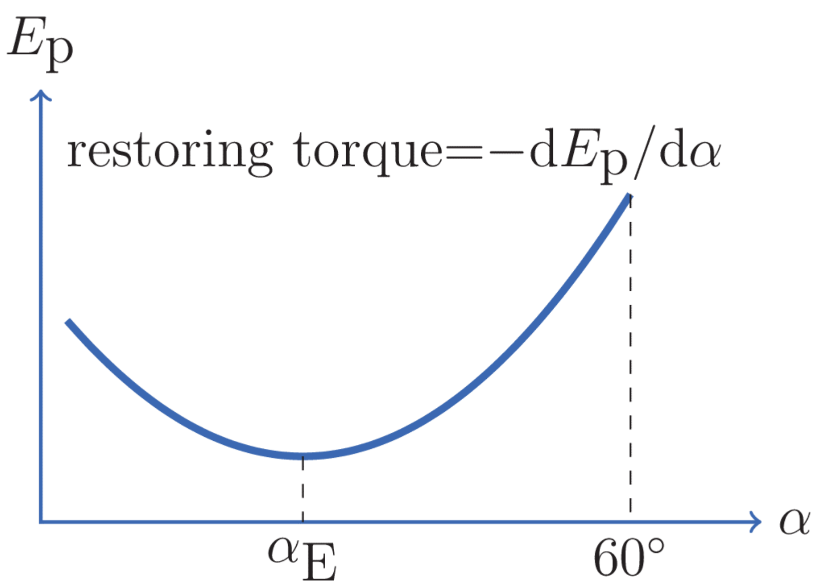
(**Full Mark = 20**)

Part	Model Answer	Marks
A1	The potential energy for $N = 2$ is: $E_{\mathrm{p}}(\alpha) = Mg \cdot y_{\mathrm{c\cdot m.(0,0)}} \times 4 + Mg \cdot \Delta y \times 2$ **(0.5 points)** - Eq. (1) where $y_{\mathrm{c\cdot m.(0,0)}} = -\frac{\sqrt{3}l}{3}\sin\left(\frac{\pi}{6}+\alpha\right)$ **(0.5 points)** - Eq. (2) is the y coordinate of center of mass of triangle (0,0), and $\Delta y = y_{\mathrm{A(0,1)}} - y_{\mathrm{A(0,0)}}$ $= -l\left[\sin\left(\frac{\pi}{3}+\alpha\right)+\sin\left(\frac{\pi}{3}-\alpha\right)\right]$ $= -\sqrt{3}l\cos\alpha$ **(0.5 points)** - Eq. (3) is the translational difference of two neighbouring triangles in y-direction. Solving Eqs. (1), (2) and (3), we obtain $E_{\mathrm{p}}(\alpha) = -\frac{2}{3}Mgl\left(4\sqrt{3}\cos\alpha + 3\sin\alpha\right)$ **(0.5 points)** - Eq. (4)	**2**
A2	 At equilibrium, the potential energy reaches a minimum, which gives: $\left.\frac{dE_{\mathrm{p}}(\alpha)}{d\alpha}\right\|_{\alpha=\alpha_{\mathrm{E}}} = 0$ **(0.5 points)** - Eq. (5) $\sqrt{3}\sin\alpha_{\mathrm{E}} + 3\cos\alpha_{\mathrm{E}} = 0$ - Eq. (6)	**1**

