Theoretical Question 2: Ray tracing and generation of entangled light

Part A. Light propagation in isotropic dielectric media

A.1 0.4 pt

Ans: $\frac{1}{\sqrt{\mu_0\epsilon}}$

Solution:

From $\vec{k}\times\vec{E} = \omega\vec{B} = \omega\mu_0\vec{H}$ and $\vec{k}\times\vec{H} = -\omega\vec{D}$, one obtains $\vec{k}\times(\vec{k}\times\vec{E}) = -\omega^2\mu_0\vec{D}$. By using the given identity $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$, one finds $\vec{k}\times(\vec{k}\times\vec{E}) = \vec{k}(\vec{k}\cdot\vec{E})-k^2\vec{E}$. Since $\vec{D}\cdot\vec{k} = 0$ and $\vec{D} = \epsilon\vec{E}$, we find $\vec{k}\times(\vec{k}\times\vec{E}) = -k^2\vec{E}$ and the relation $\vec{k}\times(\vec{k}\times\vec{E}) = -\omega^2\mu_0\vec{D}$ reduces to $-k^2\vec{E} = -\omega^2\mu_0\epsilon\vec{E}$.

Now the phase delocity is determined by $\frac{d(\vec{k}\cdot\vec{r}-\omega t)}{dt} = 0$, we find that the phase velocity $\vec{v}_p = \frac{d\vec{r}}{dt} = \frac{\omega}{k}\hat{k}$. Clearly, we have $\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\epsilon}}$. Hence $v_p = \frac{1}{\sqrt{\mu_0\epsilon}}$.

A.2 0.2 pt

Ans: $c\sqrt{\mu_0\epsilon}$

Solution:

From $v_p = \frac{1}{\sqrt{\mu_0\epsilon}} = \frac{c}{n}$, we find $n = c\sqrt{\mu_0\epsilon}$

A.3 0.4 pt

Ans: $\hat{k}$, $v_r = v_p = \frac{1}{\sqrt{\mu_0\epsilon}}$

Solution:

To find the speed of the ray, we first note that the direction of the energy flow, given by the Poynting vector $\vec{S} = \vec{E}\times\vec{H}$, is in the same direction of $\vec{k}$. The electromagnetic energy density $u = u_e + u_m$ with $u_e = \frac{1}{2}\vec{E}\cdot\vec{D}$ and $u_m = \frac{1}{2}\vec{B}\cdot\vec{H}$.

Now, from $\vec{k}\times\vec{H} = -\omega\vec{D}$, one has $\vec{D} = -\frac{1}{v_p}\hat{k}\times\vec{H}$. Hence $u_e = -\frac{1}{2v_p}\vec{E}\cdot\hat{k}\times\vec{H} = \frac{1}{2v_p}\hat{k}\cdot\vec{E}\times\vec{H}$. Similarly, from $\vec{k}\times\vec{E} = \omega\vec{B}$, we find $u_m = \frac{1}{2v_p}\vec{B}\cdot\hat{k}\times\vec{E} = \frac{1}{2v_p}\hat{k}\cdot\vec{E}\times\vec{H}$. Hence $u = \frac{1}{v_p}\hat{k}\cdot\vec{E}\times\vec{B}$. We find $v_r = S/u = v_p = \frac{1}{\sqrt{\mu_0\epsilon}}$.

Part B. Light propagation in in uniaxial dielectric media

B.1 1.5pt

Ans: $n = n_o$, $\hat{B} = \pm\hat{k}\times\hat{y} = \pm(-\cos\theta, 0, \sin\theta)$, $\hat{D} = \pm\hat{y}$ or $n = \frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta + n_e^2\cos^2\theta}}$, $\hat{B} = \pm\hat{y}$, $\hat{D} = \pm\hat{y}\times\hat{k} = \pm(\cos\theta, 0, -\sin\theta)$. For $\theta = 0$, there is only one permitted value for the refractive index

Solution:

From $\vec{k}\times\vec{E} = \omega\mu_0\vec{H}$ and $\vec{k}\times\vec{H} = -\omega\vec{D}$, one obtains $\vec{k}\times(\vec{k}\times\vec{E}) = -\omega^2\mu_0\vec{D}$. Writing out components and using $\omega = \frac{c}{n}k$, we find

$$-\cos^2\theta E_x + \cos\theta\sin\theta E_z = -\frac{n_o^2}{n^2}E_x,$$
$$-\cos^2\theta E_y - \sin^2\theta E_y = -\frac{n_o^2}{n^2}E_y,$$
$$-\sin^2\theta E_z + \cos\theta\sin\theta E_x = -\frac{n_e^2}{n^2}E_z.$$

After a bit rearrangement, we obtain

$$\left(1 - \frac{n_o^2}{n^2}\right)E_y = 0$$
$$\left(\frac{n_o^2}{n^2} - \cos^2\theta\right)E_x + \cos\theta\sin\theta E_z = 0$$
$$\cos\theta\sin\theta E_x + \left(\frac{n_o^2}{n^2} - \sin^2\theta\right)E_z = 0.$$

The vanishing of the determinant yields

$$\left(1 - \frac{n_o^2}{n^2}\right)\left[(\frac{n_o^2}{n^2} - \cos^2\theta)(\frac{n_e^2}{n^2} - \sin^2\theta) - \sin^2\theta\cos^2\theta\right] = 0. \quad (1)$$

Clearly, for a general θ, we have two solutions for n:

(1) $n = n_o$

In this case, $E_x = E_z = 0$. $\vec{E}$ is parallel to the y axis. From $\vec{k}\times\vec{E} = \omega\vec{B}$ and $\vec{k}\times(\mu_0\vec{B}) = -\omega\vec{D}$, we obtain the directions of $\vec{B}$ and $\vec{D}$ as $\hat{B} = \pm\hat{k}\times\hat{y} = \pm(-\cos\theta, 0, \sin\theta)$ and $\hat{D} = -\hat{k}\times\hat{B} = \pm(0,1,0) = \pm\hat{y}$.

