Theory Q1
Optical trap of neutral atoms (12 points),
Solution and Marking Scheme

1.1 **0.75pt**	At the instance when the seperation between charge centers is $\vec{x}$, the external field $\vec{E}$ exerts on them opposite forces $\vec{F} = \pm e\vec{E}$. After a time interval dt, the seperation is changed to $\vec{x} + d\vec{x}$, work done by the external field on the charges is thus $dW = \vec{F}d\vec{x} = \vec{F} = ed\vec{x} \cdot \vec{E} = d\vec{p} \cdot \vec{E}$ The power received by the atomic dipole $$P_{abs} = \frac{dW}{dt} = \frac{d\vec{p}}{dt} \cdot \vec{E} = \dot{\vec{p}} \cdot \vec{E}$$	**0.15** **0.3** **0.3**
1.2 **0.75pt**	Total work can be obtained by integration $$W = \int_0^{\vec{E}_0} d\vec{p} \cdot \vec{E} = \int_0^{\vec{E}_0} \alpha d\vec{E} \cdot \vec{E} = \frac{1}{2}\alpha \vec{E}_0^2 = \frac{1}{2}\vec{p}_0\vec{E}_0$$ Potential energy of the dipole is $$U_{dip} = -W = -\frac{1}{2}\vec{p}_0\vec{E}_0$$ *If the sign of U_{dip} is incorrect or the factor 1/2 is missing, students get 0pt.*	**0. 5** **0.25**

2.1 **1.0pt**	The time average of any time dependent function is denoted by $\langle f(t) \rangle = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} f(t)\, dt$ $U_{dip}(\vec{r}) = -\frac{1}{4}\alpha(\omega)\cos\varphi . E_0^2(\vec{r})$ (1) $U_{dip}(\vec{r}) = -\frac{\alpha(\omega)\cos\varphi . I(\vec{r})}{2\varepsilon_0 c}$ (2) *If student gets directly to eq. (2) – full mark (1.0pt)* *If the answer is still correct but expressed in any quantity other than those requested – 0.5 pt.*	**0.5** **0.5**

3.1 **1.0pt**	The power absorbed by the oscillator from the driving field (and re-emitted as dipole radiation) is given by $\langle P_{abs}(\vec{r}) \rangle = \langle \dot{\vec{p}}\vec{E} \rangle = -\frac{\sin\varphi . \alpha(\omega)\omega}{2} E_0^2(r)$ $\langle P_{abs}(\vec{r}) \rangle = -\frac{\sin\varphi . \alpha(\omega)\omega}{\varepsilon_0 c} I(\vec{r})$ (3) The corresponding *scattering rate* is $\Gamma_{sc}(\vec{r}) = \frac{\langle P_{abs} \rangle}{\hbar\omega} = -\frac{\alpha(\omega)\sin\varphi}{\hbar\varepsilon_0 c} I(\vec{r})$. (4)	**0.5** **0.25** **0.25**

4.1 **2.0pt**	In one dimensional Lorentz's model, we replace $\vec{E}(\vec{r},t) \to E(x,t)$. One can find the solution of the form $x = x_0 \cos(\omega t + \varphi)$ thus from the equation of motion, $\ddot{x} + \gamma_\omega \dot{x} + \omega_0^2 x = -eE_0 \cos\omega t / m_e$ $\Rightarrow x_0(\omega_0^2 - \omega^2)\cos(\omega t + \varphi) - x_0\omega\gamma_\omega \sin(\omega t + \varphi) = -eE_0 \cos\omega t / m_e$	 **0.25**

	$x_0\left\{\left[\left(\omega_0^2-\omega^2\right)\cos\varphi-\omega\gamma_\omega\sin\varphi\right]\cos\omega t-\left[\left(\omega_0^2-\omega^2\right)\sin\varphi+\omega\gamma_\omega\cos\varphi\right]\sin\omega t\right\}=$ $=-eE_0\cos\omega t/m_e$ Comparing coefficients before $\cos\omega t$ and $\sin\omega t$ on both sides, one has	**0.25**
	$\left(\omega_0^2-\omega^2\right)\cos\varphi-\omega\gamma_\omega\sin\varphi=-\dfrac{eE_0}{m_e x_0}$ $\left(\omega_0^2-\omega^2\right)\sin\varphi+\omega\gamma_\omega\cos\varphi=0$	**0.5**
	$x_0=\dfrac{eE_0/m_e}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma_\omega^2\omega^2}}$;	**0.25**
	$\sin\varphi=\dfrac{\omega\gamma_\omega}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma_\omega^2\omega^2}}$ (4) $\cos\varphi=-\dfrac{(\omega_0^2-\omega^2)}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma_\omega^2\omega^2}}$ (5)	**0.25**
	$p=-ex=-ex_0\cos(\omega t+\varphi)=\alpha E_0\cos(\omega t+\varphi)$ (6)	**0.25**
	$\alpha\left(\omega\right)=-\dfrac{e^2}{m_e\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma_\omega^2\omega^2}}$ (7) *Note: students can obtain φ via any of sin, cos, tan functions: full mark (0.25 pt)*	**0.25**