	or $$\alpha_{\mathrm{E}} = \tan^{-1}\frac{\sqrt{3}}{4}$$ **(0.5 point)** - Eq. (7)	
A3	If the total energy of the oscillation has the following form $$E(\Delta\alpha, \Delta\dot{\alpha}) = E_{\mathrm{p}} + E_{\mathrm{k}} = \frac{1}{2}K(\Delta\alpha)^2 + \frac{1}{2}I(\Delta\dot{\alpha})^2 ,$$ **(0.5 points)** - Eq. (8) where E_{p} and E_{k} are the potential and kinetic energies of the system respectively, then the motion is a simple harmonic oscillation with angular frequency $\omega = \sqrt{K/I}$. Here $\Delta\alpha = \alpha - \alpha_{\mathrm{E}}$. Under a small perturbation, the potential energy change is: $$\Delta E_{\mathrm{p}} \approx \frac{1}{2}\frac{d^2E_{\mathrm{p}}}{\mathrm{d}\alpha^2}\bigg\|_{\alpha=\alpha_{\mathrm{E}}}(\Delta\alpha)^2$$ $$= \left(\frac{1}{2}\right)\left(\frac{2}{3}Mgl\right)(4\sqrt{3}\cos\alpha_{\mathrm{E}} + 3\sin\alpha_{\mathrm{E}})(\Delta\alpha)^2$$ $$= \frac{\sqrt{57}}{3}Mgl(\Delta\alpha)^2$$ **(1 point)** - Eq. (9) The total kinetic energy of the system includes the translational kinetic energy of every plate and the rotational kinetic energy of every plate relative to its center of mass $$E_{\mathrm{k}} = \sum E_{\mathrm{k}}^{\mathrm{trans}} + \sum E_{\mathrm{k}}^{\mathrm{rot}}$$ - Eq. (10) The rotational kinetic energy is $$\sum E_{\mathrm{k}}^{\mathrm{rot}} = 4 \times \frac{1}{2}\frac{Ml^2}{12}(\Delta\dot{\alpha})^2 = \frac{1}{6}Ml^2(\Delta\dot{\alpha})^2$$ **(0.5 points)** - Eq. (11) $E_{\mathrm{k}}^{\mathrm{trans}}$ can be obtained by considering the motion of the center of mass of each triangle and setting $N = 2$. $$x_{\mathrm{c.m.}(m,n)} = m(2l\cos\alpha) + n(2l\cos\alpha)\cos\frac{\pi}{3} + \frac{l}{\sqrt{3}}\cos\left(\alpha + \frac{\pi}{6}\right),$$ $$y_{\mathrm{c.m.}(m,n)} = -n(2l\cos\alpha)\sin\frac{\pi}{3} - \frac{l}{\sqrt{3}}\sin\left(\alpha + \frac{\pi}{6}\right).$$ **(0.5 point)** Differentiating and substituting $$\sin\alpha = \frac{\sqrt{3}}{\sqrt{19}}, \cos\alpha = \frac{4}{\sqrt{19}}, \sin\left(\alpha + \frac{\pi}{6}\right) = \frac{7}{2\sqrt{19}}, \cos\left(\alpha + \frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2\sqrt{19}},$$	**5**

$$\dot{x}_{\text{c.m.}(m,n)} = -\left(2m + n + \frac{7}{6}\right)\frac{3}{\sqrt{57}} l\Delta\dot{\alpha}, \qquad \dot{y}_{\text{c.m.}(m,n)} = \frac{3(2n-1)}{2\sqrt{19}} l\Delta\dot{\alpha}.$$

$$v^2_{\text{c.m.}(m,n)} = \dot{x}^2_{\text{c.m.}(m,n)} + \dot{y}^2_{\text{c.m.}(m,n)} = \frac{(12m+6n+7)^2+27}{228} l^2(\Delta\dot{\alpha})^2,$$ **(1 point)**

$$E^{\text{trans}}_{\text{c.m.,k}} = \frac{M}{2}\left[v^2_{\text{c.m.}(0,0)} + v^2_{\text{c.m.}(0,1)} + v^2_{\text{c.m.}(1,0)} + v^2_{\text{c.m.}(1,1)}\right] = \frac{164}{57} Ml^2(\Delta\dot{\alpha})^2.$$

$$E^{\text{trans}}_{\text{k}} = E^{\text{trans}}_{\text{c.m.,k}} + E^{\text{rot}}_{\text{k}} = \frac{347}{114} Ml^2(\Delta\dot{\alpha})^2.$$ **(1 point)**

Alternatively, another way to get E_k^{trans} is based on the center of mass of the whole system:

$$E_{\text{k}} = \sum E^{\text{trans}}_{\text{c.m.,k}} + \sum E^{\text{rot}}_{\text{r.c.,k}}$$ **(0.5 points)** - Eq. (12)

where

$$E^{\text{trans}}_{\text{r.c.,k}} = \frac{M}{2}\left[v^2_{\text{r.c.}(0,0)} + v^2_{\text{r.c.}(1,0)} + v^2_{\text{r.c.}(0,1)} + v^2_{\text{r.c.}(1,1)}\right]$$ - Eq. (13)

is the translational kinetic energy relative to the center of mass of the system and

$$E^{\text{trans}}_{\text{c.m.,k}} = \frac{4M}{2} v^2_{\text{c.m.}}$$ - Eq. (14)

is the translational kinetic energy of the center of mass of the system.