(2) $(\frac{n_o^2}{n^2} - \cos^2\theta)(\frac{n_e^2}{n^2} - \sin^2\theta) - \sin^2\theta\cos^2\theta = 0$.

After rearrangement, we find $n = \frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta + n_e^2\cos^2\theta}}$. Clearly, at $\theta = 0$, $n = n_o$, there is only one refractive index. This is the direction of the optic axis.

In this case, $E_y = 0$. Hence $\vec{E}$ lies in the xz plane. Hence the relation $\vec{k}\times\vec{E} = \omega\vec{B}$ implies $\hat{B} = \pm\hat{y}$. The relation $\vec{k}\times(\mu_0\vec{B}) = -\omega\vec{D}$ implies $\hat{D} = \pm\hat{y}\times\hat{k}$.

B.2 0.8 pt

Ans: (1) when $n = n_o$, $\hat{E} = \pm\hat{y}$ and this is an ordinary ray. $\tan\alpha = 0$.

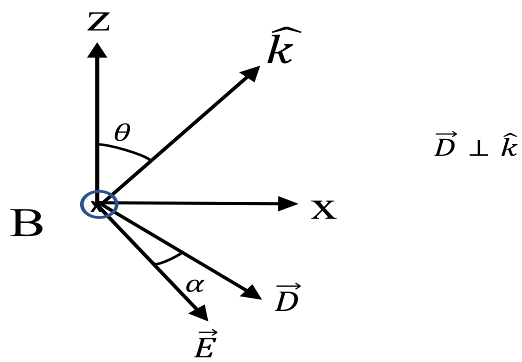
(2) when $n = \frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta + n_e^2\cos^2\theta}}$, $\hat{E} = \pm\frac{1}{\sqrt{n_e^4\cos^2\theta + n_o^4\sin^2\theta}}(-n_e^2\cos\theta, 0, n_o^2\sin\theta)$ and this is an extraordinary ray. $\tan\alpha = \frac{(n_o^2 - n_e^2)\tan\theta}{n_e^2 + n_o^2\tan^2\theta}$.

Solution:

(1) For $n = n_o$, both $\vec{E}$ and $\vec{D}$ are parallel to the y axis. This is an ordinary ray with $\tan\alpha = 0$.

(2) For $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$, $n \neq n_o$, $E_y = 0$. By substituting n back into the equations of E_x and E_z, we find that $\frac{n_o^2}{n_e^2} \sin\theta E_x + \cos\theta E_z = 0$. Hence the electric field lies in xz plane with $\hat{E} = \pm \frac{1}{\sqrt{n_e^4 \cos^2\theta + n_o^4 \sin^2\theta}} (-n_e^2 \cos\theta, 0, n_o^2 \sin\theta)$ ($\vec{B}$ points in $\mp y$ direction.). Therefore, $\vec{E}$ is not perpendicular to $\vec{k}$ and lies in the xz plan in together with $\vec{D}$ and $\vec{k}$. This is the extraordinary ray.

Since $\vec{k} \times \vec{H} = -\omega \vec{D}$, $\vec{D}$ is perpendicular to $\hat{k}$. Hence $\hat{D} = \pm(-\cos\theta, 0, \sin\theta)$. Let $\vec{B} = \hat{y}$, the relative orientation of $\vec{E}$ and $\vec{D}$ for a given θ are shown in the following figure for the case when $n_e < n_o$.



Let the angle relative to x axis be θ_1 and θ_2 for $\vec{E}$ and $\vec{D}$. We have $\tan\theta_2 = -\tan\theta$ and $\tan\theta_1 = -\frac{n_o^2}{n_e^2} \tan\theta$. Hence $\tan\alpha = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2} = \frac{(n_o^2 - n_e^2)\tan\theta}{n_e^2 + n_o^2 \tan^2\theta}$. The same result remains when $n_e > n_o$ except that $\tan\alpha < 0$, indicating that the relative orientation of $\vec{E}$ and $\vec{D}$ is reversed.

B.3 0.6 pt

Ans: $n = n_o$, $\vec{E} = \pm \hat{k} \times \hat{z} / \sin\theta$ and this is an ordinary ray.

when $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$, $\hat{E} = \pm \frac{1}{\sqrt{n_e^4 \cos^2\theta + n_o^4 \sin^2\theta}} \frac{-n_e^2 \cos\theta \hat{k} + (n_o^2 \sin^2\theta - n_e^2 \cos^2\theta)\hat{z}}{\sin\theta}$ and this is an extraordinary ray.

