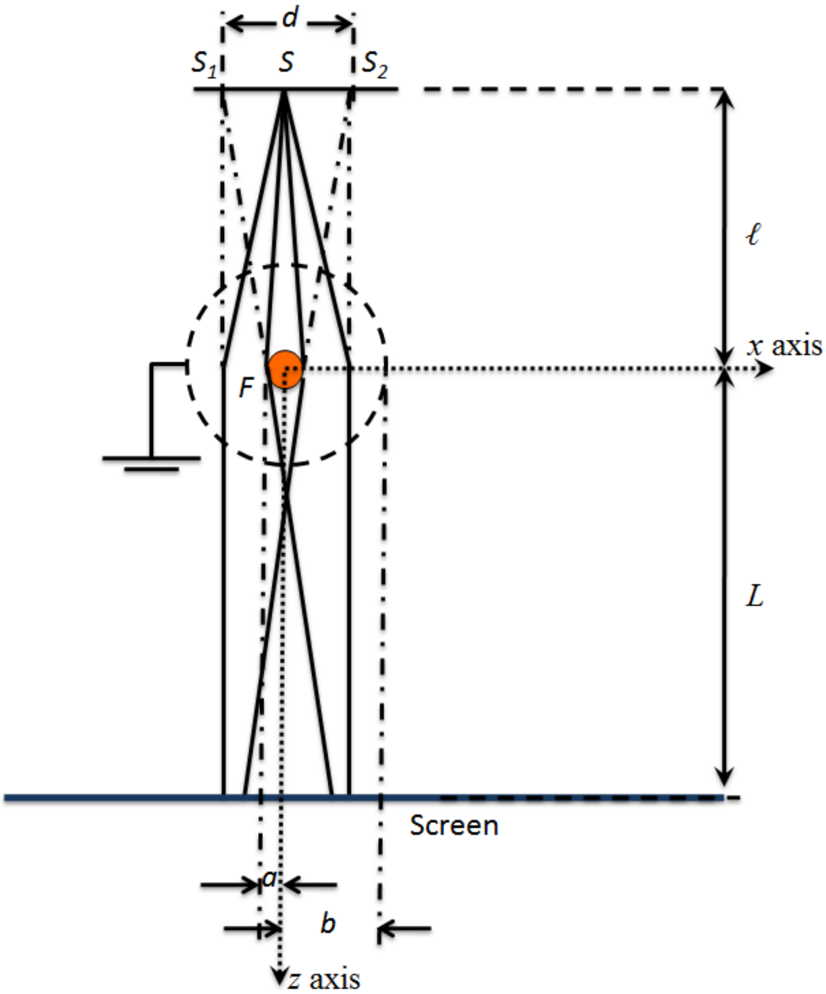
Question 2

The two-slit electron interference experiment was first performed by Möllenstedt *et al*, Merli-Missiroli and Pozzi in 1974 and Tonomura *et al* in 1989. In the two-slit electron interference experiment, a monochromatic electron point source emits particles at S that first passes through an electron "biprism" before impinging on an observational plane; S_1 and S_2 are virtual sources at distance d. In the diagram, the filament is pointing into the page. Note that it is a very thin filament (not drawn to scale in the diagram).



The electron "biprism" consists of a grounded cylindrical wire mesh with a fine filament F at the center. The distance between the source and the "biprism" is ℓ, and the distance between the distance between the "biprism" and the screen is L.

(a) (**2 points**) Taking the center of the circular cross section of the filament as the origin O, find the electric potential at any point (x,z) very near the filament in terms of V_a, a and b where V_a is the electric potential of the surface of the filament, a is the radius of the filament and b is the distance between the center of the filament and the cylindrical wire mesh. (Ignore mirror charges.)

Writing out $|E| = \frac{\lambda}{2\pi\epsilon_0 r} = -\frac{\partial}{\partial r}V(r)$

$$= -\frac{\partial}{\partial r}\frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{r}$$ (**1 point**)

Note that

$$V(r) = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{r}$$ (= 0 at the mesh)

Also at the edge of the filament, $V_a = V(r = a)$, so

$$V_a = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a}$$

Giving together

$$V(r) = V_a\frac{\ln(b/r)}{\ln(b/a)} \text{ where } r = \sqrt{x^2 + z^2}$$

(**1 point** for final expression)

(b) (**4 points**) An incoming electron plane wave with wave vector k_z is deflected by the "biprism" due to the x-component of the force exerted on the electron. Determine k_x the x-component of the wave vector due to the "biprism" in terms of the electron charge, e, v_z, V_a, k_z, a and b, where e and v_z are the charge and the z-component of the velocity of the electrons $(k_x \ll k_z)$. Note that $\vec{k} = \frac{2\pi\vec{p}}{h}$ where h is the Planck constant.

