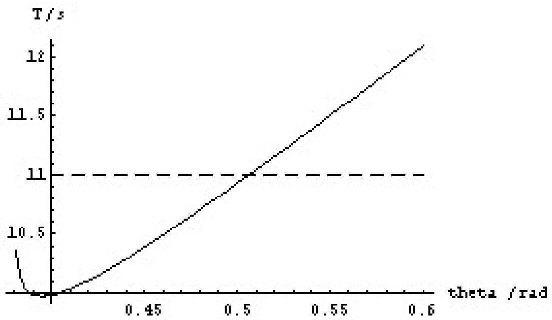
Solution to Question 2: The Rail Gun

Proper Solution (taking induced emf into consideration): (a) Let I be the current supplied by the battery in the absence of back emf. Let i be the induced current by back emf ε_b . Since $\varepsilon_b = d\phi / dt = d(BLx)/dt = BLv$, $\therefore i = Blv / R$. Net current, $I_N = I - i = I - BLv / R$. Forces parallel to rail are: Force on rod due to current is $F_c = BLI_N = BL(I - BLv / R) = BLI - B^2 L^2 v / R$. Net force on rod and young man combined is $F_N = F_c - mg \sin\theta$. (1) Newton's law: $F_N = ma = mdv / dt$. (2) Equating (1) and (2), & substituting for F_c & dividing by *m*, we obtain the acceleration $dv / dt = \alpha - v / \tau$, where $\alpha = BIL / m - g \sin\theta$ and $\tau = mR / B^2 L^2$.	 1 0.5 0.5 0.5 0.5	 3

(b)(i) Since initial velocity of rod = 0, and let velocity of rod at time t be *v(t)*, we have $$v(t) = v_\infty\left(1 - e^{-t/\tau}\right), \qquad (3)$$ where $v_\infty(\theta) = \alpha\tau = \frac{IR}{BL}\left(1 - \frac{mg}{BLI}\sin\theta\right).$	0.5	
Let t_s be the total time he spent moving along the rail, and v_s be his velocity when he leaves the rail, i.e. $$v_s = v(t_s) = v_\infty\left(1 - e^{-t_s/\tau}\right). \qquad (4)$$	0.5	
$$\therefore t_s = -\tau\ln\left(1 - v_s / v_\infty\right) \qquad (5)$$	0.5	1.5

(b) (ii) Let t_f be the time in flight: $t_f = \dfrac{2v_s \sin\theta}{g}$ (6)	0.5	
He must travel a horizontal distance w during t_f. $w = (v_s \cos\theta)t_f$ (7) $t_f = \dfrac{w}{v_s \cos\theta} = \dfrac{2v_s \sin\theta}{g}$ (8) (from (6) & (7))	0.5	
From (8), v_s is fixed by the angle θ and the width of the strait w $v_s = \sqrt{\dfrac{gw}{\sin 2\theta}}$. (9) $\therefore t_s = -\tau \ln\left(1 - \dfrac{1}{v_\infty}\sqrt{\dfrac{gw}{\sin 2\theta}}\right),$ (Substitute (9) in (5))		1.5
And $t_f = \dfrac{2\sin\theta}{g}\sqrt{\dfrac{gw}{\sin 2\theta}} = \sqrt{\dfrac{2w\tan\theta}{g}}$ (Substitute (9) in (8))	0.5	

