的 r 运动到 $r \to \infty$ (3) 从有限的 r 运动到 $r \to 0$ 对于上面回答为可能的情况，在 n, r, G, M, m, E, L 固定的情况下，请指出 r 的取值范围。（如果 r 取值范围的边界为代数方程的一个根，请指出哪个根，但不必解这个方程。）	**3.5 points** **3.5 分**

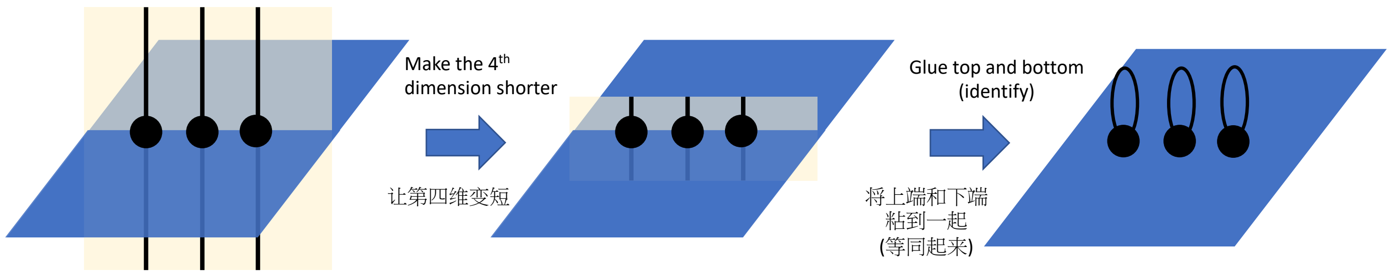
Part B. Dimensional attacks in Newtonian gravity (5 points) 牛顿引力下的维度打击 (5 分)

Suppose aliens living in 4 spatial dimensions perform dimensional attack on enemies also living in 4 spatial dimensions. The way of dimensional reduction is to reduce one space dimension into a small circle with circumference length C. In this way, in fact, there are still 4 spatial dimensions. But for small C values, seeing from far away, one cannot see the dimension of the small circle. As a result, seeing from far away, the enemies appear to be living in 3 spatial dimensions. We call such spatial regions under dimensional attack having “effectively” three spatial dimensions.

假设四维空间中的生物向同样生活在四维的敌方实施维度打击。降维的方法是，将一个空间维度缩小成周长为 C 的小圆环。这样，其实空间还有四维。但是当 C 很小时，从远处看来，看不到这个圆环代表的维度。这样，从远处看来，敌方就好像生活在三维空间一样。我们称这时遭受维度打击的空间区域“有效”维度为三维。

As illustrated in the below figure, suppose the vertical dimension is the dimension under dimensional attack. The horizontal plane denotes the remaining three-dimensional space. Before the dimensional attack, the four-dimensional space can be considered as, in a three-dimensional space, on every point there is one straight line indicating the fourth dimension. After the dimensional attack, on every point there is a small circle with circumference C indicating the fourth dimension. The circle indicates that the top and bottom endpoints of an interval are glued.

如下图所示，假设垂直方向的维度为实施维度打击的维度，水平的平面代表剩余的三维空间。维数打击前，四维空间可以看成：三维空间的每个点上都有一条直线代表第四维。维数打击后，三维空间的每个点上都有一个周长为 C 的小圆环代表第四维。圆环的含义为将一条线段的上下两个端点粘起来。



Assume that during the dimensional attack (assume the duration is short enough), apart from the sudden change of the law of gravity, a system under dimensional attack does not have additional forces exerting on it. The momentum in the 4-dimensional point of view does not change. The law of gravity can be understood in two ways: the effectively three-dimensional gravity and the more fundamental four-dimensional gravity. The values of a gravitational force computed using these two methods (using Gauss flux law) agree with each other.

假设在维度打击过程中（假设此过程持续时间足够短），除了引力定律的形式忽然改变外，被打击的系统不受额外的力作用，四维观点下的动量不变。引力定律受维度打击的影响体现为:维数打击后可以从两种观点理解引力，有效的三维引力和更基本的四维引力，通过高斯定律用这两种观点算出的引力大小相等。

B1	Consider a point particle at rest, which exerts Newtonian gravity upon other objects (whose distance is much greater than C). Find the relation between the four-dimensional Newtonian gravitational constant G_4 and the effective three-dimensional Newtonian gravitational constant G_3 (in terms of $G_4 =$ (function of G_3 and other parameters)). 考虑一个静止质点对其它物体（距离远远大于 C ）产生的牛顿引力。求四维牛顿引力常数 G_4 和三维有效牛顿引力常数 G_3 之间的关系（用 $G_4 =$（G_3及其它参数的函数） 表示）。	**3 points** **3 分**

B2	Suppose the relativistic mass-energy relation $E^2 = \|\mathbf{p}\|^2 c^2 + m^2 c^4$ applies both for three and four dimensions. Here c denotes the speed of light. Suppose before the dimensional attack, a point particle with mass m has momentum $\mathbf{p} = (p_1, p_2, p_3, p_4)$. After dimensional attack, the 4^{th} dimension (corresponding to the subscript 4 above) becomes a small circle. Calculate the three-dimensional effective mass of the point particle after the dimensional attack. 假设相对论质能关系 $E^2 = \|\mathbf{p}\|^2 c^2 + m^2 c^4$ 在三维和四维都成立，其中 c 为光速。设维度打击之前，一个质量为 m 的质点的动量为 $\mathbf{p} = (p_1, p_2, p_3, p_4)$，维度打击后，第四个空间维度（对应上述角标 4）变成小圆环，求质点在维度打击后的三维空间中的有效质量。	**1 point** **1 分**

B3	Suppose a planet (with mass m) is moving along circular orbit in 4 space dimensions, with the star (with mass M) at the center. The distance between the planet and the star is r. Now dimensional attack this system along a direction perpendicular to the plane of planet motion. Calculate the energy of the planet after the dimensional attack (gravitational potential energy plus kinetic energy in the three-dimensional effective point of view). 设质量为 m 的行星在四维空间中以圆轨道绕质量为 M 的恒星运动，与恒星距离为 r。现对此星系沿垂直行星运动平面方向进行维度打击。求维度打击后，三维空间中行星的总能量（三维有效观点下的引力势能加动能）。	**1 point** **1 分**

Part C. The Two-Dimensional Newtonian World (4 points) 二维牛顿世界 (4 分)

We further reduce spatial dimensions and consider Newtonian gravity in two dimensions ($n = 2$).
我们进一步减少空间维数，考虑二维（$n = 2$）空间中的牛顿引力。

C1	Provide formulas for the Newtonian gravitational force law and the corresponding gravitational potential energy in two dimensional. 求二维的牛顿万有引力公式及引力势能公式。	**1 point** **1 分**

C2	Provide relations for planets to move along circular orbits around the star (give algebraic equations in terms of r, G, M, m, E, L. You don't need to solve the equation). 求行星绕恒星沿圆轨道运动的所有条件（给出关于 r, G, M, m, E, L 的代数方程组即可，不必解方程）。	**1 point** **1 分**

C3	When the values of r, G, M, m, E, L are such that circular orbit is possible, and r is varied while other parameters are fixed, can the planet (1) move in a bounded range $r_1 \le r \le r_2$, where $0 < r_1 < r_2 < \infty$ (2) move from finite r to $r \to \infty$? (3) move from finite r to $r \to 0$? Just answer possible or impossible for the above three questions (may need to discuss different cases for different parameter choices). 当 n, r, G, M, m, E, L 的取值使得行星有可能形成圆轨道时，固定其它参数而 r 取不同的值时，行星是否有可能 (1) 始终在有限的 $r_1 \le r \le r_2$ 区间内运动，其中 $0 < r_1 < r_2 < \infty$ (2) 从有限的 r 运动到 $r \to \infty$ (3) 从有限的 r 运动到 $r \to 0$ 对以上三问，分别回答能或不能即可（有可能要根据参数取值分情况讨论）。	**2 point** **2 分**

Part D. The Two-Dimensional Einstein World (11 points) 二维爱因斯坦世界 (11 分)

Interestingly, in the two-dimensional case, the "gravitational" law in Einstein's general relativity is even simpler than Newtonian gravity. In general relativity, there is no gravitational force at all between massive point particles (the space outside the particles is not curved either). Instead, the only gravitational effect of a point mass is that the space around it becomes conical (like a cone).

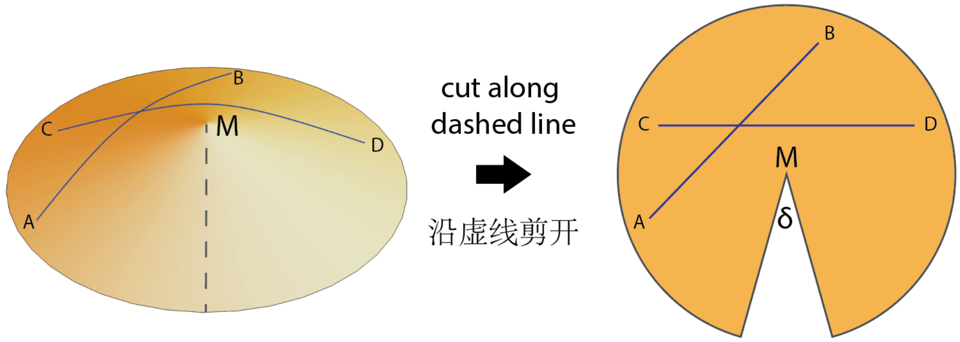
有趣的是，在二维空间情况下，爱因斯坦广义相对论中的"引力"规律比牛顿引力还要简单。广义相对论中，二维空间的质点之间根本没有相互吸引力（质点外的空间也没有弯曲）。而质量带来的效应为，一个质点周围的空间为圆锥形。

In the conical two-dimensional world, free particles and light will move along straight lines. Here straight lines are understood in the following way: if we cut the cone along a ray starting from the top vertex (not intersecting with the motion trajectory), and lay it on the plane as a circular sector, the motion trajectory is a straight line on the sector. For example, the lines AB and CD in the figure below.

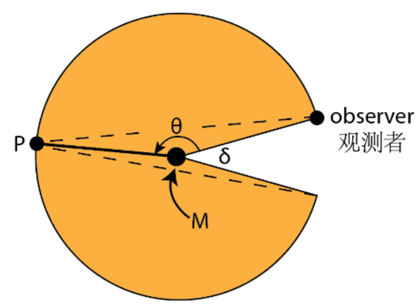
在圆锥形的二维世界中，自由粒子和光线将沿直线运动。这里的直线理解为，将圆锥沿着任意一条从顶点出发、与运动轨迹不相交的射线剪开，并在平面上摊平成为扇形后，运动轨迹在扇形上呈直线。例如下图中的直线 AB，CD。

When we cut the cone into a sector, there is a deficit angle (the angle that a sector lacks compared to a disk), denoted by δ as illustrated in the figure below. This deficit angle is proportional to the mass M of the point particle. Here we assume $\delta < \pi$.

把圆锥剪开成扇形时，扇形与圆盘相比所缺的角度（下图中的 δ）称为圆锥的缺陷角，与质点的质量 M 成正比。本题假设 $\delta < \pi$。

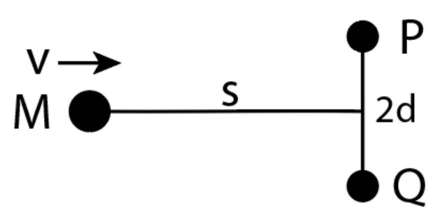


<table>
<tr><td rowspan="2">D1</td><td>Point P is moving along a circle surrounding point M (M is the center of the sector in the figure below). The angle between the observer-M line and the MP line (viewed anti-clockwise) is θ. Depending on different values of θ, sometimes the observer finds a single image of P and sometimes finds double images of P. For example, in the figure below, the observer can observe double images along the dashed lines. Find the condition for the observer to observe double images, and the angle between these two images (the angle between the two dashed lines in the figure).

点 P 围绕质点 M 做圆周运动(M 为图中扇形中心)。观测者在同一个圆周上观测 P 点的运动。观测者与 M 连线与 MP 连线（沿逆时针方向看去）的夹角为 θ。随 θ 取值不同，观测者有时看到 P 的单像，有时看到双像。例如右图，观测者可以沿图中虚线看到 P 的双像。求观测者能看到 P 双像的条件，以及双像间的夹角（即图中两虚线间的夹角）。

</td><td rowspan="2">2 points

2 分</td></tr>
</table>

<table>
<tr><td>D2</td><td>At $t = 0$, a point particle with mass M (with corresponding deficit angle δ) moves towards observers P and Q. The direction of motion is perpendicular to the PQ interval, and the speed is v (much smaller than the speed of light). The mass of P and Q are negligible. At the initial time $t = 0$, P and Q are at rest, distances MP = MQ, PQ = 2d, the distance between M and the PQ interval is s. Calculate the time t_m when P and Q meet.

质量为 M（对应角度缺陷 δ）的质点向两个质量可忽略的观测者 P、Q 运动，运动方向垂直于 PQ 连线，速率为 v（远小于光速）。在初始时刻 $t = 0$ 时，P、Q 静止，线段 MP = MQ，PQ = 2d，M 与 PQ 连线距离为s。求 P、Q 两观测者相遇的时间 t_m。

</td><td>2 points

2 分</td></tr>
</table>

D3	Consider the same setup as Problem D2. The observer P (starting from early enough time) continuously emits sound wave towards all directions. The source of sound has vibration frequency f. The speed of sound is c_s satisfying $\frac{c_s^2 - v^2\cos\delta}{c_s^2 - v^2} = \frac{13}{12}$ (use this relation to eliminate c_s from the result), and the wavelength of sound is much smaller than s and d. The media to propagate sound moves together with M (i.e. at rest with respect to M). Shortly before P and Q meet, the sound frequency that Q hears is f_r . Calculate f_r. 在与第 D2 题相同设定下，观测者 P（从时间足够早开始）持续向所有方向发出声波，声源的振动频率为 f，声波的速度 c_s 满足 $\frac{c_s^2 - v^2\cos\delta}{c_s^2 - v^2} = \frac{13}{12}$ （用此关系在结果中消去 c_s），声波的波长远小于 s 和 d。声波的传播介质跟随质点 M 运动（即与 M 相对静止）。 在 P 和 Q 即将相遇前，Q 听到的声音频率是 f_r。求 f_r。	**4 points** **4 分**

D4	Consider the same setup as Problem D3, starting from which time (i.e. find the corresponding t) on, Q starts to hear this frequency f_r? 在与 D3 题相同的设定下，从何时起（即计算其时间 t），Q 开始听到频率 f_r ？	**3 points** **3 分**

Note: In reality, although we do not note a danger under dimensional attack to two dimensions, it is still meaningful to study the physics in two dimensions. For example, in our three-dimensional universe, there probably exist one-dimensional objects called “cosmic strings”. The cosmic strings in three dimensions are similar to point particles in two dimensions. Both of them bring a deficit angle to space. The three questions in Part D corresponds to the three important observable effects of cosmic strings. Searching for cosmic strings using these three observable effects is an active interdisciplinary research direction between high energy physics and astronomy.

注：现实中，尽管我们还没有发现被维数打击降为二维的风险，研究二维的物理仍然是有意义的。例如，我们的三维宇宙中，可能存在一种叫“宇宙弦”的线状一维物体。三维空间中的宇宙弦，和二维空间中的质点类似，都会给空间带来一个缺陷角。Part D 中的三个问题，对应的就是宇宙弦的三个重要观测效应。用这三个观测效应探测宇宙弦，是高能物理和宇宙学研究中的一个活跃的交叉领域。

Pan Pearl River Delta Physics Olympiad 2020
2020 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助

Simplified Chinese Part-2 (Total 2 Problems, 60 Points)
简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 8 August 2020)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。

All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。

There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。

Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。

Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。

Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。

If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。

At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Precision measurement of the gravitational constant *G* (27 points)

问题 1：重力常数 G 的精确测量（27 分）

The precision measurement of the gravitational constant G is important because it is a fundamental constant. Besides, it can play a role in verifying (or disproving) some recent proposed versions of string theory or the existence of the fifth fundamental force.

引力常数 G 是一个基本物理常数，所以对其精确测量非常重要。此外，引力常数的精确测量也有助于验证（或排除）一些弦理论的版本或第五种基本力的存在。

Part A. Estimation of the Gravitational Field Change During the Experiment (15 points)
对实验中引力场变化的估计 （15 分）

A1	Let R and M be the radius and the mass of the Earth, respectively, express the gravitational field g in terms of R, M, and G, while ignoring the spinning of the Earth. 设 R 和 M 分别是地球的半径和质量。忽略地球自转，求引力场的强度 g，用 R，M 和 G 表示。	1 point 1 分

$$g = \frac{GM}{R^2}.$$

In this problem we present a simplified version of the latest method, which can lower the relative error of G down to 10^{-5}. As shown in Fig. 1(a), laser interferometer 1 measures the spacing between the two pendulum bobs with respect to the reference spacing between the suspension points of the pendulum, which is measured by laser interferometer 2. When the four source masses are moved from the outer position (shown in Fig. 1(a)) to the inner position (shown in Fig. 1(b)), the pendulum bob separation changes. Not pictured is the vacuum chamber that encloses the pendulums but not the source masses.

本题简化地讨论一个测量引力常数的最新实验，可以将 G 的相对误差降低到 10^{-5}。如图 1(a) 所示，激光干涉仪 1 测量两个单摆摆锤之间的距离。激光干涉仪 2 测量两个单摆悬挂点之间的距离作为参照。当四个质量源从外侧位置（如图 1(a)所示）移动到内侧位置（如图 1(b)所示）时，两个摆锤之间的距离会改变。除了质量源外，整个实验装置置于真空环境内（没有画在图中）。

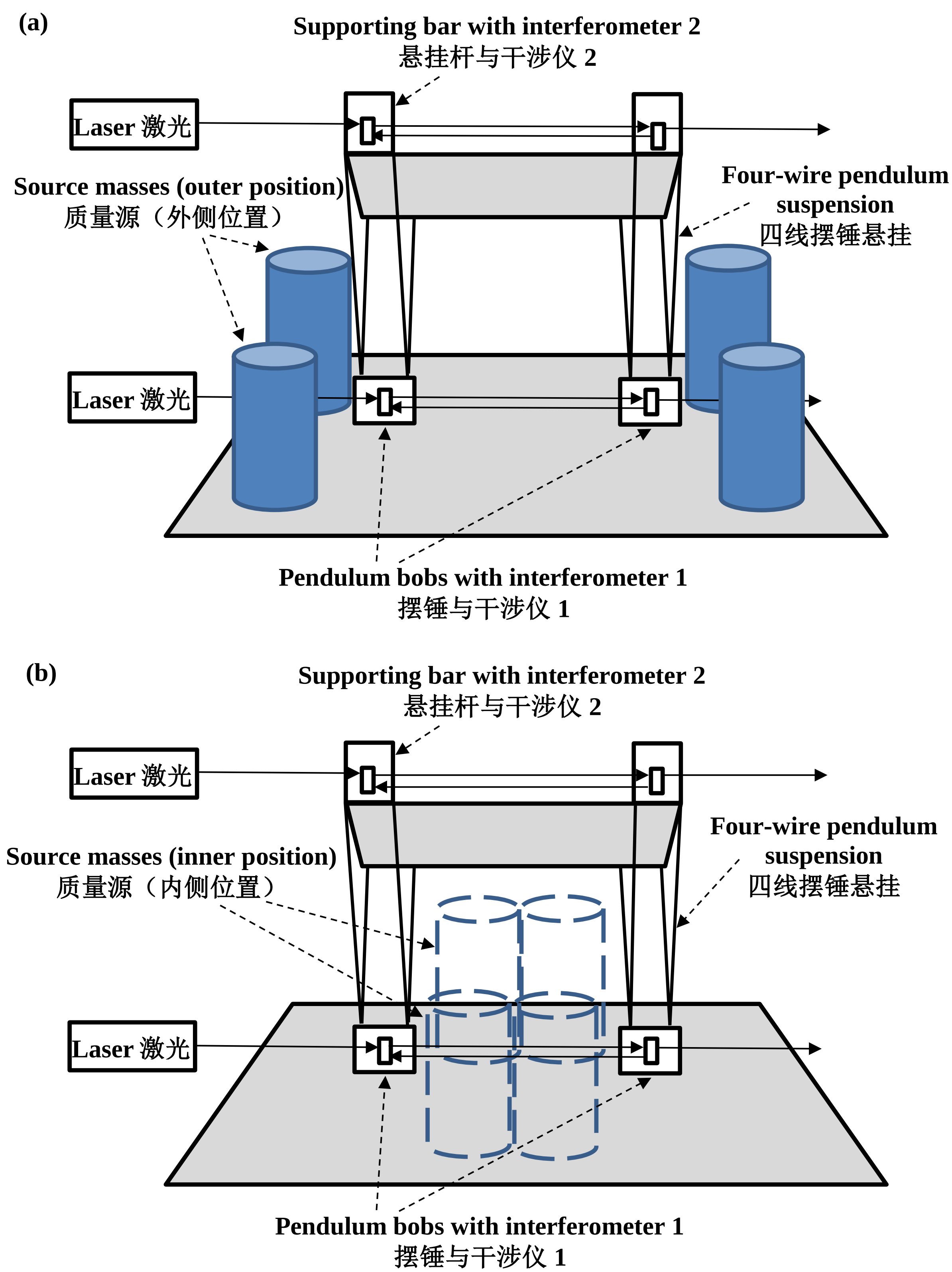


Figure 1: The setup of apparatus for measuring the gravitational constant precisely, with the source masses placed at (a) outer positions, (b) inner positions. 引力常数精确测量的实验装置，质量源置于 (a) 外侧位置，(b)内侧位置。

Figures 2 and 3 show the top and side views of the apparatus. The outer and inner positions of the source masses, and the pendulum bobs (at the ends of interferometer 1) are located symmetrically with respect to the center of the vacuum system. The length scales a_1, a_2, b, d, h and R are shown in the figures.

图 2 和图 3 是实验装置的顶视图和侧视图。质量源的外侧和内侧位置，和摆锤（在干涉仪 1 的末端）的位置相对于真空系统的中心是对称的。图中显示距离 a_1, a_2, b, d, h 和 R。

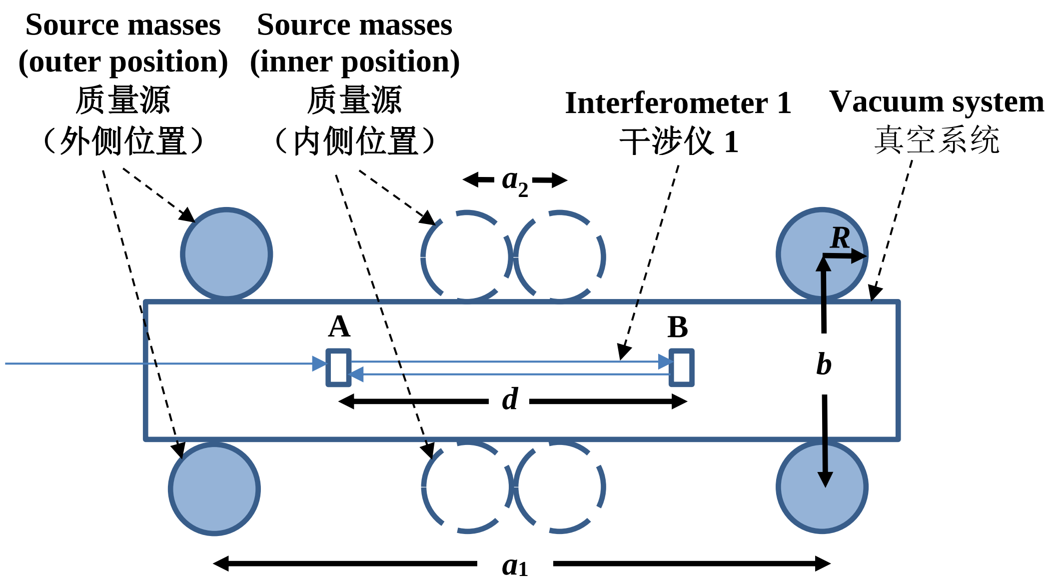


Figure 2: Top view of the apparatus. 仪器的顶视图。

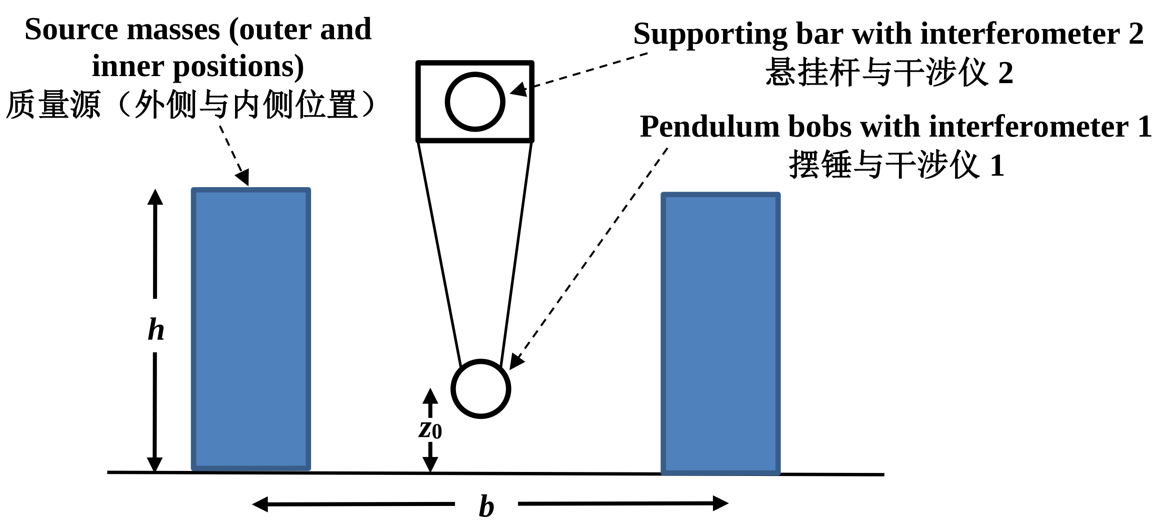


Figure 3: The side view of the apparatus. 仪器的侧视图。

It is very complicated to calculate the gravitational force of the 4 cylindrical source masses acting on pendulum bob A. Here we approximate each cylinder with uniform density, mass M, radius R and height h to be a thin wire with uniform density, mass M and height h passing through the axis of the cylinder.

计算四个圆柱形质量源对摆锤 A 的引力非常复杂。这里，我们将每个密度均匀、质量为M、半径为 R、高度为 h 的圆柱近似为沿圆柱轴向放置的细线，密度均匀、质量为 M、高度为 h 。

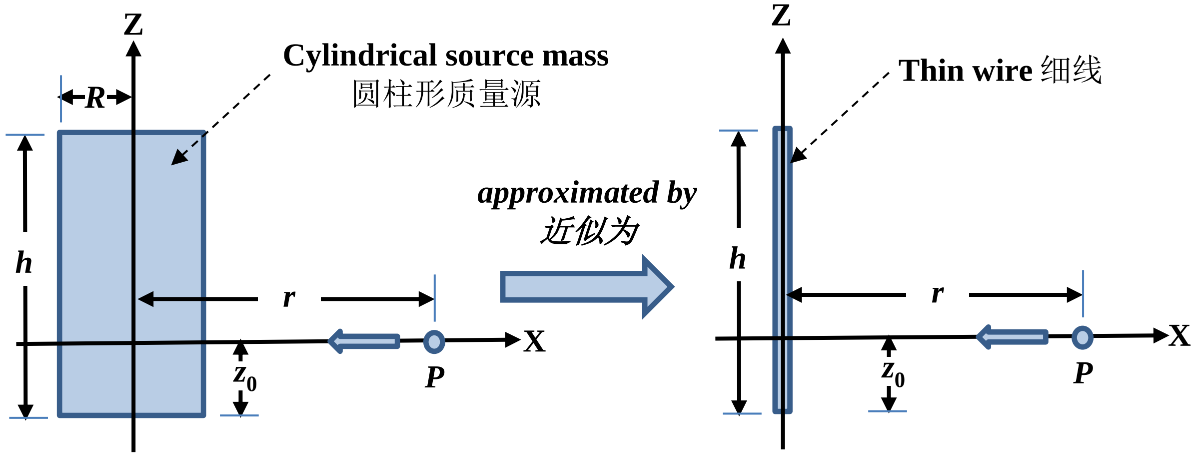


Figure 4: The X-component of the gravitational field at point P due to a cylindrical source mass is now approximated by that due to a thin wire. 计算在 P 点引力场的 X 分量时，将圆柱形质量源近似为细线。

You are provided the integral formula:
你可以使用以下积分公式：

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C,$$

A2	Derive an expression of the X-component of the gravitational field g_x at point P due to the thin wire. Express your answer in terms of G, M, h, z_0, and r. 推导由细线产生的引力场在 P 点的 X 分量 g_x。用 G, M, h, z_0 和 r 表示你的结果。	3 points 3 分

Consider an element of the plate at position z.
Distance from $P = \sqrt{r^2+z^2}$. [0.5]
Gravitational field at $P = \frac{G\rho\, dz}{r^2+z^2}$. [0.5]
X-component of the gravitational field $= \frac{G\rho\, dz}{r^2+z^2}\frac{r}{\sqrt{r^2+z^2}}$. [0.5]
Linear density: $\rho = \frac{M}{h}$.
Total X-component of the gravitational field

$$g_x = \frac{GM}{h}\int_{-z_0}^{h-z_0} dz \frac{r}{(r^2+z^2)^{\frac{3}{2}}} = \frac{GM}{h}\frac{rz}{r^2\sqrt{r^2+z^2}}\Bigg|_{-z_0}^{h-z_0}$$ [integral expression 0.5, integration result 0.5]

$$= \frac{GM}{hr}\left(\frac{h-z_0}{\sqrt{r^2+(h-z_0)^2}} + \frac{z_0}{\sqrt{r^2+z_0^2}}\right).$$ [correct answer 0.5, no penalty for ± sign]

A3	Calculate the gravitational field g_x at point P when the point is very near the thin wire. 当点 P 非常接近细线时，计算引力场在 P 点的 X 分量 g_x。	1 point 1 分

When the point is very near to the thin wire, r approaches 0. Hence
$g_x = \frac{2GM}{hr}$. [noting that the two terms in the bracket approach 1: 0.5, correct answer 0.5]

A4	Applying Gauss' law, verify the result in part A3. Write your steps in the answer sheet. 应用高斯定律，验证 A3 中的结论。在答题纸上写下过程。	1 point 1 分

Total flux of gravitational field:

$$\Phi = \left(\frac{GM}{r^2}\right)(4\pi r^2) = 4\pi GM. \qquad [0.5]$$

Applying Gauss' law to a cylindrical surface of height Δz around the thin wire,

$$\Phi = 4\pi \frac{M}{h}\Delta z = g_x(2\pi r\Delta z).$$
$$g_x = \frac{2GM}{hr}. \qquad [0.5, \text{ no penalty for } \pm \text{ sign}]$$

You are provided the following parameters:
你可以使用以下参数：

$G = 6.67 \times 10^{-11}$ Nkg^{-2}m^2	$M = 119.1$ kg	$a_1 = 0.568$ m
$a_2 = 0.166$ m	$b = 0.262$ m	$d = 0.34$ m
$h = 0.312$ m	$z_0 = 0.002$ m	$R = 0.083$ m

A5	Using the given parameters, and the thin wire approximation for the 4 cylindrical source masses, calculate the horizontal component of the gravitational field due to the 4 source masses at the position of pendulum bob A, when the source masses are located at the inner position. 使用上面给出的参数，以及用细线近似四个质量源。当质量源处于内侧位置时，计算摆锤 A 处四个质量源产生的引力场的水平分量。	3 points 3 分

For pendulum bob A, the 2 sources on the left and right in Fig. 3 are at distances

$$r_{left}^2 = \left(\frac{d}{2} - \frac{a_2}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = (0.17 - 0.083)^2 + 0.131^2 = 0.02473 \text{ m}^2,$$
$$r_{right}^2 = \left(\frac{d}{2} + \frac{a_2}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = (0.17 + 0.083)^2 + 0.131^2 = 0.08117 \text{ m}^2.$$

[know how to calculate: 0.5, correct substitution: 0.5]

The angles between the components of their gravitational fields and the axis of the interferometer are

$$\cos\theta_{left} = \frac{d - a_2}{\sqrt{(d-a_2)^2 + (b)^2}} = \frac{0.17 - 0.083}{\sqrt{(0.17 - 0.083)^2 + 0.131^2}} = 0.5532,$$
$$\cos\theta_{right} = \frac{d + a_2}{\sqrt{(d+a_2)^2 + (b)^2}} = \frac{0.17 + 0.083}{\sqrt{(0.17 + 0.083)^2 + 0.131^2}} = 0.8880.$$

[know how to calculate: 0.5, correct substitution: 0.5]

Hence the gravitational field along the interferometer axis due to the left source masses

$$g_{left} = (2)\frac{(6.67\times10^{-11})(119.1)}{(0.312)\sqrt{0.02473}}\left(\frac{0.31}{\sqrt{0.02473+0.31^2}}+\frac{0.002}{\sqrt{0.02473+0.002^2}}\right)(0.5532)$$
$$= 1.6204\times10^{-7}\ \mathrm{ms^{-2}}.$$
$$g_{right} = (2)\frac{(6.67\times10^{-11})(119.1)}{(0.312)\sqrt{0.08117}}\left(\frac{0.31}{\sqrt{0.08117+0.31^2}}+\frac{0.002}{\sqrt{0.08117+0.002^2}}\right)(0.8880)$$
$$= 1.1798\times10^{-7}\ \mathrm{ms^{-2}}.$$

Total gravitational field (in the rightward direction): $g_{total} = g_{left} + g_{right} = 2.8002\times10^{-7}\ \mathrm{ms^{-2}}$.

[know how to calculate and sum the two: 0.5, correct substitution: 0.5]

A6	Similar to part A5, calculate the horizontal component of the total gravitational field at the position of pendulum bob A, when the source masses are located at the outer position. 与 A5 部分类似，当质量源处于外侧位置时，计算摆锤 A 处总引力场的水平分量。	3 points 3 分

For pendulum bob A, the 2 sources on the left and right in Fig. 3 are at distances

$$r_{left}^2 = \left(\frac{a_1}{2}-\frac{d}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = (0.284-0.17)^2 + 0.131^2 = 0.03016\ \mathrm{m^2},$$
$$r_{right}^2 = \left(\frac{a_1}{2}+\frac{d}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = (0.284+0.17)^2 + 0.131^2 = 0.02233\ \mathrm{m^2}.$$

[know how to calculate: 0.5, correct substitution: 0.5]

The angles between the components of their gravitational fields and the axis of the interferometer are

$$\cos\theta_{left} = \frac{a_1-d}{\sqrt{(a_1-d)^2+(b)^2}} = \frac{0.284-0.17}{\sqrt{(0.284-0.17)^2+0.131^2}} = 0.6565,$$
$$\cos\theta_{right} = \frac{a_1+d}{\sqrt{(a_1+d)^2+(b)^2}} = \frac{0.284+0.17}{\sqrt{(0.284+0.17)^2+0.131^2}} = 0.9608.$$

[know how to calculate: 0.5, correct substitution: 0.5]

Hence the gravitational field along the interferometer axis due to the left source masses

$$g_{left} = (2)\frac{(6.67\times10^{-11})(119.1)}{(0.312)\sqrt{0.03016}}\left(\frac{0.31}{\sqrt{0.03016+0.31^2}}+\frac{0.002}{\sqrt{0.03016+0.002^2}}\right)(0.6565)$$
$$= 1.7016\times10^{-7}\ \mathrm{ms^{-2}}.$$
$$g_{right} = (2)\frac{(6.67\times10^{-11})(119.1)}{(0.312)\sqrt{0.02233}}\left(\frac{0.31}{\sqrt{0.02233+0.31^2}}+\frac{0.002}{\sqrt{0.02233+0.002^2}}\right)(0.9608)$$
$$= 5.7237\times10^{-7}\ \mathrm{ms^{-2}}.$$

Total gravitational field (in the leftward direction): $g_{total} = g_{left} - g_{right} = 1.1292\times10^{-7}\ \mathrm{ms^{-2}}$.

[know how to calculate and subtract: 0.5, correct substitution: 0.5]

A7	Calculate the change Δg_x of the horizontal component of the gravitational field at the position of pendulum bob A, when the source masses are moved from the outer position to the inner position. 当质量源从外侧位置移到内侧位置，计算摆锤 A 处引力场水平分量的变化Δg_x。	1 point 1 分

Change in the horizontal component of the gravitational field
$= 2.8002 \times 10^{-7} + 1.1292 \times 10^{-7} = 3.9294 \times 10^{-7}\ \mathrm{ms^{-2}}$.
[know how to add the answers of A5 and A6: 0.5, correct answer: 0.5]

In this experiment, care has to be taken to monitor the uncertainties of measurements. Since the calculation is complicated, we will simply focus on the expression derived in Part A3. Consider the case that all mass measurements have an uncertainty of 0.6 parts in 10^5, and all dimension measurements have an uncertainty of 1.4 parts in 10^5.

本实验需要小心考虑实验误差。完整计算比较复杂，为了简化，我们使用 A3 部分中推导的表达式。假设所有质量测量的误差均为 0.6×10^{-5}，所有长度测量的误差均为 1.4×10^{-5}。

Remark: The uncertainty of a physical quantity $f(x_1, x_2, \ldots)$ is given by the standard deviation of f divided by f, that is, $\sqrt{\sigma_f^2/f^2}$, where $x_1, x_2, \ldots$ are independent measurements and σ_f^2 is calculated from $\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_2^2 + \cdots$.

注：物理量 $f(x_1, x_2, \ldots)$的误差为 f 的标准差除以 f 本身，即$\sqrt{\sigma_f^2/f^2}$，其中 $x_1, x_2, \ldots$为独立测量的物理量，σ_f^2 可以用 $\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_2^2 + \cdots$ 计算。

A8	Calculate the uncertainty of g_x in Part A3. 计算 A3 部分中 g_x 的误差。	2 points 2 分

$$\sigma_{g_x}^2 = \left(\frac{2G}{hr}\right)^2 \sigma_M^2 + \left(-\frac{2GM}{h^2 r}\right)^2 \sigma_h^2 + \left(-\frac{2GM}{hr^2}\right)^2 \sigma_r^2.$$

$$\frac{\sigma_{g_x}^2}{g_x^2} = \frac{\sigma_M^2}{M^2} + \frac{\sigma_h^2}{h^2} + \frac{\sigma_r^2}{r^2} = (0.6 \times 10^{-5})^2 + (1.4 \times 10^{-5})^2 + (1.4 \times 10^{-5})^2 = 4.28 \times 10^{-10}.$$

$$\frac{\sigma_{g_x}}{g_x} = \sqrt{4.28 \times 10^{-10}} = 2.1 \times 10^{-5}.$$

[correct formula 1, correct answer 1]

Part B. Measurement of the Gravitational Field Change During the Experiment (15 points)测量实验中的引力场改变（15 分）

In Part A we estimated the gravitational field change when the source masses are moved from the outer to the inner position, assuming a certain value of the gravitational constant G. In this part we investigate how the gravitational field change can be measured experimentally, such that the gravitational constant G can be calculated. We let:

Δx = the horizontal displacement of the pendulum bob A when the source masses are moved from the outer to the inner position,
ω = the angular frequency of the natural oscillations of the pendulum bob,
m = mass of the pendulum bob.

在 Part A 中，我们假设给定引力常数 G，估算了质量源从外侧移到内侧时引力场的改变。本部分中，我们将研究如何在实验中测量引力场的改变，以及如何因此测量引力常数 G。我们定义三个量：

Δx = 当质量源从外侧移到内侧时，摆锤 A 在水平方向的偏移量，
ω = 单摆自然振动的角频率，
m = 摆锤的质量。

B1	Derive the expression of the change Δg_x in the horizontal component of the gravitational field at the position of pendulum bob A, when the source masses are moved from the outer position to the inner position. Express the result in terms of the 3 variables above. 当质量源从外侧移到内侧，求摆锤 A 处引力场水平分量的改变 Δg_x。用以上三个量表示结果。	1 point 1 分

$$\Delta g_x = g\tan\theta \approx g\theta = g\frac{\Delta x}{l} = \omega^2\Delta x.$$

[correct steps 0.5, correct answer 0.5]

The pendulum bobs are hung from the supporting bar at a distance l = 0.738 m vertically below the bar. The gravitational acceleration is g = 9.8 ms^{-2}.

摆锤从顶部悬挂下来，与悬挂杆的垂直距离为 l = 0.738 m。引力加速度是 g = 9.8 ms^{-2}。

B2	Using your result in part A7, calculate the change in the separation of the pendulum bobs. 利用 A7 部分的结果，计算摆锤位置的变化量。	1 point 1 分

$$\Delta g_x = g\frac{\Delta x}{l}$$

$$\Delta x = l\frac{\Delta g_x}{g} = (0.738)\left(\frac{3.9294\times 10^{-7}}{9.8}\right) = 29.5912\text{ nm}.$$

[correct steps 0.5, correct answer 0.5]

There are corrections to the answer in B2 because besides the pendulum motion, there are other contributions to ω^2 such as the flexing of the wires. The frequencies were found to be (0.589 8171 ± 0.000 0023) Hz for one bob and (0.589 7069 ± 0.000 0013) Hz for the other, where the number following the plus-minus sign is the standard derivation of the quantity.

B2 中的答案还没有加入校正，因为除了单摆运动之外，还有其它因素影响ω^2，例如摆线的弹性。对一个摆锤，频率的测量值为 (0.589 8171 ± 0.000 0023) Hz，对另一个摆锤，频率的测量值为 (0.589 7069 ± 0.000 0013) Hz，其中正负号后面的数字为该量的标准差。

B3	Calculate the mean and the uncertainty of the average value of the pendulum frequency. 计算单摆频率的平均值，以及单摆频率平均值的误差。	2 points 2 分

Mean:

$\frac{1}{2}(0.589\,8171 + 0.589\,7069) = 0.589\,7620$ Hz. [1]

Uncertainty:

$\frac{\sqrt{\frac{1}{4}(0.000\,0023)^2+\frac{1}{4}(0.000\,0013)^2}}{0.589\,7620} = 2.2 \times 10^{-6}$. [1]

B4	If the error of time measurement is 10^{-5} s, determine the number of periods to be measured such that the uncertainty of the measured period is 10^{-7}. 假设时间测量的绝对误差为10^{-5}秒，为了达到误差为 10^{-7} 的周期测量精度，求需要测量的周期数量。	1 point 1 分

Period:

$T = \frac{1}{0.589\,7620} = 1.6956$ s. [0.5]

Let n be the number of periods. Then

$\frac{10^{-5}}{nT} = 10^{-7}$.

$n = \frac{10^{-5}}{10^{-7}T} = 59$. [0.5]

In practice, the pendulum was swung over several hours of measurement to check whether the pendulums are stable, and repeated over the course of several months.

The horizontal displacement of the pendulum bobs can be measured with high precision using the laser interferometer (commonly known as Fabry-Pérot interferometer). The cavity of the interferometer is formed by two highly reflective mirrors separated at a distance d = 0.34 m so that the laser team traveling between them forms a standing wave. The laser wavelength is 633 nm. When the spectrum is closely examined, one finds that it consists of a sequence of peaks as shown in Fig. 5.

摆锤的水平位移可以通过激光干涉仪（法布里-珀罗干涉仪）精确测量。干涉仪腔体的两端分别放置了一块反射系数很高的镜片。两镜片相距 d = 0.34 m。激光在两镜片之间形成驻波。激光的波长为633 nm。如图 5 所示，干涉仪的光谱由一系列峰组成。

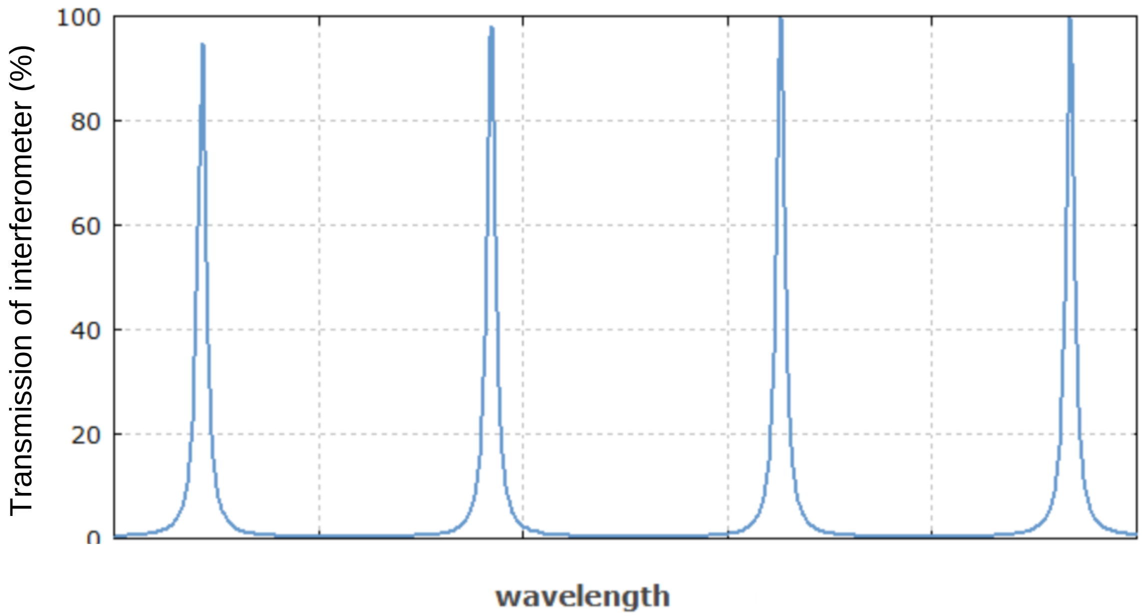


Figure 5: A sketch of the close-up of the spectrum of the interferometer. 干涉仪中光谱的细致结构示意图。

B5	Calculate the frequency separation of the neighboring peaks in the spectrum. Then calculate the wavelength separation of the neighboring peaks in the spectrum. 计算光谱中相邻两峰之间的频率差，并计算光谱中相邻两峰之间的波长差。	2 points 2 分

The peaks correspond to different modes of standing waves. Neighboring peaks correspond to standing waves differing in their traveling time back and forth the interferometer by one period. Hence the frequency separation of the neighboring peaks is given by

$$\Delta f = \frac{c}{2L} = \frac{3\times10^8}{0.68} = 441 \text{ MHz.} \qquad [1]$$

Wavelength separation:

$$\Delta\lambda = \frac{\lambda}{f}\Delta f = \frac{\lambda^2}{c}\Delta f = \frac{(633\times10^{-9})^2}{3\times10^8}(4.41\times10^8) = 5.89\times10^{-4} \text{ nm.} \qquad [1]$$

The width of the fringes in Fig. 5 is about 100 kHz.
图 5 中，条纹的宽度约为 100kHZ。

B6	Assuming that the major mechanism of power loss of the standing wave is the transmission through the mirrors, estimate the fraction of power loss from the interferometer per transmission. 假设驻波中主要的能量损失来自穿过镜面的透射。估算每次透射中能量损失占总能量的比例。	2 points 2 分

When there is no input power, the time taken by the phase angle of the standing wave mode to spread out over a range of 2π completely is

$\frac{1}{(2\pi)(100\times10^3)}$. [1]

Time for the standing wave to travel between successive transmission is

$\frac{L}{c}=\frac{0.34}{3\times10^8}$.

Hence the fraction of power loss per transmission is

$\left(\frac{0.34}{3\times10^8}\right)(2\pi)(100\times10^3)=7\times10^{-4}$. [1]

Remark: Answers without the factor of 2π are acceptable. In that case the answer will be 1×10^{-4}.

B7	Suppose the length of the interferometer changes by 1 nm. Calculate the corresponding change in the cavity frequency. 设干涉仪长度改变 1 nm，求腔体中频率的相应改变。	2 points 2 分

The nth harmonic of the cavity frequency is given by

$$f=\frac{nc}{2L}.$$

Hence the change in the cavity frequency is given by

$$\delta f=-f\frac{\delta L}{L}=-\left(\frac{3\times10^8}{633\times10^{-9}}\right)\left(\frac{1\times10^{-9}}{0.34}\right)=-1.3939\text{ MHz}.$$

[correct equation of δf: 1, correct answer 1]

When the source masses are moved from the outer to the inner position, the beat frequency of the cavity frequencies of the two interferometers changes by 125 MHz.
当质量源从外侧移到内侧时，两个干涉仪腔体中频率的拍频改变了 125 MHz。

B8	Calculate the change in the separation of the pendulum bobs. (Remark: This result will be different from that in B2 due to the approximations made in Part A.) 计算两个摆锤间距离的改变。（注：由于 Part A 中做的近似，这个结果将与 B2 中的不同。）	1 point 1 分

Change in the separation: $\Delta x=\frac{125\times10^6}{1.3939\times10^6}=89.675$ nm. [1]

References:

[1] Harold V. Parks and James E. Faller. Simple Pendulum Determination of the Gravitational Constant. *Physical Review Letters* **105**, 110801 (2010).

[2] Harold V. Parks and James E. Faller. A simple pendulum laser interferometer for determining the gravitational constant. *Philosophical Transactions of the Royal Society* **A372**, 20140024 (2014).

Problem 2: Physics in Various Dimensions (33 points) 不同维度的物理 (33 分)

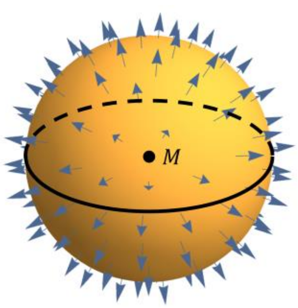
"What had she experienced? She had seen how a cruel attacker could lower the dimensions of space by one and destroy a solar system. What are dimensions?" -- Cixin LIU, *Death's End* (Translation: Ken LIU)

"她经历过什么？她刚刚看到，为了毁灭一个恒星系。残忍的攻击者把那里的空间维度降低了一维。空间维度，空间维度是什么？" —— 刘慈欣《三体・死神永生》

In this problem, we will explore 4-dimensional space, attack from 4-dimensional to 3-dimensional spaces, and the motion of celestial bodies in two dimensions. (Note: In this problem, whenever we mention the number of dimensions, we mean spatial dimensions and did not count in the time dimension. For example, when we mention 3-dimensional space, we mean the spacetime with 3 space dimensions and 1 time dimension). The setup of the problem is as follows:

本题中，我们将讨论四维空间、从四维到三维的维数攻击，以及二维空间中的天体运动问题。（注意：本题提到空间维度的数量时，并没有把时间维计算在内。例如，当我们提到三维空间时，指的是三维空间加上一维时间组成的时空。）问题设定如下：

- Let the mass of the star be M， the mass of the planet be $m \ll M$， So in the star-planet problems, the star can be considered at rest. The change of star position due to planet motion can be neglected.
- In n-dimensional space, Newtonian gravity can still be derived from Gauss flux law: For any (n-1)-dimensional closed surface enclosing the point particle M, the gravitational acceleration flux through the surface is $-A_{n-1}GM$, where A_{n-1} is the area of a $(n-1)$-dimensional unit sphere (understood as generalized area, for example, for $n=2$, $A_1 = 2\pi$ is length, for $n=3, A_2 = 4\pi$ is area, and for $n=4$, $A_3 = 2\pi^2$ is volume). The meaning of flux is: for a small area element, the flux is the dot product of the gravitational acceleration vector and the area vector. For the case when the gravitational acceleration is normal to the surface, flux is the magnitude of gravitational acceleration times this area. For example, the figure below illustrates how to derive Newtonian gravity from Gauss flux law for the case of 3 spatial dimensions.



Newtonian gravity from
Gauss flux law in 3 dimensions:

(flux through spherical surface)
$= (F/m) \times 4\pi r^2 = -A_2 GM$

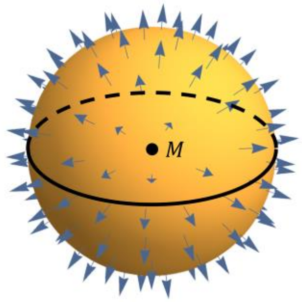
$\Rightarrow F = -GMm/r^2$

(Note: the Gauss flux law does not apply for general relativity.)

- Newton's three laws of motion still holds. Momentum conservation and angular momentum conservation laws still holds. The relativity of motion still holds.
- We approximate the stars and planets as particles, with negligible radius.

- 设恒星质量为 M，行星质量 $m \ll M$，故在行星围绕恒星运动问题中，恒星可以视为静止，行星运动对恒星位置的影响可以忽略。

- 在 n 维空间，牛顿引力可以由高斯引力通量定律推导出来，即对于包括了质点 M 的任何 $n-1$ 维闭合曲面，穿出此闭合曲面的引力加速度通量等于 $-A_{n-1}GM$，其中 A_{n-1} 为 $(n-1)$ 维单位球面的面积（理解为广义的面积，例如在 $n=2$ 情况下 $A_1=2\pi$ 为长度，$n=3$ 情况下 $A_2=4\pi$ 为面积，$n=4$ 情况下$A_3=2\pi^2$为体积）。通量的意思是：对一个小面积元，通量是引力加速度矢量点乘有向面积元。对引力加速度垂直于面积元的情况，通量为引力加速度的大小乘以面积元的面积。例如，下图中，对于三维空间，我们用高斯引理通量定律推导了牛顿引力定律。



（注意：高斯通量定律并不适用于广义相对论。）
- 牛顿三定律成立。动量守恒、角动量守恒定律成立。运动的相对性成立。
- 所有星球视为质点，其半径足够小，可以忽略。

Part A. High-Dimensional World (13 points) 高维的世界(13 分)

In the whole part A, we consider Newtonian gravity in n-dimensional ($n\geq 4$) space. Considering that the gravitational acceleration vector is parallel to the position vector, the orbit of the planet is within the 2-dimensional plane determined by the position and velocity vectors of the planet.

在整个 Part A 中，我们考虑 n ($n\geq 4$) 维空间中的牛顿引力。由于引力加速度与位置矢量共线，行星围绕恒星的运动轨迹处于行星的位置矢量与速度矢量所决定的二维平面内。

A1	Applying the Gauss flux law to a $(n-1)$-dimensional sphere, one can derive the Newtonian law of gravity in n space dimensions. Due to attraction from the star, the gravity force exerted on the planet is $F=-GMmr^\alpha$ (the direction of the force points to the star); the gravitational potential energy is $V(r)=GMm\,r^\beta/\beta$, where r is the distance between the star and the planet. Express α,β in terms of n. 将高斯引力通量定律应用于 $(n-1)$ 维球面，可以推导 n 维空间中的牛顿引力定律。由于受到恒星吸引，行星受到的引力为 $F=-GMmr^\alpha$ （力的方向指向恒星），引力势能为 $V(r)=GMm\,r^\beta/\beta$，其中 r 为行星与恒星的距离，求 α,β（用 n 表示）。	**2 point** **2 分**

Solution:

Using Gauss law, the area of a $(n-1)$ -dimensional sphere is $A_{n-1}r^{n-1}$. Thus, $F=-GMm/r^{n-1}$, i.e. $\alpha = 1-n$. (1p)

Either integrate $F=-\frac{GMm}{r^{n-1}}$, or from dimensional analysis, we get $\beta=2-n$, i.e. $V=-\frac{GMm}{(n-2)\,r^{n-2}}$. (1p)

A2	The Kepler's second law in n dimensions is: $L=mr^{\lambda}\dot{\varphi}$ is a constant (where φ is the angle between the planet-star plane and the x-axis of the motion plane, $\dot{\varphi}\equiv d\varphi/dt$). Find λ. n 维的开普勒第二定律为： $L=mr^{\lambda}\dot{\varphi}$ 为常数（φ 为行星-恒星连线与其运动平面的 x 轴之间的夹角， $\dot{\varphi}\equiv d\varphi/dt$）。求 λ。	**1 point** **1 分**

Solution:

$\lambda=2$, same as the 3-dimensional case, since the motion is restricted in 2-dimensional plane.

A3	Calculate the speed of the planet along the radius direction $\dot{r}\equiv dr/dt$. Express your result using n,r,G,M,m,E,L, where E is the energy of the planet (including kinetic energy and gravitational potential energy). 求行星的径向速度 $\dot{r}\equiv dr/dt$，用 n,r,G,M,m,E,L 表示，其中 E 为行星的能量（包括动能和引力势能）。	**3 points** **3 分**

Solution:

Kinetic energy E_K has two parts (perpendicular to each-other): along r direction and along ϕ direction. Thus, $E_K=\frac{1}{2}m\dot{r}^2+\frac{1}{2}mr^2\,\dot{\phi}^2$. (1p)

Using Kepler's law to replace $\dot{\phi}$ by L: $E_K=\frac{1}{2}m\dot{r}^2+\frac{L^2}{2mr^2}$. (0.5p)

Energy conservation: $E=E_K+V$ (0.5p)

Solve $\dot{r}$ from above: $\dot{r}=\pm\sqrt{\frac{2E}{m}+\frac{2GM}{(n-2)r^{n-2}}-\frac{L^2}{m^2r^2}}$. (1p) (#)

(If only plus sign is given in above $\pm$, deduct 0.5p.)

A4	Give the conditions for the planet to form a circular orbit (give algebraic equations using n,r,G,M,m,E,L, no need to solve these equations).	**2 points** **2 分**

	给出行星沿圆轨道运动的所有条件（给出关于 n, r, G, M, m, E, L 的代数方程组即可，不必解方程）。	

Solution: The two necessary conditions for circular orbit are:

(1) $\dot{r} = 0$ (0.5p), i.e. $\frac{2E}{m} + \frac{2GM}{(n-2)r^{n-2}} - \frac{L^2}{m^2 r^2} = 0$ (0.5p) (*)

(2) $\ddot{r} = 0$ (or equivalently gravitational force balance centripetal acceleration) (0.5p),

i.e. $\frac{GM}{r^{n-1}} - \frac{L^2}{m^2 r^3} = 0$ (0.5p) (**)

A5	For $n = 4$, when the values of r, G, M, m, E, L are such that a circular orbit is possible, and r is varied while other parameters are fixed, can the planet (1) form elliptical orbits? (2) move from finite r to $r \to \infty$? (3) move from finite r to $r \to 0$? 当 $n = 4$，并且 r, G, M, m, E, L 的取值使得行星有可能形成圆轨道时，固定其它参数而 r 取不同的值时，行星是否有可能 (1) 形成椭圆轨道 (2) 从有限的 r 运动到 $r \to \infty$ (3) 从有限的 r 运动到 $r \to 0$	**1.5 points** **1.5 分**

Solution:

For $n = 4$: equations (*) and (**) implies $E = 0$. Then a solution for one r is a solution for all values of r. Thus, for any r, we have circular orbit $\dot{r} = 0$. Thus, all (1), (2), (3) are impossible. (0.5p each)

A6	For $n > 4$, when the values of n, r, G, M, m, E, L are such that circular orbit is possible, and r is varied while other parameters are fixed, can the planet (1) form elliptical orbits? (2) move from finite r to $r \to \infty$? (3) move from finite r to $r \to 0$? For the possible cases in the above questions, please state the possible range of r given n, r, G, M, m, E, L are fixed. (If the limit of the range is one of the roots of an algebraic equation, please specify which root without the need to solve the equation explicitly.) 当 $n > 4$，并且 n, r, G, M, m, E, L 的取值使得行星有可能形成圆轨道时，固定其它参数而 r 取不同的值时，行星是否有可能	**3.5 points** **3.5 分**

	(1) 形成椭圆轨道 (2) 从有限的 r 运动到 $r \to \infty$ (3) 从有限的 r 运动到 $r \to 0$ 对于上面回答为可能的情况，在 n, r, G, M, m, E, L 固定的情况下，请指出 r 的取值范围。（如果 r 取值范围的边界为代数方程的一个根，请指出哪个根，但不必解这个方程。）	

Solution:

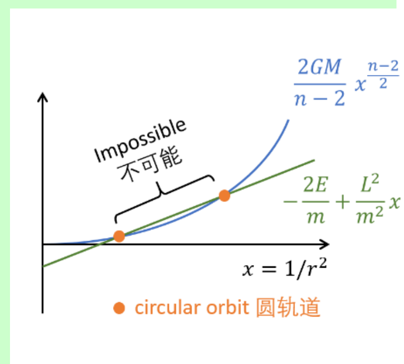
For $n > 4$: equations (*) and (**) implies $E > 0$. (0.5 p)

For (#) to make sense (positivity under the square root), we need (0.5p)

$$\frac{2E}{m} + \frac{2GM}{(n-2)r^{n-2}} - \frac{L^2}{m^2 r^2} \geq 0$$

The solutions of this inequality can be studied by letting $x = \frac{1}{r^2}$, and plot & compare the functions $\frac{2GM}{n-2} x^{(n-2)/2}$ and $\frac{L^2}{m^2} x - \frac{2E}{m}$ (0.5p).

The result is:



There are two positive r solutions of $\frac{2E}{m} + \frac{2GM}{(n-2)r^{n-2}} - \frac{L^2}{m^2 r^2} = 0$ (0.5p). Let us denote them by r_1, r_2 ($r_1 \leq r_2$). Then

(1) is impossible. (0.5p)

(2) Condition for (2) is $r < r_1$. (0.5p)

(3) Condition for (3) is $r > r_2$. (0.5p)

Part B. Dimensional attacks in Newtonian gravity (5 points) 牛顿引力下的维度打击 (5 分)

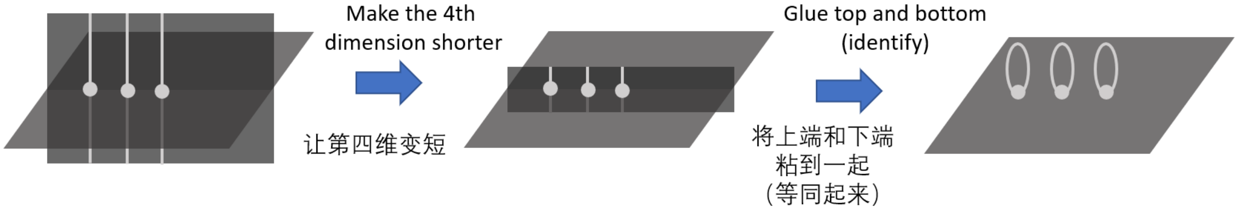
Suppose aliens living in 4 spatial dimensions perform dimensional attack on enemies also living in 4 spatial dimensions. The way of dimensional reduction is to reduce one space dimension into a small circle with circumference length C. In this way, in fact, there are still 4 spatial dimensions. But for small C values, seeing from far away, one cannot see the dimension of the small circle. As a result, seeing from far away, the enemies appear to be living in 3 spatial dimensions. We call such spatial regions under dimensional attack having "effectively" three spatial dimensions.

假设四维空间中的生物向同样生活在四维的敌方实施维度打击。降维的方法是，将一个空间维度缩小成周长为 C 的小圆环。这样，其实空间还有四维。但是当 C 很小时，从远处看来，看不到这个圆

环代表的维度。这样，从远处看来，敌方就好像生活在三维空间一样。我们称这时遭受维度打击的空间区域“有效”维度为三维。

As illustrated in the below figure, suppose the vertical dimension is the dimension under dimensional attack. The horizontal plane denotes the remaining three-dimensional space. Before the dimensional attack, the four-dimensional space can be considered as, in a three-dimensional space, on every point there is one straight line indicating the fourth dimension. After the dimensional attack, on every point there is a small circle with circumference C indicating the fourth dimension. The circle indicates that the top and bottom endpoints of an interval are glued.

如下图所示，假设垂直方向的维度为实施维度打击的维度，水平的平面代表剩余的三维空间。维数打击前，四维空间可以看成：三维空间的每个点上都有一条直线代表第四维。维数打击后，三维空间的每个点上都有一个周长为 C 的小圆环代表第四维。圆环的含义为将一条线段的上下两个端点粘起来。



Assume that during the dimensional attack (assume the duration is short enough), apart from the sudden change of the law of gravity, a system under dimensional attack does not have additional forces exerting on it. The momentum in the 4-dimensional point of view does not change. The law of gravity can be understood in two ways: the effectively three-dimensional gravity and the more fundamental four-dimensional gravity. The values of a gravitational force computed using these two methods (using Gauss flux law) agree with each other.

假设在维度打击过程中（假设此过程持续时间足够短），除了引力定律的形式忽然改变外，被打击的系统不受额外的力作用，四维观点下的动量不变。引力定律受维度打击的影响体现为:维数打击后可以从两种观点理解引力，有效的三维引力和更基本的四维引力，通过高斯定律用这两种观点算出的引力大小相等。

B1	Consider a point particle at rest, which exerts Newtonian gravity upon other objects (whose distance is much greater than C). Find the relation between the four-dimensional Newtonian gravitational constant G_4 and the effective three-dimensional Newtonian gravitational constant G_3 (in terms of $G_4 =$ (function of G_3 and other parameters)). 考虑一个静止质点对其它物体（距离远远大于 C ）产生的牛顿引力。求四维牛顿引力常数 G_4 和三维有效牛顿引力常数 G_3 之间的关系（用 $G_4 =$（G_3及其它参数的函数） 表示）。	**3 points** **3 分**

Solution:

In 4D, note that the new geometry is a cylinder. Now that $r \gg C$, the gravitational flux is approximately perpendicular to the circle (with perimeter C). The corresponding Gauss's Law is $4\pi r^2 C\,F = -2\pi^2 G_4 Mm$. (1p)

Compared with the 3D gravitational force is $F = -G_3 Mm/r^2$ (note: this 1p is given to realizing 4D force = 3D force, not knowing the 3D Newtonian gravity formula), we get $G_4 = \frac{2C}{\pi} G_3$. (1p)

[If the student guess $G_4 \propto C G_3$ with wrong proportional constant without reasons, give 1p.]

B2	Suppose the relativistic mass-energy relation $E^2 = \lvert\mathbf{p}\rvert^2 c^2 + m^2 c^4$ applies both for three and four dimensions. Here c denotes the speed of light. Suppose before the dimensional attack, a point particle with mass m has momentum $\mathbf{p} = (p_1, p_2, p_3, p_4)$. After dimensional attack, the 4th dimension (corresponding to the subscript 4 above) becomes a small circle. Calculate the three-dimensional effective mass of the point particle after the dimensional attack. 假设相对论质能关系 $E^2 = \lvert\mathbf{p}\rvert^2 c^2 + m^2 c^4$ 在三维和四维都成立，其中 c 为光速。设维度打击之前，一个质量为 m 的质点的动量为 $\mathbf{p} = (p_1, p_2, p_3, p_4)$，维度打击后，第四个空间维度（对应上述角标 4）变成小圆环，求质点在维度打击后的三维空间中的有效质量。	**1 point** **1 分**

Solution: $m_{\text{eff}} = \sqrt{m^2 + \frac{p_4^2}{c^2}}$. (1p)

B3	Suppose a planet (with mass m) is moving along circular orbit in 4 space dimensions, with the star (with mass M) at the center. The distance between the planet and the star is r. Now dimensional attack this system along a direction perpendicular to the plane of planet motion. Calculate the energy of the planet after the dimensional attack (gravitational potential energy plus kinetic energy in the three-dimensional effective point of view). 设质量为 m 的行星在四维空间中以圆轨道绕质量为 M 的恒星运动，与恒星距离为 r。现对此星系沿垂直行星运动平面方向进行维度打击。求维度打击后，三维空间中行星的总能量（三维有效观点下的引力势能加动能）。	**1 point** **1 分**

Solution:

Since gravity is much stronger in 3D, the three-dimensional energy is dominated by its potential energy $-\frac{G_3 Mm}{r}$ (1p)

It is also correct if more details are given (still 1p):

Before the dimensional attack, in 4D:

Spherical orbit: $\frac{L^2}{m} = G_4 Mm$.

The dimensional attack relates $G_4 = \frac{2C}{\pi} G_3$. Thus, $\frac{L^2}{m} = \frac{2CG_3Mm}{\pi}$.

The 3-dimensional energy is $E = -\frac{G_3Mm}{r} + \frac{CG_3Mm}{\pi r^2}$.

Remark: However, considering that G_3 receives $O(C/r)$ corrections, this form is not much more precise than $E \sim -\frac{G_3Mm}{r}$.

Part C. The Two-Dimensional Newtonian World (4 points) 二维牛顿世界 (4 分)

We further reduce spatial dimensions and consider Newtonian gravity in two dimensions ($n = 2$).

我们进一步减少空间维数，考虑二维（$n = 2$）空间中的牛顿引力。

C1	Provide formulas for the Newtonian gravitational force law and the corresponding gravitational potential energy in two dimensional. 求二维的牛顿万有引力公式及引力势能公式。	**1 point** **1 分**

Solution:

$F = -\frac{GMm}{r}$ (0.5p),

$V = GMm \log r/r_0$ or $V = GMm \log r$ (0.5p)

C2	Provide relations for planets to move along circular orbits around the star (give algebraic equations in terms of r, G, M, m, E, L. You don't need to solve the equation). 求行星绕恒星沿圆轨道运动的所有条件（给出关于 r, G, M, m, E, L 的代数方程组即可，不必解方程）。	**1 point** **1 分**

Solution:

$\frac{2E}{m} - 2GM\,(\log r + \text{const}) - \frac{L^2}{m^2r^2} = 0$ (0.5p)

(Note, here a constant is introduced to make the solution more general. The students don't have to include the constant in their answer to get this 0.5p)

$\frac{GMm}{r} = \frac{L^2}{m^2r^2}$ (0.5p)

C3	When the values of r, G, M, m, E, L are such that circular orbit is possible, and r is varied while other parameters are fixed, can the planet (1) move in a bounded range $r_1 \le r \le r_2$, where $0 < r_1 < r_2 < \infty$ (2) move from finite r to $r \to \infty$? (3) move from finite r to $r \to 0$? Just answer possible or impossible for the above three questions (may need to discuss different cases for different parameter choices). 当 n, r, G, M, m, E, L 的取值使得行星有可能形成圆轨道时，固定其它参数而 r 取不同的值时，行星是否有可能 (1) 始终在有限的 $r_1 \le r \le r_2$ 区间内运动，其中 $0 < r_1 < r_2 < \infty$ (2) 从有限的 r 运动到 $r \to \infty$ (3) 从有限的 r 运动到 $r \to 0$ 对以上三问，分别回答能或不能即可（有可能要根据参数取值分情况讨论）。	**2 point** **2 分**

Solution:

(1) Possible. Because both (2) and (3) are impossible. (1) must be possible. (Note: however, the orbit does not close to an ellipse, which needs a longer proof and is beyond the scope of this question. For those interested, one can search for Laplace-Runge-Lenz vector.) (0.5p)

However, there is one exception: if the two solutions of $\frac{2E}{m} - 2GM\,(\log r + \text{const}) - \frac{L^2}{m^2r^2} = 0$ coincide, then $r_1 = r_2$, which does not leave room for the condition $0 < r_1 < r_2 < \infty$ (0.5p)

(2) The gravitational potential is infinitely deep. Thus, for any finite E, (2) is impossible. (0.5p)

(3) Impossible. Existence of spherical orbit $\to L \neq 0$. Thus near $r \to 0, \dot{r}$ is imaginary, no physical meaning. Alternative but much more complicated explanation: repeat analysis similar to A4-A6, you will get the same conclusion that you cannot reach $r \to 0$. (0.5p)

Part D. The Two-Dimensional Einstein World (11 points) 二维爱因斯坦世界 (11 分)

Interestingly, in the two-dimensional case, the “gravitational” law in Einstein’s general relativity is even simpler than Newtonian gravity. In general relativity, there is no gravitational force at all between massive point particles (the space outside the particles is not curved either). Instead, the only gravitational effect of a point mass is that the space around it becomes conical (like a cone).

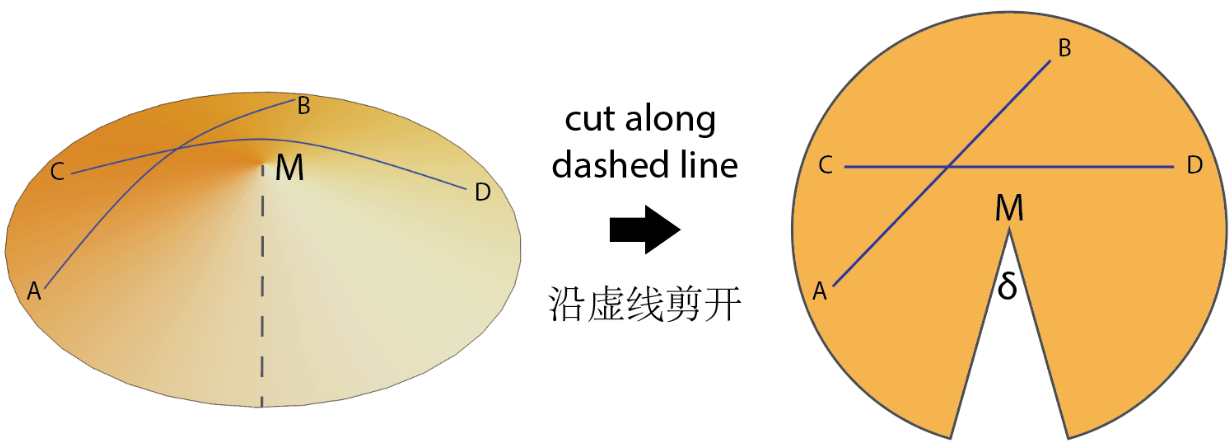
有趣的是，在二维空间情况下，爱因斯坦广义相对论中的“引力”规律比牛顿引力还要简单。广义相对论中，二维空间的质点之间根本没有相互吸引力（质点外的空间也没有弯曲）。而质量带来的效应为，一个质点周围的空间为圆锥形。

In the conical two-dimensional world, free particles and light will move along straight lines. Here straight lines are understood in the following way: if we cut the cone along a ray starting from the top vertex (not intersecting with the motion trajectory), and lay it on the plane as a circular sector, the motion trajectory is a straight line on the sector. For example, the lines AB and CD in the figure below.

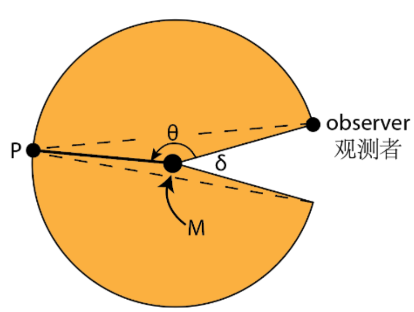
在圆锥形的二维世界中，自由粒子和光线将沿直线运动。这里的直线理解为，将圆锥沿着任意一条从顶点出发、与运动轨迹不相交的射线剪开，并在平面上摊平成为扇形后，运动轨迹在扇形上呈直线。例如下图中的直线 AB，CD。

When we cut the cone into a sector, there is a deficit angle (the angle that a sector lacks compared to a disk), denoted by δ as illustrated in the figure below. This deficit angle is proportional to the mass M of the point particle. Here we assume $\delta < \pi$.

把圆锥剪开成扇形时，扇形与圆盘相比所缺的角度（下图中的 δ）称为圆锥的缺陷角，与质点的质量 M 成正比。本题假设 $\delta < \pi$。



<table>
<tr>
<td>D1</td>
<td>Point P is moving along a circle surrounding point M (M is the center of the sector in the figure below). The angle between the observer-M line and the MP line (viewed anti-clockwise) is θ. Depending on different values of θ, sometimes the observer finds a single image of P and sometimes finds double images of P. For example, in the figure below, the observer can observe double images along the dashed lines. Find the condition for the observer to observe double images, and the angle between these two images (the angle between the two dashed lines in the figure).

点 P 围绕质点 M 做圆周运动（M 为图中扇形中心）。观测者在同一个圆周上观测 P 点的运动。观测者与 M 连线与 MP 连线（沿逆时针方向看去）的夹角为 θ。随 θ 取值不同，观测者有时看到 P 的单像，有时看到双像。例如右图，观

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<td>2 points

2 分</td>
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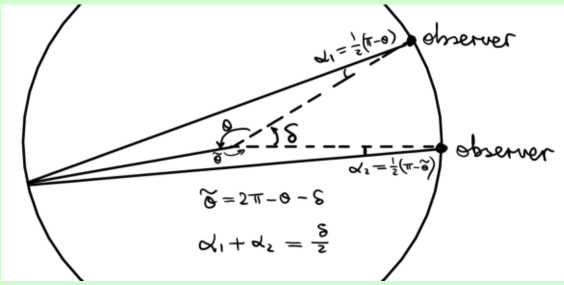
</table>

	测者可以沿图中虚线看到 P 的双像。求观测者能看到 P 双像的条件，以及双像间的夹角（即图中两虚线间的夹角）。	

Solution:

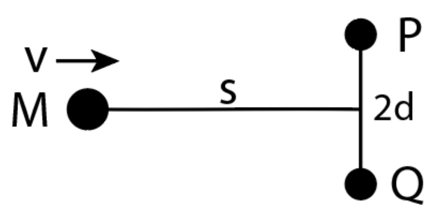
(1) When $\pi - \delta < \theta < \pi$ double images, otherwise single image (1p)



(2) Angle between the two images: the angle is $\delta/2$ (1p)

(A simpler observation is to use the inscribed angle theorem and get $\delta/2$ directly for this 1p.)

<table>
<tr>
<td>D2</td>
<td>At $t = 0$, a point particle with mass M (with corresponding deficit angle δ) moves towards observers P and Q. The direction of motion is perpendicular to the PQ interval, and the speed is v (much smaller than the speed of light). The mass of P and Q are negligible. At the initial time $t = 0$, P and Q are at rest, distances MP = MQ, PQ = 2d, the distance between M and the PQ interval is s. Calculate the time t_m when P and Q meet.

质量为 M（对应角度缺陷 δ）的质点向两个质量可忽略的观测者 P、Q 运动，运动方向垂直于 PQ 连线，速率为 v（远小于光速）。在初始时刻 $t = 0$ 时，P、Q 静止，线段 MP = MQ，PQ = 2d，M 与 PQ 连线距离为 s。求 P、Q 两观测者相遇的时间 t_m。

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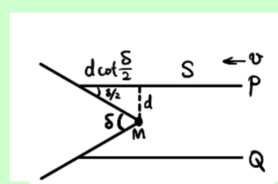
<td>2 points

2 分</td>
</tr>
</table>

Solution:

Though not necessary, it's convenient to work in the frame where M is static, and P&Q are moving. Note that the deficit angle can be drawn facing any angle (in the above examples we have drawn it facing forward, but it can also be drawn facing backward). This is not necessary either, though it will simplify the discussion. This setup or an equivalent figure deserves 1p.



The meeting time is thus $t_m = \left[s + d\cot\left(\frac{\delta}{2}\right)\right]/v$ (1p).

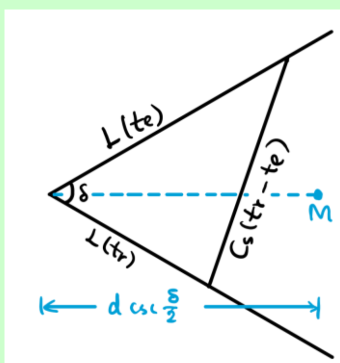
2

D3	Consider the same setup as Problem D2. The observer P (starting from early enough time) continuously emits sound wave towards all directions. The source of sound has vibration frequency f. The speed of sound is c_s satisfying $\frac{c_s^2 - v^2\cos\delta}{c_s^2 - v^2} = \frac{13}{12}$ (use this relation to eliminate c_s from the result), and the wavelength of sound is much smaller than s and d. The media to propagate sound moves together with M (i.e. at rest with respect to M). Shortly before P and Q meet, the sound frequency that Q hears is f_r . Calculate f_r. 在与第 D2 题相同设定下，观测者 P（从时间足够早开始）持续向所有方向发出声波，声源的振动频率为 f，声波的速度 c_s 满足 $\frac{c_s^2 - v^2\cos\delta}{c_s^2 - v^2} = \frac{13}{12}$ （用此关系在结果中消去 c_s），声波的波长远小于 s 和 d。声波的传播介质跟随质点 M 运动（即与 M 相对静止）。 在 P 和 Q 即将相遇前，Q 听到的声音频率是 f_r。求 f_r。	**4 points** **4 分**

Solution:

This is a Doppler effect problem. However, since both the emitter and the receiver move with respect to the media with an angle, we need to derive the corresponding formula instead of using the 1-dimensional Doppler formula.

Let the sound wave emission time be t_e, reception time be t_r (which equals to t as given in the question). Let $L(t) = v(t_m - t)$ be the distance of P (or Q) from their meeting point, as a function of t. See figure below: (Look at the black lines now. The blue lines are for Problem D4.)



From the cosine theorem: $L^2(t_e) + L^2(t_r) - 2L(t_e)L(t_r)\cos\delta = c_s^2(t_r - t_e)^2$ -- (*). (1p)

We can do two things from equation (*).

(a) Find relation between t_e and t_r. For this purpose, it is convenient to rewrite the LHS of (*) as $L^2(t_e) + L^2(t_r) - 2L(t_e)L(t_r)\cos\delta = \frac{c_s^2}{v^2}\left(L(t_e) - L(t_r)\right)^2$. From $\frac{c_s^2 - v^2\cos\delta}{c_s^2 - v^2} = \frac{13}{12}$, we get

$L(t_e) = \frac{3}{2}L(t_r)$, i.e. $t_e = (3t_r - t_m)2 = (3t_r - \frac{\left[s+d\cot\left(\frac{\delta}{2}\right)\right]}{v})/2$.

