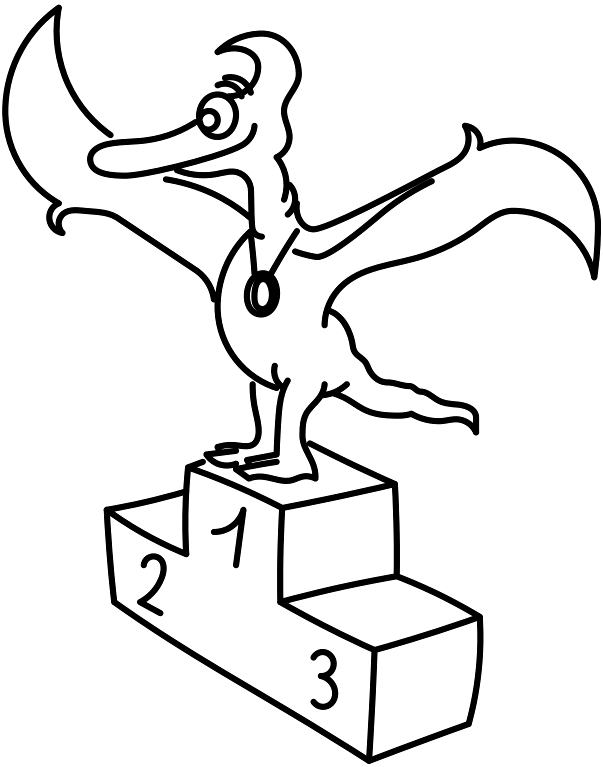
12$^{\text{th}}$ FYKOS Physics Brawl
Solutions of problems



Problem AA ... wouldn't steal a car

We're downloading (legally) a film of size 2.1 GB*; the download speed is* $350\,\mathrm{kB\cdot s^{-1}}$. *The length of the film is* 90 min. *How long after the download begins can we start watching the film to be able to watch it without pausing?* *Mirek didn't want to pay up (for faster download speed).*

The download time is $2.1\,\mathrm{GB}/350\,\mathrm{kB\cdot s^{-1}} = 100\,\mathrm{min}$. We can start watching the film 90 min before the end of the download. Therefore we have to wait at least $100\,\mathrm{min} - 90\,\mathrm{min} = 10\,\mathrm{min}$ after the download begins to watch the film continuously.

Miroslav Hanzelka
`mirek@fykos.cz`

Problem AB ... simple conveyor belt

We have a conveyor belt of mass M (by itself, without any load). Initially (at time 0 s*), there is some sand with mass m lying on the conveyor belt; the sand gradually falls off the belt with a constant rate μ (mass of sand falling per unit time, in kilograms per second), from time* 0 s *until there is no sand left on the belt. What should be the force exerted on the conveyor belt at any time t (between* 0 s *and the moment when all sand falls off) so that it'd move with* ***constant acceleration*** *a?* *Karel simplified one complicated problem.*

First, let us remember Newton's second law of motion, which tells us that $F = m_{\mathrm{tot}}a$, where F is the exerted force and m_{tot} is the total mass of the body. In our case, the total mass is time dependent, but we can express it quite simply as $m_{\mathrm{tot}} = M + m - \mu t$, the reasoning being that $M + m$ is the initial mass and μt is the mass of sand that has fallen off the belt from the start up to time t. The force we're looking for is therefore $F = (M + m - \mu t)\,a$.

Karel Kolář
`karel@fykos.cz`

Problem AC ... dropped banknote

Would it be worth it for Bill Gates to pick up a $100 banknote from the ground? Let's say that his annual income is 30,000,000,000 *Czech crowns and that by picking up the note, his average income would be stopped for 3 seconds. How much money (in Czech crowns) could he earn by picking up the banknote? In case he'd be losing money that way, give the result as a negative number. Use the exchange rate* 1 CZK = $0.046. *His income is continuous and constant.* *Matěj stole this from someone, no idea who.*

After conversion to US Dollars, the income of Bill Gates is $1,380,000,000. This value divided by the number of seconds in a year equals the average income \$43.7 $\mathrm{s^{-1}}$. That makes $131.3 lost every 3 seconds when Bill is not working. Now we can see that he shouldn't pick up a $100 banknote, because it would drop his earnings by $31.3, or 680 CZK.

