Theory Q2

Space elevator (8 points)

Solution and Marking Scheme

1 Cylindrical Space Elevator with Uniform Cross Section

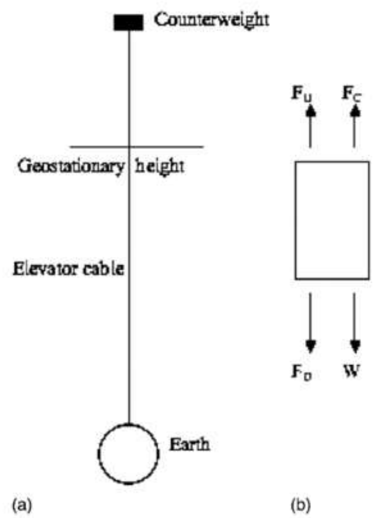
1.1
0.5pt

Consider a small element of the cylinder of thickness dr at position r, there are four forces acting on that element: gravitational $\vec{W}(r)$, centrifugal $\vec{F}_C(r)$, cable tension $\vec{F}_D = \vec{T}(r)$ at position r, tension $\vec{F}_U = \vec{T}(r+dr)$ at position $r+dr$. Positive direction is chosen from the Earth center outward. The net force must be zero, therefore:

$$-W + F_C + T(r+dr) - T(r) = 0$$

$$\Leftrightarrow -W + F_C + A.\sigma(r+dr) - A.\sigma(r) = 0,$$

0.1



Hence

$$A d\sigma = \frac{GM(A\,dr\,\rho)}{r^2} - (A\,dr\,\rho)\omega^2 r$$

$$\Rightarrow \frac{d\sigma}{dr} = GM\rho\left(\frac{1}{r^2} - \frac{r}{R_G^3}\right)$$

0.1

Note that, the tensions at the ends of the cylinder are zero. Integrating the above equation from R to R_G, one obtains the stress at R_G

$$\sigma(R_G) = GM\rho\left[\frac{1}{R} - \frac{3}{2R_G} + \frac{R^2}{2R_G^3}\right],$$

0.1

Similarly, integrating from R_G to H (the distance from the Earth center to the upper end of the cylinder), one obtains the same stress at R_G

$$\sigma(R_G) = GM\rho\left[\frac{1}{H} - \frac{3}{2R_G} + \frac{H^2}{2R_G^3}\right]$$

0.1

Equating the two above expressions, one arrives to the equation:

$$RH^2 + R^2 H - 2R_g^3 = 0,$$

from where H is determined: $H = \frac{R}{2}\left[\sqrt{1+8\left(\frac{R_G}{R}\right)^3} - 1\right] = 1.51\times 10^5$ km.

The height of the cylinder $L = H - R = \frac{R}{2}\left[\sqrt{1+8\left(\frac{R_G}{R}\right)^3} - 3\right] = 1.45\times 10^5$ km.

0.1

Note: Students can just equalize the net gravitational force and the net centrifugal force acting on the cylinder to obtain H correctly: full mark.

1.2
0.5pt

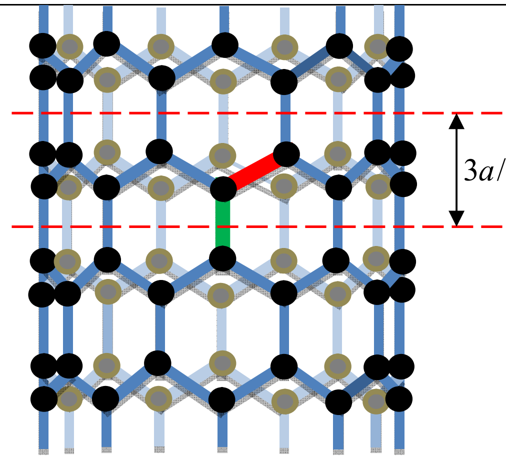
The maximal stress is determined from the requirement

	$$\frac{d\sigma}{dr} = GM\rho\left(\frac{1}{r^2} - \frac{r}{R_G^3}\right) = 0$$ which yields $r = R_G$	**0.25** **0.25**
1.3 **0.5pt**	Maximal stress is expressed by $\sigma(R_G) = GM\rho\left[\frac{1}{R} - \frac{3}{2R_G} + \frac{R^2}{2R_G^3}\right]$ (1) $\sigma(R_G) = \rho g\left[R - \frac{3R^2}{2R_G} + \frac{R^4}{2R_G^3}\right]$ (2) Numerical calculation with $\rho = 7900kg/m^3$ one obtains the ratio: $\frac{\sigma(R_G)}{5.0GA} = \frac{383\ GPa}{5.0\ \ GPa} = 76.5$, This ratio is much larger than 1, therefore steel is not suitable to build this kind of elevator. *If eq. (2) is not obtained and other correct equation like eq. (1) is derived - 0.1pt from full mark (get only 0.15pt for maximal stress).*	**0.25** **0.25**