(1p, we will not use t_e below. But the students may get t_e which also deserves 1p.)

(b) To get the Doppler effect, we need to find relation between small variations δt_e and δt_r. Varying equation (*), noting $\delta L(t) = -v\delta t$, we get

$\frac{\delta t_e}{\delta t_r} = \frac{c_s^2+v^2(2-3\cos\delta)}{c_s^2-v^2(3-2\cos\delta)} = \frac{3}{2}$. (1p)

(In the above, we have used $\frac{c_s^2-v^2\cos\delta}{c_s^2-v^2} = \frac{13}{12}$ and $L(t_e) = \frac{3}{2}L(t_r)$ in the last equal sign.)

Finally, the frequency at reception is

$f_r = \frac{\delta t_e}{\delta t_r} f = \frac{3}{2} f$. (1p)

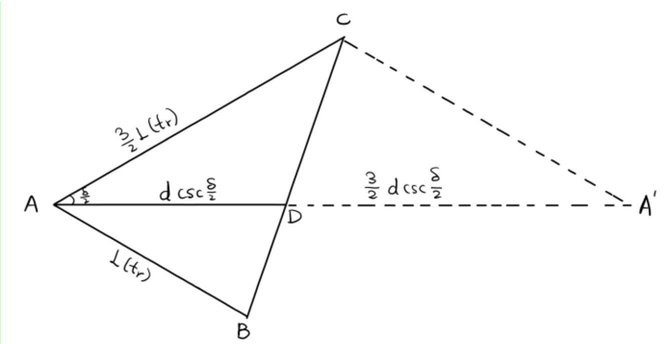
Alternatively, after getting $L(t_e) = \frac{3}{2}L(t_r)$, one may realize that the number of wave periods received is the same as the number of wave periods emitted, but in 2/3 time. Thus $f_r = \frac{3}{2}f$. This is also fully correct.

Note: if the student use 1D formula, i.e. by mistake considered the case where the media moves together with the middle of the PQ interval, then we give at most 2p out of 4p for the Doppler part: the relative velocity between the two points P and Q is $v_{\text{rel}} = 2v\sin\frac{\delta}{2}$ (1p), Corresponding frequency from Doppler effect, something like $f_r = \left(\frac{c_s+v_{\text{rel}}/2}{c_s-v_{\text{rel}}/2}\right)f$, $f_r = \left(\frac{c_s+v_{\text{rel}}}{c_s}\right)f$ or $f_r = \left(\frac{c_s}{c_s-v_{\text{rel}}}\right)f$ (1p)

D4	Consider the same setup as Problem D3, starting from which time (i.e. find the corresponding t) on, Q starts to hear this frequency f_r? 在与 D3 题相同的设定下，从何时起（即计算其时间 t），Q 开始听到频率 f_r ？	**3 points** **3 分**

Solution:

For the frequency f_r to appear, the sound has to be to the left of M (otherwise there is no deficit angle). This is illustrated as the blue lines in the above figure. Consider the marginal case that the sound crosses M. (1p)



The problem then is converted to the geometry problem above:

Given $\angle$BAD = $\angle$CAD = $\delta/2$, and |AB| = $L(t_r)$ = 3|AC|/2. Then what's the relation between the lengths |AC| and |AD|? Draw line CA' parallel to AB, and intersect the extension of AD at A'. Since $\angle$CA'D = $\frac{\delta}{2}$, |A'C| = |AC|. From the relation of similar triangles, |A'D| = 3|AD|/2. Thus, |AA'| = 5|AD|/2. Applying the cosine theorem:

$$\left(\frac{3}{2}L(t_r)\right)^2 + \left(\frac{5}{2}d\csc\frac{\delta}{2}\right)^2 - \left(\frac{3}{2}L(t_r)\right)\left(\frac{5}{2}d\csc\frac{\delta}{2}\right)\cos\frac{\delta}{2} = \left(\frac{3}{2}L(t_r)\right)^2 \quad \text{(1p)}$$

Solve this equation for $L(t_r)$, and apply the relation between $L(t_r)$ and t_r, we get

$L(t_r) = \frac{5d}{6\cos\frac{\delta}{2}\sin\frac{\delta}{2}} = \frac{5d}{3\sin\delta}$. And thus $t_r = \frac{s+d\cot\frac{\delta}{2}}{v} - \frac{5d}{3v\sin\delta}$.

Alternative approach:

$$\text{Area of ACB} = \frac{1}{2}\left(\frac{3}{2}L(t_r)\right)L(t_r)\sin\delta = \frac{3}{4}L(t_r)^2\sin\delta.$$

$$\text{Area of ACD} = \frac{1}{2}\left(\frac{3}{2}L(t_r)\right)\left(d\csc\frac{\delta}{2}\right)\sin\frac{\delta}{2} = \frac{3}{4}L(t_r)d.$$

$$\text{Area of ABD} = \frac{1}{2}L(t_r)\left(d\csc\frac{\delta}{2}\right)\sin\frac{\delta}{2} = \frac{1}{2}L(t_r)d.$$

Therefore,

$$\frac{3}{4}L(t_r)^2\sin\delta = \frac{3}{4}L(t_r)d + \frac{1}{2}L(t_r)d \quad \Rightarrow \quad L(t_r) = \frac{5d}{3\sin\delta}.$$

Using $L(t_r) = v(t_m - t)$, we obtain

$$t = \frac{s + d\cot\frac{\delta}{2}}{v} - \frac{5}{3\sin\delta}\frac{d}{v}.$$

Note: In reality, although we do not note a danger under dimensional attack to two dimensions, it is still meaningful to study the physics in two dimensions. For example, in our three-dimensional universe, there probably exist one-dimensional objects called “cosmic strings”. The cosmic strings in three dimensions are similar to point particles in two dimensions. Both of them bring a deficit angle to space. The three questions in Part D corresponds to the three important observable effects of cosmic strings. Searching for cosmic strings using these three observable effects is an active interdisciplinary research direction between high energy physics and astronomy.

注：现实中，尽管我们还没有发现被维数打击降为二维的风险，研究二维的物理仍然是有意义的。例如，我们的三维宇宙中，可能存在一种叫“宇宙弦”的线状一维物体。三维空间中的宇宙弦，和二维空间中的质点类似，都会给空间带来一个缺陷角。Part D 中的三个问题，对应的就是宇宙弦的三个重要观测效应。用这三个观测效应探测宇宙弦，是高能物理和宇宙学研究中的一个活跃的交叉领域。

Pan Pearl River Delta Physics Olympiad 2021
2021 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 4 Problems, 40 Points) 简体版卷-1（共4题，40分）
(9:30 am – 12:00 pm, 15th May 2021)

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1. [10 points]
A uniform thin rigid rod of mass m is supported by two rapidly rotating rollers, whose axes are separated by a fixed distance a. The rod is initially placed at rest symmetrically as shown in Fig.1a.
质量為 m 的均匀细刚性杆由两个快速旋转的滚筒支撑，两个滚筒的轴线距离為 a。杆最初如图 1a 所示对称放置。

(a) [5 points] Assume that the rollers rotate in opposite directions as shown in Fig.1a. The coefficient of kinetic friction between the rod and the rollers is μ. Solve for the displacement $x(t)$ of the center C of the rod from roller 1 assuming $x(0) = x_0$ and $\dot{x}(0) = 0$.
(a) [5 分] 假设滚筒以相反方向旋转，如图 1a 所示。杆和滚筒之间的动摩擦系数为 μ。假设$x(0) = x_0$ 和 $\dot{x}(0) = 0$，求杆中心 C 从滚筒 1 量度的位移 $x(t)$。

(b) [5 points] Now consider the case in which the direction of rotation of the rollers are reversed, as shown in Fig.1b. Find the displacement $x(t)$, again assuming $x(0) = x_0$ and $\dot{x}(0) = 0$.
(b) [5 分] 现在考虑使滚筒的旋转方向反向的情况，如图 1b 所示。假设$x(0) = x_0$ 和 $\dot{x}(0) = 0$，求杆中心 C 从滚筒 1 量度的位移 $x(t)$。

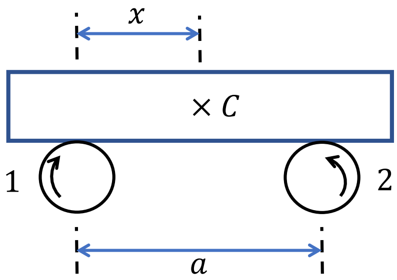


Fig. 1a

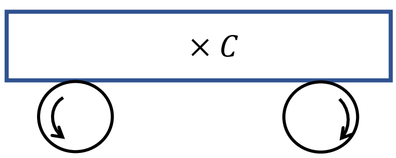
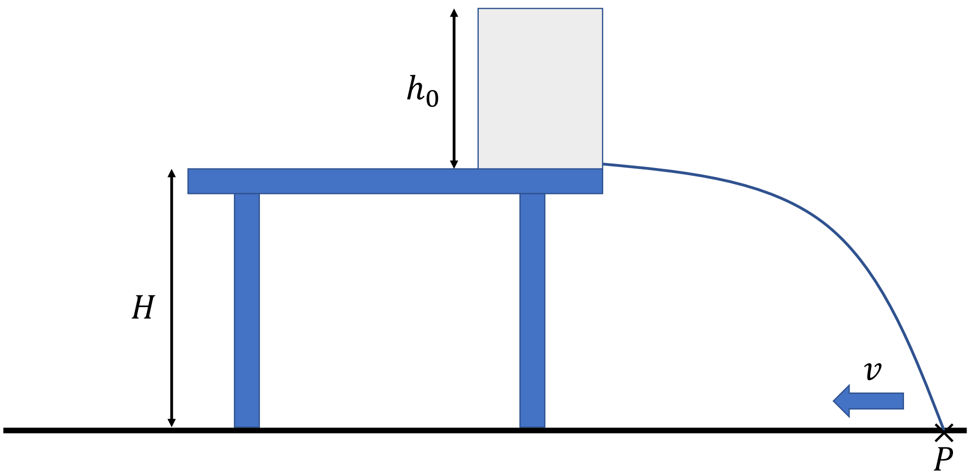


Fig. 1b

2. [10 points]

A cylindrical tank filled with liquid is placed on the side of the table. A small hole is opened on the wall of the tank near the bottom. The liquid flows out horizontally through the small hole, and hits at point P on the floor. The height of the table is H, the area of the hole is $\frac{1}{k}$ (assume $k \gg 1$) of the area of the bottom of the tank, and the initial height of the liquid in the tank is h_0. Find the velocity v at which the drop point P moves along the floor and the time T required for all liquid to flow out of the tank.

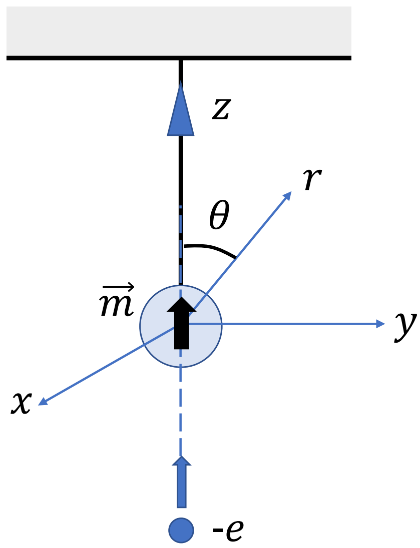
在桌边放着装有液体的圆柱形容器，容器壁靠近底部开有小孔，液体经小孔水平流出，液柱射在地板上的P点。桌面高度为H，孔的面积是容器底部面积的$\frac{1}{k}$(假设 $k \gg 1$)，原来容器中液体高h_0。求落点P沿地板移动的速率 v 及所有液体从容器中流出所需的时间 T。



3. [10 points]

A uniformly magnetized iron sphere of magnetic moment $\vec{m} = m\hat{z}$ and radius R is suspended from the ceiling by an insulating thread. A total charge $Q > 0$ is uniformly distributed throughout the iron sphere. We use the Cartesian coordinate where the origin is located at the center of the sphere, xy-plane is the horizontal plane and z-axis is pointing upward.

磁矩为 $\vec{m} = m\hat{z}$ 且半径为 R 的均匀磁化铁球通过绝缘线悬挂在天花板上。总电荷$Q > 0$ 均匀分布在整个铁球中。我们使用笛卡尔坐标，其原点位于球体的中心，xy-平面是水平面，z轴指向上方。



The magnetic field at position r due to a magnetic dipole $\vec{m}$ at origin is given by

由位於原點的磁偶极子 $\vec{m}$ 所產生的磁场， 在位置 $\vec{r}$ 处为

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi r^3}\left(\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m}\right)$$

(a) [3 points] The Poynting vector $\vec{S}(\vec{r})$ at point $\vec{r}$ is defined as
(a) [3 分] 坡印廷矢量$\vec{S}(\vec{r})$在点 $\vec{r}$ 上的定义为

$$\vec{S}(\vec{r}) = \frac{1}{\mu_0}\vec{E} \times \vec{B},$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields at point $\vec{r}$. Calculate the magnitude of the Poynting vector $\vec{S}(\vec{r})$ both inside and outside of the iron sphere.
其中 $\vec{E}$ 和 $\vec{B}$ 为点 $\vec{r}$ 的电场和磁场。计算铁球内部和外部的坡印廷矢量 $\vec{S}(\vec{r})$ 的大小。

(b) [2 points] In fact, the system has a non-zero angular momentum due to the electromagnetic field. The angular momentum L depends on the size of the sphere R, the charge Q and the magnetic dipole moment m on the sphere, and the physical constant μ_0. We shall write $L = K\mu_0^{\alpha}R^{\beta}Q^{\gamma}m^{\eta}$ where K is a dimensionless constant. Find α, β, γ and η using dimensional analysis.
(b) [2 分] 实际上，由于电磁场的存在，系统的角动量非零。角动量 L 取决于铁球的半径 R，球上的电荷 Q 和磁偶极矩 m 以及物理常数 μ_0。我们将写成 $L = K\mu_0^{\alpha}R^{\beta}Q^{\gamma}m^{\eta}$ ，其中 K 是无量纲常数。使用量纲分析找 α，β，γ 和 η。

(c) [3 points] To calculate the total angular momentum $\vec{L}$ of the system, it is given than the angular momentum density [angular momentum of the EM field per unit volume] $\vec{l}(\vec{r})$ of the electromagnetic field at a point $\vec{r}$ is
(c) [3 分] 为了计算系统的总角动量 $\vec{L}$，已知在点 $\vec{r}$ 处的电磁场角动量密度 [电磁场在每单位体积的角动量] $\vec{l}(\vec{r})$ 是

$$\vec{l}(\vec{r}) = \frac{1}{c^2}\vec{r} \times \vec{S}$$

where c is the speed of light. Calculate the total angular momentum $\vec{L}$ of the system.
其中 c 是光速。计算系统的总角动量 $\vec{L}$。

(d) [2 points] Electrons are injected into the iron sphere along the z-axis. The total amount of the charge in the sphere will reduce and the sphere will rotate. Find the angular speed of the sphere after the injection of N electrons. Assume that the moment of inertia of the iron sphere is I and each electron has charge $-e$.
(d) [2 分] 电子沿 z 轴注入铁球。球体中的总电荷量减少，并且球体将旋转。找出 N粒电子注入后球的角速度。假设铁球的惯性矩为 I，并且每个电子的电荷为 $-e$。

4. **Air Convection in Atmosphere 大气内之空气对流**

(a) [1 point] Consider a horizontal slab of air whose thickness (height) is dz. If this slab is at rest, the pressure holding it up from below must balance both the pressure from above and the weight of the slab. Use this fact to find an expression for $\frac{dP}{dz}$, the variation of pressure with altitude, in terms of the density ρ of air.
(a) [1 分] 考虑一薄块的空气，其厚度(高度)是dz。当这薄块处于静止状态时，从下方施加于薄块的压強必须平衡于从上面施加于薄块的压強和薄块的自身重量。由此,找出压強随高度变化的表达式 $\frac{dP}{dz}$，答案以空气密度 ρ 来表示。

(b) [2 points] Assume that the air is an ideal gas with molar mass M and the temperature T of the atmosphere is independent of height. Then the atmospheric pressure at height z is given by $P(z) = P(0)e^{-\lambda z}$. Find λ.

(b) [2 分] 假设空氣是摩尔质量 M 的理想氣體，而且大气的温度 T 随高度无关。因此，在高度为 z 的大气压強可以由 $P(z) = P(0)e^{-\lambda z}$ 表示。 求λ。

In practice, the atmospheric temperature depends on height. If the temperature gradient $\left|\frac{dT}{dz}\right|$ exceeds a certain critical value, convection will occur: warm, low-density air will rise, while cool, high-density air sinks. The decrease of pressure with altitude causes a rising air mass to expand adiabatically and thus to cool. The condition for convection to occur is that the rising air mass must remain warmer than the surrounding air despite this adiabatic cooling.

在实际情况下，大气的温度会随高度变化。当温度梯度 $\left|\frac{dT}{dz}\right|$ 超越一个临界值时，对流就会产生:低密度的热空气上升，高密度的冷空气则下降。随高度上升而下降的气压，使上升的空气团发生绝热膨胀，从而冷却。对流发生的条件是:上升中的气团纵然发生绝热冷却，仍须较周围的空气温暖。

(c) [2 points] Assume that the molar heat capacity of air at constant volume is $c_V = \frac{5}{2}R$. We can show that when air expands adiabatically, the temperature and pressure are related by the condition

$$\frac{dT}{dP} = a\frac{T}{P}$$

Find the constant a.

(c) [2 分] 假设空气的定容摩尔热容量为$c_V = \frac{5}{2}R$。由此可证明，当空气绝热膨胀时，温度和压強满足以下条件

$$\frac{dT}{dP} = a\frac{T}{P}$$

求常数a。

(d) [3 points] Assume that $\frac{dT}{dz}$ is just at the critical value for convection to begin, so that the temperature drop due to adiabatic expansion of the convecting air mass is the same as the temperature gradient of the surrounding air. Find a formula for $\frac{dT}{dz}$ in this case.

(d) [3 分] 假设 $\frac{dT}{dz}$ 正处于对流开始发生的临界值，以致对流空气由绝热膨胀引起的温度下降，等于周边空气的温度梯度。在此情况下,求 $\frac{dT}{dz}$ 的公式。

(e) [2 points] Calculate numerically the critical temperature gradient in part (d). Express your answer in K/km.

Data: The molar mass of the air is $M = 0.029\ kg, g = 9.8\ \mathrm{m/s^2}, R = 8.31$ J/mol/K, $T = 300$K.

(e) [2 分] 计算(d)部的临界温度梯度之数值。答案以 K/km 为单位。数值: $M = 0.029$ kg, $g = 9.8\ \mathrm{m/s^2}, R = 8.31$ J/mol/K, $T = 300$K。

~ End of Part 1 卷-1 完 ~

Pan Pearl River Delta Physics Olympiad 2021
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质量為 m 的均匀细刚性杆由两个快速旋转的滚筒支撑，两个滚筒的轴线距离為 a。杆最初如图 1a 所示对称放置。

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(a) [5 分] 假设滚筒以相反方向旋转，如图 1a 所示。杆和滚筒之间的动摩擦系数为 μ。假设$x(0) = x_0$ 和 $\dot{x}(0) = 0$，求杆中心 C 从滚筒 1 量度的位移 $x(t)$。

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(b) [5 分] 现在考虑使滚筒的旋转方向反向的情况，如图 1b 所示。假设$x(0) = x_0$ 和 $\dot{x}(0) = 0$，求杆中心 C 从滚筒 1 量度的位移 $x(t)$。

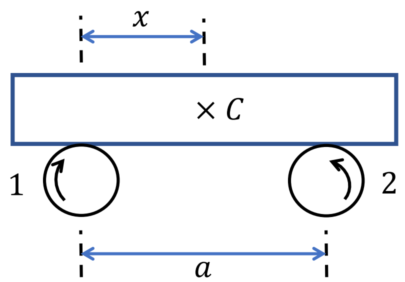


Fig. 1a

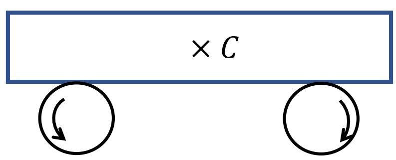
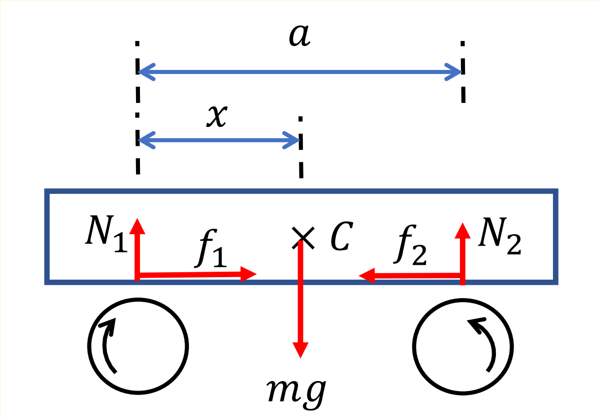


Fig. 1b

Solution:
(a) The free-body diagram of the rod is:



For equilibrium along the vertical direction, we have

$$N_1 + N_2 = mg$$
$$aN_2 = xmg$$
$$\Rightarrow N_1 = \left(1 - \frac{x}{a}\right) mg$$
$$\Rightarrow N_2 = \frac{x}{a} mg$$

The kinetic friction forces are

$$f_1 = \mu N_1 \quad and \quad f_2 = \mu N_2$$

With directions as shown in the figure.
Newton's 2nd law gives

$$m\ddot{x} = f_1 - f_2 = \frac{\mu mg}{a}(a - 2x)$$

Define $\xi = 2x - a$, we have

$$\ddot{\xi} = -\frac{2\mu g}{a}\xi$$

is a SHM.

$$\Rightarrow \xi(t) = 2x(t) - a = 2A\cos(\omega t + \phi)$$

where $\omega = \sqrt{\frac{2\mu g}{a}}$.

$$\Rightarrow x(t) = A\cos(\omega t + \phi) + \frac{a}{2}$$

At $t = 0$, $x(0) = x_0$ and $\dot{x}(0) = 0$, we have

$$x(t) = \left(x_0 - \frac{a}{2}\right)\cos\left(\sqrt{\frac{2\mu g}{a}}t\right) + \frac{a}{2}$$

(b) By reversing the direction of rotation of the rollers, we have

$$m\ddot{x} = f_2 - f_1 = \frac{\mu mg}{a}(2x - a)$$

Define $\xi = 2x - a$, we have

$$\ddot{\xi} = \frac{2\mu g}{a}\xi$$
$$\Rightarrow \xi(t) = 2Ae^{\omega t} + 2Be^{-\omega t}$$

where $\omega = \sqrt{\frac{2\mu g}{a}}$.

$$\Rightarrow x(t) = Ae^{\omega t} + Be^{-\omega t} + \frac{a}{2}$$

By the initial conditions, we have $A = B$ and

$$x(0) = x_0 = 2A + \frac{a}{2} \Rightarrow A = \frac{1}{2}\left(x_0 - \frac{a}{2}\right)$$

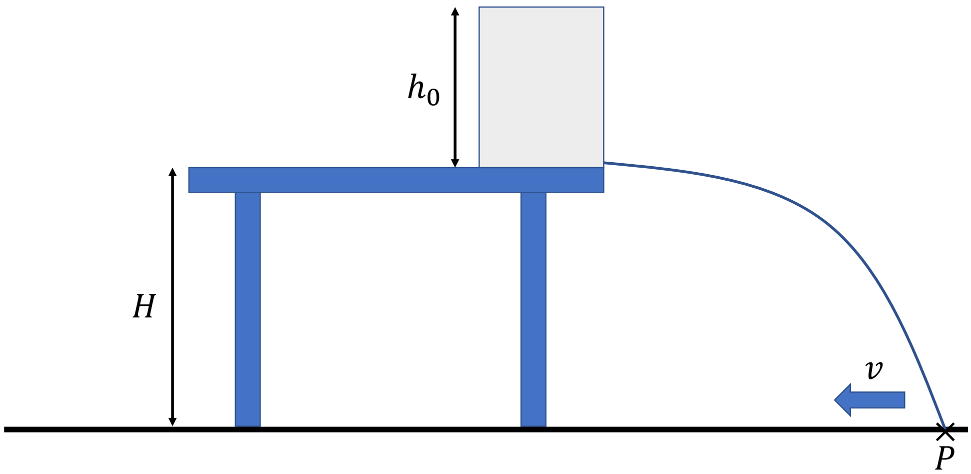
$$\Rightarrow x(t) = \left(x_0 - \frac{a}{2}\right)\frac{e^{\omega t} + e^{-\omega t}}{2} + \frac{a}{2} = \left(x_0 - \frac{a}{2}\right)\cosh \omega t + \frac{a}{2}$$

$$x(t) = \left(x_0 - \frac{a}{2}\right)\cosh\left(\sqrt{\frac{2\mu g}{a}}t\right) + \frac{a}{2}$$

2. [10 points]

A cylindrical tank filled with liquid is placed on the side of the table. A small hole is opened on the wall of the tank near the bottom. The liquid flows out horizontally through the small hole, and hits at point P on the floor. The height of the table is H, the area of the hole is $\frac{1}{k}$ (assume $k \gg 1$) of the area of the bottom of the tank, and the initial height of the liquid in the tank is h_0. Find the velocity v at which the drop point P moves along the floor and the time T required for all liquid to flow out of the tank.

在桌边放着装有液体的圆柱形容器，容器壁靠近底部开有小孔，液体经小孔水平流出，液柱射在地板上的P点。桌面高度为H，孔的面积是容器底部面积的$\frac{1}{k}$(假设 $k \gg 1$)，原来容器中液体高h_0。求落点P沿地板移动的速率 v 及所有液体从容器中流出所需的时间 T。



Solution:

(a) Assuming the when the height of the liquid inside the cylindrical tank is h, the drop of the water surface inside the tank has speed u and the liquid coming out from the hole has speed V. Since the volume of the liquid is unchanged, we have

$$V = ku$$

Bernoulli's equation gives

$$p_0 + \frac{1}{2}\rho V^2 = p_0 + \rho gh + \frac{1}{2}\rho u^2$$

$$\Rightarrow V = \frac{k\sqrt{2gh}}{\sqrt{k^2 - 1}}$$

The horizontal range of the liquid is

$$s = Vt = V \times \sqrt{\frac{2H}{g}} = \frac{k\sqrt{2gh}}{\sqrt{k^2-1}} \times \sqrt{\frac{2H}{g}} = \frac{2k\sqrt{hH}}{\sqrt{k^2-1}}$$

Therefore the position of P depends on the liquid height h. The speed of P is

$$v = \frac{ds}{dt} = \frac{2k\sqrt{H}}{\sqrt{k^2-1}}\frac{d}{dt}\sqrt{h} = \frac{k\sqrt{H}}{\sqrt{k^2-1}}\frac{1}{\sqrt{h}}\frac{dh}{dt} = \frac{k\sqrt{H}}{\sqrt{k^2-1}}\frac{u}{\sqrt{h}} = \frac{\sqrt{H}}{\sqrt{k^2-1}}\frac{V}{\sqrt{h}} = \frac{k}{k^2-1}\sqrt{2gH}$$

(b) There are 2 different ways to find T.

Method 1:

Initially, the position of P is

$$s_0 = \frac{2k\sqrt{h_0 H}}{\sqrt{k^2-1}}$$

Notice that v is independent of h, point P moves with the constant speed. When the tank is empty, P should be under the tank.

$$T = \frac{s_0}{v} = \frac{2k\sqrt{h_0 H}}{\sqrt{k^2-1}}\frac{k^2-1}{k}\frac{1}{\sqrt{2gH}} = \sqrt{\frac{2h_0(k^2-1)}{g}}$$

Method 2:

The speed of the water surface reads,

$$\frac{dh}{dt} = -u = -\frac{\sqrt{2gh}}{\sqrt{k^2-1}}$$

$$\Rightarrow \frac{dh}{\sqrt{h}} = -\sqrt{\frac{2g}{k^2-1}}dt$$

$$\Rightarrow 2\sqrt{h_0} = \sqrt{\frac{2g}{k^2-1}}T$$

$$\Rightarrow T = \sqrt{\frac{2h_0(k^2-1)}{g}}$$

****

Remark:

The calculation above is based on the assumption that $k \gg 1$ and the flow is approximately steady (i.e. the velocity of the fluid inside the tank doesn't change in time). In fact, we can generalize the calculation by considering the non-steady flow of the fluid.

We first consider the potential energy of the fluid inside the tank when the height of the fluid is h,

$$U(h) = \rho g A h \times \frac{h}{2} \Rightarrow \frac{dU}{dt} = \rho g A h \frac{dh}{dt} = -\rho g h A u = -\rho g h \left(\frac{A}{k}\right) V$$

where u is the velocity of the fluid inside the tank and V is the velocity of the fluid coming out from the hole ($u = \frac{V}{k}$).

Next, the total kinetic energy of the fluid inside the tank is (N.B. because of the conservation of mass/continuity equation, the speed of the fluid inside the tank will be the same),

$$K_{tank} = \frac{\rho u^2}{2} A h = \frac{1}{2} \rho \frac{Ah}{k^2} V^2$$

$$\Rightarrow \frac{dK_{tank}}{dt} = \frac{1}{2} \frac{\rho A}{k^2} \frac{dh}{dt} V^2 + \rho \frac{Ah}{k^2} V \frac{dV}{dt} = -\frac{1}{2} \frac{\rho A}{k^3} V^3 + \frac{\rho A}{k^2} hV \frac{dV}{dt}$$

Finally, the rate at which kinetic energy exits the tank via the hole is

$$\frac{dK_{exit}}{dt} = \frac{dm_{exit}}{dt} \frac{V^2}{2} = \frac{\rho A}{k} \frac{V^3}{2}$$

Conservation of energy implies,

$$\frac{dU}{dt} + \frac{dK_{tank}}{dt} + \frac{dK_{exit}}{dt} = 0$$

$$\Rightarrow 2gh = \left(1 - \frac{1}{k^2}\right) V^2 + \frac{2}{k} h \frac{dV}{dt}$$

We can change the independent variable form t to h using the chain rule in calculus,

$$\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} = \frac{dV}{dh}(-u) = -\frac{V}{k} \frac{dV}{dh} = -\frac{1}{2k} \frac{dV^2}{dh}$$

We obtain the ODE,

$$2gh = \left(1 - \frac{1}{k^2}\right) V^2 - \frac{1}{k^2} h \frac{dV^2}{dh}$$

$$\Rightarrow \frac{dV^2}{dh} - (k^2 - 1) \frac{V^2}{h} = -2gk^2$$

Introducing an integrating factor f such that

$$\frac{d}{dh}(fV^2) = f\frac{dV^2}{dh} + V^2\frac{df}{dh} = f\left(\frac{dV^2}{dh} - (k^2-1)\frac{V^2}{h}\right) = -2gk^2 f$$

$$\Rightarrow \frac{df}{dh} = -(k^2-1)\frac{f}{h}$$

$$\Rightarrow f = h^{1-k^2}$$

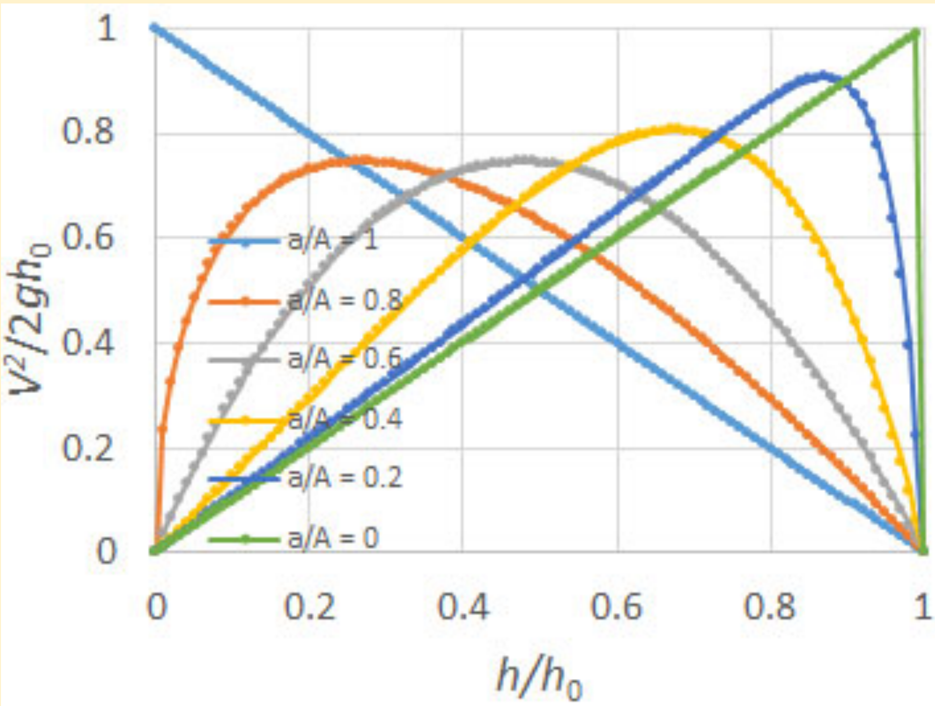
The ODE can be rewritten as

$$\frac{d}{dh}\left(h^{1-k^2}V^2\right) = -2gk^2h^{1-k^2}$$

Integrating the equation with the initial condition is that $V(h_0) = 0$, we have

$$V(h) = \sqrt{2gh}\sqrt{\frac{1-\left(\frac{h}{h_0}\right)^{k^2-2}}{1-\frac{2}{k^2}}}$$

The plot of the solution is shown below.



For the limiting case in which $k \to \infty$ (narrow opening) and $k \to 1$ (free fall of water), the solution reduces to

$$V(h, k=1) = \sqrt{2g(h_0 - h)}$$

$$V(h, k \to \infty) = \sqrt{2gh}$$

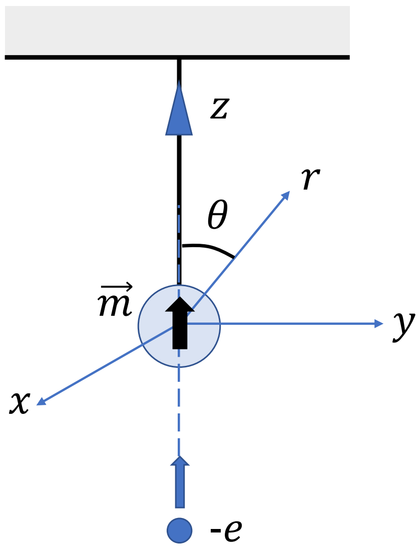
which is equal to our result obtained in part (a) in the limit of $k \to \infty$.

****

3. [10 points]
A uniformly magnetized iron sphere of magnetic moment $\vec{m} = m\hat{z}$ and radius R is suspended from the ceiling by an insulating thread. A total charge $Q > 0$ is uniformly distributed throughout the iron sphere. We use the Cartesian coordinate where the origin is located at the center of the sphere, xy-plane is the horizontal plane and z-axis is pointing upward.
磁矩为 $\vec{m} = m\hat{z}$ 且半径为 R 的均匀磁化铁球通过绝缘线悬挂在天花板上。总电荷$Q > 0$ 均匀分布在整个铁球中。我们使用笛卡尔坐标，其原点位于球体的中心，xy-平面是水平面，z轴指向上方。

The magnetic field at position r due to a magnetic dipole $\vec{m}$ at origin is given by
由位於原點的磁偶极子 $\vec{m}$ 所產生的磁场， 在位置 $\vec{r}$ 処为

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi r^3}\left(\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m}\right)$$



(a) [3 points] The Poynting vector $\vec{S}(\vec{r})$ at point $\vec{r}$ is defined as
(a) [3 分] 坡印廷矢量$\vec{S}(\vec{r})$在点 $\vec{r}$ 上的定义为

$$\vec{S}(\vec{r}) = \frac{1}{\mu_0}\vec{E} \times \vec{B},$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields at point $\vec{r}$. Calculate the magnitude of the Poynting vector $\vec{S}(\vec{r})$ both inside and outside of the iron sphere.
其中 $\vec{E}$ 和 $\vec{B}$ 为点 $\vec{r}$ 的电场和磁场。计算铁球内部和外部的坡印廷矢量 $\vec{S}(\vec{r})$ 的大小。

(b) [2 points] In fact, the system has a non-zero angular momentum due to the electromagnetic field. The angular momentum L depends on the size of the sphere R, the charge Q and the magnetic dipole moment m on the sphere, and the physical constant μ_0. We shall write $L = K\mu_0^\alpha R^\beta Q^\gamma m^\eta$ where K is a dimensionless constant. Find α, β, γ and η using dimensional analysis.
(b) [2 分] 实际上，由于电磁场的存在，系统的角动量非零。角动量 L 取决于铁球的半径 R，球上的电荷 Q 和磁偶极矩 m 以及物理常数 μ_0。我们将写成 $L = K\mu_0^\alpha R^\beta Q^\gamma m^\eta$ ，其中 K 是无量纲常数。使用量纲分析找 α，β，γ 和 η。

(c) [3 points] To calculate the total angular momentum $\vec{L}$ of the system, it is given than the angular momentum density [angular momentum of the EM field per unit volume] $\vec{l}(\vec{r})$ of the electromagnetic field at a point $\vec{r}$ is
(c) [3 分] 为了计算系统的总角动量 $\vec{L}$，已知在点 $\vec{r}$ 处的电磁场角动量密度 [电磁场在每单位体积的角动量] $\vec{l}(\vec{r})$ 是

$$\vec{l}(\vec{r}) = \frac{1}{c^2}\vec{r} \times \vec{S}$$

where c is the speed of light. Calculate the total angular momentum $\vec{L}$ of the system.
其中 c 是光速。计算系统的总角动量 $\vec{L}$。

(d) [2 points] Electrons are injected into the iron sphere along the z-axis. The total amount of the charge in the sphere will reduce and the sphere will rotate. Find the angular speed of the sphere after the injection of N electrons. Assume that the moment of inertia of the iron sphere is I and each electron has charge $-e$.
(d) [2 分] 电子沿 z 轴注入铁球。球体中的总电荷量减少，并且球体将旋转。找出 N粒电子注入后球的角速度。假设铁球的惯性矩为 I，并且每个电子的电荷为 $-e$。

Solution:

[Solution 1] Since iron is a conductor, the charges will redistribute on surface of the sphere.
(a) The electric field inside the sphere is zero and the electric field outside is

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^3}\vec{r}$$

Hence the Poynting vector inside the sphere is $\vec{S} = 0$. Outside the sphere,

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

where the magnetic field of a magnetic dipole is,

$$\vec{B} = \frac{\mu_0}{4\pi r^3}\left(\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m}\right)$$

$$\Rightarrow \vec{S} = \frac{Q}{4\pi\epsilon_0 r^3}\frac{1}{4\pi r^3}\left(\frac{3(\vec{m}\cdot\vec{r})\vec{r}\times\vec{r}}{r^2} - \vec{r}\times\vec{m}\right)$$

$$\Rightarrow \vec{S} = \frac{Qm}{16\pi^2\epsilon_0 r^5}(\hat{z}\times\hat{r})$$

$$\Rightarrow S = |\vec{S}| = \frac{Qm\sin\theta}{16\pi^2\epsilon_0 r^5}$$

(b) It is given that

$$L = K\mu_0^\alpha R^\beta Q^\gamma m^\eta$$

$$[L] = \frac{ML^2}{T}, [\mu_0] = \frac{ML}{Q^2}, [R] = L, [Q] = Q, [m] = \frac{QL^2}{T}$$

By dimensional analysis, we have

$$[Q]: -2\alpha + \gamma + \eta = 0$$

$$[M]: \alpha = 1$$

$$[L]: \alpha + \beta + 2\eta = 2$$

$$[T]: -\eta = -1$$

With 5 equations, we can get

$$\alpha = 1, \qquad \beta = -1, \qquad \gamma = 1, \qquad \eta = 1$$

And

$$L = K\frac{\mu_0 Qm}{R}$$

(c) The angular momentum density of the EM field is

$$\vec{l} = \vec{r} \times \frac{\vec{S}}{c^2} = \frac{\mu_0 Qm}{16\pi^2 r^4}\hat{r} \times (\hat{z} \times \hat{r}) = -\frac{\mu_0 Qm\sin\theta}{16\pi^2 r^4}\hat{e}_\theta$$

Because of symmetry, the total angular momentum has only the z-component, with magnitude

$$L_z = \int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta \int_R^\infty r^2 dr \left(\frac{\mu_0 Qm\sin\theta}{16\pi^2 r^4}\right)\sin\theta = \frac{\mu_0 Qm}{16\pi^2}\int_0^{2\pi} d\phi \int_0^\pi \sin^3\theta\, d\theta \int_R^\infty \frac{1}{r^2} dr$$

$$\int_0^\pi \sin^3\theta\, d\theta = -\int_0^\pi (1-\cos^2\theta) d\cos\theta = -\left(\cos\theta - \frac{1}{3}\cos^3\theta\right)\Big|_0^\pi = -\left(-2 + \frac{2}{3}\right) = \frac{4}{3}$$

$$\Rightarrow \vec{L} = \frac{\mu_0 Qm}{6\pi R}\hat{z}$$

(c) As the electrons are being injected on the sphere, the charge Q decreases, causing the electromagnetic angular momentum decreases. By the conservation of angular momentum, we have

$$\frac{\mu_0 Qm}{6\pi R} = I\omega + \frac{\mu_0(Q - Ne)m}{6\pi R}$$

$$\Rightarrow \omega = \frac{\mu_0 Nem}{6\pi RI}$$

The angular speed of the sphere is $\omega = \frac{\mu_0 Nem}{6\pi RI}$

[Solution 2] In an alternative interpretation of the question, the charges are assumed to be distributed uniformly inside the entire sphere.

(a) The electric field and magnetic field inside the sphere are

$$\vec{E} = \frac{Q\vec{r}}{4\pi\epsilon_0 R^3}$$

$$\vec{B} = \frac{2}{3}\mu_0\vec{M} = \frac{2}{3}\mu_0\frac{\vec{m}}{\frac{4}{3}\pi R^3} = \frac{\mu_0 m}{2\pi R^3}\hat{z}$$

(The derivation of the $\vec{B}$ field inside the magnetized sphere is in the end).

The Poynting vector inside the sphere becomes,

$$\vec{S}_{in} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{Qmr}{8\pi^2\epsilon_0 R^6}(\hat{r}\times\hat{z})$$

(c) The total angular momentum consists of 2 parts,

$$\vec{L} = \vec{L}_{out} + \vec{L}_{in},$$

where the angular momentum outside the sphere is

$$\vec{L}_{out} = \frac{\mu_0 Qm}{6\pi R}\hat{z}$$

Inside the sphere, we have

$$\vec{l} = \vec{r}\times\frac{\vec{S}}{c^2} = \frac{\mu_0 Qmr^2}{8\pi^2 R^6}\hat{r}\times(\hat{r}\times\hat{z}) = \frac{\mu_0 Qmr^2\sin\theta}{8\pi^2 R^6}\hat{e}_\theta$$

$$\vec{L}_{in} = \int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\, d\theta \int_R^{\infty} r^2 dr\left(-\frac{\mu_0 Qmr^2\sin\theta}{8\pi^2 R^6}\right)\sin\theta = -\frac{\mu_0 Qm}{15\pi R}\hat{z}$$

$$\Rightarrow \vec{L}_{tot} = \frac{\mu_0 Qm}{6\pi R}\hat{z} - \frac{\mu_0 Qm}{15\pi R}\hat{z} = \frac{\mu_0 Qm}{10\pi R}\hat{z}$$

(d) As the electrons are being injected on the sphere, the charge Q decreases, causing the electromagnetic angular momentum decreases. By the conservation of angular momentum, we have

1

$$\omega = \frac{\mu_0 Nem}{10\pi RI}$$

Appendix: Derivation of $\vec{B}$ field inside a uniformly magnetized sphere.

Let's consider a static magnetized sphere without electric current. Ampere's law gives

$$\vec{\nabla} \times \vec{H} = 0,$$

where $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ is the auxiliary field. Mathematically, we know that a curl-free field can be rewritten as a gradient of a scaler field ϕ_m,

$$\vec{H} = -\vec{\nabla}\phi_m,$$

Where ϕ_m is called the magnetic scaler potential.

$$\vec{\nabla} \cdot \vec{B} = \mu_0 \vec{\nabla} \cdot \left(\vec{H} + \vec{M}\right) = 0$$

$$\Rightarrow \vec{\nabla} \cdot \vec{H} = -\nabla^2 \phi_m = -\vec{\nabla} \cdot \vec{M} = \rho_m = 0. \text{ (inside the sphere)} \quad [1]$$

where $\rho_m = -\vec{\nabla} \cdot \vec{M}$ is the magnetic charge inside the sphere. However, there is a magnetic surface charge on the surface of the sphere, reads,

$$\sigma_m = \hat{r} \cdot \vec{M} = M \cos\theta.$$

Here, r and θ are sphereical coordinates. One of the boundary conditions at the surface is that the tangential component of $\vec{H}$ must be continuous,

$$\Rightarrow \phi_m(r = R^+, \theta) = \phi_m(r = R^-, \theta) \quad [2]$$

And the Gauss' law of magnetic charge gives,

$$\left.\frac{\partial \phi_m}{\partial r}\right|_{r=R^+} - \left.\frac{\partial \phi_m}{\partial r}\right|_{r=R^-} = -\sigma_m = -M\cos\theta \quad [3]$$

In other words, the magnetic charge on the surface of the sphere gives rise to a discontinuity in the radial gradient of the magnetic scaler potential at $r = R$.

The Laplace equation $\nabla^2 \phi_m = 0$ can be written in spherical coordinates,

$$\nabla^2 \phi_m = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\phi_m\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi_m}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}\phi_m = 0$$

Since the boundary conditions are independent of φ, we expect $\phi_m = \phi_m(r, \theta)$ is independent of φ.

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\phi_m\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi_m}{\partial \theta}\right) = 0$$

Next, we apply the separation of variables $\phi_m(r, \theta) = A(r)B(\theta)$, we have

1

$$\frac{d}{dr}\left(r^2\frac{d}{dr}A(r)\right)=\lambda A(r)\ \ and\ \ \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dB(\theta)}{d\theta}\right)=-\lambda B(\theta)$$

$$\Rightarrow r^2A''(r)+2rA'(r)-\lambda A(r)=0$$

Substitute $A(r)=r^n$,

The general solution of the Laplace's equation inside and outside the sphere can be written as

$$\phi_m^-(r,\theta)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)\quad for\ r\le R$$

and

$$\phi_m^+(r,\theta)=\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)\quad for\ r\ge R$$

where $\{A_l,B_l\}$ are constants can be determined by the boundary conditions and $P_l(\cos\theta)$ is the Legendre polynomials.

By the boundary conditions [2] and [3], we have

$$B_l=A_lR^{2l+1},$$

$$-\frac{2B_1}{R^3}-A_1=0,$$

$$-\frac{(l+1)B_l}{R^{l+2}}-lA_lR^{l-1}=0.\ \text{if}\ l\ne 1$$

Solving the algebraic equations, we have

$$A_l=B_l=0.\quad \text{for}\ l\ne 1$$

and

$$\Rightarrow B_1=\frac{MR^3}{3}\ \ and\ A_1=\frac{M}{3}$$

The scaler potentials are:

$$\phi_m^-(r,\theta)=\frac{Mr}{3}\cos\theta$$

$$\phi_m^+(r,\theta) = \frac{MR^3}{3r^2}\cos\theta$$

In the vacuum region outside the sphere, the magnetic field is

$$\vec{B} = \mu_0\vec{H} = -\mu_0\vec{\nabla}\phi_m^+ = \frac{\mu_0}{4\pi}\left(-\frac{\vec{m}}{r^3} + \frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5}\right)$$

where $\vec{m} = \frac{4}{3}\pi R^3\vec{M}$. Inside the magnetized sphere, $\vec{H} = -\vec{\nabla}\phi_m$ and $\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right)$

$$\Rightarrow \vec{H} = -\frac{\vec{M}}{3}$$

And

$$\vec{B} = \frac{2}{3}\mu_0\vec{M}$$

4. **Air Convection in Atmosphere 大气内之空气对流**

(a) [1 point] Consider a horizontal slab of air whose thickness (height) is dz. If this slab is at rest, the pressure holding it up from below must balance both the pressure from above and the weight of the slab. Use this fact to find an expression for $\frac{dP}{dz}$, the variation of pressure with altitude, in terms of the density ρ of air.

(a) [1 分] 考虑一薄块的空气，其厚度(高度)是dz。当这薄块处于静止状态时，从下方施加于薄块的压強必须平衡于从上面施加于薄块的压強和薄块的自身重量。由此,找出压強随高度变化的表达式 $\frac{dP}{dz}$，答案以空气密度 ρ 来表示。

(b) [2 points] Assume that the air is an ideal gas with molar mass M and the temperature T of the atmosphere is independent of height. Then the atmospheric pressure at height z is given by $P(z) = P(0)e^{-\lambda z}$. Find λ.

(b) [2 分] 假设空氣是摩尔质量 M 的理想氣體，而且大气的温度 T 随高度无关。因此，在高度为 z 的大气压強可以由 $P(z) = P(0)e^{-\lambda z}$ 表示。 求λ。

In practice, the atmospheric temperature depends on height. If the temperature gradient $\left|\frac{dT}{dz}\right|$ exceeds a certain critical value, convection will occur: warm, low-density air will rise, while cool, high-density air sinks. The decrease of pressure with altitude causes a rising air mass to expand adiabatically and thus to cool. The condition for convection to occur is that the rising air mass must remain warmer than the surrounding air despite this adiabatic cooling.

在实际情况下，大气的温度会随高度变化。当温度梯度 $\left|\frac{dT}{dz}\right|$ 超越一个临界值时，对流就会产生:低密度的热空气上升，高密度的冷空气则下降。随高度上升而下降的气压，使上升的空气团发生绝热膨胀，从而冷却。对流发生的条件是:上升中的气团纵然发生绝热冷却，仍须较周围的空气温暖。

(c) [2 points] Assume that the molar heat capacity of air at constant volume is $c_V = \frac{5}{2}R$. We can show that when air expands adiabatically, the temperature and pressure are related by the condition

$$\frac{dT}{dP} = a\frac{T}{P}$$

Find the constant a.

(c) [2 分] 假设空气的定容摩尔热容量为$c_V = \frac{5}{2}R$。由此可证明，当空气绝热膨胀时，温度和压強满足以下条件

$$\frac{dT}{dP} = a\frac{T}{P}$$

求常数a。

(d) [3 points] Assume that $\frac{dT}{dz}$ is just at the critical value for convection to begin, so that the temperature drop due to adiabatic expansion of the convecting air mass is the same as the temperature gradient of the surrounding air. Find a formula for $\frac{dT}{dz}$ in this case.

(d) [3 分] 假设 $\frac{dT}{dz}$ 正处于对流开始发生的临界值，以致对流空气由绝热膨胀引起的温度下降，等于周边空气的温度梯度。在此情况下,求 $\frac{dT}{dz}$ 的公式。

(e) [2 points] Calculate numerically the critical temperature gradient in part (d). Express your answer in K/km.

Data: The molar mass of the air is $M = 0.029\ kg$, $g = 9.8\ \mathrm{m/s^2}$, $R = 8.31$ J/mol/K, $T = 300$K.

(e) [2 分] 计算(d)部的临界温度梯度之数值。答案以 K/km 为单位。数值: $M = 0.029$ kg, $g = 9.8\ \mathrm{m/s^2}$, $R = 8.31$ J/mol/K, $T = 300$K。

Solution:

(a) Consider a horizontal slab of air whose thickness (height) is dz and the cross-sectional area is A. In equilibrium, we have

$$P(z)A = \rho g A dz + P(z+dz)A \Rightarrow \frac{dP}{dz} = -\rho g$$

(b) Using the ideal gas law, $PV = nRT$,

$$P = \left(\frac{m}{V}\right)\frac{RT}{M} \Rightarrow \rho = \frac{MP}{RT}$$

Where M is the molar mass, m is the total mass and n is the number of mole of the gas.

$$\Rightarrow \frac{dP}{dz} = -\frac{Mg}{RT}P$$

$$\Rightarrow P(z) = P(0)e^{-\lambda z}$$

Where $\lambda = \frac{Mg}{RT}$.

(c) Using the 1st law,

$$dU = dQ - dW$$

For 1 mole of gas during the adiabatic expansion, $dU = c_V dT = \frac{5}{2} RdT$, $dQ = 0$ and $dW = PdV$

$$\Rightarrow \frac{5}{2} RdT = -PdV$$

Ideal gas law $PV = RT$ implies

$$PdV + VdP = RdT$$

$$\Rightarrow -\frac{5}{2} RdT + VdP = RdT$$

$$\Rightarrow \frac{7}{2} RdT = VdP$$

$$\Rightarrow \frac{dT}{dP} = \frac{2V}{7R} = \frac{2}{7}\frac{T}{P}$$

Therefore, $a = \frac{2}{7}$.

(d) Finally,

$$\frac{dT}{dz} = \left(\frac{dT}{dP}\right)\left(\frac{dP}{dz}\right) = \frac{2}{7}\frac{T}{P}\left(-\frac{Mg}{RT}P\right) = -\frac{2Mg}{7R}$$

(e) Critical temperature gradient is

$$\frac{dT}{dz} = -\frac{2Mg}{7R} = -9.8 \text{ K/km}$$

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Simplified Chinese Part-2 (Total 2 Problems, 60 Points)
简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 15 May 2021)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。

All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。

There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。

Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。

Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。

Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。

If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。

At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Quasicrystals (28 points)
问题 1: 准晶体 (28 分)

In 2011, the Nobel Prize in Chemistry was awarded to the discovery of quasicrystals. Nowadays, quasicrystals can be found in many applications such as the hardening of steel. How do quasicrystals differ from ordinary crystals?

2011 年，准晶体的发现者荣获诺贝尔化学奖。至今，我们已可见到准晶体不同的应用，例如钢的硬化。究竟准晶体与普通晶体有何不同？

In crystals, atoms are arranged in a periodic manner. The structure of crystals is known by periodically replicating the basic unit of the arrangement of a small number of atoms (Fig. 1 (Left)).

在晶体中，原子以周期性方式排列。通过周期性地复制少量原子排列的基本单位，就可以得到晶体的结构（图 1（左））。

On the other hand, atoms in quasicrystals are arranged in an orderly manner, but the local arrangement cannot be repeated by replication (Fig. 1(Right)).

另一方面，准晶体中的原子以有序的方式排列，但是局部的排列不能通过复制而得到全部结构（图 1（右））。

However, quasicrystals are far from random. They have a “hidden order”. For example, the structure in Fig. 1(Right) can be considered as a 5-dimensional cubic structure projected onto two dimensions. To understand this idea, we will consider a 1-dimensional quasicrystal projected from a 2-dimensional square lattice in this problem.

但是，准晶体远远不是随机的。他们有一个“隐藏的规律”。例如，图 1（右）中的结构可以考虑为投影到二维空间的 5 维立方结构。为了理解这个想法，我们在本题中将考虑从二维方格投影的一维准晶体。

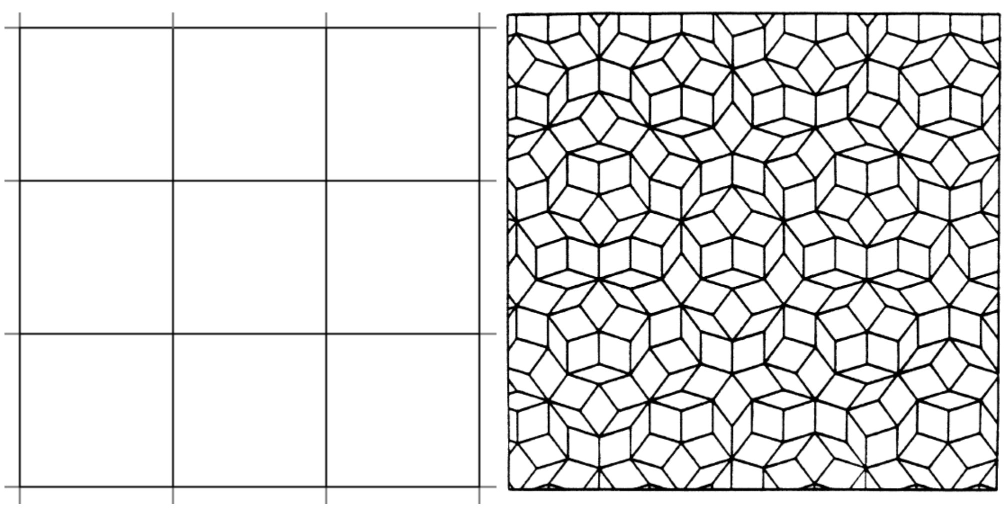


Fig. 1: (Left) A crystal (Right) A quasicrystal, known as Penrose tiling.
图 1：（左）晶体（右）称为 Penrose tiling 的准晶体。

A. Structure of Quasicrystals (14 points) 准晶体的结构（14 分）

Figure 2 shows a two-dimensional lattice in which the atoms are located at $(x_1, x_2) = (m_1 a, m_2 a)$ where m_1, m_2 are integers and a is the lattice spacing. In Section A, we assume $a = 1$. We construct a stripe defined by the condition

图 2 显示了一个二维晶格，其中原子位于$(x_1, x_2) = (m_1 a, m_2 a)$，$m_1$、$m_2$是整数，而$a$是晶格间距。在 A 部中，我们假设$a = 1$。 我们考虑一个条带，其定义为

$$\frac{x_1}{\tau} \leq x_2 < \frac{x_1}{\tau} + \tau.$$

τ is the irrational number $\tau = (1 + \sqrt{5})/2$. The inclination angle α of the strip is given by

τ 是无理数 $\tau = (1 + \sqrt{5})/2$。条带的倾角为

$$\alpha = \arctan \frac{1}{\tau}.$$

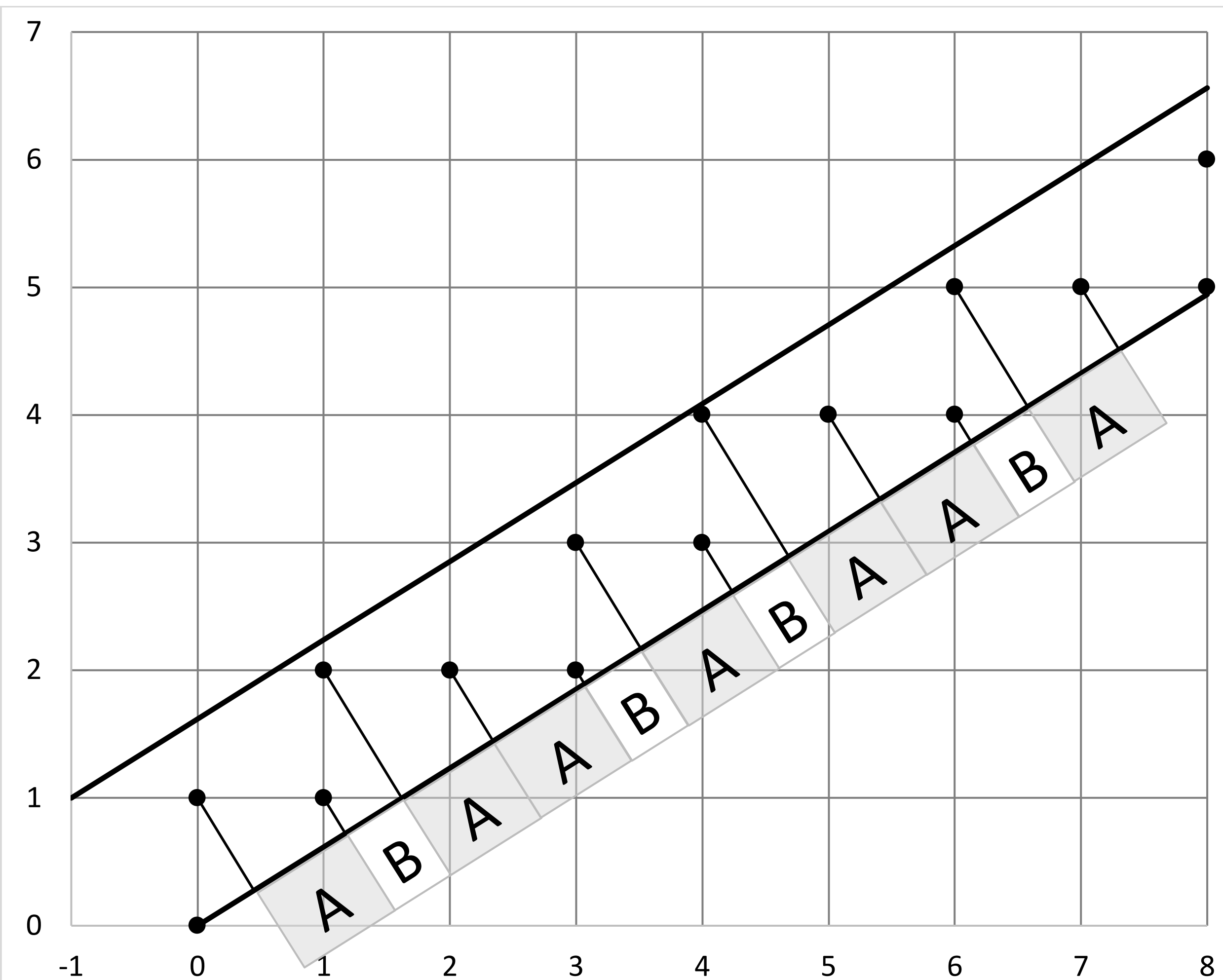


Fig. 2: Illustration of the projection method of obtaining a quasicrystal. 图 2：以投影法显示准晶体。

We project all lattice points lying within the strip to the line L_1 defined by $x_2 = x_1/\tau$. Since τ is an irrational number, the projected atomic positions on L_1 form a 1-dimensional quasicrystal. The lattice spacing now have two possible values.
我们将位于条带内的所有晶格点投影到L_1线上，L_1的定义为$x_2 = x_1/\tau$。由于τ是无理数，投影到L_1线上的原子位置便形成了一维准晶体。现在，晶格间距有两个可能的值。

A1	Calculate the lengths of the lattice spacings A and B. Write your answer as an expression containing τ. 计算晶格间距 A 和 B。答案以含τ的表达式写出。	2 points 2 分

A2	Write the position $z(m_1, m_2)$ of the atom of the quasicrystal projected from (m_1, m_2) onto the line L_1 (that is, the displacement of the atom from (0,0)). Write your answer as an expression containing τ. 写下从(m_1, m_2)投影到L_1线上的准晶体原子位置$z(m_1, m_2)$（即从(0,0)到原子的位移）。答案以含τ的表达式写出。	1 point 1 分

A3	Calculate the average lattice spacing d when the lattice length is very long. Write your answer as an expression containing τ. 计算当晶格长度很长时的平均晶格间距d。答案以含τ的表达式写出。	2 points 2 分

Note that the configuration of the lattice spacings A and B are no longer regular. It was observed that the configuration can be generated by the famous Fibonacci sequence, which uses the substitution rule
注意，晶格间距 A 和 B 的排列不再规则。我们观察到晶格的结构可以由著名的 Fibonacci sequence 生成，该序列使用替换规则

$$A \to AB, \quad B \to A.$$

For example, the configuration of the first seven spacings $ABAABABA$ in Fig. 2 is obtained by five substitutions starting from B.
例如，图 2 中前七个间距$ABAABABA$的排列是通过从 B 开始的五个替换获得的。

A4	Write the sequence after six substitutions starting from B. 写下从 B 开始六次替换后的序列。	1 point 1 分

A5	Let n_A and n_B be the number of A and B in the sequence. Write the numbers n'_A and n'_B of A and B after one substitution. 令n_A和n_B为序列中A和B的数目。写下一次替换后，A 和 B 的数目n'_A 和n'_B。	2 points 2 分

A6	Calculate the ratio n_A/n_B when the sequence is very long. 计算当序列很长时的比例n_A/n_B。	2 points 2 分

For further analysis, we introduce two mathematical notations. For a real number x, $\lfloor x \rfloor$ denotes the integer part of x, and $\mathrm{frac}(x)$ denotes the fractional part of x. For example,

为了进一步分析，我们引入两个数学符号。对于实数x，$\lfloor x \rfloor$表示 x的整数部分，而$\mathrm{frac}(x)$则表示x的小数部分。例如，

$$\lfloor 1.73 \rfloor = 1 \quad \text{and} \quad \mathrm{frac}(x) = 0.73.$$

Labeling the atom at $(0,0)$ as $j = 0$, the position of the jth atom in the quasicrystal is given by

将$(0,0)$处的原子标记为 $j = 0$，准晶体中第j个原子的位置为

$$z_j = jd + \mathrm{frac}\left(\frac{j}{\tau}\right)\Delta.$$

A7	Derive Δ as an expression containing τ. 推导Δ的表达式，式中含有τ。	2 points 2 分

A8	Calculate the coordinate (m_1, m_2) of the atom with $j = 101$. 计算$j = 101$的原子的坐标(m_1, m_2)。	2 points 2 分

B. Diffraction Pattern of Quasicrystals (14 points) 准晶体的衍射图案（14 分）

Quasicrystals were first discovered by observing their characteristic diffraction pattern.

准晶体是首先通过观察其特征衍射图案来发现的。

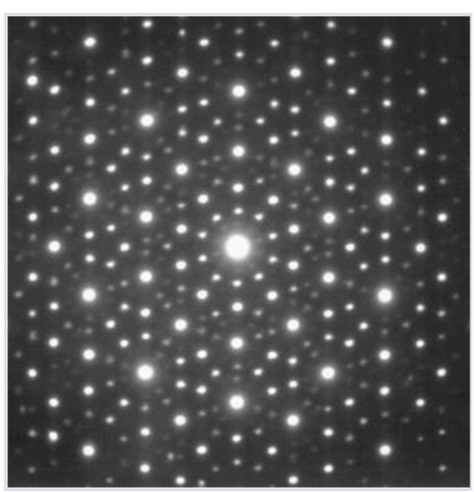


Fig. 3: Electron diffraction pattern of a quasicrystal. 图 3：准晶体的电子衍射图。

Figure 3 shows the electron diffraction pattern of a quasicrystal. A crystal is said to have n-fold symmetry if its diffraction pattern is identical if it is rotated by an angle of $2\pi/n$.

图 3 显示一种准晶体的电子衍射图。如果晶体旋转角度为$2\pi/n$后，其衍射图案相同，则该晶体具有n倍对称性。

B1	Identify the symmetries of the diffraction pattern in Fig. 3. 辨认图 3 衍射图案的对称性。	2 points 2 分

To understand how the diffraction pattern can be derived from the projection method, we first consider the diffraction pattern of a 1-dimensional crystal.

为了了解如何从投影法中得出衍射图案，我们首先考虑一维晶体的衍射图案。

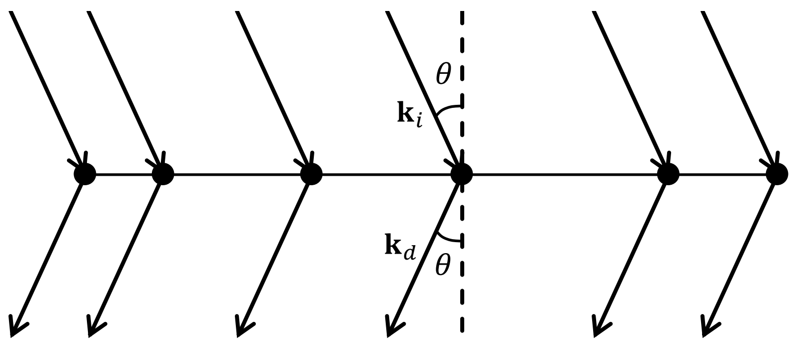


Fig. 4: Light diffraction by a lattice. 图 3：晶格的光衍射。

As shown in Fig. 4, the 1-dimensional lattice consists of N atoms. The position of atom j is x_j. A light wave with wavevector $\mathbf{k}_i$ is incident on the lattice at an angle θ with the normal direction and is diffracted at the same angle with the normal. The diffracted wave has a wavevector $\mathbf{k}_d$ with the same magnitude as $\mathbf{k}_i$. The change in the wavevector is denoted as

如图 3 所示，一维晶格由N个原子组成。 原子j的位置是x_j。光波入射到晶格中，入射波矢量为$\mathbf{k}_i$，与法线成角度θ，衍射后方向与法线成相同的角度。衍射波的波矢$\mathbf{k}_d$ 的大小与$\mathbf{k}_i$相同。 波矢的变化表示为

$$\mathbf{q} = \mathbf{k}_d - \mathbf{k}_i.$$

The magnitudes of the wavevectors are denoted as $|\mathbf{k}_d| = |\mathbf{k}_i| = k$ and $|\mathbf{q}| = q$. Note that q is a monotonic function of the diffraction angle θ, and so can represent the diffraction direction.

波矢的大小表示为$|\mathbf{k}_d| = |\mathbf{k}_i| = k$和$|\mathbf{q}| = q$。注意，$q$是衍射角$\theta$的单调函数, 因此可以表示衍射方向。

B2	Write the expression of q as a function of the diffraction angle θ. 写下q作为衍射角θ函数的表达式。	1 point 1 分

The positions and magnitudes of the diffraction peaks are given by the structure factor defined as
衍射峰的位置和大小由结构因子给出，其定义为

$$S(\mathbf{q})=\frac{1}{N}\left[\sum_{j=1}^{N}\cos(q_x x_j)\right]^2+\frac{1}{N}\left[\sum_{j=1}^{N}\sin(q_x x_j)\right]^2,$$

where q_x is the x component of $\mathbf{q}$. 其中q_x是$\mathbf{q}$的x分量。

Remark: Students who are familiar with complex numbers may use the definition

备注：熟悉复数的同学可以使用下列定义

$$S(\mathbf{q})=\frac{1}{N}\left|\sum_{j=1}^{N}\exp(iq_x x_j)\right|^2.$$

Consider a 1-dimensional periodic lattice in which $x_j=jd$. 考虑一维周期晶格，其中$x_j=jd$。

B3	What are the values of q at the peak positions of the diffraction pattern of the periodic lattice? 在周期晶格的衍射图案中，峰值位置处的q值是多少？	2 points 2 分

Now consider the case that each atom in the periodic lattice is “smeared” out to a length b, where $b<d$. This means that the density $\rho(x)$ of the lattice becomes

现在考虑以下情况：周期晶格中的每个原子都被“抹散”到长度b，其中$b<d$。这意味着晶格的密度$\rho(x)$变为

$$\rho(x)=\begin{cases}\frac{1}{b} & jd\le x\le jd+b,\\ 0 & \text{otherwise.}\end{cases}$$

The diffraction peaks do not have the same magnitude any longer. 衍射峰不再具有相同的大小。

B4	Calculate the magnitudes of the diffraction peaks at $\mathbf{q}$ of the smeared lattice. 计算抹散晶格$\mathbf{q}$处衍射峰的大小。	2 points 2 分

Now consider the quasicrystal in Part A. Since there are two incommensurate lattice spacings in the quasicrystal, its diffraction peaks are given by wavevectors with a pair of indices,

现在考虑 A 部中的准晶体。由于在准晶体中存在两种不相称的晶格间距，因此准晶体的衍射峰由具有一对指数的波矢给出，

$$q_{mn} = \frac{2\pi}{d}\left(m + \frac{n}{\tau}\right).$$

B5	Consider the phase $\phi = q_{mn} z_j$ in the structure factor of quasicrystals. Write the expression of the phase in the form 考虑准晶体中结构因子的相$\phi = q_{mn} z_j$。试写下相的表达式 $$\phi = 2\pi F + X \mathrm{frac}\left(\frac{j}{\tau}\right),$$ where F is an integer and X is a real number. 其中F是整数，X是实数。	2 points 2 分

B6	Calculate the magnitude of the diffraction peak at q_{mn}. 计算在q_{mn}处的衍射峰的大小。	2 points 2 分

B7	Find (m, n) for the highest diffraction peak in the range $0 \le n \le 3$, excluding $(m, n) = (0,0)$. Then calculate q_{mn} (in units of $2\pi/d$) and the magnitude of this peak. 找出在范围$0 \le n \le 3$内最高衍射峰(m, n)，不包括$(m, n) = (0,0)$。 然后计算q_{mn}（以$2\pi/d$为单位）和该峰值的大小。	3 points 3 分

Problem 2: Black Hole Physics (32 points)
问题 2: 黑洞物理 (32 分)

Black holes are the most mysterious objects in our universe. A black hole is surrounded by an event horizon (horizon for short). Anything that falls through the horizon into the black hole cannot escape. If a black hole is stationary, not rotating, has no electric charge, then the horizon is spherical, with radius $R = 2GE/c^4$, which can be also written as $R = 2GM/c^2$ using Einstein's energy-mass relation $E = Mc^2$. Here G, c, M and E are the Newton's constant, speed of light, and mass and energy of the black hole, respectively. The horizon area is thus $A = 4\pi R^2$. The singularity "inside" the black hole is one of the greatest mysteries in the theory of gravity, since the energy density of the singularity appears to diverge and the classical general relativity fails to operate there.

黑洞是宇宙中最神秘的天体。黑洞被事件视界 (简称视界) 环绕。通过视界掉进黑洞的东西不能逃出来。如果黑洞是稳态的，不旋转也不带电，则黑洞视界是球形的，其半径为 $R = 2GE/c^4$，利用爱因斯坦的质能关系 $E = Mc^2$，视界半径也可以写作 $R = 2GM/c^2$ 。这里 G, c, M 和 E 分别为牛顿引力常数、光速、黑洞质量和黑洞的能量。视界面积是 $A = 4\pi R^2$。黑洞内部的奇点是最神秘的引力现象之一，在奇点附近，能量密度趋于无穷大，经典广义相对论不再适用。

In the following, we will discuss the formation, thermodynamics and rotation of black holes, and how a civilization may use black holes as power plants.

在本题中，我们将讨论黑洞的形成、热力学和旋转黑洞，以及从黑洞提取能量的可能性。

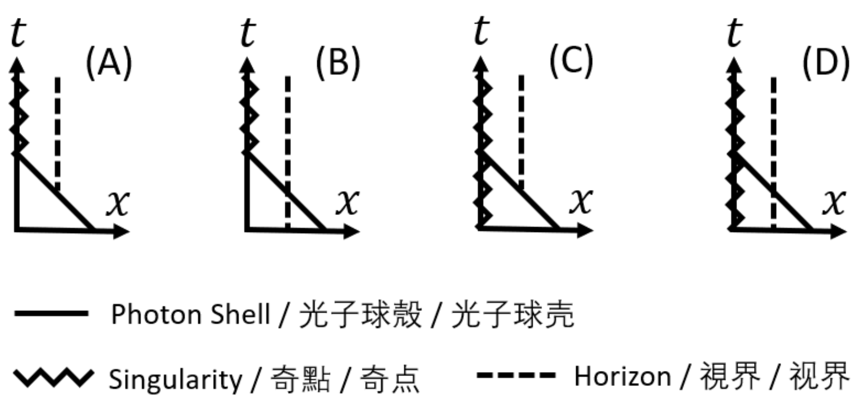
Note: To avoid the usage of general relativity, in this problem, no concepts about curved spacetime will be introduced. You do not need to think about curved spacetime when working on this problem.

注：为避免使用广义相对论知识，本题将不会涉及时空弯曲等概念。答题时不需要考虑时空弯曲。

PART A. FORMATION AND PROPERTIES 黑洞的形成和性质

We study a simple formation mechanism of black holes. Consider a spherical shell of photons (quanta of light) is moving towards the center of the shell to form a black hole. The self-interaction of the photons can be ignored.

我们研究一个简单的黑洞形成机制。考虑一个光子 (光的量子) 球壳。这个球壳中的光子向球壳中心运动，以形成黑洞。我们忽略光子之间的自相互作用。

A1	Assume that the wavelength of photons is short enough, and thus the shell is thin (this short wavelength assumption only applies for this Question A1, and may not apply for later questions), which of the following describes the formation of the horizon and the singularity of a black hole? 假设光子波长足够短，所以球壳很薄 (这个短波长近似只用在本小题，即 A1 中，后面的题中我们不再假设短波长近似)。下列哪一个图像描述了黑洞视界和奇点的形成？  	1 point 1 分

A2	Suppose the photons in the shell all have wavelength λ. Thus, the energy of each photon is $E_\gamma = hc/\lambda$, where h is the Planck's constant. To make a black hole with horizon radius R, what is the number of photons N needed? 假设球壳中所有光子的波长都是 λ。所以，每个光子的能量为 $E_\gamma = hc/\lambda$，其中 h 是普朗克常数。为了形成视界半径为 R 的黑洞，求所需的光子数 N。	1 point 1 分

Entropy of infalling photons 坍缩过程中光子的熵

A3	For a shell of N photons, the entropy can be written as $S_{N\gamma} = \alpha\, k_B^{c_1} N^{c_2}$. Here k_B is the Boltzmann's constant and α is a dimensionless constant of order 1. (In the below analytical formulae, α should be kept explicitly. In order of magnitude estimations, α can be set to 1.) Find integer numbers c_1 and c_2. N 个光子组成的球壳的熵可以写成 $S_{N\gamma} = \alpha\, k_B^{c_1} N^{c_2}$。这里 k_B 是玻尔兹曼常数，$\alpha \sim O(1)$ 是一个无量纲常数。(在下面各题的解析公式中，请保留α。在数量级估计中，可以将 α 的值取为 1). 求整数 c_1 和 c_2。	1 point 1 分

Entropy of the black hole 黑洞的熵

A4	Taking $\lambda = R$, we have the entropy of the black hole $S = S_{N\gamma}$. (If $\lambda > R$, the photons are too non-local to form a black hole. Thus, the $\lambda = R$ photons carry the largest amount of information.) Write S in terms of R, α and the constants of Nature. 设 $\lambda = R$，我们得到黑洞的熵 $S = S_{N\gamma}$。(这是因为，如果 $\lambda > R$，则光子延展范围太大，不能坍缩成黑洞。所以，$\lambda = R$ 光子可以携带最多的信息。) 请用 R, α 和基本物理常数表示 S。	1 point 1 分

A5	Based on the black hole entropy formula, which of the physical observation is *incorrect*? Choose one from the below answers. A. Black hole entropy is an interdisciplinary research direction of thermal, quantum and gravitational physics. B. The black hole entropy is an extensive quantity which scales as the volume of the black hole. C. The existence of black entropy indicates that black hole should contain many microstates. D. Black holes are gravitational systems with non-perturbative quantum effects, and are thus a key to quantum gravity. 根据黑洞熵的公式，下面哪项是错误的？请只选择一项。 A. 黑洞熵是热力学、量子理论和引力的交叉学科。 B. 黑洞熵是广延量，与黑洞的体积成正比。 C. 黑洞存在熵，所以也应该存在微观状态。 D. 黑洞是具有非微扰量子特征的引力系统，所以是通向量子引力的一把钥匙。	1 point 1 分

Part B. Entropy bounds of nature 黑洞熵限

It is conjected that black holes are the densest objects of nature, not only in energy, but also in entropy and information. Based on this observation, estimate the quantities in Part B. By estimate, you need to get the correct order of magnitude. Order one coefficients may be neglected.

物理学家猜测，黑洞不仅是世界上能量密度最大的物体，也是熵密度、信息密度最大的物体。基于这个猜测，请对 Part B 中的物理量做数量级估计。你只需要估计正确的数量级。$O(1)$ 的常数可以忽略。

B1	Nowadays, computer hard disks store information with a day-by-day increasing information density. However, to store information, enough number of states, and thus enough entropy is needed. This is understood from Boltzmann's statistical interpretation of entropy: $S = k_B \ln \Omega$, where Ω is the possible number of states of the system. Consider a spherical hard disk in the vacuum, with capacity $1\text{Tb} = 10^{12}$ bit. What is the minimal radius of this hard disk? 目前，电脑硬盘可以存储的信息密度越来越大。但是，为了存储信息，我们需要足够多的状态数，所以需要足够多的熵。这可以从玻尔兹曼熵的统计解释中看出：$S = k_B \ln \Omega$，其中 Ω 是系统可能处于的状态的数量。考虑一个真空中的球形硬盘，容量为 $1\text{Tb} = 10^{12}$比特。求硬盘的最小半径。	2 points 2 分

The Bekenstein Entropy bound 贝肯斯坦熵限

B2	Consider a clump of matter with mass M_m (and thus energy E_m) and radius R_m. When this clump of matter falls into a black hole (the black hole has existed before the clump of matter falls in), we require that R_m should be not greater than the horizon radius of the initial black hole, to make sure this clump of matter can fall in. Denote the entropy of this clump of matter as S_m. Find a universal upper bound of S_m, in terms of E_m and R_m, but independent of parameters of the black hole, or Newton's gravitational constant G. 考虑质量为 M_m (所以能量为 E_m)，半径为 R_m 的一块物质。当这块物质掉进一个黑洞时 (物质掉落前，黑洞就已经存在了)，我们要求 R_m 不大于黑洞本来的视界半径。因为这样才能确保这块物质掉进去。利用这个过程，求这块物质的熵 S_m 的普适上限。你导出的 S_m 的上限需要用 E_m 和 R_m 表示，但不依赖于黑洞的参数，也不依赖于牛顿引力常数 G。 Note: Necessary steps of derivation is required. 注：需要写出必要的推导步骤。	4 points 4 分

B3	For a 1Tb hard disk with 1nm radius, what is the minimal mass of the hard disk? 对于一个半径为 1nm，容量为 1Tb 的硬盘，硬盘的质量至少为多大？	1 point 1 分

Part C. Black hole Temperature and radiation 黑洞的温度和辐射

C1	Find the black hole temperature T in terms of horizon radius R. 求黑洞的温度 T，用视界半径 R 表示。	2 points 2 分

HAWKING RADIATION 霍金辐射

C2	According to the Stefan-Boltzmann law, an object with a temperature T should radiate. Calculate the radiation power P in terms of the horizon radius R.	2 points 2 分

	由斯蒂凡-玻尔兹曼定律，具有温度 T 的物体会发出辐射。计算辐射的功率 P，用视界半径 R 表示。 Note: the Stefan-Boltzmann constant σ can be written in more fundamental quantities as $\sigma = \frac{2\pi^5 k_B^4}{15c^2h^3}$. 注：斯蒂凡-玻尔兹曼常数 σ 可以由更基本的物理量表达：$\sigma = \frac{2\pi^5 k_B^4}{15c^2h^3}$。	

C3	Primordial black holes are a conjected type of black holes, which has existed almost from the "born" of the universe till now. Denote the mass of the primordial black hole by M_P when it has just formed in the primordial universe. For the primordial black holes that still exist now, estimate a lower bound for their M_P (ignore the accretion of the primordial black holes). 原初黑洞猜想认为，在宇宙诞生之初，就可能已经存在着一些黑洞。它们直到目前仍然存在。设 M_P 为在宇宙早期，原初黑洞刚形成时的质量。为了让原初黑洞直到现在仍然存在，求 M_P 的下限 (忽略原初黑洞的吸积)。	4 points 4 分

PART D. ROTATING BLACK HOLES 旋转的黑洞

Realistic black holes are typically rotating, due to the angular momentum conservation of in-falling matter. With rotation, the first law of thermodynamics of a black hole is $dE = TdS + \Omega dJ$, where Ω can be understood as the angular velocity of the horizon, and J the angular momentum of the black hole. In the following, we consider the $\Omega \geq 0$ parameter regime.