The center of mass of each of the 2×2 = 4 triangles always form diamond shape with lateral length $2l\cos\alpha$. The center of mass of the whole system is at the center of the diamond shape. Hence

$$v_{\text{r.c.}(0,0)} = v_{\text{r.c.}(1,1)} = \left.\frac{d(\sqrt{3}l\cos\alpha)}{d\alpha}\right|_{\alpha=\alpha_{\text{E}}} \Delta\dot{\alpha}$$

$$v_{\text{r.c.}(1,0)} = v_{\text{r.c.}(0,1)} = \left.\frac{d(l\cos\alpha)}{d\alpha}\right|_{\alpha=\alpha_{\text{E}}} \Delta\dot{\alpha}$$ - Eq. (15)

Substituting Eqs. (14) and (15) into Eq. (13), we obtain

$$E^{\text{trans}}_{\text{r.c.,k}} = 4\sin\alpha_{\text{E}}^2\, Ml^2(\Delta\dot{\alpha})^2$$ - Eq. (16)

For $E^{\text{trans}}_{\text{c.m.,k}}$,

$$v_{\text{c.m.}} = \sqrt{\left(\frac{\mathrm{d}x_{\text{c.m.}}}{\mathrm{d}\alpha}\right)^2 + \left(\frac{\mathrm{d}y_{\text{c.m.}}}{\mathrm{d}\alpha}\right)^2}\Bigg|_{\alpha=\alpha_{\text{E}}} \Delta\dot{\alpha} \qquad \text{- Eq. (17)}$$

is the velocity of the center-of-mass of the four triangular plates, with

$$x_{\text{c.m.}} = x_{\text{c.m.}(0,0)} + \frac{1}{2}\left(x_{\text{B}(0,0)} + x_{\text{A}(1,0)}\right)$$

$$= \frac{\sqrt{3}l}{3}\cos\left(\frac{\pi}{6} + \alpha\right) + \frac{3}{2}l\cos\alpha \qquad \text{- Eq. (18)}$$

$$y_{\text{c.m.}} = y_{\text{c.m.}(0,0)} + \frac{1}{2}\Delta y$$

$$= -\frac{\sqrt{3}l}{3}\sin\left(\frac{\pi}{6} + \alpha\right) - \frac{\sqrt{3}}{2}l\cos\alpha \qquad \text{- Eq. (19)}$$

Substituting Eqs. (17), (18) and (19) and into Eq. (14), we obtain

$$E_{\text{c.m.,k}}^{\text{trans}} = \left(\frac{2}{3} + 10\sin^2\alpha_E\right)Ml^2(\Delta\dot{\alpha})^2 \qquad \text{- Eq. (20)}$$

(0.5 points)

Combining Eqs. (12), (16) and (20), we obtain

$$E_{\text{k}} = E_{\text{k}}^{\text{rot}} + E_{\text{r.c.,k}}^{\text{trans}} + E_{\text{c.m.,k}}^{\text{trans}}$$

$$= \left(\frac{5}{6} + 14\sin^2\alpha_E\right)Ml^2(\Delta\dot{\alpha})^2$$

$$= \frac{347}{114}Ml^2(\Delta\dot{\alpha})^2 \qquad \text{- Eq. (21)}$$

(1.5 points)

According to Eqs. (8), (9) and (21),

$$f = \frac{1}{2\pi}\sqrt{\frac{\frac{\sqrt{57}}{3}Mgl}{\frac{347}{114}Ml^2}} = \frac{1}{2\pi}\sqrt{\frac{38\sqrt{57}}{347}\frac{g}{l}} \qquad \text{- Eq. (22)}$$

(0.5 points)

[Note 1: 0.5 point should be deducted if there are numerical mistakes, but all steps are correct.

Note 2: A rough estimate of $f \sim \sqrt{\frac{g}{l}}$ can get 0.5 points out of 5 points.]

B1 For arbitrary N, the total potential energy **3**

$$E_{\mathrm{p}} = \sum_{m,n=0}^{N-1} E_{\mathrm{p}}(m,n) \quad \text{- Eq. (23)}$$

where

$$E_{\mathrm{p}}(m,n) = \frac{1}{3} Mg\left[y_{\mathrm{A}(m,n)} + y_{\mathrm{B}(m,n)} + y_{\mathrm{C}(m,n)}\right] \quad \text{- Eq. (24)}$$

(0.5 points for Eqs. (23) and (24))

and

$$y_{\mathrm{A}(m,n)} = -nl\sin\left(\frac{\pi}{3} - \alpha\right) - nl\sin\left(\frac{\pi}{3} + \alpha\right) = -\sqrt{3}nl\cos\alpha$$

$$y_{\mathrm{B}(m,n)} = y_{\mathrm{A}(m,n)} - l\sin\alpha = -\sqrt{3}nl\cos\alpha - l\sin\alpha$$

$$y_{\mathrm{C}(m,n)} = y_{\mathrm{A}(m,n)} - l\sin\left(\frac{\pi}{3} + \alpha\right) = -\sqrt{3}nl\cos\alpha - l\sin\left(\frac{\pi}{3} + \alpha\right) \quad \text{- Eq. (25)}$$

(0.5 points for all three correct coordinates)

