Space elevator (8.0 points)

Useful mathematics formula:

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

Presently, the use of rockets is the only viable method of transporting material from Earth to Moon, Mars, and beyond. However, this method of space travel is not so efficient. A space elevator, if it could be built, would provide a completely new technology for space travel (Fig. 1). This is a long structure that is anchored at the equator and reaches a higher altitude than geostationary orbit (GEO). Geostationary orbit is a circular orbit positioned approximately $42300\,\mathrm{km}$ from the Earth's center and having a period of the same duration and direction as the rotation of the Earth. An object in this orbit will appear stationary relative to the rotating Earth. The modern ideas of the space elevator were first proposed by Artsutanov (Artsutanov, Y. et al., Science, 158, 946, 1967). However, only modest attention was paid to the subject until Pearson published an inspiring paper "The Orbital Tower: a Spacecraft Launcher Using the Earth's Rotational Energy" (Pearson J., *Acta Astronautica.* Vol. 2, p. 785, 1975). In Pearson's paper, many useful features of the space elevator were pointed out and it was made clear that for the space elevator to ever become a reality, the use of a material that is much stronger but much lighter than steel would be necessary. Due to the lack of such a material, there was little continuation of this research for many years, until the 1990s when carbon nanotubes, a new material composed of hexagonal arrays of carbon atoms, were discovered. In 2003, the Port project (http://www.port.com/) was launched to build and operate a space elevator with current technology.

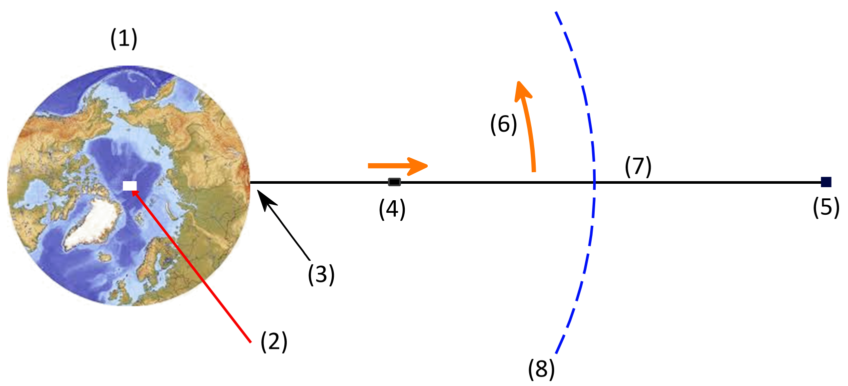

Figure 1. Space Elevator (adapted from wikipedia).(1) Earth; (2) North pole; (3) Anchored at equator; (4) Climber; (5) Counterweight; (6) Rotates with Earth; (7) Cable; (8) Geostationary orbit altitude.

In this part we will study two designs of a space elevator, mechanical properties of carbon nanotubes, and explore some applications of space elevator. You are given the mass of Earth $M = 5.98 \times 10^{24}$ kg, radius of the Earth $R = 6370$ km, geostationary orbit radius $R_G = 42300$ km, solar mass $M_S = 2 \times 10^{30}$ kg, orbital radius of the Earth around the Sun $R_E = 1.5 \times 10^8$km = 1AU (AU – the astronomical unit), the orbital speed of the Earth 30.9 km/s, and the speed of rotation of the Earth around its axis $\omega = 7.27 \times 10^{-5}$ rad/s.

1. The cylindrical space elevator with a uniform cross section (1.5 points)

Let us first consider a space elevator, which is a cylindrical wire with a uniform cross section *A* and is homogeneous with density ρ. It is a cylinder positioned vertically at the equator. Its height is greater than the height of the geostationary satellite orbit, so that the stress (force per unit area) on the bottom of the cylinder is zero. The cylinder is in tension along its entire length, with the stress adjusting itself so that each element of the cylinder is in equilibrium under the action of the gravitational, centrifugal, and tension forces.

