Theory
Question 3: Tippe Top
Solutions

Reference sheet for markers

Note: some results below were used for the previous version of part **A.10**, and are no longer needed.

Coordinate systems for convenience (note: use of matrices not needed) xyz from XYZ

$$\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{\mathbf{X}}\\ \hat{\mathbf{Y}}\\ \hat{\mathbf{Z}}\end{bmatrix}$$

123 from xyz

$$\begin{bmatrix}\hat{\mathbf{1}}\\ \hat{\mathbf{2}}\\ \hat{\mathbf{3}}\end{bmatrix} = \begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{bmatrix}$$

Position of point A from centre of mass, in xyz and 123 frames:

$$\begin{aligned}\mathbf{a} &= \alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}} \qquad (1)\\ &= \alpha R\sin\theta\hat{\mathbf{x}} + R(\alpha\cos\theta - 1)\hat{\mathbf{z}}\\ &= R\sin\theta\hat{\mathbf{1}} + R(\alpha - \cos\theta)\hat{\mathbf{3}}\end{aligned}$$

Useful products:

$$\hat{\mathbf{z}} \times \hat{\mathbf{3}} = \sin\theta\hat{\mathbf{y}} \qquad (2)$$

(3)

Note (given in question):

$$\left(\frac{\partial \mathbf{A}}{\partial t}\right)_{\mathbf{K}} = \left(\frac{\partial \mathbf{A}}{\partial t}\right)_{\widetilde{\mathbf{K}}} + \boldsymbol{\omega} \times \mathbf{A} \qquad (4)$$

Time derivatives:

$$\dot{\hat{\mathbf{3}}} = \boldsymbol{\omega} \times \hat{\mathbf{3}} \qquad (5)$$

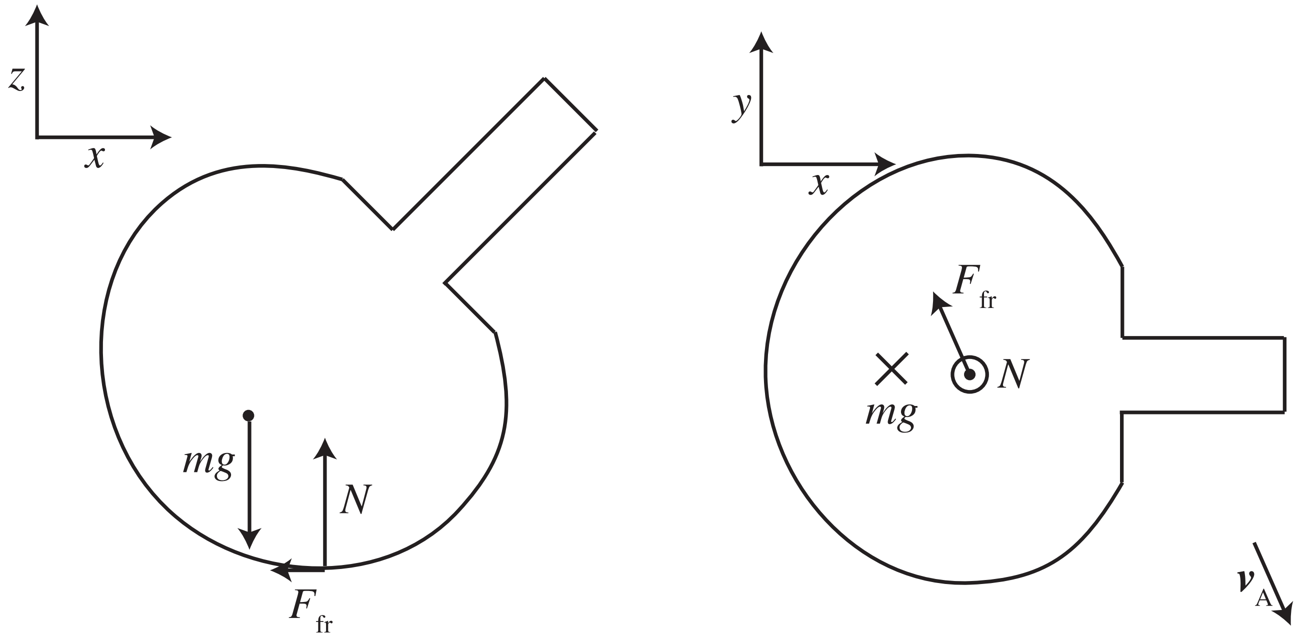
$$\dot{\hat{\mathbf{x}}} = \dot{\phi}\hat{\mathbf{y}} \qquad (6)$$

$$\dot{\hat{\mathbf{y}}} = -\dot{\phi}\hat{\mathbf{x}} \qquad (7)$$

Solutions: Tippe Top

1. (**1.0 marks**)