5.1 **1.0pt**	The power radiated due to the damping force, thus	
	$-m_e\gamma_\omega v.v=-\dfrac{1}{6\pi\varepsilon_0}\dfrac{e^2a^2}{c^3}$	**0.25**
	$\Rightarrow -m_e\gamma_\omega(\omega r)^2=-\dfrac{1}{6\pi\varepsilon_0}\dfrac{e^2(\omega^2 r)^2}{c^3}$,	**0.25**
	$\gamma_\omega=\dfrac{1}{6\pi\varepsilon_0}\dfrac{e^2\omega^2}{m_e c^3}$. (8)	**0.5**

6.1 **0.5pt**	Substituting $e^2/m_e=6\pi\varepsilon_0 c^3\gamma_\omega/\omega^2$ the on-resonance damping rate $\gamma\equiv\gamma_{\omega_0}=\left(\omega_0/\omega\right)^2\gamma_\omega$. Using Eq. (1), (4), (5) and (6) one has $\dfrac{U_{dip}\left(\vec{r}\right)}{\hbar\Gamma_{sc}\left(\vec{r}\right)}=\dfrac{-\dfrac{1}{2\varepsilon_0 c}\alpha\left(\omega\right)\cos\varphi}{-\hbar\dfrac{\alpha\left(\omega\right)\sin\varphi}{\hbar\varepsilon_0 c}}=\dfrac{1}{2}\dfrac{1}{\tan\varphi}=-\dfrac{1}{2}\dfrac{\omega_0^2-\omega^2}{\left(\omega^3/\omega_0^2\right)\gamma}$,	**0.5**

7.1 **0.5pt**	From (1), (5) and (6) one has	

	$U_{depth}=\|U_0\|=\left\|\frac{\alpha(\omega)\cos(\varphi)I(0,0)}{2\varepsilon_0 c}\right\|=\left\|\frac{\alpha(\omega)\cos(\varphi)}{2\varepsilon_0 c}\frac{2P}{\pi D_0^2}\right\|=\left\|6c^2\frac{(\omega_0^2-\omega^2)\gamma/\omega_0^2}{\left[(\omega_0^2-\omega^2)^2+\gamma^2\frac{\omega^6}{\omega_0^4}\right]}\frac{P}{D_0^2}\right\|$	**0.5**
7.2 **1.0pt**	Trap depth when $P=4mW$, laser wavelength $\lambda=985nm$, and $D_0=6\mu m$. For sodium $\lambda_0=589nm$. One has: $\omega=\frac{2\pi c}{\lambda};\omega_0=\frac{2\pi c}{\lambda_0};$ And $\gamma=\frac{1}{6\pi\varepsilon_0}\frac{e^2\omega_0^2}{m_e c^3}=\frac{2\pi e^2}{3\varepsilon_0 m_e c\lambda_0^2}=6.4\times10^7 s^{-1}$ $U_{depth}=f\,k_B T_0$ *(factors f = 3/2, 1/2,1 are all accepted)* $\Rightarrow f.T_0=4.13\mu K$	 **0.5** **0.25** **0.25**

8.1 **0.5pt**	Using linear expansion, we have $\Omega_\rho=\sqrt{\frac{4k_B fT_0}{mD_0^2}}$ and $\Omega_z=\sqrt{\frac{2k_B fT_0}{mz_R^2}}$	**0.25** **0.25**