Thus,

$$E_{\mathrm{p}}(m,n) = -\frac{1}{3} Mgl\left[3\sqrt{3}n\cos\alpha + \sin\alpha + \sin\left(\frac{\pi}{3} + \alpha\right)\right] \quad \text{- Eq. (26)}$$

and

$$E_{\mathrm{p}} = \sum_{m,n=0}^{N-1} E_{\mathrm{p}}(m,n)$$

$$= -\frac{1}{3} Mgl \sum_{m,n=0}^{N-1}\left[3\sqrt{3}n\cos\alpha + \sin\alpha + \sin\left(\frac{\pi}{3} + \alpha\right)\right] \quad \text{- Eq. (27)}$$

(0.5 points)

Using the mathematical relations

$$\sum_{m=0}^{N-1} 1 = \sum_{n=0}^{N-1} 1 = N$$

and

$$\sum_{m=0}^{N-1} m = \sum_{n=0}^{N-1} n = \frac{N(N-1)}{2} \quad \text{- Eq. (28),}$$

Eq. (27) becomes

	$$E_p = -\frac{1}{3}N^2Mgl\left[\frac{3\sqrt{3}(N-1)\cos\alpha}{2} + \sin\alpha + \sin\left(\frac{\pi}{3}+\alpha\right)\right]$$ or $$= -\frac{1}{3}N^2Mgl\left[\frac{\sqrt{3}(3N-2)\cos\alpha}{2} + \frac{3}{2}\sin\alpha\right]$$ **(1 points)** - Eq. (29) At equilibrium, $\frac{dE_p}{d\alpha} = 0$, therefore $$-\frac{3\sqrt{3}(N-1)\sin\alpha'_E}{2} + \cos\alpha'_E + \cos\left(\frac{\pi}{3}+\alpha'_E\right) = 0$$ - Eq. (30) $$\alpha'_E = \tan^{-1}\left(\frac{\sqrt{3}}{3N-2}\right)$$ **(0.5 points)** - Eq. (31) **[Remark: Increasing α lowers each triangle relative to its vertex A, but globally raises the system, i.e. the bottom tube is raised higher. When $N \to \infty$, the global displacement dominates, consequently $\alpha \to 0$.]**	
B2	Under a small perturbation, the potential energy change, according to Eq. (29) is $$\Delta E_p \approx \frac{1}{2}\frac{d^2E_p}{d\alpha^2}\bigg\|_{\alpha=\alpha'_E}(\Delta\alpha)^2 \sim N^3 \text{ or } \gamma_1 = 3$$ **(0.5 points)** - Eq. (32) **[Remark: There are N^2 triangles and the y coordinate of the total center of mass is proportional to N, hence $E_p \sim N^3$ and $\gamma_1 = 3$. Using this argument to derive the correct γ_1 can also get 0.5 points.]** The kinetic energy of a triangle includes the translational energy of its center of mass and the rotational energy about its center of mass. Hence the total kinetic energy of the N^2 triangles is $$E_k = \sum_{m,n} E_{c.m.(m,n)} + \sum_{m,n} E_{r.c.(m,n)}$$ - Eq. (33) where $$E_{r.c.(m,n)} = \frac{1}{2}\frac{Ml^2}{12}(\Delta\dot{\alpha})^2 = \frac{1}{24}Ml^2(\Delta\dot{\alpha})^2 \sim 1$$ - Eq. (34) and $$E_{c.m.(m,n)} = \frac{M}{2}v^2_{c.m.(m,n)}$$ $$= \frac{M(\Delta\dot{\alpha})^2}{2}\left[\left(\frac{dx_{c.m.(m,n)}}{d\alpha}\right)^2 + \left(\frac{dy_{c.m.(m,n)}}{d\alpha}\right)^2\right]_{\alpha=\alpha'_E}$$ **(0.5 points)** - Eq. (35)	**3**

Since

$$x_{\text{c.m.}(m,n)} = x_{\text{A}(m,n)} + \frac{\sqrt{3}l}{3}\cos\left(\frac{\pi}{6}+\alpha\right)$$

$$= (2m+n)l\cos\alpha + \frac{l}{2}\cos\alpha - \frac{\sqrt{3}l}{6}\sin\alpha$$

and

$$y_{\text{c.m.}(m,n)} = y_{\text{A}(m,n)} + \frac{\sqrt{3}l}{3}\sin\left(\frac{\pi}{6}+\alpha\right)$$

$$= \sqrt{3}nl\cos\alpha + \frac{\sqrt{3}l}{6}\cos\alpha + \frac{l}{2}\sin\alpha \qquad \text{- Eq. (36)}$$