There are several ways to work out the solution:

A charge in an electric field will experience a force and hence a change in momentum. Note that potential energy of the electron (charge = $-e_0$) is $-e_0 V(r)$. Using impulse acting on the electron due to the electric field, (**2 points**)

$$\text{Impulse} = \frac{1}{v_z}\int_{-\infty}^{\infty}(-e_0)\left(-\frac{\partial V(x,z')}{\partial x}\right)dz'|_{x=a}$$

$$= -\frac{1}{v_z}\int_{-\infty}^{\infty}\frac{-e_0 V_a x}{(x^2+z'^2)\ln\frac{b}{a}}dz'|_{x=a}$$

$$= \frac{e_0 V_a \pi}{v_z \ln\frac{b}{a}}$$

$$\Rightarrow \quad k_x = \frac{e_0 V_a \pi}{\hbar v_z \ln\frac{b}{a}}$$

(**2 points** for final expression)

The alternative solution is to write down the equations of motion for the electrons (**2 points**) and determine the deflection of the electron as it passes through the "biprism":

$$\frac{\Delta x}{\Delta z} = \frac{\lambda e}{2\epsilon_0 m v_z^2}$$

Since $V_a = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a}$,

$$\frac{\Delta x}{\Delta z} = \frac{\pi e V_a}{m v_z^2 \ln\frac{a}{b}}$$

(**2 points** for final expression)

(c) Before the point *S*, the electrons are emitted from a field emission tip and accelerated through a potential V_0. Determine the wavelength of the electron in terms of the (rest) mass *m*, charge $-e_0$ and V_0,

(i) (**2 points**) assuming relativistic effects can be ignored.

Equating the kinetic energy to eV_0 (**1 point**)

$$\frac{h}{\lambda} = \sqrt{2m|-e_0|V_0}$$

$$\lambda = \frac{h}{\sqrt{2me_0V_0}}$$

(**1 point** for final expression)

(ii) (**3 points**) taking relativistic effects into consideration.

Consider

$$E^2 = (pc)^2 + (mc^2)^2$$

$$= \left(\frac{h}{\lambda}c\right)^2 + (mc^2)^2$$

$$\frac{h^2c^2}{\lambda^2} = (mc^2 + |-e_0|V_0)^2 - (mc^2)^2$$

$$= 2mc^2e_0V_0\left(1 + \frac{e_0V_0}{2m_0c^2}\right)$$

$$\lambda = \frac{h}{\sqrt{2me_0V_0\left(1 + \frac{e_0V_0}{2m_0c^2}\right)}}$$

(**1 point** for knowing relativitic E − p relation
1 point for manipulating the equations
1 point for final expression)

(d) In Tonomura et al experiment,

v_z = $c/2$,
V_a = 10 V,
V_0 = 50 kV,
a = 0.5 μm,
b = 5 mm,
ℓ = 25 cm,
L = 1.5 m,
h = 6.6×10^{-34} Js,
electron charge, $-e = -1.6 \times 10^{-19}$ C,
mass of electron, $m = 9.1 \times 10^{-31}$ kg,
and the speed of light *in vacuo*, $c = 3 \times 10^{8}$ ms^{-1}

(i) (**2 points**) calculate the value of k_x,

Previous equation:

$$k_x = \frac{e_0 V_a \pi}{\hbar v_z \ln \frac{b}{a}}$$

Plugging the relevant numbers into the equation gives:

(**1 point** for plugging the correct values)

$$k_x = \frac{\pi}{907} \text{Å}^{-1} \; or \; 3.46 \times 10^{7} m^{-1}$$

(**1 point** for final expression)

(ii) (**2 points**) determine the fringe separation of the interference pattern on the screen,

Fringe separation is given by $\frac{1}{2}\frac{2\pi}{k_x} = 907\text{Å}$

$$\left(\begin{array}{c} \textbf{1 point} \text{ for formula, note the factor } \frac{1}{2} \\ \textbf{1 point} \text{ for final expression with units} \end{array} \right)$$

(iii) (**1 point**) If the electron wave is a spherical wave instead of a plane wave, is the fringe spacing larger, the same or smaller than the fringe spacing calculated in (ii)?

Larger. (**1 point** for the correct answer)

(iv) (**2 points**) In part (c), determine the percentage error in the wavelength of the electron using non-relativistic approximation.

Non-relativistic:

$$\frac{h}{\lambda} = \sqrt{2me_0V_0}$$
$$\lambda_{nonrel} = \frac{h}{\sqrt{2me_0V_0}}$$
$$= 5.4697 \times 10^{-12}m$$

Relativistic:

$$\lambda_{rel} = \frac{h}{\sqrt{2meV_0\left(1 + \frac{eV_0}{2m_0c^2}\right)}}$$
$$= 5.3408 \times 10^{-12}m$$

Percentage error:

$$\text{Error} = \frac{\lambda_{nonrel} - \lambda_{rel}}{\lambda_{rel}}$$
$$= 0.024$$

or 2.4 percent.

$$\left(\begin{array}{l}\textbf{1 point} \text{ for working out non} - \text{relativistic and relativistic wavelength} \\ \textbf{1 point} \text{ for final expression}\end{array}\right)$$

(v) (**2 points**) Calculate the distance *d* between the apparent double slits.

The double slit formula is given by

$$y = \frac{m\lambda(\ell + L)}{d}$$

where *m* is the order and *y* is the distance for maximum intensity from the central fringe.

In this case, since the fringe spacing is 907Å,

$$d = 1.03 \times 10^{-4}m$$

$$\left(\begin{array}{l}\textbf{1 point} \text{ for formula} \\ \textbf{1 point} \text{ for final numerical answer}\end{array}\right)$$