[Solution] *Theoretical Question 2*

Motion of an Electric Dipole in a Magnetic Field

(1) Conservation Laws

(1a) $\vec{r}_{CM} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$, $\vec{v}_{CM} = \frac{1}{2}(\vec{v}_1 + \vec{v}_2)$, $\vec{\ell} = \vec{r}_1 - \vec{r}_2$, $\vec{u} = \dot{\vec{\ell}} = \vec{v}_1 - \vec{v}_2$

Total force $\vec{F}$ on the dipole is

$$\begin{aligned}\vec{F} &= \vec{F}_1 + \vec{F}_2 = q(\vec{E} + \vec{v}_1 \times \vec{B}) + (-q)(\vec{E} + \vec{v}_2 \times \vec{B}) = q(\vec{v}_1 - \vec{v}_2) \times \vec{B} \\ &= q\dot{\vec{\ell}} \times \vec{B}\end{aligned}$$

so that

$$M\dot{\vec{v}}_{CM} = q\dot{\vec{\ell}} \times \vec{B} \qquad (M = 2m) \tag{1}$$

Computing the torque for rotation around the center of mass, we obtain

$$\begin{aligned}I\dot{\vec{\omega}} &= (\frac{\vec{\ell}}{2}) \times (q\vec{v}_1 \times \vec{B}) + (\frac{-\vec{\ell}}{2}) \times (-q\vec{v}_2 \times \vec{B}) \\ &= q\vec{\ell} \times (\vec{v}_{CM} \times \vec{B})\end{aligned} \tag{2}$$

where

$$I = \frac{1}{2} m\ell^2 \tag{3}$$

(1b) From *eq*.(1), we obtain the conservation law for the momentum:

$$\dot{\vec{P}} = 0, \quad \vec{P} = M\vec{v}_{CM} - q\vec{\ell} \times \vec{B} \tag{4}$$

From *eq*.(1) and *eq*.(2), one obtains the conservation law for the energy.

$$\dot{E} = 0, \quad E = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I\omega^2 \tag{5}$$

(1c) Using *eq*.(4) and *eq*.(2),

$$\begin{aligned}\frac{d}{dt}(\vec{r}_{CM} \times \vec{P}) \cdot \hat{B} &= (\vec{v}_{CM} \times \vec{P}) \cdot \hat{B} = -q\vec{v}_{CM} \times (\vec{\ell} \times \vec{B}) \cdot \hat{B} \\ &= q(\vec{\ell} \times \vec{B}) \times \vec{v}_{CM} \cdot \hat{B} = q(\vec{\ell} \times \vec{B}) \cdot (\vec{v}_{CM} \times \hat{B}) \\ &= q\vec{\ell} \cdot (\vec{B} \times (\vec{v}_{CM} \times \hat{B})) = -q\vec{\ell} \cdot ((\vec{v}_{CM} \times \vec{B}) \times \hat{B}) \\ &= -q\vec{\ell} \times (\vec{v}_{CM} \times \vec{B}) \cdot \hat{B} \\ &= -I\dot{\vec{\omega}} \cdot \hat{B}\end{aligned}$$

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we obtain the conservation law

$$\dot{J} = 0 \qquad J = (\vec{r}_{CM} \times \vec{P} + I\vec{\omega}) \cdot \hat{B} \tag{6}$$

for the component of the angular momentum along the direction of $\vec{B}$.

(2) Motion in a Plane Perpendicular to $\vec{B}$

(2a) Write

$$\vec{\ell} = \ell\{\cos\varphi(t)\,\hat{x} + \sin\varphi(t)\,\hat{y}\}, \quad \varphi(0) = 0, \; \dot{\varphi}(0) = \omega_0 \tag{7}$$

Note that

$$\vec{\omega} = \dot{\varphi}\,\hat{z} \tag{8}$$

From *eq*.(4), we have

$$M\vec{v}_{CM} = \vec{P} + q\ell B\,(\sin\varphi\,\hat{x} - \cos\varphi\,\hat{y}) \tag{9}$$

At $t = 0$, we have $v_{CM} = 0$, $\varphi = 0$ so that

$$\vec{P} = q\ell B\,\hat{y} \tag{10}$$

Hence from *eqs*.(9) and (10) we have

$$\dot{x}_{CM} = \left(\frac{q\ell B}{M}\right)\sin\varphi, \quad \dot{y}_{CM} = \left(\frac{q\ell B}{M}\right)(1 - \cos\varphi) \tag{11}$$