(0.5 points for correct *x* and *y*)

$$\frac{\mathrm{d}x_{\text{c.m.}(m,n)}}{\mathrm{d}\alpha} = \left[-(2m+n)\sin\alpha - \frac{1}{2}\sin\alpha - \frac{\sqrt{3}}{6}\cos\alpha\right]l$$

$$\frac{\mathrm{d}y_{\text{c.m.}(m,n)}}{\mathrm{d}\alpha} = \left[-\sqrt{3}n\sin\alpha - \frac{\sqrt{3}}{6}\sin\alpha + \frac{1}{2}\cos\alpha\right]l$$

we have

$$E_{\text{c.m.}(m,n)} = \frac{1}{2}Ml^2(\Delta\dot{\alpha})^2\begin{bmatrix}(4m^2+4n^2+4mn+2m+2n)\sin^2\alpha'_{\text{E}} \\ +\frac{2\sqrt{3}}{3}(m-n)\sin\alpha'_{\text{E}}\cos\alpha'_E+\frac{1}{3}\end{bmatrix} \qquad \text{- Eq. (37)}$$

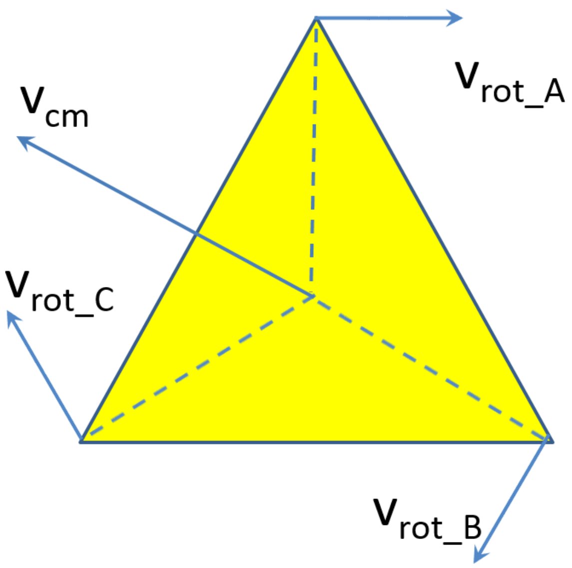
Since $\alpha'_{\text{E}} \sim \frac{1}{N}$ in Eq. (31), we have

$$E_{\text{c.m.}(m,n)} = A\cdot N^2\cdot\frac{1}{N^2} + B\cdot N\cdot\frac{1}{N} + C \sim 1 \quad \textbf{(0.5 points)} \qquad \text{- Eq. (38)}$$

According to Eqs. (33), (34) and (38), we have

$$E_{\text{k}} = \sum_{m,n} E_{\text{c.m.}(m,n)} + \sum_{m,n} E_{\text{r.c.}(m,n)} \sim N\times N\times 1 \sim N^2$$

$$\text{or } \gamma_2 = 2 \quad \textbf{(0.5 points)} \qquad \text{- Eq. (39)}$$

	[Remarks: $E_k \sim N^2$ because there are N^2 triangles, each contribute $E_{r.c.}(m,n) \sim 1$ (relative-to-center-of-mass kinetic energy) and $E_{c.m.}(m,n) \sim 1$ (center-of-mass kinetic energy).] Note that $E_{r.c.}(m,n) \sim 1$ is true for arbitrary α while $E_{c.m.}(m,n) \sim 1$ is only true for the special case of $\alpha'_E \to 0$ or $N \to \infty$. Therefore $$f'_E \sim \sqrt{\frac{E_p}{E_k}} \sim \sqrt{N}$$ or $\gamma_3 = 0.5$ **(0.5 points)** - Eq. (40)	
C1	The minimum force should act on the farthest triangle ($N-1$, $N-1$), whose motion can be decomposed into the motion of the center of mass and the rotation around the center of mass: $\vec{v} = \vec{v}_{c.m.} + \vec{v}_{rot}$. As shown in the figure, $\vec{v}_{rot}$ of vertex C makes the smallest angle relative to the direction of $\vec{v}_{c.m.}$ near $\alpha_m \equiv \pi/3$. Hence its displacement is the largest and its corresponding force is minimum, i.e. the minimum force should act on vertex C($N-1$, $N-1$). **(1 point)**  **[Remarks: A rigorous calculation is given in Appendix 3.]**	**1**