Solution: The problem has an axial symmetry so that in the plane formed by the z axis and $\hat{k}$, one can write $\vec{k} = k_z \hat{z} + k_\perp \hat{k}_\perp$ and $\vec{E} = E_z \hat{z} + E_\perp \hat{k}_\perp$, where $\hat{k}_\perp$ is perpendicular to $\hat{z}$. Clearly, we $k_z = k\cos\theta$, $k_\perp = k\sin\theta$, $E_z = E\cos\theta$, and $E_\perp = E\sin\theta$. Writing out the components for the equation: $\vec{k} \times (\vec{k} \times \vec{E}) = -\omega^2 \mu_0 \vec{D}$, we get exactly the same equations except that E_x is replaced by $E_\perp$. Hence all of the solutions are the same except $\hat{x}$ is replaced by $\hat{k}_\perp$. Since $\hat{k}_\perp \sin\theta = \hat{k} - \cos\theta \hat{z}$, we obtain that when $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$, $\hat{E} = \pm \frac{1}{\sqrt{n_e^4 \cos^2\theta + n_o^4 \sin^2\theta}} [-n_e^2 \cos\theta \frac{(\hat{k} - \cos\theta \hat{z})}{\sin\theta} + n_o^2 \sin\theta \hat{z}] =$

$\pm\frac{1}{\sqrt{n_e^4\cos^2\theta+n_o^4\sin^2\theta}}\frac{-n_e^2\cos\theta\hat{k}+(n_o^2\sin^2\theta-n_e^2\cos^2\theta)\hat{z}}{\sin\theta}$.

B.4 0.8 pt

Ans: (1) $n=n_o$, $\tan\alpha_r=0$, $v_r=\frac{c}{n_o}$, $\hat{S}=(\sin\theta,0,\cos\theta)$

(2) $n=\frac{n_on_e}{\sqrt{n_o^2\sin^2\theta+n_e^2\cos^2\theta}}$, $\tan\alpha_r=\frac{(n_o^2-n_e^2)\tan\theta}{n_e^2+n_o^2\tan^2\theta}$, $v_r=\frac{c}{n_on_e}\sqrt{\frac{n_e^4\cos^2\theta+n_o^4\sin^2\theta}{n_e^2\cos^2\theta+n_o^2\sin^2\theta}}$.

$\hat{S}=\frac{1}{\sqrt{n_e^4\cos^2\theta+n_o^4\sin^2\theta}}(n_o^2\sin\theta,0,n_e^2\cos\theta)$

(3) $n_s=\sqrt{(\hat{S}\cdot\hat{x})^2n_e^2+(\hat{S}\cdot\hat{z})^2n_o^2}$

Solution:

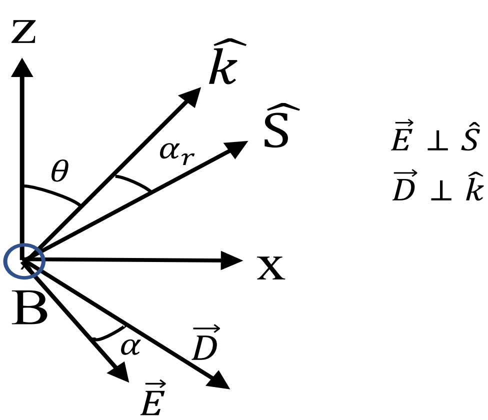
The direction of the energy flow is given by the Poynting vector, $\vec{S}=\vec{E}\times\vec{H}$. Let the energy density of EM wave be u and the ray velocity be v_r. Then $v_r=\frac{S}{u}$. Here $u=u_e+u_m$ with $u_e=\frac{1}{2}\vec{E}\cdot\vec{D}$ and $u_m=\frac{1}{2}\vec{B}\cdot\vec{H}$. There are two cases:

(i)$n=n_o$, $\vec{E}=(0,E,0)$, $\vec{D}=\epsilon\vec{E}$, $\vec{k}\times\vec{E}=\omega\mu_0\vec{H}$, $\vec{k}\times\vec{H}=-\omega\vec{D}$.

$\hat{k}$, $\vec{E}$ and $\vec{H}$ are mutually perpendicular to each other. Hence $\vec{S}$ is parallel to $\hat{k}$, i.e., $\hat{S}=(\sin\theta,0,\cos\theta)$ and $\tan\alpha_r=0$.

Now from $\vec{k}\times\vec{H}=-\omega\vec{D}$, one has $\vec{D}=-\frac{1}{v_p}\hat{k}\times\vec{H}$. Hence $u_e=-\frac{1}{2v_p}\vec{E}\cdot\hat{k}\times\vec{H}=\frac{1}{2v_p}\hat{k}\cdot\vec{E}\times\vec{H}$. Similarly, we find $u_m=\frac{1}{2v_p}\vec{H}\cdot\hat{k}\times\vec{E}=\frac{1}{2v_p}\hat{k}\cdot\vec{E}\times\vec{H}$. Hence $u=\frac{1}{v_p}\hat{k}\cdot\vec{E}\times\vec{H}$. Since $\hat{S}=\hat{k}$, we find $u=\frac{S}{v_p}$. Hence $v_r=\frac{S}{u}=v_p=\frac{\omega}{k}=\frac{c}{n_o}$.