1.1	Calculate the height of the upper end of the cylinder above the Earth's surface	0.5pt

1.2	Find the distance from the Earth's center to the point where the stress in the cylinder is maximum.	0.5pt

1.3	Find the expression for maximum stress of the cylinder in terms of ρ, R_G, R and the gravitational acceleration g. If the cylinder is made of steel whose density is 7900 $\mathrm{kg/m^3}$, tensile strength is 5.0 GPa, evaluate the ratio between the maximum stress and the tensile strength of steel. Tensile strength is the maximum stress a material can withstand.	0.5pt

2. Carbon nanotubes (2.5 points)

Calculation in the previous part shows that in order to build the space elevator, it is neccessary to have light materials with very high tensile strength. Carbon nanotubes are materials that meet such requirements because of strong chemical bondings between very light atoms. Two natural polymorphs of carbon are diamond and graphite. In diamond every carbon atom is surrounded by four nearest neighbor (NN) atoms to form a tetrahedron. Graphite has a layer structure. In each layer, carbon atoms are arranged in a hexagonal plane lattice with three NNs. Although diamond is known as the hardest materials, covalent bondings between carbon atoms in hexagonal layers of graphite is stronger than those between carbon atoms in diamond tetrahedra. Graphite is much softer than diamond because of the van der Waals bonding between carbon atoms of different layers, which is much weaker than covalent bonding.

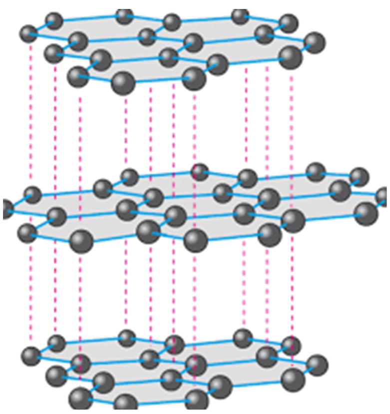

Figure 2. Graphite structure

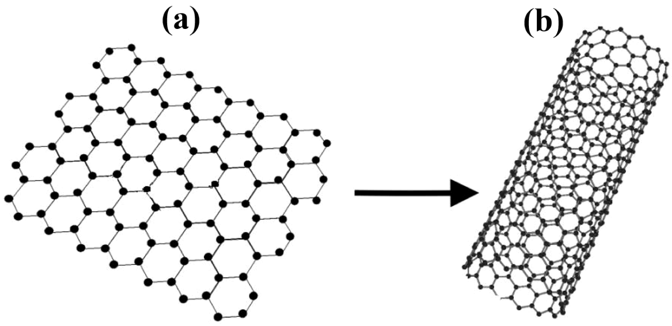

Figure 3. Graphene (a) and carbon nanotube (b).

A monatomic layer in graphite is called graphene and has monoatomic thickness. Isolated graphene sheet is not stable and has a tendency to roll up to form carbon spheres or carbon nanotubes. The hexagonal crystal lattice of graphene is depicted in Fig. 4. The distance between two NN carbon atoms is $a = 0.142\ \mathrm{nm}$ and the distance between two closest parallel bondings is $b = 0.246\ \mathrm{nm}$. Because the covalent bondings between carbon atoms in graphene are very strong, mechanical properties of carbon nanotubes are very special. They have an extremely large Young's modulus and tensile strength, as well as a very light density. Young's modulus is defined as the ratio of the stress along an axis to the strain (ratio of deformation over initial length) along that axis in the range of stress in which Hooke's law holds.

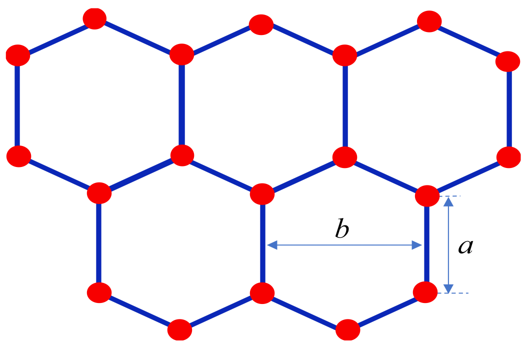

Figure 4. Graphene.