C2	At $\alpha = \alpha_\mathrm{m} \equiv \pi/3$, a small change in α will change the potential energy by: $$\Delta E_\mathrm{p}(\alpha_\mathrm{m}) = \left.\frac{dE_\mathrm{p}}{d\alpha}\right\vert_{\alpha=\alpha_\mathrm{m}} \Delta\alpha$$ $$= \frac{1}{3} N^2 Mgl \left[\left(\frac{3\sqrt{3}N}{2} - \sqrt{3}\right)\sin\alpha_\mathrm{m} - \frac{3}{2}\cos\alpha_\mathrm{m}\right]\Delta\alpha$$ $$= \frac{3}{4}(N-1)N^2 Mgl\Delta\alpha$$ **(1 point)** - Eq. (41) The displacement of $C(m,n)$ point is $$\Delta x_{\mathrm{C}(m,n)} = -\left[(2m+n)\sin\alpha_\mathrm{m} - \sin\left(\frac{\pi}{3}+\alpha_\mathrm{m}\right)\right] l\Delta\alpha$$ $$= \frac{(2m+n+1)\sqrt{3}}{2} l\Delta\alpha$$ **(0.5 points)** $$\Delta y_{\mathrm{C}(m,n)} = -\left[\sqrt{3}n\sin\alpha_\mathrm{m} - \cos\left(\frac{\pi}{3}+\alpha_\mathrm{m}\right)\right] l\Delta\alpha$$ $$= \frac{(3n+1)}{2} l\Delta\alpha$$ **(0.5 points)** For C(N-1,N-1), $\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)^2} = (3N-2)(l\Delta\alpha)$. **(1 point)** Hence $$F_\mathrm{min} = \frac{\Delta E_\mathrm{p}(\alpha_\mathrm{m})}{\Delta r_\mathrm{max}} = \frac{3(N-1)N^2}{4(3N-2)} Mg$$ **(1 point)** - Eq. (42) and $$\theta_{F_\mathrm{min}} = \tan^{-1}\left[\frac{\Delta y_{\mathrm{C}(N-1,N-1)}}{\Delta x_{\mathrm{C}(N-1,N-1)}}\right] + \pi$$ $$= -\tan^{-1}\frac{\sqrt{3}}{3} + \pi = \frac{5\pi}{6}$$ **(1 point)** - Eq. (43) **[Remarks: This θ_{F_min} is not perpendicular to the $C(N\text{-}1,N\text{-}1)$–A(0,0) direction because of the constraints of the tunes, e.g. $A(1,0)$, $A(2,0)$, $A(3,0)$, $\cdots$, are also the holding points.]**	**5**

N/A

Appendix 1:

(a) Calculation of the exact E_{p}, E_{k} and f_{E}' in Parts (C), (D) and € for arbitrary N

Under a small perturbation, the potential energy change is

$$\Delta E_{\mathrm{p}} \approx \frac{1}{2}\frac{d^2 E_{\mathrm{p}}}{d\alpha^2}\bigg|_{\alpha=\alpha_{\mathrm{E}}'} (\Delta\alpha)^2$$

$$= \frac{1}{3}N^2 Mgl\left(\frac{3\sqrt{3}N - 2\sqrt{3}}{2}\cos\alpha_{\mathrm{E}}' + \frac{3}{2}\sin\alpha_{\mathrm{E}}'\right)\frac{(\Delta\alpha)^2}{2}$$

$$= \frac{\sqrt{3(3N-2)^2+9}}{12} N^2 Mgl(\Delta\alpha)^2 \quad \text{- Eq. (44)}$$

The kinetic energy of a triangle includes the translational energy of its center of mass and the rotational energy around its center of mass. Hence the total kinetic energy of the N^2 triangles is

$$E_{\mathrm{k}} = \sum_{m,n} E_{\mathrm{c.m.}(m,n)} + \sum_{m,n} E_{\mathrm{r.c.}(m,n)} \quad \text{- Eq. (45)}$$

where

$$E_{\mathrm{r.c.}(m,n)} = \frac{1}{2}\frac{Ml^2}{12}(\Delta\dot{\alpha})^2 = \frac{1}{24}Ml^2(\Delta\dot{\alpha})^2 \quad \text{- Eq. (46)}$$

and

$$E_{\mathrm{c.m.}(m,n)} = \frac{M}{2}v_{\mathrm{c.m.}(m,n)}^2$$

$$= \frac{M(\Delta\dot{\alpha})^2}{2}\left[\left(\frac{\mathrm{d}x_{\mathrm{c.m.}(m,n)}}{\mathrm{d}\alpha}\right)^2 + \left(\frac{\mathrm{d}y_{\mathrm{c.m.}(m,n)}}{\mathrm{d}\alpha}\right)^2\right]_{\alpha=\alpha_{\mathrm{E}}'} \quad \text{- Eq. (47)}$$

