Part	Model Answer (**Full mark = 20**)	Marks
A1	The angular momentum should be $\vec{L} = r\vec{e}_r \times \vec{p} = r\vec{e}_r \times \vec{e}_\varphi \int_0^{2\pi} \frac{mv}{2\pi r} r\,d\varphi$ (1 point for the definition of angular momentum) Here $\vec{e}_r$ is the unit vector pointing from the center of the ring to the mass point on the ring and $\vec{e}_\varphi$ is the unit vector parallel to the direction of the linear velocity at the mass point. We know that $v = \omega r$, so finally we can get $\vec{L} = m\omega r^2 \vec{e}_z$, with $\vec{e}_z = \vec{e}_r \times \vec{e}_\varphi$. (1 point for the correct answer: 0.5 points for the magnitude and 0.5 points for the direction)	**2**
A2	For a current loop, the magnetic moment is defined as $\vec{M} = I\vec{A}$ The current can be expressed as $I = -ef = -e\frac{\omega}{2\pi}$ (1 point for the current expression) Finally $\vec{M} = -e\frac{\omega}{2\pi}\pi r^2 \vec{e}_z = -\frac{e\vec{L}}{2m}$ (1 point for the answer)	**2**
A3	For a current loop, under a uniform magnetic field the total torque should be $\vec{\tau} = \vec{M} \times \vec{B}$ (0.5 point for the torque definition) The work done by the magnetic field on the torque should be $W = \int_{\frac{\pi}{2}}^{\theta} \vec{\tau} \cdot \mathrm{d}\vec{\theta}' = \int_{\frac{\pi}{2}}^{\theta} -\lvert\vec{\tau}\rvert \mathrm{d}\theta' = \int_{\theta}^{\frac{\pi}{2}} \lvert\vec{M}\rvert \lvert\vec{B}\rvert \sin\theta' \mathrm{d}\theta' = \vec{M} \cdot \vec{B}$ (1.5 points for the work on the torque)	**2**

	$U = -W$ $= -\vec{M}\cdot\vec{B}$ (0.5 point for the answer) $= \frac{1}{2}e\omega r^2 B_z \cos\theta$	
A4	We assume that the magnetic field is along z direction such that $\vec{B} = B\vec{e}_z$, then in general $U = -\vec{M}\cdot\vec{B} = -M_z B$ The magnetic torque of an electron should be $M_z = \frac{-e}{2m_e}S_z$ (0.5 points for the electron torque) Thus $U = -\vec{M}\cdot\vec{B}$ $= -\frac{-e}{2m_e}S_z B$ $= \frac{\mu_B}{\hbar}S_z B$ (0.5 points for the answer) $= \frac{1}{2}\mu_B B$ Here $\mu_B = \frac{e\hbar}{2m_e}$ is the Bohr magneton. $\mu_B = 5.788\times10^{-5}\mathrm{eV\cdot T^{-1}}$	**1**
A5	Thus for spin parallel state $S_z = \frac{1}{2}\hbar$, we have $U = 5.788\times10^{-5}\mathrm{eV}$ (0.5 points) For spin anti-parallel state $S_z = -\frac{1}{2}\hbar$, we have $U = -5.788\times10^{-5}\mathrm{eV}$ (0.5 points)	**1**

B1	In the superconductivity state, electrons forming a Cooper pair have opposite spins, thus the external magnetic field cannot have any effect on the cooper pair. Thus the energy of the Cooper pair does not change. $E_S = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta$ (1 point for the answer)	**1**
B2	In the normal state, the two electrons will align their magnetic moments parallel to the external magnetic field. Therefore we have $E_{\mathrm{N}} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{2\mu_{\mathrm{B}} S_{1x} B_x}{\hbar} + \frac{2\mu_{\mathrm{B}} S_{2x} B_x}{\hbar}$ Here the potential energy of electrons should be twice as the classical estimation according to quantum mechanics. Because $S_{1x} = S_{2x} = -\frac{1}{2}\hbar$ can make the magnetic moment aligned along *x* direction, eventually we have $E_N = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\mu_{\mathrm{B}} B_x = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{e\hbar}{m_{\mathrm{e}}} B_x$ (1 point)	**1**
B3	$E_N < E_S \Rightarrow 2B_x \mu_B > 2\Delta \Rightarrow B_x > \frac{\Delta}{\mu_B}$ Thus $B_{\mathrm{P}} = \frac{\Delta}{\mu_{\mathrm{B}}} = \frac{2m_{\mathrm{e}}\Delta}{e\hbar}$ (1 points) Note: The above simple consideration for the upper critical field B_P over estimates its value. The strict derivation considering the Pauli magnetization and superconductivity condensation energy will give $B_{\mathrm{P}} = \frac{\Delta}{\sqrt{2}\mu_{\mathrm{B}}} = \sqrt{2}\frac{m_{\mathrm{e}}\Delta}{e\hbar}$.	**1**
C1	Method 1:	**3**