(ii) $n=\frac{n_on_e}{\sqrt{n_o^2\sin^2\theta+n_e^2\cos^2\theta}}$. In this case, we can tak $\vec{B}=(0,B,0)$ (negative y direction works as well). $\vec{D}$, $\vec{E}$ and $\hat{k}$ are in the xz plane and $\vec{D}$ is perpendicular to $\hat{k}$. Therefore, the angle between $\vec{S}=\frac{1}{\mu_0}\vec{E}\times\vec{B}$ and $\hat{k}$ is equal to the angle between $\vec{D}$ and $\vec{E}$, i.e., $\alpha_r=\alpha$. This is shown in the following figure when $n_e<n_o$ (for $n_e>n_o$, both α and α_r are negative, the relative orientation of $\vec{E}$ and $\vec{D}$ is reversed and ordering of $\hat{S}$ and $\hat{k}$ are switched).



Therefore, from problem (d) (ii), we get $\tan\alpha_r = \tan\alpha = \frac{(n_o^2-n_e^2)\tan\theta}{n_e^2+n_o^2\tan^2\theta}$. Now, because $u = \frac{1}{v_p}\hat{k}\cdot\vec{E}\times\vec{H} = \frac{1}{v_p}|\vec{E}\times\vec{H}|\cos\alpha$, we obtain $v_r = \frac{S}{u} = \frac{v_p}{\cos\alpha}$. Hence the phase speed v_p and the ray speed are related by $v_p = v_r\cos\alpha$. From $\tan\alpha$, one finds $\cos\alpha = \frac{n_e^2\cos^2\theta+n_o^2\sin^2\theta}{\sqrt{n_e^4\cos^2\theta+n_o^4\sin^2\theta}}$. Hence $v_r = \frac{c}{n\cos\alpha} = \frac{c}{n_o n_e}\sqrt{\frac{n_e^4\cos^2\theta+n_o^4\sin^2\theta}{n_e^2\cos^2\theta+n_o^2\sin^2\theta}}$.
Clearly, $\hat{S} = (\sin(\theta+\alpha), \cos(\theta+\alpha))$. Since $\sin\alpha = \frac{(n_o^2-n_e^2)\sin\theta\cos\theta}{\sqrt{n_e^4\cos^2\theta+n_o^4\sin^2\theta}}$ and $\cos\alpha = \frac{n_e^2 c\cos^2\theta+n_o^2\sin^2\theta}{\sqrt{n_e^4 c\cos^2\theta+n_o^4\sin^2\theta}}$, we find $\hat{S} = \frac{1}{\sqrt{n_e^4 c\cos^2\theta+n_o^4\sin^2\theta}}(n_o^2\sin\theta, 0, n_e^2\cos\theta)$.
From $n_s^2 = \left(\frac{c}{v_r}\right)^2 = n_o^2 n_e^2\frac{n_e^2\cos^2\theta+n_o^2\sin^2\theta}{n_e^4\cos^2\theta+n_o^4\sin^2\theta} = \frac{(n_o^2\sin\theta)^2 n_e^2+(n_e^2\cos\theta)n_o^2}{n_e^4\cos^2\theta+n_o^4\sin^2\theta}$, we find $n_s = (\hat{S}\cdot\hat{x})^2 n_e^2 + (\hat{S}\cdot\hat{z})^2 n_o^2$.

B.5 1.1 pt

Ans: $\bar{A} = P_1(n^2\sin^2\theta_1 - P_1)$, $\bar{B} = -2P_3(n^2\sin^2\theta_1 - P_1)$, $\bar{C} = P_2 n^2\sin^2\theta_1 - P_3^2$.
$\phi = 0$, $\tan\theta_2 = \frac{n n_e\sin\theta_1}{n_o\sqrt{n_o^2-n^2\sin^2\theta_1}}$.
$\phi = \pi/2$, $\tan\theta_2 = \frac{n n_o\sin\theta_1}{n_e\sqrt{n_e^2-n^2\sin^2\theta_1}}$.

Solution:

Let the distance along z axis between A and B be d and the point of the interface that the ray passes be the origin O. The coordinates of B and A points can be expressed as $(h_2, 0, z)$ and $(h_1, 0, d-z)$. The distances are then given by $\overline{AO} \equiv d_1 = \sqrt{h_1^2 + (d-z)^2}$ and $\overline{OB} \equiv d_2 = \sqrt{h_2^2 + z^2}$. The propagation time from A to B is determined by the ray speed v_r as $(d_1 n_{s1} + d_2 n_{s2})/c$, where n_{si} are ray indices for medium i. According to the Fermat's principle, we need to minimize the optical path length defined by $\Delta \equiv d_1 n_{s1} + d_2 n_{s2}$. According to problem (e), we have $n_{s2}^2 = (\frac{\overrightarrow{OB}}{\overline{OB}}\cdot\hat{x}_2)^2 n_e^2 + (\frac{\overrightarrow{OB}}{\overline{OB}}\cdot\hat{z}_2)^2 n_o^2$. For an isotropic medium, the ray index is simply the refractive index, i.e., $n_{s1} = n$. Using the following relations

$$\frac{\overrightarrow{OB}}{\overline{OB}}\cdot\hat{x}_2 = \cos(\phi-\theta_2) = \frac{h_2}{d_2}\cos\phi + \frac{z}{d_2}\sin\phi,$$
$$\frac{\overrightarrow{OB}}{\overline{OB}}\cdot\hat{z}_2 = \cos(\frac{\pi}{2}+\phi-\theta_2) = \sin(\theta_2-\phi) = \frac{z}{d_2}\cos\phi - \frac{h_2}{d_2}\sin\phi,$$

we find

$$\Delta = n\sqrt{h_1^2+(d-z)^2} + \sqrt{(h_2\cos\phi + z\sin\phi)^2 n_e^2 + (-h_2\sin\phi + z\cos\phi)^2 n_o^2}.$$