现实世界中的黑洞一般是旋转的。这是因为坍缩成黑洞的物体一般携带角动量，以及角动量守恒。对于转动的黑洞，黑洞的热力学第一定律为$dE = TdS + \Omega dJ$，其中 Ω 可被理解为视界的角速度，J 是黑洞的角动量。在下题中，我们考虑 $\Omega \geq 0$ 的参数区间。

D1	Now, we let the black hole to interact with a clump of matter outside the black hole. After non-adiabatic interaction, part of the matter falls into the black hole, such that the change of energy and the change of angular momentum of the black hole satisfies $dE = \lambda dJ$, where λ is a constant. Find the range for λ for the black hole to lose energy after the interaction. 现在考虑黑洞与黑洞外面的物质相互作用。经过非绝热的相互作用，部分物质掉进黑洞。这个过程中，黑洞能量和角动量的变化满足 $dE = \lambda dJ$，其中 λ 是一个常数。为了让黑洞在与物质的相互作用中能量减小，求 λ 的取值范围。	2 points 2 分

In D1 we have found out the principles of extracting energy from black holes. In practice, we study an explicit toy model of how matter extract energy from a toy "black hole" in Newtonian mechanics (i.e. no special relativity or general relativity needs to be considered). This is a simplified version of the so-called Penrose process.

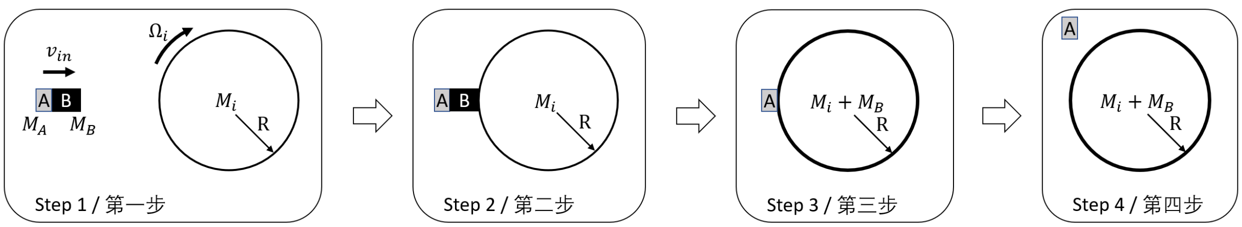
在 D1 中，我们发现了从黑洞中提取能量的一般原则。现在，我们研究一个玩具模型，来进一步理解物质如何从一个牛顿力学中的"玩具黑洞" (也就是说，不需要考虑狭义相对论和广义相对论效应) 提取能量。这是"彭罗斯过程"的一个简化版本。

Let's model the rotating black hole as a rotating sticky ring with radius R, initially with mass M_i uniformly distributed on the ring, and angular velocity Ω_i. Its center of mass is initially at rest. We neglect the gravitational effects of this ring (i.e. the ring does not source a gravitational force in our approximation).

让我们用一个半径为 R，旋转的，有粘性的环来模拟黑洞。环的初始质量为 M_i (均匀分布于环上)，角速度为 Ω_i。在初始时刻，环的质心是静止的。我们忽略环的引力效应 (也就是说，在我们的近似下，这个环并不产生引力)。

Now, consider a composite particle AB (there is a force to bind A and B, but the binding energy is negligible), where part A and part B has mass M_A and M_B, respectively. A and B are considered as point mass.

现在，考虑一个复合粒子 AB (A 和 B 之间的力把 AB 束缚在一起，但是束缚 AB 的势能可以忽略)，其中 A 部分和 B 部分分别具有质量 M_A 和 M_B。A 和 B 都可以看成质点。



Step 1: the composite particle AB moves toward the center of the black hole with an initial velocity v_{in}.

第一步：复合粒子以初始速度 v_{in} 朝着环的中心运动。

Step 2: AB stick on the black hole surface, and rotate together with the black hole.

第二步：AB 粘在黑洞表面上，并且随着黑洞转动

Step 3: B got absorbed by the black hole. To simplify the calculation, *assume* here (in D2 and D3) that after absorbing B, the black hole is still a uniform ring with radius R, and its new mass is $M_i + M_B$.

第三步：B 被黑洞吸收。为了简化计算，假设 (在 D2 和 D3 题中) B 被吸收后，黑洞仍然用一个均匀圆环表示，半径仍为 R，而环的质量变成了 $M_i + M_B$。

Step 4: At the moment B got completely absorbed, the binding between B and A disappear, and then A moves freely to the tangent direction of the black hole in the black hole reference frame.

第四步：当 B 完全被吸收的瞬间，B 和 A 之间的束缚消失了。于是，在黑洞的参考系下，A 沿着旋转的切线方向自由飞出。

Steps 2, 3, 4 happen fast enough, such that the amount of rotation of the ring during these steps is negligible.

第二、三、四步发生得足够快，这些步骤中圆环转过的角度可以忽略不计。

D2	Find the condition that the ejected kinetic energy K_{out} for particle A is greater than the initial incoming kinetic energy for the composite particle $K_{in} = \frac{1}{2}(M_A + M_B)v_{in}^2$, in the form of $\Omega_i > \cdots$ or $\Omega_i < \cdots$. 设粒子 A 的出射动能为 K_{out}，复合粒子的初始动能为 $K_{in} = \frac{1}{2}(M_A + M_B)v_{in}^2$。求 $K_{out} > K_{in}$ 的条件，用 $\Omega_i > \cdots$ 或 $\Omega_i < \cdots$ 表示。	4 points 4 分

Suppose a civilization uses a rotating black hole as a power plant, by repeatedly using the model and process described in D2, and make use of the difference in incoming and outgoing kinetic energy.

假设一个文明利用旋转黑洞来提取能量。提取能量的方法是不断使用 D2 中描述的过程，以便利用出射和入射物体的动能差。

The black hole is initially at rest with mass M and angular velocity Ω. Each time, the civilization throws composite particle AB with the same initial velocity v_{in} with respect to the civilization themselves. The composite particle AB has mass $M_A, M_B \ll M$. Again, to simplify the calculation, just as in D2, we model the black hole by a uniform ring with fixed radius R although its mass grows by absorbing B.

初始时刻，黑洞静止，质量为 M，角动量为 Ω。每一次，这个文明把一个复合粒子 AB 用相同的初速度 v_{in} (相对于这个文明自己的参考系) 扔到黑洞中。设复合粒子的质量 $M_A, M_B \ll M$。为简化计算，正如 D2 中一样，我们把黑洞简化为匀质、半径固定为 R 的圆环。由于吸收 B 粒子，圆环质量增加。

This process is repeated as long as energy can be extracted from the black hole. When the process is repeated, the civilization keeps at rest in the v_{in} direction (the horizontal direction in the figure in D2), but follows the motion of the black hole in the directions perpendicular to v_{in}.

我们重复这个过程，直到不再能从黑洞中提取能量为止。当重复这个过程的时候，这个文明在 v_{in} 方向保持静止 (即 D2 题图中的水平方向)，但在垂直于 v_{in} 的方向上跟随黑洞运动。

D3	At the moment when no net energy can be extracted from the black hole, the civilization stops throwing matter in. What is the terminal angular velocity Ω_T of the black hole when the civilization stops throwing matter in? 当不再能用这个过程从黑洞中提取能量时，这个文明停止将物体抛入黑洞。当这个文明不再向黑洞抛射物体后，求黑洞末状态的角速度 Ω_T。	6 points 6 分

~ End of Part 2 卷-2 完 ~

Pan Pearl River Delta Physics Olympiad 2021
2021 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
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Simplified Chinese Part-2 (Total 2 Problems, 60 Points)
简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 15 May 2021)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。

All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。

There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。

Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。

Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。

Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。

If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。

At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Quasicrystals (28 points)
问题 1: 准晶体 (28 分)

In 2011, the Nobel Prize in Chemistry was awarded to the discovery of quasicrystals. Nowadays, quasicrystals can be found in many applications such as the hardening of steel. How do quasicrystals differ from ordinary crystals?

2011 年，准晶体的发现者荣获诺贝尔化学奖。至今，我们已可见到准晶体不同的应用，例如钢的硬化。究竟准晶体与普通晶体有何不同？

In crystals, atoms are arranged in a periodic manner. The structure of crystals is known by periodically replicating the basic unit of the arrangement of a small number of atoms (Fig. 1 (Left)).

在晶体中，原子以周期性方式排列。通过周期性地复制少量原子排列的基本单位，就可以得到晶体的结构（图 1（左））。

On the other hand, atoms in quasicrystals are arranged in an orderly manner, but the local arrangement cannot be repeated by replication (Fig. 1(Right)).

另一方面，准晶体中的原子以有序的方式排列，但是局部的排列不能通过复制而得到全部结构（图 1（右））。

However, quasicrystals are far from random. They have a “hidden order”. For example, the structure in Fig. 1(Right) can be considered as a 5-dimensional cubic structure projected onto two dimensions. To understand this idea, we will consider a 1-dimensional quasicrystal projected from a 2-dimensional square lattice in this problem.

但是，准晶体远远不是随机的。他们有一个“隐藏的规律”。例如，图 1（右）中的结构可以考虑为投影到二维空间的 5 维立方结构。为了理解这个想法，我们在本题中将考虑从二维方格投影的一维准晶体。

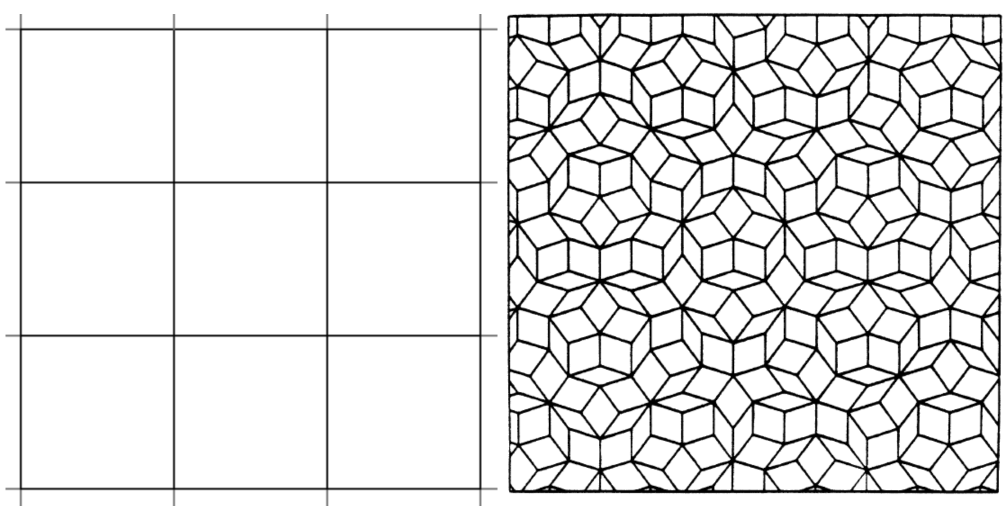


Fig. 1: (Left) A crystal (Right) A quasicrystal, known as Penrose tiling.
图 1：（左）晶体（右）称为 Penrose tiling 的准晶体。

A. Structure of Quasicrystals (14 points) 准晶体的结构（14 分）

Figure 2 shows a two-dimensional lattice in which the atoms are located at $(x_1, x_2) = (m_1 a, m_2 a)$ where m_1, m_2 are integers and a is the lattice spacing. In Section A, we assume $a = 1$. We construct a stripe defined by the condition

图 2 显示了一个二维晶格，其中原子位于$(x_1, x_2) = (m_1 a, m_2 a)$，$m_1$、$m_2$是整数，而$a$是晶格间距。在 A 部中，我们假设$a = 1$。 我们考虑一个条带，其定义为

$$\frac{x_1}{\tau} \leq x_2 < \frac{x_1}{\tau} + \tau.$$

τ is the irrational number $\tau = (1 + \sqrt{5})/2$. The inclination angle α of the strip is given by

τ 是无理数 $\tau = (1 + \sqrt{5})/2$。条带的倾角为

$$\alpha = \arctan\frac{1}{\tau}.$$

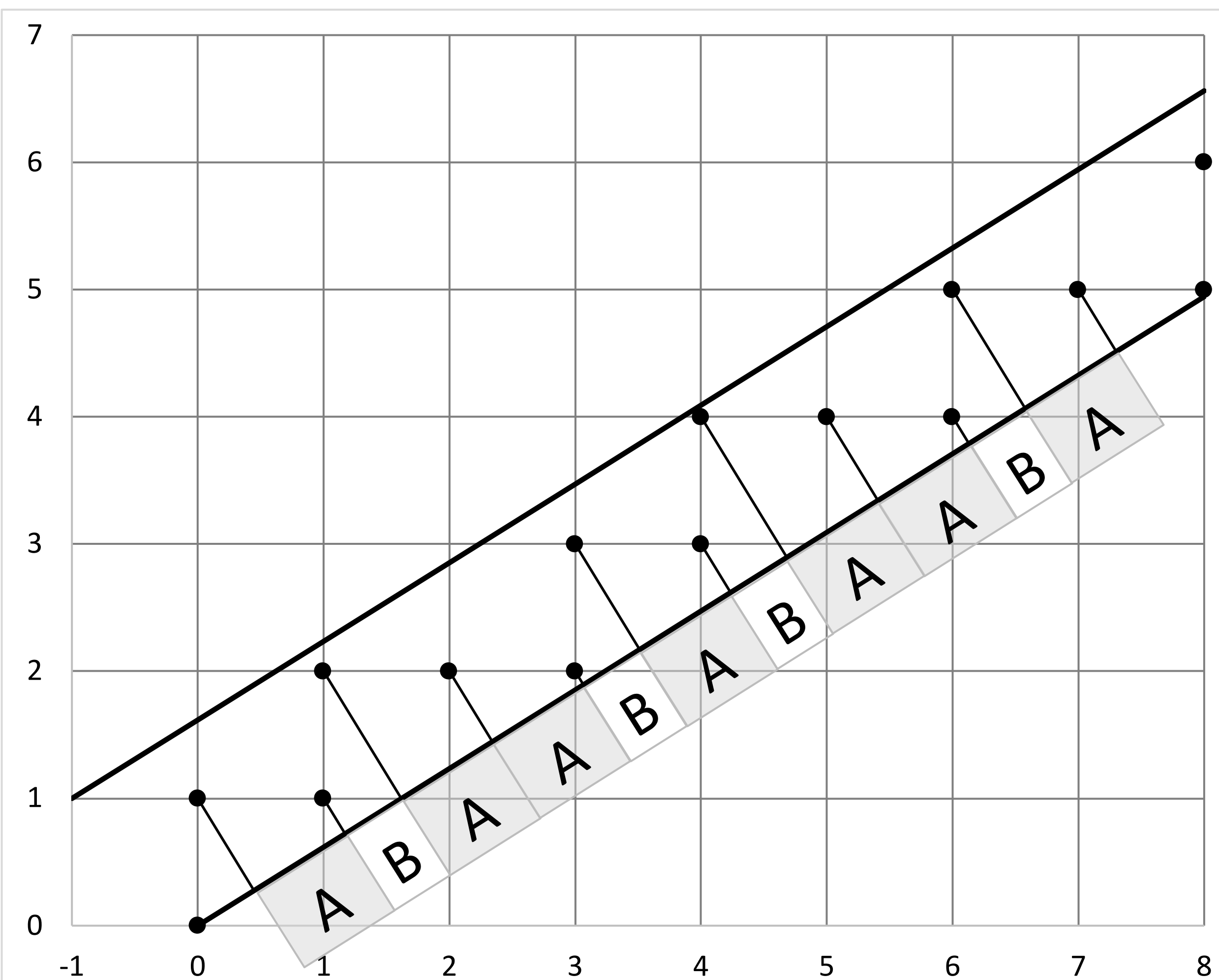


Fig. 2: Illustration of the projection method of obtaining a quasicrystal. 图 2：以投影法显示准晶体。

We project all lattice points lying within the strip to the line L_1 defined by $x_2 = x_1/\tau$. Since τ is an irrational number, the projected atomic positions on L_1 form a 1-dimensional quasicrystal. The lattice spacing now have two possible values.
我们将位于条带内的所有晶格点投影到L_1线上，L_1的定义为$x_2 = x_1/\tau$。由于τ是无理数，投影到L_1线上的原子位置便形成了一维准晶体。现在，晶格间距有两个可能的值。

A1	Calculate the lengths of the lattice spacings A and B. Write your answer as an expression containing τ. 计算晶格间距 A 和 B。答案以含τ的表达式写出。	2 points 2 分

$$A = \cos\alpha = \frac{\tau}{\sqrt{1+\tau^2}},$$
$$B = \sin\alpha = \frac{1}{\sqrt{1+\tau^2}}.$$

A2	Write the position $z(m_1, m_2)$ of the atom of the quasicrystal projected from (m_1, m_2) onto the line L_1 (that is, the displacement of the atom from (0,0)). Write your answer as an expression containing τ. 写下从(m_1, m_2)投影到L_1线上的准晶体原子位置$z(m_1, m_2)$（即从(0,0）到原子的位移）。答案以含τ的表达式写出。	1 point 1 分

$$z(m_1, m_2) = \frac{\tau m_1 + m_2}{\sqrt{1+\tau^2}}.$$

A3	Calculate the average lattice spacing d when the lattice length is very long. Write your answer as an expression containing τ. 计算当晶格长度很长时的平均晶格间距d。答案以含τ的表达式写出。	2 points 2 分

Consider the end point (m_1, m_2) of the lattice [1].
The length of the lattice approaches $\sqrt{m_1^2 + m_2^2}$.
The number of lattice points approaches $m_1 + m_2$
Furthermore, m_2 approaches m_1/τ.

$$d = \frac{\sqrt{m_1^2 + m_2^2}}{m_1 + m_2} = \frac{\sqrt{1 + 1/\tau^2}}{1 + 1/\tau} \text{ or } \frac{\sqrt{\tau^2+1}}{\tau+1} \text{ or } \frac{\sqrt{\tau^2+1}}{\tau^2}.$$

Note that the configuration of the lattice spacings A and B are no longer regular. It was observed that the configuration can be generated by the famous Fibonacci sequence, which uses the substitution rule
注意，晶格间距 A 和 B 的排列不再规则。我们观察到晶格的结构可以由著名的 Fibonacci sequence 生成，该序列使用替换规则

$$A \to AB, \quad B \to A.$$

For example, the configuration of the first seven spacings $ABAABABA$ in Fig. 2 is obtained by five substitutions starting from B.
例如，图 2 中前七个间距$ABAABABA$的排列是通过从 B 开始的五个替换获得的。

A4	Write the sequence after six substitutions starting from B. 写下从 B 开始六次替换后的序列。	1 point 1 分

$$ABAABABAABAAB$$

A5	Let n_A and n_B be the number of A and B in the sequence. Write the numbers n_A' and n_B' of A and B after one substitution. 令n_A和n_B为序列中A和B的数目。写下一次替换后，A 和 B 的数目n_A' 和n_B'。	2 points 2 分

$$n_A' = n_A + n_B,$$

$$n_B' = n_A.$$

A6	Calculate the ratio n_A/n_B when the sequence is very long. 计算当序列很长时的比例n_A/n_B。	2 points 2 分

Let $r = n_A/n_B$. Dividing the above equations,

$$r' = 1 + \frac{1}{r}.$$

When the chain is very long, r' approaches r. Hence,

$$r = 1 + \frac{1}{r}.$$

$$r^2 - r - 1 = 0.$$

$$r = \frac{1+\sqrt{5}}{2} = \tau.$$

For further analysis, we introduce two mathematical notations. For a real number x, $\lfloor x \rfloor$ denotes the integer part of x, and $\mathrm{frac}(x)$ denotes the fractional part of x. For example,

为了进一步分析，我们引入两个数学符号。对于实数x，$\lfloor x \rfloor$表示 x的整数部分，而$\mathrm{frac}(x)$则表示x的小数部分。例如，

$$\lfloor 1.73 \rfloor = 1 \quad \text{and} \quad \mathrm{frac}(x) = 0.73.$$

Labeling the atom at $(0,0)$ as $j = 0$, the position of the jth atom in the quasicrystal is given by

将$(0,0)$处的原子标记为 $j = 0$，准晶体中第j个原子的位置为

$$z_j = jd + \mathrm{frac}\left(\frac{j}{\tau}\right)\Delta.$$

A7	Derive Δ as an expression containing τ. 推导Δ的表达式，式中含有τ。	2 points 2 分

Method 1:

We can see from Fig. 2 that when $j = 1$,

$$z_1 = \frac{1}{\sqrt{\tau^2+1}}.$$

$$\mathrm{frac}\left(\frac{1}{\tau}\right) = \frac{1}{\tau}.$$

From A3,

$$d = \frac{\sqrt{\tau^2+1}}{\tau^2}.$$

Substituting into the equation,

$$\frac{1}{\sqrt{\tau^2+1}} = \frac{\sqrt{\tau^2+1}}{\tau^2} + \frac{\Delta}{\tau}.$$

$$\Delta = -\frac{1}{\tau\sqrt{\tau^2+1}}.$$

Method 2:

Note that (see the proof below)

$$m_1 = \left\lfloor \frac{j}{\tau} \right\rfloor,$$

$$m_2 = j - m_1 = j - \left\lfloor \frac{j}{\tau} \right\rfloor.$$

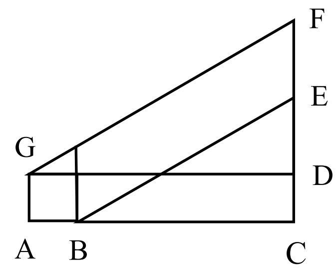
$$z_j = \frac{1}{\sqrt{1+\tau^2}}\left[\tau\left\lfloor \frac{j}{\tau} \right\rfloor + j - \left\lfloor \frac{j}{\tau} \right\rfloor\right] = \frac{j}{\sqrt{1+\tau^2}} + \frac{\tau-1}{\sqrt{1+\tau^2}}\left[\frac{j}{\tau} - \mathrm{frac}\left(\frac{j}{\tau}\right)\right]$$
$$= \frac{j}{\sqrt{1+\tau^2}} + \frac{j}{\tau^2\sqrt{1+\tau^2}} - \frac{\tau-1}{\sqrt{1+\tau^2}}\mathrm{frac}\left(\frac{j}{\tau}\right) = \frac{\sqrt{1+\tau^2}}{\tau^2}j - \frac{1}{\tau\sqrt{1+\tau^2}}\mathrm{frac}\left(\frac{j}{\tau}\right).$$
$$\Delta = -\frac{1}{\tau\sqrt{\tau^2+1}}.$$

Proof of $m_1 = \lfloor j/\tau \rfloor$:
Noting that the upper boundary passes through the point $(-1,1)$, we can calculate the number of atoms enclosed in the strip by subtracting the number of atoms in the area BCE from ACFG.

$$\text{Number of atoms in area BCE} = \sum_{i=0}^{m_1}\left(\left\lfloor\frac{i}{\tau}\right\rfloor + 1\right).$$

$$\text{Number of atoms in area GDF} = \sum_{i=0}^{m_1+1}\left(\left\lfloor\frac{i}{\tau}\right\rfloor + 1\right).$$

$$\text{Number of atoms in area ACFG} = \sum_{i=0}^{m_1+1}\left(\left\lfloor\frac{i}{\tau}\right\rfloor + 1\right) + m_1 + 2.$$



$$\text{Number of atoms in the strip} + \text{atom A} + \text{atom G} + \sum_{i=0}^{m_1}\left(\left\lfloor\frac{i}{\tau}\right\rfloor + 1\right) - \text{atom B} = \sum_{i=0}^{m_1+1}\left(\left\lfloor\frac{i}{\tau}\right\rfloor + 1\right) + m_1 + 2.$$

$$\text{Number of atoms in the strip}(= j + 1) = \sum_{i=0}^{m_1+1}\left(\left\lfloor\frac{i}{\tau}\right\rfloor + 1\right) + m_1 + 1 - \sum_{i=0}^{m_1}\left(\left\lfloor\frac{i}{\tau}\right\rfloor + 1\right)$$

$$= \left\lfloor\frac{m_1 + 1}{\tau}\right\rfloor + m_1 + 2.$$

$$j = \left\lfloor\frac{m_1 + 1}{\tau}\right\rfloor + m_1 + 1.$$

For each value of m_1, there are at most two atoms. For the atom with a higher value of m_2,

$$j = \frac{m_1 + 1}{\tau} + (m_1 + 1) - \mathrm{frac}\left(\frac{m_1 + 1}{\tau}\right) = (m_1 + 1)\tau - \mathrm{frac}\left(\frac{m_1 + 1}{\tau}\right).$$

$$\frac{j}{\tau} = m_1 + 1 - \frac{1}{\tau}\mathrm{frac}\left(\frac{m_1 + 1}{\tau}\right),$$

$$m_1 < \frac{j}{\tau} < m_1 + 1,$$

$$m_1 = \left\lfloor\frac{j}{\tau}\right\rfloor.$$

For the atom with a lower value of m_2, we note that its previous atom belongs to the group with $m_1 - 1$. Hence

$$j = \left(\left\lfloor\frac{m_1}{\tau}\right\rfloor + m_1\right) + 1 = \frac{m_1}{\tau} + m_1 + 1 - \mathrm{frac}\left(\frac{m_1}{\tau}\right) = m_1\tau + 1 - \mathrm{frac}\left(\frac{m_1}{\tau}\right).$$

$$\frac{j}{\tau} = m_1 + \frac{1}{\tau} - \frac{1}{\tau}\mathrm{frac}\left(\frac{m_1}{\tau}\right).$$

$$m_1 < \frac{j}{\tau} < m_1 + 1,$$

$$m_1 = \left\lfloor\frac{j}{\tau}\right\rfloor.$$

A8	Calculate the coordinate (m_1, m_2) of the atom with $j = 101$. 计算$j = 101$的原子的坐标(m_1, m_2)。	2 points 2 分

$$\mathrm{frac}\left(\frac{101}{\tau}\right) = \frac{101}{\tau} - \left\lfloor\frac{101}{\tau}\right\rfloor = \frac{101}{\tau} - 62.$$

$$z_{101} = 101\frac{\sqrt{\tau^2+1}}{\tau^2} - \frac{1}{\tau\sqrt{\tau^2+1}}\left(\frac{101}{\tau} - 62\right) = \frac{101}{\sqrt{\tau^2+1}} + \frac{62(\tau-1)}{\sqrt{\tau^2+1}} = \frac{39+62\tau}{\sqrt{\tau^2+1}}.$$

On the other hand,

$$z_j = m_1 A + m_2 B = \frac{\tau m_1 + m_2}{\sqrt{\tau^2+1}}.$$

Hence, $(m_1, m_2) = (62, 39)$.

B. Diffraction Pattern of Quasicrystals (14 points) 准晶体的衍射图案（14 分）

Quasicrystals were first discovered by observing their characteristic diffraction pattern.

准晶体是首先通过观察其特征衍射图案来发现的。

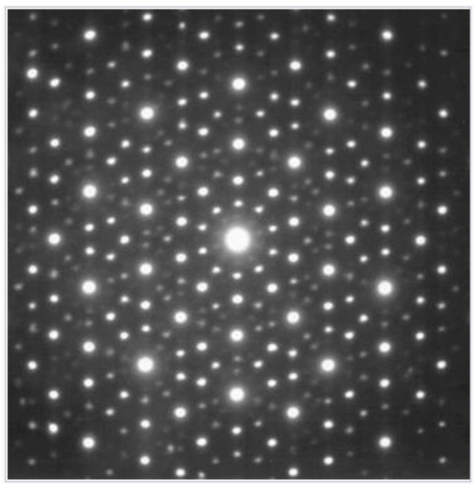


Fig. 3: Electron diffraction pattern of a quasicrystal. 图 3：准晶体的电子衍射图。

Figure 3 shows the electron diffraction pattern of a quasicrystal. A crystal is said to have n-fold symmetry if its diffraction pattern is identical if it is rotated by an angle of $2\pi/n$.

图 3 显示一种准晶体的电子衍射图。如果晶体旋转角度为$2\pi/n$后，其衍射图案相同，则该晶体具有n倍对称性。

B1	Identify the symmetries of the diffraction pattern in Fig. 3. 辨认图 3 衍射图案的对称性。	2 points 2 分

The diffraction pattern in Fig. 3 has 2-fold, 5-fold and 10-fold symmetries [2]. Note that since pentagonal structures cannot fill up the space fully, they cannot form periodic structures. See Penrose tiling in Fig. 1. The 5-fold and 10-fold symmetries are characteristics of quasicrystals.

To understand how the diffraction pattern can be derived from the projection method, we first consider the diffraction pattern of a 1-dimensional crystal.

为了了解如何从投影法中得出衍射图案，我们首先考虑一维晶体的衍射图案。

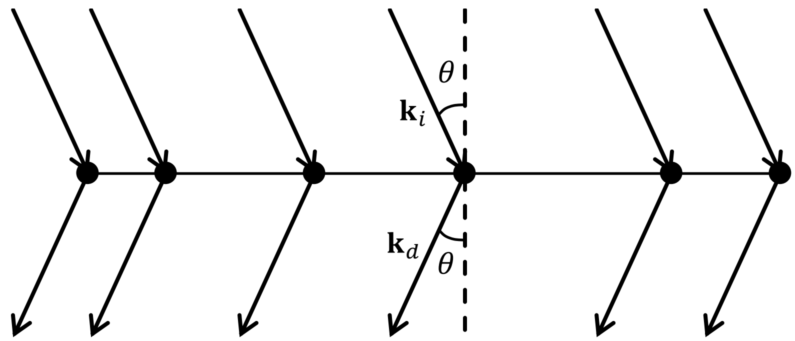


Fig. 4: Light diffraction by a lattice. 图 3：晶格的光衍射。

As shown in Fig. 4, the 1-dimensional lattice consists of N atoms. The position of atom j is x_j. A light wave with wavevector $\mathbf{k}_i$ is incident on the lattice at an angle θ with the normal direction and is diffracted at the same angle with the normal. The diffracted wave has a wavevector $\mathbf{k}_d$ with the same magnitude as $\mathbf{k}_i$. The change in the wavevector is denoted as

如图 3 所示，一维晶格由N个原子组成。 原子j的位置是x_j。光波入射到晶格中，入射波矢量为$\mathbf{k}_i$，与法线成角度θ，衍射后方向与法线成相同的角度。衍射波的波矢$\mathbf{k}_d$ 的大小与$\mathbf{k}_i$相同。 波矢的变化表示为

$$\mathbf{q} = \mathbf{k}_d - \mathbf{k}_i.$$

The magnitudes of the wavevectors are denoted as $|\mathbf{k}_d| = |\mathbf{k}_i| = k$ and $|\mathbf{q}| = q$. Note that q is a monotonic function of the diffraction angle θ, and so can represent the diffraction direction.

波矢的大小表示为$|\mathbf{k}_d| = |\mathbf{k}_i| = k$和$|\mathbf{q}| = q$。注意，$q$是衍射角$\theta$的单调函数, 因此可以表示衍射方向。

B2	Write the expression of q as a function of the diffraction angle θ. 写下q作为衍射角θ函数的表达式。	1 point 1 分

$$q = 2k\sin\theta.$$

The positions and magnitudes of the diffraction peaks are given by the structure factor defined as
衍射峰的位置和大小由结构因子给出，其定义为

$$S(\mathbf{q}) = \frac{1}{N}\left[\sum_{j=1}^{N}\cos(q_x x_j)\right]^2 + \frac{1}{N}\left[\sum_{j=1}^{N}\sin(q_x x_j)\right]^2,$$

where q_x is the x component of $\mathbf{q}$. 其中q_x是$\mathbf{q}$的x分量。

Remark: Students who are familiar with complex numbers may use the definition

备注：熟悉复数的同学可以使用下列定义

$$S(\mathbf{q}) = \frac{1}{N}\left|\sum_{j=1}^{N} \exp(iq_x x_j)\right|^2.$$

Consider a 1-dimensional periodic lattice in which $x_j = jd$. 考虑一维周期晶格，其中$x_j = jd$。

B3	What are the values of q at the peak positions of the diffraction pattern of the periodic lattice? 在周期晶格的衍射图案中，峰值位置处的q值是多少？	2 points 2 分

When $qd = 2n\pi$ for integer n,

$$S(\mathbf{q}) = \frac{1}{N}\left[\sum_{j=1}^{N} \cos(qjd)\right]^2 + \frac{1}{N}\left[\sum_{j=1}^{N} \sin(qjd)\right]^2 = \frac{1}{N}\left[\sum_{j=1}^{N} 1\right]^2 + \frac{1}{N}\left[\sum_{j=1}^{N} 0\right]^2 = N.$$

Hence, the diffraction peaks are located at $q = \frac{2n\pi}{d}$ where n is an integer.

For students using complex numbers, when $qd = 2n\pi$ for integer n,

$$S(\mathbf{q}) = \frac{1}{N}\left|\sum_{j=1}^{N} e^{iqjd}\right|^2 = \frac{1}{N}\left|\sum_{j=1}^{N} e^{i2\pi nj}\right|^2 = \frac{1}{N}\left[\sum_{j=1}^{N} 1\right]^2 = N.$$

Now consider the case that each atom in the periodic lattice is "smeared" out to a length b, where $b < d$. This means that the density $\rho(x)$ of the lattice becomes

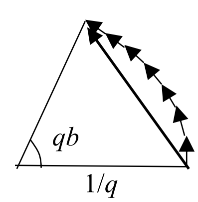
现在考虑以下情况：周期晶格中的每个原子都被"抹散"到长度b，其中$b < d$。这意味着晶格的密度$\rho(x)$变为

$$\rho(x) = \begin{cases} \frac{1}{b} & jd \le x \le jd + b, \\ 0 & \text{otherwise.} \end{cases}$$

The diffraction peaks do not have the same magnitude any longer. 衍射峰不再具有相同的大小。

B4	Calculate the magnitudes of the diffraction peaks at $\mathbf{q}$ of the smeared lattice. 计算抹散晶格$\mathbf{q}$处衍射峰的大小。	2 points 2 分

The structure factor can be calculated by the phasor method. As shown in the figure, the length of each element vector is dx, and it spans an angle qdx. Hence, the radius of the arc is $dx/qdx = 1/q$. The total angle of rotation is qb. Hence, the magnitude of the vector sum is $2(1/q)\sin qb/2$.



$$S(\mathbf{q}) = \frac{1}{N}\left(\frac{N}{b}\frac{2}{q}\sin\frac{qb}{2}\right)^2 = N\left(\frac{\sin\frac{qb}{2}}{\frac{qb}{2}}\right)^2.$$

For students using complex numbers,

$$S(\mathbf{q}) = \frac{1}{N}\left|\frac{N}{b}\int_0^b dx\, e^{iqx}\right|^2 = N\left|\frac{e^{iqb}-1}{iqb}\right|^2 = N\left(\frac{\sin\frac{qb}{2}}{\frac{qb}{2}}\right)^2.$$

Now consider the quasicrystal in Part A. Since there are two incommensurate lattice spacings in the quasicrystal, its diffraction peaks are given by wavevectors with a pair of indices,

现在考虑 A 部中的准晶体。由于在准晶体中存在两种不相称的晶格间距，因此准晶体的衍射峰由具有一对指数的波矢给出，

$$q_{mn} = \frac{2\pi}{d}\left(m + \frac{n}{\tau}\right).$$

B5	Consider the phase $\phi = q_{mn}z_j$ in the structure factor of quasicrystals. Write the expression of the phase in the form 考虑准晶体中结构因子的相$\phi = q_{mn}z_j$。试写下相的表达式 $\phi = 2\pi F + X\mathrm{frac}\left(\frac{j}{\tau}\right),$ where F is an integer and X is a real number. 其中F是整数，X是实数。	2 points 2 分

$$\phi = q_{mn}z_j = \frac{2\pi}{d}\left(m + \frac{n}{\tau}\right)\left[jd + \Delta\mathrm{frac}\left(\frac{j}{\tau}\right)\right]$$

$$= 2\pi\left(mj + n\left\lfloor\frac{j}{\tau}\right\rfloor\right) + 2\pi\left[m\frac{\Delta}{d} + n\left(1 + \frac{\Delta}{\tau d}\right)\right]\mathrm{frac}\left(\frac{j}{\tau}\right)$$

$$= 2\pi\left(mj + n\left\lfloor\frac{j}{\tau}\right\rfloor\right) + \frac{2\pi\tau(n\tau - m)}{\tau^2+1}\mathrm{frac}\left(\frac{j}{\tau}\right).$$

Hence,

$$F = mj + n\left\lfloor\frac{j}{\tau}\right\rfloor.$$

$$X = 2\pi\left[m\frac{\Delta}{d} + n\left(1 + \frac{\Delta}{\tau d}\right)\right] \quad \text{or} \quad \frac{2\pi\tau(n\tau - m)}{1+\tau^2}.$$

B6	Calculate the magnitude of the diffraction peak at q_{mn}. 计算在q_{mn}处的衍射峰的大小。	2 points 2 分

In the expression of ϕ, the first term is an integer multiple of 2π, and therefore contributes to a factor of unity upon exponentiation. In the second term, the value is uniformly and densely distributed in the interval $(0, X)$ as j covers an infinite range and τ is an irrational number. Hence, the structure factor is given by [3]

$$S(q_{mn}) = \frac{1}{N}\left|\frac{N}{X}\int_0^X dy\, e^{iy}\right| = N\left(\frac{\sin\frac{X}{2}}{\frac{X}{2}}\right)^2.$$

The same solution can be obtained by the phasor method.

B7	Find (m, n) for the highest diffraction peak in the range $0 \le n \le 3$, excluding $(m, n) = (0,0)$. Then calculate q_{mn} (in units of $2\pi/d$) and the magnitude of this peak. 找出在范围$0 \le n \le 3$内最高衍射峰(m, n)，不包括$(m, n) = (0,0)$。然后计算q_{mn}（以$2\pi/d$为单位）和该峰值的大小。	3 points 3 分

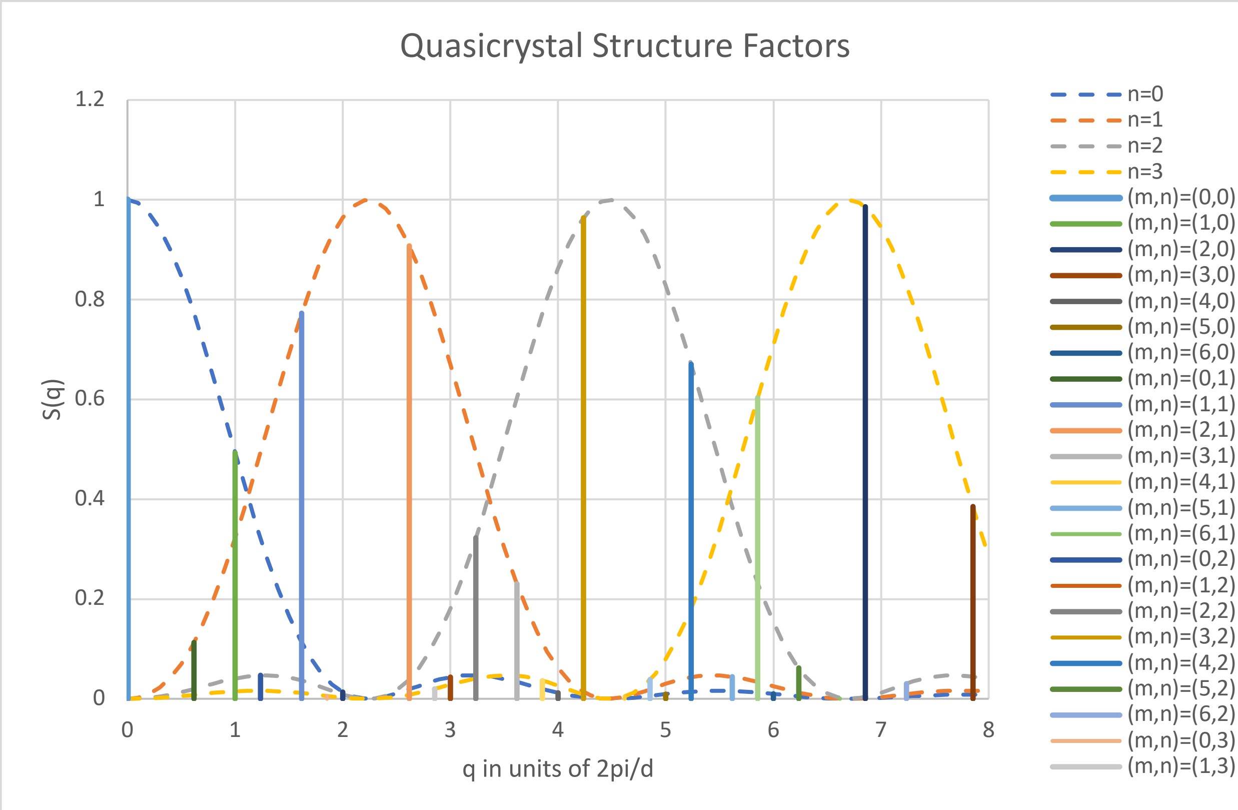
The diffraction peak is highest when X is smallest, which in turn means m/n is closest to $\tau = 1.618$. In the range $0 \le n \le 3$, this is given by $(m, n) = (5,3)$ since $5/3 = 1.667$.

$$q_{mn} = \frac{2\pi}{d}\left(m + \frac{n}{\tau}\right) = 6.854\frac{2\pi}{d}.$$

$$X = \frac{2\pi\tau(n\tau - m)}{1 + \tau^2} = -0.4100$$

$$S(q_{mn}) = N\left(\frac{\sin\frac{X}{2}}{\frac{X}{2}}\right)^2 = 0.9861N.$$

The diffraction peaks in the range $0 \le n \le 3$ are plotted in the following figure. Although the ordering of the diffraction peaks is difficult to follow, envelopes for each value of n between 0 and 3 are plotted to illustrate the hidden conditions.



References:

[1] A. N. Poddubny and E. L. Ivchenko, "Photonic Quasicrystalline and Aperiodic Structures", Physica E **42** (2010) 1871-1895.

[2] Figure 3 is reproduced from:
Tsutomu Ishimasa, Shiro Kashimoto and Ryo Maezawa, "Search and Synthesis of New Family of Quasicrystals", MRS Online Proceedings Library (OPL) , Volume 805: Symposium LL – Quasicrystals , 2003 , LL1.1

[3] Dov Levine and Paul J. Steinhardt, "Quasicrystals. I. Definition and structure", Phys. Rev. B **34**, 596-616 (1986).

Problem 2: Black Hole Physics (32 points)
问题 2: 黑洞物理 (32 分)

Black holes are the most mysterious objects in our universe. A black hole is surrounded by an event horizon (horizon for short). Anything that falls through the horizon into the black hole cannot escape. If a black hole is stationary, not rotating, has no electric charge, then the horizon is spherical, with radius $R = 2GE/c^4$, which can be also written as $R = 2GM/c^2$ using Einstein's energy-mass relation $E = Mc^2$. Here G, c, M and E are the Newton's constant, speed of light, and mass and energy of the black hole, respectively. The horizon area is thus $A = 4\pi R^2$. The singularity "inside" the black hole is one of the greatest mysteries in the theory of gravity, since the energy density of the singularity appears to diverge and the classical general relativity fails to operate there.

黑洞是宇宙中最神秘的天体。黑洞被事件视界 (简称视界) 环绕。通过视界掉进黑洞的东西不能逃出来。如果黑洞是稳态的，不旋转也不带电，则黑洞视界是球形的，其半径为 $R = 2GE/c^4$，利用爱因斯坦的质能关系 $E = Mc^2$，视界半径也可以写作 $R = 2GM/c^2$ 。这里 G, c, M 和 E 分别为牛顿引力常数、光速、黑洞质量和黑洞的能量。视界面积是 $A = 4\pi R^2$。黑洞内部的奇点是最神秘的引力现象之一，在奇点附近，能量密度趋于无穷大，经典广义相对论不再适用。

In the following, we will discuss the formation, thermodynamics and rotation of black holes, and how a civilization may use black holes as power plants.

在本题中，我们将讨论黑洞的形成、热力学和旋转黑洞，以及从黑洞提取能量的可能性。

Note: To avoid the usage of general relativity, in this problem, no concepts about curved spacetime will be introduced. You do not need to think about curved spacetime when working on this problem.

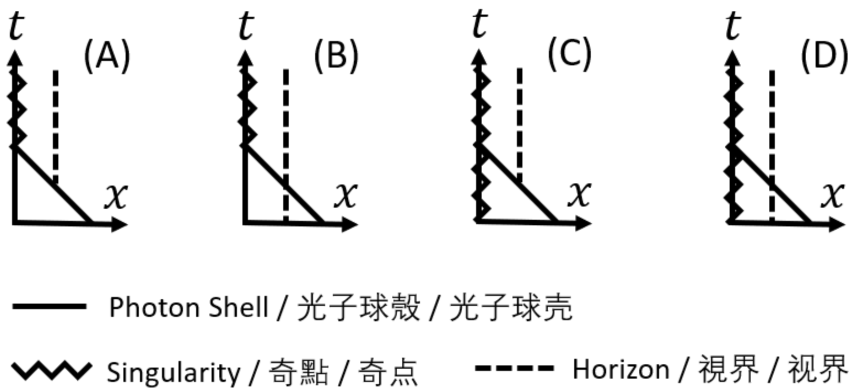
注：为避免使用广义相对论知识，本题将不会涉及时空弯曲等概念。答题时不需要考虑时空弯曲。

PART A. FORMATION AND PROPERTIES 黑洞的形成和性质

We study a simple formation mechanism of black holes. Consider a spherical shell of photons (quanta of light) is moving towards the center of the shell to form a black hole. The self-interaction of the photons can be ignored.

我们研究一个简单的黑洞形成机制。考虑一个光子 (光的量子) 球壳。这个球壳中的光子向球壳中心运动，以形成黑洞。我们忽略光子之间的自相互作用。

A1	Assume that the wavelength of photons is short enough, and thus the shell is thin (this short wavelength assumption only applies for this Question A1, and may not apply for later questions), which of the following describes the formation of the horizon and the singularity of a black hole? 假设光子波长足够短，所以球壳很薄 (这个短波长近似只用在本小题，即 A1 中，后面的题中我们不再假设短波长近似)。下列哪一个图像描述了黑洞视界和奇点的形成？	1 point 1 分

Solution: A

A2	Suppose the photons in the shell all have wavelength λ. Thus, the energy of each photon is $E_\gamma = hc/\lambda$, where h is the Planck's constant. To make a black hole with horizon radius R, what is the number of photons N needed? 假设球壳中所有光子的波长都是 λ。所以，每个光子的能量为 $E_\gamma = hc/\lambda$，其中 h 是普朗克常数。为了形成视界半径为 R 的黑洞，求所需的光子数 N。	1 point 1 分

Solution: $N = \frac{E}{E_\gamma} = \frac{c^3 R\lambda}{2Gh}$

Entropy of infalling photons 坍缩过程中光子的熵

A3	For a shell of N photons, the entropy can be written as $S_{N\gamma} = \alpha\, k_B^{c_1} N^{c_2}$. Here k_B is the Boltzmann's constant and α is a dimensionless constant of order 1. (In the below analytical formulae, α should be kept explicitly. In order of magnitude estimations, α can be set to 1.) Find integer numbers c_1 and c_2. N 个光子组成的球壳的熵可以写成 $S_{N\gamma} = \alpha\, k_B^{c_1} N^{c_2}$。这里 k_B 是玻尔兹曼常数，$\alpha \sim O(1)$ 是一个无量纲常数。(在下面各题的解析公式中，请保留α。在数量级估计中，可以将 α 的值取为 1). 求整数 c_1 和 c_2。	1 point 1 分

Solution: $c_1 = 1$ from dimensional analysis; $c_2 = 1$ from that entropy of photons is an extensive quantity.

Entropy of the black hole 黑洞的熵

A4	Taking $\lambda = R$, we have the entropy of the black hole $S = S_{N\gamma}$. (If $\lambda > R$, the photons are too non-local to form a black hole. Thus, the $\lambda = R$ photons carry the largest amount of information.) Write S in terms of R, α and the constants of Nature. 设 $\lambda = R$，我们得到黑洞的熵 $S = S_{N\gamma}$。(这是因为，如果 $\lambda > R$，则光子延展范围太大，不能坍缩成黑洞。所以，$\lambda = R$ 光子可以携带最多的信息。) 请用 R, α 和基本物理常数表示 S。	1 point 1 分

Solution: $S = \alpha\, k_B N = \frac{\alpha\, k_B\, R^2 c^3}{2Gh}$.

A5	Based on the black hole entropy formula, which of the physical observation is *incorrect*? Choose one from the below answers. A. Black hole entropy is an interdisciplinary research direction of thermal, quantum and gravitational physics. B. The black hole entropy is an extensive quantity which scales as the volume of the black hole. C. The existence of black entropy indicates that black hole should contain many microstates. D. Black holes are gravitational systems with non-perturbative quantum effects, and are thus a key to quantum gravity. 根据黑洞熵的公式，下面哪项是错误的？请只选择一项。 A. 黑洞熵是热力学、量子理论和引力的交叉学科。 B. 黑洞熵是广延量，与黑洞的体积成正比。 C. 黑洞存在熵，所以也应该存在微观状态。 D. 黑洞是具有非微扰量子特征的引力系统，所以是通向量子引力的一把钥匙。	1 point 1 分

Solution: B (BH entropy scales as area, not volume, which is very special from other entropies.)

A is correct because the formula contains k_B (thermal), h (quantum), c (special relativity) and G (general relativity)

C is correct as a usual interpretation of entropy indicates.

D is correct because h is in the denominator.

Part B. Entropy bounds of nature 黑洞熵限

It is conjected that black holes are the densest objects of nature, not only in energy, but also in entropy and information. Based on this observation, estimate the quantities in Part B. By estimate, you need to get the correct order of magnitude. Order one coefficients may be neglected.

物理学家猜测，黑洞不仅是世界上能量密度最大的物体，也是熵密度、信息密度最大的物体。基于这个猜测，请对 Part B 中的物理量做数量级估计。你只需要估计正确的数量级。$O(1)$ 的常数可以忽略。

B1	Nowadays, computer hard disks store information with a day-by-day increasing information density. However, to store information, enough number of states, and thus enough entropy is needed. This is understood from Boltzmann's statistical interpretation of entropy: $S = k_B \ln \Omega$, where Ω is the possible number of states of the	2 points 2 分

	system. Consider a spherical hard disk in the vacuum, with capacity 1Tb $= 10^{12}$ bit. What is the minimal radius of this hard disk? 目前，电脑硬盘可以存储的信息密度越来越大。但是，为了存储信息，我们需要足够多的状态数，所以需要足够多的熵。这可以从玻尔兹曼熵的统计解释中看出：$S = k_B \ln \Omega$，其中 Ω 是系统可能处于的状态的数量。考虑一个真空中的球形硬盘，容量为 1Tb $= 10^{12}$比特。求硬盘的最小半径。	

Solution:

$\frac{S}{k_B} = \ln \Omega = 1Tb = 10^{12}$.

(Note: $\ln \Omega = 1Tb$, instead of $\Omega = 1Tb$, since 1Tb hard disk can represent 2^{1Tb} states. Here log base 2 or base e is ignored since it's order of magnitude estimate.)

$R = \sqrt{2Gh\, 10^{12}/c^3} = 6 \times 10^{-29}$m

The Bekenstein Entropy bound 贝肯斯坦熵限

B2	Consider a clump of matter with mass M_m (and thus energy E_m) and radius R_m. When this clump of matter falls into a black hole (the black hole has existed before the clump of matter falls in), we require that R_m should be not greater than the horizon radius of the initial black hole, to make sure this clump of matter can fall in. Denote the entropy of this clump of matter as S_m. Find a universal upper bound of S_m, in terms of E_m and R_m, but independent of parameters of the black hole, or Newton's gravitational constant G. 考虑质量为 M_m (所以能量为 E_m)，半径为 R_m 的一块物质。当这块物质掉进一个黑洞时 (物质掉落前，黑洞就已经存在了)，我们要求 R_m 不大于黑洞本来的视界半径。因为这样才能确保这块物质掉进去。利用这个过程，求这块物质的熵 S_m 的普适上限。你导出的 S_m 的上限需要用 E_m 和 R_m 表示，但不依赖于黑洞的参数，也不依赖于牛顿引力常数 G。 Note: Necessary steps of derivation is required. 注：需要写出必要的推导步骤。	4 points 4 分

Solution: We use subscript i to denote quantities of the initial black hole, subscript f to denote quantities of the final black hole after matter falls in. Thus, the conservation of energy and non-decrease of entropy reads:

$$E_f = E_i + E_m \quad \Rightarrow \quad E_m = \frac{c^4}{2G}(R_f - R_i).$$

$$S_f \geq S_i + S_m$$

Thus, $S_m \le \frac{\alpha k_B c^3}{2Gh}(R_f - R_i)(R_f + R_i) = \frac{\alpha k_B}{hc} E_m(2R_i + \frac{2G}{c^4} E_m)$

In addition, we have $R_i \ge R_m$. For each possible initial black hole R_i, there is a corresponding bound. We should take the tightest bound in all this bounds by taking $R_i = R_m$ (note: this is not directly inserting $R_i \ge R_m$ to the above equation, because the direction of the inequality sign is different.) Thus,

$$S_m \le \frac{\alpha k_B}{hc} E_m \left(2R_m + \frac{2G}{c^4} E_m\right)$$

Further, as we mentioned, black holes are densest objects in energy. For the clump of matter to be at most as dense as black holes, we have $\frac{2GE_m}{c^4} \le R_m$. Thus,

$$S_m \le \frac{3\alpha k_B}{hc} E_m R_m$$

B3	For a 1Tb hard disk with 1nm radius, what is the minimal mass of the hard disk? 对于一个半径为 1nm，容量为 1Tb 的硬盘，硬盘的质量至少为多大？	1 point 1 分

Solution: $10^{12}/(\frac{3c}{h} \times 10^{-9}\mathrm{m})$ = 7×10^{-22}kg

Part C. Black hole Temperature and radiation 黑洞的温度和辐射

C1	Find the black hole temperature T in terms of horizon radius R. 求黑洞的温度 T，用视界半径 R 表示。	2 points 2 分

Solution: We already know the black hole energy and entropy. From the first law of thermodynamics,

$$T = \frac{dE}{dS} = \frac{hc}{2\alpha\, k_B R}.$$

HAWKING RADIATION 霍金辐射

C2	According to the Stefan-Boltzmann law, an object with a temperature T should radiate. Calculate the radiation power P in terms of the horizon radius R. 由斯蒂凡-玻尔兹曼定律，具有温度 T 的物体会发出辐射。计算辐射的功率 P，用视界半径 R 表示。 Note: the Stefan-Boltzmann constant σ can be written in more fundamental quantities as $\sigma = \frac{2\pi^5 k_B^4}{15c^2h^3}$. 注：斯蒂凡-玻尔兹曼常数 σ 可以由更基本的物理量表达：$\sigma = \frac{2\pi^5 k_B^4}{15c^2h^3}$。	2 points 2 分

Solution: $P = 4\pi R^2 \sigma T^4 = hc^2\pi^6/(30\alpha^4 R^2)$.

C3	Primordial black holes are a conjected type of black holes, which has existed almost from the "born" of the universe till now. Denote the mass of the primordial black hole by M_P when it has just formed in the primordial universe. For the primordial black holes that still exist now, estimate a lower bound for their M_P (ignore the accretion of the primordial black holes). 原初黑洞猜想认为，在宇宙诞生之初，就可能已经存在着一些黑洞。它们直到目前仍然存在。设 M_P 为在宇宙早期，原初黑洞刚形成时的质量。为了让原初黑洞直到现在仍然存在，求 M_P 的下限 (忽略原初黑洞的吸积)。	4 points 4 分

Solution:

There is a mass bound of primordial black holes, because if the primordial black holes are too light, then these black holes energy are all radiated away by Hawking radiation during the history of the universe. Thus, we consider the black holes which are just Hawking-radiated to zero mass now. They are the lightest primordial black holes and we calculate their M_P.

$$\frac{dM}{dt} = \frac{P}{c^2} = -\frac{h\pi^6 c^4}{120\alpha^4 G^2 M^2}$$

$M(t) = \left(\frac{h\pi^6 c^4}{40\alpha^4 G^2}(t_0 - t)\right)^{1/3}$, where $t_0 \sim 10^{10}$ years is the age of the universe (which we hope that you know).

At $t = 0$, $M_P > M(0) = 1.4 \times 10^{13}$kg.

PART D. ROTATING BLACK HOLES 旋转的黑洞

Realistic black holes are typically rotating, due to the angular momentum conservation of in-falling matter. With rotation, the first law of thermodynamics of a black hole is $dE = TdS + \Omega dJ$, where Ω can be understood as the angular velocity of the horizon, and J the angular momentum of the black hole. In the following, we consider the $\Omega \geq 0$ parameter regime.

现实世界中的黑洞一般是旋转的。这是因为坍缩成黑洞的物体一般携带角动量，以及角动量守恒。对于转动的黑洞，黑洞的热力学第一定律为$dE = TdS + \Omega dJ$，其中 Ω 可被理解为视界的角速度，J 是黑洞的角动量。在下题中，我们考虑 $\Omega \geq 0$ 的参数区间。

D1	Now, we let the black hole to interact with a clump of matter outside the black hole. After non-adiabatic interaction, part of the matter falls into the black hole, such that the change of energy and the change of angular momentum of the black hole satisfies $dE = \lambda dJ$, where λ is a constant. Find the range for λ for the black hole to lose energy after the interaction. 现在考虑黑洞与黑洞外面的物质相互作用。经过非绝热的相互作用，部分物质掉进黑洞。这个过程中，黑洞能量和角动量的变化满足 $dE = \lambda dJ$，其中 λ 是一个常数。为了让黑洞在与物质的相互作用中能量减小，求 λ 的取值范围。	2 points 2 分

Solution: Insert it to the first law, we have

$dE = \frac{TdS}{(1-\Omega/\lambda)}$. From $dS > 0$ (non-adiabaticity), the condition $dE < 0$ implies $0 < \lambda < \Omega$.

In D1 we have found out the principles of extracting energy from black holes. In practice, we study an explicit toy model of how matter extract energy from a toy “black hole” in Newtonian mechanics (i.e. no special relativity or general relativity needs to be considered). This is a simplified version of the so-called Penrose process.

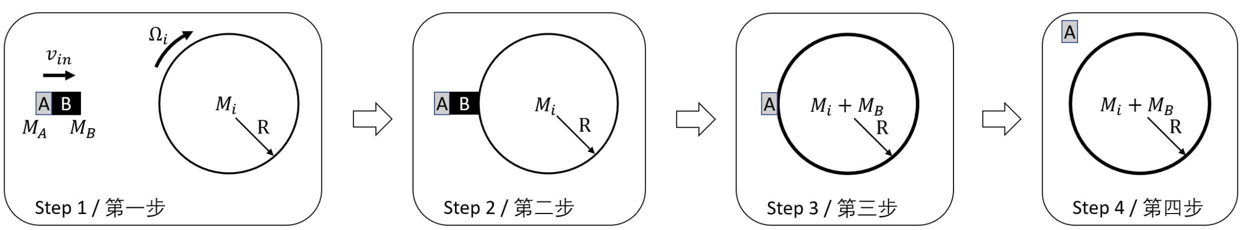
在 D1 中，我们发现了从黑洞中提取能量的一般原则。现在，我们研究一个玩具模型，来进一步理解物质如何从一个牛顿力学中的 “玩具黑洞” (也就是说，不需要考虑狭义相对论和广义相对论效应) 提取能量。这是 “彭罗斯过程” 的一个简化版本。

Let’s model the rotating black hole as a rotating sticky ring with radius R, initially with mass M_i uniformly distributed on the ring, and angular velocity Ω_i. Its center of mass is initially at rest. We neglect the gravitational effects of this ring (i.e. the ring does not source a gravitational force in our approximation).

让我们用一个半径为 R，旋转的，有粘性的环来模拟黑洞。环的初始质量为 M_i (均匀分布于环上)，角速度为 Ω_i。在初始时刻，环的质心是静止的。我们忽略环的引力效应 (也就是说，在我们的近似下，这个环并不产生引力)。

Now, consider a composite particle AB (there is a force to bind A and B, but the binding energy is negligible), where part A and part B has mass M_A and M_B, respectively. A and B are considered as point mass.

现在，考虑一个复合粒子 AB (A 和 B 之间的力把 AB 束缚在一起，但是束缚 AB 的势能可以忽略)，其中 A 部分和 B 部分分别具有质量 M_A 和 M_B。A 和 B 都可以看成质点。



Step 1: the composite particle AB moves toward the center of the black hole with an initial velocity v_{in}.

第一步：复合粒子以初始速度 v_{in} 朝着环的中心运动。

Step 2: AB stick on the black hole surface, and rotate together with the black hole.

第二步：AB 粘在黑洞表面上，并且随着黑洞转动

Step 3: B got absorbed by the black hole. To simplify the calculation, *assume* here (in D2 and D3) that after absorbing B, the black hole is still a uniform ring with radius R, and its new mass is $M_i + M_B$.

第三步：B 被黑洞吸收。为了简化计算，假设 (在 D2 和 D3 题中) B 被吸收后，黑洞仍然用一个均匀圆环表示，半径仍为 R，而环的质量变成了 $M_i + M_B$。

Step 4: At the moment B got completely absorbed, the binding between B and A disappear, and then A moves freely to the tangent direction of the black hole in the black hole reference frame.

第四步：当 B 完全被吸收的瞬间，B 和 A 之间的束缚消失了。于是，在黑洞的参考系下，A 沿着旋转的切线方向自由飞出。

Steps 2, 3, 4 happen fast enough, such that the amount of rotation of the ring during these steps is negligible.

第二、三、四步发生得足够快，这些步骤中圆环转过的角度可以忽略不计。

D2	Find the condition that the ejected kinetic energy K_{out} for particle A is greater than the initial incoming kinetic energy for the composite particle $K_{in} = \frac{1}{2}(M_A + M_B)v_{in}^2$, in the form of $\Omega_i > \cdots$ or $\Omega_i < \cdots$. 设粒子 A 的出射动能为 K_{out}，复合粒子的初始动能为 $K_{in} = \frac{1}{2}(M_A + M_B)v_{in}^2$。求 $K_{out} > K_{in}$ 的条件，用 $\Omega_i > \cdots$ 或 $\Omega_i < \cdots$ 表示。	4 points 4 分

Solution:

$$v_f = \frac{M_A+M_B}{M_i+M_A+M_B} v_{in}$$

$$\Omega_f = \frac{M_i}{M_i+M_A+M_B} \Omega_i$$

After the ejection of particle A, particle A has $v_{out} = (v_x, v_y)$, where $v_x = v_f$ and $v_y = \Omega_f R$.

Thus, the requirement $K_{out} > K_{in}$ can be expressed as

$$\Omega_i > \frac{v_{in}}{M_i R} \sqrt{\frac{(M_A+M_B)(M_i^2+M_B^2+2M_iM_A+2M_iM_B+M_AM_B)}{M_A}}$$

Suppose a civilization uses a rotating black hole as a power plant, by repeatedly using the model and process described in D2, and make use of the difference in incoming and outgoing kinetic energy.

假设一个文明利用旋转黑洞来提取能量。提取能量的方法是不断使用 D2 中描述的过程，以便利用出射和入射物体的动能差。

The black hole is initially at rest with mass M and angular velocity Ω. Each time, the civilization throws composite particle AB with the same initial velocity v_{in} with respect to the civilization themselves. The composite particle AB has mass $M_A, M_B \ll M$. Again, to simplify the calculation, just as in D2, we model the black hole by a uniform ring with fixed radius R although its mass grows by absorbing B.

初始时刻，黑洞静止，质量为 M，角动量为 Ω。每一次，这个文明把一个复合粒子 AB 用相同的初速度 v_{in} (相对于这个文明自己的参考系) 扔到黑洞中。设复合粒子的质量 $M_A, M_B \ll M$。为简化计算，正如 D2 中一样，我们把黑洞简化为匀质、半径固定为 R 的圆环。由于吸收 B 粒子，圆环质量增加。

This process is repeated as long as energy can be extracted from the black hole. When the process is repeated, the civilization keeps at rest in the v_{in} direction (the horizontal direction in the figure in D2), but follows the motion of the black hole in the directions perpendicular to v_{in}.

我们重复这个过程，直到不再能从黑洞中提取能量为止。当重复这个过程的时候，这个文明在 v_{in} 方向保持静止 (即 D2 题图中的水平方向)，但在垂直于 v_{in} 的方向上跟随黑洞运动。

D3	At the moment when no net energy can be extracted from the black hole, the civilization stops throwing matter in. What is the terminal angular velocity Ω_T of the black hole when the civilization stops throwing matter in? 当不再能用这个过程从黑洞中提取能量时，这个文明停止将物体抛入黑洞。当这个文明不再向黑洞抛射物体后，求黑洞末状态的角速度 Ω_T。	6 points 6 分

Solution:

Denote the total matter A, B the civilization has thrown up to a time be M_Σ, and $M_B = \gamma(M_A + M_B)$. At this time, the black hole mass is $M + \gamma M_\Sigma$.

Thus, for the matter that the civilization throws each time, $M_A = (1-\gamma)dM_\Sigma$, $M_B = \gamma dM_\Sigma$.

The differential momentum conservation equation in the horizontal direction is

$$v_{BH} + dv_{BH} = \frac{dM_\Sigma}{M+\gamma M_\Sigma} v_{in} + \frac{M+\gamma M_\Sigma + dM_\Sigma}{M+\gamma M_\Sigma} v_{BH}.$$

Thus, $v_{BH} = (1-x)\, v_{in}$, where $x \equiv \left(\frac{M}{M+\gamma M_\Sigma}\right)^{\frac{1}{\gamma}}$.

The differential angular momentum conservation equation is

$$(M + \gamma M_\Sigma + dM_\Sigma)(\Omega_{BH} + d\Omega_{BH}) = (M + \gamma M_\Sigma)\Omega_{BH}.$$

Thus, $\Omega_{BH} = x\,\Omega$.

At the time the civilization decides to stop throwing matter in,

$$0 = dE = \frac{1}{2}\frac{M_A}{M}(1-x)^2 v_{in}^2 dM_\Sigma + \frac{1}{2}\frac{M_A}{M} R^2\Omega^2 x^2 dM_\Sigma - \frac{1}{2} v_{in}^2 dM_\Sigma$$

Solve this equation, only taking the $x > 0$ solution x_+, we get

$$x_+ = \frac{1 + \sqrt{1 + \left(1 + \frac{R^2\Omega^2}{v_{in}^2}\right)\frac{M_B}{M_A}}}{1 + \frac{R^2\Omega^2}{v_{in}^2}}, \text{ and } \Omega_T = x_+ \Omega.$$

Note:

1. The argument of black hole entropy here is a bit hand-waving. One should use quantum theory of vacuum fluctuation around black hole geometry in general relativity to actually calculate Hawking radiation and black hole entropy.

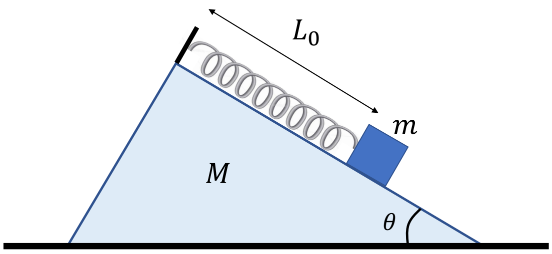
2. Throughout the solutions, we have used $\alpha \sim 1$ to estimate order of magnitudes. In fact, $\alpha = 4\pi^2$. Thus, the order of magnitudes estimation is off by this amount. If you are interested, you can easily insert $\alpha = 4\pi^2$ to get more precise estimations in the above questions.

Pan Pearl River Delta Physics Olympiad 2022
2022 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 4 Problems, 40 Points) 简体版卷-1（共4题，40分）
(9:30 am – 12:00 pm, 14th May 2022)

Please fill in your final answers to all problems on the answer sheet. 请在答题纸上填上各题的最后答案。 At the end of the competition, please submit the answer sheet only. Question papers and working sheets will not be collected. 比赛结束时，请只交回答题纸，题目纸和草稿纸将不会收回。

1. [10 points] A wooden wedge of mass M is placed on a smooth table. A small wooden block of mass m is connected to the top of the wooden wedge with a spring of elastic constant k. Suppose the natural length of the spring is L_0, and there is no friction between two wooden blocks. Now the small block is released at rest at distance L_0 from the top of the wooden wedge (as shown in the picture) and slides down freely. Find
(a) [1pt] The equilibrium position of the small block m (i.e. the position where the acceleration of m vanishes) measured from the top of the wooden wedge.
(b) [2pt] The horizontal distance travelled by the wooden wedge when the small block m reaches its equilibrium position.
(c) [1pt] The oscillating amplitude of the small block m along the slope of the wedge.
(d) [6pt] The period of oscillation of the small block m along the slope of the wedge.

1. [10 分] 一块楔形木块，质量为 M ，被置放于一光滑之桌面上，另一质量为 m 之小木块以一弹力常数为 k 的弹簧系于楔形木块之顶端。设弹簧之原长为 L_0，且两木块之间无摩擦。今将小木块自离木块顶端 L_0 处静止释放（如图所示），让其自由滑下。试求



(a) [1 分] 小木块 m 之平衡位置(m 的加速度为零的位置)离楔形木块顶端之距离。
(b) [2 分] 小木块 m 到达其平衡位置时，楔形木块移动之水平距离。
(c) [1 分] 小木块 m 沿楔形木块斜面之振幅。
(d) [6 分] 小木块 m 来回振荡之周期。

2. [10 points] A system of 3 energy levels, $E_1 = 0, E_2 = \epsilon$, and $E_3 = 10\epsilon$ ($\epsilon > 0$) is populated by $N \gg 1$ particles at temperature T. The particles populate the energy levels according to the classical Boltzmann distribution law.
(a) [2pt] What is the average number of particles, N_3, with energy E_3?
(b) [2pt] What is the average energy of a particle at temperature T?
(c) [2pt] At sufficiently low temperature T_c, only energy levels E_1, E_2 are populated. Calculate the order of magnitude of the characteristic temperature T_c.
(d) [2pt] Calculate the molar specific heat at constant volume C_v at low temperature $k_B T \ll \epsilon$.
(e) [2pt] Calculate the molar specific heat at constant volume C_v at high temperature $k_B T \gg \epsilon$.

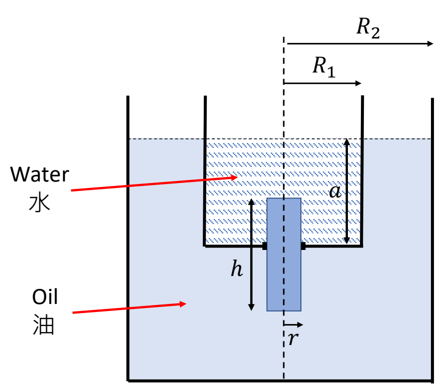
2. [10 分] 一个由 3 个能级 $E_1 = 0$、$E_2 = \epsilon$ 和 $E_3 = 10\epsilon$ ($\epsilon > 0$) 组成的系统在温度 T 下由 $N \gg 1$ 个粒子填充。这些粒子根据经典 Boltzmann 分布定律填充能级。
(a) [2 分] 具有能量 E_3 的粒子的平均数量 N_3 是多少？
(b) [2 分] 粒子在温度 T 下的平均能量是多少？
(c) [2 分] 在足够低的温度 T_c 下，系统仅填充 E_1、E_2 两个能级。计算特征温度 T_c 的数量级。
(d) [2 分] 计算低温 $k_B T \ll \epsilon$ 下等体积摩尔比热 C_v。
(e) [2 分] 计算高温 $k_B T \gg \epsilon$ 下等体积摩尔比热 C_v。

3. [10 points] One cylindrical vessel of radius R_1 is fixed inside another cylindrical vessel of radius R_2, as shown in the figure. In the bottom of the small vessel, there is a small hole with a bushing and a wooden cylinder of radius r and height $h = 21$cm is inserted. The wooden cylinder can only move vertically relative to bushing without friction. Water is poured into the small vessel to a height of $a = 30$cm, and oil is poured into the large vessel to the same level. And the wooden cylinder is in equilibrium.
Given that the water density is $\rho_w = 1000\text{kg/m}^3$, the oil density is $\rho_o = 790\text{kg/m}^3$ and the wooden cylinder density is $\rho = 600\text{kg/m}^3$.

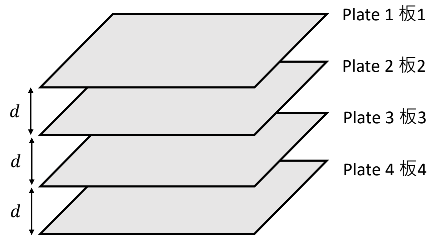
(a) [5 pt] Find the ratio of the length of the wooden cylinder immersed in the water to its total length h.
(b) [5 pt] Find the condition between ρ_w, ρ_o, r, R_1 and R_2 such that the equilibrium of the wooden cylinder is stable. (Hint: You need to consider the finite size effect of R_1, R_2 and r)

3. [10 分] 一个半径为 R_1 的圆柱容器固定在另一个半径为R_2 的圆柱容器内，如图所示。在小容器的底部有一个带衬套的小孔，插入半径为 r、高为 $h = 21$cm 的木圆柱。木圆柱只能相对于衬套垂直移动而无摩擦。将水倒入小容器至 $a = 30$cm 的高度，将油倒入大容器至同一高度。并且木圆柱处于平衡状态。
假设水的密度为 $\rho_w = 1000\text{kg/m}^3$，油的密度为 $\rho_o = 790\text{kg/m}^3$，木圆柱的密度为 $\rho = 600\text{kg/m}^3$。

(a) [5 分] 木圓柱在水中的部分與全長 h 的比例?
(b) [5 分] 找出 ρ_w, ρ_o, r, R_1 和 R_2 之间的条件，使得木圆柱的平衡是稳定的。（提示：您需要考虑 R_1、R_2 和 r 的有限尺寸效应）



4. [10 points] Four square conducting plates of area A are arranged at an even spacing d as shown in the diagram. (Assume that $A \gg d^2$)
4. [10 分] 如图所示，四块面积为 A 的方形导电板以等间距 d 排列。（假设$A \gg d^2$）



We perform the following steps to the system:
Step 1: Plate 1 and 4 are first connected to a voltage source of magnitude V_0, with plate 1 positive; Plate2 and 3 are connected with a wire.
Step 2: Remove the voltage source between plate 1 and 4.
Step 3: Remove the wire between plate 2 and 3.
Step 4: Finally, plate 1 and 4 are connected by a wire.

我们对系统执行以下步骤：

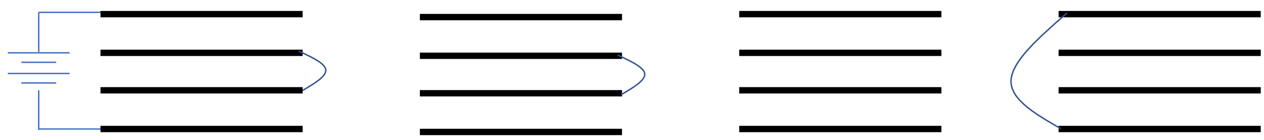
步骤 1：板 1 和 4 首先连接到幅度为V_0 的电压源，板 1 为正极；板 2 和板 3 用电线连接。

步骤 2：移除板 1 和板 4 之间的电压源。

步骤 3：拆下板 2 和 3 之间的电线。

步骤 4：最后，板 1 和 4 用电线连接。

The steps are summarized in the diagrams below.
下图总结了这些步骤。



(a) [6pt] Find the potential difference ΔV_{12}, ΔV_{23} and ΔV_{34} at step 4, where $\Delta V_{ij} = V_i - V_j$ is the potential difference between plate i and j.
(a) [6 分] 求步骤 4 中的电位差 ΔV_{12}、ΔV_{23} 和 ΔV_{34}，其中 $\Delta V_{ij} = V_i - V_j$ 为板 i 和 j 之间的电位差。

(b) [4pt] What is the net electrostatic force acting on the plate 1 at step 4?
(b) [4 分] 求步骤 4 作用在板 1 上的净静电力是多少？

~ End of Part 1 卷-1 完 ~

Pan Pearl River Delta Physics Olympiad 2022
2022 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 4 Problems, 40 Points) 简体版卷-1（共4题，40分）
(9:30 am – 12:00 pm, 7th February 2022)

Please fill in your final answers to all problems on the answer sheet.
请在答题纸上填上各题的最后答案。
At the end of the competition, please submit the answer sheet only. Question papers and working sheets will not be collected.
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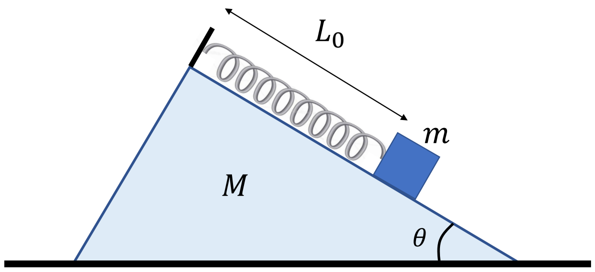
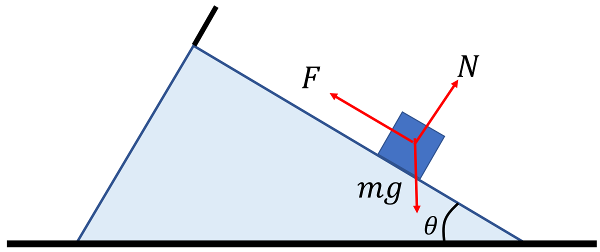
1. [10 points] A wooden wedge of mass M is placed on a smooth table. A small wooden block of mass m is connected to the top of the wooden wedge with a spring of elastic constant k. Suppose the natural length of the spring is L_0, and there is no friction between two wooden blocks. Now the small block is released at rest at distance L_0 from the top of the wooden wedge (as shown in the picture) and slides down freely. Find
(a) [1pt] The equilibrium position of the small block m measured from the top of the wooden wedge.
(b) [2pt] The horizontal distance travelled by the wooden wedge when the small block m reaches its equilibrium position, i.e. the position where the acceleration of m vanishes.
(c) [1pt] The oscillating amplitude of the small block m along the slope of the wedge.
(d) [6pt] The period of oscillation of the small block m along the slope of the wedge.

1. [10 分] 一塊楔形木塊，質量為 M ，被置放於一光滑之桌面上，另一質量為 m 之小木塊以一彈力常數為 k 的彈簧繫於楔形木塊之頂端。設彈簧之原長為 L_0，且兩木塊之間無摩擦。今將小木塊自離木塊頂端 L_0 處靜止釋放（如圖所示），讓其自由滑下。試求



(a) [1 分] 小木块 m 之平衡位置离楔形木块顶端之距离。
(b) [2 分] 小木块 m 到达其平衡位置时（m 的加速度为零的位置），楔形木块移动之水平距离。
(c) [1 分] 小木块 m 沿楔形木块斜面之振幅。
(d) [6 分] 小木块 m 来回振荡之周期。

Solution:
(a) There are 3 forces acting on the block m.



At equilibrium,

$$F = mg\sin\theta = k(l_{eq} - l_0)$$
$$\Rightarrow l_{eq} = \frac{mg\sin\theta}{k} + l_0$$

(b) Let the horizontal position of wedge and small block be X and x respectively. Initially, $x = X = 0$. At equilibrium position, we have

$$mx + MX = 0$$

From part (a),

$$x - X = (l - l_0)\cos\theta = \frac{mg}{k}\sin\theta\cos\theta$$

Solve 2 equations, we get

$$X = -\frac{mg\sin\theta}{k}\frac{m}{m+M}\cos\theta$$

The horizontal distance travelled by the wedge is $\frac{mg\sin\theta}{k}\frac{m}{m+M}\cos\theta$.