Substituting $\psi(x)=\left(\frac{2\lambda}{\pi}\right)^{\frac{1}{4}}\mathrm{e}^{-\lambda x^2}$ into the $F(\psi)$, we have

$$F(\psi)=\sqrt{\frac{2\lambda}{\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\lambda x^2}\left[-\alpha\mathrm{e}^{-\lambda x^2}-\frac{\hbar}{4m_\mathrm{e}}\left(-2\lambda e^{-\lambda x^2}+4\lambda^2x^2\mathrm{e}^{-\lambda x^2}\right)+\frac{e^2B_z^2x^2}{m_\mathrm{e}}\mathrm{e}^{-\lambda x^2}\right]\mathrm{d}x$$

$$=\sqrt{\frac{2\lambda}{\pi}}\int_{-\infty}^{+\infty}\left[\left(-\alpha+\frac{\hbar^2\lambda}{2m_\mathrm{e}}\right)\mathrm{e}^{-2\lambda x^2}+\left(\frac{e^2B_z^2}{m_e}-\frac{\hbar^2\lambda^2}{m_e}\right)x^2\mathrm{e}^{-2\lambda x^2}\right]\mathrm{d}x$$

$$=-\alpha+\frac{\hbar^2\lambda}{2m_\mathrm{e}}+\left(\frac{e^2B_z^2}{m_\mathrm{e}}-\frac{\hbar^2\lambda^2}{m_\mathrm{e}}\right)\cdot\frac{1}{4\lambda}$$

$$=-\alpha+\frac{\hbar^2\lambda}{4m_\mathrm{e}}+\frac{e^2B_z^2}{4\lambda m_\mathrm{e}}$$

(1.5 points for the correct expression of $F(\psi)$ as a function of λ)

We can treat $F(\psi)$ as a function of λ. Thus we have

$$F(\psi)=-\alpha+\frac{\hbar^2\lambda}{4m_\mathrm{e}}+\frac{e^2B_z^2}{4\lambda m_\mathrm{e}},\ \text{and}\ \frac{\mathrm{d}F}{\mathrm{d}\lambda}=\frac{\hbar^2}{4m_\mathrm{e}}-\frac{e^2B_z^2}{4m_\mathrm{e}\lambda^2}.$$

$F(\psi)$ takes the minimum value when $\frac{\mathrm{d}F}{\mathrm{d}\lambda}=0$ and $\frac{\mathrm{d}^2F}{\mathrm{d}\lambda^2}>0$, thus

$\frac{\hbar^2}{4m_\mathrm{e}}-\frac{e^2B_z^2}{4m_\mathrm{e}\lambda^2}=0$ (0.5 point for the way to minimize $F(\psi)$)

Finally, we can get

$\lambda=\frac{eB_z}{\hbar}$ (1 point for the correct answer)

We can check that $\frac{d^2F}{d\lambda^2}>0$ when $\lambda=\frac{eB_z}{\hbar}$, which guarantees that $F(\psi)$ takes the minimum value when $\lambda=\frac{eB_z}{\hbar}$.