From conservation of energy, i.e. *Eq*.(5), we have

$$\frac{1}{2}I\dot{\varphi}^2 + \frac{(q\ell B)^2}{M}(1 - \cos\varphi) = \frac{1}{2}I\omega_0^2$$

$$\therefore \quad \dot{\varphi}^2 + \frac{1}{2}\omega_c^2(1 - \cos\varphi) = \omega_0^2 \tag{12}$$

where

$$\omega_c^2 = \frac{4(q\ell B)^2}{MI} = \left(\frac{2qB}{m}\right)^2 \tag{13}$$

In order to make a full turn, $\dot{\varphi}$ can not become zero so that

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$$\omega_0^2 > \omega_c^2 \;\Rightarrow\; |\omega_0| > \omega_c = \frac{2qB}{m} \tag{14}$$

(2b) From *Eq*.(6), we have

$$x_{CM} P + I\omega = J \tag{15}$$

where P is the magnitude of $\vec{P}$.

At $t = 0$, we have $J = I\omega_0$ so that

$$x_{CM} P + I\omega = I\omega_0 \tag{16}$$

From *eq*.(12), one can see that $\omega_0^2 \geq \omega^2$ so that $x_{CM} \geq 0$. Thus x_{CM} reaches a maximum d_m when ω takes its minimum value.

When $\omega_0 < \omega_c$, the minimum value of ω is $-\omega_0$ so that

$$d_m = \frac{2I}{P}\omega_0 = \left(\frac{m\omega_0}{qB}\right)\ell, \quad \omega_0 < \omega_c \tag{17}$$

When $\omega_0 > \omega_c$, the minimum value of ω is $\sqrt{\omega_0^2 - \omega_c^2}$ so that

$$d_m = \left(\frac{I}{P}\right)\left(\omega_0 - \sqrt{\omega_0^2 - \omega_c^2}\right) = \frac{m}{2qB}\left(\omega_0 - \sqrt{\omega_0^2 - \omega_c^2}\right)\ell, \quad \omega_0 > \omega_c \tag{18}$$

When $\omega_0 = \omega_c$, $\omega^2 = \frac{1}{2}\omega_c^2(1 + \cos\varphi) = \omega_c^2 \cos^2\frac{\phi}{2}$

$$\therefore \quad \dot{\varphi} = \omega_c \cos\frac{\phi}{2}$$

When φ is close to π, let $\varphi = \pi - 2\varepsilon$ then

$$\dot{\varepsilon} = -\frac{1}{2}\omega_c \sin\varepsilon \approx -\frac{1}{2}\omega_c \varepsilon$$

$$\therefore \quad \varepsilon \sim e^{-\omega_c t/2}$$

so that it will take $t \to \infty$ for $\varepsilon \to 0$, i.e. for φ to reach π. Hence

$$d_m = \left(\frac{I}{P}\right)\omega_c = \left(\frac{m\omega_c}{2qB}\right)\ell, \quad \omega_0 = \omega_c \tag{19}$$

(2c) Tension on the rod comes from three sources:

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(i) Coulomb force $= \dfrac{1}{4\pi\varepsilon_0}\dfrac{q^2}{\ell^2}$ (20)

Positive value means compression on the rod.

(ii) Centrifugal force due to rotation of the rod $= -\dfrac{1}{2}m\omega^2\ell$ (21)

(iii) Magnetic force on the particles due to the motion of the center of the mass

$$= q\vec{v}_{CM}\times\vec{B}\cdot(-\hat{\ell}) = q\vec{v}_{CM}\cdot\hat{\ell}\times\vec{B}$$

Taking the square of both sides of *eq*.(4) and using the initial condition for the value of P^2, we obtain

$$\frac{1}{2}Mv_{CM}^2 = q\ell\vec{v}_{CM}\cdot\hat{\ell}\times\vec{B} = \frac{1}{2}I(\omega_0^2-\omega^2) \qquad (22)$$

Combining the three forces, we have

$$\textit{tension on the rod} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{\ell^2} - \frac{1}{2}m\ell\omega^2 + \frac{1}{4}m\ell(\omega_0^2-\omega^2) \qquad (23)$$

A positive value means compression on the rod.