The minimum occurs when $\frac{d\Delta}{dz} = 0$. We obtain

$$n\frac{z-d}{\sqrt{h_1^2+(d-z)^2}} + \frac{(h_2\sin\phi\cos\phi(n_e^2-n_o^2) + z(n_e^2\sin^2\phi + n_o^2\cos^2\phi)}{\sqrt{(h_2\cos\phi+z\sin\phi)^2 n_e^2 + (-h_2\sin\phi+z\cos\phi)^2 n_o^2}} = 0.$$

Recognizing $\frac{d-z}{\sqrt{h_1^2+(d-z)^2}} = \sin\theta_1$, moving the second term to the left and taking square of the equation, we obtain

$$n^2 \sin^2\theta_1 = \frac{(P_3 - P_1 \tan\theta_2)^2}{P_1 \tan^2\theta_2 - 2P_3 \tan\theta_2 + P_2},$$

where $P_1 = n_o^2 \cos^2\phi + n_e^2 \sin^2\phi$, $P_2 = n_o^2 \sin^2\phi + n_e^2 \cos^2\phi$, and $P_3 = (n_o^2 - n_e^2)\sin\phi\cos\phi$. By expanding the above equation out, we find

$$P_1(n^2\sin^2\theta_1 - P_1)\tan^2\theta_2 - 2P_3(n^2\sin^2\theta_1 - P_1)\tan\theta_1 + P_2 n^2 \sin^2\theta_1 - P_3^2 = 0.$$

Hence $\bar{A} = P_1(n^2\sin^2\theta_1 - P_1)$, $\bar{B} = -2P_3(n^2\sin^2\theta_1 - P_1)$, and $\bar{C} = P_2 n^2\sin^2\theta_1 - P_3^2$.
For $\phi = 0$, we have $P_3 = 0$, $P_1 = n_o^2$, and $P_2 = n_e^2$. We find $n_o^2(n^2\sin^2\theta_1 - n_o^2)\tan^2\theta_2 + n_e^2 n^2 \sin^2\theta_1 = 0$. Hence $\tan\theta_2 = \frac{n n_e \sin\theta_1}{n_o\sqrt{n_o^2 - n^2\sin^2\theta_1}}$.
For $\phi = \pi/2$, we have $P_3 = 0$, $P_1 = n_e^2$, and $P_2 = n_o^2$. We find $n_e^2(n^2\sin^2\theta_1 - n_e^2)\tan^2\theta_2 + n_o^2 n^2\sin^2\theta_1 = 0$. Hence $\tan\theta_2 = \frac{n n_o \sin\theta_1}{n_e\sqrt{n_e^2 - n^2\sin^2\theta_1}}$.

Part C. Entanglement of light

C.1 0.8 pt

Ans:(1) $\omega = \omega_1 \pm \omega_2$, $\vec{k} = \vec{k}_1 \pm \vec{k}_2$
(2) $\hbar\omega = \hbar\omega_1 \pm \hbar\omega_2$, $\hbar\vec{k} = \hbar\vec{k}_1 \pm \hbar\vec{k}_2$ represents the energy conservation and momentum conservation of photons.
(3) Splitting of photon: Energy conservation $\omega = \omega_1 + \omega_2$, momentum conservation: $\vec{k} = \vec{k}_1 + \vec{k}_2$.

Solution:
For a light wave with frequency ω and $\vec{k}$, the corresponding polarization density and the electric field are in the form of $\vec{A}\cos(\omega t - \vec{k}\cdot\vec{r})$, which can be rewritten as $\frac{\vec{A}}{2}(e^{i(\omega t - \vec{k}\cdot\vec{r})} + e^{-i(\omega t - \vec{k}\cdot\vec{r})})$. By substituting the above form into the equation $P_i^{NL} = \sum_j \sum_k \chi_{ijk}^{(2)} E_j E_k$ and equating the relevant exponents, we find all possible relations are

$$\omega = \omega_1 + \omega_2, \vec{k} = \vec{k}_1 + \vec{k}_2.$$
$$\text{or } \omega = \omega_1 - \omega_2, \vec{k} = \vec{k}_1 - \vec{k}_2,$$

where we have made use of the fact that the frequency is positive. The meaning for the these relations is clear if one recall that the energy and momentum of a photon is given by $\hbar\omega$ and $\hbar\vec{k}$. The relation of $\hbar\omega = \hbar\omega_1 + \hbar\omega_2, \hbar\vec{k} = \hbar\vec{k}_1 + \hbar\vec{k}_2$ represents the energy and momentum conservations when a photon with (ω, $\vec{k}$) is annihilated and split into two photons with (ω_1, $\vec{k}_1$) and (ω_2, $\vec{k}_2$), while the relation of $\hbar\omega = \hbar\omega_1 - \hbar\omega_2, \hbar\vec{k} = \hbar\vec{k}_1 - \hbar\vec{k}_2$ represents the energy and momentum conservations when a photon with (ω_1, $\vec{k}_1$) is annihilated and split into two photons with (ω, $\vec{k}$) and (ω_2, $\vec{k}_2$).

