Solutions of Problem No. 1

Optical fiber

1.a. At both sides of the point O (outside and inside the fiber), according to Snell law, we have:

$$n_0 \boldsymbol{sin}\,\theta_i = n_1 \boldsymbol{sin}\,\theta_1 \quad (1)$$

where θ_1 is the value of angle θ at point O *inside* the fiber.

The light trajectory lays in the *xOz* plane. Because the refraction index *n* varies along *x* direction, we divide *Ox* axis into small elements *dx*, so that in each of these elements *n* can be considered as constant. We have, then:

$$n\,\boldsymbol{sin}\,i = (n+dn).\boldsymbol{sin}(i+di) \quad (2)$$

where *i* is the angle between the light trajectory and *x* direction. Because $\theta + i = \frac{\pi}{2}$, then

$$n\,\boldsymbol{cos}\,\theta = (n+dn).\boldsymbol{cos}(\theta+d\theta) \quad (3)$$

Thus, at each point of coordinate *x* on the light trajectory, we have:

$$n\,\boldsymbol{cos}\,\theta = n_1\sqrt{1-\alpha^2x^2}\,\boldsymbol{cos}\,\theta = n_1\,\boldsymbol{cos}\,\theta_1 \quad (4)$$

Because

$$\boldsymbol{cos}\,\theta_1 = \sqrt{1-\boldsymbol{sin}^2\theta_1} = \sqrt{1-\frac{\boldsymbol{sin}^2\theta_i}{n_1^2}} \quad (5)$$

we have

$$n\,\boldsymbol{cos}\,\theta = n_1\,\boldsymbol{cos}\,\theta_1 = n_1\sqrt{1-\frac{\boldsymbol{sin}^2\theta_i}{n_1^2}} = \sqrt{n_1^2-\boldsymbol{sin}^2\theta_i}$$

Then

$$n\,\boldsymbol{cos}\,\theta = C = \sqrt{n_1^2-\boldsymbol{sin}^2\theta_i} \quad (6)$$

1.2. Because $\frac{dx}{dz} = x' = \boldsymbol{tan}\,\theta$, from (6) we have:

$$n_1\sqrt{1-\alpha^2x^2}\,\boldsymbol{cos}\,\theta = n_1\sqrt{1-\alpha^2x^2}\left(1+\boldsymbol{tan}^2\theta\right)^{-\frac{1}{2}} = C \quad (7)$$

Squaring the two sides, we obtain:

$$\left(1-\alpha^2x^2\right)\left(1+\boldsymbol{tan}^2\theta\right)^{-1} = \frac{C^2}{n_1^2}$$

and

$$1+x'^2 = (1-\alpha^2x^2)\frac{n_1^2}{C^2} \quad (8)$$

After derivating the two sides of (8) versus *z*, we get:

$$x'' + \frac{\alpha^2n_1^2}{C^2}x = 0 \quad (9)$$

Because $n = n_1\sqrt{1-\alpha^2x^2}$ and

- $n = n_1$ at x=0
- $n = n_2$ at x=a

we get

$$\alpha = \frac{\sqrt{n_1^2 - n_2^2}}{a.n_1}$$

Finally, we get the equation for x''

$$x'' + \frac{n_1^2 - n_2^2}{a^2 (n_1^2 - \sin^2 \theta_i)} \cdot x = 0 \qquad (10)$$

1.c. The equation for the light trajectory is obtained by solving (10). This is an equation similar to that for an harmonic oscillation, which solution can be written right away

$$x = x_0 \sin(pz + q) \qquad (11)$$

with

$$p = \frac{1}{a}\sqrt{\frac{n_1^2 - n_2^2}{n_1^2 - \sin^2 \theta_i}}$$

The parameters p and q are determined from the boundary conditions:

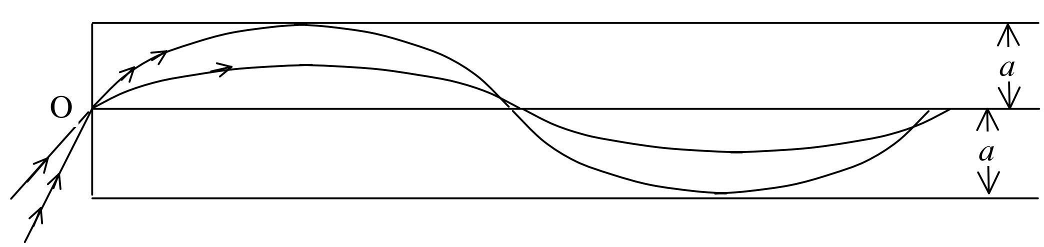
- at z=0, x=0, hence q=0
- at z=0 inside the fiber, $x' = \frac{dx}{dz} = \tan\theta_1$, then

$$x_0 = \frac{\tan\theta_1}{p} = \frac{a.\sin\theta_i}{\sqrt{n_1^2 - n_2^2}} \qquad (12)$$

The equation for the trajectory of the light inside the fiber is:

$$x = \frac{a \sin\theta_i}{\sqrt{n_1^2 - n_2^2}} . \sin\left(\sqrt{\frac{n_1^2 - n_2^2}{n_1^2 - \sin^2 \theta_i}} . \frac{z}{a}\right) \qquad (13)$$

1.d. Here is a sketch of the trajectories of two rays entering the fiber at O, under different incident angles.



2.a. The condition for the light to propagate along the fiber is that $x_0 \le a$. This means that:

$$\frac{a\,\sin\theta_i}{\sqrt{n_1^2 - n_2^2}} \le a$$

or:

$$\sin\theta_i \le \sqrt{n_1^2 - n_2^2} \qquad (14)$$

Thus the incident angle θ_i must not exceed $\theta_{i\,M}$, with

$$\sin\theta_{iM} = \sqrt{n_1^2 - n_2^2} = 0.344 \qquad (14a)$$

or:

$$\theta_i \le \theta_{i\,M} = Arc\,\sin\left(\sqrt{n_1^2 - n_2^2}\right) = Arc\,\sin 0.344 = 0.351\,\text{rad} = 20.13^{\circ}$$

2.b. The crossing points of the light beam with Oz axis must satisfy the condition $pz = k\pi$, with k - an integer. The z coordinates of these points are:

$$z = \frac{k\pi}{p} = k\pi a\sqrt{\frac{n_1^2 - \sin^2\theta_i}{n_1^2 - n_2^2}} \qquad (15)$$

except for θ_i =0.

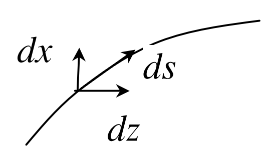
3.a. The rays entering the fiber at different incident angles have different trajectories. As a consequence, the propagation speeds of the rays along the fiber should be different.

The light trajectories are sinusoidal as given in (13). Let us calculate the time τ it takes the light to propagate from point O to its first crossing point with Oz axis. This is twice the time it takes the light to propagate from point O to its position most distant from Oz axis.

The time required for the light to travel a small segment ds along its trajectory is

$$dt = \frac{n}{c}ds = \frac{n}{c}\sqrt{dx^2 + dz^2} = \frac{n}{c}\sqrt{1 + \frac{dz^2}{dx^2}}.dx$$

$$= \frac{n}{c}\sqrt{1 + \left(\frac{1}{\tan\theta}\right)^2}.dx = \frac{n}{c}\frac{dx}{\sin\theta}$$