Method 2:

	$$F(\psi)=\int_{-\infty}^{+\infty}\psi\left(-\alpha\psi-\frac{\hbar^2}{4m_e}\frac{d^2\psi}{dx^2}+\frac{e^2B_z^2x^2}{m_e}\psi\right)dx$$ $$=\int_{-\infty}^{+\infty}\psi\left(-\alpha-\frac{\hbar^2}{4m_e}\frac{d^2}{dx^2}+\frac{e^2B_z^2x^2}{m_e}\right)\psi dx \quad \text{(1 point)}$$ $$=\int_{-\infty}^{+\infty}\psi\tilde{H}\psi dx$$ In this way, for normalized wave function ψ the $F(\psi)$ is simply the energy expectation $\langle\tilde{H}\rangle$, the eigenvalue of the Hamiltonian $$\tilde{H}=-\frac{\hbar^2}{4m_e}\frac{d^2}{dx^2}+\frac{e^2B_z^2}{m_e}x^2-\alpha$$ The first two terms correspond to the quantum simple harmonic oscillator Hamiltonian. Thus the ground state energy should be $$F_{\min}=\frac{1}{2}\hbar\omega-\alpha$$ Here $\omega=\frac{eB_z}{m_e}$ and ground state wave function becomes $$\psi=\left(\frac{2m_e\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{m_e\omega}{\hbar}x^2}$$ $$=\left(\frac{2eB_z}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{eB_z}{\hbar}x^2} \quad \text{(1 point)}$$ Therefore, we have $$\lambda=\frac{eB_z}{\hbar}. \quad \text{(1 point)}$$	
C2	From Part (C1) we know $F_{\min}(\psi)=\frac{\hbar eB_z}{2m_e}-\alpha$. At the critical value for B_z, it makes the energy difference zero. It means that the critical value B_z satisfies $$\frac{\hbar eB_z}{2m_e}-\alpha=0. \quad \text{(1 point for this equation)}$$ Consequently,	2

	$B_z = \frac{2m_e\alpha}{e\hbar}$. (1 point for the correct answer)	
D1	$E_{\rm I} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - \frac{2\mu_{\rm B}S_{1z}B_z}{\hbar} + \frac{2\mu_{\rm B}S_{2z}B_z}{\hbar}$ Here $S_{1z} = \frac{1}{2}\hbar$, $S_{2z} = -\frac{1}{2}\hbar$ $E_{\rm I} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - \frac{2\mu_{\rm B}S_{1z}B_z}{\hbar} + \frac{2\mu_{\rm B}S_{2z}B_z}{\hbar}$ $= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - \frac{2\mu_{\rm B}B_z}{2} - \frac{2\mu_{\rm B}B_z}{2}$ (1 point) $= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - 2\mu_{\rm B}B_z$ $= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - \frac{e\hbar}{m_e}B_z$	1
D2	In the normal state, the electrons will align the magnetic moment parallel to the total magnetic field, thus $E_{\|\|} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{2\mu_{\rm B}\vec{S}_1\cdot\vec{B}_1}{\hbar} + \frac{2\mu_{\rm B}\vec{S}_2\cdot\vec{B}_2}{\hbar}$ For electron 1, $\vec{B}_1 = (B_x, 0, -B_z)$ For electron 2, $\vec{B}_2 = (B_x, 0, B_z)$ Therefore, $\vec{S}_1 = -\frac{1}{2}\hbar\left(\frac{B_x}{\sqrt{B_x^2+B_z^2}}, 0, \frac{-B_z}{\sqrt{B_x^2+B_z^2}}\right)$ and $\vec{S}_2 = -\frac{1}{2}\hbar\left(\frac{B_x}{\sqrt{B_x^2+B_z^2}}, 0, \frac{B_z}{\sqrt{B_x^2+B_z^2}}\right)$ can make the their magnetic moments parallel to the total magnetic field respectively. (1 point for the correct expression of spins: 0.5 points for each respectively) Finally $E_{\|\|} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\mu_{\rm B}\sqrt{B_x^2+B_z^2} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{e\hbar}{m_e}\sqrt{B_x^2+B_z^2}$ (1 point for the answer)	2
D3	$E_{\|\|} < E_{\rm Ising} \Rightarrow 2\mu_{\rm B}\sqrt{B_x^2+B_z^2} > 2\Delta + 2\mu_{\rm B}B_z \Rightarrow B_x > \frac{\sqrt{\Delta^2 + 2\Delta\mu_{\rm B}B_z}}{\mu_{\rm B}}$ (1 points)	1

	Another correct expression is: $B_I > \dfrac{2m_e\sqrt{\Delta^2 + \dfrac{e\hbar}{m_e}\Delta B_z}}{e\hbar}$.	