C.2 0.8 pt

Ans: $\mathbf{o} \to \mathbf{o} + \mathbf{o}$, $\mathbf{e} \to \mathbf{e} + \mathbf{e}$

Solution:

For the collinear case, the phase matching conditions become $\omega = \omega_1 + \omega_2$, $\frac{n_i(\omega)\omega}{c} = \frac{n_j(\omega_1)\omega_1}{c} + \frac{n_k(\omega_2)\omega_2}{c}$, where i, j, and k are indices of either **o** or **e**. Assuming that $\omega_1 \geq \omega_2$, one can solve ω_1 as $\omega_1 = \omega - \omega_2$. We obtain

$$n_i(\omega) - n_j(\omega_1) = \frac{\omega_2}{\omega}\left[n_k(\omega_2) - n_j(\omega_1)\right]. \tag{2}$$

Clearly, because $\omega > \omega_1 \geq \omega_2$, if $i = j = k$, $n_i(\omega) - n_j(\omega_1) > 0$ and $n_k(\omega_2) - n_j(\omega_1) \leq 0$, the above equation cannot be satisfied. For other cases, because there is no relation between n_o and n_e, the phase matching conditions can be satisfied. Hence only $\mathbf{o} \to \mathbf{o}+\mathbf{o}$ and $\mathbf{e} \to \mathbf{e}+\mathbf{e}$ are not possible.

C.3 1.5 pt

Ans: (1) $M = \frac{K_o[1-N_e(\Omega_e,\theta)\cot\theta]+K_e}{2K_eK_o}$, $E = -N_e/2M$ and $F = -(\Omega - \Omega_e)(\frac{1}{u_o} - \frac{1}{u_e}) + \frac{N_e^2}{4M}$

(2) the angle between the axis of the cone and z' is $N/K_o = -\frac{2K_eN_e}{K_o[1-N_e(\Omega_e,\theta)\cot\theta]+K_e}$

(3) the angle of cone is about $\frac{\sqrt{L/M}}{K_o} = -\frac{(\Omega-\Omega_e)}{MK_o}(\frac{1}{u_o} - \frac{1}{u_e}) + \frac{N_e^2}{4M^2K_o}$.

Solution:

To satisfy the phase matching condition, we expand the angular frequencies ω_1 and ω_2 into $\omega_1 = \Omega_e + \nu$ and $\omega_2 = \Omega_o + \nu'$. Clearly, because $\Omega_e + \Omega_o = \Omega_p$, to satisfy $\omega_1 + \omega_2 = \omega$, $\nu' = -\nu$. Similarly, the conditions for the wavevectors, $\vec{k} = \vec{k}_1 + \vec{k}_2$, can be written as $k_z = k = K_p = k_{1z} + k_{2z}$ and $\vec{k}_{2\perp} = -\vec{k}_{1\perp} \equiv \vec{q}_\perp$. For the **o** light ray, we have $k_{2\perp}^2 + k_{2z}^2 = k_2^2$ with $k_2 = \frac{n_o(\omega_2)\omega_2}{c}$. One finds that $k_{2z} = \sqrt{k_2^2 - k_{2\perp}^2} = k_2 - \frac{k_{2\perp}^2}{2k_2}$. Expanding the dependence of ω_2 in k_2 to ν, we obtain

$$k_2 = \frac{n_o(\omega_2)\omega_2}{c} = \frac{n_o(\Omega_o)\Omega_o}{c} + \frac{dk_2}{d\omega_2}(\omega_2 - \Omega_o) = K_o - \frac{\nu}{u_o},$$

where u_o is the group velocity for the ordinary ray. Hence to the second order of corrections, we get

$$k_{2z} = K_o - \frac{\nu}{u_o} - \frac{q_\perp^2}{2K_o}.$$

Similarly, for the **e** light ray, we have $k_{1\perp}^2 + k_{1z}^2 = k_1^2$ with $k_1 = \frac{n_e(\omega_1,\theta_p)\omega_1}{c}$. One finds that $k_{1z} = \sqrt{k_1^2 - k_{1\perp}^2} = k_1 - \frac{k_{1\perp}^2}{2k_1}$. The expansion of k_1 is different from that for k_2 due to its angle dependence. Let the spherical angles for $\vec{k}_1$ be θ_1 and ϕ_1. We have

$$k_1 = \frac{n_e(\omega_1,\theta_1)\omega_1}{c} = \frac{n_e(\Omega_e,\theta)\Omega_e}{c} + \frac{dk_1(\Omega_e,\theta)}{d\Omega_e}(\omega_1 - \Omega_e) + \frac{\Omega_e}{c}\frac{dn_e(\Omega_e,\theta)}{d\theta}(\theta_1 - \theta) + \cdots$$