   Free body diagrams:



   Note: the direction of $\mathbf{F}_f$ must be opposite to the direction of $\mathbf{v}_A$, but is otherwise unimportant. Sum of forces:

$$\mathbf{F}_{\text{ext}} = (N - mg)\hat{z} + \mathbf{F}_f \quad \text{(sufficient for full marks)} \tag{8}$$
$$= (N - mg)\hat{z} - \frac{\mu_k N}{|v_A|}\mathbf{v_A}$$

   Sketched $\mathbf{v_A}$ must be in opposite direction to $\mathbf{F}_f$ on xy diagram.

2. (**0.8 marks**)

   Sum of torques:

$$\boldsymbol{\tau}_{\text{ext}} = \mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F}_f) \tag{9}$$
$$= (\alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}}) \times (N\hat{\mathbf{z}} + F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}})$$
$$= \alpha RN\hat{\mathbf{3}} \times \hat{\mathbf{z}} + \alpha R(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}) \times (F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}}) - R\hat{\mathbf{z}} \times (F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}})$$
$$= -\alpha RN\sin\theta\hat{\mathbf{y}} + \alpha R\sin\theta F_{f,y}\hat{\mathbf{z}} + \alpha R\cos\theta F_{f,x}\hat{\mathbf{y}} - \alpha R\cos\theta F_{f,y}\hat{\mathbf{x}} - RF_{f,x}\hat{\mathbf{y}} + RF_{f,x}\hat{\mathbf{x}}$$
$$= RF_{f,y}(1 - \alpha\cos\theta)\hat{\mathbf{x}} + [RF_{f,x}(\alpha\cos\theta - 1) - \alpha RN\sin\theta]\,\hat{\mathbf{y}} + \alpha R\sin\theta F_{f,y}\hat{\mathbf{z}} \tag{10}$$

3. (**0.4 marks**)

   Motion at A satisfies

$$\mathbf{v_A} = \dot{\mathbf{s}} + \boldsymbol{\omega} \times \mathbf{a} \tag{11}$$

   where $\boldsymbol{\omega}$ is the total angular velocity of the top in the centre of mass frame (this is deteremined in the next part). Want to show that $\mathbf{v_A} \cdot \hat{\mathbf{z}} = 0$.

   To show this, take time derivative of contact condition in XYZ or xyz frame (note: either is suitable, as we only need the $\hat{\mathbf{z}}$ component, and $\hat{\mathbf{z}} = \hat{\mathbf{Z}}$).

Contact condition:

$$
\begin{aligned}
(\mathbf{s} + \mathbf{a}) \cdot \hat{\mathbf{z}} &= 0 \quad \text{at all times} \qquad (12)\\
\Rightarrow \frac{d}{dt}(\mathbf{s} + \mathbf{a}) \cdot \hat{\mathbf{z}} &= 0 \quad \text{at all times}
\end{aligned}
$$

Note we only care about the z-component, and $(\boldsymbol{\omega} \times \hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} = 0$. Then, using 11, 1, and 5,

$$
\begin{aligned}
\mathbf{v_A} \cdot \hat{\mathbf{z}} &= (\dot{\mathbf{s}} + \boldsymbol{\omega} \times \mathbf{a}) \cdot \hat{\mathbf{z}} \\
&= \left(\dot{\mathbf{s}} + \alpha R \boldsymbol{\omega} \times \hat{\mathbf{3}}\right) \cdot \hat{\mathbf{z}} \\
&= \left(\dot{\mathbf{s}} + \alpha R \frac{d\hat{\mathbf{3}}}{dt}\right) \cdot \hat{\mathbf{z}} \\
&= (\dot{\mathbf{s}} + \dot{\mathbf{a}}) \cdot \hat{\mathbf{z}} = 0 \qquad (13)
\end{aligned}
$$