(c) Therefore, total time is: $T = t_s + t_f = -\tau \ln\left(1 - \frac{1}{v_\infty}\sqrt{\frac{gw}{\sin 2\theta}}\right) + \sqrt{\frac{2w\tan\theta}{g}}$ The values of the parameters are: B=10.0 T, I= 2424 A, L=2.00 m, R=1.0 Ω, g=10 m/s^2, m=80 kg, and w=1000 m. Then $\tau = \frac{mR}{B^2L^2} = \frac{(80)(1.0)}{(10.0)^2(2.00)^2} = 0.20$ s. $v_\infty(\theta) = \frac{2424}{(10.0)(2.00)}\left(1 - \frac{(80)(10)}{(10.0)(2.00)(2424)}\sin\theta\right)$ $= 121(1 - 0.0165\sin\theta)$ So, $T = t_s + t_f = -0.20\ln\left(1 - \frac{100}{v_\infty}\frac{1}{\sqrt{\sin 2\theta}}\right) + 14.14\sqrt{\tan\theta}$ By plotting *T* as a function of *θ*, we obtain the following graph:  Note that the lower bound for the range of θ to plot may be determined by the condition $v_s / v_\infty < 1$ (or the argument of ln is positive), and since mg/BLI is small (0.0165), $v_\infty \approx IR/BL$ (= 121 m/s), we have the condition $\sin(2\theta) > 0.68$, i.e. $\theta > 0.37$. So one may start plotting from $\theta = 0.38$. From the graph, for θ within the range (~0.38, 0.505) radian the time *T* is within 11 s.	Labeling: 0.1 each axis Unit: 0.1 each axis Proper Range in θ: 0.3 lower limit (more than 0.37, less than 0.5), 0.2 upper limit (more than 0.5 and less than 0.6) Proper shape of curve: 0.2 Accurate intersection at $\theta = 0.5$: 0.4	1.5

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(d)
However, there is another constraint, i.e. the length of rail D. Let D_s be the distance travelled during the time interval t_s

$$D_s = \int_0^{t_s} v(t)dt = v_\infty \int_0^{t_s} \left(1 - e^{-t/\tau}\right)dt = v_\infty \left(t + \tau e^{-\beta t}\right)_0^{t_s} = v_\infty \left[t_s - \tau\left(1 - e^{-\beta t}\right)\right] = v_\infty t_s - v(t_s)\tau$$

i.e.

$$D_s = -\tau\left[v_\infty(\theta)\ln\left(1 - \frac{1}{v_\infty(\theta)}\sqrt{\frac{gw}{\sin 2\theta}}\right) + \sqrt{\frac{gw}{\sin 2\theta}}\right]$$

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0.5

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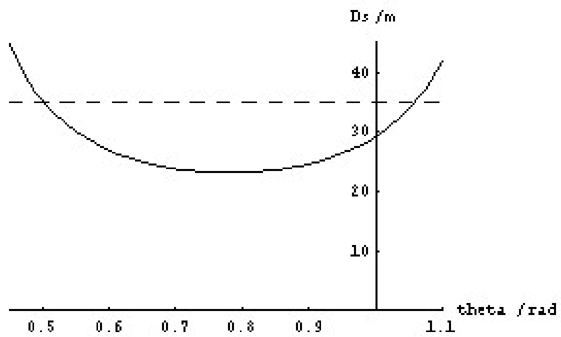
The graph below shows D_s as a function of θ.



It is necessary that $D_s \le D$, which means θ must range between .5 and1.06 radians.

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Labeling:
0.1 each axis

Unit:
0.1 each axis

Proper Range in θ:
0.3 lower limit (more than 0.4, less than 0.49),
0.2 upper limit (more than 0.51 and less than 1.1)

Proper shape of curve: 0.2

Accurate intersection at $\theta = 0.5$: 0.4

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In order to satisfy both conditions, θ must range between 0.5 & 0.505 radians.

(Remarks: Using the formula for t_f , t_s & D, we get

At $\theta = 0.507$, t_f =10.540, t_s = 0.466, giving T = 11.01 s, & D = 34.3 m
At $\theta = 0.506$, t_f =10.527, t_s = 0.467, giving T = 10.99 s, & D = 34.4 m
At $\theta = 0.502$, t_f =10.478, t_s = 0.472, giving T = 10.95 s, & D =34.96 m
At $\theta = 0.50$, t_f =10.453, t_s = 0.474, giving T = 10.93 s, & D =35.2 m,
So the more precise angle range is between 0.502 to 0.507, but students are not expected to give such answers.
To 2 sig fig T = 11 s. Range is 0.50 to 0.51 (in degree: 28.6^0 to 29.2^0 or 29^0)

