Theoretical Question 2

Strong Resistive Electromagnets

Resistive electromagnets are magnets with coils made of a normal metal such as copper or aluminum. Modern strong resistive electromagnets can provide steady magnetic fields higher than 30 tesla. Their coils are typically built by stacking hundreds of thin circular plates made of copper sheet metal with lots of cooling holes stamped in them; there are also insulators with the same pattern. When voltage is applied across the coil, current flows through the plates along a helical path to generate high magnetic fields in the center of the magnet.

In this question we aim to assess if a cylindrical coil (or *solenoid*) of many *turns* can serve as a magnet for generating high magnetic fields. As shown in Fig. 1, the center of the magnet is at O. Its cylindrical coil consists of N turns of copper wire carrying a current I uniformly distributed over the cross section of the wire. The coil's mean diameter is D and its length along the axial direction x is ℓ. The wire's cross section is rectangular with width a and height b. The turns of the coil are so tightly wound that the plane of each turn may be taken as perpendicular to the x axis and $\ell = Na$. In Table 1, data specifying physical dimensions of the coil are listed.

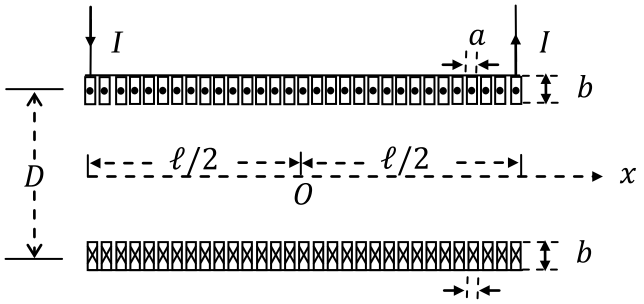


Figure 1

Table 1
$\ell = 12.0$ cm
$D = 6.0$ cm
$a = 2.0$ mm
$b = 5.0$ mm

In assessing if such a magnet can serve to provide high magnetic fields, two limiting factors must not be overlooked. One is the mechanical rigidity of the coil to withstand large Lorentz force on the field-producing current. The other is that the enormous amount of Joule heat generated in the wire must not cause excessive temperature rise. We shall examine these two factors using simplified models.

The Appendix at the end of the question lists some mathematical formulae and physical data which may be used if necessary.

Part A. Magnetic Fields on the Axis of the Coil

Assume $b \ll D$ so that one may regard the wire as a thin strip of width a. Let O be the origin of x coordinates. The direction of the current flow is as shown in Fig. 1.

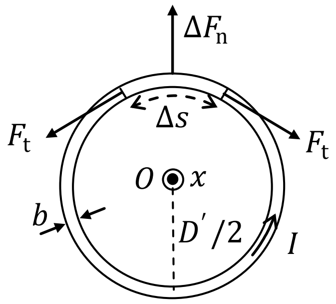
(a) Find the x-component $B(x)$ of the magnetic field on the axis of the coil as a function of x when the steady current passing through the coil is I. [1.0 point]

(b) Find the steady current I_0 passing through the coil if $B(0)$ is 10.0 T. Use data given in Table 1 when computing numerical values. [0.4 point]

Part B. The Upper Limit of Current

In Part B, we assume **the length ℓ of the coil is infinite** and $b \ll D$**.** Consider the turn of the coil located at $x = 0$. The magnetic field exerts Lorentz force on the current passing through the turn. Thus, as Fig. 2 shows, a wire segment of length Δs is subject to a normal force ΔF_n which tends to make the turn expand.

Figure 2



(c) Suppose that, when the current is I, the mean diameter of the expanded coil remains at a constant value D' larger than D, as shown in Fig. 2.

Find the outward normal force per unit length $\Delta F_n / \Delta s$. [1.2 point]

Find the tension F_t acting along the wire. [0.6 point]

(d) Neglect the coil's acceleration during the expansion. Assume the turn will break when the wire's *unit elongation* (i.e. tensile strain or fractional change of the length) is 60 % and *tensile stress* (i.e. tension per unit cross sectional area of the unstrained wire) is $\sigma_b = 455$ MPa. Let I_b be the current at which the turn will break and B_b the corresponding magnetic field at the center O.

Find an expression for I_b and then calculate its value. [0.8 point]

Find an expression for B_b and then calculate its value. [0.4 point]

Part C. The Rate of Temperature Rise

When the current I is 10.0 kA and the temperature T of the coil is 293 K, assume that the resistivity, the specific heat capacity at constant pressure, and the mass density of the wire of the coil are, respectively, given by $\rho_e = 1.72 \times 10^{-8}\ \Omega \cdot \mathrm{m}$, $c_p = 3.85 \times 10^2\ \mathrm{J/(kg \cdot K)}$ and $\rho_m = 8.98 \times 10^3\ \mathrm{kg \cdot m^{-3}}$.

