II.1. Consider a shell of width dr which is at the distance r from the centre of the star. Let ρ be the density of star. Gravitational Potential energy of shell is

$$dE_G = -G\frac{(4\pi r^3\rho/3)(4\pi r^2 dr\rho)}{r} \qquad \text{0.5 mark}$$

$$E_G = \int_0^R dE_G = -\frac{3}{5}\frac{GM^2}{R} \qquad \text{0.3 mark}$$

It is negative, i.e. gravitational force is radially inward. 0.2 mark

II.2. Total energy of the star $E = E_G + E_e$. 0.2 mark
At equilibrium, at $R = R_{wd}$

$$\frac{dE}{dR} = 0 \qquad \text{0.8 mark}$$

$$\frac{dE_G}{dR} = -\frac{dE_e}{dR}$$

$$\frac{3}{5}\frac{GM^2}{R_{wd}^2} = \frac{\hbar^2\pi^3}{10m_e 4^{2/3}}\left(\frac{3}{\pi}\right)^{7/3}\frac{2N_e^{5/3}}{R_{wd}^3}$$

$$R_{wd} = \frac{\hbar^2\pi^3}{6Gm_e 4^{2/3}}\left(\frac{3}{\pi}\right)^{7/3}\frac{2N_e^{5/3}}{M^2} \qquad \text{1.0 marks}$$

II.3. Since all the hydrogen is ionized, the number of protons $(N_p) = N_e$. Also $m_p \gg m_e$. Hence

$$N_e = N_p \approx \frac{M}{m_p} \qquad \text{0.5 mark}$$

$$R_{wd} = \frac{\hbar^2\pi^3}{6Gm_e 4^{2/3}}\left(\frac{3}{\pi}\right)^{7/3}\frac{2}{M^{1/3}m_p^{5/3}}$$

$$R_{wd} = 2.28 \times 10^4\,\text{km} \qquad \text{1.0 mark}$$

II.4. If r_{sep} is average separation between electrons then

$$N_e \times \frac{4}{3}\pi r_{sep}^3 \approx \frac{4}{3}\pi R_{wd}^3 \qquad \text{0.2 mark}$$

$$r_{sep}^3 = \frac{R_{wd}^3}{N_e} \approx R_{wd}^3\frac{m_p}{M} \qquad \text{0.2 mark}$$

$$r_{sep} = 2.13 \times 10^{-12}\ \text{m} \qquad \text{0.6 mark}$$

II.5. For a particle confined in a box of length r_{sep}, its de Broglie wavelength λ_{dB} for the ground state can be written as

$$\lambda_{dB} = 2r_{sep} \quad \boxed{0.2 \text{ mark}}$$

$$\text{and momentum } p = \frac{h}{\lambda_{dB}} \quad \boxed{0.3 \text{ mark}}$$

$$v \approx \frac{h}{2m_e r_{sep}} \quad \boxed{0.2 \text{ mark}}$$

$$= 1.08 \times 10^8 \text{ m.s}^{-1} \quad \boxed{0.3 \text{ mark}}$$

Correction: If one takes relativistic momentum

$$\lambda_{dB} = 2r_{sep} \quad \boxed{0.2 \text{ mark}}$$

$$p = \frac{h}{\lambda_{dB}} = \frac{h}{2r_{sep}} = \frac{m_e v}{\sqrt{1 - v^2/c^2}} \quad \boxed{0.3 \text{ mark}}$$

$$v = \frac{h}{\sqrt{4m_e^2 r^2 + \frac{h^2}{c^2}}} \quad \boxed{0.2 \text{ mark}}$$

$$= 1.06 \times 10^8 \text{ m.s}^{-1} \quad \boxed{0.3 \text{ mark}}$$

II.6. Similar to part II.2, at equilibrium

$$\frac{dE_G}{dR} = -\frac{dE_e^{rel}}{dR} \quad \boxed{0.3 \text{ mark}}$$

$$\frac{3}{5}\frac{GM^2}{R^2} = \frac{\pi^2}{4^{4/3}}\left(\frac{3}{\pi}\right)^{5/3}\frac{\hbar c}{R^2}N_e^{4/3} \quad \boxed{0.4 \text{ mark}}$$

For critical mass

$$M_c = \frac{3(5^3\pi)^{1/2}}{16m_p^2}\left(\frac{\hbar c}{G}\right)^{3/2} \quad \boxed{0.8 \text{ mark}}$$

Alternatively: Since the total energy

$$E_G + E_e^{rel} = \left(-\frac{3}{5}GM^2 + \frac{\pi^2}{4^{2/3}}\left(\frac{3}{\pi}\right)^{5/3}\hbar c N_e^{4/3}\right)\frac{1}{R}$$

must be minimized for equilibrium, one can argue that if coefficient of $1/R$ is positive then star would collapse otherwise it would expand.

II.7. For $M > M_c$

Expand	
Contract	✓

$\boxed{0.5 \text{ mark}}$

II.8.

$$M_c = 1.36 \times 10^{31} \text{ kg} \quad \boxed{1.0 \text{ mark}}$$

$$= 6.8\ M_S \quad \boxed{0.5 \text{ mark}}$$