Matěj Mezera
`m.mezera@fykos.cz`

Problem AD ... zero to hundred

One of the performance characteristics of a car is the time needed to accelerate from $0\,\mathrm{km\cdot h^{-1}}$ *to* $100\,\mathrm{km\cdot h^{-1}}$*. Consider a car that can do it in* $4.0\,\mathrm{s}$*. Find the distance travelled by the car while accelerating. Suppose that the acceleration of the car is constant.*

Karel was wondering about accelerating.

The distance s covered by a uniformly accelerating object during time t can be expressed as $s = at^2/2$, where a denotes the (constant) acceleration. In our case, the acceleration is

$$a = \Delta v/\Delta t = 100/4/3.6\,\mathrm{m\cdot s^{-2}} = 6.94\,\mathrm{m\cdot s^{-2}}\,.$$

We can now find the distance

$$s = \frac{1}{2}\Delta v \cdot \Delta t \doteq 55.6\,\mathrm{m}\,.$$

Assuming the acceleration is constant, the car travels a distance of $56\,\mathrm{m}$.

Karel Kolář
`karel@fykos.cz`

Problem AE ... bathing

When Dano fills his bathtub, he first turns on the hot water with temperature $t_1 = 52\,{}^\circ\mathrm{C}$*. Because the hot water is too hot, he then mixes in some cold water with temperature* $t_2 = 18\,{}^\circ\mathrm{C}$*. He wants to achieve a temperature* $t = 42\,{}^\circ\mathrm{C}$ *(which is not ideal, but some people like it hot). What is the ratio of water volumes* $K = V_2/V_1$ *needed to achieve this temperature? Neglect the heat capacity of the bathtub and any thermal exchange with the surroundings. Volume* V_1 *labels the water with temperature* t_1*.*

Karel likes to have a bath.

We could write down the calorimetry formula, but can manage to solve this problem just with a simple argument. Let us assume that the specific heat of water is constant in the temperature range from t_2 to t_1 and that the volume of the water also doesn't depend on the temperature. Then, the final temperature after mixing can be written as the average of cold and hot water temperatures weighted by their respective volumes,

$$t = \frac{V_1 t_1 + V_2 t_2}{V_1 + V_2}\,.$$

Through simple manipulations, we rewrite the equation using the ratio V_2/V_1,

$$t\left(V_1 + V_2\right) = V_1 t_1 + V_2 t_2 \qquad \Rightarrow \qquad K = \frac{V_2}{V_1} = \frac{t - t_1}{t_2 - t} = \frac{5}{12} \doteq 0.42\,.$$

The volume ratio is $K = 0.42$.

Karel Kolář
`karel@fykos.cz`

Problem AF ... comparison of the calorific value

What is the difference between heat of combustion of one litre of diesel fuel and one litre of petrol (gasoline)? The heat of combustion of diesel is $H_\mathrm{d} = 42.6\,\mathrm{MJ{\cdot}kg^{-1}}$ *and that of petrol is* $H_\mathrm{p} = 43.6\,\mathrm{MJ{\cdot}kg^{-1}}$*. The density of petrol is* $\varrho_\mathrm{p} = 740\,\mathrm{kg{\cdot}m^{-3}}$*, the density of diesel is* $\varrho_\mathrm{d} = 840\,\mathrm{kg{\cdot}m^{-3}}$*. The answer must contain two information:* ***which fuel*** *has the higher heat of combustion per litre and by* ***how much****.*

Karel was thinking about the efficiency of diesel fuel.