(c) (Method 1) The maximum distance travelled by the block m is the distance between the initial and the equilibrium position,

$$A = |l_0 - l_{eq}| = \frac{mg}{k}\sin\theta$$

(Method 2)
Assume the maximum distance travelled by the block m is l_m. From the energy conservation,

$$mg(l_m - l_0)\sin\theta = \frac{1}{2}k(l_m - l_0)^2$$
$$\Rightarrow l_m - l_0 = \frac{2mg}{k}\sin\theta$$

And the amplitude is given by the maximum distance from the equilibrium position,

$$A = l_m - l_{eq} = \frac{mg}{k}\sin\theta$$

(d) Let the vertical and horizontal acceleration of mass m be a_y and a_x and the horizontal acceleration of the wedge M is A_x. We have

$$ma_x + MA_x = 0. \quad (1)$$
$$\frac{a_x - A_x}{a_y} = \cot\theta \quad (2)$$
$$-N\sin\theta + F\cos\theta = MA_x \quad (3)$$
$$mg - N\cos\theta - F\sin\theta = ma_y \quad (4)$$
$$F = k(l - l_0) = \frac{k}{\cos\theta}(x - X) = \frac{kx}{\cos\theta}\left(1 + \frac{m}{M}\right) \quad (5)$$

From eqtn (1) and (2),

$$a_y = (a_x - A_x)\tan\theta = a_x\left(1+\frac{m}{M}\right)\tan\theta$$

Substitute into (4),

$$mg - N\cos\theta - F\sin\theta = ma_x\left(1+\frac{m}{M}\right)\tan\theta \quad (6)$$

From (3) and (6), we can eliminate N and

$$mg\sin\theta - F = ma_x\left(1+\frac{m}{M}\right)\frac{\sin^2\theta}{\cos\theta} - MA_x\cos\theta = ma_x\left[\left(1+\frac{m}{M}\right)\frac{\sin^2\theta}{\cos\theta} + \cos\theta\right]$$

Substitute F from eqtn (5),

$$mg\sin\theta - \frac{kx}{\cos\theta}\left(1+\frac{m}{M}\right) = ma_x\left[\left(1+\frac{m}{M}\right)\frac{\sin^2\theta}{\cos\theta} + \cos\theta\right]$$

We can find a term $a_x \propto -x$ in the expression which gives rise the SHM. The angular frequency is

$$\omega^2 = \frac{\frac{k}{\cos\theta}\left(1+\frac{m}{M}\right)}{m\left[\left(1+\frac{m}{M}\right)\frac{\sin^2\theta}{\cos\theta} + \cos\theta\right]} = \frac{k\left(1+\frac{m}{M}\right)}{m[\left(1+\frac{m}{M}\right)\sin^2\theta + \cos^2\theta]}$$

$$\Rightarrow \omega^2 = \frac{k(M+m)}{m[M+m\sin^2\theta]}$$

$$\Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m(M+m\sin^2\theta)}{k(M+m)}}$$

We can check the result in several limits:

(1) As $M \to \infty, \omega^2 \to \frac{k}{m}$

(2) As $\theta \to 0$,

$$\omega^2 \to \frac{k}{m}\left(1+\frac{m}{M}\right) = \frac{k}{\left(\frac{mM}{m+M}\right)}$$

(3) As $\theta \to \frac{\pi}{2}$,

$$\omega^2 = \frac{k}{m}$$

(Method 2) Consider the motion of the wedge M. Fundamentally, the motion of M is complicated: it is pulled by the spring and at the same time also pressed by m. But effectively, since it is under harmonic motion, we can consider the spring and m together as an effective spring with spring constant k_{eff}. The angular frequency is $\omega^2 = k_{\text{eff}}/M$, and from energy conservation,

$\frac{1}{2}k_{\text{eff}}X^2 = \frac{1}{2}M\,V^2$ (7), where V is the velocity of the wedge at the equilibrium position.

Here V can be solved by energy conservation and momentum conservation of the system. Let the velocity of m with respect to M be $\vec{v}_{\text{r}}$, then $\vec{v}_{\text{r}}$ satisfies $v_{\text{ry}} = v_{\text{rx}}\tan\theta$.

Let the velocity of m with respect to the ground be $\vec{v}$. Then, $v_x = v_{\text{rx}} - V$, $v_y = v_{\text{ry}}$.

Momentum conservation of the x-direction: $MV = mv_x \quad \Rightarrow \quad v_x = \frac{M}{m}V$, $v_y = \frac{M+m}{m}V\tan\theta$.

Energy conservation: $\frac{1}{2}mg \times \frac{mg\sin^2\theta}{k} = \frac{1}{2}m\left(v_x^2 + v_y^2\right) + \frac{1}{2}MV^2$.

Inserting v_x and v_y into the above equation, solve $\frac{1}{2}MV^2$ and then use eqtn (7) to solve k_{eff}, we have

$k_{\text{eff}} = \frac{kM(M+m)}{m(M+m\sin^2\theta)}$. Thus, the frequency is $\omega^2 = \frac{k_{\text{eff}}}{M} = \frac{k(M+m)}{m[M+m\sin^2\theta]}$.

2. [10 points] A system of 3 energy levels, $E_1 = 0, E_2 = \epsilon$, and $E_3 = 10\epsilon$ ($\epsilon > 0$) is populated by $N \gg 1$ particles at temperature T. The particles populate the energy levels according to the classical Boltzmann distribution law.
(a) [2pt] What is the average number of particles, N_3, with energy E_3?
(b) [2pt] What is the average energy of a particle at temperature T?
(c) [2pt] At sufficiently low temperature T_c, only energy levels E_1, E_2 are populated. Calculate the order of magnitude of the characteristic temperature T_c.
(d) [2pt] Calculate the molar specific heat at constant volume C_v at low temperature $T \ll \epsilon$.
(e) [2pt] Calculate the molar specific heat at constant volume C_v at high temperature $T \gg \epsilon$.

2. [10 分] 一个由 3 个能级 $E_1 = 0$、$E_2 = \epsilon$ 和 $E_3 = 10\epsilon$ ($\epsilon > 0$) 组成的系统在温度 T 下由 $N \gg 1$ 个粒子填充。这些粒子根据经典 Boltzmann 分布定律填充能级。

(a) [2 分] 具有能量 E_3 的粒子的平均数量 N_3 是多少？
(b) [2 分] 粒子在温度 T 下的平均能量是多少？
(c) [2 分] 在足够低的温度 T_c 下，仅填充 E_1、E_2 兩个能级。计算特征温度 T_c 的数量级。
(d) [2 分] 计算低温 $T \ll \epsilon$ 下等体积摩尔比热 C_v。
(e) [2 分] 计算高温 $T \gg \epsilon$ 下等体积摩尔比热 C_v。

Solution:
(a) We have

$$N_1 + N_2 + N_3 = N$$
$$\frac{N_2}{N_1} = e^{-\frac{\epsilon}{kT}}$$
$$\frac{N_3}{N_1} = e^{-\frac{10\epsilon}{kT}}$$
$$\Rightarrow N_1 = \frac{N}{1 + e^{-\frac{\epsilon}{kT}} + e^{-\frac{10\epsilon}{kT}}}$$

And

$$N_3 = \frac{Ne^{-\frac{10\epsilon}{kT}}}{1 + e^{-\frac{\epsilon}{kT}} + e^{-\frac{10\epsilon}{kT}}}$$

(b) The average energy of a particle is

$$E = \frac{\epsilon e^{-\frac{\epsilon}{kT}} + 10\epsilon e^{-\frac{10\epsilon}{kT}}}{1 + e^{-\frac{\epsilon}{kT}} + e^{-\frac{10\epsilon}{kT}}}$$

(c) If E_3 is unpopulated, we have $N_3 < 1$. At the characteristic temperature T_c,

$$N_3 = 1$$
$$\Rightarrow Ne^{-\frac{10\epsilon}{kT_c}} = 1 + e^{-\frac{\epsilon}{kT_c}} + e^{-\frac{10\epsilon}{kT_c}}$$
$$\Rightarrow N = e^{\frac{10\epsilon}{kT_c}} + e^{\frac{9\epsilon}{kT_c}} + 1$$

If $N \gg 1$,

$$\Rightarrow N \approx e^{\frac{10\epsilon}{kT_c}}$$
$$\Rightarrow T_c \approx \frac{10\epsilon}{k \ln N} = O\left(\frac{\epsilon}{k \ln N}\right)$$

(d) The molar specific heat is given by

$$C_V = N_A \frac{\partial E}{\partial T} = (kN_A \epsilon^2 \beta^2) \frac{\left(e^{-\beta\epsilon} + 100e^{-10\beta\epsilon} + 81e^{-11\beta\epsilon}\right)}{(1 + e^{-\beta\epsilon} + e^{-10\beta\epsilon})^2}$$

At low temperature $T \ll \epsilon$,

$$C_V \approx \left(\frac{N_A \epsilon^2}{kT^2}\right) e^{-\frac{\epsilon}{kT}}$$

(e) At high temperature $T \gg \epsilon$ (i.e. $\beta\epsilon \ll 1$),

$$C_V \approx (kN_A \epsilon^2 \beta^2) \frac{(1 + 100 + 81)}{(3)^2} = \frac{182}{9} \frac{N_A \epsilon^2}{kT^2}$$

3. [10 points] One cylindrical vessel of radius R_1 is fixed inside another cylindrical vessel of radius R_2, as shown in the figure. In the bottom of the small vessel, there is a small hole with a bushing and a wooden cylinder of radius r and height $h = 21\text{cm}$ is inserted. The wooden cylinder can only move vertically relative to bushing without friction. Water is poured into the small vessel to a height of $a = 30\text{cm}$, and oil is poured into the large vessel to the same level. And the wooden cylinder is in equilibrium.

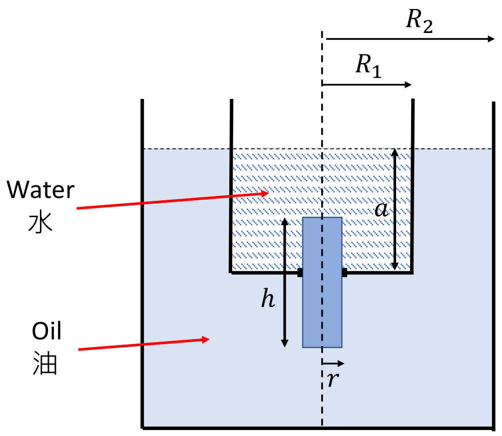
Given that the water density is $\rho_w = 1000\text{kg/m}^3$, the oil density is $\rho_o = 790\text{kg/m}^3$ and the wooden cylinder density is $\rho = 600\text{kg/m}^3$.

(a) [5 pt] Find the fraction of the wooden cylinder is in the water?
(b) [5 pt] Find the condition between ρ_w, ρ_o, r, R_1 and R_2 such that the equilibrium of the wooden cylinder is stable. (Hint: You need to consider the finite size effect of R_1, R_2 and r)

3. [10 分] 一个半径为 R_1 的圆柱容器固定在另一个半径为R_2 的圆柱容器内，如图所示。在小容器的底部有一个带衬套的小孔，插入半径为 r 、高为 $h = 21\text{cm}$ 的木圆柱。木圆柱只能相对于衬套垂直移动而无摩擦。将水倒入小容器至 $a = 30\text{cm}$ 的高度，将油倒入大容器至同一高度。并且木圆柱处于平衡状态。

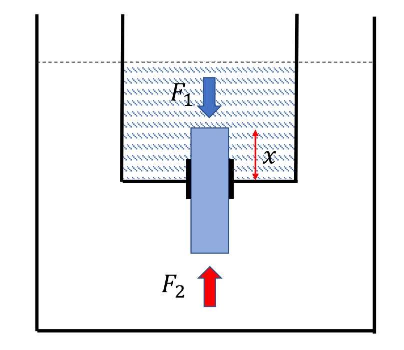
假设水的密度为 $\rho_w = 1000\text{kg/m}^3$，油的密度为 $\rho_o = 790\text{kg/m}^3$，木圆柱的密度为 $\rho = 600\text{kg/m}^3$。

(a) [5 分] 木圓柱在水中的部分與全長的比例?
(b) [5 分] 找出 ρ_w, ρ_o, r, R_1 和 R_2 之间的条件，使得木圆柱的平衡是稳定的。（提示：您需要考虑 R_1 、R_2 和 r 的有限尺寸效应）



Solution:

(a) At equilibrium, we have



$$P_1 + \rho g h = P_2$$

$$P_1 = P_{atm} + \rho_w g(a - x)$$

$$P_2 = P_{atm} + \rho_o g(a + h - x)$$

$$\Rightarrow \rho_w(a - x) + \rho h = \rho_0(a + h - x)$$

$$\Rightarrow (\rho_o - \rho_w)x = -\rho h + \rho_0(a + h) - \rho_w a$$

$$\Rightarrow x = \frac{(\rho - \rho_0)h + (\rho_w - \rho_o)a}{\rho_w - \rho_o}$$

$$\text{Fraction of the cylinder in the water} = \frac{x}{h} = \frac{(\rho - \rho_0) + (\rho_w - \rho_o)\frac{a}{h}}{\rho_w - \rho_o} = \frac{-190 + 210 \times \frac{30}{21}}{210}$$

$$= -\frac{19}{21} + \frac{30}{21} = \frac{11}{21}$$

(b) If $x \to x + \Delta x$ ($\Delta x > 0$), the water level will rise and the oil level will drop.

The rise of the water level is

$$\pi R_1^2(a + \Delta a_w) - \pi r^2(x + \Delta x) = \pi R_1^2 a - \pi r^2 x$$

$$\Rightarrow \Delta a_w = \frac{r^2}{R_1^2}\Delta x$$

Similarly,

$$\pi(R_2^2 - R_1^2)\Delta a_0 = \pi r^2 \Delta x \Rightarrow \Delta a_o = \frac{r^2}{R_2^2 - R_1^2}\Delta x$$

The net force acting on the cylinder is

$$F_{net} = (P_1 + \rho g h - P_2)\pi r^2$$

$$P_1 = P_{atm} + \rho_w g(a + \Delta a_w - x - \Delta x)$$

$$P_2 = P_{atm} + \rho_o(a - \Delta a_o + h - x - \Delta x)$$

$$\Rightarrow F_{net} = \rho_w g(a + \Delta a_w - x - \Delta x) + \rho g h - \rho_o g(a - \Delta a_0 + h - x - \Delta x)$$

$$= \rho_w g(\Delta a_w - \Delta x) + \rho_0 g(\Delta a_0 + \Delta x) = \rho_w g\left(\frac{r^2}{R_1^2} - 1\right)\Delta x + \rho_0 g\left(\frac{r^2}{R_2^2 - R_1^2} + 1\right)\Delta x$$

$$= \left(\rho_w\left(\frac{r^2}{R_1^2} - 1\right) + \rho_0\left(\frac{r^2}{R_2^2 - R_1^2} + 1\right)\right)g\Delta x$$

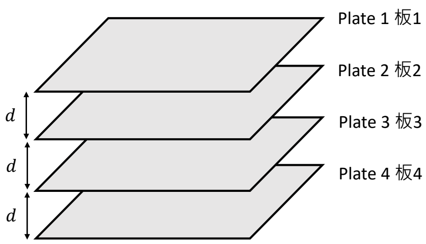
The force will push the cylinder into the equilibrium position if $F_{net} > 0$

$$\Rightarrow \rho_w\left(\frac{r^2}{R_1^2} - 1\right) + \rho_0\left(\frac{r^2}{R_2^2 - R_1^2} + 1\right) > 0$$

$$\Rightarrow \rho_w\left(\frac{R_1^2 - r^2}{R_1^2}\right) < \rho_0\left(\frac{R_2^2 - R_1^2 + r^2}{R_2^2 - R_1^2}\right)$$

4. [10 points] Four square metal plates of area A are arranged at an even spacing d as shown in the diagram. (Assume that $A \gg d^2$)

4. [10 分] 如图所示，四块面积为 A 的方形金属板以等间距 d 排列。（假设$A \gg d^2$）



We perform the following steps to the system:
Step 1: Plate 1 and 4 are first connected to a voltage source of magnitude V_0, with plate 1 positive; Plate2 and 3 are connected with a wire.
Step 2: Remove the voltage source between plate 1 and 4.
Step 3: Remove the wire between plate 2 and 3.
Step 4: Finally, plate 1 and 4 are connected by a wire.

我们对系统执行以下步骤：

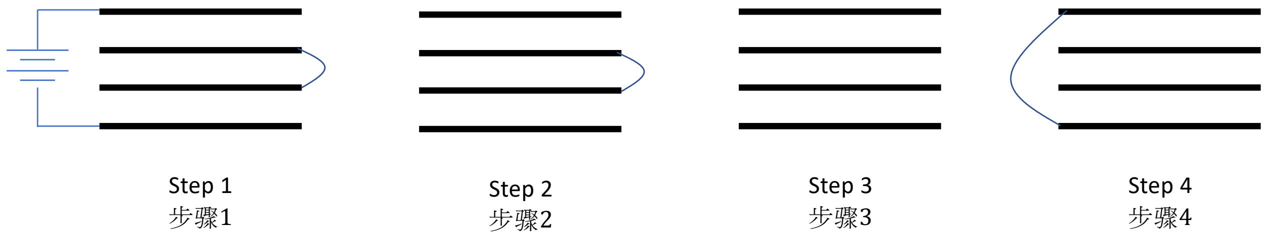
步骤 1：板 1 和 4 首先连接到幅度为V_0 的电压源，板 1 为正极；板 2 和板 3 用电线连接。

步骤 2：移除板 1 和板 4 之间的电压源。

步骤 3：拆下板 2 和 3 之间的电线。

步骤 4：最后，板 1 和 4 用电线连接。

The steps are summarized in the diagrams below.
下图总结了这些步骤。



(a) [6pt] Find the potential difference ΔV_{12}, ΔV_{23} and ΔV_{34} at step 4, where $\Delta V_{ij} = V_i - V_j$ is the potential difference between plate i and j.

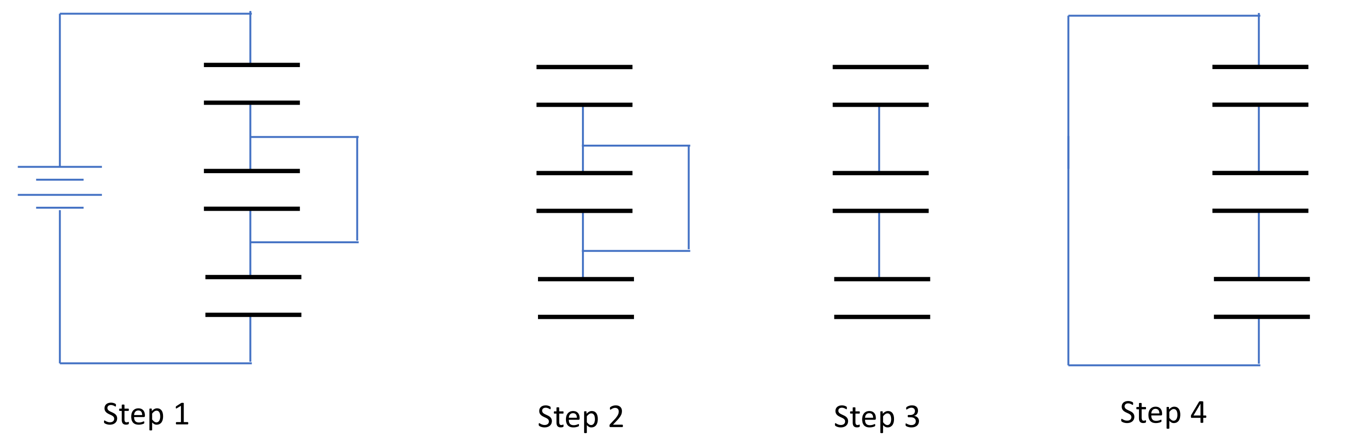
(a) [6 分] 求步骤 4 中的电位差 ΔV_{12}、ΔV_{23} 和 ΔV_{34}，其中 $\Delta V_{ij} = V_i - V_j$ 为板 i 和 j 之间的电位差。

(b) [4pt] What is the net electrostatic force acting on the plate 1 at step 4?

(b) [4 分] 求步骤 4 作用在板 1 上的净静电力是多少？

Solution:

We treat the plates as three capacitors in series. Each has an identical capacitance C. The figure below then show the 4 steps.



Since C_2 is shorted initially, effectively there are only two capacitors in series. The voltage drop across C_1 and C_3 is $V_0/2$. The top plate of C_1 will then have a positive charge of $q_0 = \frac{CV_0}{2}$. Note that this means that the bottom plate of C_1 will have a negative charge of $-q_0$. Removing the voltage source and then the wire across C_2 will not change the charges or potential drops across the other two capacitors.

Shorting the top plate of C_1 with the bottom plate of C_3 will make a difference. Positive charge will flow out of the top plate of C_1 into the bottom plate of C_3. Also, negative charge will flow out of the bottom plate of C_1 into top plate of C_2. The result is that C_1 will acquire a potential difference of V_1, C_2 a potential difference of V_2 and C_3 a potential difference of V_3.

Let the final charge on the top plate of each capacitor be q_1, q_2 and q_3 respectively.

Kirchhoff's loop rule implies,

$$V_1 + V_2 + V_3 = 0$$

By symmetry, we have

$$V_1 = V_3$$

$$\Rightarrow 2V_1 = -V_2$$

By charge conservation between the bottom plate of C_1 and the top plate of C_2, we have

$$-q_0 = -q_1 + q_2 \Rightarrow -\frac{1}{2}V_0 = -V_1 + V_2$$

$$\Rightarrow V_2 = -\frac{1}{3}V_0$$

And

$$V_1 = \frac{1}{6}V_0.$$

Final answer:

$$\Delta V_{12} = \frac{1}{6}V_0, \qquad \Delta V_{23} = -\frac{1}{3}V_0, \quad \Delta V_{34} = \frac{1}{6}V_0$$

(b) The charge on the top plate is

$$q_1 = C\Delta V_{12} = \frac{1}{6}\frac{\epsilon_0 A}{d}\,V_0$$

And the electric field inside the top capacitor is

$$E_1 = \frac{\Delta V_{12}}{d} = \frac{V_0}{6d}$$

The electric force acts on the top plate is

$$F = \frac{1}{2} E_1 \times q_1 = \frac{1}{2} \frac{V_0}{6d} \frac{\epsilon_0 A}{6d} V_0 = \frac{1}{72} \frac{\epsilon_0 A}{d^2} V_0^2$$

2

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Simplified Chinese Part-2 (Total 2 Problems, 60 Points)
简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 14 May 2022)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。

All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。

There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。

Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。

Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。

Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。

If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。

At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Spatiotemporal varying electric permittivity (30 points)
问题 1: 时空变介电常数 (30 分)

In electromagnetism, dielectric media, e.g. a block of glass, is represented by a permittivity different from the one of vacuum. The reflection and refraction for electromagnetic waves on a slab of dielectric material can be derived from considering dispersion relationship and matching boundary condition.
在电磁学中，介电材料(例如一块玻璃)可以用特定的介电常数表示，其数值不同于真空介电常数。在给定色散关系和匹配边界条件下，可以导出电磁波在介电材料板上的反射和折射。

In the question, we would like to investigate the Fabry-Pérot resonance from a slab of dielectric medium in the first step and the analog concept when the permittivity of the material becomes inhomogeneous in the time domain instead of the spatial domain.
在这个问题中，我们第一步研究电介质板的法布里-珀罗共振现象，以及当材料的介电常数在时域而不是空间域中变得不均匀时的类似概念。

Figure 1 shows the schematic diagrams for light entering a block of dielectric medium at normal incidence. The left diagram shows the case for a block of infinite thickness, in which the light only undergoes one instance of reflection and refraction at the interface. The right diagram shows the case for a finite thickness, in which the light ray undergoes multiple reflections within the slab. The spatial and temporal axes are the horizontal and vertical ones respectively.
图 1 显示了光线以垂直入射方式进入电介质板的示意图。左图显示了无限厚度板的情况，其中光在界面处仅经历一次反射和折射。右图显示了有限厚度板的情况，其中光线在板内经历了多次反射。空间轴和时间轴分别是图中水平轴和垂直轴。

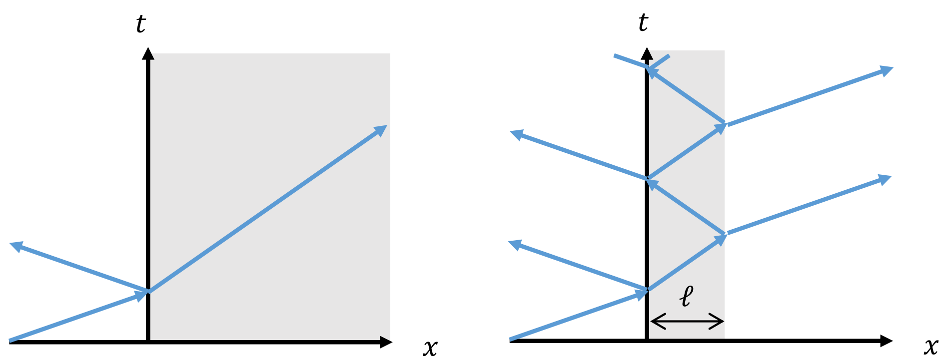


Fig. 1: (Left) Schematic diagram for a light ray entering an interface between vacuum and dielectric medium (Right) Schematic diagram for a light ray entering a slab of dielectric medium undergoing multiple reflections.
图 1：（左）光线进入真空和电介质之间的界面的示意图，（右）光线进入电介质经过多次反射的示意图。

PART A. Fabry-Pérot resonance 法布里-珀罗共振现象 (13 points)

To obtain the amount of light being reflected and refracted from a block of dielectric medium, we start from the Maxwell's equations:
为了获得光经过一块电介质板后的反射和折射强度，我们从麦克斯韦方程开始：

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}=0,\qquad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\qquad \nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t},$$

where the curl and div operators are defined by
其中 curl 运算符定义为

$$\nabla \times \boldsymbol{E} = \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

The constitutive relationship for the dielectric material or vacuum in relating the different field variables are expressed as
在介电材料或真空中，不同场变量的本构关系为

$$\boldsymbol{D} = \epsilon(x)\boldsymbol{E}$$
$$\boldsymbol{B} = \mu_0 \boldsymbol{H}$$

In this part, the electric permittivity profile $\epsilon(x)$ only varies in x but not in y, z and t. All the materials are non-magnetic, having the same value of magnetic permeability μ_0 of vacuum. We only consider electromagnetic waves which propagate in the x-direction.
在本部分中，介电常数曲线 $\epsilon(x)$ 仅随着x变化，而与y、z 和 t 无关。 所有材料都是非磁性的，具有相同的真空磁导率 μ_0值。我们只考虑在 x 方向传播的电磁波。

A1	We consider an electromagnetic wave propagating only in the x-direction specified by $\boldsymbol{E} = \hat{y}E_y(x,t)$, $\boldsymbol{H} = \hat{z}H_z(x,t)$. Reduce the Maxwell's equations to two differential equations only on E_y and H_z. Each differential equation is first order in both temporal dimension t and spatial dimension x. 我们仅考虑由 $\boldsymbol{E} = \hat{y}E_y(x,t)$，$\boldsymbol{H} = \hat{z}H_z(x,t)$指定的在$x$方向传播的电磁波。请将麦克斯韦方程组简化为$E_y$和$H_z$上的两个微分方程式。每个微分方程式在时间维度 t 和空间维度x上都具有一阶。	2 Points 2 分
A2	For a sinusoidal wave propagating in the positive x-direction in a medium of constant permittivity ϵ_1 (e.g. in a glass), it has a form $E_y = E_{in}\cos(k_1 x - \omega t)$. Find the dispersion relationship between k_1 and ω 对于在介电常数为 ϵ_1 的介质（例如在玻璃中）中沿 x 正方向传播的正弦波，它的形式为 $E_y = E_{in}\cos(k_1 x - \omega t)$。 求$k_1$与 ω 的色散关系。	2 Points 2 分

Now, we consider an interface between vacuum and a dielectric medium (see left panel of Fig. 1). A fixed incident electromagnetic wave $E_y = E_{in}\cos(k_0 x - \omega t)$ propagates in vacuum to the right hand side before entering the dielectric medium. The permittivity $\epsilon(x)$ in this case is a step function, being ϵ_0 for $x < 0$ and ϵ_1 for $x \geq 0$.
现在，我们考虑真空和电介质之间的界面(见图 1 的左图)。一个固定的入射电磁波 $E_y = E_{in}\cos(k_0 x - \omega t)$ 在进入玻璃之前在真空中向右方传播。在这种情况下，介电常数 $\epsilon(x)$ 是一个阶跃函数，当$x < 0$ 为 ϵ_0，当 $x \geq 0$ 为 ϵ_1。

A3	Suppose the transmitted waves is represented by $E_y = E_t\cos(k_1 x - \omega t + \phi_t)$ (where $0 \leq \phi_t < \pi$). Find E_t and ϕ_t in terms of E_{in}, ω and the permittivities ϵ_0, ϵ_1 假设传输的波由 $E_y = E_t\cos(k_1 x - \omega t + \phi_t)$ 表示，其中$0 \leq \phi_t < \pi$。求 E_t 和 ϕ_t ，用 E_{in}、ω 和介电常数 ϵ_0, ϵ_1来表示。	5 Points 5 分

Now, we consider a plate of dielectric medium with a finite thickness ℓ, with a fixed electromagnetic wave shining normally to the plate. There will be multiple reflections and transmissions between the two interfaces of the dielectric medium (see right panel of Fig. 1). The incident wave is still fixed as $E_y = E_{in}\cos(k_0 x - \omega t)$ while the total transmitted waves after the whole slab of dielectric medium is now specified by $E_y = E_t^{(\mathrm{tot})}\cos\left(k_0 x - \omega t + \phi_t^{(\mathrm{tot})}\right)$, by summing all the multiple reflections and transmissions
现在，我们考虑具有有限厚度 ℓ 的电介质板，即一个固定的电磁波正入射到电介质板。在电介质的两个界面之间会有多次反射和透射(如图 1 的右图)。入射波仍然固定为$E_y = E_{in}\cos(k_0 x - \omega t)$，而穿过整个电介质板的透射波通过对所有的多次反射和透射求和，可以写成为 $E_y = E_t^{(\mathrm{tot})}\cos\left(k_0 x - \omega t + \phi_t^{(\mathrm{tot})}\right)$

A4	Derive the maximum value for $\left\|E_t^{(\mathrm{tot})}\right\|$ when we can choose an optimal value of ℓ . Derive the condition for such ℓ. 当我们可以选择 ℓ 的最佳值时， 导出 $\left\|E_t^{(\mathrm{tot})}\right\|$ 的最大值以及 ℓ 的满足条件。	4 Points 4 分

PART B. Time-varying permittivity 时变介电常数 (14 points)

We have considered a boundary of the permittivity profile in the spatial domain x. Now we go to the complementary problem. Suppose the electric permittivity is constant in all spatial dimensions but varies in temporal dimension t. For example, materials like $\mathrm{LiNbO_3}$ will have its permittivity changes with time when a DC voltage is applied across the material through the Kerr electro-optic effect.
我们已经考虑了空间域 x 中介电常数分布的边界。 现在我们转到其互补问题。 假设介电常数在所有空间维度上都是恒定的，但在时间维度 t 上变化。例如，当通过克尔电光效应在材料上施加直流电压时，LiNbO3 等材料的介电常数会随时间发生变化。

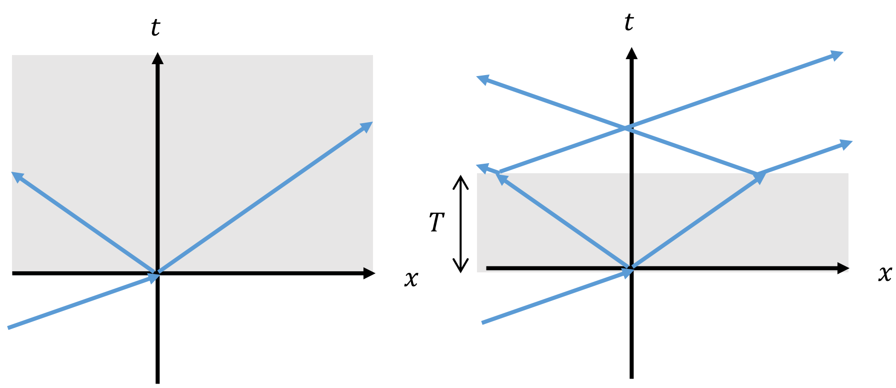


Fig. 2: (Left) Schematic diagram for a light ray entering a time-boundary in which the permittivity changes from ϵ_1 to ϵ_2 (Right) Schematic diagram for a light ray going through two time-boundaries. The permittivity changes from ϵ_1 to ϵ_2 at $t = 0$ and changes back to ϵ_1 at $t = T$.
图 2：（左）光线进入时间边界的示意图，其中介电常数从 ϵ_1 变为 ϵ_2（右）光线穿过两个时间边界的示意图。 介电常数在 $t = 0$ 时从 ϵ_1 变为 ϵ_2，并在 $t = T$ 时变回 ϵ_1。

In a similar fashion, left panel of Fig. 2 shows the trajectory for a light ray originally propagating in a homogeneous medium in the positive direction. Then, the permittivity suddenly changes from ϵ_1 to another value ϵ_2 at $t = 0$. In the right panel, we now have a "slab" of such time-varying material in the time domain. The permittivity changes back to ϵ_1 at $t = T$. The blue arrows show the light rays. At every boundary, the ray splits in a forward and backward propagating ray. As the initial ray is propagating to the right, we call the left propagating wave as "reflection/backward propagating" and the right propagating wave as "transmission/forward propagating" after a time boundary. Note that the rays can only propagate in the positive time direction.
以类似的方式，图 2 的左图显示了最初在一个均质介质中沿正方向传播的光线的轨迹。 然后，介电常数在 $t = 0$ 时突然从 ϵ_1 变为另一个值 ϵ_2。 在右侧面板中，我们在时域中有一个这种随时间变化的材料的"板"。 介电常数在 $t = T$ 时变回 ϵ_1。 蓝色箭头表示光线。 在每个边界处，光线分裂为向前和向后传播的光线。 由于初始光线向右传播，我们称左传播波为"反射/后向传播"，右传播波在时间边界后称为"透射/前向传播"。请注意，光线只能在正时间方向上前进。

To be specific, consider an electromagnetic waves propagates in an infinite block of dielectric medium originally. Suddenly, at time equal to zero, the dielectric constant in the whole space changes from ϵ_1 to ϵ_2. We call $t = 0$ as a temporal boundary. See left panel of Fig. 2.
具体来说，考虑电磁波最初在无限大的电介质中传播。 突然，在时间等于 0 时，在整个空间介电常数从 ϵ_1 变为 ϵ_2 。 我们称 $t = 0$ 为时间边界。参见图 2 的左侧。

B1	What are the continuity conditions in this case across the time boundary? 在这种情况下，跨越时间边界的连续性条件是什么？	3 Points 3 分

Initially, the electromagnetic waves is specified by $\boldsymbol{E} = \hat{y}E_1 \cos(k_1 x - \omega t)$, a forward propagating waves in the positive *x* direction. After the permittivity has changed, there is a backward propagating waves (along the negative x direction) and forward propagating

waves (along the positive x direction). They are defined as the temporal reflection and temporal transmission waves.
最初，电磁波为 $\boldsymbol{E} = \hat{y}E_1 \cos(k_1 x - \omega t)$，即在正 x 方向上的前向传播波。介电常数改变后，有向后传播的波（沿负 x 方向）和向前传播的波（沿正 x 方向）。它们被定义为时间反射波和时间传输波。

B2	The angular frequency has changed from ω to another value ω' across the time-boundary. Express ω' in terms of ω and the permittivity values. 角频率穿过时间边界時会由 ω 变为另一个值 ω'。用 ω 和介电常数值表示 ω'。	3 Points 3 分
B3	Express the amplitude of the electric field of the backward propagating wave and the forward propagating wave. 求后向传播波和前向传播波的电场幅度。	3 Points 3 分

Suppose the whole medium goes back to a dielectric medium of permittivity of ϵ_1 after time $t = T$ (see right hand side of Fig. 2).
假设在时间 $t = T$ 之后，整个介质回到介电常数为 ϵ_1 的电介质（参见图 2 的右侧）。

B4	Find out the amplitude of the backward and forward propagating waves finally. What is the condition to get minimal reflection? 求出最后的后向传播波和前向传播波的电场幅度。获得最小反射的条件是什么？	5 Points 5 分

PART C. Spatiotemporal permittivity 时空变介电常数 (5 points)

In the final part, we consider the case that the permittivity has to be written as a function in both x and t, i.e. $\epsilon(x,t)$. For a simple case, the permittivity function consists of two constants: ϵ_1 in the white region and ϵ_2 in the gray region, a so-called spatiotemporal boundary depicted in Fig. 3. The spatiotemporal boundary is defined by a straight line $x = -ut$
在最后一部分，我们考虑介电常数必须写为 x 和 t 的函数的情况，即 $\epsilon(x,t)$。对于一个简单的情况，介电常数函数由两个常数组成：白色区域中的 ϵ_1 和灰色区域中的 ϵ_2，即所谓的时空边界，如图 3 所示。时空边界由直线 $x = -ut$ 定义。

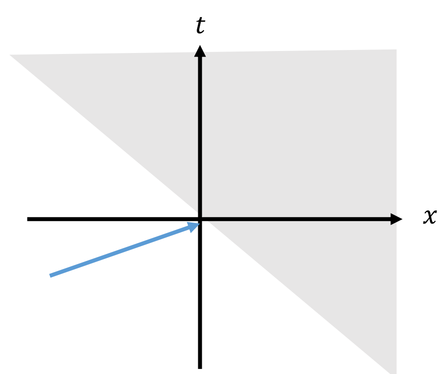


Fig. 3: A spatiotemporal boundary. In the white (grey) region, the permittivity is ϵ_1 (ϵ_2)
图 3：时空边界。在白色（灰色）区域，介电常数为 ϵ_1 (ϵ_2)

C1	What are the continuity conditions in this case across the spatiotemporal boundary? Hint: You may need to consider a coordinate transformation. 在这种情况下，跨越时空边界的连续性条件是什么？提示：您可能需要考虑坐标变换。	3 Points 3 分

Problem 2: Fractal Dimensions of Networks (30 points)
问题 2: 网络的分数维数 (30 分)

A line is one-dimensional, a plane is two-dimensional, and the volume of a ball is three-dimensional. Is the dimension of an object always an integer? What's the dimension of a coastline? What's the dimension of the Internet or social network? The dimension of complicated objects is an interdisciplinary study between physics and many other sciences. Here, we will use "box counting dimension" to discuss the dimension of fractal and complex network objects.
直线是一维物体，平面是二维的，而球体是三维的。物体的维度总是整数吗？海岸线是多少维的，互联网、朋友圈又是多少维的？复杂物体的维度，是物理学和多个学科的交叉学科。这里，我们将使用“计盒维数”讨论分形几何和复杂网络的维度问题。

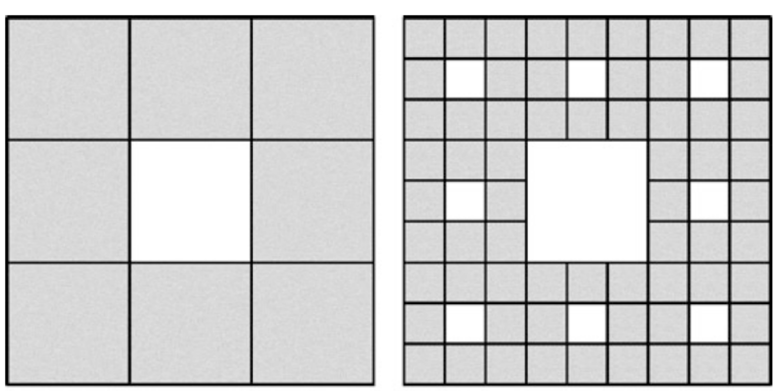
Based on this, we discuss the "renormalization group" of a complex network. Renormalization group describes how physics theories vary as a function of scales, and thus is "theory of theory". Renormalization group is first discovered in the quantum field theory of high energy particle physics and condensed matter physics. Here, complex networks made by vertices and edges, is probably the simplest example to introduce renormalization group.
以此为基础，我们讨论复杂网络的“重整化群”问题。重整化群是物理理论随着尺度变化而变化的现象，是“理论中的理论”。重整化群在高能粒子物理和凝聚态物理的量子场论描述中最先被发现。而只由点和线组成的复杂网络，或许是介绍重整化群的最简单的例子。

<u>Part A. Fractals and Box counting dimensions 分形和计盒维数 (7 points)</u>

We cover an image using squares with side length no more than s. Let $B(s)$ be the least number of squares to cover the whole image. Then the box counting dimension d_B is
我们用边长不超过 s 的正方形来覆盖整个图形。设 $B(s)$ 是可以覆盖整个图形所用正方形的最少数目。则计盒维数 d_B 为

$$d_B \equiv \lim_{s\to 0} \frac{\log B(s)}{\log(1/s)} \ .$$

In this problem we assume the above limit exists. For example, the Sierpinski carpet is the infinite-time iteration of the following figure (in the following figure we only displayed the first two iterations). Through infinite iteration, self-similar patterns emerge in the following figure. Such complex objects are known as fractals. In each iteration, in the nine squares, the center one is removed. What's the box counting dimension of the Sierpinski carpet?
本题中，我们假设以上极限是存在的。例如，谢尔宾斯基毯是下图的无穷迭代（下图中之显示了迭代的前两次）。通过无穷迭代，下图产生了自相似结构。这样的复杂物体叫做分形。每次迭代中，九个正方形里，正中的正方形空缺。谢尔宾斯基毯的计盒维数是多少呢？



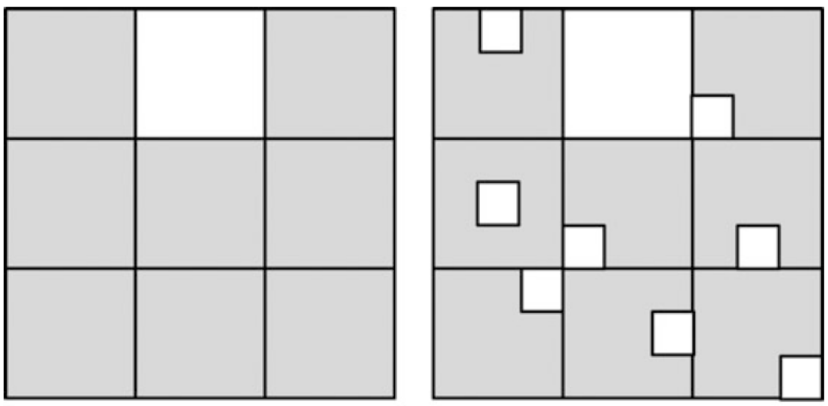
Here, after each iteration, we can use squares to cover the image, with side length s which is equal to the side length of the dark box. Let the length of the whole Sierpinski carpet be L. Then $B\left(s=\frac{L}{3}\right)=8$, $B\left(s=\frac{L}{9}\right)=64, \cdots, B\left(s=\frac{L}{3^n}\right)=8^n$. Note that here the limit

$s \to 0$ is equivalent to the limit $n \to \infty$. Thus, the box counting dimension of the Sierpinski carpet is
这里，在每次迭代后，我们可以用边长 s 等于深色方块的正方形来覆盖图形。假设整个谢尔宾斯基毯的长度为 L，则 $B\left(s=\frac{L}{3}\right)=8, B\left(s=\frac{L}{9}\right)=64, \cdots, B\left(s=\frac{L}{3^n}\right)=8^n$。注意到 $s \to 0$ 的极限就是 $n \to \infty$ 的极限，故谢尔宾斯基毯的计盒维数为

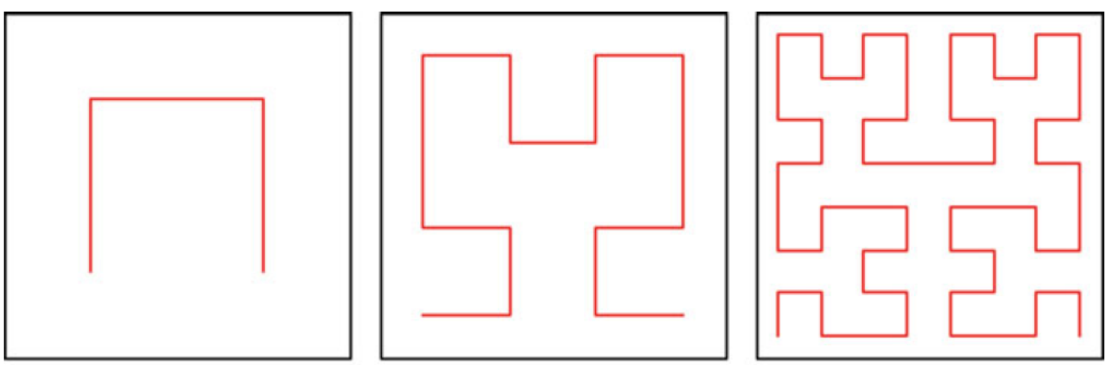
$$d_B=\lim_{s\to 0}\frac{\log B(s)}{\log(1/s)}=\lim_{n\to\infty}\frac{\log(8^n)}{\log(3^n/L)}=\frac{\log 8}{\log 3}.$$

(Note that in the sense of taking limits, whichever unit we use, $\log L$ is negligible compared to $\log 3^n$ in the above equation.)
（注意到，在取极限的意义下，无论取什么单位，上式中 $\log L$ 和 $\log 3^n$ 相比都可以忽略。）

A1	The randomized Sierpinski carpet is the infinite-time iteration of the following figure (in the following figure we only displayed the first two iterations). In each iteration, a random square among the nine is removed. Calculate the box counting dimension of the randomized Sierpinski carpet. 随机化谢尔宾斯基毯是下图的无穷迭代（下图中只显示了迭代的前两次）。每次迭代中，九个正方形随机空缺一个。求随机化谢尔宾斯基毯的计盒维数。	2 Points 2 分

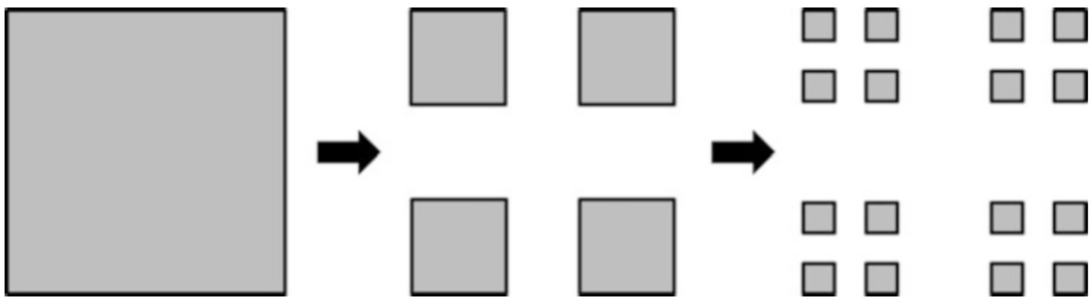


A2	The Hilbert curve is the infinite-time iteration of the following figure (in the following figure we only displayed the first three iterations). Calculate the box counting dimension of the Hilbert curve. 希尔伯特曲线是下图的无穷迭代（下图中只显示了迭代的前三次）。求希尔伯特曲线的计盒维数。	2 Points 2 分



A3	The below is one form of Cantor set, defined as the infinite-time iteration of the following figure (in the following figure we only displayed the first three iterations). Calculate the box counting dimension of the Cantor set. 一种形式的康托尔集是下图的无穷迭代（下图中只显示了迭代的前三次）。求康托尔集的计盒维数。	3 Points 3 分

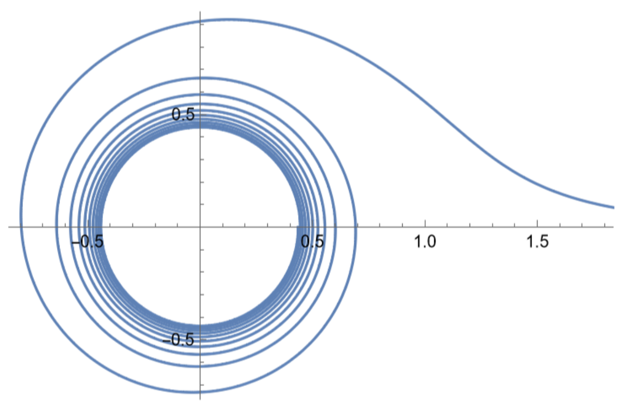
2



Part B. Intersection Between a Spiral and the Positive x-Axis 螺旋线与正x轴的交点 (9 points)

Consider a spiral $r(\theta) = \theta^{-\alpha}$ $(\alpha > 0)$ in polar coordinates. Consider all the intersections between this spiral and the positive x-axis (i.e., the ray $\theta = 0$ starting from the origin). We will call them "intersections" for short. What's the box counting dimension of these intersections? We will solve this problem step-by-step in this part. (Note: in the calculation, we will allow $\theta \to +\infty$. In the below figure, we have not shown the large θ behavior.)

考虑极坐标下的螺旋线 。考虑该螺旋线与 x 轴正方向（即从原点出发的 $\theta = 0$ 射线）的所有交点（本部分 B 中，简称其为 "交点"）。这些交点的计盒维数是多少？我们将分几个小题来解决这个问题。（注：在计算中我们将允许 $\theta \to +\infty$，下图中并没有显示出 θ 取很大值时的行为。）



B1	We denote the θ value of all the intersections by $\theta = \lambda j$, where λ is a constant, j is non-negative consecutive integers. Calculate λ and calculate the range of j. 所有交点的 θ 值可以记作 $\theta = \lambda j$，其中 λ 为常数，j 为可连续取值的非负整数。求 λ 的值，以及 j 的取值范围。	1 Points 1 分
B2	Calculate the horizontal axis x_j of the intersection labelled by j, as a function of j and α. 求交点 j 的横坐标 x_j，用 j 和 α 表示。	1 Points 1 分

The distance between neighbor intersections 相邻交点的间距

B3	Given a very small length s. Suppose J is the smallest number which satisfying the following condition. 给定一个足够小的长度 s，设 J 满足如下条件的最小的数字： $x_j - x_{j+1} \le s$ for all (对于所有) $j \ge J$ Calculate J, in terms of α and s. 求 J 的值，用 α 和 s 表示。	3 Points 3 分

Box Counting Dimension 计盒数量

B4	For small enough s, calculate $B(s)$ in terms of α and s. Note that here we use one-dimensional interval with length no more than s (instead of two-dimensional boxes as given in Part A) as “boxes” to cover the intersections. 对足够小的 s，求 $B(s)$，用 α 和 s 表示。注意，这里我们用长度至多为 s 的一维线段（而不是 A 部分中的二维正方形）作为“盒子”来覆盖这些交点。	3 Points 3 分
B5	Calculate the box counting dimension of the intersections d_B in terms of α and s. 求交点的计盒维数 d_B，用 α 和 s 表示。	1 Points 1 分

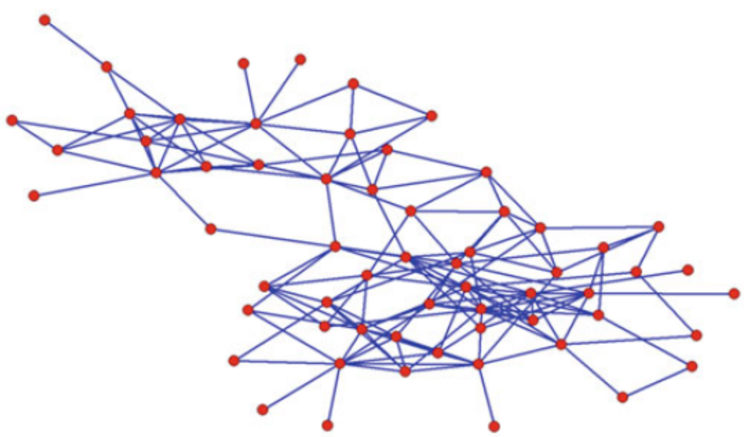
Part C The Box Counting Dimension of a Complex Network 复杂网络的计盒维数 (4 points)

In the following, we will discuss network formed by vertices and their connections. We can use a generalized version of box counting dimension to analyze the feature of the network. Now the generalized “box” is no-longer square in the geometrical sense, but rather we require that the distance of any two vertices within a box to be no more than s (i.e., a vertex in a box needs to travel at most s edges to connect to any other given vertex). All the boxes can cover all the vertices. A vertex can only be in one box. The box counting dimension is calculated from the minimal box required.

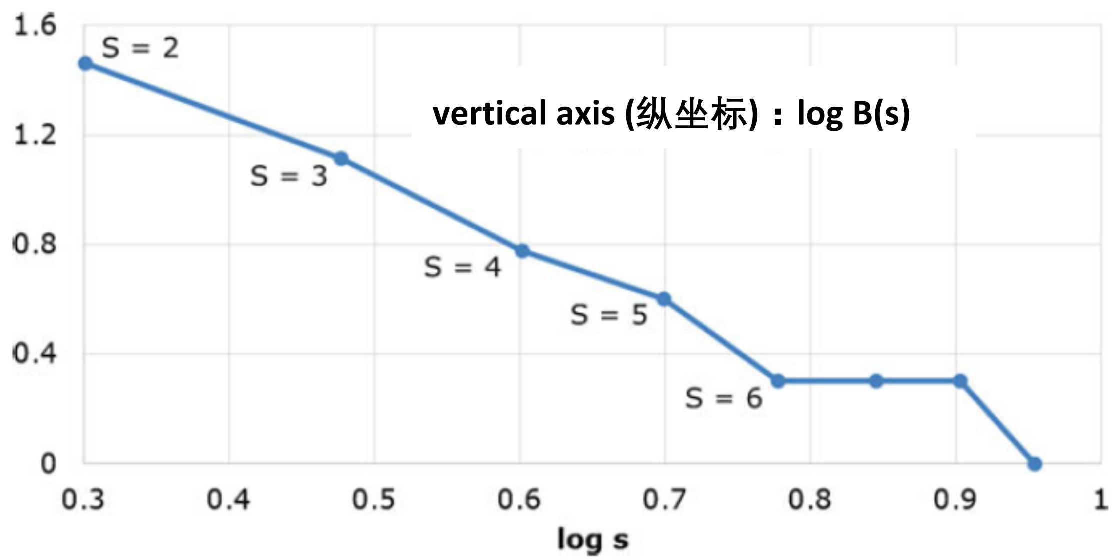
下面，我们将讨论节点和节点之间的连接组成的网络。我们可以用推广的计盒维数来分析网络的特征。这时，推广的“盒子”不再是几何意义上的正方形，而是要求盒子里每两个节点间的距离不多于 s（也就是说一个节点最多通过 s 条边就可以连接到盒子里的任一其它节点）。所有的盒子能覆盖所有节点，不同的盒子里包含的节点不能有交集。计盒维数可从满足以上条件的最少盒子数来计算。

海豚的社交网络 THE SOCIAL NETWORK OF DOLPHINS

The below figure is a dolphin social network recorded by researchers:
下图为研究人员记录一个海豚群组的社交网络：



After analyze, we get
经过分析上图，得到



C1	Suppose the slope of the plot at $s \to 0$ (in a complex network, since $s \geq 1$, the $s \to 0$ limit can only be understood in a sense of continuation) shows the same trend as the slope formed by the $s = 3$ and the $s = 2$ data points (for a fractal complex network, this slope should be a constant). Estimate the box counting dimension of the Dolphin network using the $s = 3$ and the $s = 2$ data points. 假设 $s \to 0$ 的曲线斜率（在复杂网络中，由于$s \geq 1$，$s \to 0$ 的极限只能在数学延拓的意义上定义）与 $s = 3$ 和 $s = 2$ 两个数据点连线的斜率体现出来的趋势相同（对于分形复杂网络，这个斜率应该是个常数），请用$s = 3$ 和 $s = 2$ 两个数据点估计海豚网络的计盒维数。	2 Points 2 分

THE SCALE OF THE FRACTAL NETWORK 分形网络的尺度

C2	Assuming the complex network has number of vertices $N_0 \gg 1$ with a fractal structure and box counting dimension d_B. Estimate the average distance $\bar{r}$ between nodes using N_0 and d_B (as an estimate, we ignore the difference between average distance and maximal distance, and ignore $O(1)$ constants in the limit of an infinite network). 设复杂网络的节点个数为 $N_0 \gg 1$，网络具有分形结构，计盒维数为 d_B，请用 N_0 和 d_B 估计节点之间的平均距离 $\bar{r}$（作为估计，我们忽略平均距离和最大距离的区别，也忽略在网络无穷大极限下的 $O(1)$ 常数）。	2 Points 2 分

Part D The renormalization of Complex Networks 复杂网络的重整化 (10 points)

When calculating the box counting dimension, we use boxes to contain vertices. Now we can construct a new complex network: the vertices of the new network corresponds to the boxes of the original network. If in the original network, a vertex in a box connects to at least one vertex in another box, then the two boxes of the new network (considered as two vertices in the new network) is connected. For example,

在计算计盒维数的时候，我们用盒子把节点"装"起来。这时，我们可以构造一个新的复杂网络：新网络的节点是原复杂网络的盒子；如果旧网络一个盒子里的一个节点至少连接到另一个盒子里的任何节点，则新网络里这两个盒子（即新网络里的两个节点）相连。例如：

D1	For a fractal network, let the box counting dimension of the original network be d_B, after renormalization with box size s, calculate the box counting dimension of the new network. 对分形网络，设原网络的计盒维数为 d_B，在进行盒子尺度为 s 的重整化后，求新网络的计盒维数。	2 Points 2 分

Adding Long Range Connections to a Network 向网络添加长程连接

Many complex networks in our real life do not look like a fractal network. For example, you may have heard the "six degrees of separation", that through at most six people, you can use "friend of friend of friend of friend of friend of friend" connection to know any person in the world. Thus, we often feel "what a small world"!

我们现实世界中的很多复杂网络看上去并不像分形网络。例如，你可能听说过"六度分隔理论"，就是说，最多通过六个人，你能以"朋友的朋友的朋友的朋友的朋友的朋友"的方式认识世界上任何一个人。因此，我们经常惊呼"世界太小了"。

Usually in a fractal network there are too few long-range connections, not enough to have the "small world" feature. To describe a "small world", we randomly add long-range connections to a network: For any pair of vertices with distance r ($r > 1$), we randomly add connections to these vertices with probability $p(r) = Ar^{-\alpha}$ ($\alpha > 0$). For large enough r, these newly added connections dominate the connections of the new network.

通常的分形网络中这样的长程连接非常少，不足以描述"小世界"。为了描述这样的"小世界"现象，我们向一个分形网络中随机添加一些长程连接：对于任何距离为 r 的两个节点（$r > 1$），我们以概率 $p(r) = A\,r^{-\alpha}$ $(\alpha > 0)$ 的概率连接这两个节点。对于足够大的 r，新添加的这些连接是新网络的主要连接方式。

D2	Now, we perform renormalization of this network with box size s. After renormalization, the new distance r_s of the new network is related to the original distance r of the original network by $r_s = r/s$. After renormalization, for the new network, at sufficiently large distances, calculate the probability $p_s(r_s)$ that two vertices are connected, as a function of A, s, α, d_B, r_s. (Here d_B is the fractal dimension of the original network before adding long range connection. Before adding long range connections, we can consider the network in the box as a sub-network similar to the whole network.) 现在，我们对网络做盒子尺度为 s 的重整化。重整化后，新网络上的距离 r_s 与旧网络上的距离 r 的关系为 $r_s = r/s$。求重整化后，新网络在足够大距离上，两个节点之间的连接概率 $p_s(r_s)$，用 A, s, α, d_B, r_s 表示。（这里 d_B 为添加长程链接前，原网络的计盒维数。在添加长程连接前，可将盒子里的网络看成是一个与整个网络相似的子网络。）	4 Points 4 分

The Fixed Point of a Renormalization Group 重整化群的不动点

D3	For a large enough network G, we can apply the renormalization procedure R repeatedly, to get $R(G)$, $R(R(G))$, $R(R(\ldots R(G)\ldots))$ $R(G), R(R(G)), R(R(\ldots R(G)\ldots))$. This repeated renormalization procedure is known as the renormalization group. If after sufficiently many operations, the statistical property of the network no longer changes (in this problem the probably of long-range connections no longer changes), we call the network after many renormalization the "fixed point" of renormalization group (here, the "point" in fixed point means that, in the space of all networks, each network is considered as a point). Since infinite iteration is difficult technically, we alternatively take one renormalization with the $s \to \infty$ (but still $s \ll \bar{r}$) limit. For different values of α, after adding long-range connections with probability $p(r) = A\,r^{-\alpha}$ $(\alpha > 0)$, calculate the expression of $p(r)$ on the fixed point (i.e., determine all possible cases of $p_s(r_s)$ as $r_s \to \infty$). You may use an identity $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. 对于足够大的网络 G，我们可以反复进行重整化操作 R，得到 $R(G), R(R(G)), R(R(\ldots R(G)\ldots))$。这种反复的重整化操作叫做重整化群。如果进行足够多次重整化操作后，网络的统计性质不再改变（本题中体现为长程连接的概率不再改变），我们称这样做过很多次重整化的网络为重整化群的"不动点"。（这里点的意思是，所有网络组成的空间中，每个网络看成是其中一个点）。由于无穷次迭代在技术上较困难，我们也可以取单次 $s \to \infty$ (但是仍然满足 $s \ll \bar{r}$) 的极限来代替无穷次迭代的操作。根据 α 的不同取值，求以概率 $p(r) = A\,r^{-\alpha}$ $(\alpha > 0)$ 添加长程连接之后，复杂网络的不动点上 $p(r)$ 的表达式 (即在 $r_s \to \infty$ 的极限下计算所有 $p_s(r_s)$ 的极限情况)。你可能会用到 $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$。	4 Points 4 分

~ End of Part 2 卷-2 完 ~

Problem 1: Spatiotemporal varying electric permittivity (30 points)

问题 1: 时变介电常数 (30 分)

In electromagnetism, dielectric media, e.g. a block of glass, is represented by a permittivity different from the one of vacuum. The reflection and refraction for electromagnetic waves on a slab of dielectric material can be derived from considering dispersion relationship and matching boundary condition.

在电磁学中，不同于真空介质，例如一块玻璃，其介电材料可以用特定的介电常数表示。在给定色散关系和匹配边界条件下，可以导出电磁波在介电材料板上的反射和折射。

In the question, we would like to investigate the Fabry-Pérot resonance from a slab of dielectric medium in the first step and the analog concept when the permittivity of the material becomes inhomogeneous in the time domain instead of the spatial domain.

在这个问题中，我们第一步研究电介质板的法布里-珀罗共振现象，以及当材料的介电常数在时域而不是空间域中变得不均匀时类似的概念。

Figure 1 shows the schematic diagrams for light entering a block of dielectric medium at normal incidence. The left diagram shows the case for a block of infinite thickness, in which the light only undergoes one instance of reflection and refraction at the interface. The right diagram shows the case for a finite thickness, in which the light ray undergoes multiple reflections within the slab. The spatial and temporal axes are the horizontal and vertical ones respectively.

图 1 显示了光线以垂直入射方式进入电介质板的示意图。左图显示了无限厚度板的情况，其中光在界面处仅经历一次反射和折射。右图显示了有限厚度板的情况，其中光线在板内经历了多次反射。空间轴和时间轴分别是图中水平轴和垂直轴。

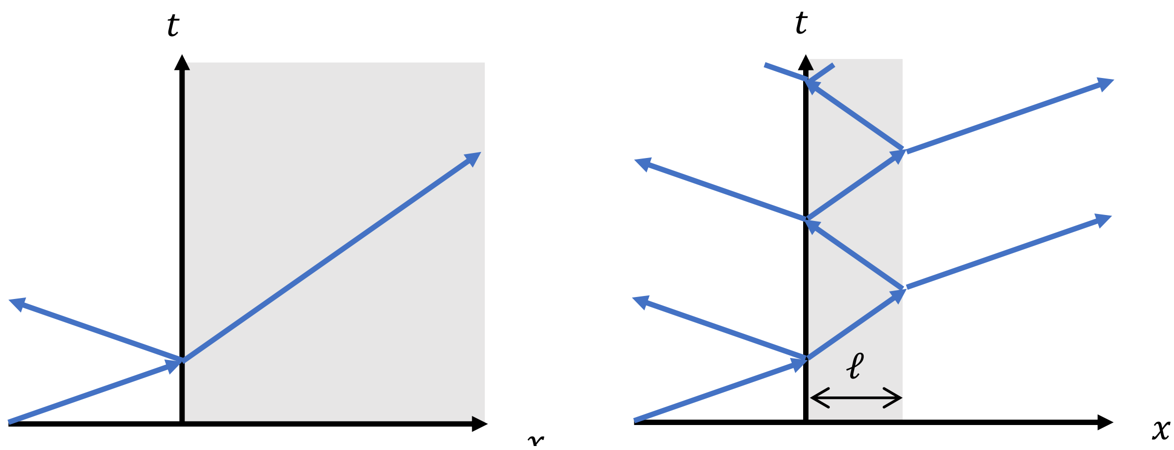


Fig. 1: (Left) Schematic diagram for a light ray entering an interface between vacuum and dielectric medium (Right) Schematic diagram for a light ray entering a slab of dielectric medium undergoing multiple reflections.

图 1：（左）光线进入真空和电介质的示意图，（右）光线进入电介质经过多次反射的示意图。

A. Fabry-Pérot resonance (13 points)

To obtain the amount of light being reflected and refracted from a block of dielectric medium, we start from the Maxwell's equations:

为了获得光经过一块电介质板后的反射和折射强度，我们从麦克斯韦方程开始：

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t},$$

where the curl and div operators are defined by

其中 curl 运算符定义为

$$\nabla \times \boldsymbol{E} = \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_z}{\partial y}\right)$$

The constitutive relationship for the dielectric material or vacuum in relating the different field variables are expressed as

介电材料或真空与不同场变量相关的本构关系为

$$\boldsymbol{D} = \epsilon(x)\boldsymbol{E}$$
$$\boldsymbol{B} = \mu_0\boldsymbol{H}$$

In this part, the electric permittivity profile $\epsilon(x)$ only varies in x but not in y, z and t. All the materials are non-magnetic, having the same value of magnetic permeability μ_0 of vacuum. We only consider electromagnetic waves which propagate in the x-direction.

在本部分中，介电常数曲线 $\epsilon(x)$ 仅随着 x 变化，而与 y、z 和 t 无关。 所有材料都是非磁性的，具有相同的真空磁导率μ_0值。 我们只考虑在 x 方向传播的电磁波。

A1	We consider an electromagnetic wave propagating only in the x-direction specified by $\boldsymbol{E} = \hat{y}E_y(x,t)$, $\boldsymbol{H} = \hat{z}H_z(x,t)$. Reduce the Maxwell's equations to two differential equations only on E_y and H_z. Each differential equation is first order in both temporal dimension t and spatial dimension x. 我们仅考虑由$\boldsymbol{E} = \hat{y}E_y(x,t)$，$\boldsymbol{H} = \hat{z}H_z(x,t)$指定的在$x$方向传播的电磁波。请将麦克斯韦方程组简化为$E_y$和$H_z$上的**两个**微分方程式。**每个**分方程式在时间维度 t 和空间维度 x 上都具有一阶。	2 Points 2 分

Answer:

$$\partial_x E_y = -\partial_t B_z = -\mu_0 \partial_t H_z$$
$$\partial_x H_z = -\partial_t D_y = -\epsilon(x)\partial_t E_y$$

A2	For a sinusoidal wave propagating in the positive *x*-direction in a medium of constant permittivity ϵ_1 (e.g. in a glass), it has a form $E_y = E_{in}\cos(k_1 x - \omega t)$. Find the dispersion relationship between k_1 and ω 对于在介电常数为 ϵ_1 的介质（例如在玻璃中）中沿 x 正方向传播的正弦波，它的形式为 $E_y = E_{in}\cos(k_1 x - \omega t)$。 求$k_1$与 ω 的色散关系。	2 Points 2 分

Answer:

Let also $H_z = H_{in}\cos(k_1 x - \omega t)$, we have

$$-k_1 E_y = -\omega\mu_0 H_z$$
$$-k_1 H_z = -\omega\epsilon_1 E_y$$
$$\Rightarrow k_1 = \omega\sqrt{\mu_0\epsilon_1}, \qquad H_z = \frac{k_1}{\omega\mu_0}E_y = \frac{\sqrt{\epsilon_1}}{\sqrt{\mu_0}}E_y \triangleq \frac{E_y}{\eta_1}$$

Now, we consider an interface between vacuum and a dielectric medium (see left panel of Fig. 1). A fixed incident electromagnetic wave $E_y = E_{in}\cos(k_0 x - \omega t)$ propagates in vacuum to the right hand side before entering the dielectric medium. The permittivity $\epsilon(x)$ in this case is a step function, being ϵ_0 for $x < 0$ and ϵ_1 for $x \geq 0$.

现在，我们考虑真空和电介质之间的界面(见图 1 的左图)。一个固定的入射电磁波 $E_y = E_{in}\cos(k_0 x - \omega t)$ 在进入玻璃之前在真空中传播到右手边。在这种情况下，介电常数 $\epsilon(x)$ 是一个阶跃函数，当$x < 0$ 为 ϵ_0，当 $x \geq 0$ 为 ϵ_1。

A3	Suppose the transmitted waves is represented by $E_y = E_t\cos(k_1 x - \omega t + \phi_t)$. Find E_t and ϕ_t in terms of E_{in}, ω and the permittivities ϵ_0, ϵ_1 假设传输的波由 $E_y = E_t\cos(k_1 x - \omega t + \phi_t)$ 表示。 根据 E_{in}、ω 和其他介电常数 ϵ_0, ϵ_1，求 E_t 和 ϕ_t	5 Points 5 分

Answer:

Let also the reflected wave as $E_y = E_r \cos(k_0 x - \omega t + \phi_r)$, $k_i = \omega\sqrt{\mu_0 \epsilon_i}$, $\eta_i = \sqrt{\mu_0}/\sqrt{\epsilon_i}$.
For $x \leq 0$,

$$E_y = E_{in} \cos(k_0 x - \omega t) + E_r \cos(-k_0 x - \omega t + \phi_r)$$
$$H_z = \frac{E_{in}}{\eta_0} \cos(k_0 x - \omega t) - \frac{E_r}{\eta_0} \cos(-k_0 x - \omega t + \phi_r)$$

For $x \geq 0$,

$$E_y = E_t \cos(k_1 x - \omega t + \phi_t)$$
$$H_z = \frac{E_t}{\eta_1} \cos(-k_1 x - \omega t + \phi_t)$$

On the boundary, the continuous variables are E_y and H_z

$$E_{in} \cos(\omega t) + E_r \cos(\omega t - \phi_r) = E_t \cos(\omega t - \phi_t)$$
$$\frac{E_{in}}{\eta_0} \cos(\omega t) - \frac{E_r}{\eta_0} \cos(\omega t - \phi_r) = \frac{E_t}{\eta_1} \cos(\omega t - \phi_t)$$

Expand into $\cos(\omega t)$ and $\sin(\omega t)$, we obtain the following equations:

$$E_{in} + E_r \cos\phi_r = E_t \cos\phi_t$$
$$(E_{in} - E_r \cos\phi_r)/\eta_0 = E_t \cos\phi_t /\eta_1$$
$$E_r \sin\phi_r = E_t \sin\phi_t$$
$$-E_r \sin\phi_r /\eta_0 = E_t \sin\phi_t /\eta_1$$

Therefore,

$$E_t = \frac{2\eta_1}{\eta_0 + \eta_1} E_{in}$$
$$E_r = \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} E_{in}$$
$$\phi_r = \phi_t = 0$$

We can also define transmission and reflection coefficients t and r by

$$t = \frac{2\eta_1}{\eta_0 + \eta_1} = \frac{2\sqrt{\epsilon_0}}{\sqrt{\epsilon_0} + \sqrt{\epsilon_1}}$$
$$r = \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} = \frac{\sqrt{\epsilon_0} - \sqrt{\epsilon_1}}{\sqrt{\epsilon_0} + \sqrt{\epsilon_1}}$$

Now, we consider a plate of dielectric medium with a finite thickness ℓ, with a fixed electromagnetic wave shining normally to the plate. There will be multiple reflections and transmissions between the two interfaces of the dielectric medium (see right panel of Fig. 1). The incident wave is still fixed as $E_y = E_{in} \cos(k_0 x - \omega t)$ while the total transmitted waves after the whole slab of dielectric medium is now specified by $E_y = E_t^{(\mathrm{tot})} \cos\left(k_0 x - \omega t + \phi_t^{(\mathrm{tot})}\right)$, by summing all the multiple reflections and transmissions.

现在，我们考虑具有有限厚度 l 的电介质板，即一个固定的电磁波正入射到电介质板。在电介质的两个界面之间会有多次反射和透射(如图 1 的右图)。入射波仍然固定为

$E_y = E_{in}\cos(k_0 x - \omega t)$， 而整个电介质板之后的透射波可以写成为 $E_y = E_t^{(\text{tot})}\cos\left(k_0 x - \omega t + \phi_t^{(\text{tot})}\right)$，通过对所有多次反射和透射求和。

A4	Derive the maximum value for $\left\|E_t^{(\text{tot})}\right\|$ when we can choose an optimal value of ℓ . Derive the condition for such ℓ. 当我们可以选择 ℓ 的最佳值时， 导出 $\left\|E_t^{(\text{tot})}\right\|$ 的最大值以及ℓ满足的条件。	4 Points 4 分

Answer:

From last part, it is actually more convenient to use complex number notation, e.g.

$$E_r\cos\phi_r + iE_r\sin\phi_r \to e_r$$

Then

$$e_t = \frac{2\eta_1}{\eta_0+\eta_1}e_{in}$$
$$e_r = \frac{\eta_1-\eta_0}{\eta_1+\eta_0}e_{in}$$

From vacuum to dielectric medium:

$$\mathcal{t} = \frac{2\eta_1}{\eta_0+\eta_1}, \qquad \mathcal{r} = \frac{\eta_1-\eta_0}{\eta_1+\eta_0}$$

From dielectric medium to vacuum:

$$\mathcal{t}' = \frac{2\eta_0}{\eta_0+\eta_1}, \qquad \mathcal{r}' = \frac{\eta_0-\eta_1}{\eta_1+\eta_0} = -\mathcal{r}$$

Total transmission (summing the multiple scattering processes in right panel of Fig. 1) is

$$\begin{aligned}\mathcal{t}_{tot} &= \mathcal{t}e^{ik_1\ell}\mathcal{t}' + \mathcal{t}e^{ik_1\ell}\mathcal{r}'\,e^{ik_1\ell}\mathcal{r}'\,e^{ik_1\ell}\mathcal{t}' + \cdots \\ &= \frac{\mathcal{t}\mathcal{t}'e^{ik_1\ell}}{1-\mathcal{r}'^2e^{2ik_1\ell}}\end{aligned}$$

Therefore, the maximum value of $\left|E_t^{(\text{tot})}\right|$ is

$$\frac{4\eta_0\eta_1}{(\eta_0+\eta_1)^2}\frac{1}{1-\left(\frac{\eta_0-\eta_1}{\eta_1+\eta_0}\right)^2} = \frac{4\eta_0\eta_1}{4\eta_0\eta_1} = 1$$

The condition is

$$k_1\ell = m\pi$$

for an arbitrary integer m.

B. Time-varying permittivity (14 points)

We have considered a boundary of the permittivity profile in the spatial domain x. Now we go to the complementary problem. Suppose the electric permittivity is constant in all spatial dimensions but varies in temporal dimension t. For example, materials like $\mathrm{LiNbO_3}$ will have its permittivity changes when a DC voltage is applied across the material through the Kerr electro-optic effect.

我们已经考虑了空间域 x 中介电常数分布的边界。 现在我们转到其互补问题。 假设介电常数在所有空间维度上都是恒定的，但在时间维度 t 上变化。例如，当通过克尔电光效应在材料上施加直流电压时，LiNb03 等材料的介电常数会发生变化。

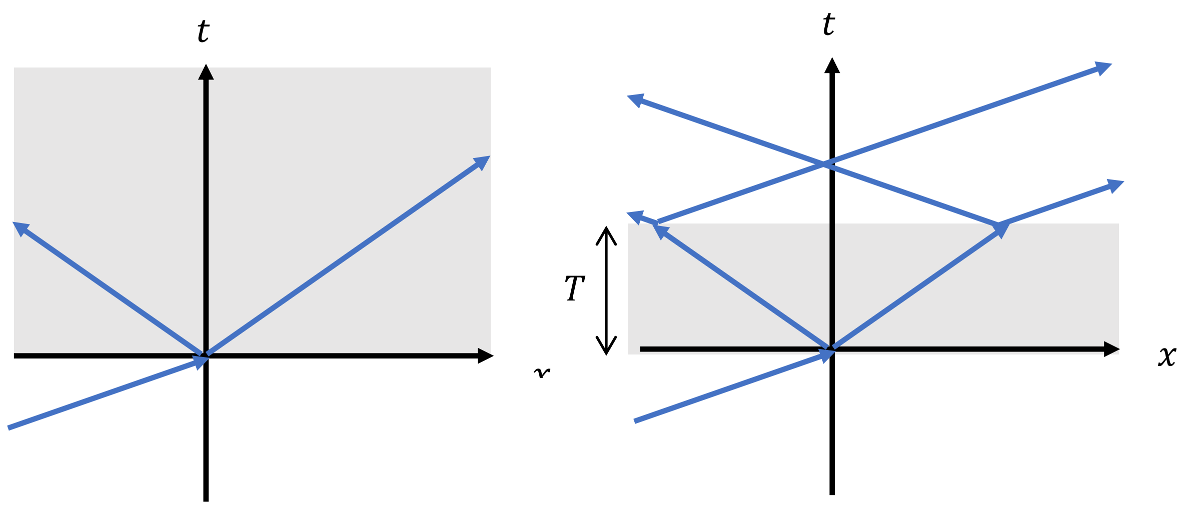


Fig. 2: (Left) Schematic diagram for a light ray entering a time-boundary in which the permittivity changes from ϵ_1 to ϵ_2 (Right) Schematic diagram for a light ray going through two time-boundaries. The permittivity changes from ϵ_1 to ϵ_2 at $t = 0$ and changes back to ϵ_1 at $t = T$.

图 2：（左）光线进入时间边界的示意图，其中介电常数从 ϵ_1 变为 ϵ_2（右）光线穿过两个时间边界的示意图。 介电常数在 t=0 时从 ϵ_1 变为 ϵ_2，并在 t=T 时变回 ϵ_1。

In a similar fashion, left panel of Fig. 2 shows the trajectory for a light ray originally propagating in a homogeneous medium in the positive direction. Then, the permittivity suddenly changes from ϵ_1 to another value ϵ_2 at $t = 0$. In the right panel, we now have a “slab” of such time-varying material in the time domain. The permittivity changes back to ϵ_1 at $t = T$. The blue arrows show the light rays. At every boundary, the ray splits in a forward and backward propagating ray. As the initial ray is propagating to the right, we call the left propagating wave as “reflection/backward propagating” and the right propagating wave as “transmission/forward propagating” after a time boundary. Note that the rays can only to in the positive time direction.

以类似的方式，图 2 的左图显示了最初在一个均质介质中沿正方向传播的光线的轨迹。然后，介电常数在 t=0 时突然从 ϵ_1 变为另一个值 ϵ_2。 在右侧面板中，我们在时域中有一个这种随时间变化的材料的“板”。 介电常数在 t=T 时变回 ϵ_1。 蓝色箭头表示光线。 在每个边界处，光线分裂为向前和向后传播的光线。 由于初始光线向右传播，我们称左传播波为“反射/后向传播”，右传播波在时间边界后称为“透射/前向传播”。请注意，光线只能在正时间方向上前进。

To be specific, consider an electromagnetic waves propagates in an infinite block of dielectric medium originally. Suddenly, at time equal to zero, the dielectric constant in the whole space changes from ϵ_1 to ϵ_2. We call $t = 0$ as a temporal boundary. See left panel of Fig. 2.

具体来说，考虑电磁波最初在无限大的电介质中传播。 突然，在时间等于 0 时，在整个空间介电常数从 ϵ_1 变为 ϵ_2。 我们称 t=0 为时间边界。参见图 2 的左侧。

B1	What are the continuity conditions in this case across the time boundary? 在这种情况下，跨越时间边界的连续性条件是什么？	3 Points 3 分

Answer:

From the Maxwell’s equations,

$$\partial_x E_y = -\partial_t B_z$$
$$\partial_x H_z = -\partial_t D_y$$

under the similar principle to consider a spatial boundary, a temporal boundary requires continuity of D_y and B_z.

Initially, the electromagnetic waves is specified by $\boldsymbol{E} = \hat{y}E_1\cos(k_1x - \omega t)$, a forward propagating waves in the positive x direction. After the permittivity has changed, there is a backward propagating waves (along the negative x direction) and forward propagating waves (along the positive x direction). They are the defined as the temporal reflection and temporal transmission waves.