9.1 **0.5pt**	Mean potential enery $U(z_0)=const+\frac{1}{2}m\Omega_z^2 z_0^2.$ To estimate the particle momentum, we assume $p\sim\Delta p,\Delta z\sim z_0$. The uncertainty principle is written now $p\sim\frac{\hbar}{z_0}.$ Kinetic energy $K=\frac{p^2}{2m}=\frac{\hbar^2}{2mz_0^2}.$ Total energy of the particle $E=\frac{1}{2}m\Omega_z^2 z_0^2+\frac{\hbar^2}{2mz_0^2}+const$ Minimal energy corresponds to the energy balance $\frac{1}{2}m\Omega_z^2 z_0^2=\frac{\hbar^2}{2mz_0^2}\Rightarrow z_0=\sqrt{\frac{\hbar}{m\Omega_z}}.$ *If the student followed a correct analysis any obtained correct answer upto some multiplication factor: full mark* *If the student obtained correct answer using dimensional analysis: only 0.1 pt is granted*	 **0.2** **0.1** **0.1** **0.1**
9.2 **0.25pt**	Insert the expression of the cloud size $z_0=\sqrt{\frac{\hbar}{m\Omega_z}}$ to the energy expression $E_{\min}=\frac{1}{2}m\Omega_z^2 z_0^2+\frac{\hbar^2}{2mz_0^2}+const$ one obtains $E_{\min}=\hbar\Omega_z+const.$ *If the student obtained the answer* $E_{min}=\frac{\hbar\Omega_z}{2}$ *by using* $E_n=\hbar\Omega_z\left(n+\frac{1}{2}\right)$ *: full mark*	 **0.25**
9.3	From the uncertainty principle, the particle velocity therefore is estimated to be	**0.25**

0.25pt	$$mv_z = \frac{\hbar}{z_0} = \sqrt{m\hbar\Omega_z} \Rightarrow v_z = \sqrt{\frac{\hbar\Omega_z}{m}}.$$ Alternative estimation is constructed from kinetic energy: $\frac{1}{2}mv_z^2 = K = \frac{1}{2}\hbar\Omega_z \Rightarrow v_z = \sqrt{\frac{\hbar\Omega_z}{m}}$	

10.1 0.5pt	For the three dimendional trap, one has: $z_0 = \sqrt{\frac{\hbar}{m\Omega_z}}$. Similarly for x, y coordinates $x_0 = y_0 = \sqrt{\frac{\hbar}{m\Omega_\rho}}$ and thus $\rho_0 = \sqrt{x_0^2 + y_0^2} = \sqrt{\frac{2\hbar}{m\Omega_\rho}}$. The condensate aspect ratio: $\frac{z_0}{\rho_0} = \sqrt{\frac{\Omega_\rho}{2\Omega_z}}$. *Student may use either x_0, y_0 or ρ_0 in estimating the radial size of the cloud. Correct answers upto multiplication factor: full mark*	**0.2** **0.2** **0.1**
10.2 0.5pt	$$v_z = \sqrt{\frac{\hbar\Omega_z}{m}},$$ $$v_x = v_y = \sqrt{\frac{\hbar\Omega_\rho}{m}}, \Rightarrow v_\rho = \sqrt{v_x^2 + v_y^2} \sim \sqrt{\frac{2\hbar\Omega_\rho}{m}},$$ $$\frac{v_\rho}{v_z} \sim \sqrt{\frac{2\Omega_\rho}{\Omega_z}}.$$ *Student may use either v_x, v_y or v_ρ in estimating expansion velocity in the radial direction. Correct answers upto some multiplication factor: full mark*	**0.25** **0.25**
10.3 0.5pt	After the time t, the sizes of the condensate cloud are: $z_L = z_0 + v_z t \approx v_z t \qquad \rho_L = \rho_0 + v_\rho t \approx v_\rho t$. The cloud aspect ratio after the time t, $\frac{z_L}{\rho_L} \approx \frac{v_z}{v_\rho} \sim \sqrt{\frac{\Omega_z}{2\Omega_\rho}} \ll 1$. *Correct final answers upto some multiplication factor: full mark*	**0. 25** **0.25**
10.4 0.5pt	Due to isotropic nature of thermal cloud, described by the Maxwell distribution: $$v_{T,z} = v_{T,\rho} \Rightarrow \frac{v_{T,\rho}}{v_{T,z}} \approx 1.$$ one can easily find $z_{T,L} = z_0 + v_z t \approx v_z t$, $\rho_{T,L} = \rho_0 + v_\rho t \approx v_\rho t$. After a very long time, the aspect ratio of the thermal cloud therefore: $\rho_{T,L} : z_{T,L} \sim 1$ *Note: students use different velocities (arithmetrical, rms, projection….etc.) to estimate the expansion of the cloud, as long as they give the correct ratio $\rho_L : z_T \sim 1$, full mark of this sub question is granted. In this question, the correct multiplication factor is requested. For incorrect multiplication factor: zero mark*	**0.2** **0.2** **0.1**