Since we know the heats of combustion per unit mass, all we have to do is multiply these values by densities of respective fuels to get the heats of combustion per unit volume. The only catch in this problem is that the densities are given in kilograms per cubic meter, so we have to use the conversion $1\,\mathrm{l} = 10^{-3}\,\mathrm{m^3}$. After multiplication, we obtain

$$E_\mathrm{d} = H_\mathrm{d}\varrho_\mathrm{d} \doteq 35.8\,\mathrm{MJ{\cdot}l^{-1}}\,, \quad E_\mathrm{p} = H_\mathrm{p}\varrho_\mathrm{p} \doteq 32.3\,\mathrm{MJ{\cdot}l^{-1}}\,,$$
$$E_\mathrm{d} - E_\mathrm{p} = (H_\mathrm{d}\varrho_\mathrm{d} - H_\mathrm{p}\varrho_\mathrm{p}) \doteq 3.5\,\mathrm{MJ{\cdot}l^{-1}}$$

Diesel has higher heat of combustion per litre than gasoline, namely by $3.5\,\mathrm{MJ{\cdot}l^{-1}}$.

Karel Kolář
`karel@fykos.cz`

Problem AG ... conveyor belt travelling problem

At the Dubai airport, there are conveyor belts for walking; the velocity of a conveyor belt is v*. Matěj needs to get to the terminal B27. He can either go directly to the terminal without using conveyor belts or take an alternative path where he gets to walk on conveyor belts. The second path is 1/3 longer, but half of its length is covered by conveyor belts. After a few minutes of hard calculations, he finds out that both paths take the same time, so he decides to take the moving walkway because it is more fun. Your task is to find Matěj's walking speed* u *(assume that this speed is constant).* *Matěj loves walking on conveyor belts.*

First, let us denote the length of the direct path by s. Then, this path takes time

$$t = \frac{s}{u}\,.$$

The other path consists of two parts. The first part has length $\frac{2}{3}s$ and is travelled at velocity u; the second part has length $\frac{2}{3}s$ and is travelled at velocity $v+u$. The total time can be expressed as

$$t = \frac{2}{3}\frac{s}{u} + \frac{2}{3}\frac{s}{v+u}\,.$$

Because the two times are equal, we can solve the equation for u,

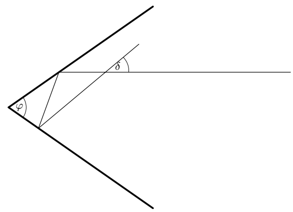
$$\frac{s}{u} = \frac{2s}{3u} + \frac{2s}{3(v+u)}\,,$$
$$\frac{1}{3u} = \frac{2}{3(v+u)}\,,$$
$$u = v\,.$$

Matěj's walking speed is the same as the speed of a conveyor belt.

Matěj Mezera
`m.mezera@fykos.cz`

Problem AH ... crossed mirrors

A ray of light is being reflected by two planar mirrors (see figure). The angle between the mirrors is φ. How does the angle δ (see figure) depend on the angle φ? *Daniel was studying optics.*



Denoting the angles between the ray and the first and second mirror α and β respectively, we have the following relation for the left triangle $\alpha + \beta + \varphi = \pi$. For the right triangle, we get

$$2\left(\frac{\pi}{2} - \alpha\right) + 2\left(\frac{\pi}{2} - \beta\right) + \delta = \pi\,,$$

from which we can easily express $\delta = 2(\alpha + \beta) - \pi$. The last step is to substitute for $\alpha + \beta$ from the first expression, which leads to $\delta = \pi - 2\varphi$.

Jáchym Bártík
tuaki@fykos.cz

Problem BA ... antimatterianism

Perhaps you've heard stories of so-called breatharians and pranists, who only need to breath to "suck out" the life energy from their surroundings. However, what if someone wanted to survive only on antimatter? That is, it's apparently impossible, but we're looking for an estimate of the minimum mass of antimatter such a human would have to carry inside to survive until natural death. Consider a 20-year-old adult who's expected to die in 60 years (i.e. at the age of 80). The power he needs to survive is $P = 100\,\mathrm{W}$. He has plenty of matter available; also let's assume that he consumes the energy with 100 % efficiency.

Karel was wondering how much a breatharian steals from the fridge at night.

The well-known (and often misunderstood) equation $E = mc^2$ says that matter (or antimatter) with mass m has an equivalent energy of mc^2. Reactions of antimatter with matter will lead to annihilation and release the energy of the antimatter and also that of the matter. So when we have antimatter with mass m and plenty of matter, we can gain energy $2mc^2$ (the reaction ratio is 1 to 1).