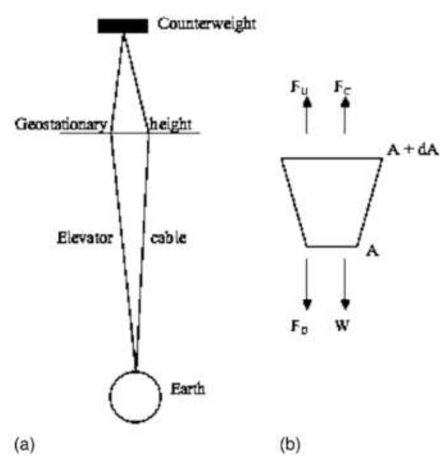
2 Carbon Nanotubes

2.1 **0.25pt**	Expand exponential function in series, and limit to the lowest power of x, one has $V = V_0\left(-1 + \frac{4x^2}{a^2}\right)$ and gets $P = -V_0$ and $Q = \frac{4V_0}{a^2}$.	**0.1** **0.15**
2.2 **0.25pt**	$F = -\frac{dV}{dx} = -\frac{8V_0}{a^2}x$ then $k = \frac{8V_0}{a^2} = 313Nm^{-1}$.	**0.1** **0.15**
2.3 **0.5pt**	Young's modulus of the carbon nanotube. Denote d the diameter of the carbon nanotube, one has $d = 27b/\pi$. $E_1 = \frac{stress\ \sigma}{strain\ \varepsilon} = \frac{F/A}{x/a} = \frac{kx/A}{x/a} = \frac{ka}{A} = \frac{32V_0}{a\pi d^2}$ $E\ =\ NE_1 = 342\,\text{GPa}$	**0.25** **0.25**

2.4 **0.5pt**	$V_0 = \frac{1}{2}kx_{\max}^2 \Rightarrow x_{\max} = \sqrt{\frac{2V_0}{k}} = \frac{1}{2}a$ $= 0.071\,\text{nm}$	**0.25** **0.25**
2.5 **0.5pt**	Tensile strength of the carbon nanotube, $\sigma_0 = E\frac{x_{\max}}{a} = E/2 = 171\,\text{GPa}$.	**0.5**

2.6 **0.5pt**	Volume $\frac{\pi d^2}{4}\times\frac{3a}{2}$ contains 18 carbon atoms, therefore the density of the carbon nanotube, $\rho=\frac{2\times 27\times 12\times 10^{-3}}{N_A\times\frac{\pi d^2}{4}\times\frac{3a}{2}}=1440\,\mathrm{kg/m^3}$. 	**0.25** **0.25**

3 Tapered Space Elevator with Uniform Stress

3.1 **0.5pt**	The solution to this section is analogous to that given in the previous section, however, now one has to take into account the fact that the stress σ is constant, but the cross section area A varies along the tower. $\sigma dA=\frac{GM\left(A\,dr\,\rho\right)}{r^2}-\left(A\,dr\,\rho\right)\omega^2 r$ $\Rightarrow\frac{dA}{A}=\frac{\rho g R^2}{\sigma}\left(\frac{1}{r^2}-\frac{r}{R_G^3}\right)dr$ where $g=GM/R^2$ is gravitational acceleration at the Earth surface. By integration one can obtain the tower cross section as:  $A(h)=A_S\exp\left[\frac{\rho g R^2}{\sigma}\left(\frac{1}{R}+\frac{R^2}{2R_G^3}-\frac{1}{R+h}-\frac{(R+h)^2}{2R_G^3}\right)\right]$	**0.25** **0.25**
3.2 **0.5pt**	Using the condition A(H)=A(R)=A_S one arrives to the equation $R\,H^2+R^2H-2R_G^3=0$, which allows to determine $H=\frac{R}{2}\left[\sqrt{1+8\left(\frac{R_G}{R}\right)^3}-1\right]=151000\,\mathrm{km}.$	**0.25** **0.25**
3.3 **0.5pt**	The ratio $\frac{A_G}{A_S}=\exp[\frac{R}{2L_C}\{\left(\frac{R}{R_G}\right)^3-3\left(\frac{R}{R_G}\right)+2\}]=1.623$ where $L_C=\frac{\sigma}{\rho g}$	**0.5**
3.4 **1.0pt**	Net force exerted on the counterweight must be zero $\frac{GMm_C}{\left[R_G+h_C\right]^2}+A\left(R_G+h_C\right).\sigma=m_C\omega^2\left[R_G+h_C\right]$, replacing $A\left(R_G+h\right)$ from the equation for cross section area, one can determine the counterweight mass.	**0.5**

	$$m_C = \frac{\rho A_S L_C \exp\left[\frac{R^2}{2L_C R_G^3}\left(\frac{2R_G^3+R^3}{R} - \frac{2R_G^3+\left(R_G+h_C\right)^3}{R_G+h_C}\right)\right]}{\frac{R^2\left(R_G+h_C\right)}{R_G^3}\left[1-\left(\frac{R_G}{R_G+h_C}\right)^3\right]}.$$	**0.50**