4. (**0.8 marks**)

Total angular velocity $\boldsymbol{\omega}$ of top is the sum of three distinct rotations:

$$
\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{2}} + \dot{\phi}\hat{\mathbf{z}} + \dot{\psi}\hat{\mathbf{3}}
$$

Use transformations shown in figure 3 or otherwise to transform into xyz or 123 frame:

$$
\boldsymbol{\omega} = \dot{\psi}\sin\theta\hat{\mathbf{x}} + \dot{\theta}\hat{\mathbf{y}} + (\dot{\psi}\cos\theta + \dot{\phi})\hat{\mathbf{z}} \qquad (14)
$$

$$
\boldsymbol{\omega} = -\dot{\phi}\sin\theta\hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{2}} + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{3}} \qquad (15)
$$

5. (**1.0 marks**)

Where $\mathbf{I}$ is the inertia tensor

$$
\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3, \end{bmatrix}
$$

we have

$$
\begin{aligned}
E_T &= K_T + K_R + U_G \\
&= \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}\boldsymbol{\omega} + \frac{1}{2}m\dot{\mathbf{s}}^2 + mgR(1 - \alpha\cos\theta)
\end{aligned}
$$

From 11,

$$
\begin{aligned}
\dot{\mathbf{s}} &= \mathbf{v_A} - \boldsymbol{\omega} \times \mathbf{a} \\
&= \mathbf{v_A} - (\dot{\theta}\hat{\mathbf{2}} + \dot{\phi}\hat{\mathbf{z}} + \dot{\psi}\hat{\mathbf{3}}) \times (\alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}}) \\
&= v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} - \left(\dot{\theta}\alpha R\hat{\mathbf{1}} - \dot{\theta}R\hat{\mathbf{z}} + \dot{\phi}\alpha R\hat{\mathbf{z}} \times \hat{\mathbf{3}} - \dot{\psi}R\hat{\mathbf{3}} \times \hat{\mathbf{z}}\right) \\
&= \left(v_x + \dot{\theta}R(1 - \alpha\cos\theta)\right)\hat{\mathbf{x}} + \left(v_y - R\sin\theta(\alpha\dot{\phi} + \dot{\psi})\right)\hat{\mathbf{y}} + \dot{\theta}\alpha R\sin\theta\hat{\mathbf{z}}
\end{aligned}
$$

using 2. Thus

$$
\begin{aligned}
E_T = {} & \frac{1}{2}\left[I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2\right] \\
& + \frac{m}{2}\left[\left(v_x + \dot{\theta}R(1 - \alpha\cos\theta)\right)^2 + \left(v_y - R\sin\theta(\alpha\dot{\phi} + \dot{\psi})\right)^2 + \dot{\theta}^2\alpha^2R^2\sin^2\theta\right] + mgR(1 - \alpha\cos\theta)
\end{aligned}
$$

6. (**0.4 marks**)

From 10,

$$\frac{d\mathbf{L}}{dt} \cdot \hat{\mathbf{z}} = \sum \boldsymbol{\tau} \cdot \hat{\mathbf{z}} = \alpha R \sin\theta F_{f,y} \tag{16}$$

7. (**1.4 marks**)

Changes in energy: $h = \mathbf{s} \cdot \hat{\mathbf{z}}$ increases, so $\dot{U}_G > 0$.

At start and end (phases **I** and **V**) there is little translation so $K_T \sim 0$ at **I** and **V**. Thus, energy transfer is from K_R to U_G.