最初，电磁波由 $\boldsymbol{E} = \hat{y}E_1\cos(k_1x - \omega t)$ 指定，即在正 x 方向上的前向传播波。介电常数改变后，有向后传播的波（沿负 x 方向）和向前传播的波（沿正 x 方向）。 它们被定义为时间反射波和时间传输波。

B2	The radial frequency has changed from ω to another value ω' across the time-boundary. Express ω' in terms of ω and the permittivity values. 径向频率在时间边界上已从 ω 变为另一个值 ω'。 用 ω 和介电常数值表示 ω'。	3 Points 3 分

Answer:

For $t \leq 0$,

$$E_y = E_1\cos(k_1x - \omega t)$$
$$H_z = \frac{E_1}{\eta_1}\cos(k_1x - \omega t)$$

For $t \geq 0$,

$$E_y = E_r\cos(k_2x - \omega' t - \phi_r) + E_t\cos(k_2x - \omega' t + \phi_t)$$
$$H_z = -\frac{E_r}{\eta_2}\cos(k_2x - \omega' t - \phi_r) + \frac{E_t}{\eta_2}\cos(k_2x - \omega' t + \phi_t)$$

where $k_2 = \omega'\sqrt{\mu_0\epsilon_2}$.

Continuity gives

$$\epsilon_1E_1\cos(k_1x) = \epsilon_2E_r\cos(k_2x - \phi_r) + \epsilon_2E_t\cos(k_2x + \phi_t)$$
$$\frac{E_1}{\eta_1}\cos(k_1x) = -\frac{E_r}{\eta_2}\cos(k_2x - \phi_r) + \frac{E_t}{\eta_2}\cos(k_2x + \phi_t)$$

The condition can only be satisfied by setting $k_2 = k_1$. This is in analogy to the spatial case that ω is conserved across a spatial boundary. For a time-boundary. k is conserved.

Therefore

$$\omega' = \sqrt{\epsilon_1}\omega/\sqrt{\epsilon_2}$$

From now on, I just call $k = k_1 = k_2 = \omega\sqrt{\mu_0\epsilon_1}$

B3	Express the amplitude of the backward propagating wave and the forward propagating wave. 求后向传播波和前向传播波的幅度。	3 Points 3 分

Answer:

From last part, the continuity is derived as

Continuity gives

$$\epsilon_1 E_1 \cos(kx) = \epsilon_2 E_r \cos(kx - \phi_r) + \epsilon_2 E_t \cos(kx + \phi_t)$$
$$\frac{E_1}{\eta_1}\cos(kx) = -\frac{E_r}{\eta_2}\cos(kx - \phi_r) + \frac{E_t}{\eta_2}\cos(kx + \phi_t)$$

We therefore have $\phi_t = \phi_r = 0$ and

$$\epsilon_1 E_1 = \epsilon_2 E_r + \epsilon_2 E_t$$
$$\frac{E_1}{\eta_1} = \frac{E_r}{\eta_2} + \frac{E_t}{\eta_2}$$

So, we have

$$E_r = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \frac{\eta_2}{\eta_1}\right)E_1 = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}}\right)E_1$$
$$E_t = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\eta_2}{\eta_1}\right)E_1 = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}}\right)E_1$$

Suppose the whole medium goes back to a dielectric medium of permittivity of ϵ_1 after time $t = T$ (see right hand side of Fig. 2).

假设在时间 $t = T$ 之后，整个介质回到介电常数为 ϵ_1 的电介质 （参见图 2 的右侧）。

B4	Find out the amplitude of the backward and forward propagating waves finally. What is the condition to get minimal reflection? 最后求出后向传播波和前向传播波的幅度。 获得最小反射的条件是什么？	5 Points 5 分

Answer:

We use a tuple to represent the backward and forward propagating waves and use complex quantities. The different processes (see right hand side of Fig. 2) can be written as the following step by step.

At $t = 0$

$$(0,1) \to \left(\frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \frac{\eta_2}{\eta_1}\right), \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\eta_2}{\eta_1}\right)\right)$$

At $t = T^-$, the two waves propagate and become

$$\left(\frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \frac{\eta_2}{\eta_1}\right) e^{-i\omega' T}, \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\eta_2}{\eta_1}\right) e^{i\omega' T}\right)$$

At $t = T$, we have two processes:

$$(0,1) \to \left(\frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} - \frac{\eta_1}{\eta_2}\right), \frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} + \frac{\eta_1}{\eta_2}\right)\right)$$
$$(1,0) \to \left(\frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} + \frac{\eta_1}{\eta_2}\right), \frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} - \frac{\eta_1}{\eta_2}\right)\right)$$

As the system is linear, the final waves become

$$\left(\frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \frac{\eta_2}{\eta_1}\right) e^{-i\omega' T}, \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\eta_2}{\eta_1}\right) e^{i\omega' T}\right)$$
$$\to \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \frac{\eta_2}{\eta_1}\right) e^{-i\omega' T}\left(\frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} - \frac{\eta_1}{\eta_2}\right), \frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} + \frac{\eta_1}{\eta_2}\right)\right)$$
$$+ \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\eta_2}{\eta_1}\right) e^{i\omega' T}\left(\frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} + \frac{\eta_1}{\eta_2}\right), \frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} - \frac{\eta_1}{\eta_2}\right)\right)$$

$$= \left(\frac{i}{2}\sin(\omega' T)\left(-\frac{\epsilon_1\eta_1}{\epsilon_2\eta_2} + \frac{\epsilon_2\eta_2}{\epsilon_1\eta_1}\right), \cos(\omega' T) + \frac{i}{2}\sin(\omega' T)\left(\frac{\epsilon_1\eta_1}{\epsilon_2\eta_2} + \frac{\epsilon_2\eta_2}{\epsilon_1\eta_1}\right)\right)$$
$$= \left(\frac{i}{2}\sin(\omega' T)\left(-\frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} + \frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}\right), \cos(\omega' T) + \frac{i}{2}\sin(\omega' T)\left(\frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} + \frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}\right)\right)$$

The condition to get minimal reflection is $\omega' T = m\pi$.

C. Spatiotemporal permittivity (5 points)

In the final part, we consider the case that the permittivity has to be written as a function in both x and t, i.e. $\epsilon(x,t)$. For a simple case, the permittivity function consists of two constants: ϵ_1 in the white region and ϵ_2 in the gray region, a so-called spatiotemporal boundary depicted in Fig. 3. The spatiotemporal boundary is defined by a straight line $x = -ut$

在最后一部分，我们考虑介电常数必须写为 x 和 t 的函数的情况，即 ϵ(x,t)。 对于一个简单的情况，介电常数函数由两个常数组成：白色区域中的 ϵ_1 和灰色区域中的 $\epsilon 2$，即所谓的时空边界，如图 3 所示。时空边界由直线 $x = -ut$ 定义。

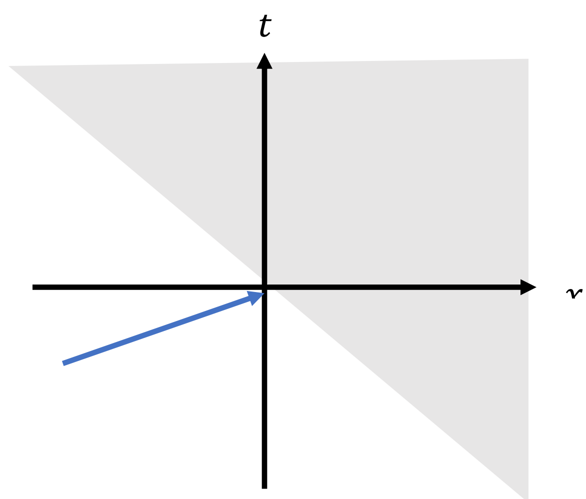


Fig. 3: A spatiotemporal boundary. In the gray (white) region, the permittivity is ϵ_1 (ϵ_2)

图 3：时空边界。 在灰色（白色）区域，介电常数为 ϵ_1 (ϵ_2)

C1	What are the continuity conditions in this case across the spatiotemporal boundary? Hint: You may need to consider a coordinate transformation. 在这种情况下，跨越时空边界的连续性条件是什么？提示：您可能需要考虑坐标变换。	3 Points 3 分

Answer:

We change the coordinate from (x,t) to (q,τ)

$$q = q(x,t)$$
$$\tau = x + ut$$

As $\epsilon(x,t) = \epsilon(\tau)$, we would like to observe how the Maxwell's equations connect field quantities at different τ.

$$\partial_x = \frac{\partial q}{\partial x}\partial_q + \partial_\tau$$
$$\partial_t = \frac{\partial q}{\partial x}\partial_q + u\partial_\tau$$

From the Maxwell's equations,

$$\partial_x E_y + \partial_t B_z = 0$$
$$\partial_x H_z + \partial_t D_y = 0$$

we have

$$\frac{\partial q}{\partial x}\left(\partial_q E_y + \partial_q B_z\right) + \partial_\tau\left(E_y + uB_z\right) = 0$$
$$\frac{\partial q}{\partial x}\left(\partial_q H_z + \partial_q D_y\right) + \partial_\tau\left(H_z + uD_y\right) = 0$$

The continuity variables are

$$E_y + uB_z$$
$$H_z + uD_y$$

being continuous across the boundary

A more specific way:

Define $u = c_1 \cot\theta$ so that the boundary is described by

$$c_1 \cos\theta \;\, t + \sin\theta \;\, x = 0$$

We change coordinate from (x,t) to fictitious (x',t') by

$$\begin{pmatrix} x' \\ c_1 t' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ c_1 t \end{pmatrix}$$

$$\begin{pmatrix} \partial_x \\ \dfrac{1}{c_1}\partial_t \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \partial_{x'} \\ \dfrac{1}{c_1}\partial_{t'} \end{pmatrix}$$

From the Maxwell's equations,

$$\partial_x E_y + \partial_t B_z = 0$$
$$\partial_x H_z + \partial_t D_y = 0$$

$$\cos\theta\,\partial_{x'} E_y + \sin\theta\,\frac{1}{c_1}\partial_{t'} E_y + c_1\left(-\sin\theta\,\partial_{x'} B_z + \cos\theta\,\frac{1}{c_1}\partial_{t'} B_z\right) = 0$$
$$\cos\theta\,\partial_{x'} H_z + \sin\theta\,\frac{1}{c_1}\partial_{t'} H_z + c_1\left(-\sin\theta\,\partial_{x'} D_y + \cos\theta\,\frac{1}{c_1}\partial_{t'} D_y\right) = 0$$

$$\partial_{x'}\left(\cos\theta\,E_y - \sin\theta\,c_1 B_z\right) + \partial_{t'}\left(\sin\theta\,\frac{1}{c_1}E_y + \cos\theta\,B_z\right) = 0$$
$$\partial_{x'}\left(\cos\theta\,H_z - \sin\theta\,c_1 D_y\right) + \partial_{t'}\left(\sin\theta\,\frac{1}{c_1}H_z + \cos\theta\,D_y\right) = 0$$

Since ϵ is a function of t' and independent of x',

The continuity variables are

$$\sin\theta\,\frac{1}{c_1}E_y + \cos\theta\,B_z$$
$$\sin\theta\,\frac{1}{c_1}H_z + \cos\theta\,D_y$$

or equivalently

$$E_y + uB_z$$
$$H_z + uD_y$$

being continuous across the boundary

Problem 2: Fractal Dimensions of Networks (30 points)
问题 2: 网络的分数维数 (30 分)

A line is one-dimensional, a plane is two-dimensional, and the volume of a ball is three-dimensional. Is the dimension of an object always an integer? What's the dimension of a coastline? What's the dimension of the Internet or social network? The dimension of complicated objects is an interdisciplinary study between physics and many other sciences. Here, we will use "box counting dimension" to discuss the dimension of fractal and complex network objects.

直线是一维物体，平面是二维的，而球体是三维的。物体的维度总是整数吗？海岸线是多少维的，互联网、朋友圈又是多少维的？复杂物体的维度，是物理学和多个学科的交叉学科。这里，我们将使用 "计盒维数" 讨论分形几何和复杂网络的维度问题。

Based on this, we discuss the "renormalization group" of a complex network. Renormalization group describes how physics theories vary as a function of scales, and thus is "theory of theory". Renormalization group is first discovered in the quantum field theory of high energy particle physics and condensed matter physics. Here, complex networks made by vertices and edges, is probably the simplest example to introduce renormalization group.

以此为基础，我们讨论复杂网络的 "重整化群" 问题。重整化群是物理理论随着尺度变化而变化的现象，是 "理论中的理论" 。重整化群在高能粒子物理和凝聚态物理的量子场论描述中最先被发现。而只由点和线组成的复杂网络，或许是介绍重整化群的最简单的例子。

PART A. FRACTALS AND BOX COUNTING DIMENSIONS 分形和计盒维数

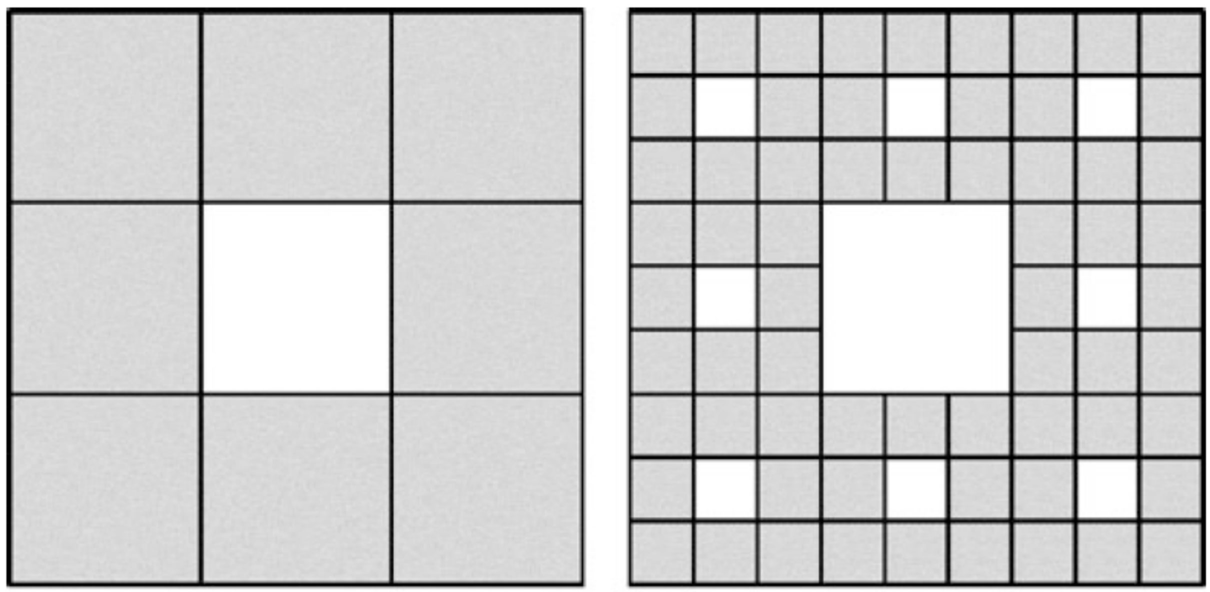
We cover an image using squares with side length no more than s. Let $B(s)$ be the least number of squares to cover the whole image. Then the box counting dimension d_B is

我们用边长不超过 s 的正方形来覆盖整个图形。设 $B(s)$ 是可以覆盖整个图形所用正方形的最少数目。则计盒维数 d_B 为

$$d_B \equiv \lim_{s\to 0}\frac{\log B(s)}{\log(1/s)} .$$

In this problem we assume the following limit exists. For example, the Sierpinski carpet is the infinite-time iteration of the following figure (in the following figure we only displayed the first two iterations). Through infinite iteration, self-similar patterns emerge in the following figure. Such complex objects is known as fractals. In each iteration, in the nine squares, the center one is removed. What's the box counting dimension of the Sierpinski carpet?

本题中，我们假设以上极限是存在的。例如，谢尔宾斯基毯是下图的无穷迭代（下图中之显示了迭代的前两次）。通过无穷迭代，下图产生了自相似结构。这样的复杂物体叫做分形。每次迭代中，九个正方形里，正中的正方形空缺。谢尔宾斯基毯的计盒维数是多少呢？



Here, after each iteration, we can use squares to cover the image, with side length s which is equal to the side length of the dark box. Let the length of the whole Sierpinski carpet be L. Then $B\left(s=\frac{L}{3}\right)=8$, $B\left(s=\frac{L}{9}\right)=64, \cdots, B\left(s=\frac{L}{3^n}\right)=8^n$. Note that here the limit $s\to 0$ is equivalent to the limit $n\to\infty$. Thus, the box counting dimension of the Sierpinski carpet is

这里，在每次迭代后，我们可以用边长 s 等于深色方块的正方形来覆盖图形。假设整个谢尔宾斯基毯的长度为 L，则 $B\left(s=\frac{L}{3}\right)=8, B\left(s=\frac{L}{9}\right)=64, \cdots, B\left(s=\frac{L}{3^n}\right)=8^n$。注意到 $s\to 0$ 的极限就是 $n\to\infty$ 的极限，故谢尔宾斯基毯的计盒维数为

$$d_B=\lim_{s\to 0}\frac{\log B(s)}{\log(1/s)}=\lim_{n\to\infty}\frac{\log(8^n)}{\log(3^n/L)}=\frac{\log 8}{\log 3}.$$

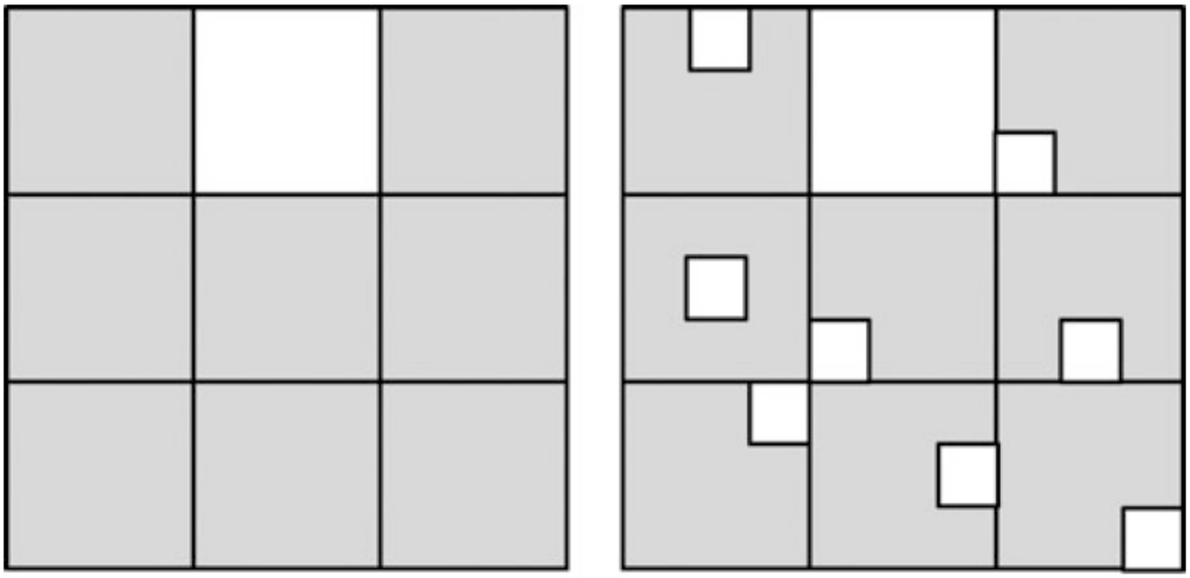
(Note that in the sense of taking limits, whichever unit we use, $\log L$ is negligible compared to $\log 3^n$ in the above equation.)

（注意到，在取极限的意义下，无论取什么单位，上式中 $\log L$ 和 $\log 3^n$ 相比都可以忽略。）

A1. (2P) BOX COUNTING DIMENSION OF A RANDOMIZED SIERPINSKI CARPET 随机化谢尔宾斯基毯的计盒维数

The randomized Sierpinski carpet is the infinite-time iteration of the following figure (in the following figure we only displayed the first two iterations). In each iteration, a random square among the nine is removed. Calculate the box counting dimension of the randomized Sierpinski carpet.

随机化谢尔宾斯基毯是下图的无穷迭代（下图中只显示了迭代的前两次）。每次迭代中，九个正方形随机空缺一个。求随机化谢尔宾斯基毯的计盒维数。

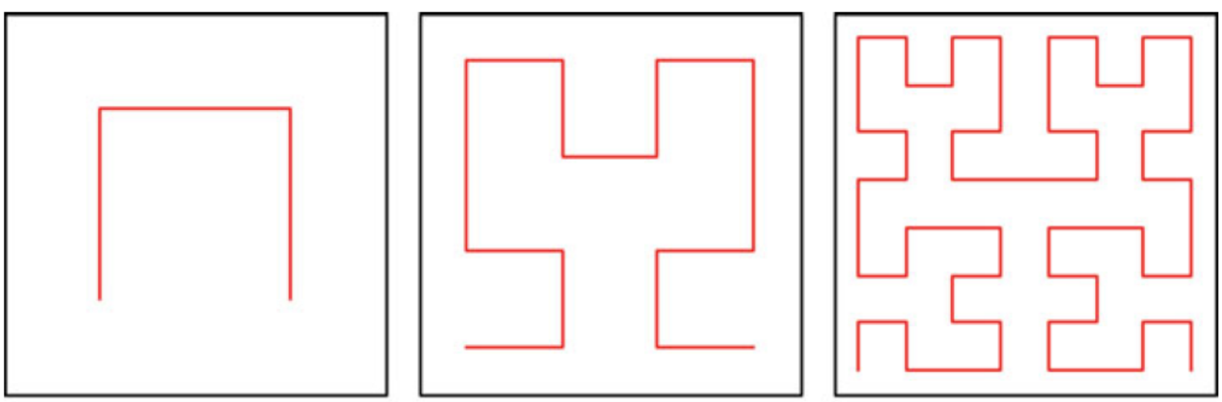


Solution: $\log 8/\log 3$ ($\simeq 1.893$, either the precise or approximate numerical number is fine). There is no difference between the above example (the position of the squire does not enter the calculation).

A2. (2P) BOX COUNTING DIMENSION OF A HILBERT CURVE 希尔伯特曲线的计盒维数

The Hilbert curve is the infinite-time iteration of the following figure (in the following figure we only displayed the first three iterations). Calculate the box counting dimension of the Hilbert curve.

希尔伯特曲线是下图的无穷迭代（下图中只显示了迭代的前三次）。求希尔伯特曲线的计盒维数。



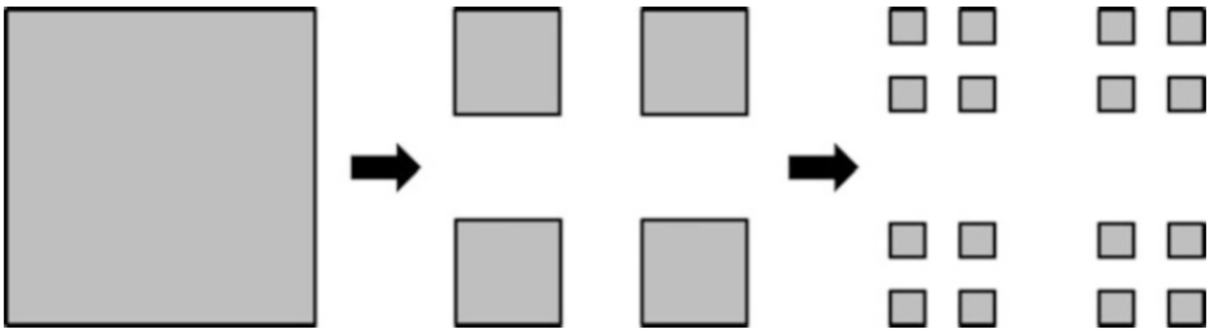
Solution: 2.

Because the Hilbert curve is space-filling. It passes through any given point. (This may not be easy to prove, but it's straightforward to see that for any given point, the curve will pass by close enough to this point.) Thus, no matter how small s is, we always need to put boxes all over space to cover the whole curve.

A3. (3P) BOX COUNTING DIMENSION OF THE CANTOR SET 康托尔集的计盒维数

The below is one form of Cantor set, defined as the infinite-time iteration of the following figure (in the following figure we only displayed the first two iterations)

一种形式的康托尔集是下图的无穷迭代（下图中只显示了迭代的前三次）。求康托尔集的计盒维数。

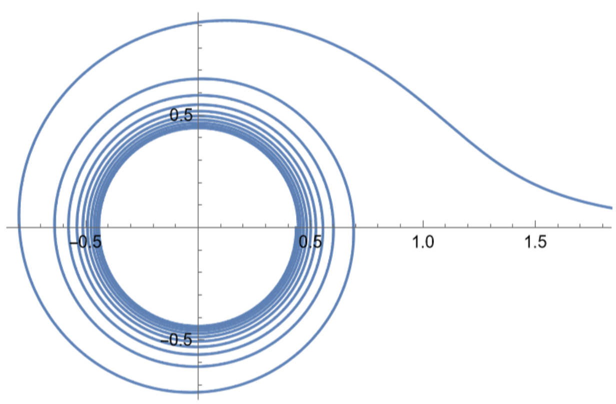


Solution: $B\left(s=\frac{L}{3^n}\right)=4^n$ (1p). Thus $d_B=\frac{\log 4}{\log 3}\simeq 1.262$ (2p)

PART B. INTERSECTION BETWEEN A SPIRAL AND THE POSITIVE X-AXIS 螺旋线与正x轴的交点

Consider a spiral in the polar coordinate. Consider all the intersections between this spiral and the positive x-axis (i.e., the ray $\theta=0$ starting from the origin). We will call them "intersections" for short. What's the box counting dimension of these intersections? We will solve this problem step-by-step in this part. (Note: in the calculation, we will allow $\theta\to+\infty$. In the below figure, we have not shown the large θ behavior.)

考虑极坐标下的螺旋线$r(\theta)=\theta^{-\alpha}$ $(\alpha>0)$。考虑该螺旋线与x轴正方向（即从原点出发的$\theta=0$射线）的所有交点（本部分 B 中，简称其为"交点"）。这些交点的计盒维数是多少？我们将分几个小题来解决这个问题。（注：在计算中我们将允许$\theta\to+\infty$，下图中并没有显示出θ取很大值时的行为。）



B1. (1P) THE VALUE OF θ OF THE INTERSECTIONS 交点的 θ 值

We denote the θ value of all the intersections by $\theta = \lambda j$, where λ is a constant, j is non-negative consecutive integers. Calculate λ and calculate the range of j.

所有交点的 θ 值可以记作 $\theta = \lambda j$，其中 λ 为常数，j 为可连续取值的非负整数。求 λ 的值，以及 j 的取值范围。

Solution: $j = 1,2,\cdots$ (i.e. $j \neq 0$), $\lambda = 2\pi$, $\theta = 2\pi j$.

B2. (1P) THE HORIZONTAL AXIS OF THE INTERSECTIONS 交点的横坐标

Calculate the horizontal axis x_j of the intersection labelled by j, as a function of j and α.

求交点 j 的横坐标 x_j，用 j 和 α 表示。

Solution: $x_j = r\cos\theta = r = (2\pi j)^{-\alpha}$.

B3. (3P) THE DISTANCE BETWEEN NEIGHBOR INTERSECTIONS 相邻交点的间距

Given a very small length s. Suppose J is the smallest number which satisfying the following condition.

给定一个足够小的长度 s，设 J 满足如下条件的最小的数字：

$$x_j - x_{j+1} \leq s \text{ for all (对于所有) } j \geq J$$

Calculate J, in terms of α and s.

求 J 的值，用 α 和 s 表示。

Solution:

Since s is sufficiently small, J must be sufficiently large. (0.5p)

By definition, J satisfies $(2\pi J)^{-\alpha} - [2\pi(J+1)]^{-\alpha} \leq s$ (1p)

Divide $(2\pi J)^{-\alpha}$ and Taylor expand the expression, we have $J \geq \frac{1}{2\pi}\left(\frac{2\pi\alpha}{s}\right)^{\frac{1}{\alpha+1}}$ (1p)

Since J is the smallest number satisfying this inequality, $J = \frac{1}{2\pi}\left(\frac{2\pi\alpha}{s}\right)^{\frac{1}{\alpha+1}} = \left(\frac{\alpha}{(2\pi)^{\alpha}s}\right)^{\frac{1}{\alpha+1}}$ (0.5p)

B4. (3P) BOX COUNTING 计盒数量

For small enough s, calculate $B(s)$ in terms of α and s. Note that here we use one-dimensional interval with length no more than s (instead of two-dimensional boxes as given in Part A) as "boxes" to cover the intersections.

对足够小的 s，求 $B(s)$，用 α 和 s 表示。注意，这里我们用长度至多为 s 的一维线段（而不是 A 部分中的二维正方形）作为“盒子”来覆盖这些交点。

For $1 \le j \le J$, each of the intersection point needs to be covered by one box, since these points are far apart. For $j > J$, we need x_J/s boxes to cover the entire $0 < x < x_J$ range. Thus $B(s) = J + \frac{x_J}{s}$ (2p, one for each term)

Note: since J is large, answers such as $B(s) = (J-1) + \frac{x_J}{s}$ or $B(s) = J + \frac{x_{J+1}}{s}$ can be considered equally right and get full marks.

Insert the expressions for J and x_J, we have

$B(s) = \left[\frac{1}{2\pi}(2\pi\alpha)^{\frac{1}{\alpha+1}} + (2\pi\alpha)^{-\frac{\alpha}{\alpha+1}}\right] s^{-\frac{1}{\alpha+1}}$ (1p, where coefficient 0.5p and power dependence of s 0.5p)

B5. (1P) 计盒维数 BOX COUNTING DIMENSION

Calculate the box counting dimension of the intersections d_B in terms of α and s.

求交点的计盒维数 d_B，用 α 和 s 表示。

Take log of B4 in the small s limit, we have $d_B = 1/(1+\alpha)$.

PART C 复杂网络的计盒维数 THE BOX COUNTING DIMENSION OF A COMPLEX NETWORK

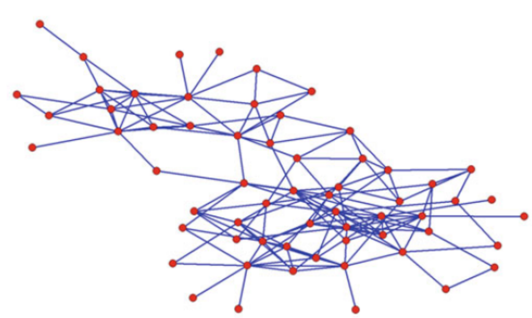
In the following, we will discuss network formed by vertices and their connections. We can use a generalized version of box counting dimension to analyze the feature of the network. Now the generalized "box" is no-longer square in the geometrical sense, but rather we require that the distance of any two vertices within a box to be no more than s (i.e., a vertex in a box needs to travel at most s edges to connect to any other given vertex). All the boxes can cover all the vertices. A vertex can only be in one box. The box counting dimension is calculated from the minimal box required.

下面，我们将讨论节点和节点之间的连接组成的网络。我们可以用推广的计盒维数来分析网络的特征。这时，推广的"盒子"不再是几何意义上的正方形，而是要求盒子里每两个节点间的距离不多于 s（也就是说一个节点最多通过 s 条边就可以连接到盒子里的任一其它节点）。所有的盒子能覆盖所有节点，不同的盒子里包含的节点不能有交集。计盒维数可从满足以上条件的最少盒子数来计算。

C1 (2P) 海豚的社交网络 THE SOCIAL NETWORK OF DOLPHINS

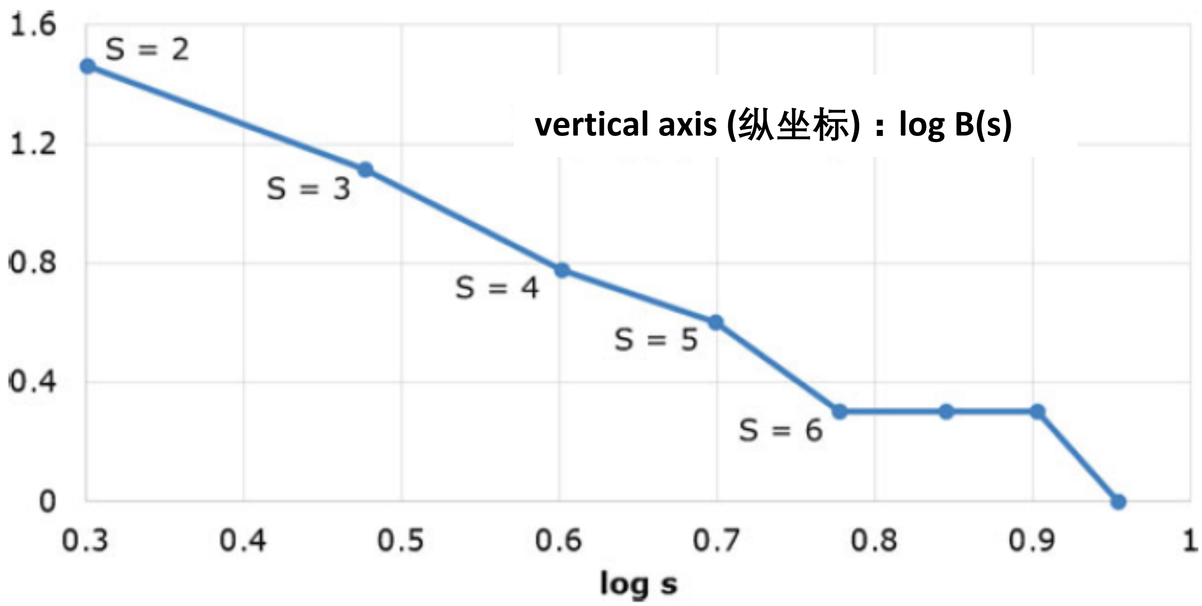
The below figure is a dolphin social network recorded by researchers:

下图为研究人员记录一个海豚群组的社交网络：



After analyze, we get

经过分析上图，得到



Suppose the slope of the plot at $s \to 0$ (in a complex network, since $s \geq 1$, the $s \to 0$ limit can only be understood in a sense of continuation) shows the same trend as the slope formed by the $s = 3$ and the $s = 2$ data points (for a fractal complex network, this slope should be a constant). Estimate the box counting dimension of the Dolphin network using the $s = 3$ and the $s = 2$ data points.

假设 $s \to 0$ 的曲线斜率（在复杂网络中，由于$s \geq 1$，$s \to 0$ 的极限只能在数学延拓的意义上定义）与 $s = 3$ 和 $s = 2$ 两个数据点连线的斜率体现出来的趋势相同（对于分形复杂网络，这个斜率应该是个常数），请用 $s = 3$ 和 $s = 2$ 两个数据点估计海豚网络的计盒维数。

Solution: We note that there is an unknown constant term in the box counting dimension. This constant term can be eliminated by taking the slope of the curve.

As we have assumed that the $s \to 0$ limit exists for calculating d_B, we have

$-d_B = \lim_{s\to 0} \frac{\log B(s)}{\log s} = \lim_{s\to 0} \lim_{\Delta s\to 0} \frac{\log B(s+\Delta s)}{\log(s+\Delta s)} = \lim_{s\to 0} \lim_{\Delta s\to 0} \frac{\log B(s+\Delta s)-\log B(s)}{\log(s+\Delta s)-\log s}$ (Proportional theorem)

From our assumption, we can use the slope of the $s = 3$ and $s = 2$ data points to estimate this slope. Thus,

$$d_B \simeq -\frac{\log B(3) - \log B(2)}{\log 3 - \log 2} \simeq 2.0$$

Answers between 1.5 ~ 2.5 are acceptable, which accounts for numerical errors when extracting data from figure.

C2 (2P) 分形网络的尺度 THE SCALE OF THE FRACTAL NETWORK

Assuming the complex network has number of vertices $N_0 \gg 1$ with a fractal structure and box counting dimension d_B. Estimate the average distance $\bar{r}$ between nodes using N_0 and d_B (as an estimate, we ignore the difference between average distance and maximal distance, and ignore $O(1)$ constants in the limit of an infinite network).

设复杂网络的节点个数为 $N_0 \gg 1$，网络具有分形结构，计盒维数为 d_B，请用 N_0 和 d_B 估计节点之间的平均距离 $\bar{r}$（作为估计，我们忽略平均距离和最大距离的区别，也忽略在网络无穷大极限下的 $O(1)$ 常数）。

Solution: From the experience of the previous question, we noted that d_B can be estimated from the slope of the $(\log(s), \log B(s))$ curve. Take $s \sim O(1)$, from the fractal structure of the network, the nodes in the box $B(s)$ should behave similar to the whole network. Thus,

$$d_B \sim \frac{\log B(s) - \log B(s = \bar{r})}{\log \bar{r} - \log s} \sim \frac{\log B(s)}{\log \bar{r}}$$

Note that $\log B(s = \bar{r}) = O(1)$ and $\log s = O(1)$ are neglected in the last step of the above equation. Since we have taken $s \sim O(1)$, $B(s) \sim N_0$. Thus, $\bar{r} \sim N_0^{1/d_B}$.

Alternative Solution: The students may get an intuitive solution $\bar{r} \sim N_0^{1/d_B}$ directly, by imaging $\bar{r}$ and N_0 as the length and volume of a d_B-dimensional cube. Though mathematically this is not rigorously following our definitions, we can also give full marks to this solution.

PART D 复杂网络的重整化 THE RENORMALIZATION OF COMPLEX NETWORKS

When calculating the box counting dimension, we use boxes to contain vertices. Now we can construct a new complex network: the vertices of the new network corresponds to the boxes of the original network. If in the original network, a vertex in a box connects to at least one vertex in another box, then the two boxes of the new network (considered as two vertices in the new network) is connected. For example,

在计算计盒维数的时候，我们用盒子把节点“装”起来。这时，我们可以构造一个新的复杂网络：新网络的节点是原复杂网络的盒子；如果旧网络一个盒子里的一个节点至少连接到另一个盒子里的任何节点，则新网络里这两个盒子（即新网络里的两个节点）相连。例如：

D1 (2P) 重整化后的计盒维数 THE BOX COUNTING DIMENSION AFTER RENORMALIZATION

For a fractal network, let the box counting dimension of the original network be d_B, after renormalization with box size s, calculate the box counting dimension of the new network.

对分形网络，设原网络的计盒维数为 d_B，在进行盒子尺度为 s 的重整化后，求新网络的计盒维数。

Solution: d_B (because of self-similarity. Or one can calculate using slope, taking boxes with size larger than s.)

D2 (4P) 向网络添加长程连接 ADDING LONG RANGE CONNECTIONS TO A NETWORK

Many complex networks in our real life do not look like a fractal network. For example, you may have heard the “six degrees of separation”, that through at most six people, you can use “friend of friend of friend of friend of friend of friend” connection to know any person in the world. Thus, we often feel “what a small world”!

我们现实世界中的很多复杂网络看上去并不像分形网络。例如，你可能听说过 “六度分隔理论” ，就是说，最多通过六个人，你能以 “朋友的朋友的朋友的朋友的朋友的朋友” 的方式认识世界上任何一个人。因此，我们经常惊呼 “世界太小了” 。

Usually in a fractal network there are too few long-range connections, not enough to have the “small world” feature. To describe a “small world”, we randomly add long-range connections to a network: For any pair of vertices with distance r ($r > 1$), we randomly add connections to these vertices with probability $p(r) = Ar^{-\alpha}$ $(\alpha > 0)$. For large enough r, these newly added connections dominate the connections of the new network.

通常的分形网络中这样的长程连接非常少，不足以描述 “小世界” 。为了描述这样的 “小世界” 现象，我们向一个分形网络中随机添加一些长程连接：对于任何距离为 r 的两个节点（$r > 1$），我们以概率 $p(r) = A\,r^{-\alpha}$ $(\alpha > 0)$ 的概率连接这两个节点。对于足够大的 r， 新添加的这些连接是新网络的主要连接方式。

Now, we perform renormalization of this network with box size s. After renormalization, the new distance r_s of the new network is related to the original distance r of the original network by $r_s = r/s$. After renormalization, for the new network, at sufficiently large distances, calculate the probability $p_s(r_s)$ that two vertices are connected, as a function of A, s, α, d_B, r_s. (Here d_B is the fractal dimension of the original network before adding long range connection. Before adding long range connections, we can consider the network in the box as a sub-network similar to the whole network.)

现在，我们对网络做盒子尺度为 s 的重整化。重整化后，新网络上的距离 r_s 与旧网络上的距离 r 的关系为 $r_s = r/s$。求重整化后，新网络在足够大距离上，两个节点之间的连接概率 $p_s(r_s)$，用 A, s, α, d_B, r_s 表示。（这里 d_B 为添加长程链接前，原网络的计盒维数。在添加长程连接前，可将盒子里的网络看成是一个与整个网络相似的子网络。）

Solution:

Consider the old network. Let the number of nodes in a typical renormalization box be N_B. Since $s \gg 1$, we have $N_B = s^{d_B}$. (1p)

For two such boxes, the probability of no connection between two boxes is $[1 - p(r)]^{N_B^2}$ (2p). Note here 2 in N_B^2 is because you can first calculate the probability between one box and any single node in the other box. This factor of 2 deserves 1 point.

Thus, $p_s(r_s) = 1 - (1 - A(sr_s)^{-\alpha})^{s^{2d_B}}$

D3 (4P) 重整化群的不动点 THE FIXED POINT OF A RENORMALIZATION GROUP

For a large enough network G, we can apply the renormalization procedure R repeatedly, to get $R(G)$, $R(R(G))$, $R(R(\ldots R(G) \ldots))$ $R(G), R(R(G)), R(R(\ldots R(G) \ldots))$. This repeated renormalization procedure is known as the renormalization group. If after sufficiently many operations, the statistical property of the network no longer changes (in this problem the probably of long-range connections no longer changes), we call the network after many renormalization the "fixed point" of renormalization group (here, the "point" in fixed point means that, in the space of all networks, each network is considered as a point). Since infinite iteration is difficult technically, we alternatively take one renormalization with the $s \to \infty$ (but still $s \ll \bar{r}$) limit. For different values of α, after adding long-range connections with probability $p(r) = A\, r^{-\alpha}$ $(\alpha > 0)$, calculate the expression of $p(r)$ on the fixed point (i.e., determine all possible cases of $p_s(r_s)$ as $r_s \to \infty$). You may use an identity $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

对于足够大的网络 G，我们可以反复进行重整化操作 R，得到 $R(G), R(R(G)), R(R(\ldots R(G) \ldots))$。这种反复的重整化操作叫做重整化群。如果进行足够多次重整化操作后，网络的统计性质不再改变（本题中体现为长程连接的概率不再改变），我们称这样做过很多次重整化的网络为重整化群的"不动点"。（这里点的意思是，所有网络组成的空间中，每个网络看成是其中一个点）。由于无穷次迭代在技术上较困难，我们也可以取单次 $s \to \infty$ (但是仍然满足 $s \ll \bar{r}$) 的极限来代替无穷次迭代的操作。根据 α 的不同取值，求以概率 $p(r) = A\, r^{-\alpha}$ $(\alpha > 0)$ 添加长程连接之后，复杂网络的不动点上 $p(r)$ 的表达式 (即在 $r_s \to \infty$ 的极限下计算所有 $p_s(r_s)$ 的极限情况)。你可能会用到 $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$。

Solution:

Note that $(1 - A(sr_s)^{-\alpha})^{s^{2d_B}} = [(1 - A(sr_s)^{-\alpha})^{s^\alpha}]\hat{}(s^{2d_B-\alpha}) \simeq \exp\left(-A\, r_s^{-\alpha} s^{2d_B-\alpha}\right)$. (1p)

Thus, $p_s \simeq 1 - \exp\left(-A\, r_s^{-\alpha} s^{2d_B-\alpha}\right)$ has three cases:

When $2d_B - \alpha > 0$, $p_s \to 1$ (complete graph fixed point). (1p)

When $2d_B - \alpha < 0$, $p_s \to 0$ (fractal fixed point). (1p)

When $2d_B - \alpha = 0$, $p_s \to 1 - \exp(-A\, r^{-2d_B})$ (small-world fixed point). (1p)

Note: the names of the fixed points just indicate the physical meaning. The students only need to write down the different cases and probabilities (each 0.5p), no need to write the names of the fixed points.

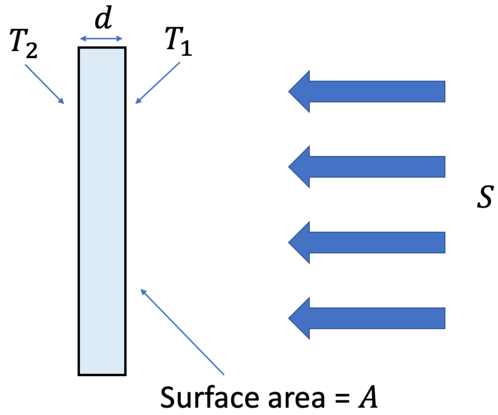
References:

[1] "Fractal Dimensions of Networks", by Eric Rosenberg, Springer Press.

[2] "Small-World to Fractal Transition in Complex Networks: A Renormalization Group Approach", Rozenfeld, Song and Makse, Physical Review Letters 104, 025701 (2010).

Pan Pearl River Delta Physics Olympiad 2022
2022 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 4 Problems, 40 Points) 简体版卷-1（共4题，40分）
(9:30 am – 12:00 pm, 29th Jan 2023)

Please fill in your final answers to all problems on the answer sheet. 请在答题纸上填上各题的最后答案。 At the end of the competition, please submit the answer sheet only. Question papers and working sheets will not be collected. 比赛结束时，请只交回答题纸，题目纸和草稿纸将不会收回。



1. [10 points] Consider a satellite that has a shape of a plate of surface area A and thickness $d \ll \sqrt{A}$. The satellite can convert solar energy into electrical energy and charge the onboard batteries making use of the temperature difference. The solar energy flux density [solar power per unit area] at the position of the satellite is S and the satellite is facing towards Sun. Assuming that the emissivity of both sides of the satellite is ϵ and the temperatures of the satellite on two sides are T_1 and T_2 respectively and σ is the Stefan-Boltzmann constant.
1. [10 分] 考虑一个人造卫星，其形状为表面积为 A 且厚度为$d \ll \sqrt{A}$ 的平板。卫星可以利用温差将太阳能转化为电能，为星载电池充电。人造卫星所在位置的太阳能通量密度[单位面积的太阳能功率]为S，卫星正对着太阳。假设整颗卫星的发射率（又称辐射率）为ϵ，卫星两侧的温度分别为 T_1 和 T_2 ，σ 是 Stefan-Boltzmann 常数。

(a) [2] What is the net heat flux [energy per second] **absorbed** by the bright side (the side facing towards Sun) of the satellite?
(a) [2] 卫星的亮面（面向太阳的一面）吸收的净热通量 [每秒能量] 是多少？

(b) [1] What is the net heat flux [energy per second] **released** from the dark side of the satellite?
(b) [1] 从卫星暗面释放的净热通量 [每秒能量] 是多少？

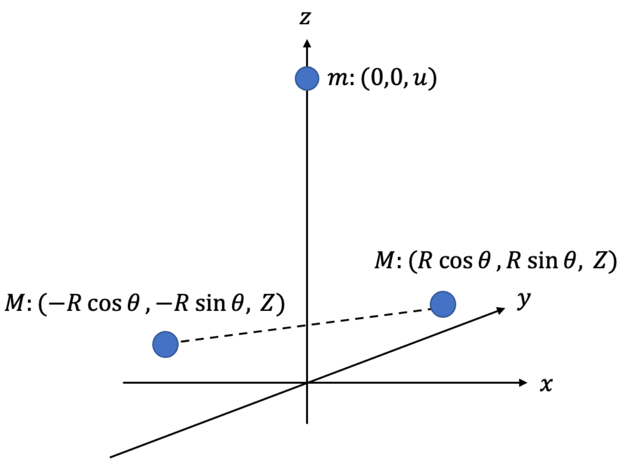
(c) [1] What is value of the emissivity ϵ to get the theoretically maximal charging power P_{max}?
(c) [1] 可得到理论上最大充电功率 P_{max} 的发射率 ϵ 的值是多少？

(d) [3] Find a condition for the temperature T_1 in order to get the theoretically maximal charging power P_{max} provided by the satellite. Express the condition in term of the dimensionless variable $x = \frac{\sigma T_1^4}{S}$. **You don't need to solve the equation in this part.**
(d) [3] 求温度T_1的条件，使得卫星能達到理论上最大充电功率 P_{max}。 用无量纲变量 $x = \frac{\sigma T_1^4}{S}$ 表示你的答案。 你不需要解这部分的方程。

(e) [3] What is the theoretically maximal charging power $P_{\max}$ provided by the satellite? Calculate the numerical value of $\frac{P_{\max}}{AS}$. Your answer should be correct to at least 2 significant figures.
(e) [3] 卫星可提供的理论上最大充电功率P_{max}是多少？ 计算 $\frac{P_{\max}}{AS}$ 的数值。 你的答案至少应正确到 2 位有效数字。

2. [10 points] Binary-sun solar system: Consider a binary pair of identical suns of mass M orbiting in the $x-y$ plane in an orbit centred at the origin. The gravitational constant is G. Now add a planet of mass m with an initial condition on the z axis above the center of mass of the two suns and with a velocity along the z direction. By the symmetry of the system, the small planet will remain on the z axis, suns will have equal z coordinates and the center of mass of two suns will also remain on the z axis.

2. [10 分]双星太阳系：考虑两个质量均为 M 的相同太阳在 $x-y$ 平面中以原点为中心的轨道运行。万有引力常数为 G。现在添加一个质量 m 的行星，其初始条件位于两个太阳质心上方的 z 轴上，速度在 z 方向上。通过系统的对称性，小行星将保持在 z 轴上，两个太阳的 z 坐标相同，其质心也维持在 z 轴上。



We use the Cartesian coordinates to describe the dynamics of the system: the coordinate of the planet $(0,0,u)$, and the coordinates of two suns are $(\pm R\cos\theta, \pm R\sin\theta, Z)$.

我们用笛卡尔坐标来描述系统的动力学：行星坐标为$(0,0,u)$，两个太阳的坐标为$(\pm R\cos\theta, \pm R\sin\theta, Z)$。

(a) [0.5] What is the total kinetic energy, T, of the system? Express the answer in terms of R, θ, Z, u and their time derivative.

(a) [0.5] 系统的总动能 T 是多少？用 R, θ, Z, u 及其时间导数表示答案。

(b) [0.5] What is the total potential energy, V, of the system? Express the answer in terms of R, θ, Z, u and their time derivative.

(b) [0.5] 系统的总势能 V 是多少？用 R, θ, Z, u 及其时间导数表示答案。

(c) [0.5] What is the total linear momentum, P, of the system? Express the answer in terms of R, θ, Z, u and their time derivative.

(c) [0.5] 系统的总线性动量 P 是多少？用 R, θ, Z, u 及其时间导数表示答案。

(d) [0.5] What is the total angular momentum, L, about the z axis of the system? Express the answer in terms of R, θ, Z, u and their time derivative.

(d) [0.5] 关于系统 z 轴的总角动量 L 是多少？用 R, θ, Z, u 及其时间导数表示答案。

From now on, we introduce the dynamical variables $q(t) = u - Z$ and the center of mass coordinate of the system $Q(t) = \frac{mu+2MZ}{m+2M}$.

从现在开始，我们引入动力学变量$q(t) = u - Z$ 和系统的质心坐标 $Q(t) = \frac{mu+2MZ}{m+2M}$.。

(e) [2] Find the equation of motion for $q(t)$. Express your answer in terms of $\ddot{q}, q, R$ and given physical parameters.

(e) [2] 找出 $q(t)$ 的运动方程。用 $\ddot{q}, q, R$ 和给定的物理参数表达你的答案。

(f) [2] Find the equation of motion for $R(t)$. Express your answer in terms of $\ddot{R}, R, q, L$ and given physics parameters.

(f) [2] 求出 $R(t)$ 的运动方程。用 $\ddot{R}, R, q, L$ 和给定的物理参数表达你的答案。

(g) [2] In the limit of small planetary mass $m \ll M$ we can ignore the effect of the planet on the motion of the suns. Find the explicit solution for the motion of the suns $R(t)$ for orbits with small eccentricity $\epsilon \ll 1$. Write your solution as circular motion plus a term proportional to e, i.e. $R(t) = R_0 + \epsilon R_1(t)$ where R_0 is the radius of the circular orbit. You can assume the initial condition $R(0) = R_0(1+\epsilon)$.

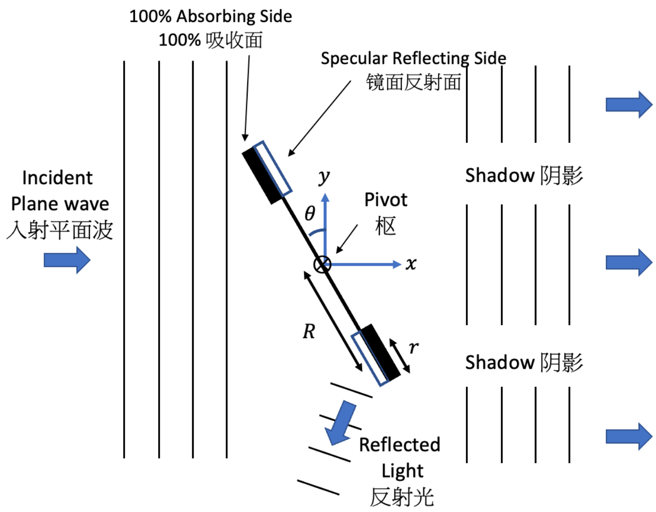
(g) [2] 在小行星质量 $m \ll M$ 的极限下，我们可以忽略行星对太阳运动的影响。对于小偏心率 $\epsilon \ll 1$ 的轨道，找到太阳运动 $R(t)$ 的显式解。将你的答案写成圆周运动加上与 ϵ 成比例的项，即 $R(t) = R_0 + \epsilon R_1(t)$，其中 R_0 是圆形轨道的半径。你可以假设初始条件 $R(0) = R_0(1+\epsilon)$。

(h) [2] Obtain the equation of motion for $q(t)$ in the limit $m \ll M$. You can see that $q(t)$ is a nonlinear oscillator driven by a nonlinear force term.

(h) [2] 求 $q(t)$ 在极限$m \ll M$时的运动方程。你可以看到 $q(t)$ 是一个由非线性力项驱动的非线性振荡器。

3. [10 points] A massless rod can rotate without friction about the pivot point at its center. Light, propagating as a plane wave, propagates from left to right, along the x axis. The electric field of the light is given by

3. [10 分] 一根无质量的杆可以绕其中心的枢轴点无摩擦地旋转。 平面光波沿 x 轴从左向右传播。 光的电场由下式给出



$$\vec{E}(x,t) = E_0\hat{y}\cos(kx - \omega t)$$

where $\vec{k} = k\hat{x}$ and E_0 is a real number. The angle between the rod and $\hat{y}$ is denoted by θ.

其中 $\vec{k} = k\hat{x}$ 且 E_0 为实数。杆和 $\hat{y}$ 之间的角度用 θ 表示。

Cantered at the ends of the rod are disks, each with one side perfectly mirror with 100% reflection and the other side with 100% absorbing. The disks are oriented so that light in the upper part of the rob (above the pivot) always strikes an absorptive surface, while in the lower part, it strikes a reflective surface. Each of the disks have mass m and radius r. Assume that the distance R from the pivot to the center of each disk satisfies $R \gg r$.

在杆的两端是圆盘，每个圆盘的一侧是 100% 反射的完美镜面，另一侧 100% 吸收。圆盘的方向使得杆上部（枢轴上方）的光总是照射到吸收面，而在下部，它照射到反射面。每个圆盘的质量为 m，半径为 r。假设枢轴点到每个圆盘中心的距离R 满足$R \gg r$。

The Poynting vector, which describes the energy flux density (i.e. the energy per unit area per unit time) is given by

描述能量通量密度（即每单位时间每单位面积的能量）的坡印亭矢量由下式给出

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}),$$

where μ_0 is the vacuum permeability. The momentum density $\vec{p}$ carried by the EM wave is

其中 μ_0 是真空磁导率。 光波携带的动量密度 $\vec{p}$ 为

$$\vec{p} = \frac{1}{c^2}\vec{S},$$

where c is the speed of light in vacuum.

其中 c 是真空中的光速。

(a) [1] What are the frequency f, wavelength λ, and magnetic field $\vec{B}(x,t)$ of the light?

(a) [1] 光的频率 f、波长 λ 和磁场 $\vec{B}(x,t)$ 分别是什么？

(b) [1] What is the time-averaged Poynting vector of the incident light?

(b) [1] 入射光的时间平均坡印廷矢量是什么？

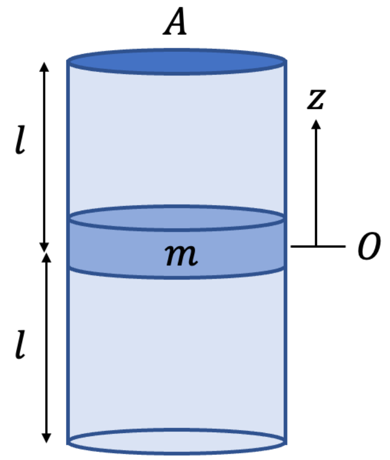
(c) [6] What is the total torque which is delivered by the light to the system of rod plus disks around the pivot point at a given angle θ? What is the average torque over a full rotation of the rod?

(c) [6] 在给定角度 θ 下，光传递到围绕枢轴点的杆加圆盘系统的总扭矩是多少？杆旋转一圈的平均扭矩是多少？

(d) [2] Find the average angular acceleration of the rod over a revolution.

(d) [2] 求杆旋转一圈的平均角加速度。

4. [10 points] A vertical, insulated and sealed cylinder with a cross-sectional area A, and an insulating piston of mass m inside, whose thickness is negligible compared with the length of the cylinder. At the beginning, the piston is fixed in the center of the cylinder, which divides the cylinder into two air chambers with the same length l, as shown in the figure. Assume that the upper and lower gas chambers of the cylinder each contain n moles of monatomic ideal gas with temperature T_0. In the following problems, it can be assumed that there is little friction between the piston and the cylinder wall, and that l is much larger than the distance traveled by the piston ($\gg z$).



4. [10 分] 有一垂直豎立的密閉絕熱圓筒，截面積為 A，內裝有一質量為m的絕熱活塞，其厚度和圓筒的長度相比，可忽略不計。起始時，活塞被固定在圓筒的中央，將圓筒分隔成兩個長度同為 l 的氣室，如圖所示。設圓筒的上下兩氣室各含有溫度為T_0 和n 摩耳的單原子分子理想氣體。在下列的問題中，可以假定活塞與圓筒壁之間的摩擦力很少，且 l 遠大於活塞所移動的距離 ($l \gg z$)。

(a) [4] Release the piston from rest at time $t = 0$ so that it can move freely up and down. Find the trajectory of the piston $z(t)$. You can neglect the friction between the piston and the cylinder in the part.

(a) [4] 在時間 $t = 0$時從靜止中釋放活塞，使其能自由上下運動， 找出活塞的軌跡 $z(t)$。在这部分，你可以忽略活塞和气缸之间的摩擦。

(b) [2] How does the temperature of the upper and lower chambers in the cylinder change with the position of the piston, z?

(b) [2] 圓筒內上下兩氣室的溫度如何隨活塞位置 z 的變動而改變?

(c) [4] Although there is little friction between the piston and the cylinder, the piston will eventually come to rest after a long time. Find the position of the piston, z_f, when it rests and the temperature, T_f, of the gas in the cylinder at that time. . We can assume that the heat capacity of the cylinder and the piston is negligible, the temperature of the upper and lower chambers will eventually come to the same because of the movement of the piston and all heat lost due to friction will transfer into the internal energy of the gas.

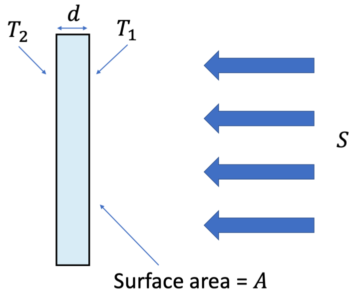
(c) [4] 活塞與圓筒壁間的摩擦力雖然很少，但經過一段長時間後，終會使活塞靜止下來。試求活塞最後靜止時的位置 z_f 和其時筒內氣體的溫度 T_f？我們假設圓筒壁和活塞的熱容量可忽略不計，且由於活塞的運動，使上下氣室溫度最後趨於一致，而且由于摩擦而损失的所有热量都将转化为气体的内能。

~ End of Part 1 卷-1 完 ~

Pan Pearl River Delta Physics Olympiad 2022
2022 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 4 Problems, 40 Points) 简体版卷-1（共4题，40分）
(9:30 am – 12:00 pm, 29th Jan 2023)

Please fill in your final answers to all problems on the answer sheet.
请在答题纸上填上各题的最后答案。
At the end of the competition, please submit the answer sheet only. Question papers and working sheets will not be collected.
比赛结束时，请只交回答题纸，题目纸和草稿纸将不会收回。

1. [10 points] Consider a satellite that has a shape of a plate of surface area A and thickness $d \ll \sqrt{A}$. The satellite can convert solar energy into electrical energy and charge the onboard batteries making use of the temperature difference. The solar energy flux density [solar power per unit area] at the position of the satellite is S and the satellite is facing towards Sun. Assuming that the emissivity of both sides of the satellite is ϵ and the temperatures of the satellite on two sides are T_1 and T_2 respectively and σ is the Stefan-Boltzmann constant.
1. [10 分] 考虑一个人造卫星，其形状为表面积为 A 且厚度为$d \ll \sqrt{A}$ 的平板。卫星可以利用温差将太阳能转化为电能，为星载电池充电。人造卫星所在位置的太阳能通量密度[单位面积的太阳能功率]为S，卫星正对着太阳。假设整颗卫星的发射率（又称辐射率）为ϵ，卫星两侧的温度分别为 T_1 和 T_2 ，σ 是 Stefan-Boltzmann 常数。



(a) [2] What is the net heat flux [energy per second] **absorbed** by the bright side (the side facing towards Sun) of the satellite?
(a) [2] 卫星的亮面（面向太阳的一面）**吸收**的净热通量 [每秒能量] 是多少？
(b) [1] What is the net heat flux [energy per second] **released** from the dark side of the satellite?
(b) [1] 从卫星暗面**释放**的净热通量 [每秒能量] 是多少？
(c) [1] What is value of the emissivity ϵ to get the theoretically maximal charging power P_{max}?
(c) [1] 可得到理论上最大充电功率 P_{max} 的发射率 ϵ 的值是多少？
(d) [3] Find a condition for the temperature T_1 in order to get the theoretically maximal charging power P_{max} provided by the satellite. Express the condition in term of the dimensionless variable $x = \frac{\sigma T_1^4}{S}$. **You don't need to solve the equation in this part.**
(d) [3] 求温度T_1的条件，使得卫星能達到理论上最大充电功率 P_{max}。 用无量纲变量 $x = \frac{\sigma T_1^4}{S}$ 表示你的答案。 你不需要解这部分的方程。
(e) [3] What is the theoretically maximal charging power $P_{\max}$ provided by the satellite? Calculate the numerical value of $\frac{P_{\max}}{AS}$. Your answer should be correct to at least 2 significant figures.
(e) [3] 卫星可提供的理论上最大充电功率P_{max}是多少？ 计算 $\frac{P_{\max}}{AS}$ 的数值。 你的答案至少应正确到 2 位有效数字。

Solution:
(a)

$$\dot{Q}_1 = \epsilon A(S - \sigma T_1^4)$$

1

(b)

$$\dot{Q}_2 = \epsilon\sigma A T_2^4$$

(c) The power is given by

$$P = \dot{Q}_1 - \dot{Q}_2 = \epsilon A(S - \sigma T_1^4 - \sigma T_2^4) \qquad (1)$$

For maximal power, the satellite should be a Carnot engine,

$$\frac{\dot{Q}_1}{T_1} = \frac{\dot{Q}_2}{T_2}$$

$$\Rightarrow \frac{\epsilon A(S - \sigma T_1^4)}{T_1} = \frac{\epsilon\sigma A T_2^4}{T_2}$$

$$\Rightarrow T_2^3 = \frac{S}{\sigma T_1} - T_1^3$$

Sub. Into eqtn (1),

$$P = \epsilon A(S - \sigma T_1^4 - \sigma T_2^4) = \epsilon A\left(S - \sigma T_1^4 - \sigma\left(\frac{S}{\sigma T_1} - T_1^3\right)^{4/3}\right)$$

Obviously, we can get maximal power if $\epsilon = 1$.

(d) Define $x = \frac{\sigma T_1^4}{S}$, we have

$$\frac{P_{max}}{AS} = 1 - x - \left(\frac{1-x}{x^{\frac{1}{4}}}\right)^{\frac{4}{3}} = 1 - x - \frac{(1-x)^{4/3}}{x^{1/3}}$$

We can get max. power P if x satisfies the following equation:

$$\frac{d}{dx}\left(\frac{P_{max}}{AS}\right) = -1 - \frac{4}{3}\frac{(1-x)^{\frac{1}{3}}}{x^{1/3}} + \frac{(1-x)^{\frac{4}{3}}}{3x^{4/3}} = 0$$

(e) We can solve the equation numerically and we get:

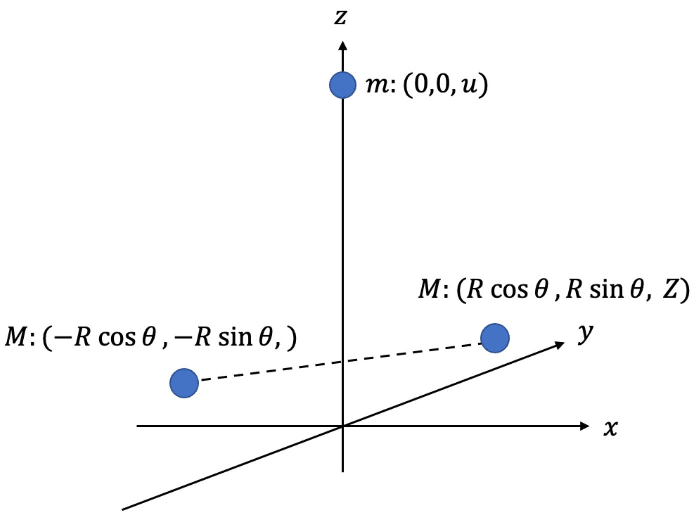
$$x \approx 0.751$$

The corresponding value of power is

$$\frac{P_{max}}{AS} = 0.077$$

2. [10 points] Binary-sun solar system: Consider a binary pair of identical suns of mass M orbiting in the $x-y$ plane in an orbit centred at the origin. The gravitational constant is G. Now add a planet of mass m with an initial condition on the z axis above the center of mass of the two suns and with a velocity along the z direction. By the symmetry of the system, the small planet will remain on the z axis, suns will have equal z coordinates and the center of mass of two suns will also remain on the z axis.

2. [10 分]双星太阳系：考虑两个质量均为 M 的相同太阳在 $x-y$ 平面中以原点为中心的轨道运行。万有引力常数为 G。现在添加一个质量 m 的行星，其初始条件位于两个太阳质心上方的 z 轴上，速度在 z 方向上。通过系统的对称性，小行星将保持在 z 轴上，两个太阳的 z 坐标相同，其质心也维持在 z 轴上。



We use the Cartesian coordinates to describe the dynamics of the system: the coordinate of the planet $(0,0,u)$, and the coordinates of two suns are $(\pm R\cos\theta\,, \pm R\sin\theta\,, Z)$.

我们用笛卡尔坐标来描述系统的动力学：行星坐标为$(0,0,u)$，两个太阳的坐标为$(\pm R\cos\theta\,, \pm R\sin\theta\,, Z)$。

(a) [0.5] What is the total kinetic energy, T, of the system? Express the answer in terms of R, θ, Z, u and their time derivative.

(a) [0.5] 系统的总动能 T 是多少？用 $R, \theta\,, Z, u$ 及其时间导数表示答案。

(b) [0.5] What is the total potential energy, V, of the system? Express the answer in terms of R, θ, Z, u and their time derivative.

(b) [0.5] 系统的总势能 V 是多少？用 $R, \theta\,, Z, u$ 及其时间导数表示答案。

(c) [0.5] What is the total linear momentum, P, of the system? Express the answer in terms of R, θ, Z, u and their time derivative.

(c) [0.5] 系统的总线性动量 P 是多少？用 $R, \theta\,, Z, u$ 及其时间导数表示答案。

(d) [0.5] What is the total angular momentum, L, about the z axis of the system? Express the answer in terms of R, θ, Z, u and their time derivative.

(d) [0.5] 关于系统 z 轴的总角动量 L 是多少？用 $R, \theta\,, Z, u$ 及其时间导数表示答案。

From now on, we introduce the dynamical variables $q(t) = u - Z$ and the center of mass coordinate of the system $Q(t) = \frac{mu+2MZ}{m+2M}$.

从现在开始，我们引入动力学变量$q(t) = u - Z$ 和系统的质心坐标 $Q(t) = \frac{mu+2MZ}{m+2M}$.。

(e) [2] Find the equation of motion for $q(t)$. Express your answer in terms of $\ddot{q}, q, R$ and given physical parameters.

(e) [2] 找出 $q(t)$ 的运动方程。用 $\ddot{q}, q, R$ 和给定的物理参数表达你的答案。

(f) [2] Find the equation of motion for $R(t)$. Express your answer in terms of $\ddot{R}, R, q, L$ and given physics parameters.

(f) [2] 求出 $R(t)$ 的运动方程。用 $\ddot{R}, R, q, L$ 和给定的物理参数表达你的答案。

(g) [2] In the limit of small planetary mass $m \ll M$ we can ignore the effect of the planet on the motion of the suns. Find the explicit solution for the motion of the suns $R(t)$ for orbits with small eccentricity $\epsilon \ll 1$. Write your solution as circular motion plus a term proportional to e, i.e. $R(t) = R_0 + \epsilon R_1(t)$ where R_0 is the radius of the circular orbit. You can assume the initial condition $R(0) = R_0(1+\epsilon)$.

(g) [2] 在小行星质量 $m \ll M$ 的极限下，我们可以忽略行星对太阳运动的影响。对于小偏心率 $\epsilon \ll 1$ 的轨道，找到太阳运动 $R(t)$ 的显式解。将你的答案写成圆周运动加上与 ϵ 成比例的项，即 $R(t) = R_0 + \epsilon R_1(t)$，其中 R_0 是圆形轨道的半径。你可以假设初始条件 $R(0) = R_0(1+\epsilon)$。

(h) [2] Obtain the equation of motion for $q(t)$ in the limit $m \ll M$. You can see that $q(t)$ is a nonlinear oscillator driven by a nonlinear force term.

(h) [2] 求 $q(t)$ 在极限$m \ll M$时的运动方程。你可以看到 $q(t)$ 是一个由非线性力项驱动的非线性振荡器。

Solution:

(a) The total kinetic energy is

$$T = M\dot{R}^2 + M\dot{Z}^2 + MR^2\dot{\theta}^2 + \frac{1}{2}m\dot{u}^2$$

(b) The total potential energy is

$$V = -\frac{GM^2}{2R} - \frac{2GMm}{\sqrt{R^2 + (u-Z)^2}}$$

(c) The total linear momentum is

$$P = m\dot{u} + 2M\dot{Z}$$

(d) The total angular momentum is

$$L = 2MR^2\dot{\theta}$$

(e) We can rewrite u and Z in term of q and Q,

$$u = Q + \frac{2M}{m+2M}q$$

$$Z = Q - \frac{m}{m+2M}q$$

The EOM for the planet is

$$m\ddot{u} = -\frac{2GMm}{(R^2+q^2)^{3/2}}q$$

Since

$$u = Q + \frac{2M}{m+2M}q \Rightarrow \ddot{u} = \frac{2M}{m+2M}\ddot{q}$$

$$\Rightarrow \frac{2mM}{m+2M}\ddot{q} = -\frac{2GMm}{(R^2+q^2)^{3/2}}q$$

$$\Rightarrow \mu\ddot{q} = -\frac{2GMm}{(R^2+q^2)^{3/2}}q$$

where the reduced mass of the system μ is defined by:

$$\frac{1}{\mu} = \frac{1}{2M} + \frac{1}{m}$$

(f) The total energy can be rewritten as,

$$E = T + V = M\dot{R}^2 + M\dot{Z}^2 + MR^2\dot{\theta}^2 + \frac{1}{2}m\dot{u}^2 - \frac{GM^2}{2R} - \frac{2GMm}{\sqrt{R^2 + (u - Z)^2}}$$

$$E = M\dot{R}^2 + M\left(\dot{Q} - \frac{m}{m + 2M}\dot{q}\right)^2 + MR^2\dot{\theta}^2 + \frac{1}{2}m\left(\dot{Q} + \frac{m}{m + 2M}\dot{q}\right)^2 - \frac{GM^2}{2R} - \frac{2GMm}{\sqrt{R^2 + q^2}}$$

$$E = M\dot{R}^2 + \frac{1}{2}(m + 2M)\dot{Q}^2 + \frac{Mm}{m + 2M}\dot{q}^2 + MR^2\dot{\theta}^2 - \frac{GM^2}{2R} - \frac{2GMm}{\sqrt{R^2 + q^2}}$$

$$E = M\dot{R}^2 + \frac{1}{2}(m + 2M)\dot{Q}^2 + \frac{1}{2}\mu\dot{q}^2 + MR^2\dot{\theta}^2 - \frac{GM^2}{2R} - \frac{2GMm}{\sqrt{R^2 + q^2}}$$

$$\Rightarrow E = M\dot{R}^2 + \frac{P^2}{2(m + 2M)} + \frac{1}{2}\mu\dot{q}^2 + MR^2\left(\frac{L}{2MR^2}\right)^2 - \frac{GM^2}{2R} - \frac{2GMm}{\sqrt{R^2 + q^2}}$$

$$\Rightarrow E = \frac{P^2}{2(m + 2M)} + M\dot{R}^2 - \frac{GM^2}{2R} + \frac{L^2}{4MR^2} + \frac{1}{2}\mu\dot{q}^2 - \frac{2GMm}{\sqrt{R^2 + q^2}}$$

where the momentum and angular momentum

$$p = m\dot{u} + 2M\dot{Z} = m\left(\dot{Q} + \frac{2M}{m + 2M}\dot{q}\right) + 2M\left(\dot{Q} - \frac{2M}{m + 2M}\dot{q}\right) = (m + 2M)\dot{Q}$$

$$L = 2MR^2\dot{\theta}$$

are conserved.

The constant energy gives

$$\frac{dE}{dt} = 0 \Rightarrow 2M\dot{R}\ddot{R} + \frac{GM^2}{2R^2}\dot{R} - \frac{L^2}{2MR^3}\dot{R} + \mu\dot{q}\ddot{q} + \frac{GMm}{(R^2 + q^2)^{\frac{3}{2}}}\left(2R\dot{R} + +2q\dot{q}\right) = 0$$

$$\Rightarrow \left(2M\ddot{R} - \frac{L^2}{2MR^3} + \frac{GM^2}{2R^2} + \frac{2GMmR}{(R^2+q^2)^{\frac{3}{2}}}\right)\dot{R} + \left(\mu\ddot{q} + \frac{2GMmq}{(R^2+q^2)^{\frac{3}{2}}}\right)\dot{q} = 0$$

$$\Rightarrow 2M\ddot{R} - \frac{L^2}{2MR^3} + \frac{GM^2}{2R^2} + \frac{2GMmR}{(R^2 + q^2)^{\frac{3}{2}}} = 0$$

(g) In the limit $\frac{m}{M} \to 0$,

$$2M\ddot{R} - \frac{L^2}{2MR^3} + \frac{GM^2}{2R^2} + \frac{2GMmR}{(R^2 + q^2)^{\frac{3}{2}}} = 0$$

$$\Rightarrow \frac{2}{M}\ddot{R} - \frac{L^2}{2M^3R^3} + \frac{G}{2R^2} + \frac{2GR}{(R^2+q^2)^{\frac{3}{2}}}\frac{m}{M} \approx \frac{2}{M}\ddot{R} - \frac{L^2}{2M^3R^3} + \frac{G}{2R^2} = 0$$

The orbit, $R(t) = R_0 + \epsilon R_1(t)$,

$$\frac{2\epsilon}{M}\ddot{R}_1 - \frac{L^2}{2M^3R_0^3}\left(1 - \frac{3\epsilon R_1}{R_0}\right) + \frac{G}{2R_0^2}\left(1 - \frac{2\epsilon R_1}{R_0}\right) = 0$$

$$\Rightarrow \frac{L^2}{2M^3R_0^3} = \frac{G}{2R_0^2} \Rightarrow R_0 = \frac{L^2}{GM^3}$$

And

$$\frac{2\epsilon}{M}\ddot{R}_1 = -\frac{L^2}{2M^3R_0^3}\frac{3R_1}{R_0} + \frac{G}{2R_0^2}\frac{2\epsilon R_1}{R_0} = -\left(\frac{G\epsilon}{2R_0^3}\right)R_1$$

$$\Rightarrow \ddot{R}_1 = -\frac{GM}{4R_0^3}R_1$$

$$\Rightarrow R_1 = A\cos\left(\sqrt{\frac{GM}{4R_0^3}}t + \phi\right)$$

The radial coordinate of the suns are

$$R_1(t) = R_0\left(1 + A\epsilon\cos\left(\sqrt{\frac{GM}{4R_0^3}}t + \phi\right)\right) = R_0\left(1 + \epsilon\cos\left(\sqrt{\frac{GM}{4R_0^3}}t\right)\right)$$

(h) In the limit $m \ll M$, $\mu \approx m$ and

$$(R^2+q^2)^{-\frac{3}{2}} = \left(R_0^2\left(1 + \epsilon\cos\left(\sqrt{\frac{GM}{4R_0^3}}t\right)\right)^2 + q^2\right)^{-\frac{3}{2}} \approx (R_0^2+q^2)^{-\frac{3}{2}}\left(1 + \frac{2R_0^2}{R_0^2+q^2}\epsilon\cos\left(\sqrt{\frac{GM}{4R_0^3}}t\right)\right)^{-\frac{3}{2}}$$

$$\approx \frac{1}{(R_0^2+q^2)^{\frac{3}{2}}} - \frac{3R_0^2}{(R_0^2+q^2)^{5/2}}\epsilon\cos\left(\sqrt{\frac{GM}{4R_0^3}}t\right)$$

Finally, we obtain the EOM for $q(t)$,

$$m\ddot{q} + \frac{2GMm}{(R_0^2+q^2)^{\frac{3}{2}}}q - \frac{6GMmR_0^2\epsilon}{\left((R_0^2+q^2)^{\frac{5}{2}}\right)}\cos\left(\sqrt{\frac{GM}{4R_0^3}}t\right) = 0$$

The dynamics of this system given by the EOM for $q(t)$ turns out to be complex and chaotic even in the limit $m \ll M$. If you want to learn more about the complex dynamic (chaos) of this system, you can search the "Slitnikov Problem" in the internet.