Power P over time $t = 60\,\mathrm{yr} \doteq 1.9 \cdot 10^9\,\mathrm{s}$ equals energy $E = Pt$, which in combination with $E = 2mc^2$ leads to

$$m = \frac{Pt}{2c^2} \doteq 1 \cdot 10^{-6}\,\mathrm{kg} = 1\,\mathrm{mg}\,.$$

This is a minuscule amount in comparison to the weight of an average human body. However, to obtain 1 mg of antimatter in current particle accelerators, we would have to let them run for about 100 million years. Which is a lot.

Jakub Šafin
xellos@fykos.cz

Problem BB ... interplanetary pendulum

Let's denote the ratio of Earth's mass to Moon's mass by a. Next, let's denote the respective ratio of radii by b. $a > 1 \& b > 1$. Find the ratio of periods of the same pendulum on Earth's surface and on Moon's surface (in terms of a and b only).

Matěj loves the song Organism Do Evolve.[1]

The period of a simple pendulum can be expressed as

$$T = 2\pi\sqrt{\frac{l}{g}}\,.$$

Was this a physical, not mathematical, pendulum, the right-hand side would just be multiplied by a constant dependent on the moment of inertia. The only thing we need to know here is the ratio of gravitational accelerations on the surface of the Earth and the Moon. Gravitational acceleration is given by the formula

$$g = G\frac{M}{R^2}\,.$$

So, the ratio of those two gravitational accelerations is $\frac{g_\mathrm{E}}{g_\mathrm{M}} = \frac{GM_\mathrm{E}R_\mathrm{M}^2}{GM_M R_\mathrm{E}^2} = \frac{a}{b^2}$, where the subscripts stand for the Earth and the Moon.

Then, the ratio of the periods of the given pendulum is

$$\frac{T_\mathrm{E}}{T_\mathrm{M}} = \sqrt{\frac{b^2}{a}} = \frac{b}{\sqrt{a}}\,.$$

For Earth and Moon, this amounts to approximately 0.41.

Matěj Mezera
m.mezera@fykos.cz

Problem BC ... charged balls

There are two rigid (inelastic) strings of negligible weight with lengths $l = 1.0\,\mathrm{m}$ attached to the same point at height $h = 3.0\,\mathrm{m}$ above the ground. First, we connect one small metal ball with mass m at the free end of one string. The charge of the ball is $Q = 2.8\,\mathrm{\mu C}$. Then, we attach an identical ball with the same charge at the free end of the other string. The balls electrically repel each other; they reach an equilibrium position when their height above the ground increases by $\Delta h = 3.0\,\mathrm{cm}$. What is the mass of one ball? *Danka is fascinated by charged balls.*

After reaching the equilibrium position, the angle between the strings is α. The tensile force of each string responsible for keeping each ball stable compensates the effect of gravitational and electrostatic forces. The gravitational force acting on each ball is

$$F_\mathrm{g} = mg\,.$$

The repulsive electrostatic force depends on the charge of the balls and also on their mutual distance d,

$$F_e = \frac{1}{4\pi\varepsilon}\frac{Q^2}{d^2}\,.$$

[1] https://www.youtube.com/watch?v=dx-Sy5M3bME

The distance d follows from the geometry of the problem

$$\left(\frac{d}{2}\right)^2 = l^2 - (l - \Delta h)^2 \qquad \Rightarrow \qquad d = 2\sqrt{2l\Delta h - (\Delta h)^2}\,.$$

Since the tensile force compensates the force resulting from F_g and F_e, it must have the opposite direction, so the angle α can be expressed as

$$\tan\alpha = \frac{F_e}{F_g}\,.$$

Other expressions for this angle follow from simple geometry:

$$\cos\alpha = \frac{l - \Delta h}{l}\,, \quad \sin\alpha = \frac{d}{2l}\,.$$

So,

$$\tan\alpha = \frac{d}{2(l - \Delta h)}\,.$$

Using the formulas above, we can write

$$\frac{d}{2(l - \Delta h)} = \frac{F_e}{F_g}\,,$$

$$\frac{d}{2(l - \Delta h)} = \frac{Q^2}{4\pi\varepsilon m g d^2}\,.$$

Finally, the mass m is

$$m = \frac{Q^2(l - \Delta h)}{2\pi\varepsilon g d^3}\,,$$

$$m = \frac{Q^2(l - \Delta h)}{16\pi\varepsilon g\left(2l\Delta h - (\Delta h)^2\right)^{\frac{3}{2}}} \doteq 120\,\mathrm{g}\,.$$

The mass of each small metal ball is approximately $120\,\mathrm{g}$.