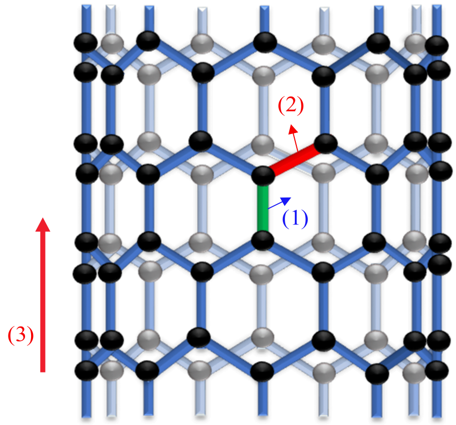

Figure 5. An illustration of a carbon nanotube with 9 carbon-carbon parallel bondings. Note: In this problem, there are 27 carbon-carbon parallel bondings. (1) parallel bond; (2) slanted bond; (3) tube axis.

Now we examine some mechanical properties of a carbon nanotube having 27 carbon-carbon bondings parallel to the tube axis (for an illustration, see Figure 5). The bonding between two carbon atoms can be described by the Morse potential $V(x) = V_0(e^{-4\frac{x}{a}} - 2e^{-2\frac{x}{a}})$. Here $a = 0.142\,nm$ is the equilibrium distance between two NN carbon atoms, $V_0 = 4.93eV$ is the bonding energy, and x is the displacement of the atom from the equilibrium position. Hereafter, we approximate the Morse potential by a quadratic potential $V(x) = P + Qx^2$. All non-nearest-neighbor interactions are neglected. In this approximation, one can propose that carbon atoms are bonded through "springs" with the spring constant k. Changes in angles between bonds are neglected.

2.1	Find coefficients P and Q in term of a and V_0	0.25pt

2.2	Calculate the value of the spring constant k.	0.25pt

2.3	Calculate the value of the Young's modulus of the carbon nanotube.	0.5pt

In order to estimate the tensile strength, we assume that when the "spring" connecting carbon atoms has the maximum extension x_{max} the harmonic potential energy equals to the bonding energy.

2.4	Calculate the value of the maximum extension x_{max} of the spring.	0.5pt

2.5	Estimate the tensile strength σ_0 of the carbon nanotube.	0.5pt

2.6	Given that the molar mass of carbon is 12 g, estimate the density of the carbon nanotube.	0.5pt

3. The tapered space elevator with a uniform stress (2.5 points)

In the previous section, the density and the tensile strength of carbon nanotubes have been evaluated theoretically. These evaluated values indeed depend on the specific structure of carbon nanotubes. Nevertheless, the idea of space elevator construction is truly feasible. Now we will study a new space elevator design of the so-called tapered tower whose cross section varies with height in such a way that both the stress σ and mass density ρ are uniform over the entire tower length. The tower has axial symmetry and is positioned vertically at the equator; its height is greater than the height of the geostationary satellite orbit. Denote the cross sectional area of the tapered tower on the Earth surface by A_S and at geostationary height A_G.

3.1	Find the cross section $A(h)$ as a function of distance h up the tower from the ground.	0.5pt

3.2	The tower is designed symmetrically so that the cross sections at the two ends are equal, find the distance from the center of the Earth to the upper end of the tower.	0.5pt

3.3	The taper ratio is defined as A_G/A_S. Find the taper ratio of the tower made of carbon nanotubes with tensile strength 130 GPa and density $1300\ \mathrm{kg/m^3}$.	0.5pt

3.4	We can considerably shorten the length of the elevator by terminating it at the upper end by a counterweight of the appropriate mass. Let h_C be the height of the tower relative to the geostationary height, and find the relation of mass m_C of the counterweight and h_C.	1.0pt

4. Applications: launching payload into orbit and spacecraft to the other planets (1.5 points)

The main application of space elevator is the use of the tower's rotational energy to launch payload into orbit or send spacecraft to the other planets. It is very easy to get payload into space: we simply have to make it ride up the elevator to an altitude *r* and release it from rest. For simplicity in the calculations, let us assume that the motion of the tower occurs in the plane of Earth's orbit.

4.1	Find the critical height r_C up the tower, measured from Earth's center, at which the object would have to be released from rest to escape Earth's gravity.	0.5pt

Building a tower of greater height than r_C is necessary if we wish to use it to launch spacecraft on voyages to other planets. Given that the tower height is 107000 km from Earth's center.

4.2	Find the minimal and the maximal distances from the Sun that a spacecraft released from rest from the top of the tower can reach. Give your answers in astronomical units. We neglect the Earth's gravitational attraction at this height.	1.0pt