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0.5

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2.5

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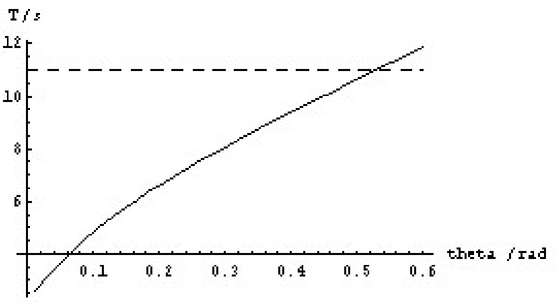
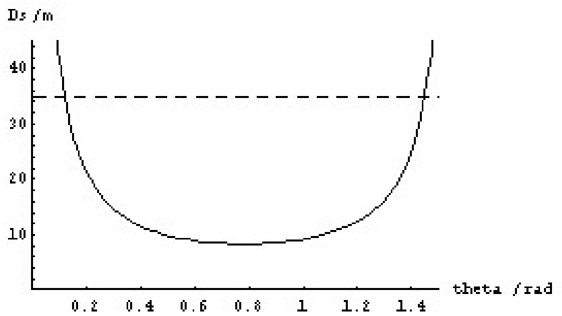
<u>Alternate Solution (Not taking induced emf into consideration)</u>:		
If induced emf is not taken into account, there is no induced current, so the net force acting on the combined mass of the young man and rod is		
$$F_N = BIL - mg\sin\theta \ .$$	0.2 BIL 0.2 $mg\sin\theta$	
And we have instead $$dv/dt = \alpha,$$ where $$\alpha = BIL/m - g\sin\theta \ .$$		
$$\therefore v(t) = \alpha t$$	0.1	
and $$\therefore v_s = v(t_s) = \alpha t_s$$	0.2	
$$t_f = \frac{2v_s\sin\theta}{g} = \frac{2\alpha t_s\sin\theta}{g} \ .$$		
Therefore, $$w = (v_s\cos\theta)t_f = \frac{\alpha^2 t_s^{\ 2}\sin 2\theta}{g} \ ,$$		
giving $$t_s = \frac{1}{\alpha}\sqrt{\frac{gw}{\sin 2\theta}}$$	0.5	
and $$t_f = \sqrt{\frac{2w\tan\theta}{g}} \ .$$	0.5	
Hence, $$T = t_s + t_f = \frac{1}{\alpha}\sqrt{\frac{gw}{\sin 2\theta}} + \sqrt{\frac{2w\tan\theta}{g}} = \frac{\sqrt{wg}}{\alpha}\frac{\left[1 + 2\left(\frac{\alpha}{g}\right)\sin\theta\right]}{\sqrt{\sin 2\theta}} \ .$$ where $\alpha = BIL/m - g\sin\theta$.		
The values of the parameters are: B=10.0 T, I= 2424 A, L=2.00m, R=1.0 Ω, g=10 m/s^2, m=80 kg, and w=1000 m. Then,		
$$T = \frac{100}{\alpha}\frac{[1 + 0.20\alpha\sin\theta]}{\sqrt{\sin 2\theta}}$$ where $\alpha = 606 - 10\sin\theta$.	0.3	2

 For θ within the range (~0, 0.52) radian the time T is within 11 s.	Labeling: 0.1 each axis Unit: 0.1 each axis Proper Range in θ: 0.1 lower limit (more than 0, less than 0.5), 0.2 upper limit (more than 0.52 and less than 0.8) Proper shape of curve: 0.2 Accurate intersection at $\theta = 0.52$: 0.4	1.3
However, there is another constraint, i.e. the length of rail D. Let D_s be the distance travelled during the time interval t_s $$D_s = \frac{gw}{2\alpha \sin 2\theta} = \frac{5000}{\alpha \sin 2\theta}$$ which is plotted below  It is necessary that $D_s \leq D$, which means θ must range between 0.11 and 1.43 radians.	Labeling: 0.1 each axis Unit: 0.1 each axis Proper Range in θ: 0.1 lower limit (more than 0.08, less than 0.11), 0.1 upper limit (more than 0.52 and less than 1.5) Proper shape of curve: 0.2 Accurate intersection at $\theta = 0.11$: 0.4	1.2
In order to satisfy both conditions, θ must range between 0.11 & 0.52 radians.		0.5