3. [10 points] A massless rod can rotate without friction about the pivot point at its center. Light, propagating as a plane wave, propagates from left to right, along the x axis. The electric field of the light is given by

3. [10 分] 一根无质量的杆可以绕其中心的枢轴点无摩擦地旋转。 平面光波沿 x 轴从左向右传播。 光的电场由下式给出

$$\vec{E}(x,t) = E_0 \hat{y} \cos(kx - \omega t)$$

where $\vec{k} = k\hat{x}$ and E_0 is a real number. The angle between the rod and $\hat{y}$ is denoted by θ.
其中 $\vec{k} = k\hat{x}$ 且 E_0 为实数。杆和 $\hat{y}$ 之间的角度用 θ 表示。

Cantered at the ends of the rod are disks, each with one side perfectly mirror with 100% reflection and the other side with 100% absorbing. The disks are oriented so that light in the upper part of the rob (above the pivot) always strikes an absorptive surface, while in the lower part, it strikes a reflective surface. Each of the disks have mass m and radius r. Assume that the distance R from the pivot to the center of each disk satisfies $R \gg r$.
在杆的两端是圆盘，每个圆盘的一侧是 100% 反射的完美镜面，另一侧 100% 吸收。圆盘的方向使得杆上部（枢轴上方）的光总是照射到吸收面，而在下部，它照射到反射面。每个圆盘的质量为 m，半径为 r。假设枢轴点到每个圆盘中心的距离R 满足$R \gg r$。

The Poynting vector, which describes the energy flux density (i.e. the energy per unit area per unit time) is given by
描述能量通量密度（即每单位时间每单位面积的能量）的坡印亭矢量由下式给出

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}),$$

where μ_0 is the vacuum permeability. The momentum density $\vec{p}$ carried by the EM wave is
其中 μ_0 是真空磁导率。 光波携带的动量密度 $\vec{p}$ 为

$$\vec{p} = \frac{1}{c^2}\vec{S},$$

where c is the speed of light in vacuum.
其中 c 是真空中的光速。

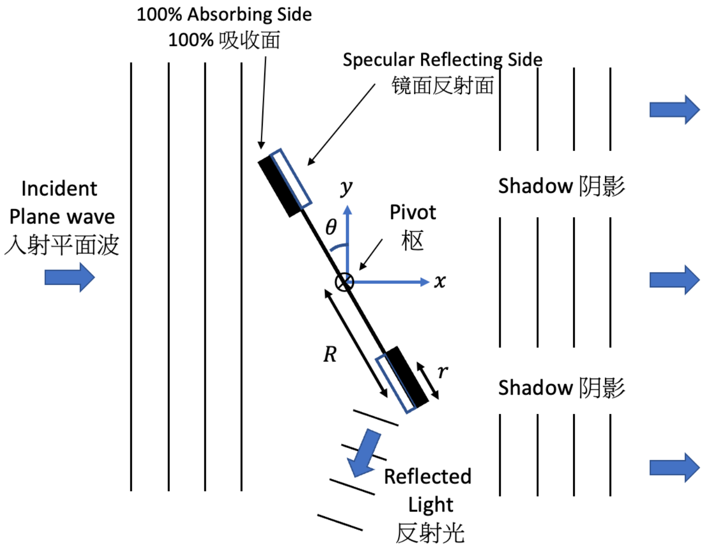
(a) [1] What are the frequency f, wavelength λ, and magnetic field $\vec{B}(x,t)$ of the light?
(a) [1] 光的频率 f、波长 λ 和磁场 $\vec{B}(x,t)$ 分别是什么？

(b) [1] What is the time-averaged Poynting vector of the incident light?
(b) [1] 入射光的时间平均坡印廷矢量是什么？

(c) [6] What is the total torque which is delivered by the light to the system of rod plus disks around the pivot point at a given angle θ? What is the average torque over a full rotation of the rod?
(c) [6] 在给定角度 θ 下，光传递到围绕枢轴点的杆加圆盘系统的总扭矩是多少？杆旋转一圈的平均扭矩是多少？

(d) [2] Find the average angular acceleration of the rod over a revolution.
(d) [2] 求杆旋转一圈的平均角加速度。

1



Solution:

(a) $f = \frac{\omega}{2\pi}$, $\lambda = \frac{c}{f} = \frac{2\pi}{k}$ and $\vec{B}(x,t) = \frac{E_0}{c}\hat{z}\, cos\,(kx - \omega t)$

(b) $\vec{E}$ and $\vec{B}$ are perpendicular and in phase. The time average of $\cos^2 \omega t$ is $\frac{1}{2}$, so that

$$\langle \vec{S} \rangle = \frac{E_0^2}{2c\mu_0}$$

(c) The force exerted on the disk in the direction normal to its surface is given by the total momentum per second transferred by the light in this direction. The momentum transferred per unit area per second is $p_n c \cos\theta = pc\cos^2\theta$, where p_n is the component of momentum density of the wave along the normal to the disk surface. The total momentum transfer per second to the absorbing disk (i.e. the force) is then

$$F_{abs} = (pc\cos^2\theta)(\pi r^2) = \frac{S}{c}\pi r^2 \cos^2\theta = \frac{1}{2c^2\mu_0}E_0^2 \pi r^2 \cos^2\theta$$

For the reflecting surface, the corresponding total momentum transfer is twice as much,

$$F_{ref} = \frac{1}{c^2\mu_0}E_0^2 \pi r^2 \cos^2\theta$$

(d) The torque is given by the sum of $\vec{r} \times \vec{F}$. The net torque is

$$\vec{\tau} = \frac{1}{2c^2\mu_0}E_0^2 \pi r^2 R \cos^2\theta\, \hat{z}$$

Taking time average over one full rotation in θ,

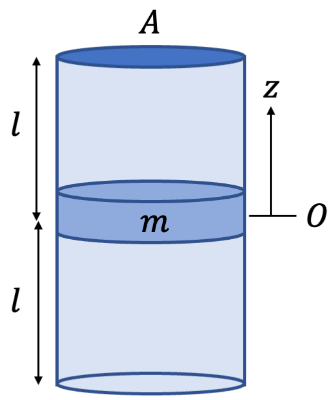
$$\langle \vec{\tau} \rangle = \frac{1}{4c^2\mu_0}E_0^2 \pi r^2 R\hat{z}$$

(e) Since $\tau = I\alpha$ where I is the moment of inertia of the system about the pivot. In the limit $R \gg r$, we have $I = 2mR^2$ and

$$\langle\ddot{\theta}\rangle = \frac{E_0^2\pi r^2}{8mRc^2\mu_0}$$

4. [10 points] A vertical, insulated and sealed cylinder with a cross-sectional area A, and an insulating piston of mass m inside, whose thickness is negligible compared with the length of the cylinder. At the beginning, the piston is fixed in the center of the cylinder, which divides the cylinder into two air chambers with the same length l, as shown in the figure. Assume that the upper and lower gas chambers of the cylinder each contain n moles of monatomic ideal gas with temperature T_0. In the following problems, it can be assumed that there is little friction between the piston and the cylinder wall, and that l is much larger than the distance traveled by the piston ($\gg z$).

4. [10 分] 有一垂直豎立的密閉絕熱圓筒，截面積為 A，內裝有一質量為m的絕熱活塞，其厚度和圓筒的長度相比，可忽略不計。起始時，活塞被固定在圓筒的中央，將圓筒分隔成兩個長度同為 l 的氣室，如圖所示。設圓筒的上下兩氣室各含有溫度為T_0 和n 摩耳的單原子分子理想氣體。在下列的問題中，可以假定活塞與圓筒壁之間的摩擦力很少，且 l 遠大於活塞所移動的距離 ($l \gg z$)。



(a) [4] Release the piston from rest at time $t = 0$ so that it can move freely up and down. Find the trajectory of the piston $z(t)$. You can neglect the friction between the piston and the cylinder in the part.

(a) [4] 在時間 $t = 0$時從靜止中釋放活塞，使其能自由上下運動， 找出活塞的軌跡 $z(t)$。在这部分，你可以忽略活塞和气缸之间的摩擦。

(b) [2] How does the temperature of the upper and lower chambers in the cylinder change with the position of the piston, z?

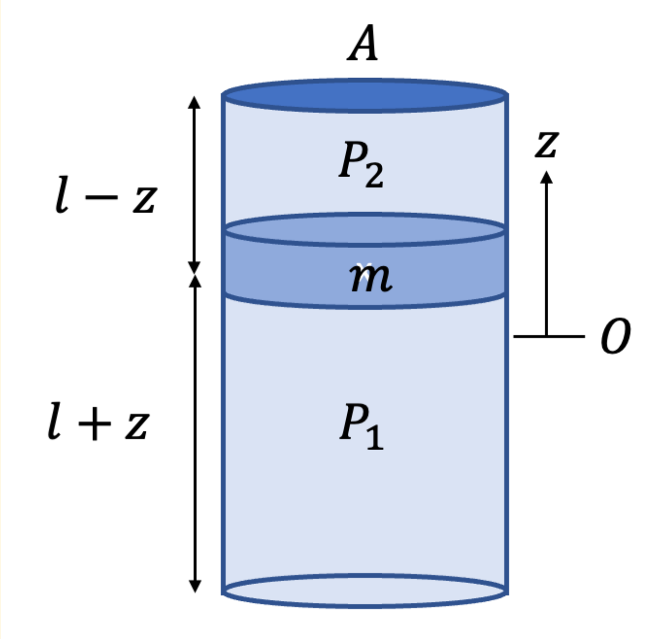
(b) [2] 圓筒內上下兩氣室的溫度如何隨活塞位置 z 的變動而改變?

(c) [4] Although there is little friction between the piston and the cylinder, the piston will eventually come to rest after a long time. Find the position of the piston, z_f, when it rests and the temperature, T_f, of the gas in the cylinder at that time. . We can assume that the heat capacity of the cylinder and the piston is negligible, the temperature of the upper and lower chambers will eventually come to the same because of the movement of the piston and all heat lost due to friction will transfer into the internal energy of the gas.

(c) [4] 活塞與圓筒壁間的摩擦力雖然很少，但經過一段長時間後，終會使活塞靜止下來。試求活塞最後靜止時的位置 z_f 和其時筒內氣體的溫度 T_f？我们假設圓筒壁和活塞的熱容量可忽略不計，且由於活塞的運動，使上下氣室溫度最後趨於一致，而且由于摩擦而损失的所有热量都将转化为气体的内能。

Solution:

(a) The process is adiabatic and we have $PV^\gamma = C$ where $\gamma = \frac{C_p}{C_V} = \frac{5}{3}$.



From the figure, we have

$$P_1 V_1^\gamma = P_2 V_2^\gamma = P_0 V_0^\gamma$$

$$\Rightarrow P_1 (l + z)^\gamma = P_2 (l - z)^\gamma = P_0 l^\gamma$$

$$P_1 = P_0 \left(\frac{1}{1 + \frac{z}{l}} \right)^\gamma \approx P_0 \left(1 - \frac{\gamma z}{l}\right)$$

$$P_2 \approx P_0 \left(1 + \frac{\gamma z}{l}\right) = P_0 \left(1 + \frac{5z}{3l}\right)$$

Newton's 2nd law gives

$$-P_0 A \left(1 + \frac{\gamma z}{l}\right) + P_0 A \left(1 - \frac{\gamma z}{l}\right) - mg = -\frac{2P_0 A\gamma}{l} z - mg = m\ddot{z}$$

$$\Rightarrow \ddot{z} = -\frac{2P_0 A\gamma}{ml} z - g$$

$$\Rightarrow \ddot{z} = -\omega^2 \left(z + \frac{g}{\omega^2}\right)$$

where $\omega = \sqrt{\frac{2P_0 A\gamma}{ml}}$. The general solution of $z(t)$ is:

$$z(t) = A\cos(\omega t + \phi) - \frac{g}{\omega^2}$$

From the initial conditions $z(0) = \dot{z}(0) = 0$,

$$\Rightarrow A\cos\phi - \frac{g}{\omega^2} = 0$$

$$-A\omega\sin\phi = 0$$

$$\Rightarrow z(t) = -\frac{g}{\omega^2}(1-\cos\omega t)$$

(b)

$$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} = T_0V_0^{\gamma-1}$$

$$\Rightarrow T_1 = T_0\left(\frac{1}{1+\frac{z}{l}}\right)^{\gamma-1} \approx T_0\left(1-\frac{(\gamma-1)z}{l}\right)$$

$$T_2 = T_0\left(1+\frac{(\gamma-1)z}{l}\right) = T_0\left(1+\frac{2z}{3l}\right)$$

(c) At equilibrium z_0, both chambers have the same temperature T and the pressure satisfies

$$(P_1-P_2)A = mg$$

$$\Rightarrow \left(\frac{nRT}{A(l+z_0)} - \frac{nRT}{A(l-z_0)}\right)A = mg$$

$$\Rightarrow \frac{1}{l+z_0} - \frac{1}{l-z_0} = \frac{mg}{nRT} \quad (1)$$

By energy conservation, we have

$$2\times\left(\frac{3}{2}nRT_0\right) = 2\times\left(\frac{3}{2}nRT\right) + mgz_0$$

$$\Rightarrow T = T_0 - \frac{mg}{3nR}z_0 \quad (2)$$

From equations (1) and (2), we can eliminate T and get

$$\frac{1}{l+z_0} - \frac{1}{l-z_0} = -\frac{2z_0}{l^2-z_0^2} = \frac{mg}{nRT} = \frac{mg}{nR\left(T_0 - \frac{mg}{3nR}z_0\right)}$$

$$\Rightarrow l^2 - z_0^2 = -\frac{2nR}{mg}z_0\left(T_0 - \frac{mg}{3nR}z_0\right)$$

$$\Rightarrow z_0^2 - \frac{6nRT_0}{5mg}z_0 + \frac{3}{5}l^2 = 0$$

$$\Rightarrow z_0 = \left(\frac{3nRT_0}{5mg}\right) \pm \sqrt{\left(\frac{3nRT_0}{5mg}\right)^2 + \frac{3l^2}{5}}$$

Since $z_0 < 0$, we get

$$z_0 = \left(\frac{3nRT_0}{5mg}\right) - \sqrt{\left(\frac{3nRT_0}{5mg}\right)^2 + \frac{3l^2}{5}}$$

Note: In the limit when $\left(\frac{3nRT_0}{5mg}\right)^2 \gg \frac{3l^2}{5}$ $\left(\text{i. e. } \frac{nRT_0}{mg} \gg l\right)$, we have

$$z_0 = \left(\frac{3nRT_0}{5mg}\right)\left(1 - \sqrt{1 + \frac{3l^2}{5}\left(\frac{5mg}{3nRT_0}\right)^2}\right) \approx -\frac{1}{2}\left(\frac{3nRT_0}{5mg}\right)\frac{3l^2}{5}\left(\frac{5mg}{3nRT_0}\right)^2 = -\frac{3l^2}{10}\frac{5mg}{3nRT_0} = -\frac{mgl^2}{2nRT_0}$$

This result is justified if

$$|z_0| = \frac{mgl^2}{2nRT_0} \ll l \Rightarrow l \ll \frac{2nRT_0}{mg} \sim \frac{nRT_0}{mg}$$

is valid at high temperature $T_0 \gg \frac{mgl}{nR}$ and $z \ll l$.

And the temperature is

$$T = \frac{4}{5}T_0 + \sqrt{\left(\frac{T_0}{5}\right)^2 + \frac{1}{15}\left(\frac{mgl}{nR}\right)^2} = T_0\left(\frac{4}{5} + \frac{1}{5}\sqrt{1 + \frac{5}{3T_0^2}\left(\frac{mgl}{nR}\right)^2}\right) \approx T_0\left(1 + \frac{m^2g^2l^2}{6n^2R^2T_0^2}\right)$$

~ End of Part 1 卷-1 完 ~

Pan Pearl River Delta Physics Olympiad 2023
2023 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
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Simplified Chinese Part-2 (Total 2 Problems, 60 Points)
简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 29 January 2023)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。

All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。

There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。

Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。

Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。

Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。

If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。

At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Vacuum bubbles (28 points)
问题 1: 真空泡泡 (28 分)

Is our vacuum stable? We don't know. It's possible that we do not live in the true vacuum. Rather, we live in a false vacuum which can decay into true vacuum by emerging and expanding bubbles. To describe such a possibility, we will make use of a space-time dependent "scalar field" $\phi(t,x,y,z)$, which takes a real value at every space-time point. (Similar to height on a map, which takes a real value at every point on the x-y plane, while a scalar field takes a real value for any given t,x,y,z. Also, in a full quantum theory, we have to distinguish operators and numbers, but here we will assume the scalar field only take real number values.)

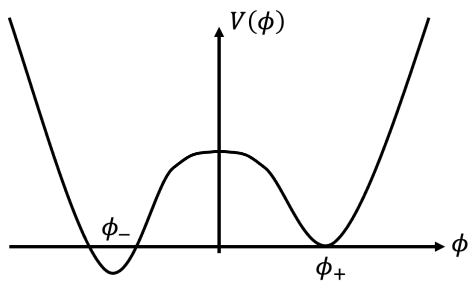
我们的真空是稳定的吗？我们不知道。有可能我们并不是生活在一个稳定的真真空（既真的"真空"）里面。我们可能生活在一个假真空里，而假真空可以通过自发产生和膨胀的泡泡衰变到真真空。为了描述这种可能，我们将使用一个依赖于时空点到"标量场"$\phi(t,x,y,z)$：它在每个时空点上取一个实数。（这有点像地图上的高度，在x-y平面的每个点上取一个实数。不过标量场是在每个t,x,y,z时空点上取一个实数。另外，在完整的量子理论中，我们需要考虑算符和数的区别，但这里我们假设标量场的取值仅是普通的实数。）

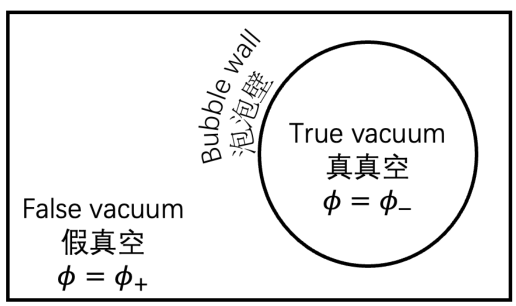
The scalar field satisfies the following equation of motion: $\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial z^2}+\frac{dV(\phi)}{d\phi}=0$, where $V(\phi)$ is the potential density of the field, and we will call it potential for short in this problem. The energy density of the scalar field is $\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dy}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2+V(\phi)$.

标量场满足运动方程$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial z^2}+\frac{dV(\phi)}{d\phi}=0$，其中$V(\phi)$是场的势能密度，我们将简称它为标量场的势能。标量场的能量密度是$\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dy}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2+V(\phi)$。

We consider the following potential: the false vacuum has field value $\phi=\phi_+$ where $V(\phi_+)=0$, and the true vacuum has field value $\phi=\phi_-$, where $V(\phi_-)$ is slightly negative. In the left panel of the following figure, we plot the shape of the potential. The right panel is an example of the false and true vacuum in position space.

我们考虑如下势能：假真空处标量场取值是$\phi=\phi_+$，满足$V(\phi_+)=0$；真真空处标量场取值是$\phi=\phi_-$，$V(\phi_-)$取一个接近 0 的负值。下图（左）是势能的函数形式，下图（右）是位置空间中假真空和真真空的一个例子。





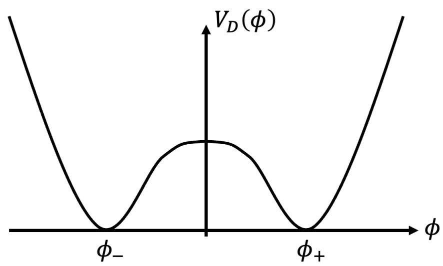
In this problem, we will use natural units and set the speed of light in vacuum $c=1$ (by redefining the time unit as the time that light traveled over unit length). In this unit, when an object is at rest, its energy equals its mass by the famous formula $E=mc^2=m$.

本题中，我们将使用自然单位并定义真空中的光速$c=1$（既定义时间的单位为光穿越单位长度的时间）。在自然单位制中，物体的静止能量等于它的质量，这就是著名的公式$E=mc^2=m$。

A. DOMAIN WALL 畴壁

Before coming to the asymmetric potential which generates vacuum bubbles, let us consider a symmetric potential as follows:

在我们研究非对称的势能以及它生成的真空泡泡前，我们先考虑如下一个对称的势能：



Let's find a static solution which is homogeneous along the y and z directions, known as a domain wall. The potential of the domain wall $V(\phi) = V_D(\phi)$ is illustrated above, with two minima $V_D(\phi_\pm) = 0$. The domain wall can be used as the local approximation of the bubble wall.

我们将找一种在 y 和 z 方向均匀的 “畴壁” 解。上图是畴壁解对应的势能 $V(\phi) = V_D(\phi)$，它具有两个最小值 $V_D(\phi_\pm) = 0$。畴壁可以作为泡泡壁的局域近似。

A1. SIMPLIFY THE EQUATION OF MOTION 化简运动方程

A1	Given the static and homogeneity (independency of y and z) conditions, write down the simplified equation of motion for ϕ. 根据静态，以及均匀（不依赖于y 和 z 方向）的条件，写出ϕ化简后的运动方程。	2 points 2 分

A2. THE DOMAIN WALL PROFILE 畴壁上标量场取值的空间变化

A2	Express $\frac{d\phi}{dx}$ in terms of $V_D(\phi)$. 请把 $\frac{d\phi}{dx}$ 用 $V_D(\phi)$ 表达出来。	3 points 3 分

A3. THE DOMAIN WALL TENSION 畴壁的张力 (2')

A3	The tension of the domain wall (energy density of the wall for unit area in the y and z directions) is $\sigma = \int_{\phi_-}^{\phi_+} f(\phi)d\phi$. Find $f(\phi)$ in terms of $V_D(\phi)$. 畴壁的张力（既在 y、z 方向单位面积上，畴壁的能量密度）是 $\sigma = \int_{\phi_-}^{\phi_+} f(\phi)d\phi$。请把 $f(\phi)$用 $V_D(\phi)$ 表达出来。	2 points 2 分

Note: to avoid propagation of possible errors, in the later part of the questions, please still use the domain wall tension σ where applicable, instead of using the integral expression that you obtain.

注：为避免潜在的错误传播，在本题后面的部分中，当用到畴壁张力时，请仍使用符号 σ，而不是这里你求出的积分表达式。

B. BUBBLE WALL 泡泡壁

If we only look at a small part, a bubble wall can be approximated as a domain wall. But globally, the bubble can be approximated to be spherical with radius R. Let's assume that R is large enough, such that the thickness of the bubble wall is much smaller than R (thin wall approximation). Inside and outside the bubble, $\phi \to \phi_\pm$ exponentially quickly.
如果我们只看一泡泡壁的一小部分的话，泡泡壁上的一小块可以用畴壁来近似。但是整体上，真空泡泡可以近似为球形的，具有半径 R。假设 R 足够大，泡泡壁的厚度远远小于 R（薄壁近似）。在泡泡壁的内部和泡泡壁的外部，ϕ 指数快地趋向于 $\phi_\pm$。

At the moment when a bubble is nucleated, the bubble is static, and the bubble nucleation and motion create negligible amount of radiation or other dissipations.
在真空泡泡产生的时刻，泡泡是静止的。泡泡产生的过程带来的辐射或其它耗散可以忽略。

B1. THE ENERGY ON THE BUBBLE WALL 泡泡壁的能量 (1')

B1	At the momentum of bubble nucleation, calculate the energy E_W carried by the bubble wall using R and the bubble tension σ. 在真空泡泡产生的时刻，利用 R 和泡泡壁的张力 σ 计算泡泡壁的能量 E_W。	1 point 1 分

B2. FALSE AND TRUE VACUA 假真空和真真空

B2	For a spherical bubble to appear, there must be an energy density difference between $V(\phi_\pm)$. Thus, to write down a potential to model bubble nucleation, we consider the potential $V(\phi) = V_D(\phi) + \frac{\epsilon}{\phi_+ - \phi_-}(\phi - \phi_+)$. In the thin wall approximation, we are only interested in leading order results in ϵ (the lowest order in Taylor expansion which contains ϵ). Calculate ϵ using σ and R. 为了让球形真空泡泡能够出现，$V(\phi_\pm)$ 的取值必须不同。所以，为了对真空泡泡的产生过程建立势能模型，我们考虑势能 $V(\phi) = V_D(\phi) + \frac{\epsilon}{\phi_+ - \phi_-}(\phi - \phi_+)$。在薄壁近似下，我们只感兴趣 ϵ 零头阶效应（既泰勒展开中含有 ϵ 的最低阶）。利用 σ 和 R 计算 ϵ。	2 points 2 分

B3. BUBBLE MOTION 泡泡的运动

B3	At the moment of bubble nucleation, calculate the acceleration a of the bubble wall in terms of σ and R. 在真空泡泡产生的瞬间，利用 σ 和 R 计算泡泡的加速度 a。	2 points 2 分

B4. BEYOND NEWTONIAN MECHANICS 超越牛顿力学

B4	When the speed of the bubble wall is close to the speed of light, Newtonian mechanics breaks down and special relativity should be used instead. In special relativity, the kinetic energy of a moving object is $E_K = (\gamma - 1)m$, where $\gamma \equiv \frac{1}{\sqrt{1-v^2}}$.. Calculate the time needed from the nucleation of the bubble to that the speed of the bubble wall to reach 0.6. 当真空泡泡的速度接近光速，牛顿力学不再适用，我们应该使用狭义相对论。在狭义相对论里，运动物体的动能为 $E_K = (\gamma - 1)m$，其中 $\gamma \equiv \frac{1}{\sqrt{1-v^2}}$。计算真空泡泡从产生到泡泡壁速度达到 0.6 所需要的时间。 Hint: you may need the mathematical relation $\frac{d\sqrt{x^2-1}}{dx} = \frac{x}{\sqrt{x^2-1}}$. 提示：你可能需要数学关系 $\frac{d\sqrt{x^2-1}}{dx} = \frac{x}{\sqrt{x^2-1}}$。	4 points 4 分

C. NUCLEATION RATE OF THE BUBBLE 泡泡的产生率

What's the probability for a bubble to appear? It can be shown that the nucleation rate Γ of the bubble, i.e., the probability for a bubble to appear in unit volume and during unit time, can be written as $\Gamma \simeq Ae^{-S_E/\hbar}$, where A and $\hbar$ are constants, and S_E is a "Euclidean action", which can be calculated with the following procedure:

真空泡泡产生的概率是多少？可以证明，泡泡的产生率 Γ，即单位时间单位体积，一个泡泡产生的概率，可以由 $\Gamma \simeq Ae^{-S_E/\hbar}$ 计算，其中 A 和 $\hbar$ 是常数，S_E 是一个 "欧氏作用量"，由以下步骤计算：

(1) We "rotate" our physical time t to "Euclidean time" $\tau = it$ (where $i^2 = -1$).

(1) 将物理时间 t"旋转"到欧氏时间 $\tau = it$（其中$i^2 = -1$）。

(2) The real-time and imaginary time field configurations are related by $\phi(t = 0, x, y, z) = \phi(\tau = 0, x, y, z)$.

(2) 实数时间和虚数时间到场位形由 $\phi(t = 0, x, y, z) = \phi(\tau = 0, x, y, z)$ 联系起来。

(3) Given the above time boundary condition, find a 4-dimensional rotational symmetric solution of the Euclidean equation of motion $\frac{\partial^2\phi}{\partial\tau^2} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{dV(\phi)}{d\phi} = 0$.

(3) 给定上述时间边界条件，找到欧氏运动方程 $\frac{\partial^2\phi}{\partial\tau^2} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{dV(\phi)}{d\phi} = 0$ 的四维旋转对称的解。

(4) Insert the solution to the Euclidean action $S_E = \int dt\, d^3x \left[\frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{1}{2}\left(\frac{d\phi}{dy}\right)^2 + \frac{1}{2}\left(\frac{d\phi}{dz}\right)^2 + V(\phi)\right]$ to find Γ.

(4) 将找到的解带入欧氏作用量 $S_E = \int dt\, d^3x \left[\frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{1}{2}\left(\frac{d\phi}{dy}\right)^2 + \frac{1}{2}\left(\frac{d\phi}{dz}\right)^2 + V(\phi)\right]$ 来求出 Γ。

Let us do this calculation in this part. Note that a 4-dimensional unit ball with radius r has "volume" $\frac{\pi^2}{2}r^4$ and surface "area" $2\pi^2 r^3$.
让我们在这一部分中做上述计算。注意，四维球的"体积"为 $\frac{\pi^2}{2}r^4$，球面面积为 $2\pi^2 r^3$。

C1. THE EUCLIDEAN EQUATION OF MOTION 欧氏运动方程

Since we are to look for a 4-dimentional rotational symmetric solution of the Euclidean equation of motion, it is convenient to use $\rho = \sqrt{\tau^2 + x^2 + y^2 + z^2}$ as the variable for equation solving. Assuming that $\phi = \phi(\rho)$ only depends on ρ (i.e., 4-dimensional rotational symmetric), the general Euclidean equation of motion can be written as $\frac{d^2\phi}{d\rho^2} + f(\rho)\frac{d\phi}{d\rho} - \frac{dV(\phi)}{d\phi} = 0$.
为了寻找运动方程的四维转动不变解，使用 $\rho = \sqrt{\tau^2 + x^2 + y^2 + z^2}$ 作为解方程的变量比较方便。假设$\phi = \phi(\rho)$只依赖于ρ（这就是四维转动不变的含义），一般的欧氏运动方程可以写成 $\frac{d^2\phi}{d\rho^2} + f(\rho)\frac{d\phi}{d\rho} - \frac{dV(\phi)}{d\phi} = 0$。

C1	Find the expression of $f(\rho)$. 求 $f(\rho)$。 Hint: if the formal calculation needs too much calculus, you can consider an example: by tuning the form of $V(\phi)$, one can obtain a solution $\phi = \rho^2$. Then $f(\rho)$ can be solved by this example (and this form $f(\rho)$ will apply for all forms of $V(\phi)$, not limited to this special form of solution). 提示：如果进行普适的计算需要太多微积分，你可以考虑一个例子：通过调整 $V(\phi)$ 的形式，我们得到一个解 $\phi = \rho^2$。这时，$f(\rho)$ 可以从这个例子里解出来（之后这个 $f(\rho)$ 的形式对所有 $V(\phi)$ 都适用，不仅限于这个特殊解）。	2 points 2 分

C2. THE EUCLIDEAN ACTION 欧氏作用量

C2	Write the Euclidean action S_E as an integral of ρ from $\rho = 0$ to $\rho \to \infty$. 以对 ρ 的积分（从$\rho = 0$ 积到 $\rho \to \infty$）的形式写出欧氏作用量 S_E。	2 points 2 分

C3. QUALITATIVE INSPECTION 定性讨论 (4')

Before trying to solve the Euclidean equation of motion, let us first see how it behaves. If you are not familiar with this Euclidean equation of motion, you can consider the following analogy: consider ϕ as the position of a particle, and ρ as an effective time variable. In this case, answer the following questions (choose one from the options):

在解欧氏运动方程之前，我们先看看方程的性质。如果你不熟悉欧氏运动方程，你可以用如下类比来理解：把 ϕ 类比为一个粒子的位置，把 ρ 类比为适用于这个粒子的有效时间变量。在这种情况下，回答以下问题（单项选择题）：

C3-1	What's the nature of $f(\rho)\frac{d\phi}{d\rho}$? 以下哪项是 $f(\rho)\frac{d\phi}{d\rho}$ 的性质？ (A) friction (i.e. decelerate the particle) 阻力（即让粒子减速） (B) anti-friction (i.e. accelerate the particle) 推力（即让粒子加速） (C) friction for $\frac{d\phi}{d\rho} > 0$ and anti-friction otherwise 在 $\frac{d\phi}{d\rho} > 0$ 情况下是阻力，否则是推力 (D) friction for $\frac{d\phi}{d\rho} < 0$ and anti-friction otherwise 在 $\frac{d\phi}{d\rho} < 0$ 情况下是阻力，否则是推力	1 point 1 分

C3-2	What's the force that drives the motion of the "particle" ϕ? 驱动粒子 ϕ 运动的力是哪个？ (A) V (B) $-V$ (C) $dV/d\phi$ (D) $-dV/d\phi$	1 point 1 分

C3-3	Where is the "starting point" of ϕ at $\rho = 0$? (Here δ is extremely small but finite) 在 $\rho = 0$，ϕ 的 “起始位置” 在哪里？（其中 δ 是一个非常小但有限的数） (A) $\phi_- - \lvert O(\delta)\rvert$ (B) ϕ_- (C) $\phi_- + \lvert O(\delta)\rvert$ (D) $\phi_+ - \lvert O(\delta)\rvert$ (E) ϕ_+ (F) $\phi_+ + \lvert O(\delta)\rvert$	1 point 1 分

C3-4	Where is the "end point" of ϕ at $\rho \to \infty$? 在 $\rho \to \infty$，ϕ 的 “最终位置” 在哪里？ (A) $\phi_- - \lvert O(\delta)\rvert$ (B) ϕ_- (C) $\phi_- + \lvert O(\delta)\rvert$ (D) $\phi_+ - \lvert O(\delta)\rvert$ (E) ϕ_+ (F) $\phi_+ + \lvert O(\delta)\rvert$	1 point 1 分

C4. THE BUBBLE NUCLEATION RATE 泡泡产生率

C3-4	Express S_E in terms of R and σ. 用 R 和 σ 写出 S_E。	4 points 4 分

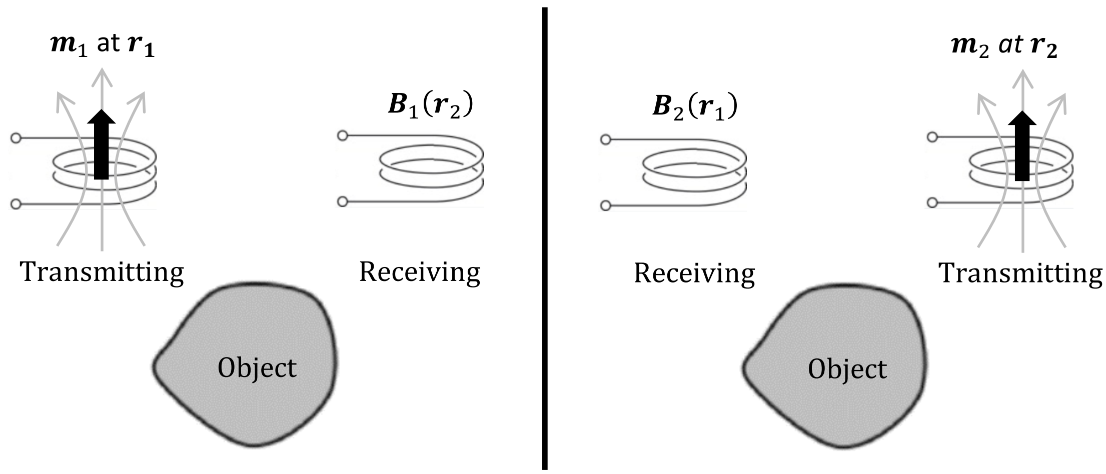
Problem 2: Lorentz reciprocity (32 points)
问题 2: 洛伦兹互易性 (32 分)

Lorentz reciprocity is a fundamental principle in electromagnetism with important application in antenna design theory. It states that the receiving and transmitting capabilities of an antenna are identical. On the other hand, reciprocity can be broken by using magnetic materials under an external magnetic field with strong magneto-optical effect. The study of Lorentz reciprocity can be extended to nearly zero frequency at magnetostatics, shown in the figure below, with two current coils and an arbitrary object fixed in locations. When one current coil works in transmitting mode, another one works in receiving mode. If we model the two current coils as magnetic dipole moments $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ at locations $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, generating magnetic flux densities $\boldsymbol{B}_1(\boldsymbol{r})$ and $\boldsymbol{B}_2(\boldsymbol{r})$ in transmitting mode, respectively. Then, the ***reciprocity relationship*** can be expressed as

$$\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$$

洛伦兹互易性是电磁学的基本原理，在天线设计理论中具有重要应用。 它指出天线的接收和发射能力是相同的。 另一方面，在外磁场下，使用具有强磁光效应的磁性材料可以破坏互易性。 洛伦兹互易的研究可以扩展到静磁学，如下图所示，有两个电流环和一个固定在某个位置的物体。 当一个电流环在发射模式工作时，另一个电流环在接收模式工作。 如果我们将两个电流环设为位置 $\boldsymbol{r}_1$ 和 $\boldsymbol{r}_2$ 处的磁偶极子 $\boldsymbol{m}_1$ 和 $\boldsymbol{m}_2$，分别在发射模式下产生磁通密度 $\boldsymbol{B}_1(\boldsymbol{r})$ 和 $\boldsymbol{B}_2(\boldsymbol{r})$。 那么，***互易关系***可以表示为

$$\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$$



The magnetic field $\boldsymbol{B}(\boldsymbol{r})$ generated by a magnetic dipole $\boldsymbol{m}_i$ located at $\boldsymbol{r}_i$ is given by,

在位置 $\boldsymbol{r}_i$ 处的磁偶极子 $\boldsymbol{m}_i$ 所产生的磁场 $\boldsymbol{B}_i(\boldsymbol{r})$ 由下式给出，

$$\boldsymbol{B}_i(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\boldsymbol{r}-\boldsymbol{r}_i)(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{m}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^5} - \frac{\boldsymbol{m}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}\right)$$

In this problem, we will first establish the reciprocity relationship and will investigate how it can be broken by imposing a constant velocity on the object.

本题中，我们将首先建立互易关系，并研究如何通过对物体施加恒定速度来打破它。

A. ESTABLISHING MAGNETOSTATIC RECIPROCITY AND NON-RECIPROCITY 建立静磁互易性和非互易性

For magnetostatics, we have vector potential $\boldsymbol{A}$, magnetic flux density $\boldsymbol{B}$, magnetic field strength $\boldsymbol{H}$ and impressed current density $\boldsymbol{J}$. These fields satisfy the Ampère's law

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$

with material response

$$\nabla \times \boldsymbol{A} = \boldsymbol{B} = \mu \boldsymbol{H}$$

where $\mu(\boldsymbol{r})$ is the isotropic magnetic permeability profile for the material, i.e. the object in the figure.

对于静磁学，我们有矢量势 $\boldsymbol{A}$、磁通密度 $\boldsymbol{B}$、磁场强度 $\boldsymbol{H}$ 和外加电流密度 $\boldsymbol{J}$。这些场满足安培定律

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$

关于物质反应的表达式为

$$\nabla \times \boldsymbol{A} = \boldsymbol{B} = \mu \boldsymbol{H}$$

其中 $\mu(\boldsymbol{r})$ 是材料的各向同性磁导率分布，即图中的物体。

We want to establish the reciprocity relationship for magnetostatics when we have a magnetic dipole moment $\boldsymbol{m}_1$ at location $\boldsymbol{r}_1$ in one case and a magnetic dipole $\boldsymbol{m}_2$ at $\boldsymbol{r}_2$ in another case. The two dipoles generate magnetic fields $\boldsymbol{B}_1(\boldsymbol{r})$ and $\boldsymbol{B}_2(\boldsymbol{r})$, respectively. The two cases, labeled by $i = 1,2$, have the current density $\boldsymbol{J}_i = \nabla \times \boldsymbol{M}_i$ and magnetization $\boldsymbol{M}_i = \boldsymbol{m}_i\delta(\boldsymbol{r}-\boldsymbol{r}_i)$ for the two dipoles.

在一种情况下$\boldsymbol{r}_1$处有一个磁偶极子 $\boldsymbol{m}_1$并产生磁场 $\boldsymbol{B}_1(\boldsymbol{r})$而在另一种情况下$\boldsymbol{r}_2$处有一个磁偶极子 $\boldsymbol{m}_2$并产生磁场 $\boldsymbol{B}_2(\boldsymbol{r})$。对于由 $i = 1,2$ 索引的两个偶极子，我们有电流密度$\boldsymbol{J}_i = \nabla \times \boldsymbol{M}_i$和磁化强度 $\boldsymbol{M}_i = \boldsymbol{m}_i \delta(\boldsymbol{r} - \boldsymbol{r}_i)$。以下我们想要建立静磁学的互易关系。

Here, we also give some formulas for these differential operators:

在这里，我们提供一些关于微分算子的公式：

$$
\begin{aligned}
&\nabla \times \boldsymbol{A} = \hat{x}(\partial_y A_z - \partial_z A_y) + \hat{y}(\partial_z A_x - \partial_x A_z) + \hat{z}(\partial_x A_y - \partial_y A_x) \\
&\nabla \cdot \boldsymbol{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z \\
&\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{B} - \boldsymbol{A} \cdot \nabla \times \boldsymbol{B} \\
&\int \nabla \cdot \boldsymbol{A} dV = \int \boldsymbol{A} \cdot d\boldsymbol{a} \\
&\int \nabla \times \boldsymbol{A} dV = -\int \boldsymbol{A} \times d\boldsymbol{a} \\
&\int \boldsymbol{B} \cdot \nabla \times \boldsymbol{A} dV = \int \boldsymbol{A} \cdot \nabla \times \boldsymbol{B} dV + \int \boldsymbol{A} \times \boldsymbol{B} \cdot d\boldsymbol{a}
\end{aligned}
$$

and Kronecker delta function $\delta(\boldsymbol{r})$ is defined by

$$\delta(\boldsymbol{r}) = \begin{cases} \infty & \text{if } \boldsymbol{r} = \boldsymbol{0} \\ 0 & \text{otherwise} \end{cases}$$

which satisfies $\int \delta(\boldsymbol{r}) dV = 1$ when we integrate a volume V enclosing the origin. $\boldsymbol{a}$ is defined as the closed area enclosing volume V.

当我们对包含原点的一个体积 V 进行积分时，Kronecker delta 函数 $\delta(\boldsymbol{r})$ 定義為:

$$\delta(\boldsymbol{r}) = \begin{cases} \infty & \text{if } \boldsymbol{r} = \boldsymbol{0} \\ 0 & \text{除此以外} \end{cases}$$

而且满足 $\int \delta(\boldsymbol{r}) dV = 1$。 $\boldsymbol{a}$ 定义为包围体积 V 的封闭区域。

A1. PROVING THE RECIPROCITY RELATIONSHIP 证明互易关系

A1	Prove the reciprocity relationship $\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$. Hint: you may consider $\nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2)$. 证明互易关系 $\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$。 提示：你可以考虑 $\nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2)$。	3 points 3 分

If the material conducts electricity with an electrical conductivity σ, we have to add an additional term to the current density $\boldsymbol{J}$ due to the free current through

$$\boldsymbol{J} \to \boldsymbol{J} + \sigma \boldsymbol{E}$$

in the Ampère's law stated previously. Suppose now we move the conductor by a constant velocity $\boldsymbol{v}$. There will be a Lorentz force on the free charges proportional to $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$. It further updates the additional term in the current density through

$$\boldsymbol{J} \to \boldsymbol{J} + \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}),$$

which may upset the reciprocity relationship.

如果材料以电导率 σ 导电，由于通过的自由电流，我们必须为在此前安培定律中的电流密度 $\boldsymbol{J}$ 添加一个附加项

$$\boldsymbol{J} \to \boldsymbol{J} + \sigma \boldsymbol{E}.$$

假设现在我们以恒定速度 $\boldsymbol{v}$ 移动导体。自由电荷上将存在与 $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$成比例的洛伦兹力，这进一步更新电流密度中的附加项以使破坏互易关系变的可能：

$$\boldsymbol{J} \to \boldsymbol{J} + \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}).$$

A2. BREAKING RECIPROCITY RELATIONSHIP 打破互易关系

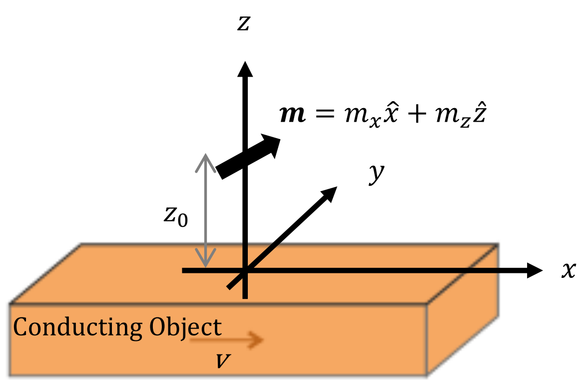
A2	Express the possibly non-zero $\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) - \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$ as a volume integral in terms of the vector potentials $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ and the conductor velocity $\boldsymbol{v}$. 将可能非零的 $\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) - \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$表示为根据矢量势$\boldsymbol{A}_1$ 和$\boldsymbol{A}_2$ 以及导体速度$\boldsymbol{v}$ 的体积积分。	3 points 3 分

You can complete part B and C without part A.
你可以在没有 A 部分的情况下完成 B 部分和 C 部分。

B. MAGNETIC DIPOLES ON A MOVING PERFECT CONDUCTOR 运动理想导体上的磁偶极子

In this part, we first obtain the magnetic field of a single magnetic dipole on a moving conductor, which is a perfect conductor (i.e. the electrical conductivity $\sigma = \infty$) here, occupying $z \le 0$ and is moving with a velocity v in the positive x-direction. This is defined as the laboratory frame, as shown in the figure below. The magnetic dipole, situated at $(x, y, z) = (0, 0, z_0)$ on top of the conductor, has a magnetic moment $\boldsymbol{m} = m_x\hat{x} + m_z\hat{z}$ with zero component in the y-direction.

在这一部分中，我们首先得到运动导体上单个磁偶极子的磁场，这里是理想导体 (即电导率 $\sigma = \infty$)，占据 z⩽0 并且在正 x 方向上以速度 v 运动。 这被定义为实验室坐标系，如下图所示。 位于导体顶部 $(x, y, z) = (0, 0, z_0)$ 处的磁偶极子写为 $\boldsymbol{m} = m_x\hat{x} + m_z\hat{z}$，在 y 方向上没有分量。



In the moving frame at a velocity $\boldsymbol{v} = v\hat{x}$ with respect to the laboratory frame, the object is simply a perfect conductor at rest, giving us a convenience to find the magnetic fields generated by the magnetic dipole. The coordinates in the moving frame, denoted as (x', y', z', t'), is transformed from the coordinates in the laboratory frame (x, y, z, t) through the Lorentz transformation:

在相对于实验室坐标系以速度 $\boldsymbol{v} = v\hat{x}$ 运动的坐标系中，物体只是一个静止的理想导体，使我们更方便地找到磁偶极子产生的磁场。 移动坐标系中的坐标，表示为 (x', y', z', t')，由实验室坐标系(x, y, z, t) 中的坐标通过洛伦兹变换转换而来：

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \qquad ct' = \gamma(ct - vx/c)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ and c is the speed of light.

其中 $\gamma = 1/\sqrt{1 - v^2/c^2}$，$c$ 是光速。

The magnetic and electric fields ($\boldsymbol{B}'$ and $\boldsymbol{E}'$) in the moving frame are related to the fields ($\boldsymbol{B}$ and $\boldsymbol{E}$) in the laboratory frame by the Lorentz transformation,

移动坐标系中的磁场和电场（$\boldsymbol{B}'$和 $\boldsymbol{E}'$）与实验室坐标系中的场（$\boldsymbol{B}$ 和 $\boldsymbol{E}$）相关，可由洛伦兹变换给出:

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} B_x \\ \gamma B_y \\ \gamma B_z \end{pmatrix} + \frac{v\gamma}{c^2}\begin{pmatrix} 0 \\ E_z \\ -E_y \end{pmatrix}, \qquad \begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} E_x \\ \gamma E_y \\ \gamma E_z \end{pmatrix} - v\gamma\begin{pmatrix} 0 \\ B_z \\ -B_y \end{pmatrix}$$

We further simplify the problem by removing the conductor at the moment.

我们现在通过移除导体来进一步简化问题。

B1. MAGNETIC FIELD OF A MOVING MAGNETIC DIPOLE IN FREESPACE 自由空间中运动磁偶极子的磁场

B1	What is the magnetic field $\boldsymbol{B}'(x',y',z',t')$ from the dipole for an observer moving at velocity v with respect to the laboratory frame? In this part, we only consider a magnetic dipole $\boldsymbol{m}=m_x\hat{x}$ pointing in the x-direction and there is no conductor below the dipole yet. Please express your answer in the coordinates of the moving frame. 请找出相对于实验室坐标系以速度v 移动的观察者的偶极子磁场 $\boldsymbol{B}'(x',y',z',t')$。 在这一部分中，我们只考虑指向 x 方向的磁偶极子$\boldsymbol{m}=m_x\hat{x}$ 并且偶极子下方还没有导体。 请用移动坐标系的坐标表达你的答案。	3 point 3 分

Next, we introduce the perfect conductor below the dipole. The perfect metal has a planar interface at $z=0$ while the dipole is placed at a distance z_0 above the metal. The metal is being moved at a velocity v in the positive x-direction.

接下来，我们引入偶极子下方的理想导体。 理想导体在 z=0 处有一个平面界面，而偶极子位于金属上方 z_0 处。 金属在正 x 方向上以速度 v 移动。

B2. MAGNETIC FIELD OF A MAGNETIC DIPOLE ON A MOVING CONDUCTOR (I) 运动导体上磁偶极子的磁场（1）

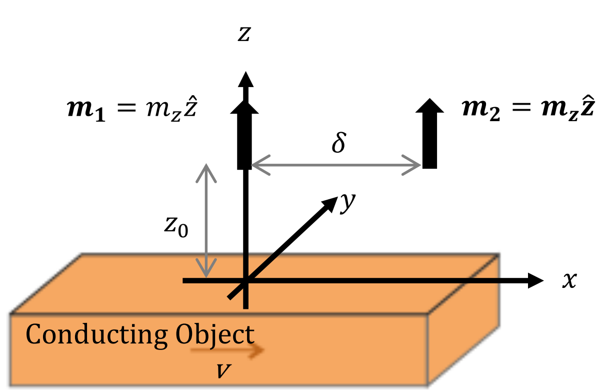
B2	Please express the magnetic field of the magnetic dipole $\boldsymbol{m}=m_x\hat{x}$ on top of a moving perfect conductor (i.e $\boldsymbol{B}(x,y,0^+)$) in terms of the laboratory frame coordinate. Please also verify the boundary condition of the magnetic field in the moving frame at which the conductor at rest. Hint: adopt the method of images and assume both electric and magnetic fields inside a perfect conductor is zero at a nearly-zero frequency. 请用实验室坐标系表示在移动中的理想导体上磁偶极子$\boldsymbol{m}=m_x\hat{x}$ 的磁场(即 $\boldsymbol{B}(x,y,0^+)$)。 还请验证导体静止时移动坐标系中的磁场边界条件。提示：采用镜像法。假定在接近零频率时完美导体里电场和磁场为零。	4 points 4 分

B3. MAGNETIC FIELD OF A MAGNETIC DIPOLE ON A MOVING CONDUCTOR (2) 运动导体上磁偶极子的磁场（2）

B3	What is the magnetic field from the magnetic dipole on top of a moving perfect conductor (i.e $\boldsymbol{B}(x,y,0^+)$) in the laboratory frame if the magnetic dipole above the moving conductor is changed to $\boldsymbol{m}=m_z\hat{z}$, pointing in the positive z direction? 如果运动导体上方的磁偶极子转为$\boldsymbol{m}=m_z\hat{z}$，指向正 z 方向，那么请找出实验室坐标系中磁偶极子在移动中的理想导体上的磁场(即 $\boldsymbol{B}(x,y,0^+)$)？	3 points 3 分

Now, we consider the two situations: one with magnetic dipole moment $\boldsymbol{m}_1$ at $\boldsymbol{r}_1=(0,0,z_0)$ and another magnetic dipole moment $\boldsymbol{m}_2$ at $\boldsymbol{r}_2=(\delta,0,z_0)$. When one of them is turned on, another one is turned off.

现在，我们考虑两种情况：一种是 $\boldsymbol{r}_1=(0,0,z_0)$ 处的磁偶极矩 $\boldsymbol{m}_1$，另一种是 $\boldsymbol{r}_2=(\delta,0,z_0)$ 处的磁偶极矩 $\boldsymbol{m}_2$。 当其中一个打开时，另一个被关闭。



Now, we define the two problems to solve next. For the *first* problem, suppose the two magnetic dipoles moments $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ point in the z-direction with same size m_z, as shown in the above figure. Dipole moment $\boldsymbol{m}_1$ imposes a magnetic field of z-component $B_z(\delta)$ on $\boldsymbol{r}_2 = (x_2, 0, z_0)$ and dipole $\boldsymbol{m}_2$ imposes magnetic field $B_z(-\delta)$ on $\boldsymbol{r}_1 = (x_1, 0, z_0)$ according to the same function $B_z(x - x_i)$. We then define a reciprocity figure-of-merit

$$\mathcal{R} = \frac{B_z(\delta) - B_z(-\delta)}{B_z(\delta) + B_z(-\delta)}.$$

When $\mathcal{R} = 0$, reciprocity is satisfied. Reciprocity is broken when $\mathcal{R}$ deviates from value zero.

现在，我们定义接下来要解决的两个问题。对于第一个问题，假设两个磁偶极子$\boldsymbol{m}_1$和$\boldsymbol{m}_2$都指向 z 方向，大小m_z相同，如上图所示。 偶极子 $\boldsymbol{m}_1$ 根据函数 $B_z(x - x_i)$ 将 z 分量 $B_z(\delta)$ 的磁场施加到 $\boldsymbol{r}_2 = (x_2, 0, z_0)$上，偶极子 $\boldsymbol{m}_2$ 将磁场 $B_z(-\delta)$ 施加到 $\boldsymbol{r}_1 = (x_1, 0, z_0)$ 上。 然后我们定义互易功值

$$\mathcal{R} = \frac{B_z(\delta) - B_z(-\delta)}{B_z(\delta) + B_z(-\delta)}.$$

当$\mathcal{R} = 0$时，满足互易性。 当 $\mathcal{R}$ 偏离零值时，互易性被打破。

For the *second* problem, suppose we change the pointing direction for $\boldsymbol{m}_1$ to the positive x-direction with magnitude remaining the same. The magnitude and direction of $\boldsymbol{m}_2$ are not changed. In this case, the reciprocity merit is defined as $\mathcal{R} = \left(B_{1z}(\delta) - B_{2x}(-\delta)\right)/\left(B_{1z}(\delta) + B_{2x}(-\delta)\right)$.

对于第二个问题，假设我们将 $\boldsymbol{m}_1$的指向更改为正 x 方向而大小保持不变。$\boldsymbol{m}_2$方向和大小不变。 在这种情况下，互易功值定义为 $\mathcal{R} = \left(B_{1z}(\delta) - B_{2x}(-\delta)\right)/\left(B_{1z}(\delta) + B_{2x}(-\delta)\right)$。

B4. RECIPROCITY MERIT FOR DIPOLES ON A PERFECT CONDUCTOR 理想导体上偶极子的互易功值

B4	Find the reciprocity merit $\mathcal{R}$ for the two defined problems about magnetic dipoles on the moving perfect conductor. 求出关于以上两个问题运动理想导体上磁偶极子的互易功值$\mathcal{R}$。	3 points 3 分

C. MAGNETIC DIPOLES ON A MOVING CONDUCTOR WITH FINITE CONDUCTIVITY 具有有限电导率的移动导体上的磁偶极子

In this part, we move to a more realistic situation that the conductor is a metal. It has a large but finite conductivity σ (in unit of $\Omega^{-1}m^{-1}$), deviating from the perfect conductor condition. Current density in the conductor is given by $\boldsymbol{J} = \sigma\boldsymbol{E}$. We also assume that the current on the conductor surface is confined by a skin depth d of small thickness so that the electric and magnetic fields cannot penetrate beyond the skin depth from the conductor surface. Then, the surface current density can be written as $(\sigma d)\boldsymbol{E}$. We further take the approximation that d is just a constant. We only consider the two dipoles pointing in the positive z-direction with same size m_z in this part.

在这一部分中，我们转向一个更现实的情况，即金属导体具有较大但有限的电导率 σ（单位为 $\Omega^{-1}m^{-1}$），偏离理想导体条件。 导体中的电流密度由 $\boldsymbol{J} = \sigma\boldsymbol{E}$ 给出。 我们还假设电流只在导体表面厚度较小的趋肤深度 d 内流动，电场和磁场不能从导体表面穿透超过趋肤深度。 因此表面电流密度可以写成 $(\sigma d)\boldsymbol{E}$。我们进一步假定 d 为一个常数。 在这部分我们只考虑两个偶极子都指向正 z 方向，大小 m_z相同。

Again, we need to solve the magnetic field from only one dipole at $(0,0,z_0)$ first. In fact, the surface current profile generated on the surface of conductor cannot be easily solved without adopting a numerical solver. Instead, we can approach the problem by extending the method of image as an approximation. In this case, we would like to have a point-like multipolar source at the image position $(0,0,-z_0)$ in order to give as closely as possible the same reflected field generated from the surface current. For the current case of $\boldsymbol{m} = m_z\hat{z}$, we put an image magnetic dipole with given form of magnetic moment $m_x^{(r)}\hat{x} + m_z^{(r)}\hat{z}$ and electric moment $p_y^{(r)}\hat{y}$ at the same location $(0,0,-z_0)$. The mirrored magnetic dipole is now relaxed to have both magnetic and electric components while neglecting the higher order multipoles. The size of these dipole moments are yet to be determined.

同样，我们需要先求解 $(0,0,z_0)$ 处只有一个偶极子的磁场。 实际上，导体表面产生的表面电流分布不采用数值求解器是无法精确求解的。 这里，我们尝试通过电像法来近似解决这个问题。 在这种情况下，我们希望在镜像位置 $(0,0,-z_0)$ 处有一个点状多极源，以便尽可能地提供相同的从表面电流生成的反射场。 对于当前 $\boldsymbol{m} = m_z\hat{z}$ 的情况，我们将镜像磁偶极子设置在$(0,0,-z_0)$ 处，它拥有给定的表达式：磁矩为 $m_x^{(r)}\hat{x} + m_z^{(r)}\hat{z}$ 、电矩为 $p_y^{(r)}\hat{y}$。 镜像磁偶极子现在同时具有磁和电分量，同时忽略高阶多极子。 这些偶极矩的大小有待确定。

C1. GENERALIZED METHOD OF IMAGES 广义电像法

C1	Find the magnetic field $\boldsymbol{B}'(x',y',0^+)$ and electric field $\boldsymbol{E}'(x',y',0^+)$ on the conductor surface in the moving frame. Express your answer in m_z, $m_x^{(r)}$, $m_z^{(r)}$ and $p_y^{(r)}$. You can use either the moving frame or laboratory frame coordinates. Do not need to solve the mirrored dipole moments yet. 求移动坐标系导体表面的磁场$\boldsymbol{B}'(x',y',0^+)$ 和电场$\boldsymbol{E}'(x',y',0^+)$。 请用$m_z$、$m_x^{(r)}$ 、$m_z^{(r)}$ 和$p_y^{(r)}$ 表达你的答案。可以使用移动坐标或实验室坐标表达你的答案。 暂时不需要求解镜像偶极矩。	5 points 5 分

C2. FINDING THE MIRRORED DIPOLE MOMENTS 计算镜像偶极矩

C2	Find $m_x^{(r)}$, $m_z^{(r)}$ and $p_y^{(r)}$ in response to a given m_z. The mirror dipole gives the same reflected field generated by the surface current on the conductor. As approximation, only apply the boundary condition (in the moving frame) on the surface current along the y-direction, which is the dominant current than the one along the x-direction. It may be useful to express the answers in term of the dimensionless parameter $\kappa = \mu_0 v\gamma\sigma d \gg 1$. 用给定的$m_z$表示 $m_z^{(r)}$、$m_z^{(r)}$ 和 $p_y^{(r)}$。 镜像偶极子给出了与导体表面电流产生的相同的反射场。 作为近似，请仅将边界条件（在移动坐标系中）应用于沿 y 方向的表面电流，该电流比沿 x 方向的电流占主导地位。可以考虑用无量纲参数$\kappa = \mu_0 v\gamma\sigma d \gg 1$ 表达答案。	5 points 5 分

C3. RECIPROCITY MERIT FOR DIPOLES ON A CONDUCTOR OF FINITE CONDUCTIVITY
有限电导率导体上偶极子的互易功值

C3	Find the reciprocity merit $\mathcal{R}$ for the two identical dipoles m_z displaced by δ (with $\boldsymbol{r}_1 = (0,0,z_0)$, $\boldsymbol{r}_2 = (\delta,0,z_0)$)in the x-direction. 找出在 x 方向上位移δ （$\boldsymbol{r}_1 = (0,0,z_0)$, $\boldsymbol{r}_2 = (\delta,0,z_0)$)）的两个相同偶极子 m_z 的互易功价值$\mathcal{R}$。	3 point 3 分

~~ END of Paper 2 卷 2 完 ~~

Pan Pearl River Delta Physics Olympiad 2023
2023 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
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Simplified Chinese Part-2 (Total 2 Problems, 60 Points)
简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 29 January 2023)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。

All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。

There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。

Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。

Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。

Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。

If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。

At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Vacuum bubbles (28 points)
问题 1: 真空泡泡 (28 分)

Is our vacuum stable? We don't know. It's possible that we do not live in the true vacuum. Rather, we live in a false vacuum which can decay into true vacuum by emerging and expanding bubbles. To describe such a possibility, we will make use of a space-time dependent "scalar field" $\phi(t,x,y,z)$, which takes a real value at every space-time point. (Similar to height on a map, which takes a real value at every point on the x-y plane, while a scalar field takes a real value for any given t,x,y,z. Also, in a full quantum theory, we have to distinguish operators and numbers, but here we will assume the scalar field only take real number values.)

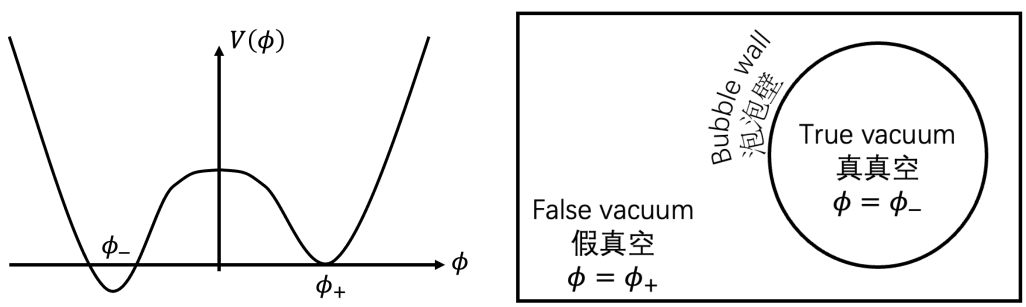
我们的真空是稳定的吗？我们不知道。有可能我们并不是生活在一个稳定的真真空（既真的 “真空” ）里面。我们可能生活在一个假真空里，而假真空可以通过自发产生和膨胀的泡泡衰变到真真空。为了描述这种可能，我们将使用一个依赖于时空点到 “标量场” $\phi(t,x,y,z)$：它在每个时空点上取一个实数。（这有点像地图上的高度，在 x-y 平面的每个点上取一个实数。不过标量场是在每个t,x,y,z时空点上取一个实数。另外，在完整的量子理论中，我们需要考虑算符和数的区别，但这里我们假设标量场的取值仅是普通的实数。）

The scalar field satisfies the following equation of motion: $\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial z^2}+\frac{dV(\phi)}{d\phi}=0$, where $V(\phi)$ is the potential density of the field, and we will call it potential for short in this problem. The energy density of the scalar field is $\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dy}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2+V(\phi)$.

标量场满足运动方程 $\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial z^2}+\frac{dV(\phi)}{d\phi}=0$，其中$V(\phi)$是场的势能密度，我们将简称它为标量场的势能。标量场的能量密度是 $\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dy}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2+V(\phi)$。

We consider the following potential: the false vacuum has field value $\phi=\phi_+$ where $V(\phi_+)=0$, and the true vacuum has field value $\phi=\phi_-$, where $V(\phi_-)$ is slightly negative. In the left panel of the following figure, we plot the shape of the potential. The right panel is an example of the false and true vacuum in position space.

我们考虑如下势能：假真空处标量场取值是 $\phi=\phi_+$，满足 $V(\phi_+)=0$；真真空处标量场取值是$\phi=\phi_-$， $V(\phi_-)$ 取一个接近 0 的负值。下图（左）是势能的函数形式，下图（右）是位置空间中假真空和真真空的一个例子。



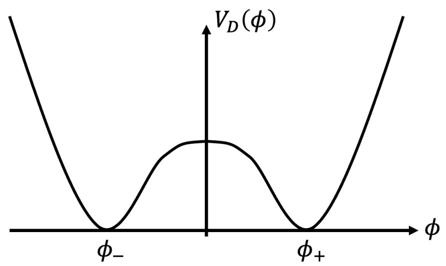
In this problem, we will use natural units and set the speed of light in vacuum $c=1$ (by redefining the time unit as the time that light traveled over unit length). In this unit, when an object is at rest, its energy equals its mass by the famous formula $E=mc^2=m$.

本题中，我们将使用自然单位并定义真空中的光速 $c=1$（既定义时间的单位为光穿越单位长度的时间）。在自然单位制中，物体的静止能量等于它的质量，这就是著名的公式 $E=mc^2=m$。

A. DOMAIN WALL 畴壁

Before coming to the asymmetric potential which generates vacuum bubbles, let us consider a symmetric potential as follows:

在我们研究非对称的势能以及它生成的真空泡泡前，我们先考虑如下一个对称的势能：



Let's find a static solution which is homogeneous along the y and z directions, known as a domain wall. The potential of the domain wall $V(\phi) = V_D(\phi)$ is illustrated above, with two minima $V_D(\phi_\pm) = 0$. The domain wall can be used as the local approximation of the bubble wall.

我们将找一种在 y 和 z 方向均匀的 "畴壁" 解。上图是畴壁解对应的势能 $V(\phi) = V_D(\phi)$，它具有两个最小值 $V_D(\phi_\pm) = 0$。畴壁可以作为泡泡壁的局域近似。

A1. SIMPLIFY THE EQUATION OF MOTION 化简运动方程

A1	Given the static and homogeneity (independency of y and z) conditions, write down the simplified equation of motion for ϕ. 根据静态，以及均匀（不依赖于y 和 z 方向）的条件，写出ϕ化简后的运动方程。	2 points 2 分

Solution: $\frac{d^2\phi}{dx^2} = \frac{dV_D(\phi)}{d\phi}$. [Note, if the student did not convert ∂ to d, we also consider it as correct. Also, since in Part A, $V = V_D$, both are correct.]

A2. THE DOMAIN WALL PROFILE 畴壁上标量场取值的空间变化

A2	Express $\frac{d\phi}{dx}$ in terms of $V_D(\phi)$. 请把 $\frac{d\phi}{dx}$ 用 $V_D(\phi)$ 表达出来。	3 points 3 分

Solution: $\frac{d^2\phi}{dx^2} = \frac{dV_D}{d\phi} = \frac{dV_D}{dx}\frac{dx}{d\phi}$, thus $\frac{1}{2}\frac{d}{dx}\left(\left(\frac{d\phi}{dx}\right)^2\right) = \frac{dV_D}{dx}$. Since far away from the domain wall, we have boundary condition $\frac{d\phi}{dx} = 0$ and $V_D = 0$, we have $\left(\frac{d\phi}{dx}\right)^2 = 2V_D$.

A3. THE DOMAIN WALL TENSION 畴壁的张力 (2')

A3	The tension of the domain wall (energy density of the wall for unit area in the y and z directions) is $\sigma = \int_{\phi_-}^{\phi_+} f(\phi)d\phi$. Find $f(\phi)$ in terms of $V_D(\phi)$. 畴壁的张力（既在 y、z 方向单位面积上，畴壁的能量密度）是 $\sigma = \int_{\phi_-}^{\phi_+} f(\phi)d\phi$。请把 $f(\phi)$用 $V_D(\phi)$ 表达出来。	2 points 2 分

Note: to avoid propagation of possible errors, in the later part of the questions, please still use the domain wall tension σ where applicable, instead of using the integral expression that you obtain.

注：为避免潜在的错误传播，在本题后面的部分中，当用到畴壁张力时，请仍使用符号 σ，而不是这里你求出的积分表达式。

Solution: $\sigma = \int_{-\infty}^{\infty}\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V_D(\phi)\right]dx = \int_{\phi_-}^{\phi_+}[2V_D(\phi)]\frac{dx}{d\phi}d\phi = \int_{\phi_-}^{\phi_+}[2V_D(\phi)]^{1/2}d\phi$

$$f(\phi) = \sqrt{2V_D(\phi)}$$

B. BUBBLE WALL 泡泡壁

If we only look at a small part, a bubble wall can be approximated as a domain wall. But globally, the bubble can be approximated to be spherical with radius R. Let's assume that R is large enough, such that the thickness of the bubble wall is much smaller than R (thin wall approximation). Inside and outside the bubble, $\phi \to \phi_\pm$ exponentially quickly.
如果我们只看一泡泡壁的一小部分的话，泡泡壁上的一小块可以用畴壁来近似。但是整体上，真空泡泡可以近似为球形的，具有半径 R。假设 R 足够大，泡泡壁的厚度远远小于 R（薄壁近似）。在泡泡壁的内部和泡泡壁的外部，ϕ 指数快地趋向于 $\phi_\pm$。

At the moment when a bubble is nucleated, the bubble is static, and the bubble nucleation and motion create negligible amount of radiation or other dissipations.
在真空泡泡产生的时刻，泡泡是静止的。泡泡产生的过程带来的辐射或其它耗散可以忽略。

B1. THE ENERGY ON THE BUBBLE WALL 泡泡壁的能量 (1')

B1	At the momentum of bubble nucleation, calculate the energy E_W carried by the bubble wall using R and the bubble tension σ. 在真空泡泡产生的时刻，利用 R 和泡泡壁的张力 σ 计算泡泡壁的能量 E_W。	1 point 1 分

Solution: $E_W = 4\pi R^2\sigma$.

B2. FALSE AND TRUE VACUA 假真空和真真空

B2	For a spherical bubble to appear, there must be an energy density difference between $V(\phi_\pm)$. Thus, to write down a potential to model bubble nucleation, we consider the potential $V(\phi) = V_D(\phi) + \frac{\epsilon}{\phi_+ - \phi_-}(\phi - \phi_+)$. In the thin wall approximation, we are only interested in leading order results in ϵ (the lowest order in Taylor expansion which contains ϵ). Calculate ϵ using σ and R. 为了让球形真空泡泡能够出现，$V(\phi_\pm)$ 的取值必须不同。所以，为了对真空泡泡的产生过程建立势能模型，我们考虑势能 $V(\phi) = V_D(\phi) + \frac{\epsilon}{\phi_+ - \phi_-}(\phi - \phi_+)$。在薄壁近似下，我们只感兴趣 ϵ 零头阶效应（既泰勒展开中含有 ϵ 的最低阶）。利用 σ 和 R 计算 ϵ。	2 points 2 分

Solution:

The energy density inside the bubble is $-\epsilon$.

From energy conservation, $E_W = \frac{4}{3}\pi R^3\epsilon$,

thus $\epsilon = \frac{3\sigma}{R}$.

Note: though the introduction of the linear term modifies the minima of the potential a little, but the modification multiplying the energy density will be $O(\epsilon^2)$ and thus neglected.

B3. BUBBLE MOTION 泡泡的运动

B3	At the moment of bubble nucleation, calculate the acceleration a of the bubble wall in terms of σ and R. 在真空泡泡产生的瞬间，利用 σ 和 R 计算泡泡的加速度 a。	2 points 2 分

Solution:

Consider bubble expansion $R \to R + \delta R$.

Since volume expands faster than surface, the energy obtained by the bubble wall from this expansion is $\delta E_W = \frac{4}{3}\pi(R+\delta R)^3\epsilon - 4\pi(R+\delta R)^2\sigma = 4\pi R\sigma\,\delta R$

Since the force $F = \frac{\delta E_W}{\delta R} = ma = E_W a$ (in the unit $c = 1$)

Acceleration $a = \frac{\delta E_W/\delta R}{E_W} = \frac{1}{R}$.

B4. BEYOND NEWTONIAN MECHANICS 超越牛顿力学

B4	When the speed of the bubble wall is close to the speed of light, Newtonian mechanics breaks down and special relativity should be used instead. In special relativity, the kinetic energy of a moving object is $E_K = (\gamma - 1)m$, where $\gamma \equiv \frac{1}{\sqrt{1-v^2}}$.. Calculate the time needed from the nucleation of the bubble to that the speed of the bubble wall to reach 0.6. 当真空泡泡的速度接近光速，牛顿力学不再适用，我们应该使用狭义相对论。在狭义相对论里，运动物体的动能为 $E_K = (\gamma - 1)m$，其中 $\gamma \equiv \frac{1}{\sqrt{1-v^2}}$。计算真空泡泡从产生到泡泡壁速度达到 0.6 所需要的时间。 Hint: you may need the mathematical relation $\frac{d\sqrt{x^2-1}}{dx} = \frac{x}{\sqrt{x^2-1}}$. 提示：你可能需要数学关系 $\frac{d\sqrt{x^2-1}}{dx} = \frac{x}{\sqrt{x^2-1}}$。	4 points 4 分

Solution:

The rest + kinetic energy of the bubble is $E_W = \gamma m = \frac{1}{\sqrt{1-v^2}} 4\pi r^2 \sigma$, where $r = r(t)$ is the time-dependent radius of the bubble (to be distinguished from the initial radius of the bubble R.

Thus, energy conservation at time t yields $\frac{1}{\sqrt{1-v^2}} 4\pi r^2 \sigma = \frac{4}{3}\pi r^3 \epsilon$,

i.e., $\frac{1}{\sqrt{1-v^2}} = \frac{r}{R}$, i.e., $\frac{dr}{dt} = \frac{\sqrt{r^2-R^2}}{r}$, i.e., $\frac{rdr}{\sqrt{r^2-R^2}} = d\sqrt{r^2 - R^2} = dt$.

Given the initial condition $r(t = 0) = R$, we have $r^2 - R^2 = t^2$.

(Note, after a complicated calculation (it's more complicated if you use relativistic force), you get a surprisingly simple result. This is not a coincidence. In fact, this is an analytical continuation of a Euclidean 4-sphere. We will see a little bit of this in Part C.)

$v = \frac{dr}{dt} = \frac{t}{\sqrt{R^2+t^2}} = 0.6$, $t = 0.75R$.

C. NUCLEATION RATE OF THE BUBBLE 泡泡的产生率

What's the probability for a bubble to appear? It can be shown that the nucleation rate Γ of the bubble, i.e., the probability for a bubble to appear in unit volume and during unit time, can be written as $\Gamma \simeq Ae^{-S_E/\hbar}$, where A and $\hbar$ are constants, and S_E is a "Euclidean action", which can be calculated with the following procedure:

真空泡泡产生的概率是多少？可以证明，泡泡的产生率 Γ，即单位时间单位体积，一个泡泡产生的概率，可以由 $\Gamma \simeq Ae^{-S_E/\hbar}$ 计算，其中 A 和 $\hbar$ 是常数，S_E 是一个 “欧氏作用量” ，由以下步骤计算：

(1) We "rotate" our physical time t to "Euclidean time" $\tau = it$ (where $i^2 = -1$).

(1) 将物理时间 t“旋转”到欧氏时间 $\tau = it$（其中$i^2 = -1$）。

(2) The real-time and imaginary time field configurations are related by $\phi(t = 0, x, y, z) = \phi(\tau = 0, x, y, z)$.

(2) 实数时间和虚数时间到场位形由 $\phi(t = 0, x, y, z) = \phi(\tau = 0, x, y, z)$ 联系起来。

(3) Given the above time boundary condition, find a 4-dimensional rotational symmetric solution of the Euclidean equation of motion $\frac{\partial^2\phi}{\partial\tau^2}+\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{dV(\phi)}{d\phi}=0$.

(3) 给定上述时间边界条件，找到欧氏运动方程 $\frac{\partial^2\phi}{\partial\tau^2}+\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{dV(\phi)}{d\phi}=0$ 的四维旋转对称的解。

(4) Insert the solution to the Euclidean action $S_E=\int dt\, d^3x\left[\frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dy}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2+V(\phi)\right]$ to find Γ.

(4) 将找到的解带入欧氏作用量 $S_E=\int dt\, d^3x\left[\frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dy}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2+V(\phi)\right]$ 来求出 Γ。

Let us do this calculation in this part. Note that a 4-dimensional unit ball with radius r has "volume" $\frac{\pi^2}{2}r^4$ and surface "area" $2\pi^2r^3$.
让我们在这一部分中做上述计算。注意，四维球的“体积”为 $\frac{\pi^2}{2}r^4$，球面面积为 $2\pi^2r^3$。

C1. THE EUCLIDEAN EQUATION OF MOTION 欧氏运动方程

Since we are to look for a 4-dimentional rotational symmetric solution of the Euclidean equation of motion, it is convenient to use $\rho=\sqrt{\tau^2+x^2+y^2+z^2}$ as the variable for equation solving. Assuming that $\phi=\phi(\rho)$ only depends on ρ (i.e., 4-dimensional rotational symmetric), the general Euclidean equation of motion can be written as $\frac{d^2\phi}{d\rho^2}+f(\rho)\frac{d\phi}{d\rho}-\frac{dV(\phi)}{d\phi}=0$.
为了寻找运动方程的四维转动不变解，使用 $\rho=\sqrt{\tau^2+x^2+y^2+z^2}$ 作为解方程的变量比较方便。假设$\phi=\phi(\rho)$只依赖于ρ（这就是四维转动不变的含义），一般的欧氏运动方程可以写成 $\frac{d^2\phi}{d\rho^2}+f(\rho)\frac{d\phi}{d\rho}-\frac{dV(\phi)}{d\phi}=0$。

C1	Find the expression of $f(\rho)$. 求 $f(\rho)$。 Hint: if the formal calculation needs too much calculus, you can consider an example: by tuning the form of $V(\phi)$, one can obtain a solution $\phi=\rho^2$. Then $f(\rho)$ can be solved by this example (and this form $f(\rho)$ will apply for all forms of $V(\phi)$, not limited to this special form of solution). 提示：如果进行普适的计算需要太多微积分，你可以考虑一个例子：通过调整 $V(\phi)$ 的形式，我们得到一个解 $\phi=\rho^2$。这时，$f(\rho)$ 可以从这个例子里解出来（之后这个 $f(\rho)$ 的形式对所有 $V(\phi)$ 都适用，不仅限于这个特殊解）。	2 points 2 分

Solution:

Since the above equation is a general one, we can design a potential, such that the solution of the Euclidean equation is $\phi=\rho^2=\tau^2+x^2+y^2+z^2$. We have

$$\frac{\partial^2\phi}{\partial\tau^2}=\frac{\partial^2\phi}{\partial x^2}=\frac{\partial^2\phi}{\partial y^2}=\frac{\partial^2\phi}{\partial z^2}=2$$

On the other hand, $\frac{d^2\phi}{d\rho^2}=2$, $\frac{d\phi}{d\rho}=2\rho$ and

$$8=\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\frac{d^2\phi}{d\rho^2}+f(\rho)\frac{d\phi}{d\rho}=2+2\rho f(\rho)$$

Then $2\rho\times f(\rho)=6, f(\rho)=3/\rho$.

C2. THE EUCLIDEAN ACTION 欧氏作用量

C2	Write the Euclidean action S_E as an integral of ρ from $\rho=0$ to $\rho\to\infty$. 以对 ρ 的积分（从$\rho=0$ 积到 $\rho\to\infty$）的形式写出欧氏作用量 S_E。	2 points 2 分

Solution: For 4-dimensional space, we have $d\tau dxdydz=2\pi^2\rho^3d\rho$ and

$S_E = 2\pi^2 \int_0^\infty \rho^3 \left(\frac{1}{2}\left(\frac{d\phi}{d\rho}\right)^2 + V\right) d\rho.$

C3. QUALITATIVE INSPECTION 定性讨论 (4')

Before trying to solve the Euclidean equation of motion, let us first see how it behaves. If you are not familiar with this Euclidean equation of motion, you can consider the following analogy: consider ϕ as the position of a particle, and ρ as an effective time variable. In this case, answer the following questions (choose one from the options):

在解欧氏运动方程之前，我们先看看方程的性质。如果你不熟悉欧氏运动方程，你可以用如下类比来理解：把 ϕ 类比为一个粒子的位置，把 ρ 类比为适用于这个粒子的有效时间变量。在这种情况下，回答以下问题（单项选择题）：

C3-1	What's the nature of $f(\rho)\frac{d\phi}{d\rho}$? 以下哪项是 $f(\rho)\frac{d\phi}{d\rho}$ 的性质？ (A) friction (i.e. decelerate the particle) 阻力（即让粒子减速） (B) anti-friction (i.e. accelerate the particle) 推力（即让粒子加速） (C) friction for $\frac{d\phi}{d\rho} > 0$ and anti-friction otherwise 在 $\frac{d\phi}{d\rho} > 0$ 情况下是阻力，否则是推力 (D) friction for $\frac{d\phi}{d\rho} < 0$ and anti-friction otherwise 在 $\frac{d\phi}{d\rho} < 0$ 情况下是阻力，否则是推力	1 point 1 分

C3-2	What's the force that drives the motion of the "particle" ϕ? 驱动粒子 ϕ 运动的力是哪个？ (A) V (B) $-V$ (C) $dV/d\phi$ (D) $-dV/d\phi$	1 point 1 分

C3-3	Where is the "starting point" of ϕ at $\rho = 0$? (Here δ is extremely small but finite) 在 $\rho = 0$，ϕ 的“起始位置”在哪里？（其中 δ 是一个非常小但有限的数） (A) $\phi_- - \lvert O(\delta)\rvert$ (B) ϕ_- (C) $\phi_- + \lvert O(\delta)\rvert$ (D) $\phi_+ - \lvert O(\delta)\rvert$ (E) ϕ_+ (F) $\phi_+ + \lvert O(\delta)\rvert$	1 point 1 分

C3-4	Where is the "end point" of ϕ at $\rho \to \infty$? 在 $\rho \to \infty$，ϕ 的“最终位置”在哪里？ (A) $\phi_- - \lvert O(\delta)\rvert$ (B) ϕ_- (C) $\phi_- + \lvert O(\delta)\rvert$ (D) $\phi_+ - \lvert O(\delta)\rvert$ (E) ϕ_+ (F) $\phi_+ + \lvert O(\delta)\rvert$	1 point 1 分

Solution: 1. A; 2. C; 3. C; 4. E

Note: For (C3-3) the solution is C instead of B. One can see it from two ways: First, consider x=y=z=0. In this case, $\rho = 0$ denotes the bubble center after bubble nucleation. Then from continuity of ϕ, the field value at the bubble center should have a slightly (by an exponentially small margin) larger field value than ϕ_-. The second way is considering the effective particle and the friction force. For (C3-4), at infinity, consider $\tau \to 0$, then it corresponds to spatial infinity, thus $\phi = \phi_+$ (spatial infinity makes it exact).

C4. THE BUBBLE NUCLEATION RATE 泡泡产生率

C3-4	Express S_E in terms of R and σ. 用 R 和 σ 写出 S_E。	4 points 4 分

Solution:

For the integral $S_E = 2\pi^2 \int_0^\infty \rho^3 \left(\frac{1}{2}\left(\frac{d\phi}{d\rho}\right)^2 + V\right) d\rho$

Note that at $\rho \to \infty$, the contribution vanishes. Thus, the integral breaks up to

(1) The contribution of the domain wall (replacing x with ρ):

(surface area) x (tension) = $2\pi^2 R^3 \sigma$.

Note: you may notice that the dimension of the problem here is different from the problem above. But at the domain wall, ρ is a constant and all the above calculation of the domain wall tension carries over to here.

(2) The contribution of the vacuum energy inside the bubble.

(volume) x (vacuum energy density) $= -\frac{1}{2}\pi^2 R^4 \epsilon$

Summing the two terms up and inserting $\epsilon = \frac{3\sigma}{R}$, we get

$S_E = \frac{1}{2}\pi^2 R^3 \sigma$.