4 Applications

4.1 **0.5pt**	An object can leave the Earth if its energy at the distance r satisfies $E = \frac{m(\omega r)^2}{2} - \frac{GMm}{r} \geq 0$ from which $r_C = \left(2GM/\omega^2\right)^{\frac{1}{3}} = 53200km$	**0.25**
	In order to launch an object, the upper end of the tower must locate above the distance r_C.	**0.25**

4.2 **1.0pt**	We denote the Earth orbital velocity as v_E, the spacecraft velocity when it's released from the tower top as $v_1 = \omega h_0$. The spacecraft can reach the furthest distance from the Sun if $\vec{v}_1$ is parallel to $\vec{v}_E$. The spacecaft velocity relative to the Sun is $v_E + v_1$. The Earth orbital radius R_E also is the smallest distance from the sun (if one neglects the tower length compared to the radius of the Earth's orbit). r_2 is the apogee distance of the spacecraft from the Sun, v_2 is its velocity at apogee. Angular momentum and energy convervation laws read	
	$m\left(v_E+v_1\right)R_E = mv_2 r_2$	**0.1**
	$\frac{1}{2}m\left(v_E+v_1\right)^2 - \frac{GM_S m}{R_E} = \frac{1}{2}mv_2^2 - \frac{GM_S m}{r_2}$	**0.1**
	Here the energy term $-\frac{GMm}{h_0}$ due the earth's gravity is neglected. Eliminating v_2 one has	
	$\left[\left(v_E+\omega h_0\right)^2 - \frac{2GM_S}{R_E}\right]r_2^2 + 2GM_S r_2 - \left(v_E+\omega h_0\right)^2 R_E^2 = 0$	**0.1**
	from which $r_{Max} = r_2 = \frac{\left(v_E+\omega h_0\right)^2 R_E^2}{2GM_S - \left(v_E+\omega h_0\right)^2 R_E}$.	**0.1**
	Numerical calculation gives r_2=5.3AU, that covers Jupiter's orbit.	**0.1**
	Similarly, for the spacecraft to approach as close as possible to the Sun, the released velocity $\vec{v}_1$ must be antiparallel to $\vec{v}_E$. The spacecaft velocity relative to the Sun is $v_E - v_1$, r_2 is the perigee distance of the spacecraft from the Sun, v_2 is its velocity at perigee. The previous angular momentum and energy convervation laws still hold,	
	$m\left(v_E-v_1\right)R_E = mv_2 r_2$	**0.1**

	$\frac{1}{2}m\left(v_E - v_1\right)^2 - \frac{GM_S m}{R_E} = \frac{1}{2}mv_2^2 - \frac{GM_S m}{r_2}$	**0.1**
	Here the energy term $-\frac{GMm}{h_0}$ due the earth's gravity is neglected. Eliminating v_2 one has	
	$\left[\left(v_E - \omega h_0\right)^2 - \frac{2GM_S}{R_E}\right]r_2^2 + 2GM_S r_2 - \left(v_E - \omega h_0\right)^2 R_E^2 = 0$	**0.1**
	from which $r_{\min} = r_2 = \frac{\left(v_E - \omega h_0\right)^2 R_E^2}{2GM_S - \left(v_E - \omega h_0\right)^2 R_E}$.	**0.1**
	Numerical calculation gives $r_{\min} = 0.43\,\text{AU}$, meaning the Mercury's orbit is within our reach.	**0.1**

References

[1] Artsutanov, Y. Kosmos na elektrovoze. *Komsomolskaya Pravda* July 31 (1960); contents described in Lvov *Science* **158**, 946–947 (1967).

[2] Pearson, J. The Orbital Tower: a Spacecraft Launcher Using the Earth's Rotational Energy. *Acta Astronautica* **2**, 785 (1975)

[3] Aravind, P. K. The physics of the space elevator. *American Journal of Physics* **75**, 125 (2007).

[4] Bochníček, Z. A Carbon Nanotube Cable for a Space Elevator**.** *The Physics Teacher* **51**, 462 (2013).