Since

$$x_{\mathrm{c.m.}(m,n)} = x_{\mathrm{A}(m,n)} + \frac{\sqrt{3}l}{3}\cos\left(\frac{\pi}{6}+\alpha\right)$$

$$= (2m+n)l\cos\alpha + \frac{l}{2}\cos\alpha - \frac{\sqrt{3}l}{6}\sin\alpha$$

and

$$y_{\mathrm{c.m.}(m,n)} = y_{\mathrm{A}(m,n)} - \frac{\sqrt{3}l}{3}\sin\left(\frac{\pi}{6}+\alpha\right)$$

$$= -\sqrt{3}nl\cos\alpha - \frac{\sqrt{3}l}{6}\cos\alpha - \frac{l}{2}\sin\alpha$$ - Eq. (48)

Hence,

$$\frac{\mathrm{d}x_{\text{c.m.}(m,n)}}{\mathrm{d}\alpha} = \left[-(2m+n)\sin\alpha - \frac{1}{2}\sin\alpha - \frac{\sqrt{3}}{6}\cos\alpha\right] l$$

$$\frac{\mathrm{d}y_{\text{c.m.}(m,n)}}{\mathrm{d}\alpha} = \left[-\sqrt{3}n\sin\alpha + \frac{\sqrt{3}}{6}\sin\alpha - \frac{1}{2}\cos\alpha\right] l$$

We have

$$E_{\text{c.m.}(m,n)} = \frac{1}{2}Ml^2(\Delta\dot{\alpha})^2 \begin{bmatrix} (4m^2+4n^2+4mn+2m+2n)\sin^2\alpha'_{\mathrm{E}} \\ +\frac{2\sqrt{3}}{3}(m-n)\sin\alpha'_{\mathrm{E}}\cos\alpha'_{\mathrm{E}} + \frac{1}{3} \end{bmatrix}$$ - Eq. (49)

and

$$E_k = \sum_{m,n} E_{\text{c.m.}(m,n)} + \sum_{m,n} E_{\text{r.c.}(m,n)}$$

$$= \left[\frac{1}{6}(11N-1)(N-1)\sin^2\alpha'_{\mathrm{E}} + \frac{5}{24}\right] N^2Ml^2(\Delta\dot{\alpha})^2$$

$$= \left[\frac{(11N-1)(N-1)}{2(3N-2)^2+6} + \frac{5}{24}\right] N^2Ml^2(\Delta\dot{\alpha})^2$$ - Eq. (50)

With Eqs. (44) and (50), we have

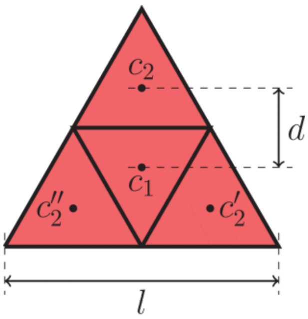
$$f'_{\mathrm{E}} = \frac{1}{2\pi}\sqrt{\frac{\frac{\sqrt{3(3N-2)^2+9}}{12}N^2Mgl}{\left[\frac{(11N-1)(N-1)}{2(3N-2)^2+6} + \frac{5}{24}\right]N^2Ml^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\sqrt{3(3N-2)^2+9}}{\left[\frac{12(11N-1)(N-1)}{(3N-2)^2+3}+5\right]}\frac{g}{l}}$$ - Eq. (51)

(b) <u>Center of mass movement of the whole system</u>

According to Eq. (48), we have

$$x_{\text{c.m.(sys.)}}(\alpha) = \frac{\sum_{m,n} x_{\text{c.m.}(m,n)}}{N^2}$$

	$$= \frac{\sum_{m,n}\left[(2m+n)l\cos\alpha + \frac{l}{2}\cos\alpha - \frac{\sqrt{3}l}{6}\sin\alpha\right]}{N^2}$$ $$= \left(\frac{3N-2}{2}\right) l\cos\alpha - \frac{\sqrt{3}l}{6}\sin\alpha$$ and $$y_{\text{c.m.}(m,n)}(\alpha) = \frac{\sum_{m,n} y_{\text{c.m.}(m,n)}}{N^2}$$ $$= -\frac{\sum_{m,n}\left[\sqrt{3}nl\cos\alpha + \frac{\sqrt{3}l}{6}\cos\alpha + \frac{l}{2}\sin\alpha\right]}{N^2}$$ $$= -\left(\frac{3N-2}{6}\right)\sqrt{3}l\cos\alpha - \frac{l\sin\alpha}{2}$$ - Eq. (52) Eq. (52) is the trajectory of the center of mass for the whole system, which is not a straight line.	
	Appendix 2: <u>Calculation of the moment of inertia of a triangular plate</u>  An equilateral triangle with lateral length l can be divided into four small equilateral triangles with lateral length $l/2$. For the central small triangle centered at c_1, its moment of inertia is $$I_1 = \beta\frac{M}{4}\left(\frac{l}{2}\right)^2$$ - Eq. (53) For the non-central small triangle centered at c_2, c_2' and c_2'', $$I_2 = I_1 + \frac{M}{4}d^2$$ - Eq. (54) where $d = \sqrt{3}l/6$ is the distance between the centers of triangles 1 and 2. The second term is from the parallel-axis theorem. The moment of inertia of the whole triangle is the sum of the moment of inertia of the four sub-triangles:	**N/A**

$$\beta M l^2 = 4 \times \beta \frac{M}{4}\left(\frac{l}{2}\right)^2 + 3 \times \frac{M}{4} d^2 \qquad \text{- Eq.(55)}$$