Here $\frac{n_e(\Omega_e,\theta)\Omega_e}{c} = K_e$, $\frac{dk_1(\Omega_e,\theta)}{d\Omega_e}$ is $1/u_e$ with u_e being the group velocity for the extraordinary ray and is given by

$$\frac{dk_1(\Omega_e,\theta)}{d\Omega_e} = \frac{n_e(\Omega_e,\theta)}{c} + \frac{\Omega_e}{c}\frac{dn_e(\Omega_e,\theta)}{d\Omega_e}.$$

Because $\frac{dn_e(\Omega_e,\theta)}{d\theta} = \frac{n_o n_e(n_e^2-n_o^2)\sin\theta\cos\theta}{(n_o^2\sin^2\theta+n_e^2\cos^2\theta)^{3/2}} = n_e(\Omega_e,\theta)N_e(\Omega_e,\theta)$, we find $N_e(\Omega_e,\theta) = \frac{(n_e^2-n_o^2)\sin\theta\cos\theta}{n_o^2\sin^2\theta+n_e^2\cos^2\theta}$. Note that for $n_e < n_o$, $N_e(\Omega_e,\theta) < 0$. To find $\delta\theta = \theta_1 - \theta$, we note that for any $\vec{k}_\alpha$, one has (cf. Fig. 2(a))

$$\hat{k}_\alpha \cdot \widehat{OA} = \cos\theta_\alpha = \cos\theta\cos\psi_\alpha + \sin\theta\sin\psi_\alpha\cos\phi_\alpha.$$

Since $\sin\psi_1 = |\vec{k}_{\perp,1}|/|\vec{k}_1| = q_\perp/k_1 \ll 1$ and $\cos\psi_1 = \sqrt{1-\sin^2\psi_1} = 1 - 1/2\sin^2\psi_1 + \cdots$, to the second order, we can replace k_1 by K_e and obtain

$$\hat{k}_1 \cdot \widehat{OA} = \cos\theta_1 = \cos\theta\left[1 - \frac{1}{2}\frac{q_\perp^2}{K_e^2} + \cdots\right] + \sin\theta\left[\frac{q_\perp}{K_e} + \cdots\right]\cos\phi_1.$$

On the other hand, $\cos\theta_1 = \cos\theta + \frac{d\cos\theta}{d\theta}(\theta_1-\theta) + \cdots = \cos\theta - \sin\theta(\theta_1-\theta) + \cdots$. Comparing this equation to the equaton for $\hat{k}_1 \cdot \widehat{OA}$, we obtain

$$\theta_1 - \theta = \frac{1}{2}\frac{q_\perp^2}{K_e^2}\cot\theta - \frac{q_\perp}{K_e}\cos\phi_1 \cdots = \frac{1}{2}\frac{q_\perp^2}{K_e^2}\cot\theta + \frac{q_{x'}}{K_e} + \cdots$$

Putting all together, we find

$$k_{1z} = K_e + \frac{1}{u_e}(\Omega - \Omega_e) + N_e(\Omega_e,\theta)q_{x'} + \frac{q_\perp^2}{2K_e}\left[N_e(\Omega_e,\theta)\cot\theta - 1\right] + \cdots.$$

The above equation when combined with the equation of k_{1z} and the relation $K_p = k_{1z} + k_{2z}$, we find

$$(\Omega - \Omega_e)(\frac{1}{u_e} - \frac{1}{u_o}) + N_e(\Omega_e,\theta)q_{x'} + q_\perp^2\left\{\frac{K_o\left[N_e(\Omega_e,\theta)\cot\theta - 1\right] - K_e}{2K_eK_o}\right\} = 0.$$

Because $n_e < n_o$, $N_e(\Omega_e, \theta) < 0$. The above equation can be rewritten in the form

$$M\left[q_{x'} - \frac{N_e}{2D}\right]^2 + Mq_{y'}^2 = -(\Omega - \Omega_e)(\frac{1}{u_o} - \frac{1}{u_e}) + \frac{N_e^2}{4M}.$$

Here $D = \frac{K_o[1-N_e(\Omega_e,\theta)\cot\theta]+K_e}{2K_eK_o} > 0$. Hence $E = -N_e/2M > 0$ $(N_e < 0)$ and $L = -(\Omega - \Omega_e)(\frac{1}{u_o} - \frac{1}{u_e}) + \frac{N_e^2}{4M}$. Clearly, the cone axis formed by $\vec{k}_2$ is characterized by $\vec{q}_\perp$. We find that the angle between the axis of the cone and z' is $\tan^{-1}(N/k_{1z})$, which is about $N/k_{1z} \approx N/K_o = -\frac{2K_eN_e}{K_o[1-N_e(\Omega_e,\theta)\cot\theta]+K_e}$. The angle of the cone is given by $\sin^{-1}\frac{\sqrt{L/M}}{k_2} \approx \frac{\sqrt{L/M}}{K_o} = -\frac{(\Omega-\Omega_e)}{MK_o}(\frac{1}{u_o} - \frac{1}{u_e}) + \frac{N_e^2}{4M^2K_o}$.