From (6), we have

$$dt = \frac{n_1^2\left(1 - \alpha^2 x^2\right)}{c.\sqrt{\sin^2\theta_i - n_1^2\alpha^2 x^2}}.dx$$

and

$$\frac{\tau}{2} = \int_0^{x_0} dt = \frac{n_1^2}{c}\left[\int_0^{x_0}\frac{dx}{\sqrt{\sin^2\theta_i - n_1^2\alpha^2x^2}} - \alpha^2\int_0^{x_0}\frac{x^2dx}{\sqrt{\sin^2\theta_i - n_1^2\alpha^2x^2}}\right] \qquad (16)$$

$$= \frac{n_1^2}{c}\left[I_1 - \alpha^2 I_2\right]$$

where

$$I_1 = \frac{1}{n_1\alpha} Arc\,\boldsymbol{sin} \frac{n_1 \alpha x}{\boldsymbol{sin}\,\theta_{\rm i}}\Bigg|_0^{x_0} = \frac{\pi a}{2\sqrt{n_1^2 - n_2^2}} \tag{17}$$

$$I_2 = \frac{-x\sqrt{\boldsymbol{sin}^2\,\theta_{\rm i} - n_1^2\alpha^2 x^2}}{2n_1^2\alpha^2}\Bigg|_0^{x_0} + \frac{\boldsymbol{sin}^2\,\theta_{\rm i}.Arc\,\boldsymbol{sin}\dfrac{n_1\alpha x}{\boldsymbol{sin}\,\theta_{\rm i}}}{2n_1^3\alpha^3}\Bigg|_0^{x_0} = \frac{\pi\,\boldsymbol{sin}^2\,\theta_{\rm i}}{4n_1^3\alpha^3} \tag{18}$$

Using (16), (17), (18), we obtain

$$\tau = \frac{\pi a.n_1^2}{c\sqrt{n_1^2 - n_2^2}}\left(1 - \frac{\boldsymbol{sin}^2\,\theta_{\rm i}}{2n_1^2}\right) \tag{19}$$

The propagation speed along the fiber is $v = \frac{z}{\tau}$, where z is the coordinate of the first crossing point, which is determined by (15) for $k = 1$. Because z and τ depend on the incident angle $\theta_{\rm i}$, v also depends on $\theta_{\rm i}$.

For $\theta_{\rm i} = \theta_{\rm iM}$, from (14a), we get

$$v_{\rm M} = \frac{\pi a n_2}{\sqrt{n_1^2 - n_2^2}} . \frac{2c\sqrt{n_1^2 - n_2^2}}{\pi a n_1^2}\left(1 + \frac{n_2^2}{n_1^2}\right)^{-1} = \frac{2cn_2}{n_1^2 + n_2^2} \tag{20}$$

and

$$v_{\rm M} = \frac{2 \times 2.998 \times 10^8 \times 1.460}{1.500^2 + 1.460^2} = 1.998 \times 10^8 \text{ m/s} \tag{20a}$$

The propagation speed of the light along the Oz axis is

$$v = \frac{c}{n_1} \tag{21}$$

because the refraction index is n_1 on the axis of the fiber.

The numerical value is

$$v_0 = \frac{2.998 \times 10^8}{1.5} = 1.999 \times 10^8 \text{ m/s} \tag{21a}$$

3.b. If the beam of the light pulses is formed by rays converging at O, then the rays with different incident angles has different propagation speeds. The two rays of incident angles $\theta_i = 0$ and $\theta_i = \theta_{\rm iM}$ arrive to the plane z with a time delay

$$\Delta t = \frac{z}{v_{\rm M}} - \frac{z}{v_0} = \frac{z}{c}.\frac{(n_1 - n_2)^2}{2n_2} \tag{22}$$

This means that a very short light pulse becomes a pulse of finite width Δt given by (22) at the plane z. If two consecutive pulses enter the fiber with a delay greater than Δt, then at the plane z, they are separated. Hence the repetition frequency of the pulses must not exceed the maximal value:

$$f_M = (\Delta t)^{-1} = \frac{2.c.n_2}{z.(n_1 - n_2)^2} \tag{23}$$

If $z = 1000\ \text{m}$, then

$$f_M = \frac{2 \times 2.998 \times 10^8 \times 1.460}{1000 \times (1.500 - 1.460)^2} = 547.1\text{MHz}$$