Thus

$$\beta = \frac{1}{12} \qquad \text{- Eq. (56)}$$

Appendix 3: The minimum force corresponds to the maximum displacement of the exerting point of this force.

N/A

Consider the position of vertices A, B, C of a triangle (m,n) :

$$x_{\mathrm{A}(m,n)} = (2m+n)\cos\alpha_{\mathrm{m}}\, l$$

$$y_{\mathrm{A}(m,n)} = -\sqrt{3}n\cos\alpha_{\mathrm{m}}\, l$$

$$x_{\mathrm{B}(m,n)} = (2m+n+1)\cos\alpha_{\mathrm{m}}\, l$$

$$y_{\mathrm{B}(m,n)} = -(\sqrt{3}n\cos\alpha_{\mathrm{m}} + \sin\alpha_{\mathrm{m}})l$$

$$x_{\mathrm{C}(m,n)} = \left[(2m+n)\cos\alpha_{\mathrm{m}} + \cos\left(\frac{\pi}{3}+\alpha_{\mathrm{m}}\right)\right] l$$

$$y_{\mathrm{C}(m,n)} = -\left[\sqrt{3}n\cos\alpha_{\mathrm{m}} + \sin\left(\frac{\pi}{3}+\alpha_{\mathrm{m}}\right)\right] l \qquad \text{- Eq. (57)}$$

Taking derivatives on α on the above coordinates we get

$$\Delta x_{\mathrm{A}(m,n)} = -(2m+n)\sin\alpha_{\mathrm{m}}\, l\Delta\alpha = -\frac{(2m+n)\sqrt{3}}{2} l\Delta\alpha$$

$$\Delta y_{\mathrm{A}(m,n)} = \sqrt{3}n\sin\alpha_{\mathrm{m}}\,(l\Delta\alpha) = \frac{3n}{2} l\Delta\alpha$$

$$\Delta x_{\mathrm{B}(m,n)} = -(2m+n+1)\sin\alpha_{\mathrm{m}}\, l\Delta\alpha = -\frac{(2m+n+1)\sqrt{3}}{2} l\Delta\alpha$$

$$\Delta y_{\mathrm{B}(m,n)} = -(-\sqrt{3}n\sin\alpha_{\mathrm{m}} + \cos\alpha_{\mathrm{m}})l\Delta\alpha = \frac{3n-1}{2} l\Delta\alpha$$

$$\Delta x_{\mathrm{C}(m,n)} = \left[-(2m+n)\sin\alpha_{\mathrm{m}} - \sin\left(\frac{\pi}{3}+\alpha_{\mathrm{m}}\right)\right] l\Delta = -\frac{(2m+n+1)\sqrt{3}}{2} l\Delta\alpha$$

$$\Delta y_{\mathrm{C}(m,n)} = -\left[-\sqrt{3}n\sin\alpha_{\mathrm{m}} + \cos\left(\frac{\pi}{3}+\alpha_{\mathrm{m}}\right)\right] l\Delta\alpha = \frac{(3n+1)}{2} l\Delta\alpha \qquad \text{- Eq. (58)}$$

For $\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, we have

$$\Delta r_{\mathrm{A}(m,n)} = \sqrt{3m^2+3n^2+3mn}(l\Delta\alpha)$$

$$\Delta r_{\mathrm{B}(m,n)} = \sqrt{3m^2+3n^2+3mn+3m+1}(l\Delta\alpha)$$

$$\Delta r_{\mathrm{C}(m,n)} = \sqrt{3m^2+3n^2+3mn+3m+3n+1}(l\Delta\alpha) \qquad \text{- Eq. (59)}$$

Thus we find

$$\Delta r_{\mathrm{C}(m,n)} > \Delta r_{\mathrm{B}(m,n)} > \Delta r_{\mathrm{A}(m,n)} \qquad \text{- Eq. (60)}$$

Therefore, we should choose point C of the triangle $(N-1, N-1)$ to obtain

$$\Delta r_{\max} = (3N-2)l\Delta\alpha \qquad \text{- Eq. (61)}$$

so that the force is minimal.