C.4 0.8pt

Ans: $P(\alpha,\beta) = \frac{1}{2}\sin^2(\alpha+\beta)$, $P(\alpha,\beta_\perp) = \frac{1}{2}\cos^2(\alpha+\beta)$, $P(\alpha_\perp,\beta) = \frac{1}{2}\cos^2(\alpha+\beta)$, $P(\alpha_\perp,\beta_\perp) = \frac{1}{2}\sin^2(\alpha+\beta)$

Solution:

For a-photon, let the electric field along the polarizer and perpendicular to the polarization represented by $|\alpha_x\rangle$ and $|\alpha_y\rangle$. Here α_x and α_x are essentially the electric field amplitudes in appropriate units. The electric fields (the states) along $\hat{x}'$ and $\hat{y}'$ can be written as

$$|\hat{x}'_a\rangle = \cos\alpha|\alpha_x\rangle - \sin\alpha|\alpha_y\rangle,$$
$$|\hat{y}'_a\rangle = \sin\alpha|\alpha_x\rangle + \cos\alpha|\alpha_y\rangle.$$

Similarly, for b-photon, we have

$$|\hat{x}'_b\rangle = \cos\beta|\beta_x\rangle - \sin\beta|\beta_y\rangle,$$
$$|\hat{y}'_b\rangle = \sin\beta|\beta_x\rangle + \cos\beta|\beta_y\rangle.$$

Hence we obtain

$$|\hat{x}'_a\rangle|\hat{y}'_b\rangle = (\cos\alpha|\alpha_x\rangle - \sin\alpha|\alpha_y\rangle)(\sin\beta|\beta_x\rangle + \cos\beta|\beta_y\rangle),$$
$$|\hat{y}'_a\rangle|\hat{x}'_b\rangle = (\sin\alpha|\alpha_x\rangle + \cos\alpha|\alpha_y\rangle)(\cos\beta|\beta_x\rangle - \sin\beta|\beta_y\rangle).$$

The state of the entangled photon pair can be written as

$$\begin{aligned}
& \frac{1}{\sqrt{2}}(|\hat{x}'_a\rangle|\hat{y}'_b\rangle + |\hat{y}'_a\rangle|\hat{x}'_b\rangle) \\
= & \frac{1}{\sqrt{2}}[(\cos\alpha\sin\beta + \sin\alpha\cos\beta)(|\alpha_x\rangle|\beta_x\rangle - |\alpha_y\rangle|\beta_y\rangle) \\
+ & (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(|\alpha_x\rangle|\beta_y\rangle - |\alpha_y\rangle|\beta_x\rangle)] \\
= & \frac{1}{\sqrt{2}}[\sin(\alpha+\beta)(|\alpha_x\rangle|\beta_x\rangle - |\alpha_y\rangle|\beta_y\rangle) + \cos(\alpha+\beta)(|\alpha_x\rangle|\beta_y\rangle - |\alpha_y\rangle|\beta_x\rangle)]
\end{aligned}$$

From the above equation, we obtain

$$
\begin{aligned}
P(\alpha,\beta) &= \frac{1}{2}\sin^2(\alpha+\beta),\\
P(\alpha_\perp,\beta_\perp) &= \frac{1}{2}\sin^2(\alpha+\beta),\\
P(\alpha,\beta_\perp) &= \frac{1}{2}\cos^2(\alpha+\beta),\\
P(\alpha_\perp,\beta) &= \frac{1}{2}\cos^2(\alpha+\beta).
\end{aligned}
$$

C.5 0.5pt

Ans: $S = |\cos 2(\alpha-\beta) - \cos 2(\alpha-\beta')| + |\cos 2(\alpha'-\beta) + \cos 2(\alpha'-\beta')|$

$S = 2\sqrt{2}$. $S > 2$ indicates that it is not consistent with classical theories.

Solution:

One first realizes that $E(\alpha,\beta) = \frac{P(\alpha,\beta)+P(\alpha_\perp,\beta_\perp)-P(\alpha,\beta_\perp)-P(\alpha_\perp,\beta)}{P(\alpha,\beta)+P(\alpha_\perp,\beta_\perp)+P(\alpha,\beta_\perp)+P(\alpha_\perp,\beta)}$. Using expressions for P, we find

$$
\begin{aligned}
& E(\alpha,\beta) = \sin^2(\alpha+\beta) - \cos^2(\alpha+\beta)\\
= \;& (\sin\alpha\cos\beta + \cos\alpha\sin\beta)^2 - (\cos\alpha\cos\beta - \sin\alpha\sin\beta)^2\\
= \;& -(\cos^2\alpha - \sin^2\alpha)(\cos^2\beta - \sin^2\beta) + 4\sin\alpha\sin\beta\cos\alpha\cos\beta\\
= \;& \sin(2\alpha)\sin(2\beta) - \cos(2\alpha)\cos(2\beta) = -\cos 2(\alpha-\beta).
\end{aligned}
$$

Hence $S = |\cos 2(\alpha-\beta) - \cos 2(\alpha-\beta')| + |\cos 2(\alpha'-\beta) + \cos 2(\alpha'-\beta')|$. For $\alpha = \frac{\pi}{4}$, $\alpha' = 0$, $\beta = -\frac{\pi}{8}$, $\beta' = \frac{\pi}{8}$, we find $S = |-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}| + |\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}| = 2\sqrt{2} > 2$. Hence classical theories do not apply.