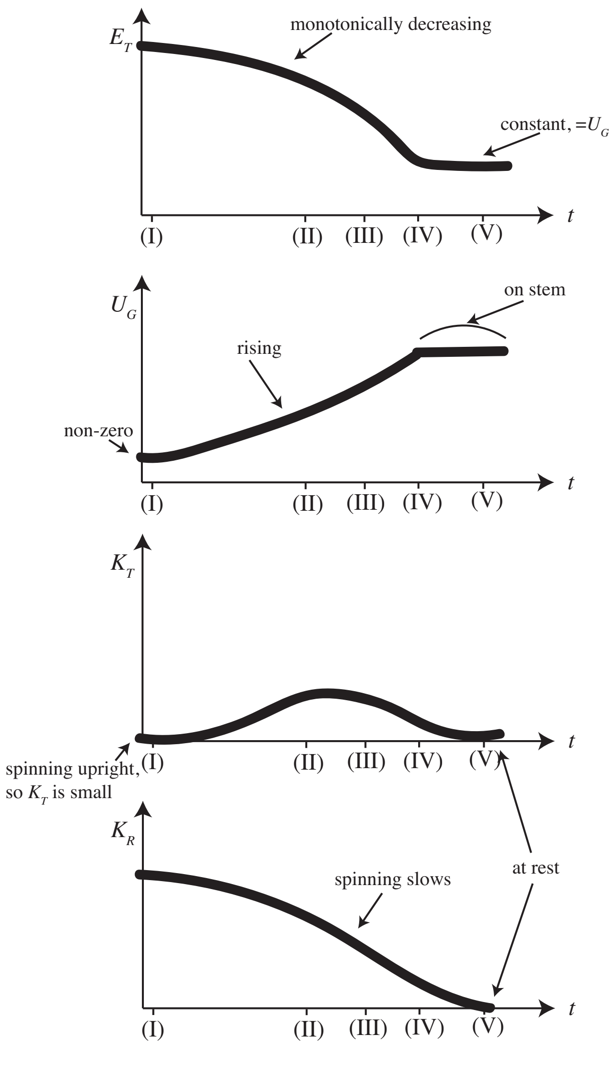
Normal force does no work. Frictional force does work at point A. Direction is $-\mathbf{v_A}$:

$$W = \int \mathbf{F}_f \cdot \mathbf{v_A}\ dt < 0$$
$$\Rightarrow \frac{d}{dt} E_T = -\mu_k N |\mathbf{v_A}|$$

Thus $\mathbf{F}_f$ decreases the total energy monotonically.

16 implies only the $\mathbf{F}_f . \hat{\mathbf{y}}$ acts to decrease $\mathbf{L} \cdot \hat{\mathbf{z}}$. Energy transfer from K_R to U_G, caused by component of frictional force in $\hat{\mathbf{y}}$ direction, so component of resultant torque is in the $\mathbf{a} \times \hat{\mathbf{y}}$ direction.

8. (**2.0 marks**)



Expectation (see figure):

- E_T: monotonically decreasing
- K_R: monotonically decreasing; zero at V
- K_T: zero at I and V; higher between; close to zero at IV
- U_G: flat at start and finish; higher at end; increases from I to IV then flat; increase roughly at same time that K_{rot} decreases

9. (**0.5 marks**)

From 15,

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = I_1 \left(-\dot{\phi} \sin\theta \hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{2}} \right) + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{3}} \tag{17}$$

Taking cross product with $\hat{\mathbf{3}}$:

$$\begin{aligned} \mathbf{L} \times \hat{\mathbf{3}} &= I_1 \left(\dot{\phi} \sin\theta \hat{\mathbf{2}} + \dot{\theta}\hat{\mathbf{1}} \right) \\ &= I_1(\boldsymbol{\omega} \times \hat{\mathbf{3}}) \end{aligned} \tag{18}$$

10. (**1.7 marks**)

About any axis through the centre of mass,

$$\frac{d\mathbf{L}}{dt} \neq 0 \Leftrightarrow \tau_{\text{ext}} \neq 0$$

External torque given by 9,

$$\begin{aligned} \boldsymbol{\tau}_{\text{ext}} &= \mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F}_f) \\ &\Rightarrow \tau_{\text{ext}} \cdot \mathbf{a} = 0 \\ &\frac{d\mathbf{L}}{dt} \cdot \mathbf{a} = 0 \end{aligned}$$

Thus, angular momentum in the direction of $\mathbf{a}$ must be constant, so $\mathbf{v} = \mathbf{a}$.

To demonstrate this mathematically, 5, 10, 18 allow

$$\begin{aligned} -\dot{\lambda} &= \frac{d\mathbf{L}}{dt} \cdot \mathbf{a} + \alpha R \mathbf{L} \cdot \frac{d\hat{3}}{dt} \\ &= (\mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F_f})) \cdot \mathbf{a} + \frac{\alpha R}{I_1} \mathbf{L} \cdot (\boldsymbol{\omega} \times \mathbf{L}) \\ &= 0 \end{aligned}$$