Reference: "Fate of the false vacuum: Semiclassical theory", by Sidney Coleman, Phys. Rev. D **15**, 2929 (1977).

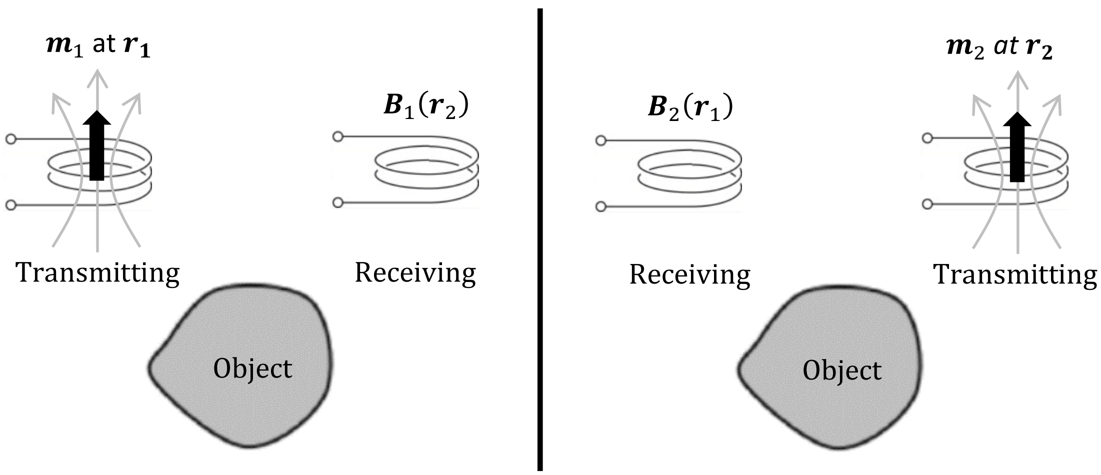
Problem 2: (32 points)
问题 1: (32 分)

Lorentz reciprocity is a fundamental principle in electromagnetism with important application in antenna design theory. It states that the receiving and transmitting capabilities of an antenna are identical. On the other hand, reciprocity can be broken by using magnetic materials under an external magnetic field with strong magneto-optical effect. The study of Lorentz reciprocity can be extended to nearly zero frequency at magnetostatics, shown in the figure below, with two current coils and an arbitrary object fixed in locations. When one current coil works in transmitting mode, another one works in receiving mode. If we model the two current coils as magnetic dipole moments $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ at locations $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, generating magnetic flux densities $\boldsymbol{B}_1(\boldsymbol{r})$ and $\boldsymbol{B}_2(\boldsymbol{r})$ in transmitting mode, respectively. Then, the ***reciprocity relationship*** can be expressed as

$$\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$$

洛伦兹互易性是电磁学的基本原理，在天线设计理论中具有重要应用。 它指出天线的接收和发射能力是相同的。 另一方面，在外磁场下，使用具有强磁光效应的磁性材料可以破坏互易性。 洛伦兹互易的研究可以扩展到静磁学，如下图所示，有两个电流环和一个固定在某个位置的物体。 当一个电流环在发射模式工作时，另一个电流环在接收模式工作。 如果我们将两个电流环设为位置 $\boldsymbol{r}_1$ 和 $\boldsymbol{r}_2$ 处的磁偶极子 $\boldsymbol{m}_1$ 和 $\boldsymbol{m}_2$，分别在发射模式下产生磁通密度 $\boldsymbol{B}_1(\boldsymbol{r})$ 和 $\boldsymbol{B}_2(\boldsymbol{r})$。 那么，***互易关系***可以表示为

$$\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$$



The magnetic field $\boldsymbol{B}(\boldsymbol{r})$ generated by a magnetic dipole $\boldsymbol{m}_i$ located at $\boldsymbol{r}_i$ is given by,

在位置 $\boldsymbol{r}_i$ 处的磁偶极子 $\boldsymbol{m}_i$ 所产生的磁场 $\boldsymbol{B}_i(\boldsymbol{r})$ 由下式给出，

$$\boldsymbol{B}_i(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\boldsymbol{r}-\boldsymbol{r}_i)(\boldsymbol{r}-\boldsymbol{r}_i)\cdot \boldsymbol{m}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^5} - \frac{\boldsymbol{m}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}\right)$$

In this problem, we will first establish the reciprocity relationship and will investigate how it can be broken by imposing a constant velocity on the object.

本题中，我们将首先建立互易关系，并研究如何通过对物体施加恒定速度来打破它。

A. ESTABLISHING MAGNETOSTATIC RECIPROCITY AND NON-RECIPROCITY 建立静磁互易性和非互易性

For magnetostatics, we have vector potential $\boldsymbol{A}$, magnetic flux density $\boldsymbol{B}$, magnetic field strength $\boldsymbol{H}$ and impressed current density $\boldsymbol{J}$. These fields satisfy the Ampère's law

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$

with material response

$$\nabla \times \boldsymbol{A} = \boldsymbol{B} = \mu \boldsymbol{H}$$

where $\mu(\boldsymbol{r})$ is the isotropic magnetic permeability profile for the material, i.e. the object in the figure.

对于静磁学，我们有矢量势 $\boldsymbol{A}$、磁通密度 $\boldsymbol{B}$、磁场强度 $\boldsymbol{H}$ 和外加电流密度 $\boldsymbol{J}$。这些场满足安培定律

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$

关于物质反应的表达式为

$$\nabla \times \boldsymbol{A} = \boldsymbol{B} = \mu \boldsymbol{H}$$

其中 $\mu(\boldsymbol{r})$ 是材料的各向同性磁导率分布，即图中的物体。

We want to establish the reciprocity relationship for magnetostatics when we have a magnetic dipole moment $\boldsymbol{m}_1$ at location $\boldsymbol{r}_1$ in one case and a magnetic dipole $\boldsymbol{m}_2$ at $\boldsymbol{r}_2$ in another case. The two dipoles generate magnetic fields $\boldsymbol{B}_1(\boldsymbol{r})$ and $\boldsymbol{B}_2(\boldsymbol{r})$, respectively. The two cases, labeled by $i = 1,2$, have the current density $\boldsymbol{J}_i = \nabla \times \boldsymbol{M}_i$ and magnetization $\boldsymbol{M}_i = \boldsymbol{m}_i \delta(\boldsymbol{r} - \boldsymbol{r}_i)$ for the two dipoles.

在一种情况下$\boldsymbol{r}_1$处有一个磁偶极子 $\boldsymbol{m}_1$并产生磁场 $\boldsymbol{B}_1(\boldsymbol{r})$而在另一种情况下$\boldsymbol{r}_2$处有一个磁偶极子 $\boldsymbol{m}_2$并产生磁场 $\boldsymbol{B}_2(\boldsymbol{r})$。对于由 $i = 1,2$ 索引的两个偶极子，我们有电流密度 $\boldsymbol{J}_i = \nabla \times \boldsymbol{M}_i$和磁化强度 $\boldsymbol{M}_i = \boldsymbol{m}_i \delta(\boldsymbol{r} - \boldsymbol{r}_i)$。以下我们想要建立静磁学的互易关系。

Here, we also give some formulas for these differential operators:

在这里，我们提供一些关于微分算子的公式：

$$\begin{aligned}
&\nabla \times \boldsymbol{A} = \hat{x}(\partial_y A_z - \partial_z A_y) + \hat{y}(\partial_z A_x - \partial_x A_z) + \hat{z}(\partial_x A_y - \partial_y A_x) \\
&\nabla \cdot \boldsymbol{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z \\
&\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{B} - \boldsymbol{A} \cdot \nabla \times \boldsymbol{B} \\
&\int \nabla \cdot \boldsymbol{A} dV = \int \boldsymbol{A} \cdot d\boldsymbol{a} \\
&\int \nabla \times \boldsymbol{A} dV = -\int \boldsymbol{A} \times d\boldsymbol{a} \\
&\int \boldsymbol{B} \cdot \nabla \times \boldsymbol{A} dV = \int \boldsymbol{A} \cdot \nabla \times \boldsymbol{B} dV + \int \boldsymbol{A} \times \boldsymbol{B} \cdot d\boldsymbol{a}
\end{aligned}$$

and Kronecker delta function $\delta(\boldsymbol{r})$ is defined by

$$\delta(\boldsymbol{r}) = \begin{cases} \infty & \text{if } \boldsymbol{r} = \boldsymbol{0} \\ 0 & \text{otherwise} \end{cases}$$

which satisfies $\int \delta(\boldsymbol{r}) dV = 1$ when we integrate a volume V enclosing the origin. $\boldsymbol{a}$ is defined as the closed area enclosing volume V.

当我们对包含原点的一个体积 V 进行积分时，Kronecker delta 函数 $\delta(\boldsymbol{r})$ 定義為:

$$\delta(\boldsymbol{r}) = \begin{cases} \infty & \text{if } \boldsymbol{r} = \boldsymbol{0} \\ 0 & \text{除此以外} \end{cases}$$

而且满足 $\int \delta(\boldsymbol{r}) dV = 1$。 $\boldsymbol{a}$ 定义为包围体积 V 的封闭区域。

A1. PROVING THE RECIPROCITY RELATIONSHIP 证明互易关系

A1	Prove the reciprocity relationship $\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$. Hint: you may consider $\nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2)$. 证明互易关系 $\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$。 提示：你可以考虑 $\nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2)$。	3 points 3 分

Solution:

In magnetostatics, we have

$$\nabla \times \boldsymbol{A} = \boldsymbol{B} = \mu \boldsymbol{H}$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$

where the free currents are from impressed current coils which can be described as magnetization $\boldsymbol{M}$ through

$$\boldsymbol{J} = \nabla \times \boldsymbol{M}$$

For two different sets of fields and currents labeled by subscript 1 and 2 (with same medium μ), we have

$$\begin{aligned}&\nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2 - \boldsymbol{H}_2 \times \boldsymbol{A}_1)\\ &= \boldsymbol{J}_1 \cdot \boldsymbol{A}_2 - \boldsymbol{H}_1 \cdot \boldsymbol{B}_2 - \boldsymbol{J}_2 \cdot \boldsymbol{A}_1 + \boldsymbol{H}_2 \cdot \boldsymbol{B}_1\\ &= \boldsymbol{J}_1 \cdot \boldsymbol{A}_2 - \boldsymbol{J}_2 \cdot \boldsymbol{A}_1 - \mu \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 + \mu \boldsymbol{H}_2 \cdot \boldsymbol{H}_1\end{aligned}$$

$$\Rightarrow \int \nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2 - \boldsymbol{H}_2 \times \boldsymbol{A}_1) dV = \int \boldsymbol{J}_1 \cdot \boldsymbol{A}_2 dV - \int \boldsymbol{J}_2 \cdot \boldsymbol{A}_1 dV$$

There are only close-loop currents, i.e. made of magnetic dipoles. For each magnetic dipole, we have

$$\boldsymbol{B}_{\mathrm{i}}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\boldsymbol{r}-\boldsymbol{r}_i)(\boldsymbol{r}-\boldsymbol{r}_i)\cdot \boldsymbol{m}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^5} - \frac{\boldsymbol{m}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}\right)$$

As integration volume grows to infinity, the integral $\int \nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2 - \boldsymbol{H}_2 \times \boldsymbol{A}_1) dV$ (with Gauss's law) decays to zero by

$$|\boldsymbol{H}| \propto 1/r^3, \qquad |\boldsymbol{A}| \propto 1/r^2, \qquad |\boldsymbol{H} \times \boldsymbol{A}| \propto 1/r^5$$

Then

$$\int \boldsymbol{J}_1 \cdot \boldsymbol{A}_2 dV = \int \boldsymbol{J}_2 \cdot \boldsymbol{A}_1 dV$$

By substituting $\boldsymbol{J} = \nabla \times \boldsymbol{M}$, we have

$$\int \boldsymbol{M}_1 \cdot \boldsymbol{B}_2 dV - \int \boldsymbol{M}_2 \cdot \boldsymbol{B}_1 dV = -\int \boldsymbol{M}_1 \times \boldsymbol{A}_2 \cdot d\boldsymbol{a} + \int \boldsymbol{M}_2 \times \boldsymbol{A}_1 \cdot d\boldsymbol{a}$$

Localized sources imply the right hand side goes to zero. Finally, from two point dipoles $\boldsymbol{m}_1$ at $\boldsymbol{r}_1$ and $\boldsymbol{m}_2$ at $\boldsymbol{r}_2$ for the two magnetization $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, we arrive

$$\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$$

If the material conducts electricity with an electrical conductivity σ, we have to add an additional term to the current density $\boldsymbol{J}$ due to the free current through

$$\boldsymbol{J} \to \boldsymbol{J} + \sigma \boldsymbol{E}$$

in the Ampère's law stated previously. Suppose now we move the conductor by a constant velocity $\boldsymbol{v}$. There will be a Lorentz force on the free charges proportional to $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$. It further updates the additional term in the current density through

$$\boldsymbol{J} \to \boldsymbol{J} + \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}),$$

which may upset the reciprocity relationship.

如果材料以电导率 σ 导电，由于通过的自由电流，我们必须为在此前安培定律中的电流密度 $\boldsymbol{J}$ 添加一个附加项

$$\boldsymbol{J} \to \boldsymbol{J} + \sigma \boldsymbol{E}.$$

假设现在我们以恒定速度 $\boldsymbol{v}$ 移动导体。自由电荷上将存在与 $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$成比例的洛伦兹力，这进一步更新电流密度中的附加项以使破坏互易关系变的可能：

$$\boldsymbol{J} \to \boldsymbol{J} + \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}).$$

A2. BREAKING RECIPROCITY RELATIONSHIP 打破互易关系

A2	Express the possibly non-zero $\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) - \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$ as a volume integral in terms of the vector potentials $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ and the conductor velocity $\boldsymbol{v}$. 将可能非零的 $\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) - \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$表示为根据矢量势$\boldsymbol{A}_1$ 和$\boldsymbol{A}_2$ 以及导体速度$\boldsymbol{v}$ 的体积积分。	3 points 3 分

Solution:

The Ampère's law becomes

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Then, the additional part to $\nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2 - \boldsymbol{H}_2 \times \boldsymbol{A}_1)$ is

$$\sigma(\boldsymbol{E}_1 \cdot \boldsymbol{A}_2 - \boldsymbol{E}_2 \cdot \boldsymbol{A}_1) + \sigma(\boldsymbol{v} \times \boldsymbol{B}_1 \cdot \boldsymbol{A}_2 - \boldsymbol{v} \times \boldsymbol{B}_2 \cdot \boldsymbol{A}_1)$$

The volume integration of the first term goes to zero for $\nabla \times \boldsymbol{E}_i = \boldsymbol{0}$ and decay of the boundary terms as r approaching infinity.

The second term is

$$\begin{aligned} &\sigma \boldsymbol{v} \cdot (\boldsymbol{B}_1 \times \boldsymbol{A}_2 - \boldsymbol{B}_2 \times \boldsymbol{A}_1) \\ &= \sigma \boldsymbol{v} \cdot \big((\nabla \times \boldsymbol{A}_1) \times \boldsymbol{A}_2 - (\nabla \times \boldsymbol{A}_2) \times \boldsymbol{A}_1\big) \end{aligned}$$

Therefore

$$\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) - \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2) = \int \sigma \boldsymbol{v} \cdot \big(\boldsymbol{A}_2 \times (\nabla \times \boldsymbol{A}_1) - \boldsymbol{A}_1 \times (\nabla \times \boldsymbol{A}_2)\big) dV$$

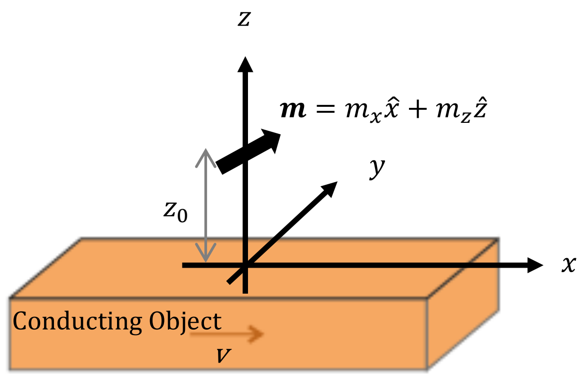
You can complete part B and C without part A.

你可以在没有 A 部分的情况下完成 B 部分和 C 部分。

B. MAGNETIC DIPOLES ON A MOVING PERFECT CONDUCTOR 运动理想导体上的磁偶极子

In this part, we first obtain the magnetic field of a single magnetic dipole on a moving conductor, which is a perfect conductor (i.e. the electrical conductivity $\sigma = \infty$) here, occupying $z \leq 0$ and is moving with a velocity v in the positive *x*-direction. This is defined as the laboratory frame, as shown in the figure below. The magnetic dipole, situated at $(x, y, z) = (0,0,z_0)$ on top of the conductor, has a magnetic moment $\boldsymbol{m} = m_x \hat{x} + m_z \hat{z}$ with zero component in the y-direction.

在这一部分中，我们首先得到运动导体上单个磁偶极子的磁场，这里是理想导体 (即电导率 $\sigma = \infty$)，占据 z≤0 并且在正 *x* 方向上以速度 v 运动。 这被定义为实验室坐标系，如下图所示。 位于导体顶部 $(x, y, z) = (0,0,z_0)$ 处的磁偶极子写为 $\boldsymbol{m} = m_x \hat{x} + m_z \hat{z}$，在 *y* 方向上没有分量。



In the moving frame at a velocity $\boldsymbol{v} = v\hat{x}$ with respect to the laboratory frame, the object is simply a perfect conductor at rest, giving us a convenience to find the magnetic fields generated by the magnetic dipole. The coordinates in the moving frame, denoted as (x', y', z', t'), is transformed from the coordinates in the laboratory frame (x, y, z, t) through the Lorentz transformation:

在相对于实验室坐标系以速度 $\boldsymbol{v} = v\hat{x}$ 运动的坐标系中，物体只是一个静止的理想导体，使我们更方便地找到磁偶极子产生的磁场。 移动坐标系中的坐标，表示为 (x', y', z', t')，由实验室坐标系(x, y, z, t) 中的坐标通过洛伦兹变换转换而来：

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \qquad ct' = \gamma(ct - vx/c)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ and c is the speed of light.

其中 $\gamma = 1/\sqrt{1 - v^2/c^2}$，$c$ 是光速。

The magnetic and electric fields ($\boldsymbol{B}'$ and $\boldsymbol{E}'$) in the moving frame are related to the fields ($\boldsymbol{B}$ and $\boldsymbol{E}$) in the laboratory frame by the Lorentz transformation,

移动坐标系中的磁场和电场（$\boldsymbol{B}'$和 $\boldsymbol{E}'$）与实验室坐标系中的场（$\boldsymbol{B}$ 和 $\boldsymbol{E}$）相关，可由洛伦兹变换给出:

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} B_x \\ \gamma B_y \\ \gamma B_z \end{pmatrix} + \frac{v\gamma}{c^2}\begin{pmatrix} 0 \\ E_z \\ -E_y \end{pmatrix}, \qquad \begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} E_x \\ \gamma E_y \\ \gamma E_z \end{pmatrix} - v\gamma\begin{pmatrix} 0 \\ B_z \\ -B_y \end{pmatrix}$$

We further simplify the problem by removing the conductor at the moment.

我们现在通过移除导体来进一步简化问题。

B1. MAGNETIC FIELD OF A MOVING MAGNETIC DIPOLE IN FREESPACE 自由空间中运动磁偶极子的磁场

B1	What is the magnetic field $\boldsymbol{B}'(x', y', z', t')$ from the dipole for an observer moving at velocity v with respect to the laboratory frame? In this part, we only consider a magnetic dipole $\boldsymbol{m} = m_x\hat{x}$ pointing in the x-direction and there is no conductor below the dipole yet. Please express your answer in the coordinates of the moving frame. 请找出相对于实验室坐标系以速度v 移动的观察者的偶极子磁场 $\boldsymbol{B}'(x', y', z', t')$。 在这一部分中，我们只考虑指向 x 方向的磁偶极子$\boldsymbol{m} = m_x\hat{x}$ 并且偶极子下方还没有导体。 请用移动坐标系的坐标表达你的答案。	3 point 3 分

Solution:

The B-field for a magnetic dipole $\boldsymbol{m} = m_x\hat{x}$ held at $z_0\hat{z}$ is

$$\frac{\mu_0}{4\pi}\left(\frac{3(\boldsymbol{r} - z_0\hat{z})(\boldsymbol{r} - z_0\hat{z}) \cdot \boldsymbol{m}}{|\boldsymbol{r} - z_0\hat{z}|^5} - \frac{\boldsymbol{m}}{|\boldsymbol{r} - z_0\hat{z}|^3}\right)$$

For m_x dipole, we have

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{\mu_0 m_x}{4\pi(x^2+y^2+(z-z_0)^2)^{5/2}} \begin{pmatrix} 2x^2-y^2-(z-z_0)^2 \\ 3xy \\ 3x(z-z_0) \end{pmatrix}$$

In the moving frame, the fields transform to

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} B_x \\ \gamma B_y \\ \gamma B_z \end{pmatrix}$$
$$= \frac{\mu_0 m_x}{4\pi(\gamma^2(x'+vt')^2+y'^2+(z'-z_0)^2)^{5/2}} \begin{pmatrix} 2\gamma^2(x'+vt')^2-y'^2-(z'-z_0)^2 \\ 3\gamma^2(x'+vt')y' \\ 3\gamma^2(x'+vt')(z'-z_0) \end{pmatrix}$$

where

$$x=\gamma(x'+vt'), y=y', z=z', ct=\gamma(ct'+vx'/c)$$

with $\gamma=1/\sqrt{1-v^2/c^2}$. We also have $x'=-vt'+x/\gamma$. Full marks if the answer is given in non-relativistic limit $\gamma\to 1$.

Next, we introduce the perfect conductor below the dipole. The perfect metal has a planar interface at $z=0$ while the dipole is placed at a distance z_0 above the metal. The metal is being moved at a velocity v in the positive x-direction.

接下来，我们引入偶极子下方的理想导体。 理想导体在 z=0 处有一个平面界面，而偶极子位于金属上方 z_0 处。 金属在正 x 方向上以速度 v 移动。

B2. MAGNETIC FIELD OF A MAGNETIC DIPOLE ON A MOVING CONDUCTOR (I) 运动导体上磁偶极子的磁场（1）

B2	Please express the magnetic field of the magnetic dipole $\boldsymbol{m}=m_x\hat{x}$ on top of a moving perfect conductor (i.e $\boldsymbol{B}(x,y,0^+)$) in terms of the laboratory frame coordinate. Please also verify the boundary condition of the magnetic field in the moving frame at which the conductor at rest. Hint: adopt the method of images and assume both electric and magnetic fields inside a perfect conductor is zero at a nearly-zero frequency. 请用实验室坐标系表示在移动中的理想导体上磁偶极子$\boldsymbol{m}=m_x\hat{x}$ 的磁场(即 $\boldsymbol{B}(x,y,0^+)$)。 还请验证导体静止时移动坐标系中的磁场边界条件。提示：采用镜像法。假定在接近零频率时完美导体里电场和磁场为零。	4 points 4 分

Solution:

We adopt the method of images by having a magnetic dipole m_x at mirror location at $-z_0\hat{z}$. In the moving frame with velocity v with respect to the laboratory frame, the conductor is at rest, and we can apply the normal boundary condition $B'_z=0$ on the surface of the perfect conductor.

In the laboratory frame (dipole at rest), the total field is

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{\mu_0 m_x}{4\pi}\frac{1}{(x^2+y^2+(z-z_0)^2)^{5/2}} \begin{pmatrix} 2x^2-y^2-(z-z_0)^2 \\ 3xy \\ 3x(z-z_0) \end{pmatrix} + \frac{\mu_0 m_x}{4\pi}\frac{1}{(x^2+y^2+(z+z_0)^2)^{5/2}} \begin{pmatrix} 2x^2-y^2-(z+z_0)^2 \\ 3xy \\ 3x(z+z_0) \end{pmatrix}$$

for $z\geq 0$.

The total field in the moving frame is

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} B_x \\ \gamma B_y \\ \gamma B_z \end{pmatrix} = \frac{\mu_0 m_x}{4\pi} \frac{1}{(\gamma^2(x'+vt')^2 + y'^2 + (z'-z_0)^2)^{5/2}} \begin{pmatrix} 2\gamma^2(x'+vt')^2 - y'^2 - (z'-z_0)^2 \\ 3\gamma^2(x'+vt')y' \\ 3\gamma^2(x'+vt')(z'-z_0) \end{pmatrix} + \frac{\mu_0 m_x}{4\pi} \frac{1}{(\gamma^2(x'+vt')^2 + y'^2 + (z'+z_0)^2)^{5/2}} \begin{pmatrix} 2\gamma^2(x'+vt')^2 - y'^2 - (z'+z_0)^2 \\ 3\gamma^2(x'+vt')y' \\ 3\gamma^2(x'+vt')(z'+z_0) \end{pmatrix}$$

for $z' \geq 0$.

At $z' = 0^+$, we have

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \frac{\mu_0 m_x}{2\pi} \frac{1}{(\gamma^2(x'+vt')^2 + y'^2 + z_0^2)^{5/2}} \begin{pmatrix} 2\gamma^2(x'+vt')^2 - y'^2 - z_0^2 \\ 3\gamma^2(x'+vt')y' \\ 0 \end{pmatrix}$$

satisfying boundary condition on surface of conductor $B'_z = 0$ in justifying the solution from the method of image.

The magnetic field in the laboratory frame (for the case of no electric fields in the laboratory frame) is

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} B'_x \\ B'_y/\gamma \\ B'_z/\gamma \end{pmatrix} = \frac{\mu_0 m_x}{2\pi} \frac{1}{(x^2 + y^2 + z_0^2)^{5/2}} \begin{pmatrix} (2x^2 - y^2 - z_0^2)/\gamma \\ 3xy \\ 0 \end{pmatrix}$$

Full marks are given if answers are expressed in either laboratory frame or moving frame coordinates.

For a perfect metal, the mirror dipole of (m_x, m_y, m_z) is $(m_x, m_y, -m_z)$ by definition. However, it is required to verify the boundary condition in the moving frame as stated.

B3. MAGNETIC FIELD OF A MAGNETIC DIPOLE ON A MOVING CONDUCTOR (2) 运动导体上磁偶极子的磁场（2）

B3	What is the magnetic field from the magnetic dipole on top of a moving perfect conductor (i.e $\boldsymbol{B}(x, y, 0^+)$) in the laboratory frame if the magnetic dipole above the moving conductor is changed to $\boldsymbol{m} = m_z\hat{z}$, pointing in the positive z direction? 如果运动导体上方的磁偶极子转为$\boldsymbol{m} = m_z\hat{z}$，指向正 z 方向，那么请找出实验室坐标系中磁偶极子在移动中的理想导体上的磁场(即 $\boldsymbol{B}(x, y, 0^+)$)？	3 points 3 分

Solution:

For a m_z dipole, we have the magnetic field in the laboratory frame as

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{\mu_0 m_z}{4\pi(x^2 + y^2 + (z-z_0)^2)^{5/2}} \begin{pmatrix} 3x(z-z_0) \\ 3y(z-z_0) \\ -x^2 - y^2 + 2(z-z_0)^2 \end{pmatrix}$$

Putting a mirror dipole $-m_z$ results the total field as

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{\mu_0 m_z}{4\pi(x^2 + y^2 + (z-z_0)^2)^{5/2}} \begin{pmatrix} 3x(z-z_0) \\ 3y(z-z_0) \\ -x^2 - y^2 + 2(z-z_0)^2 \end{pmatrix} - \frac{\mu_0 m_z}{4\pi(x^2 + y^2 + (z+z_0)^2)^{5/2}} \begin{pmatrix} 3x(z+z_0) \\ 3y(z+z_0) \\ -x^2 - y^2 + 2(z+z_0)^2 \end{pmatrix}$$

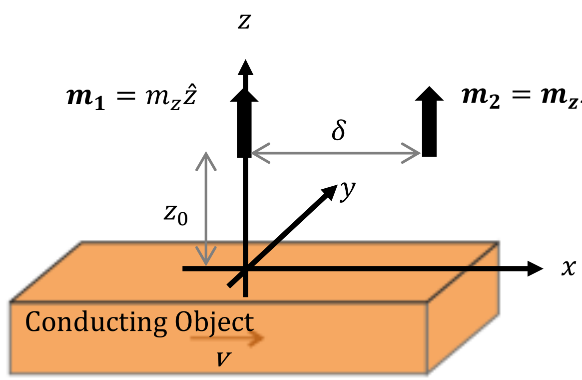
At $z = 0$, we have

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = -\frac{3z_0\mu_0 m_z}{2\pi(x^2+y^2+z_0^2)^{5/2}}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

In principle, the fields are transformed to the moving frame at which the boundary condition is verified, as we have done in part B.2, but we have skipped here for brevity.

Now, we consider the two situations: one with magnetic dipole moment $\boldsymbol{m}_1$ at $\boldsymbol{r}_1 = (0,0,z_0)$ and another magnetic dipole moment $\boldsymbol{m}_2$ at $\boldsymbol{r}_2 = (\delta, 0, z_0)$. When one of them is turned on, another one is turned off.

现在，我们考虑两种情况：一种是 $\boldsymbol{r}_1 = (0,0,z_0)$ 处的磁偶极矩 $\boldsymbol{m}_1$，另一种是 $\boldsymbol{r}_2 = (\delta, 0, z_0)$ 处的磁偶极矩 $\boldsymbol{m}_2$。当其中一个打开时，另一个被关闭。



Now, we define the two problems to solve next. For the *first* problem, suppose the two magnetic dipoles moments $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ point in the z-direction with same size m_z, as shown in the above figure. Dipole moment $\boldsymbol{m}_1$ imposes a magnetic field of z-component $B_z(\delta)$ on $\boldsymbol{r}_2 = (x_2, 0, z_0)$ and dipole $\boldsymbol{m}_2$ imposes magnetic field $B_z(-\delta)$ on $\boldsymbol{r}_1 = (x_1, 0, z_0)$ according to the same function $B_z(x - x_i)$. We then define a reciprocity figure-of-merit

$$\mathcal{R} = \frac{B_z(\delta) - B_z(-\delta)}{B_z(\delta) + B_z(-\delta)}.$$

When $\mathcal{R} = 0$, reciprocity is satisfied. Reciprocity is broken when $\mathcal{R}$ deviates from value zero.

现在，我们定义接下来要解决的两个问题。对于第一个问题，假设两个磁偶极子$\boldsymbol{m}_1$和$\boldsymbol{m}_2$都指向 z 方向，大小m_z相同，如上图所示。偶极子 $\boldsymbol{m}_1$ 根据函数 $B_z(x - x_i)$ 将 z 分量 $B_z(\delta)$ 的磁场施加到 $\boldsymbol{r}_2 = (x_2, 0, z_0)$上，偶极子 $\boldsymbol{m}_2$ 将磁场 $B_z(-\delta)$ 施加到 $\boldsymbol{r}_1 = (x_1, 0, z_0)$ 上。然后我们定义互易功值

$$\mathcal{R} = \frac{B_z(\delta) - B_z(-\delta)}{B_z(\delta) + B_z(-\delta)}.$$

当$\mathcal{R} = 0$时，满足互易性。当 $\mathcal{R}$ 偏离零值时，互易性被打破。

For the *second* problem, suppose we change the pointing direction for $\boldsymbol{m}_1$ to the positive x-direction with magnitude remaining the same. The magnitude and direction of $\boldsymbol{m}_2$ are not changed. In this case, the reciprocity merit is defined as $\mathcal{R} = \big(B_{1z}(\delta) - B_{2x}(-\delta)\big)/\big(B_{1z}(\delta) + B_{2x}(-\delta)\big)$.

对于第二个问题，假设我们将 $\boldsymbol{m}_1$的指向更改为正 x 方向而大小保持不变。$\boldsymbol{m}_2$方向和大小不变。在这种情况下，互易功值定义为 $\mathcal{R} = \big(B_{1z}(\delta) - B_{2x}(-\delta)\big)/\big(B_{1z}(\delta) + B_{2x}(-\delta)\big)$。

B4. RECIPROCITY MERIT FOR DIPOLES ON A PERFECT CONDUCTOR 理想导体上偶极子的互易功值

B4	Find the reciprocity merit $\mathcal{R}$ for the two defined problems about magnetic dipoles on the moving perfect conductor. 求出关于以上两个问题运动理想导体上磁偶极子的互易功值$\mathcal{R}$。	3 points 3 分

Solution:

For the first case, the magnetic field from dipole 1 is

$$B_z(x,0,z_0) = -\frac{\mu_0 m_z}{4\pi|x|^5}x^2 - \frac{\mu_0 m_z}{4\pi(x^2+4z_0^2)^{5/2}}(-x^2+8z_0^2)$$

which is even in x. Then, we have $\mathcal{R}=0$.

For the second case, the magnetic field from dipole 1 on dipole 2 is

$$B_{1z}(\delta) = \frac{\mu_0 m_x}{4\pi}\frac{6xz_0}{(x^2+4z_0^2)^{5/2}}$$

The magnetic field from dipole 2 on dipole 1 is

$$B_{2x}(-\delta) = -\frac{\mu_0 m_z}{4\pi}\frac{6xz_0}{(x^2+4z_0^2)^{5/2}}$$

with $m_z = m_x$.

They have the same values and hence $\mathcal{R}=0$.

In fact, for a moving perfect metal, although breaking the time-reversal symmetry, is still not able to generate the non-reciprocal coupling. We need dissipation with a finite conductivity.

C. MAGNETIC DIPOLES ON A MOVING CONDUCTOR WITH FINITE CONDUCTIVITY 具有有限电导率的移动导体上的磁偶极子

In this part, we move to a more realistic situation that the conductor is a metal. It has a large but finite conductivity σ (in unit of $\Omega^{-1}m^{-1}$), deviating from the perfect conductor condition. Current density in the conductor is given by $\boldsymbol{J}=\sigma\boldsymbol{E}$. We also assume that the current on the conductor surface is confined by a skin depth d of small thickness so that the electric and magnetic fields cannot penetrate beyond the skin depth from the conductor surface. Then, the surface current density can be written as $(\sigma d)\boldsymbol{E}$. We further take the approximation that d is just a constant. We only consider the two dipoles pointing in the positive z-direction with same size m_z in this part.

在这一部分中，我们转向一个更现实的情况，即金属导体具有较大但有限的电导率 σ（单位为 $\Omega^{-1}m^{-1}$），偏离理想导体条件。 导体中的电流密度由 $\boldsymbol{J}=\sigma\boldsymbol{E}$ 给出。 我们还假设电流只在导体表面厚度较小的趋肤深度 d 内流动，电场和磁场不能从导体表面穿透超过趋肤深度。 因此表面电流密度可以写成 $(\sigma d)\boldsymbol{E}$。我们进一步假定 d 为一个常数。 在这部分我们只考虑两个偶极子都指向正 z 方向，大小 m_z相同。

Again, we need to solve the magnetic field from only one dipole at $(0,0,z_0)$ first. In fact, the surface current profile generated on the surface of conductor cannot be easily solved without adopting a numerical solver. Instead, we can approach the problem by extending the method of image as an approximation. In this case, we would like to have a point-like multipolar source at the image position $(0,0,-z_0)$ in order to give as closely as possible the same reflected field generated from the surface current. For the current case of $\boldsymbol{m}=m_z\hat{z}$, we put an image magnetic dipole with given form of magnetic moment $m_x^{(r)}\hat{x}+m_z^{(r)}\hat{z}$ and electric moment $p_y^{(r)}\hat{y}$ at the same location $(0,0,-z_0)$. The mirrored magnetic dipole is now relaxed to have both magnetic and electric components while neglecting the higher order multipoles. The size of these dipole moments are yet to be determined.

同样，我们需要先求解 $(0,0,z_0)$ 处只有一个偶极子的磁场。 实际上，导体表面产生的表面电流分布不采用数值求解器是无法精确求解的。 这里，我们尝试通过电像法来近似解决这个问题。 在这种情况下，我们希望在镜像位置 $(0,0,-z_0)$ 处有一个点状多极源，以便尽可能地提供相同的从表面电流生成的反射场。 对于当前 $\boldsymbol{m}=m_z\hat{z}$ 的情况，我们将镜像磁偶极子设置在$(0,0,-z_0)$ 处，它拥有给定的表达式：磁矩为 $m_x^{(r)}\hat{x}+m_z^{(r)}\hat{z}$ 、电矩为 $p_y^{(r)}\hat{y}$。 镜像磁偶极子现在同时具有磁和电分量，同时忽略高阶多极子。 这些偶极矩的大小有待确定。

C1. GENERALIZED METHOD OF IMAGES 广义电像法

C1	Find the magnetic field $\boldsymbol{B}'(x',y',0^+)$ and electric field $\boldsymbol{E}'(x',y',0^+)$ on the conductor surface in the moving frame. Express your answer in m_z, $m_x^{(r)}$, $m_z^{(r)}$ and $p_y^{(r)}$. You can use either the moving frame or laboratory frame coordinates. Do not need to solve the mirrored dipole moments yet. 求移动坐标系导体表面的磁场$\boldsymbol{B}'(x',y',0^+)$ 和电场$\boldsymbol{E}'(x',y',0^+)$。 请用$m_z$、$m_x^{(r)}$、$m_z^{(r)}$ 和$p_y^{(r)}$ 表达你的答案。可以使用移动坐标或实验室坐标表达你的答案。 暂时不需要求解镜像偶极矩。	5 points 5 分

Solution:

The magnetic field comes from the magnetic dipole m_z at $(0,0,z_0)$, magnetic dipole $m_x^{(r)}\hat{x}+m_z^{(r)}\hat{z}$ at $(0,0,-z_0)$ and a small contribution from electric dipole $p_y^{(r)}$ at $(0,0,-z_0)$ due to the change of reference frame.

The dipole fields on metal surface in the laboratory frame are

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{\mu_0 m_z}{4\pi(x^2+y^2+(z-z_0)^2)^{5/2}}\begin{pmatrix} 3x(z-z_0) \\ 3y(z-z_0) \\ -x^2-y^2+2(z-z_0)^2 \end{pmatrix} + \frac{\mu_0 m_z^{(r)}}{4\pi(x^2+y^2+(z+z_0)^2)^{5/2}}\begin{pmatrix} 3x(z+z_0) \\ 3y(z+z_0) \\ -x^2-y^2+2(z+z_0)^2 \end{pmatrix}$$
$$+\frac{\mu_0 m_x^{(r)}}{4\pi(x^2+y^2+(z+z_0)^2)^{5/2}}\begin{pmatrix} 2x^2-y^2-(z+z_0)^2 \\ 3xy \\ 3x(z+z_0) \end{pmatrix}$$

at $z=0$, giving

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{\mu_0}{4\pi(x^2+y^2+z_0^2)^{5/2}}\begin{pmatrix} 3xz_0\left(m_z^{(r)}-m_z\right)+(2x^2-y^2-z_0^2)m_x^{(r)} \\ 3yz_0\left(m_z^{(r)}-m_z\right)+3xym_x^{(r)} \\ (2z_0^2-x^2-y^2)\left(m_z^{(r)}+m_z\right)+3xz_0m_x^{(r)} \end{pmatrix}$$

We also have

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \frac{\mu_0 c^2 p_y^{(r)}}{4\pi(x^2+y^2+z_0^2)^{5/2}}\begin{pmatrix} 3xy \\ -x^2+2y^2-z_0^2 \\ 3yz_0 \end{pmatrix}$$

In the moving frame, the fields are transformed by

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} B_x \\ \gamma B_y \\ \gamma B_z \end{pmatrix} + \frac{v\gamma}{c^2}\begin{pmatrix} 0 \\ E_z \\ -E_y \end{pmatrix}$$
$$\begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} E_x \\ \gamma E_y \\ \gamma E_z \end{pmatrix} - v\gamma\begin{pmatrix} 0 \\ B_z \\ -B_y \end{pmatrix}$$

Then,

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} / \frac{\mu_0}{4\pi(x^2+y^2+z_0^2)^{5/2}} = \begin{pmatrix} 3xz_0\left(m_z^{(r)} - m_z\right) + (2x^2 - y^2 - z_0^2)m_x^{(r)} \\ 3yz_0\gamma\left(m_z^{(r)} - m_z + v\gamma p_y^{(r)}\right) + 3xym_x^{(r)} \\ (2z_0^2 - x^2 - y^2)\gamma\left(m_z^{(r)} + m_z\right) + 3xz_0 m_x^{(r)} - (-x^2 + 2y^2 - z_0^2)v\gamma^2 p_y^{(r)} \end{pmatrix}$$

$$\begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} / \frac{\mu_0}{4\pi(x^2+y^2+z_0^2)^{5/2}} = \begin{pmatrix} 3xyc^2 p_y^{(r)} \\ (-x^2 + 2y^2 - z_0^2)\gamma c^2 p_y^{(r)} - v\gamma(2z_0^2 - x^2 - y^2)\left(m_z^{(r)} + m_z\right) - 3xz_0 v\gamma m_x^{(r)} \\ 3y\gamma\left(z_0 c^2 p_y^{(r)} + z_0 v\left(m_z^{(r)} - m_z\right) + xvm_x^{(r)}\right) \end{pmatrix}$$

Full marks will be given for fields expressed either in lab frame or moving frame coordinates.

C2. FINDING THE MIRRORED DIPOLE MOMENTS 计算镜像偶极矩

C2	Find $m_x^{(r)}$, $m_z^{(r)}$ and $p_y^{(r)}$ in response to a given m_z. The mirror dipole gives the same reflected field generated by the surface current on the conductor. As approximation, only apply the boundary condition (in the moving frame) on the surface current along the y-direction, which is the dominant current than the one along the x-direction. It may be useful to express the answers in term of the dimensionless parameter $\kappa = \mu_0 v\gamma\sigma d \gg 1$. 用给定的$m_z$表示 $m_x^{(r)}$、$m_z^{(r)}$ 和 $p_y^{(r)}$。 镜像偶极子给出了与导体表面电流产生的相同的反射场。 作为近似，请仅将边界条件（在移动坐标系中）应用于沿 y 方向的表面电流，该电流比沿 x 方向的电流占主导地位。可以考虑用无量纲参数$\kappa = \mu_0 v\gamma\sigma d \gg 1$ 表达答案。	5 points 5 分

Solution:

The surface current is given by

$$\boldsymbol{J}'_s = d\,\sigma\,\boldsymbol{E}' = \hat{z} \times \boldsymbol{B}'/\mu_0$$

at $z' = 0$. The second equality comes from Ampère's law with the assumption that the field decays to negligible values beyond skin depth. The dominant current is along the y-direction. We have $B'_x = \mu_0 d\,\sigma\,E'_y$:

$$\begin{aligned} &3xz_0\left(m_z^{(r)} - m_z\right) + (2x^2 - y^2 - z_0^2)m_x^{(r)} \\ &= \kappa\left((-x^2 + 2y^2 - z_0^2)c^2 p_y^{(r)}/v - (2z_0^2 - x^2 - y^2)\left(m_z^{(r)} + m_z\right) - 3xz_0 m_x^{(r)}\right) \end{aligned}$$

where $\kappa = \mu_0 v\gamma\sigma d$. Equating coefficients for the different powers of x and y give

$$2m_x^{(r)} = \kappa\left(m_z^{(r)} + m_z - c^2 p_y^{(r)}/v\right)$$

$$-m_x^{(r)} = \kappa\left(m_z^{(r)} + m_z + 2c^2 p_y^{(r)}/v\right)$$
$$m_z^{(r)} - m_z = -\kappa m_x^{(r)}$$

and hence

$$\frac{m_x^{(r)}}{m_z} = \frac{2\kappa}{1+\kappa^2} \cong \frac{2}{\kappa}$$
$$\frac{m_z^{(r)}}{m_z} = \frac{1-\kappa^2}{1+\kappa^2} \cong -1 + \frac{2}{\kappa^2}$$
$$\frac{p_y^{(r)}}{m_z} = -\frac{v}{c^2}\frac{2}{1+\kappa^2} \cong -\frac{v}{c^2}\frac{2}{\kappa^2}$$

where

$$\kappa = \mu_0 v \gamma \sigma d.$$

We have assumed a large κ limit so that the material deviates a bit from the perfect metal.

C3. RECIPROCITY MERIT FOR DIPOLES ON A CONDUCTOR OF FINITE CONDUCTIVITY
有限电导率导体上偶极子的互易功值

C3	Find the reciprocity merit $\mathcal{R}$ for the two identical dipoles m_z displaced by δ (with $\boldsymbol{r}_1 = (0,0,z_0)$, $\boldsymbol{r}_2 = (\delta, 0, z_0)$)in the x-direction. 找出在 x 方向上位移δ （$\boldsymbol{r}_1 = (0,0,z_0), \boldsymbol{r}_2 = (\delta, 0, z_0)$）的两个相同偶极子 m_z 的互易功价值$\mathcal{R}$。	3 point 3 分

Solution:

The magnetic field has contributions from the original m_z dipole and the mirror magnetic dipole:

$$B_z(x,y,z) = \frac{\mu_0 m_z}{4\pi(x^2+y^2+(z-z_0)^2)^{5/2}}(-x^2-y^2+2(z-z_0)^2) + \frac{\mu_0 m_z^{(r)}}{4\pi(x^2+y^2+(z+z_0)^2)^{5/2}}(-x^2-y^2+2(z+z_0)^2)$$
$$+ \frac{\mu_0 m_x^{(r)}}{4\pi(x^2+y^2+(z+z_0)^2)^{5/2}}\left(3x(z+z_0)\right)$$

$$B_z(\delta,0,z_0)$$
$$= \frac{\mu_0 m_z}{4\pi(\delta^2+4z_0^2)^{5/2}}\left(-\frac{(\delta^2+4z_0^2)^{5/2}}{|\delta|^3} + \frac{m_z^{(r)}}{m_z}(8z_0^2-\delta^2) + \frac{m_x^{(r)}}{m_z}6z_0\delta\right)$$

Then

$$\mathcal{R} = \frac{B_z(\delta,0,z_0) - B_z(-\delta,0,z_0)}{B_z(\delta,0,z_0) + B_z(-\delta,0,z_0)}$$
$$= sign(\delta)\frac{\frac{m_x^{(r)}}{m_z}6|\delta/z_0|^4}{-(4+|\delta/z_0|^2)^{5/2} + \frac{m_z^{(r)}}{m_z}(8-|\delta/z_0|^2)|\delta/z_0|^3}$$

When the material is approaching a perfect metal, we have $m_z^{(r)} \cong -m_z$ to be substituted into the denominator, giving

$$\mathcal{R} \cong -sign(\delta)\frac{m_x^{(r)}}{m_z}\frac{6|\delta/z_0|^4}{(4+|\delta/z_0|^2)^{5/2} + (8-|\delta/z_0|^2)|\delta/z_0|^3}$$

By further substituting last part's answer, we obtain

$$\mathcal{R} \cong -\frac{sign(\delta)}{\kappa}\frac{12|\delta/z_0|^4}{(4+|\delta/z_0|^2)^{5/2} + (8-|\delta/z_0|^2)|\delta/z_0|^3}$$

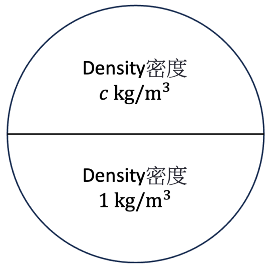
Full marks can be given without the final substitution.

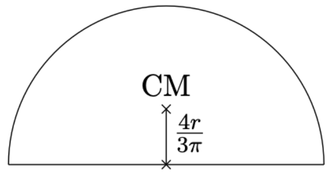
Reference: J. Prat-Camps, P. Maurer, G. Kichmair, and O. Romero-Isart, Phys. Rev. Lett. 121, 213903 (2018).

Pan Pearl River Delta Physics Olympiad 2024
2024 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST. 香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 4 Problems, 40 Points) 简体版卷-1（共4题，40分）(9:30 am – 12:00 pm, 18th Feb 2024)

Please fill in your final answers to all problems on the answer sheet.
请在答题纸上填上各题的最后答案。
At the end of the competition, please submit the answer sheet only. Question papers and working sheets will not be collected.
比赛结束时，请只交回答题纸，题目纸和草稿纸将不会收回。

1. [10 points] We consider an inhomogeneous cylinder whose have are mode of two materials of different densities. The cylinder has radius r and length L and its cross section is shown in Fig. 1a. The bottom half made of a material whose density is 1 kg/m3 while the upper half is mode of a material whose density is c kg/m3, where c is a parameter $0 < c < 1$. In the problem, we can use the fact that for a half-cylinder of radius r, the center of mass (CM) is located at a distance of $\frac{4r}{3\pi}$ from the axis of the half-cylinder, as shown in Fig. 1b.





1. [10 分] 我们考虑一个不均匀的圆柱体，其由两种密度不同的材料构成。圆柱体的半径为 r，长度为L，其横截面如图 1a 所示。下半部分由密度为1 kg/m^3 的材料制成，而上半部分由密度为c kg/m^3 的材料制成，其中c是一个参数，$0 < c < 1$。在问题中，我们可以利用这样一个事实，对于一个半径为 r 的半圆柱体，其质心 (CM)位于半圆柱体轴线距离为 $\frac{4r}{3\pi}$ 的位置，如图 1b 所示。

(ai) [2] Compute the total mass M of the entire inhomogeneous cylinder, and the distance d between the geometrical center and the center of mass of the entire cylinder. Express the answer is terms of r, L, c.
(aii) [2] Compute the moment of inertia I of the entire inhomogeneous cylinder with respect to its geometrical axis. Express the answers in terms of r, L, c.

(b) [2] The geometrical axis of the cylinder is fixed in a horizontal position, but the cylinder is free to rotate without any friction around the axis. If the cylinder oscillates around its stable equilibrium position with small amplitude, calculate the period of oscillation of the cylinder. Express the answer in terms of M, I, r, d and the gravitational acceleration g.

Now we assume that the cylinder is completely free to move on a horizontal table under the gravity. We assume that the coefficient of static friction between the cylinder and the table is infinite, such that the cylinder cannot slide. Suppose that at time $t = 0$, the cylinder is in its equilibrium position with an initial angular velocity ω_0.
(ci) [2] If ω_0 is sufficiently small, the cylinder will undergo a period motion around its stable equilibrium. What is the period of oscillation if the amplitude of the oscillation is small? Express the answer in terms of M, I, r, d, g.
(cii) [2] What is the minimum value of ω_0 that allows the cylinder to roll forever in the same direction. Express the answer in terms of M, I, r, d.

(ai) [2] 计算整个不均匀圆柱体的总质量 M 和几何中心与质心之间的距离 d。用r，L，c 表示答案。
(aii) [2] 计算整个不均匀圆柱体相对于其几何轴的转动惯量 I。用 r，L，c 表示答案。

(b) [2] 圆柱体的几何轴固定在水平位置，但圆柱体围绕几何轴作自由旋转，没有任何摩擦。如果圆柱体在其稳定平衡位置周围振荡，并且振幅很小，计算圆柱体的振荡周期。用M，I，r，d 和重力加速度 g 表示答案。

现在我们假设圆柱体在重力下完全自由地在水平枱面上移动。我们假设圆柱体和枱面之间的静摩擦系数是无限大的，因此圆柱体无法滑动。假设在时刻 $t = 0$ 时，圆柱体处于其平衡位置，并具有初始角速度 ω_0。
(ci) [2] 如果 ω_0 足够小，圆柱体将围绕其稳定平衡进行周期运动。如果振幅很小，振荡周期是多少？用 M，I，r，d，g表示答案。
(cii) [2] 让圆柱体永远向同一方向滚动的最小 ω_0值是多少？用M，I，r，d 表示答案。

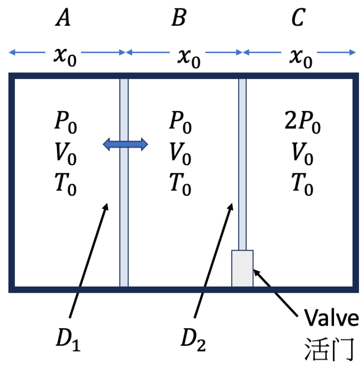
2. [10 points] A closed container is divided into three compartments, A, B, and C, by two partitions, D_1 and D_2, as shown in Figure 3. Each compartment is filled with the same monoatomic ideal gas with pressure P, volume V, and absolute temperature T as shown in the figure. The mass of partition D_1 is m, which can slide freely without friction, while partition D_2 is fixed and has a small valve on it. Now, the valve on partition D_2 is opened, allowing the gases in compartments B and C to mix and the entire system to reach equilibrium while maintaining a constant temperature T_0.
(a) [3] What are the pressures and volumes of the gases in compartments A, B, and C after the entire system reaches equilibrium?
(b) [3] How much total heat is absorbed by the gases in compartments B and C during the process of the entire system reaching equilibrium?
(c) [4] Calculate the change in entropy ΔS of the entire system during the process of reaching equilibrium.

2. [10 分] 一封闭的容器以D_1 和 D_2 两隔板分隔成 A、B、和 C 三室，三室中各充以相同的单原子理想气体，其压强 P、体积 V、和绝对温度 T 分别如图三所示。图中 D_1 隔板的质量为 m，可以无摩擦地自由滑动；D_2 隔板固定，其上设有一小活门。现打开 D_2 隔板上的小活门，使 B 和 C 二室中的气体得以混合，并且让整个系统在维持等温 T_0 的情况下达到平衡状态。

(a) [3] 整个系统在达到平衡状态后，A、B、和 C 三室中的气体压强和体积各为何？

(b) [3] 在整个系统达到平衡状态的过程中，B 和 C 两室中的气体合计吸热多少？

(c) [4] 在整个系统达到平衡状态的过程中，计算整个系统的熵的变化量 ΔS。



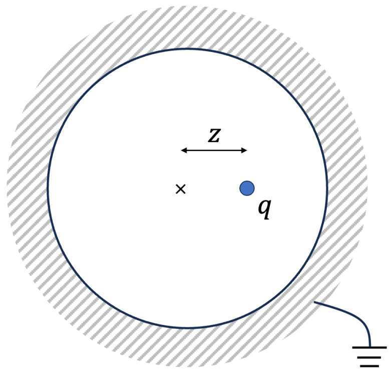
3. [10 points] Trapped Ball

A ball (modelled as a point charge of magnitude $q > 0$) of mass m is trapped in a spherical cavity of radius R carved from an infinite grounded conductor. The charge is at a distance z from the center. In the problem, the gravity can be ignored.

(a) [2] Sketch the electric field lines inside the spherical cavity.

(b) [4] Find the electric force $F(z)$ acting on the ball in terms of q, z and R.

(c) [4] If the ball is released at the center with a very small speed, find the speed of the ball v when it is at a distance $R/2$ from the center of the conductor. Express the answer in terms of q, m, R.



3. [10 分] 困在球内的小球

一颗质量为 m 的小球（假设为一个大小为 $q > 0$ 的点电荷）被困在一个由无限大的接地导体挖空而成的半径为 R 的球形空腔中。该电荷距离中心的距离为z。在这个问题中，可以忽略重力。

(a) [2] 描绘球形空腔内的电场线。

(b) [4] 求作用于小球的电力 $F(z)$，用 q ，z和 R 表示答案。

(c) [4] 如果小球在中心以极小的速度释放，求当小球距导体中心距离为$R/2$ 时的速度v。用 q ，m ，R 表示答案。

4. [10 points] **<u>In this question, all answers cannot be expressed in terms of any trigonometrical functions.</u>**

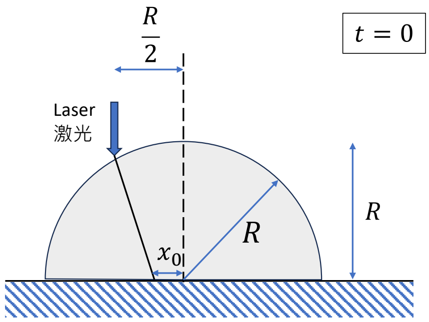
An ice hemisphere with radius R and refractive index n lies on a warm flat table and melts slowly. The rate of heat transfer from the table to the ice is proportional to the area of contact between them. It is known that the ice hemisphere completely melts in time T_0. Throughout the process, a laser beam incident on the ice from above. The beam is vertically incident at a distance of $R/2$ from the axis of symmetry (see figure).

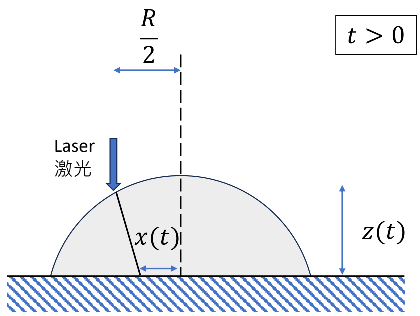
Assume that the temperature of the ice and the surrounding atmosphere are 0°C and remains constant during the melting process. The laser beam does not transfer energy to the ice. The melting water immediately flows off the table, and the ice does not move along the table.

(a) [4] What is the position of the point on the table, $x_0 = x(t = 0)$, where the beam hit at time $t = 0$? Express the answer in terms of n and R.

(b) [3] What is the height of the ice $z(t)$ as a function of time t? Express the answer in terms of R and T_0.

(c) [3] What is the position of the point on the table, $x(t)$, where the beam hit for $t \geq 0$? Express the answer in terms of n, R, T_0 and t.





4. [10 分] <u>在这个问题中，所有答案都不能用任何三角函数来表达。</u>

一个半径 R 和折射率 n 的冰半球位于一个温暖的平面桌子上，并缓慢融化。桌子和冰之间的传热速率与它们的接触面积成正比。已知冰半球在时间 T_0 内完全融化。在整个过程中，激光束从上方照射到冰上。光束从垂直于对称轴距离为 $R/2$ 的位置入射（见图）。

假设冰和周围大气的温度都是0°C，并且在融化过程中保持不变。激光束不会向冰传递能量。溶化了的水立即流到桌子外，冰不会沿桌子移动。

(a) [4] 当$t = 0$ 时，光束击中桌子上的点 $x_0 = x(t = 0)$ 的位置在哪里？用 n 和 R 表示答案。

(b) [3] 冰的高度 $z(t)$ 作为时间 t 的函数是多少？用 R 和 T_0 表示答案。

(c) [3] 对于 $t \geq 0$，光束击中桌子上的点 $x(t)$ 的位置在哪里？用n ，R ，T_0 和 t 表示答案。

~ End of Part 1 卷-1 完 ~

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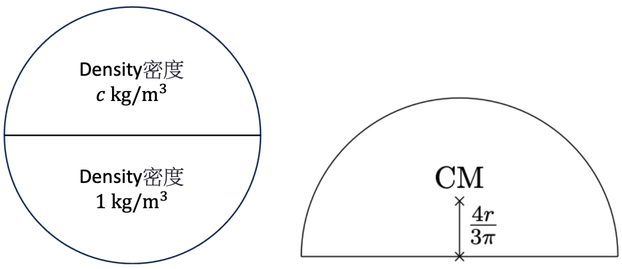

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(ci) [2] If ω_0 is sufficiently small, the cylinder will undergo a period motion around its stable equilibrium. What is the period of oscillation if the amplitude of the oscillation is small? Express the answer in terms of M, I, r, d, g.
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(cii) [2] 让圆柱体永远向同一方向滚动的最小 ω_0值是多少？用M，I，r，d 表示答案。

Solution:
(ai)

$$M = \frac{\pi r^2 L}{2}(1+c)$$

$$d = \frac{\frac{\pi r^2 L}{2}\left(\frac{4r}{3\pi} - \frac{4r}{3\pi}c\right)}{\frac{\pi r^2 L}{2}(1+c)} = \frac{4r}{3\pi}\left(\frac{1-c}{1+c}\right)$$

(aii) The moment of inertia of a cylinder of mass M and radius r is $\frac{1}{2}Mr^2$

$$I = \frac{1}{2}\left(\frac{\pi r^2 L}{2}\right)(1+c)r^2 = \frac{\pi r^4 L}{4}(1+c)$$

(b) Newton's 2nd law gives,

$$\tau = I\ddot{\theta}$$

The torque w.r.t. the geometrical axis is

$$\tau = -Mgd\sin\theta$$
$$I\ddot{\theta} = -Mgd\sin\theta$$
$$\ddot{\theta} = -\frac{Mgd}{I}\sin\theta$$
$$\Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{Mgd}}$$

(ci) We consider the equation of motion w.r.t. the contact point on the ground,

$$\tau = I_{con}\ddot{\theta}$$

Where the torque is the same as in part b,

$$\tau = -Mgd\sin\theta \approx -Mgd\theta$$

The moment of inertia w.r.t. the contact is given by

$$I_{con} = I_{cm} + M(r-d)^2$$
$$I = I_{cm} + Md^2$$
$$\Rightarrow I_{con} = I + M((r-d)^2 - d^2) = I + M(r^2 - 2rd)$$

$$\Rightarrow \ddot{\theta} = -\frac{Mgd}{I_{con}}\theta$$

The period

$$T = 2\pi\sqrt{\frac{I_{con}}{Mgd}} = 2\pi\sqrt{\frac{I + M(r^2-2rd)}{Mgd}}$$

You can see that as $d \to 0$, the period $T \to \infty$ which is expected.

(Method 2):
The Lagrangian is

$$L = T - U = \frac{1}{2}I_{con}\dot{\theta}^2 - \frac{1}{2}Mgd\theta^2$$

The EOM is given by,

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0 \Rightarrow I_{con}\ddot{\theta} + Mgd\theta = 0$$
$$\Rightarrow \ddot{\theta} = -\frac{Mgd}{I_{con}}\theta$$
$$\Rightarrow T = 2\pi\sqrt{\frac{I_{con}}{Mgd}}\,2\pi\sqrt{\frac{I + M(r^2-2rd)}{Mgd}}$$

(cii) Since the cylinder can only roll without sliding, the total energy is conserved. In order to escape from the oscillation, the cylinder must have enough kinetic energy to overcome the gravitational PE when its CM is directly above the geometrical axis.

$$\frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega_0^2 + Mg(r-d) = Mg(r+d)$$

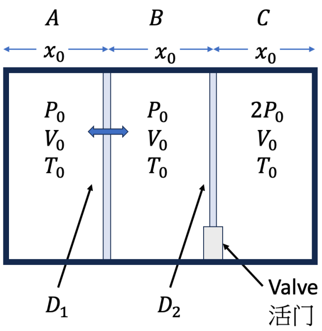
$$\Rightarrow \frac{1}{2}M\omega_0^2(r-d)^2 + \frac{1}{2}(I - Md^2)\omega_0^2 = 2Mgd$$

$$\Rightarrow \omega_0 = \sqrt{\frac{4Mgd}{M(r-d)^2 + (I - Md^2)}}$$

We can see that as $d \to 0, \omega_0 \to 0$. It implies that there is no stable equilibrium in this limit and the cylinder will move forever given any infinitesimal initial push.

2. [10 points] A closed container is divided into three compartments, A, B, and C, by two partitions, D_1 and D_2, as shown in Figure 3. Each compartment is filled with the same monoatomic ideal gas with pressure P, volume V, and absolute temperature T as shown in the figure. The mass of partition D_1 is m, which can slide freely without friction, while partition D_2 is fixed and has a small valve on it. Now, the valve on partition D_2 is opened, allowing the gases in compartments B and C to mix and the entire system to reach equilibrium while maintaining a constant temperature T_0.
(a) [3] What are the pressures and volumes of the gases in compartments A, B, and C after the entire system reaches equilibrium?
(b) [3] How much total heat is absorbed by the gases in compartments B and C during the process of the entire system reaching equilibrium?
(c) [4] Calculate the change in entropy ΔS of the entire system during the process of reaching equilibrium.

一封闭的容器以D_1 和 D_2 两隔板分隔成 A、B、和 C 三室，三室中各充以相同的单原子理想气体，其压强 P、体积 V、和绝对温度 T 分别如图三所示。图中 D_1 隔板的质量为 m，可以无摩擦地自由滑动；D_2 隔板固定，其上设有一小活门。现打开 D_2 隔板上的小活门，使 B 和 C 二室中的气体得以混合，并且让整个系统在维持等温 T_0 的情况下达到平衡状态。
(a) [3] 整个系统在达到平衡状态后，A、B、和 C 三室中的气体压强和体积各为何？
(b) [3] 在整个系统达到平衡状态的过程中，B 和 C 两室中的气体合计吸热多少？
(c) [4] 在整个系统达到平衡状态的过程中，计算整个系统的熵的变化量 ΔS。



Solution:
(a) Let the initial number density of gas in compartment A, B and C be n_A, n_B, n_C, After open the have in D_2 and reach the equilibrium, the densities become n'_A, n'_B, n'_C respectively. In the final state, the pressure in all compartments are p_f and temperature T_0. We have

$$p_f V_A = n'_A R T_0$$
$$p_f V'_B = n'_B R T_0$$
$$p_f V'_C = n'_C R T_0$$

Add 3 equations together,

$$p_f(V'_A + V'_B + V'_C) = 3V_0 p_f = (n'_A + n'_B + n_C{}')RT_0 = (n_A + n_B + n_C)RT_0$$

$$\Rightarrow p_f = \frac{1}{3V_0}(n_A + n_B + n_C)RT_0 = \frac{1}{3V_0}(P_0V_0 + P_0V_0 + 2P_0V_0) = \frac{4}{3}P_0$$

For A, since $n'_A = n_A$,

$$p_f V'_A = n'_A R T_0 = n_A R T_0 \Rightarrow V'_A = \frac{3}{4}V_0$$

$$V'_C = V_0$$

$$V'_B = 3V_0 - V_0 - \frac{3}{4}V_0 = \frac{5}{4}V_0$$

(b) (Method 1) Since the entire system is maintained at the same temperature T_0, we have

$$\Delta Q_A + \Delta Q_{BC} = 0$$

For compartment A, since the internal energy doesn't change,

$$\Delta Q_A = \Delta W = \int_{V_0}^{\frac{3}{4}V_0} p_A dV_A = n_A R T_0 \int_{V_0}^{\frac{3}{4}V_0} \frac{1}{V_A} dV_A = P_0 V_0 \ln\frac{3}{4}$$

$$\Rightarrow \Delta Q_{BC} = -\Delta Q_A = P_0 V_0 \ln\frac{4}{3} \approx 0.288\, P_0 V_0$$

(Method 2) Since the gas is maintained at constant temperature T_0, the internal energy of the (ideal) gas doesn't change.

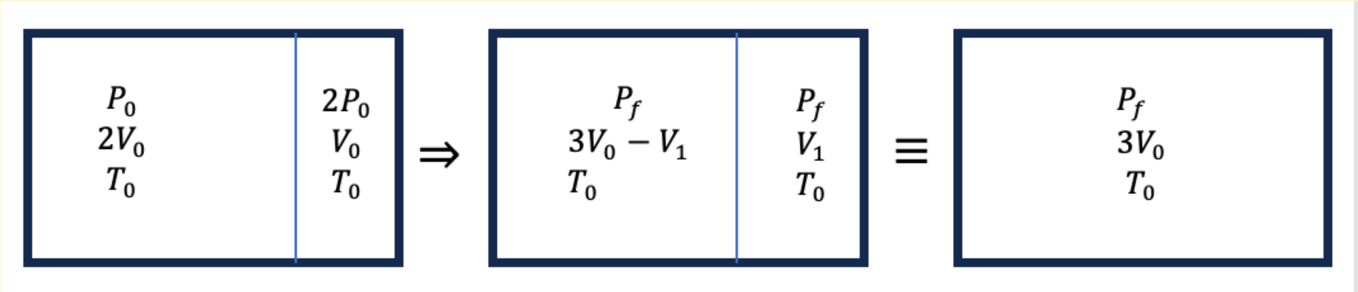
$$\Delta Q = \int_{V_0}^{\frac{5}{4}V_0} p_B dV_B = \int_{V_0}^{\frac{5}{4}V_0} p_A dV_B = \int_{V_0}^{\frac{5}{4}V_0} \frac{N_A k T_0}{2V_0 - V_B} dV_B = N_A k T_0 \ln\frac{4}{3} = P_0 V_0 \ln\frac{4}{3} \approx 0.288\, P_0 V_0$$

(c) Notice that the gas flow between compartment B and C is irreversible therefore we can't simply apply the formula

$$dS = \frac{dQ}{T}$$

(Method 1) One way we can do is to construct a reversible path which connect the initial state to the final state. We imagine there is a partition between A+B and C which can move slowly. The gas in C can expand so that the pressure decreases from $2P_0$ to P_f. We would like to determine V_1 such that the

$$2P_0 V_0 = \frac{4}{3} P_0 V_1 \Rightarrow V_1 = \frac{3}{2} V_0$$



We can imagine there is a partition between A+B and C and move slowly,

$$\Delta S = \frac{dQ}{T_0} = \frac{pdV}{T_0} = \frac{(p_C - p_{AB}) dV_C}{T_0}$$

$$\Rightarrow \Delta S = \int_{V_0}^{\frac{3}{2}V_0} \left(\frac{2Nk}{V_C} - \frac{2Nk}{3V_0 - V_C}\right) dV_C = 2Nk \int_{V_0}^{\frac{3}{2}V_0} \left(\frac{1}{V_C} - \frac{1}{3V_0 - V_C}\right) dV_C = 2Nk\left(\ln\frac{3}{2} + \ln\frac{3}{4}\right) = 2Nk \ln\frac{9}{8} = 2\frac{P_0 V_0}{T_0}\ln\frac{9}{8}$$
$$\approx 0.236 \frac{P_0 V_0}{T_0}$$

(Method 2) To get the entropy of BC, we apply the Sackur-Tetrode equation for the ideal gas,

$$S = Nk\left[\ln\left(\frac{V}{N}\left(\frac{4\pi m U}{3Nh^2}\right)^{3/2}\right) + \frac{5}{2}\right] = Nk\left[\ln\left(\frac{V}{N}\left(\frac{2\pi m k T_0}{h^2}\right)^{3/2}\right) + \frac{5}{2}\right] = Nk\left(\ln\frac{V T_0^{\frac{3}{2}}}{N} + c\right) = -Nk \ln P + C$$

when T_0 is constant.

Initial entropy

$$S_i = -2\frac{P_0 V_0}{T_0} \ln P_0 - \frac{2P_0 V_0}{T_0} \ln 2P_0$$

Final state:

$$S_f = -4Nk \ln P_f = -4Nk \ln\frac{4}{3} P_0$$

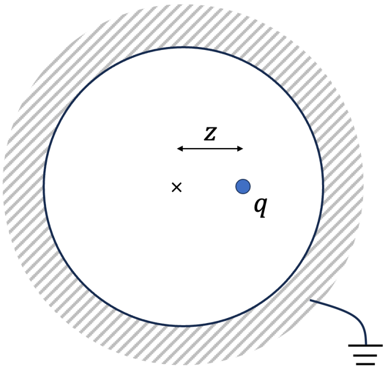
$$\Delta S = S_f - S_i = -\frac{4P_0 V_0}{T_0} \ln\frac{4}{3} P_0 + 2\frac{P_0 V_0}{T_0} \ln P_0 + \frac{2P_0 V_0}{T_0} \ln 2P_0 = \frac{P_0 V_0}{T_0}\left(-4\ln\frac{4}{3} P_0 + 2\ln P_0 + 2\ln 2P_0\right)$$
$$= \frac{P_0 V_0}{T_0}\left(-4\ln\frac{4}{3} + 2\ln 2\right) = 2\frac{P_0 V_0}{T_0}\left(\ln\frac{9}{8}\right) \approx 0.236 \frac{P_0 V_0}{T_0}$$

3. [10 points] Trapped Ball
A ball (modelled as a point charge of magnitude $q > 0$) of mass m is trapped in a spherical cavity of radius R carved from an infinite grounded conductor. The charge is at a distance z from the center. In the problem, the gravity can be ignored.

(a) [2] Sketch the electric field lines inside the spherical cavity.
(b) [4] Find the electric force $F(z)$ acting on the ball in terms of q, z and R.
(c) [4] If the ball is released at the center with a very small speed, find the speed of the ball v when it is at a distance $R/2$ from the center of the conductor. Express the answer in terms of q, m, R.

3. [10 分] 困在球内的小球
一颗质量为 m 的小球（假设为一个大小为 $q > 0$ 的点电荷）被困在一个由无限大的接地导体挖空而成的半径为 R 的球形空腔中。该电荷距离中心的距离为z。在这个问题中，可以忽略重力。

(a) [2] 描绘球形空腔内的电场线。
(b) [4] 求作用于小球的电力 $F(z)$，用 q ，z和 R 表示答案。
(c) [4] 如果小球在中心以极小的速度释放，求当小球距导体中心距离为$R/2$ 时的速度v。用 q ，m ，R 表示答案。



Solution:
(a) Inside the conductor, the electric field is zero. Electric field lines should terminate perpendicular to the conductor. Due to accumulation of surface charges on the wall of the cavity, the electric field strength on the 'right hand side' of the charge should be stronger than that on the 'left hand side' and hence the density of electric field lines should be higher. Finally, note that as the charge is positive, the electric field lines should point out from the charge. With these considerations in mind, we have the electric field lines shown in Fig. (a):

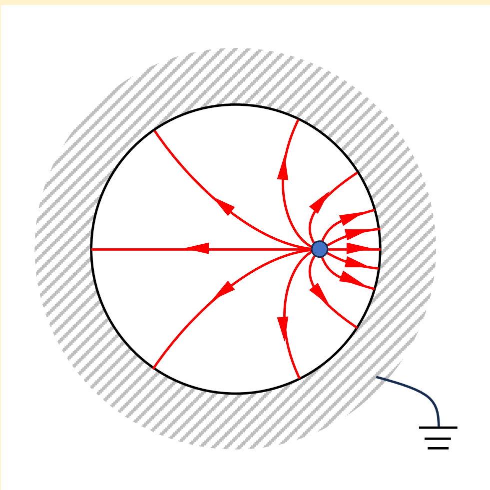


Figure (a)

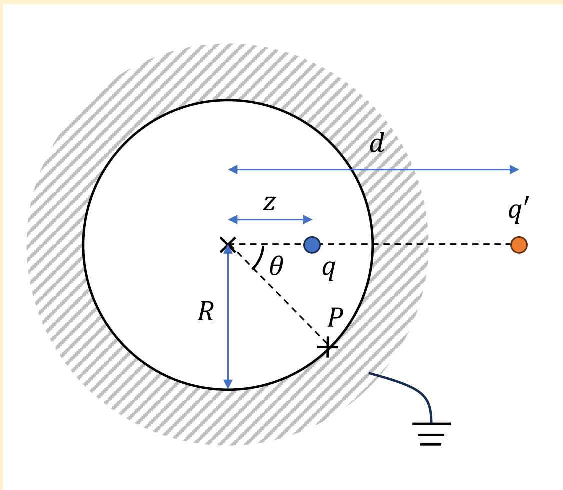


Figure (b)

(b) To solve this problem, we apply the method of image charges, with an image charge of magnitude $q' < 0$ located at d from the center of the spherical cavity as shown in Fig. (b). Consider the potential $\phi_P(\theta)$ at a point P on the wall:

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{z^2+R^2-2zR\cos\theta}}+\frac{q'}{\sqrt{d^2+R^2-2dR\cos\theta}}\right)=\phi_P(\theta)=0$$
$$\Rightarrow \frac{q^2}{z^2+R^2-2zR\cos\theta}=\frac{{q'}^2}{d^2+R^2-2dR\cos\theta}$$
$$\Rightarrow q^2(d^2+R^2)-{q'}^2(z^2+R^2)+2R({q'}^2z-dq^2)\cos\theta=0$$

For this to be satisfied by all θ, we need

$$\begin{cases} {q'}^2z-dq^2=0 \\ q^2(d^2+R^2)-{q'}^2(z^2+R^2)=0 \end{cases}$$

Solving gives

$$d=\frac{R^2}{z} \quad \text{and} \quad q'=-\frac{qR}{z}$$

This is the almost the same work we did when we solved the image charge problem for a charge outside a grounded conducting sphere. This gives our force (attractive) to be

$$|F|=\frac{1}{4\pi\varepsilon_0}\frac{|qq'|}{(d-z)^2}=\frac{1}{4\pi\varepsilon_0}\frac{q^2Rz}{(R^2-z^2)^2}$$

(ci) By the work-energy theorem, the velocity at $z=R$ is given by

$$\frac{1}{2}m(v^2-0^2)=\int_0^{R/2}F(z)\,\mathrm{d}z$$

$$v=\sqrt{\frac{2}{m}\int_0^{R/2}F(z)\,\mathrm{d}z}=\sqrt{\frac{q^2R}{2\pi\epsilon_0 m}\int_0^{R/2}\frac{z}{(R^2-z^2)^2}dz}=\sqrt{\frac{q^2}{2\pi\epsilon_0 mR}\int_0^{1/2}\frac{u}{(1-u^2)^2}du}=\sqrt{\frac{1}{12\pi\varepsilon_0}\frac{q^2}{mR}}$$

4. [10 points] <u>In this question, all answers cannot be expressed in terms of any trigonometrical functions.</u>

An ice hemisphere with radius R and refractive index n lies on a warm flat table and melts slowly. The rate of heat transfer from the table to the ice is proportional to the area of contact between them. It is known that the ice hemisphere completely melts in time T_0. Throughout the process, a laser beam incident on the ice from above. The beam is vertically incident at a distance of $R/2$ from the axis of symmetry (see figure).

Assume that the temperature of the ice and the surrounding atmosphere are 0°C and remains constant during the melting process. The laser beam does not transfer energy to the ice. The melting water immediately flows off the table, and the ice does not move along the table.

(a) [4] What is the position of the point on the table, $x_0=x(t=0)$, where the beam hit at time $t=0$? Express the answer in terms of n and R.

(b) [3] What is the height of the ice $z(t)$ as a function of time t? Express the answer in terms of R and T_0.

(c) [3] What is the position of the point on the table, $x(t)$, where the beam hit for $t\geq 0$? Express the answer in terms of n, R, T_0 and t.

4. [10 分] <u>在这个问题中，所有答案都不能用任何三角函数来表达。</u>

一个半径 R 和折射率 n 的冰半球位于一个温暖的平面桌子上，并缓慢融化。桌子和冰之间的传热速率与它们的接触面积成正比。已知冰半球在时间 T_0 内完全融化。在整个过程中，激光束从上方照射到冰上。光束从垂直于对称轴距离为 $R/2$ 的位置入射（见图）。

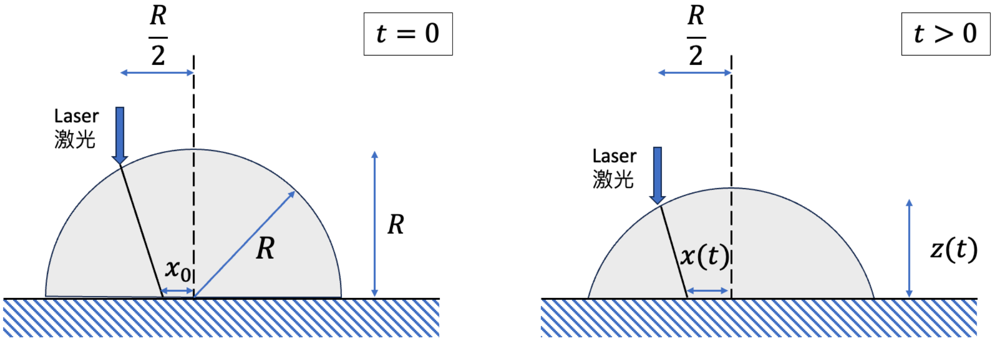
假设冰和周围大气的温度都是$0°C$，并且在融化过程中保持不变。激光束不会向冰传递能量。溶化了的水立即流到桌子外，冰不会沿桌子移动。

(a) [4] 当$t = 0$时，光束击中桌子上的点 $x_0=x(t=0)$ 的位置在哪里？用 n 和 R 表示答案。

(b) [3] 冰的高度 $z(t)$ 作为时间 t 的函数是多少？用 R 和 T_0 表示答案。

(c) [3] 对于 $t\geq 0$，光束击中桌子上的点 $x(t)$ 的位置在哪里？用n ，R ，T_0 和 t 表示答案。

1



Solution:

(a) Let's find the point on the table where the beam hits at the very first moment $t = 0$. Consider Figure 5, it is easy to find that the angle of incidence of the beam on the hemisphere is $\alpha = 30°$, i.e. $\sin\alpha = \frac{1}{2}$.