(e) Find an expression for the *power density* (i.e. power per unit volume) of heat generation in the coil and then calculate its value. Use data in Table 1. [0.5 point]

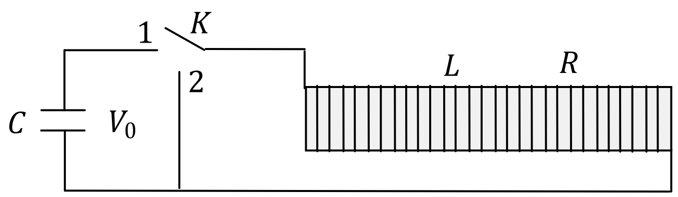
(f) Let $\dot{T}$ be the time rate of change of temperature in the coil. Find an expression for $\dot{T}$ and then calculate its value. [0.5 point]

Part D. A Pulsed-Field Magnet

If the large current needed for a strong magnet lasts only for a short time, the temperature rise caused by excessive Joule heating may be greatly reduced. This idea is employed in a *pulsed-field* magnet.

Thus, as shown in Fig. 3, a capacitor bank of capacitance C charged initially to a potential V_0 is used to drive the current I through the coil. The circuit is equipped with a switch K. The inductance L and resistance R of the circuit are assumed to be *entirely* due to the coil. The construct and dimensions of the coil are the same as given in Fig. 1 and Table 1. Assume R, L, and C to be independent of temperature and the magnetic field is the same as that of an infinite solenoid with $\ell \to \infty$.

Figure 3



(g) Find expressions for the inductance L and resistance R. [0.6 point]

Calculate the values of L and R. Use data given in Table 1. [0.4 point]

(h) At time $t = 0$, the switch K is thrown to position 1 and the current starts flowing. For $t \geq 0$, the charge $Q(t)$ on the positive plate of the capacitor and the current $I(t)$ entering the positive plate are given by

$$Q(t) = \frac{CV_0}{\sin\theta_0} e^{-\alpha t} \sin(\omega t + \theta_0), \tag{1}$$

$$I(t) = \frac{dQ}{dt} = \left(\frac{-\alpha}{\cos\theta_0}\right) \frac{CV_0}{\sin\theta_0} e^{-\alpha t} \sin\omega t, \tag{2}$$

in which α and ω are positive constants and θ_0 is given by

$$\tan\theta_0 = \frac{\omega}{\alpha}, \qquad 0 < \theta_0 < \frac{\pi}{2}. \tag{3}$$

Note that, if $Q(t)$ is expressed as a function of a new variable $t' \equiv (t + \theta_0/\omega)$, then $Q(t')$ and its time derivative $I(t)$ are identical in form except for an overall constant factor. The time derivative of $I(t)$ may therefore be obtained similarly without further differentiations.

Find α and ω in terms of R, L, and C. [0.8 point]

Calculate the values of α and ω when C is 10.0 mF. [0.4 point]

(i) Let I_{m} be the maximum value of $|I(t)|$ for $t > 0$. Find an expression for I_{m}. [0.6 point]
If $C = 10.0$ mF, what is the maximum value $V_{0\mathrm{b}}$ of the initial voltage V_0 of the capacitor bank for which I_{m} will not exceed I_{b} found in Problem (d)? [0.4 point]

(j) Suppose the switch K is moved instantly from position 1 to 2 when the absolute value of the current $|I(t)|$ reaches I_{m}. Let ΔE be the total amount of heat dissipated in the coil from $t = 0$ to ∞ and ΔT the corresponding temperature increase of the coil. Assume the initial voltage V_0 takes on the maximum value $V_{0\mathrm{b}}$ obtained in Problem (i) and the electromagnetic energy loss is only in the form of heat dissipated in the coil.
Find an expression for ΔE and then calculate its value. [1.0 point]
Find an expression for ΔT and then calculate its value. Note that the value for ΔT must be compatible with the assumption of constant R and L. [0.4 point]

Appendix

1. $\displaystyle\int_0^L \frac{dx}{(D^2 + x^2)^{3/2}} = \frac{1}{D^2}\left\{\frac{L}{(D^2 + L^2)^{1/2}}\right\}$

2. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

3. permeability of free space $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$

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