According to Snell's law of refraction

$$\sin\beta = \frac{\sin\alpha}{n} = \frac{1}{2n} \Rightarrow \tan\beta = \frac{1}{\sqrt{4n^2-1}}$$

From the pink triangle with angle γ in Figure 5 it is easy to find its horizontal leg $\frac{\sqrt{3}R}{2}\tan\gamma$, which means the point of incidence of the beam on the table: it is located at a distance x_0 from the center of hemisphere O:

$$x_0 = \frac{R}{2} - \frac{\sqrt{3}}{2}R\tan\gamma$$

$$\tan\gamma = \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\sqrt{4n^2-1}-\sqrt{3}}{\sqrt{12n^2-3}+1}$$

$$x_0 = \frac{R}{2}\left(1 - \frac{\sqrt{12n^2-3}-3}{\sqrt{12n^2-3}+1}\right) = \frac{2R}{\sqrt{12n^2-3}+1}$$

(b) At time t, the base in contact with the table is a circle with radius r. Within a time interval dt, the amount of heat absorbed is

$$\mathrm{d}Q = K\pi r^2 dt$$

for some constant K. The volume of the ice melt is

$$dQ = Ldm = L\rho\pi r^2 dz$$

Where L and ρ are the latent heat and density of the ice respectively. Therefore we have

$$K\pi r^2 dt = -L\rho\pi r^2 dz$$

$$\Rightarrow \frac{dz}{dt} = -\frac{K}{L\rho} \text{ is a constant}$$

i.e. the melting uniformly reduces the thickness of the ice, we have

$$z(t) = R\left(1 - \frac{t}{T_0}\right)$$

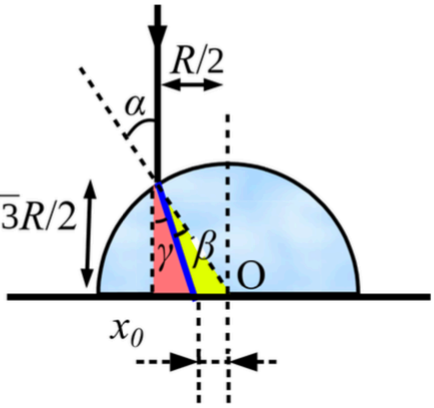
(c) At the end, when the base of the ice has radius $r \le \frac{R}{2}$, the beam will obviously hit the table vertically (i.e. $x(t) = \frac{R}{2}$). It corresponds to the case when the thickness of the ice reduces to $z(t) = \left(1 - \frac{\sqrt{3}}{2}\right)R$. It will happen at the time $t = \frac{\sqrt{3}}{2}T_0$. Therefore, for $t > \frac{\sqrt{3}}{2}T_0$, we have $x(t) = \frac{R}{2}$

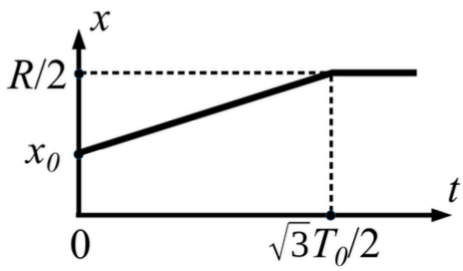
To get $x(t)$ for $t < \frac{\sqrt{3}}{2}T_0$, we can consider the similar triangles.

$$\frac{\frac{R}{2} - x_0}{\frac{R}{2} - x(t)} = \frac{\frac{\sqrt{3}}{2}R}{R\left(\frac{\sqrt{3}}{2} - \frac{t}{T_0}\right)}$$

$$\Rightarrow \left(\frac{R}{2} - x_0\right)\left(\frac{\sqrt{3}}{2} - \frac{t}{T_0}\right) = \frac{\sqrt{3}}{2}\left(\frac{R}{2} - x\right)$$

$$\Rightarrow x(t) = \frac{R}{2} - \left(\frac{R}{2} - x_0\right)\left(1 - \frac{2}{\sqrt{3}}\frac{t}{T_0}\right) = \frac{R}{2} - \frac{R}{2}\left(1 - \frac{4}{\sqrt{12n^2-3}+1}\right)\left(1 - \frac{2}{\sqrt{3}}\frac{t}{T_0}\right)$$





~ **End of Part 1 卷-1 完** ~

Pan Pearl River Delta Physics Olympiad 2024

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Simplified Chinese Part-2 (Total 2 Problems, 60 Points)

简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 18 February 2024)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。

All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。

There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。

Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。

Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。

Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。

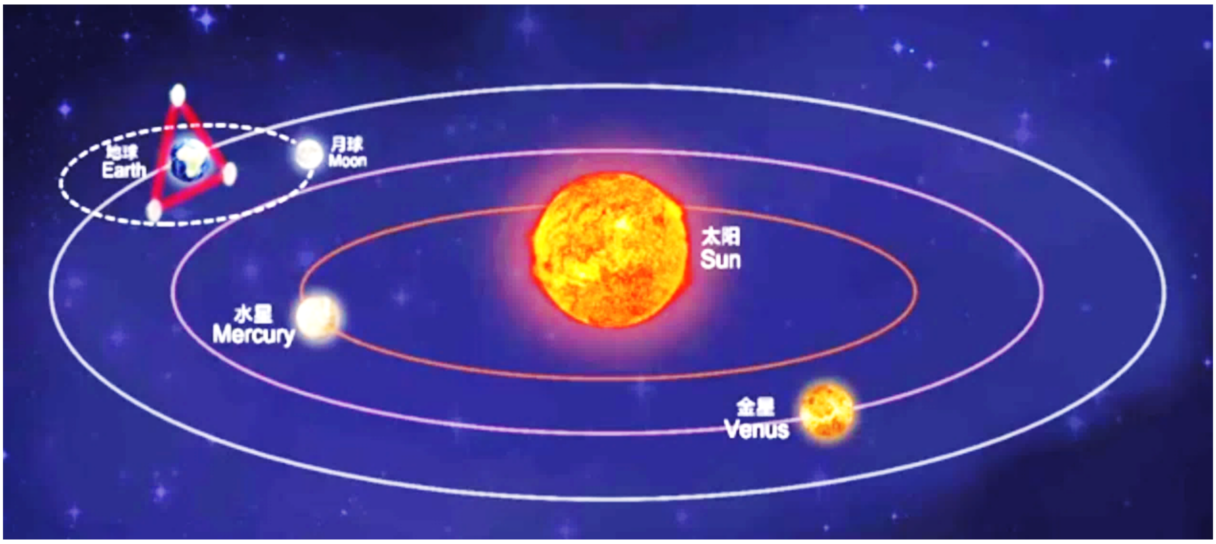
If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。

At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Spatiotemporal varying electric permittivity (30 points)
问题 1: 时空变介电常数 (30 分)

The discovery of gravitational waves initiated an era of gravitational wave astronomy. In addition to the ground-based gravitational wave observatories, gravitational wave observatories based on laser interference between satellites are also planned, for example, the Taiji and Tianqin programs in China and LISA in Europe. Here, we study a simplified version similar to the Tianqin program.

引力波的发现，开启了引力波天文学时代。除了在地面上建设引力波天文台，目前，通过卫星之间激光干涉的空间引力波计划也在筹划之中，例如中国的太极计划、天琴计划，和欧洲的 LISA。这里，我们考虑类似天琴引力波探测计划的一个简化版本。



As illustrated in this figure, we consider three satellites surrounding the earth following circular orbits. They form an equilateral triangle. They form an interferometry in the nearly vacuum environment near the earth. From the change of interference patterns, the change of space distance is measured to detect gravitational waves. Here we will study the error sources for Tianqin to reach its desired measurement precision.

我们考虑如图所示，环绕地球呈等边三角形的三颗卫星按圆轨道运动，在地球周围接近真空的环境中组成激光干涉仪。通过激光的干涉条纹变化，来感知时空距离随时间的变化，探测引力波。本题将讨论，为了达到引力波探测精度，需要考虑的误差来源。

In this problem, we will use the physical constants and satellite parameters including:

在本题中将用到的物理参数和卫星技术参数包括：

Newton's gravitational constant 牛顿万有引力常数 $G = 6.67 \times 10^{-11} \mathrm{m}^3/(\mathrm{kg\ s}^2)$

Planck's constant 普朗克常数 $h = 6.626 \times 10^{-34} \mathrm{m}^2 \mathrm{kg/s}$

Vacuum Permeability 真空磁导率 $\mu_0 = 1.257 \times 10^{-6}\ \mathrm{kg\ m\ s^{-2}\ A^{-2}}$

The mass of the earth 地球质量 $M = 5.97 \times 10^{24}\ \mathrm{kg}$

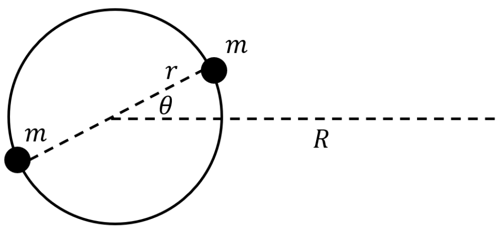
The radius of the earth 地球半径 $r = 6.37 \times 10^6 \mathrm{m}$

The distance from a satellite to the center of the earth 卫星轨道与地心的距离 $R = 10^8\ \mathrm{m}$

The laser wavelength used by the satellite 卫星使用激光波长 $\lambda = 1064\ \mathrm{nm}$

The size of the optical system of the satellite 卫星光学系统尺度 $D = 0.1\ \mathrm{m}$

Part A: Gravitational fluctuations on the orbit of the satellite 卫星轨道上的引力扰动

<table>
<tr><td>A1</td><td>Here we only consider gravity from the earth and consider the earth as a homogeneous ideal ball. Give the periodicity T of a satellite rotating around the earth. Please use second as the unit and give three significant figures.
在仅考虑地球引力，且设地球是均匀理想球体的情况下，求卫星绕地球转动的周期 T，请给出以秒为单位的具体数值，精确到三位有效数字。</td><td>2 Points
2 分</td></tr>
<tr><td>A2</td><td>Since the shape and density of the earth is inhomogeneous, the satellite will feel an additional acceleration δa in addition to the uniform circular motion. To simplify the calculation, let us model the inhomogeneity of the earth as follows: Consider an ideal ball with mass $M-2m$. Two additional point masses (each has mass m) are put to diametrically opposite points on the equator of the earth. Assume that the satellite orbit and the earth are in the same plane, with angle θ between them. Give the precise formula to calculate δa.
A2. 由于地球形状与密度的不均匀性，会对卫星产生除匀速圆周运动之外的额外加速度δa。为简化计算，将地球的不均匀性建模为：质量为 $M-2m$ 的理想球体，其赤道对径点上放置两个额外的质点，每个质点质量为 m。且假设卫星的轨道与地球赤道在同一平面内，与两质点夹角为θ。由此给出 δa 的精确计算公式。

</td><td>2 Points
2 分</td></tr>
<tr><td>A3</td><td>Calculate all the possible periodicities for δa. Please use second as the unit and give three significant figures.
A3. 求 δa 随时间变化所有可能周期的数值，请给出以秒为单位的具体数值，精确到三位有效数字。</td><td>2 Points
2 分</td></tr>
<tr><td>A4</td><td>In the $R \gg r$ limit, give the leading order expression (the lowest nonzero order in the Taylor expansion of r/R) for δa.
求在 $R \gg r$ 极限下，δa 的领头阶（对r/R进行泰勒展开的最低的非零阶）表达式。</td><td>2 Points
2 分</td></tr>
<tr><td>A5</td><td>Estimate the typical value of δa (an error within two orders of magnitude will be considered as correct).
估计 δa 的典型数值（误差在两个量级之内可视为正确）。</td><td>3 Points
3 分</td></tr>
<tr><td>A6</td><td>In satellite experiments, we are interested in the gravitational waves with a particular periodicity (such as periods between 1-1000 seconds). Thus, if the periodicity of the gravitational fluctuation is too long, it will not interfere the gravitational wave measurement. Assume the satellite is co-rotating in the same direction with the spinning direction of the earth. In the Taylor expansion of δa, calculate the component with period closest to 1000s. Denote this component as δa_{1000}. Estimate the value of δa_{1000} / δa for $\theta = \pi/3$. (an error within two orders of magnitude will be considered as correct)
A6. 在卫星实验上，我们感兴趣特定变化周期（例如周期为 1-1000 秒）的引力波信号。所以，变化周期太慢的引力扰动并不对引力波测量造成干扰。设卫星与地球自转方向相同，求 δa 的泰勒展开中周期最接近 1000s 的分量 δa_{1000} 与 δa 的比例 δa_{1000} / δa 。请给出该比例 δa_{1000} / δa 当 $\theta = \pi/3$ 时的数值，误差在两个量级内可视为正确。</td><td>3 Points
3 分</td></tr>
</table>

Part B: Free electrons from the solar wind 太阳风中的自由电子

Consider the laser signal between the satellites. Although the space between satellites is close to the vacuum, but it is not the absolute vacuum. In particular, solar wind will introduce free electrons. Let the number density of the free electrons be N_e, the electric charge of an electron be e，the electron mass be m_e. And we ignore other media apart from these electrons.

考虑卫星之间的激光信号。虽然卫星之间的环境真空度较高，但并不是绝对的真空。特别地，太阳风会带来自由电子。设自由电子的数密度为 N_e，电子电量为e，电子质量为 m_e，且忽略卫星之间除自由电子以外其它的介质。

B1	Assume that the electrons move freely in the electric field produced by the laser. Calculate the acceleration of the electron $d\mathbf{v}_e/dt$ as a function of the electric field $\mathbf{E}$ produced by the laser. 设电子在激光产生的电场中自由运动，求电子的加速度 $d\mathbf{v}_e/dt$ 与激光产生的电场 $\mathbf{E}$ 之间的关系。	1 Points 1 分
B2	Calculate the time dependence of the electric current, $d\mathbf{J}/dt$, from the free electrons. 求自由电子带来的电流随时间的变化 $d\mathbf{J}/dt$。	2 Points 2 分
B3	Calculate the phase speed of the laser v_p in the environment of the free electrons (since v_p is very close to the speed of light, the higher order difference between v_p and the speed of light can be ignored). Hint: from the Maxwell equations, one can derive that $\frac{\partial^2\mathbf{E}}{\partial t^2} - c^2\nabla^2\mathbf{E} + c^2\,\mu_0\frac{d\mathbf{J}}{dt} = 0$. B3. 求激光在自由电子中运动的相速度 v_p（由于 v_p 足够接近光速， v_p 与光速差别的高阶项可以忽略）。 提示：由麦克斯韦方程组，可以推出 $\frac{\partial^2\mathbf{E}}{\partial t^2} - c^2\nabla^2\mathbf{E} + c^2\,\mu_0\frac{d\mathbf{J}}{dt} = 0$.	3 Points 3 分
B4	Let $N_e = 10\ \mathrm{cm}^{-3}$. Calculate the phase error of the laser between two satellites. In other words, if there were no free electrons, the laser waveform arrived at a satellite is $\cos\theta$. Now with free electrons, the same wave form at the same moment changes into $\cos(\theta+\delta\theta)$. Calculate the value of $\delta\theta$. 设 $N_e = 10\ \mathrm{cm}^{-3}$，求两个卫星之间，由自由电子引起的激光相位误差。即假设没有自由电子，到达卫星波形为 $\cos\theta$ ，有自由电子的情况下，在同一时刻到达卫星波形为 $\cos(\theta+\delta\theta)$ ，求 $\delta\theta$ 的数值。	3 Points 3 分

Part C: Shot noise 散粒噪声

Any precision measurements are limited by the uncertainty principle of quantum mechanics. Assume that every photon's arrival time at the detector can be considered as independent stochastic processes. Also, in actual experiments, phase error of the laser is more important. But here for simplicity, here we only estimate photon number errors.

再精确的测量手段，都要受到量子力学的制约。设卫星干涉仪中，激光中每个光子到达探测器的时间都是独立的随机事件。另外，实验中其实更关心激光的相位误差，但是这里我们为简便起见，仅估计光子数误差。

C1	During a certain period of time, the average photon number in the laser is N. In this case, the error in the photon number measurement in the laser is $\Delta N = N^\alpha$. Find α. 某段时间内，激光中平均包含 N 个光子。此时对激光中光子数的测量的误差为 $\Delta N = N^\alpha$。求 α。	1 Points 1 分
C2	If we request that in one second, the relative error of photon number measurement is $\frac{\Delta N}{N} < 3\times10^{-6}$. Calculate the minimal power of laser P_{rec} that the satellite should receive. (3 points) 若要求在一秒钟时间内，光子数测量的相对误差为 $\frac{\Delta N}{N} < 3\times10^{-6}$，求卫星接收到的激光最低功率 P_{rec}。	3 Points 3 分
C3	Assume the laser arrived at a satellite is emitted from the other satellite from the three-satellite system. Estimate: in an ideal case, what is the minimal emission power of laser P_{emit} from the other satellite (can be considered to be correct if the order-of-magnitude is correct). 设到达卫星的激光是由三卫星系统中，另一个卫星上的激光器发射的。估计理想状况下，激光的最低发射功率 P_{emit}（量级正确即可视为正确）。	3 Points 3 分

Problem 2: Metric-modified geodesic and heat conduction (30 points)
问题 2: 度量修正的测地线和热传导

Solving physics, such as wave propagation, geodesics, and thermal conduction, on a curved surface in 3D requires a thorough understanding of metrics and differential geometry. However, there can be significant simplifications for systems with spatial symmetry or by adopting coordinate transformation. In this question, we will go through two problems for physics on a curved surface. The first one is light propagating on a curved surface. Figure 1 (a) shows a circular cone with height 5 mm and a base diameter of $2\rho_0 = 10$ mm, joining to a flat surface. The flat surface has a circular hole of the same diameter so that as a whole, there is only one single surface with the cone part indicating the 'curved space.' The entire surface, including the flat surface and the cone, have a very thin surface so that light can be effectively confined on such a surface. We have assumed the cone is joint smoothly to the flat surface. The second question, being illustrated later, is about steady-state thermal conduction on a hemispherical surface.

在 3D 曲面上解决物理问题，如波传播、测地线和热传导，需要对度量和微分几何有深刻的理解。然而，对于具有空间对称性的系统或采用坐标变换的情况，可以进行显著的简化。在这个问题中，我们将讨论在曲面上解决的两个物理问题。第一个问题是光在曲面上传播。图 1（a）显示了一个高度为 5 mm、底直径为$2\rho_0 = 10$ mm 的圆锥体与一个平面相连接。平面上有一个相同直径的圆孔，以便整体上只有一个单一的表面，圆锥部分表示“曲面”。整个表面，包括平面和圆锥体，都有非常薄的表面，以便光可以有效地限制在这样的表面上。我们假设锥体与平坦表面平稳连接。第二个问题将在后面进行阐述，它涉及到在半球面上的稳态热传导。

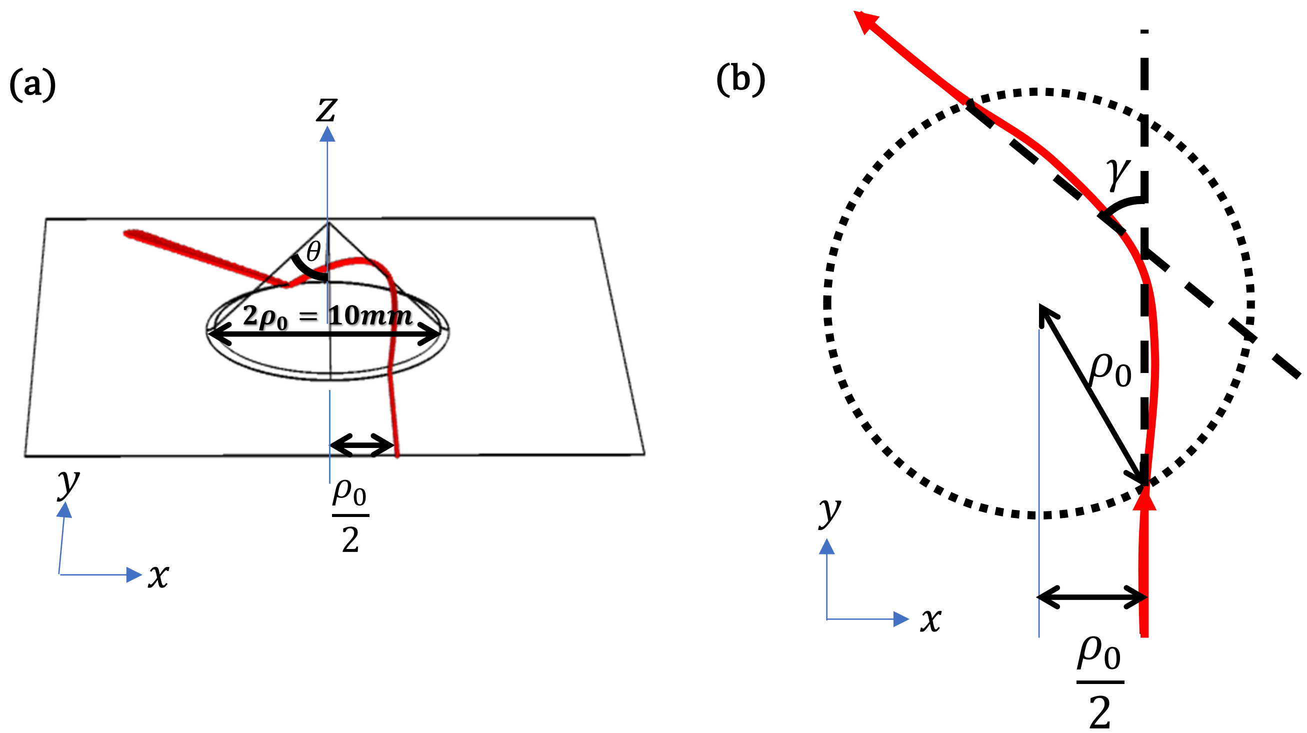


Figure 1(a) depicts a curved surface created by connecting a circular cone to a flat surface with a hole of the same size as the cone's base. In Figure 1(b), we present a top view of this surface. Light confined to such a surface originates on the flat surface at the bottom, undergoes bending due to the cone, and exits in a different direction.

图 1(a)展示了由连接到一个底部与圆锥体底部相同大小的圆孔的平面上的圆锥体所定义的曲面。图 1(b)是该曲面的俯视图。被限制在这样的曲面上的光从底部的平面上开始，因圆锥体的弯曲而改变方向，最终以不同的方向出射。

A. GEODESIC ON A ROTATIONAL SYMMETRIC CURVED SURFACE 旋转对称曲面上的测地线

In mechanics, we are aware that when a system exhibits rotational symmetry, we can simplify the derivation of dynamics by applying the conservation of angular momentum. For instance, we can employ the conservation of angular momentum to derive Kepler's laws. In the current scenario, we consider a normalized angular momentum, denoted as L, which is defined as:

在力学中，我们知道当系统具有旋转对称性时，我们可以使用角动量守恒来简化问题。例如，我们可以使用角动量守恒

来推导开普勒定律。在当前情况下，我们考虑一个归一化的角动量，记为 L，其定义如下：

$$L = \hat{z} \cdot \boldsymbol{\rho} \times \frac{d\boldsymbol{\rho}}{ds} = \rho^2 \frac{d\phi}{ds}.$$

Here, $\boldsymbol{\rho}$ represents the projected position vector on the two-dimensional x-y plane, given by $\boldsymbol{\rho} = x\hat{x} + y\hat{y} = \hat{x}\rho\cos\phi + \hat{y}\rho\sin\phi$, with the projected cone center as the origin. ρ is the magnitude of vector $\boldsymbol{\rho}$ and s is the arc length along the path of light on the surface.

在这里，$\boldsymbol{\rho}$ 代表了在二维 x-y 平面上的投影位置矢量，由$\boldsymbol{\rho} = x\hat{x} + y\hat{y} = \hat{x}\rho\cos\phi + \hat{y}\rho\sin\phi$给出，其中投影锥体中心为原点。$\rho$是矢量$\boldsymbol{\rho}$ 的大小。s 是沿着曲面上光的路径的弧长。

A1	Given that the infinitesimal arc length on the cone satisfies $ds^2 = dx^2 + dy^2 + dz^2$ and $z = z(\rho)$ is the height at that point, prove the geodesic on the cone satisfies 假设圆锥体上的无穷小弧长满足$ds^2 = dx^2 + dy^2 + dz^2$，其中$z = z(\rho)$是该点的高度，请证明圆锥体上的测地线满足 $$\rho'(\phi)^2 = \frac{\sin^2\theta}{L^2}\rho^2(\rho^2 - L^2)$$ Instead of using arc length s to parametrize the geodesic, we have used ϕ for parametrization. 这里，我们使用ϕ来参数化测地线，而不是使用弧长 s。	3 points
A2	For a light starting on the flat surface with a perpendicular distance of $\rho_0/2$ to the origin, what will be minimal ρ the light can go. 对于从平面上距离原点垂直距离为 $\rho_0/2$ 的入射光线，它能够到达的最小 ρ 是多少？	2 points
A3	What is the deflection angle γ by comparing the entering and exit rays on the flat surface? Hint: you may need to use $\int \frac{dx}{x\sqrt{x^2-1}} = \tan^{-1}\sqrt{x^2-1} + c$ 通过比较光线在平面表面的入射和出射角度，计算偏转角度 γ。 提示：你可能需要用 $\int \frac{dx}{x\sqrt{x^2-1}} = \tan^{-1}\sqrt{x^2-1} + c$	7 points

B. HEAT CONDUCTION ON A SPHERICAL SURFACE (I) 球面上的热传导（I）

A usual trick is to search for a coordinate transform from the curved surface (represented by the Cartesian coordinates (x, y, z)) to a 2-dimensional coordinates system $X - Y$ plane so that the physics on the (X, Y) just looks like a flat plane. For a unit spherical surface, such a map is the stereographic projection

一個有用的技巧是寻找一个坐标变换，将曲面（由笛卡尔坐标(x, y, z) 表示）映射到一个二维的 $X - Y$ 坐标上，使得在 $X - Y$ 坐标上的物理现象看起来就像一个平面。对于一个单位球面，这样的映射是立体投影。

$$(X, Y) = \left(\frac{x}{z+1}, \frac{y}{z+1}\right) = (\rho\cos\phi, \rho\sin\phi)$$

Suppose now we consider heat conduction problem on such a spherical surface, i.e. a very thin shell of spherical surface. The steady-state heat conduction has the temperature profile satisfying the Laplace equation

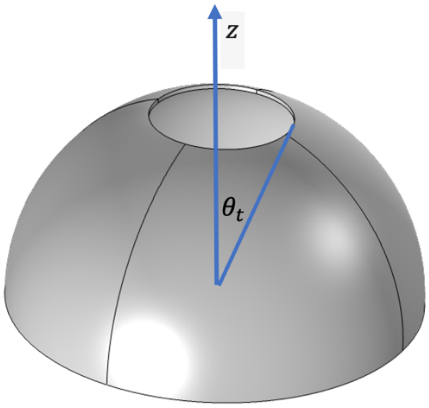
现在我们考虑在这样一个球面上的热传导问题，即一个非常薄的球面壳体。稳态热传导具有满足拉普拉斯方程(Laplace equation)的温度分布。

$$\nabla^2 T(\theta, \phi) = 0$$

while temperature profile is independent of radial distance r in spherical coordinate (r, θ, ϕ). The spherical surface is at radius $r = 1$.

其中温度分布与径向距离 r 无关。这个球面的半径为 $r = 1$。

B1	Prove T satisfies Laplace equation on the (X, Y) coordinate: 证明 T 在 (X, Y) 坐标上满足 Laplace 方程: $\frac{1}{\rho}\partial_\rho(\rho\partial_\rho T) + \frac{1}{\rho^2}\partial_\phi^2 T = 0$ Hint: For convenience, we are given the Laplacian in spherical and cylindrical coordinates as 为了方便起见，我们提供了球坐标和柱坐标下的 Laplacian 方程式。 In spherical coordinate (在球坐标下) (r, θ, ϕ): $\nabla^2 f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\,\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 f$ cylindrical coordinate (在柱坐标下) (ρ, ϕ, z): $\nabla^2 f = \frac{1}{\rho}\partial_\rho(\rho\partial_\rho f) + \frac{1}{\rho^2}\partial_\phi^2 f + \partial_z^2 f$	4 points

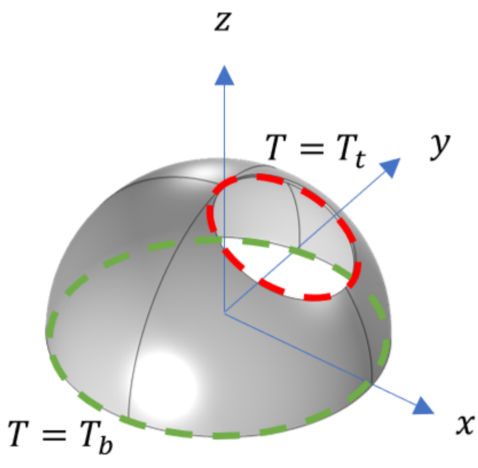


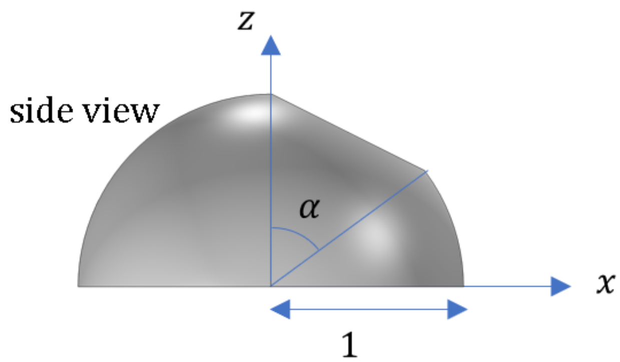
Now, the above figure gives the thin shell in the shape of lamp shade, which is in a hemi-spherical surface with a circular opening at the top. The whole shape still has a rotational symmetry about the vertical z-axis. The bottom of the lamp shade is kept at temperature T_b (at $\theta = \frac{\pi}{2}$ for spherical polar coordinate) and the top is kept at temperature T_t (at $\theta = \theta_t$).

现在，上图给出了一个薄壳，呈灯罩形状，是一个顶部有一个圆形开口的半球面。整个形状仍然具有旋转对称性。灯罩的底部保持在温度 T_b（在球坐标下的 $\theta = \frac{\pi}{2}$ 处），顶部保持在温度 T_t（在$\theta = \theta_t$ 处）。

B2	Solve the temperature profile, as a function of θ, with such rotational symmetry. 解出具有旋转对称性的温度分布 $T(\theta)$，作为 θ 的函数。	4 points

C. HEAT CONDUCTION ON A SPHERICAL SURFACE (II) 球面上的热方程(II)





Now, we consider the top opening is tilted about the y-axis in breaking rotational symmetry. Suppose the top opening is still a circle on the spherical surface passing through $(x, y, z) = (0,0,1)$ and $(\sin\alpha, 0, \cos\alpha)$ as diameter and its normal is on the x-z plane.

现在，我们考虑顶部开口围绕 y 轴倾斜，破坏了旋转对称性。假设球面顶部开口仍然是一个圆，通过由 $(x, y, z) = (0,0,1)$ 和 $(\sin\alpha, 0, \cos\alpha)$ 作为直径，并且其法向量位于 x-z 平面上。

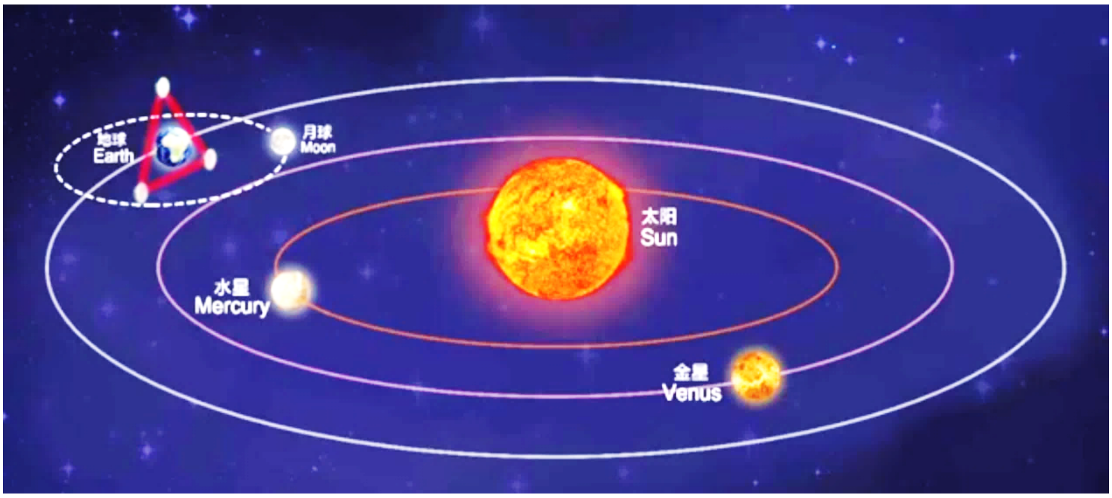
C1	Determine where the top opening is mapped on (X, Y) plane through the stereographic projection map. 通过立体投影映射，确定顶部开口在 (X, Y) 平面上的映射位置。 Hint: the answer is still a circle in X and Y coordinates. 提示：答案在 X 和 Y 坐标上仍然是一个圆。	2 points
C2	Solve the temperature profile $T(X, Y)$ when the bottom opening is kept at temperature T_b and the top is kept at temperature T_t. You can leave your answer in terms of X and Y coordinates. Hint: In the stereographic projected domain X-Y plane, Laplace equation is satisfied and you can further use general method of image, like the one used in solving electrostatic problem by putting two point charges on the X-axis with undetermined charges. 解出温度分布函数 $T(X, Y)$。底部开口保持温度T_b，顶部开口保持温度T_t。您可以用 X 和 Y 坐标表示您的答案。 提示：Laplace 方程仍然满足在立体投影后的 X-Y 平面上。你可以使用在解决静电问题時的镜像电荷法，在 X 轴上放置两个待确定电荷的电荷。	8 points 8 分

~ ENF OF Paper 2 卷二完 ~

Problem 1: Error estimates of a gravitational wave experiment (30 points)

问题 1: 引力波测量中的误差估计 (30 分)

The discovery of gravitational waves initiated an era of gravitational wave astronomy. In addition to the ground-based gravitational wave observatories, gravitational wave observatories based on laser interference between satellites are also planned, for example, the Taiji and Tianqin programs in China and LISA in Europe. Here, we study a simplified version similar to the Tianqin program.
引力波的发现，开启了引力波天文学时代。除了在地面上建设引力波天文台，目前，通过卫星之间激光干涉的空间引力波计划也在筹划之中，例如中国的太极计划、天琴计划，和欧洲的 LISA。这里，我们考虑类似天琴引力波探测计划的一个简化版本。



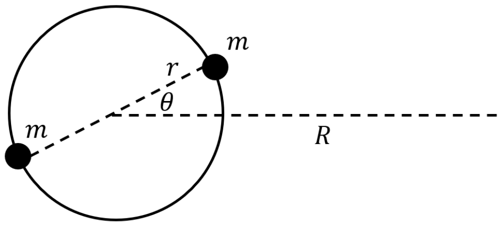
As illustrated in this figure, we consider three satellites surrounding the earth following circular orbits. They form an equilateral triangle. They form an interferometry in the nearly vacuum environment near the earth. From the change of interference patterns, the change of space distance is measured to detect gravitational waves. Here we will study the error sources for Tianqin to reach its desired measurement precision.
我们考虑如图所示，环绕地球呈等边三角形的三颗卫星按圆轨道运动，在地球周围接近真空的环境中组成激光干涉仪。通过激光的干涉条纹变化，来感知时空距离随时间的变化，探测引力波。本题将讨论，为了达到引力波探测精度，需要考虑的误差来源。

In this problem, we will use the physical constants and satellite parameters including:
在本题中将用到的物理参数和卫星技术参数包括：

Newton's gravitational constant 牛顿万有引力常数 $G = 6.67 \times 10^{-11} \mathrm{m}^3/(\mathrm{kg\ s}^2)$
Planck's constant 普朗克常数 $h = 6.626 \times 10^{-34} \mathrm{m}^2 \mathrm{kg/s}$
Vacuum Permeability 真空磁导率 $\mu_0 = 1.257 \times 10^{-6}\ \mathrm{kg\ m\ s^{-2}\ A^{-2}}$
The mass of the earth 地球质量 $M = 5.97 \times 10^{24}\ \mathrm{kg}$
The radius of the earth 地球半径 $r = 6.37 \times 10^6 \mathrm{m}$
The distance from a satellite to the center of the earth 卫星轨道与地心的距离 $R = 10^8\ \mathrm{m}$
The laser wavelength used by the satellite 卫星使用激光波长 $\lambda = 1064\ \mathrm{nm}$
The size of the optical system of the satellite 卫星光学系统尺度 $D = 0.1\ \mathrm{m}$

Part A: Gravitational fluctuations on the orbit of the satellite 卫星轨道上的引力扰动

A1	Here we only consider gravity from the earth and consider the earth as a homogeneous ideal ball. Give the periodicity T of a satellite rotating around the earth. Please use second as the unit and give three significant figures. 在仅考虑地球引力，且设地球是均匀理想球体的情况下，求卫星绕地球转动的周期 T，请给出以	2 Points 2 分

	秒为单位的具体数值，精确到三位有效数字。	
A2	Since the shape and density of the earth is inhomogeneous, the satellite will feel an additional acceleration δa in addition to the uniform circular motion. To simplify the calculation, let us model the inhomogeneity of the earth as follows: Consider an ideal ball with mass $M-2m$. Two additional point masses (each has mass m) are put to diametrically opposite points on the equator of the earth. Assume that the satellite orbit and the earth are in the same plane, with angle θ between them. Give the precise formula to calculate δa. A2. 由于地球形状与密度的不均匀性，会对卫星产生除匀速圆周运动之外的额外加速度δa。为简化计算，将地球的不均匀性建模为：质量为 $M-2m$ 的理想球体，其赤道对径点上放置两个额外的质点，每个质点质量为 m。且假设卫星的轨道与地球赤道在同一平面内，与两质点夹角为θ。由此给出 δa 的精确计算公式。  	2 Points 2 分
A3	Calculate all the possible periodicities for δa. Please use second as the unit and give three significant figures. A3. 求 δa 随时间变化所有可能周期的数值，请给出以秒为单位的具体数值，精确到三位有效数字。	2 Points 2 分
A4	In the $R \gg r$ limit, give the leading order expression (the lowest nonzero order in the Taylor expansion of r/R) for δa. 求在 $R \gg r$ 极限下，δa 的领头阶（对r/R进行泰勒展开的最低的非零阶）表达式。	2 Points 2 分
A5	Estimate the typical value of δa (an error within two orders of magnitude will be considered as correct). 估计 δa 的典型数值（误差在两个量级之内可视为正确）。	3 Points 3 分
A6	In satellite experiments, we are interested in the gravitational waves with a particular periodicity (such as periods between 1-1000 seconds). Thus, if the periodicity of the gravitational fluctuation is too long, it will not interfere the gravitational wave measurement. Assume the satellite is co-rotating in the same direction with the spinning direction of the earth. In the Taylor expansion of δa, calculate the component with period closest to 1000s. Denote this component as δa_{1000}. Estimate the value of δa_{1000} / δa for $\theta = \pi/3$. (an error within two orders of magnitude will be considered as correct) A6. 在卫星实验上，我们感兴趣特定变化周期（例如周期为 1-1000 秒）的引力波信号。所以，变化周期太慢的引力扰动并不对引力波测量造成干扰。设卫星与地球自转方向相同，求 δa 的泰勒展开中周期最接近 1000s 的分量 δa_{1000} 与 δa 的比例 δa_{1000} / δa 。请给出该比例 δa_{1000} / δa 当 $\theta = \pi/3$ 时的数值，误差在两个量级内可视为正确。	3 Points 3 分

A1 Solution:

$\frac{v^2}{R} = \frac{GM}{R^2}, v = \sqrt{\frac{GM}{R}},$

$T = \frac{2\pi R}{v} = 3.15 \times 10^5$s.

A2. Solution：

$$\delta a_x = -\frac{G(M-2m)}{R^2} - \frac{Gm\,(R - r\cos\theta)}{(R^2+r^2-2\,R\,r\cos\theta)^{\frac{3}{2}}} - \frac{Gm\,(R + r\cos\theta)}{(R^2+r^2+2\,R\,r\cos\theta)^{\frac{3}{2}}}$$

$$\delta a_y = \frac{Gm\,r\sin\theta}{(R^2+r^2-2\,R\,r\cos\theta)^{\frac{3}{2}}} - \frac{Gm\,r\sin\theta}{(R^2+r^2+2\,R\,r\cos\theta)^{\frac{3}{2}}}$$

A3. Solution:

Note that the above expression has a symmetry of $\cos\theta \leftrightarrow -\cos\theta$. Thus in θ, the period is π (i.e., 1/2 of the period of the 2π rotation). In time, there are two possibilities: the earth co-rotate or counter-rotate with the satellite. Note that the earth period is a day = $86400s$. Thus, the two possible periods are

$\frac{1/2}{\frac{1}{86400}\pm\frac{1}{3.147\times10^5}} = 3.39 \times 10^4$ s (counterrotate), or 5.95×10^4 s (corotate).

A4: Solution:

Taylor-expand to second order (note that the first order in r/R result cancels):

$\delta a_x \simeq \frac{3Gmr^2(1-3\cos^2\theta)}{R^4}$.

$\delta a_y \simeq \frac{3Gmr^2\sin 2\theta}{R^4}$.

A5: Solution:

There can be many reasonable ways to estimate. For example, one can consider the density difference between the rock and the sea. Thus m can be estimated using

$10^3\mathrm{kg\ m^{-3}}$ × (average depth of the sea $3.5 \times 10^3 m$) × (10% of the earth surface area) $\sim 1.8 \times 10^{20}$. Thus, for a typical value of θ, for which $2 - 8\cos^2\theta$ doesn't cancel to extremely small values,

$\delta a \sim 10^{-9}\ \mathrm{m/s^2}$ (Note: the same order of magnitude can be obtained using precise modeling of the earth. Answers ranging from $10^{-7} \sim 10^{-11}$ can be considered correct.

A6: Solution:

Considering that in the co-rotating case, the period of the system is 119098s. To extract the component with 1000s period, Taylor-expand to the $(\cos\theta)^{120}$ term (note: there is no $(\cos\theta)^{119}$ term. Expanding to $(\cos\theta)^{118}$ is equally fine).
Then for n=120:

Note that the $\frac{3}{2}\left(\frac{3}{2}+1\right)\cdots\left(\frac{3}{2}+(n-1)\right)$ almost cancels the 1/n! in the Taylor expansion (the difference is at the order $\sqrt{n}$, which is within the error allowance).

$$\frac{\delta a_{1000}}{\delta a} = \frac{\frac{2^{n+1}\ R^n r^n \cos^n\theta}{R^{2n+2}}}{\delta a} = 10^{-141}$$

Note: taking n=118 the result is 10^{-139}, also considered correct. Also note that in reality (when we go beyond the current toy model), the higher order multiples will dominate δa_{1000}.

Part B: Free electrons from the solar wind 太阳风中的自由电子

Consider the laser signal between the satellites. Although the space between satellites is close to the vacuum, but it is not the absolute vacuum. In particular, solar wind will introduce free electrons. Let the number density of the free electrons be N_e, the electric charge of an electron be e，the electron mass be m_e. And we ignore other media apart from these electrons.

考虑卫星之间的激光信号。虽然卫星之间的环境真空度较高，但并不是绝对的真空。特别地，太阳风会带来自由电

子。设自由电子的数密度为 N_e，电子电量为e，电子质量为 m_e，且忽略卫星之间除自由电子以外其它的介质。

B1	Assume that the electrons move freely in the electric field produced by the laser. Calculate the acceleration of the electron $d\mathbf{v}_e/dt$ as a function of the electric field $\mathbf{E}$ produced by the laser. 设电子在激光产生的电场中自由运动，求电子的加速度 $d\mathbf{v}_e/dt$ 与激光产生的电场 $\mathbf{E}$ 之间的关系。	1 Points 1 分
B2	Calculate the time dependence of the current from the free electrons. 求自由电子带来的电流随时间的变化 $d\mathbf{J}/dt$。	2 Points 2 分
B3	Calculate the phase speed of the laser v_p in the environment of the free electrons (since v_p is very close to the speed of light, the higher order difference between v_p and the speed of light can be ignored). Hint: from the Maxwell equations, one can derive that $\frac{\partial^2\mathbf{E}}{\partial t^2} - c^2\nabla^2\mathbf{E} + c^2\,\mu_0\frac{d\mathbf{J}}{dt} = 0$. B3. 求激光在自由电子中运动的相速度 v_p（由于 v_p 足够接近光速， v_p 与光速差别的高阶项可以忽略）。 提示：由麦克斯韦方程组，可以推出 $\frac{\partial^2\mathbf{E}}{\partial t^2} - c^2\nabla^2\mathbf{E} + c^2\,\mu_0\frac{d\mathbf{J}}{dt} = 0$.	3 Points 3 分
B4	Let $N_e = 10\ \mathrm{cm}^{-3}$. Calculate the phase error of the laser between two satellites. In other words, if there were no free electrons, the laser waveform arrived at a satellite is $\cos\theta$. Now with free electrons, the same wave form at the same moment changes into $\cos(\theta+\delta\theta)$. Calculate the value of $\delta\theta$. 设 $N_e = 10\ \mathrm{cm}^{-3}$，求两个卫星之间，由自由电子引起的激光相位误差。即假设没有自由电子，到达卫星波形为 $\cos\theta$ ，有自由电子的情况下，在同一时刻到达卫星波形为 $\cos(\theta+\delta\theta)$ ，求 $\delta\theta$ 的数值。	3 Points 3 分

B1: Solution:

$$\frac{d\mathbf{v}_e}{dt} = -e\frac{\mathbf{E}}{m_e}$$

B2: Solution:

$$\frac{d\mathbf{J}}{dt} = -e\,N_e\frac{d\mathbf{v}_e}{dt} = \frac{N_e e^2\mathbf{E}}{m_e}$$

B3: Solution:

From $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \nabla\times\mathbf{B} = \mu_0\,\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$,

$\frac{\partial^2\mathbf{E}}{\partial t^2} - c^2\nabla^2\mathbf{E} + c^2\,\mu_0\frac{d\mathbf{J}}{dt} = 0$ (where $\nabla\times(\nabla\times\mathbf{E}) = \nabla^2\mathbf{E} - \nabla(\nabla\cdot\boldsymbol{E}) \simeq \nabla^2\mathbf{E}$)

It's solution in frequency space is

$\omega^2 = c^2k^2 + c^2\mu_0\frac{N_e e^2}{m_e}$.

Thus, the phase velocity

$v_p = \frac{\omega}{k} = \sqrt{c^2 + c^2\mu_0\frac{N_e e^2}{m_e k^2}} \simeq \sqrt{c^2 + c^2\mu_0\frac{N_e e^2\lambda^2}{4\pi^2\,m_e}} \simeq c\left(1 + \frac{\mu_0 N_e e^2\lambda^2}{8\pi^2\,m_e}\right)$

B4: Solution:

The wave can be written as $\cos(kx - \omega t + \theta)$, where θ is the initial phase at the emitter $t = 0, x = 0$. The receiver has a distance $x = \sqrt{3}\,R$ and the arrival time is $t = \sqrt{3}\,R/c$.

With free electrons, $k = \frac{\omega}{c}\left(1 - \frac{\mu_0 N_e e^2\lambda^2}{8\pi^2\,m_e}\right)$. Thus inserting x and t,

$$\delta\theta = -\frac{\omega}{c} \times \frac{\mu_0 N_e e^2 \lambda^2}{8\pi^2 m_e} \times \sqrt{3}\, R = -\frac{\mu_0 N_e e^2 \lambda}{4\pi\, m_e} \times \sqrt{3}\, R = 5.19 \times 10^{-6}$$

Part C: Shot noise 散粒噪声

Any precision measurements are limited by the uncertainty principle of quantum mechanics. Assume that every photon's arrival time at the detector can be considered as independent stochastic processes. Also, in actual experiments, phase error of the laser is more important. But here for simplicity, here we only estimate photon number errors.

再精确的测量手段，都要受到量子力学的制约。设卫星干涉仪中，激光中每个光子到达探测器的时间都是独立的随机事件。另外，实验中其实更关心激光的相位误差，但是这里我们为简便起见，仅估计光子数误差。

C1	During a certain period of time, the average photon number in the laser is N. In this case, the error in the photon number measurement in the laser is $\Delta N = N^{\alpha}$. Find α. C1. 某段时间内，激光中平均包含 N 个光子。此时对激光中光子数的测量的误差为 $\Delta N = N^{\alpha}$。求 α。	1 Points 1 分
C2	If we request that in one second, the relative error of photon number measurement is $\frac{\Delta N}{N} < 3 \times 10^{-6}$. Calculate the minimal power of laser P_{rec} that the satellite should receive. (3 points) C2. 若要求在一秒钟时间内，光子数测量的相对误差为 $\frac{\Delta N}{N} < 3 \times 10^{-6}$，求卫星接收到的激光最低功率 P_{rec}。	3 Points 3 分
C3	Assume the laser arrived at a satellite is emitted from the other satellite from the three-satellite system. Estimate: in an ideal case, what is the minimal emission power of laser P_{emit} from the other satellite (can be considered to be correct if the order-of-magnitude is correct). 设到达卫星的激光是由三卫星系统中，另一个卫星上的激光器发射的。估计理想状况下，激光的最低发射功率 P_{emit}（量级正确即可视为正确）。	3 Points 3 分

C1: Solution：$\alpha = 1/2$.

C2: Solution:

We thus need the minimal number of photon $N = 1.11 \times 10^{11}$ per second.

The energy of each photon is $E = h\,\nu = h\frac{c}{\lambda} = 1.87 \times 10^{-19} J$

Thus the minimal power at reception is $P_{\text{rec}} = 2 \times 10^{-8}\text{W}$.

C3: Solution 1: Using Gaussian laser beam and the formula

$$\frac{P_{\text{rec}}}{P_{\text{emit}}} = \left(\frac{\pi\, D^2}{4\, R\, \lambda}\right)^2 = 5.45 \times 10^{-9}\,.$$

Thus, $P_{\text{emit}} \sim 3.67\text{W}$

Solution 2: Estimation from the uncertainty principle

Within the satellite optics system diameter D, for each photon, the momentum uncertainty of the laser is $\Delta p \sim h/D$. Thus the minimal spread angle is λ/D. Over a distance R, the minimal radius of the spot is $\lambda\, R/D$. Considering the receiver radius is at most D, too. Thus,

$$\frac{P_{\text{rec}}}{P_{\text{emit}}} \sim \left(\frac{D}{R\frac{\lambda}{D}}\right)^2 \sim 8.83 \times 10^{-9}\,.$$

Thus, $P_{\text{emit}} \sim 2.26\text{W}$

Problem 2: Metric-modified geodesic and heat conduction (30 points)
问题 2: 度量修正的测地线和热传导

Solving physics, such as wave propagation, geodesics, and thermal conduction, on a curved surface in 3D requires a thorough understanding of metrics and differential geometry. However, there can be significant simplifications for systems with spatial symmetry or by adopting coordinate transformation. In this question, we will go through two problems for physics on a curved surface. The first one is light propagating on a curved surface. Figure 1 (a) shows a circular cone with height 5 mm and a base diameter of $2\rho_0 = 10$ mm, joining to a flat surface. The flat surface has a circular hole of the same diameter so that as a whole, there is only one single surface with the cone part indicating the 'curved space.' The entire surface, including the flat surface and the cone, have a very thin surface so that light can be effectively confined on such a surface. We have assumed the cone is joint smoothly to the flat surface. The second question, being illustrated later, is about steady-state thermal conduction on a hemispherical surface.

在 3D 曲面上解决物理问题，如波传播、测地线和热传导，需要对度量和微分几何有深刻的理解。然而，对于具有空间对称性的系统或采用坐标变换的情况，可以进行显著的简化。在这个问题中，我们将讨论在曲面上解决的两个物理问题。第一个问题是光在曲面上传播。图 1（a）显示了一个高度为 5 mm、底直径为$2\rho_0 = 10$ mm 的圆锥体与一个平面相连接。平面上有一个相同直径的圆孔，以便整体上只有一个单一的表面，圆锥部分表示“曲面”。整个表面，包括平面和圆锥体，都有非常薄的表面，以便光可以有效地限制在这样的表面上。我们假设锥体与平坦表面平稳连接。第二个问题将在后面进行阐述，它涉及到在半球面上的稳态热传导。

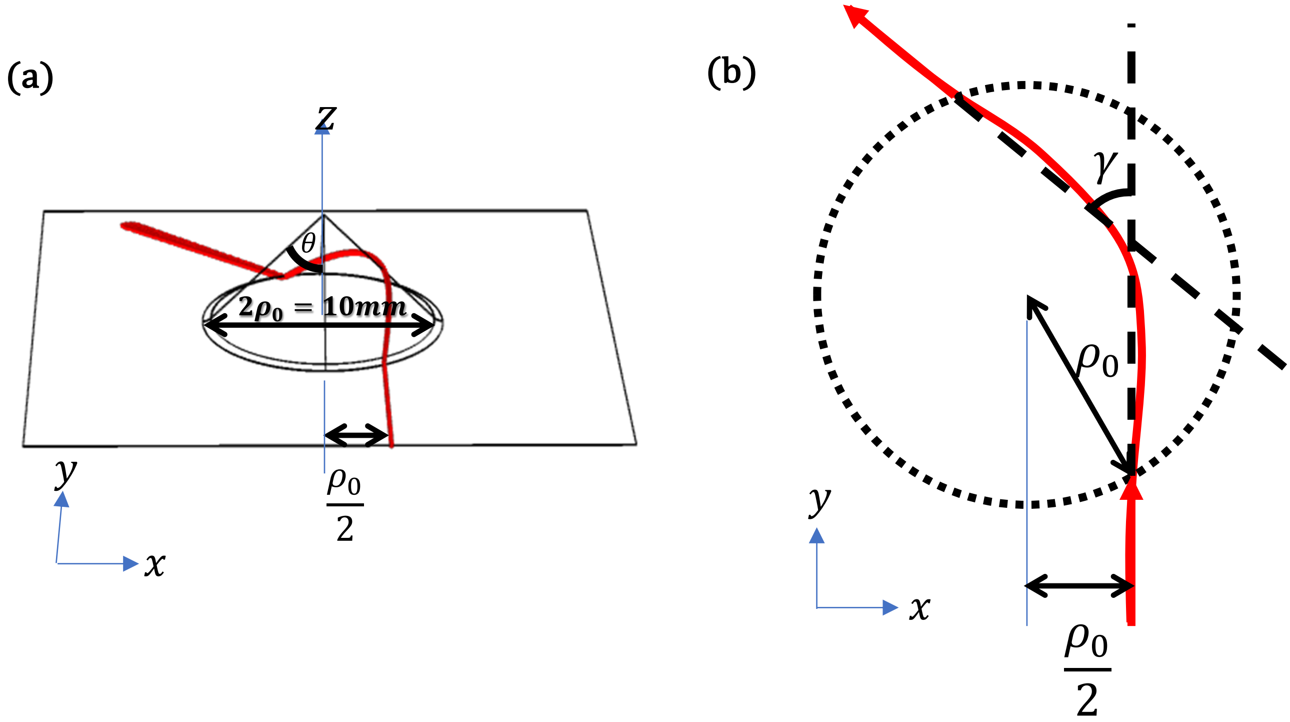


Figure 1(a) depicts a curved surface created by connecting a circular cone to a flat surface with a hole of the same size as the cone's base. In Figure 1(b), we present a top view of this surface. Light confined to such a surface originates on the flat surface at the bottom, undergoes bending due to the cone, and exits in a different direction.

图 1(a)展示了由连接到一个底部与圆锥体底部相同大小的圆孔的平面上的圆锥体所定义的曲面。图 1(b)是该曲面的俯视图。被限制在这样的曲面上的光从底部的平面上开始，因圆锥体的弯曲而改变方向，最终以不同的方向出射。

A. GEODESIC ON A ROTATIONAL SYMMETRIC CURVED SURFACE 旋转对称曲面上的测地线

In mechanics, we are aware that when a system exhibits rotational symmetry, we can simplify the derivation of dynamics by applying the conservation of angular momentum. For instance, we can employ the conservation of angular momentum to derive Kepler's laws. In the current scenario, we consider a normalized angular momentum, denoted as L, which is defined as:

在力学中，我们知道当系统具有旋转对称性时，我们可以使用角动量守恒来简化问题。例如，我们可以使用角动量守恒来推导开普勒定律。在当前情况下，我们考虑一个归一化的角动量，记为 L，其定义如下：

$$L = \hat{z} \cdot \boldsymbol{\rho} \times \frac{d\boldsymbol{\rho}}{ds} = \rho^2 \frac{d\phi}{ds}.$$

Here, $\boldsymbol{\rho}$ represents the projected position vector on the two-dimensional x-y plane, given by $\boldsymbol{\rho} = x\hat{x} + y\hat{y} = \hat{x}\rho\cos\phi + \hat{y}\rho\sin\phi$, with the projected cone center as the origin. ρ is the magnitude of vector $\boldsymbol{\rho}$ and s is the arc length along the path of light on the surface.
在这里，$\boldsymbol{\rho}$ 代表了在二维 x-y 平面上的投影位置矢量，由$\boldsymbol{\rho} = x\hat{x} + y\hat{y} = \hat{x}\rho\cos\phi + \hat{y}\rho\sin\phi$给出，其中投影锥体中心为原点。$\rho$是矢量$\boldsymbol{\rho}$ 的大小。s 是沿着曲面上光的路径的弧长。

A1	Given that the infinitesimal arc length on the cone satisfies $ds^2 = dx^2 + dy^2 + dz^2$ and $z = z(\rho)$ is the height at that point, prove the geodesic on the cone satisfies 假设圆锥体上的无穷小弧长满足$ds^2 = dx^2 + dy^2 + dz^2$，其中$z = z(\rho)$是该点的高度，请证明圆锥体上的测地线满足 $$\rho'(\phi)^2 = \frac{\sin^2\theta}{L^2}\rho^2(\rho^2 - L^2)$$ Instead of using arc length s to parametrize the geodesic, we have used ϕ for parametrization. 这里，我们使用ϕ来参数化测地线，而不是使用弧长 s。	3 points
A2	For a light starting on the flat surface with a perpendicular distance of $\rho_0/2$ to the origin, what will be minimal ρ the light can go. 对于从平面上距离原点垂直距离为 $\rho_0/2$ 的入射光线，它能够到达的最小 ρ 是多少？	2 points
A3	What is the deflection angle γ by comparing the entering and exit rays on the flat surface? Hint: you may need to use $\int \frac{dx}{x\sqrt{x^2-1}} = \tan^{-1}\sqrt{x^2-1} + c$ 通过比较光线在平面表面的入射和出射角度，计算偏转角度 γ。 提示：你可能需要用 $\int \frac{dx}{x\sqrt{x^2-1}} = \tan^{-1}\sqrt{x^2-1} + c$	7 points

A1: Coordinates:

$$x = \rho\cos\phi\,, \qquad y = \rho\sin\phi\,, \qquad z = z(\rho)$$
$$\Rightarrow \frac{dx}{ds} = \frac{d\rho}{ds}\cos\phi - \rho\sin\phi\frac{d\phi}{ds}, \qquad \frac{dy}{ds} = \frac{d\rho}{ds}\sin\phi + \rho\cos\phi\frac{d\phi}{ds}$$

Along the light path, the arc length elapsed in a small section is governed by

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= (\cos\phi\, d\rho - \rho\sin\phi\, d\phi)^2 + (\sin\phi\, d\rho + \rho\cos\phi\, d\phi)^2 + z'(\rho)^2(d\rho)^2 \\ &= (1 + z'(\rho)^2)(d\rho)^2 + \rho^2(d\phi)^2 \end{aligned}$$

Angular momentum:

$$L = \hat{z}\cdot\boldsymbol{\rho}\times\frac{d\boldsymbol{\rho}}{ds} = \rho\cos\phi\frac{dy}{ds} - \rho\sin\phi\frac{dx}{ds} = \rho^2\frac{d\phi}{ds}$$
$$\Rightarrow \frac{\rho^4}{L^2} = \left(\frac{ds}{d\phi}\right)^2 = (1 + \cot^2\theta)\rho'(\phi)^2 + \rho^2 = \csc^2\theta\,\rho'(\phi)^2 + \rho^2$$
$$\Rightarrow \rho'(\phi)^2 = \sin^2\theta\,\rho^2\left(\frac{\rho^2}{L^2} - 1\right)$$

A2: The trajectory on the flat surface can be described by

$$x = \frac{\rho_0}{2}, \qquad y = s + s_0,$$

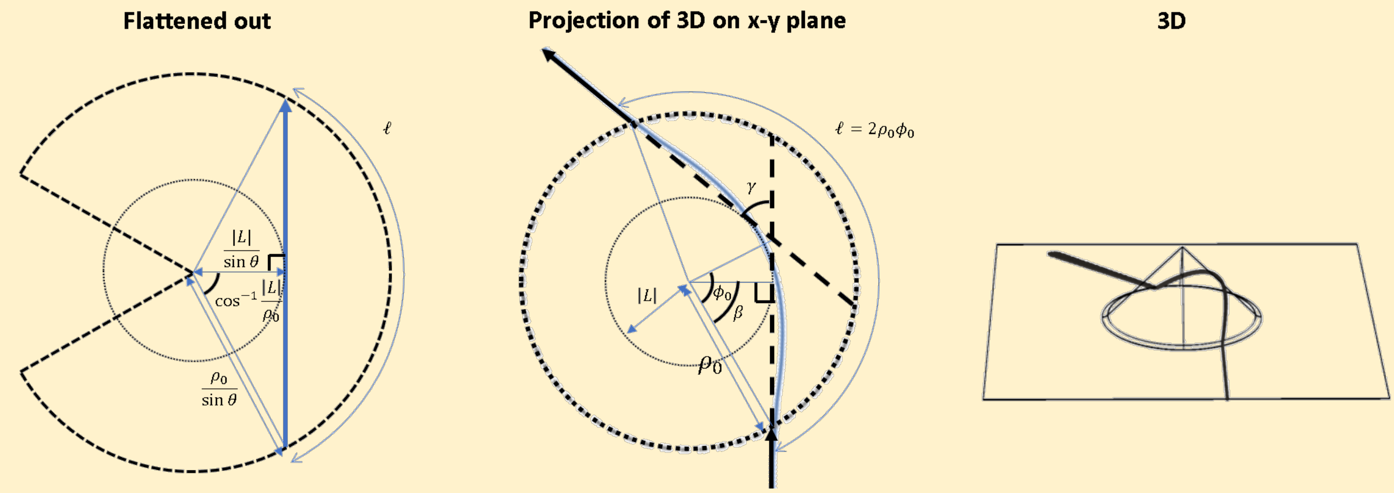
where s_0 is a constant. Then the normalized angular momentum is

$$L = x\frac{dy}{ds} - y\frac{dx}{ds} = \frac{\rho_0}{2} \quad ,$$

which is independent of s as expected on the flat surface before entering the cone. We recognize that L has also the meaning of perpendicular distance of the entering ray. Then from A1,

$$\rho'(\phi)^2 \geq 0 \Rightarrow \rho \geq |L| = \frac{\rho_0}{2}$$

A3: Solution: We consider that L is a constant of motion so that the entering and exit rays have the same perpendicular distance $|L|$. We define $2\phi_0$ as the angle elapsed for the ray in the region of the cone and $2\beta = 2\cos^{-1}(|L|/\rho_0)$ as the angle elapsed for the ray if it has not been deflected (see middle panel).



(Method 1)

To calculate ϕ_0, we flatten out the cone to a flat surface. The cone becomes a sector of radius $\rho_0/\sin\theta$. The nearest projected distance $|L|$ (in A2) now becomes $|L|/\sin\theta$ (see left panel). Therefore, the elapsed angle subtends a distance on the edge of cone as

$$\ell = \frac{2\rho_0}{\sin\theta}\cos^{-1}\frac{|L|}{\rho_0}$$

Back to the projected view (middle panel), we have the same distance $\ell = 2\rho_0\phi_0$ so that we have obtained

$$\phi_0 = \frac{1}{\sin\theta}\cos^{-1}\frac{|L|}{\rho_0} = \frac{\beta}{\sin\theta}$$

(Method 2) Alternatively, we can consider the ray has its ρ gradually decreasing when it enters the cone until the nearest distance, $|L|$ from A2, for the first half of the ray trajectory, we choose the negative square root from A1:

$$\frac{d\rho}{d\phi} = -\sin\theta\,\rho\sqrt{(\rho/|L|)^2 - 1}$$

Now, we can integrate to get ϕ_0

$$\phi_0 = \int d\phi = -\frac{1}{\sin\theta}\int \frac{d\rho/|L|}{(\rho/|L|)\sqrt{(\rho/|L|)^2 - 1}} = \left[-\frac{\tan^{-1}\sqrt{\rho^2/L^2 - 1}}{\sin\theta}\right]_{\rho=\rho_0}^{\rho=|L|}$$

$$= \frac{\tan^{-1}\sqrt{\rho_0^2/L^2 - 1}}{\sin\theta} = \frac{1}{\sin\theta}\cos^{-1}\frac{|L|}{\rho_0}$$

After calculating ϕ_0:

Then the deflection angle γ is governed by

$$\gamma = 2(\phi_0 - \beta) = 2\left(\frac{1}{\sin\theta} - 1\right)\beta.$$

Now, substituting values for our particular example, $|L| = \rho_0/2$ and $\theta = \pi/4$, we have

$$\beta = \cos^{-1}\frac{|L|}{\rho_0} = \frac{\pi}{3} = 60°.$$

$$\gamma = \left(\sqrt{2} - 1\right)\frac{2\pi}{3} = 49.7°.$$

as the deflection angle.

Reference: The experiment and the flattened-out model are depicted in Phys. Rev. Appl. 11, 034072 (2019).

B. HEAT CONDUCTION ON A SPHERICAL SURFACE (I) 球面上的热传导（I）

A usual trick is to search for a coordinate transform from the curved surface (represented by the Cartesian coordinates (x, y, z)) to a 2-dimensional coordinates system $X - Y$ plane so that the physics on the (X, Y) just looks like a flat plane. For a unit spherical surface, such a map is the stereographic projection

一個有用的技巧是寻找一个坐标变换，将曲面（由笛卡尔坐标(x, y, z) 表示）映射到一个二维的 $X - Y$ 坐标上，使得在 $X - Y$ 坐标上的物理现象看起来就像一个平面。对于一个单位球面，这样的映射是立体投影。

$$(X, Y) = \left(\frac{x}{z+1}, \frac{y}{z+1}\right) = (\rho\cos\phi, \rho\sin\phi)$$

Suppose now we consider heat conduction problem on such a spherical surface, i.e. a very thin shell of spherical surface. The steady-state heat conduction has the temperature profile satisfying the Laplace equation

现在我们考虑在这样一个球面上的热传导问题，即一个非常薄的球面壳体。稳态热传导具有满足拉普拉斯方程(Laplace equation)的温度分布。

$$\nabla^2 T(\theta, \phi) = 0$$

while temperature profile is independent of radial distance r in spherical coordinate (r, θ, ϕ). The spherical surface is at radius $r = 1$.

其中温度分布与径向距离 r 无关。这个球面的半径为 $r = 1$。

B1	Prove T satisfies Laplace equation on the (X, Y) coordinate: 证明 T 在 (X, Y) 坐标上满足 Laplace 方程: $$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho T) + \frac{1}{\rho^2}\partial_\phi^2 T = 0$$ Hint: For convenience, we are given the Laplacian in spherical and cylindrical coordinates as 为了方便起见，我们提供了球坐标和柱坐标下的 Laplacian 方程式。 In spherical coordinate (在球坐标下) (r, θ, ϕ): $$\nabla^2 f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 f$$ cylindrical coordinate (在柱坐标下) (ρ, ϕ, z): $$\nabla^2 f = \frac{1}{\rho}\partial_\rho(\rho\partial_\rho f) + \frac{1}{\rho^2}\partial_\phi^2 f + \partial_z^2 f$$	4 points

Solution:

For the stereographic projection, a θ is mapped to a ρ and ϕ is unaltered, serving both the azimuthal coordinate for both the spherical coordinate and the stereographic projected coordinate. Then,

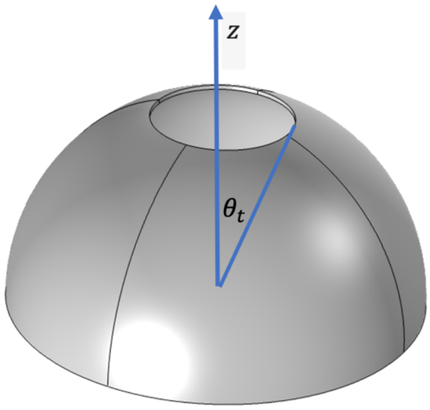
$$\rho = \frac{\sin\theta}{\cos\theta + 1} = \tan\frac{\theta}{2}$$
$$\Rightarrow \frac{\partial\rho}{\partial\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}, \qquad \sin\theta\frac{\partial\rho}{\partial\theta} = \rho$$
$$\Rightarrow \sin\theta\,\partial_\theta = \rho\partial_\rho$$

For a r-independent T profile, it satisfies the Laplace equation at $r = 1$ as

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta T) + \frac{1}{\sin^2\theta}\partial_\phi^2 T = 0 \Rightarrow \sin\theta\,\partial_\theta(\sin\theta\,\partial_\theta T) + \partial_\phi^2 T = 0.$$

which is now transformed to

$$\rho\partial_\rho(\rho\partial_\rho T) + \partial_\phi^2 T = 0. \quad \Rightarrow \frac{1}{\rho}\partial_\rho(\rho\partial_\rho T) + \frac{1}{\rho^2}\partial_\phi^2 T = 0$$



Now, the above figure gives the thin shell in the shape of lamp shade, which is in a hemi-spherical surface with a circular opening at the top. The whole shape still has a rotational symmetry about the vertical z-axis. The bottom of the lamp shade is kept at temperature T_b (at $\theta = \frac{\pi}{2}$ for spherical polar coordinate) and the top is kept at temperature T_t (at $\theta = \theta_t$).

现在，上图给出了一个薄壳，呈灯罩形状，是一个顶部有一个圆形开口的半球面。整个形状仍然具有旋转对称性。灯罩的底部保持在温度 T_b（在球坐标下的 $\theta = \frac{\pi}{2}$ 处），顶部保持在温度 T_t（在$\theta = \theta_t$ 处）。

B2	Solve the temperature profile, as a function of θ, with such rotational symmetry. 解出具有旋转对称性的温度分布 $T(\theta)$，作为 θ 的函数。	4 points

Solution:

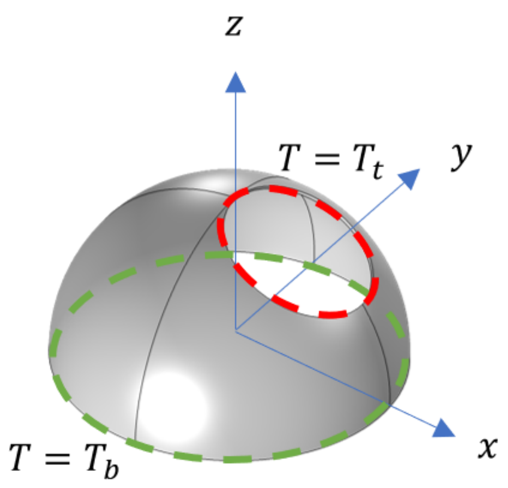
With rotational symmetric, T is independent of ϕ:

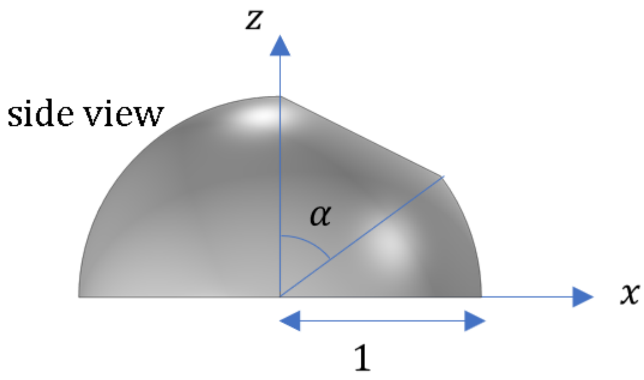
$$\frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho T\right) = 0.$$

Then

$$T = \alpha \ln \rho + \beta = (T_t - T_b)\frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\tan\frac{\theta_t}{2}\right)} + T_b$$

C. HEAT CONDUCTION ON A SPHERICAL SURFACE (II) 球面上的热方程(II)





Now, we consider the top opening is tilted about the y-axis in breaking rotational symmetry. Suppose the top opening is still a circle on the spherical surface passing through $(x, y, z) = (0,0,1)$ and $(\sin\alpha, 0, \cos\alpha)$ as diameter and its normal is on the x-z plane.

现在，我们考虑顶部开口围绕 y 轴倾斜，破坏了旋转对称性。假设球面顶部开口仍然是一个圆，通过由 $(x, y, z) = (0,0,1)$ 和 $(\sin\alpha, 0, \cos\alpha)$ 作为直径，并且其法向量位于 x-z 平面上。

C1	Determine where the top opening is mapped on (X, Y) plane through the stereographic projection map. 通过立体投影映射，确定顶部开口在 (X, Y) 平面上的映射位置。 Hint: the answer is still a circle in X and Y coordinates. 提示：答案在 X 和 Y 坐标上仍然是一个圆。	2 points
C2	Solve the temperature profile $T(X, Y)$ when the bottom opening is kept at temperature T_b and the top is kept at temperature T_t. You can leave your answer in terms of X and Y coordinates. Hint: In the stereographic projected domain X-Y plane, Laplace equation is satisfied and you can further use general method of image, like the one used in solving electrostatic problem by putting two point charges on the X-axis with undetermined charges. 解出温度分布函数 $T(X, Y)$。底部开口保持温度T_b，顶部开口保持温度T_t。您可以用 X 和 Y 坐标表示您的答案。 提示：Laplace 方程仍然满足在立体投影后的 X-Y 平面上。你可以使用在解决静电问题時的镜像电荷法，在 X 轴上放置两个待确定电荷的电荷。	8 points 8 分

C1: The two points of a diameter:

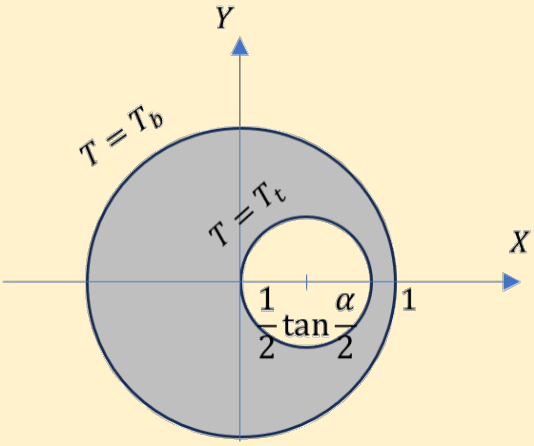
$$(x, y, z) = (0,0,1) \to (X, Y) = (0,0)$$

$$(x, y, z) = (\sin\alpha, 0, \cos\alpha) \to (X, Y) = \left(\tan\frac{\alpha}{2}, 0\right)$$

Mapped circle has center $(X, Y) = \left(\frac{1}{2}\tan\frac{\alpha}{2}, 0\right)$ with diameter $\frac{1}{2}\tan\frac{\alpha}{2}$, or written as

$$\left(X - \frac{1}{2}\tan\frac{\alpha}{2}\right)^2 + Y^2 = \frac{1}{4}\tan^2\frac{\alpha}{2}$$

C2:



The Laplace equation is satisfied on the X-Y plane. $T = T_b$ on the unit circle with center at origin. $T = T_t$ on a smaller circle passing through origin and with center at $\left(a = \frac{1}{2}\tan\frac{\alpha}{2}, 0\right)$.

Equivalently to our thermal conduction problem, we can treat T as an electrostatic potential also satisfying the Laplace equation. Here, we adopt the general method of image to obtain T in (X, Y) space. Assume that we have two point charges on the X-axis with undetermined charges at the moment. We let the distance between them is $2A$ and the middle of the two coordinates is at X_0 on the X-axis. Then, the potential for such a system is

$$T = c_1 \ln\left(\frac{(X - X_0 + A)^2 + Y^2}{(X - X_0 - A)^2 + Y^2}\right) + c_2.$$

An equipotential (same value of T) contour can be written as a circle with center on the X-axis:

$$\frac{(X - X_0 + A)^2 + Y^2}{(Y - X_0 - A)^2 + Y^2} = K \Rightarrow \left(X - \left(X_0 + \frac{K+1}{K-1}A\right)\right)^2 + Y^2 = \left(\left(\frac{K+1}{K-1}\right)^2 - 1\right)A^2$$

where K is an arbitrary number depending on the value of T. Now, we need the inner and outer circles in (X, Y) space map to different equipotential lines. Then, we have following equations for the centres and radii,

$$X_0 + \frac{K_1+1}{K_1-1}A = a, \quad \left(\frac{K_1+1}{K_1-1}\right)^2 - 1 = \frac{a^2}{A^2},$$
$$X_0 + \frac{K_2+1}{K_2-1}A = 0, \quad \left(\frac{K_2+1}{K_2-1}\right)^2 - 1 = \frac{1}{A^2},$$

Eliminating K_1 and K_2, we have

$$(a - X_0)^2 = A^2 + a^2, \qquad X_0^2 = A^2 + 1,$$

in which X_0 and A can be solved as

$$X_0 = \frac{1}{2a}, \qquad A = \pm\frac{\sqrt{1 - 4a^2}}{2a}$$

We let

$$\mu = \frac{1 + \sqrt{1 - 4a^2}}{2a} > 1, \qquad \mu^{-1} = \frac{1 - \sqrt{1 - 4a^2}}{2a} < 1$$

Then

$$T = c_1 \ln\left(\frac{(X-\mu)^2 + Y^2}{(X-\mu^{-1})^2 + Y^2}\right) + c_2$$

We probe the c_1 and c_2 coefficients by

$$(X,Y) = (0,0) \Rightarrow T_t = 4c_1 \ln\mu + c_2$$
$$(X,Y) = (1,0) \Rightarrow T_b = 2c_1 \ln\mu + c_2$$
$$\Rightarrow c_1 = \frac{T_t - T_b}{2\ln\mu}, c_2 = 2T_b - T_t$$

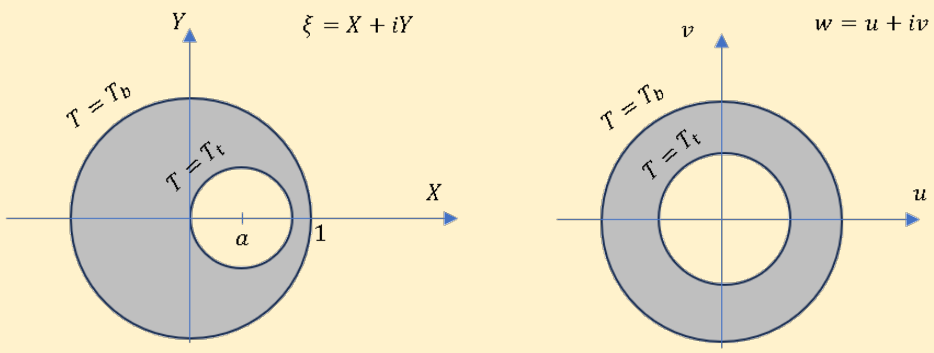
Therefore

$$\begin{aligned} T &= \frac{T_t - T_b}{2\ln\mu} \ln\left(\frac{(X-\mu)^2 + Y^2}{(X-\mu^{-1})^2 + Y^2}\right) + 2T_b - T_t \\ &= \frac{T_t - T_b}{2\ln\mu} \ln\left(\frac{\rho^2 - 2\mu\rho\cos\phi + \mu^2}{\rho^2 - 2\mu^{-1}\rho\cos\phi + \mu^{-2}}\right) + 2T_b - T_t \\ &= \frac{T_t - T_b}{2\ln\mu} \ln\left(\frac{\tan^2\frac{\theta}{2} - 2\mu\,\tan\frac{\theta}{2}\cos\phi + \mu^2}{\tan^2\frac{\theta}{2} - 2\mu^{-1}\tan\frac{\theta}{2}\cos\phi + \mu^{-2}}\right) + 2T_b - T_t \end{aligned}$$

where $\mu = \frac{1+\sqrt{1-4a^2}}{2a}$, $a = \frac{1}{2}\tan\frac{\alpha}{2}$.

Alternative solution:

If you know conformal map which maps circle to circle and preserves Laplace equation, we can adopt a conformal map (coordinate transformation) from (X,Y) to (u,v) so that the two circles are concentric.



By writing complex $\xi = X + iY$ and $w = u + iv$, we seek

$$w = \frac{b\xi + c}{\xi + d}$$

which maps the unit circle to unit circle and the inner circle to another circle with common center at $w = 0$

$$\frac{b+c}{1+d} = 1, \qquad \frac{-b+c}{-1+d} = -1$$
$$\frac{0+c}{0+d} + \frac{2ab+c}{2a+d} = 0$$

By solving b, c and d from the above, we obtain the conformal map as

$$w = \frac{\mu\xi - 1}{\mu - \xi}$$

or

$$w = \frac{\mu - \xi}{\mu\xi - 1}$$

We choose the first solution which maps the inner circle to a circle with radius less than one, just for convenience. The radius of the inner circle is now mapped to a circle with radius $|(0-1)/(\mu-0)| = \mu^{-1} < 1$.

From previous experience in B2 in solving ϕ-independent Laplace equation, we have

$$\begin{aligned}
T &= (T_t - T_b)\frac{\ln|w|}{\ln \mu^{-1}} + T_b \\
&= \frac{T_t - T_b}{2\ln \mu^{-1}} \ln\left|\frac{\mu\xi - 1}{\mu - \xi}\right|^2 + T_b \\
&= \frac{T_t - T_b}{2\ln \mu^{-1}} \ln\left|\frac{\xi - \mu^{-1}}{\mu - \xi}\right|^2 + 2T_b - T_t \\
&= \frac{T_t - T_b}{2\ln \mu} \ln\left|\frac{\xi - \mu}{\xi - \mu^{-1}}\right|^2 + 2T_b - T_t \\
&= \frac{T_t - T_b}{2\ln \mu} \ln\left(\frac{(X-\mu)^2 + Y^2}{(X-\mu^{-1})^2 + Y^2}\right) + 2T_b - T_t
\end{aligned}$$