Daniela Pittnerová
`daniela@fykos.cz`

Problem BD ... the new dimension

Use dimensional analysis to determine the formula for dynamic viscosity η of a material using only the Boltzmann constant k_B, absolute temperature T, collision radius r and molecular mass m; let's denote the dimensionless constant of the formula by C.
Hint: *The SI unit of dynamic viscosity is* Pa·s.

Tomáš beheld the beauty of nature without constants.

We are looking for a dynamic viscosity formula in the form $\eta = C k_\mathrm{B}^\alpha T^\beta m^\gamma r^\delta$. Writing down the SI base units of individual physical quantities, we get the equation

$$\frac{\mathrm{kg}}{\mathrm{m{\cdot}s}} = C\left(\frac{\mathrm{kg{\cdot}m^2}}{\mathrm{s^2{\cdot}K}}\right)^\alpha \mathrm{K}^\beta \mathrm{kg}^\gamma \mathrm{m}^\delta\,.$$

By comparing powers of respective units on the left-hand and right-hand side, we obtain a system of four linear equations

$$
\begin{aligned}
1 &= \alpha + \gamma \,, \\
0 &= -\alpha + \beta \,, \\
-1 &= 2\alpha + \delta \,, \\
-1 &= -2\alpha \,.
\end{aligned}
$$

This system can be easily solved to obtain the desired formula

$$\eta = C \frac{\sqrt{k_\mathrm{B} T m}}{r^2} \,.$$

As you can see, we have obtained a valid formula for dynamic viscosity without any complex derivations. This method will never determine the value of the constant C, but that can be measured experimentally.

Tomáš Hrbek
`tomash@fykos.cz`

Problem BE ... repulsive cube

Consider a solid homogeneous metal cube with side of length a. The electrical resistance between two opposite faces of the cube is R. What will be the resistance of a cube from the same material in the same configuration, but with sides of length b? The cube is connected to the circuit using two perfectly conducting plates in perfect contact with the two opposite faces of the cube.

Karel was casting dice.

We are going to solve this problem with just a little bit of insight. On one hand, the resistance depends linearly on the length of the conducting body, therefore it would be multiplied by b/a if we only extended the cube in one direction; on the other hand, it is inversely proportional to the cross section of the body, so extending the body in the two remaining dimensions will multiply the resistance by $(a/b)^2$.

Thus, the resistance of a cube with side length b is given by

$$R' = \frac{a}{b} R \,,$$

where R is the resistance of a cube with side length a. Expansion of a cube leads to a decrease of its resistance.

Karel Kolář
`karel@fykos.cz`

Problem BF ... bubble bath for Janap

Janap wanted to dive into a bubble bath with temperature at least $T_2 = 35\,^\circ\mathrm{C}$. However, the maximum temperature of tap water was only $T_1 = 18\,^\circ\mathrm{C}$. Janap thought that it would be interesting to heat some water by dropping it from an airplane at certain altitude h and converting its kinetic energy to heat. Assume that the water is at rest when dropped from the airplane and that all the kinetic energy acquired during the fall is used to heat up the water. What should *the altitude of the airplane be to heat the water just right? Assume that the gravitational acceleration is $g = 9.81\,\mathrm{kg\cdot m^{-2}}$. You can also make use of some of these constants: specific heat of water $c = 4{,}200\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$, and gravitational constant $G = 6.67\cdot 10^{-11}\,\mathrm{N\cdot kg^{-2}\cdot m^2}$, Earth's radius $R_\mathrm{Z} = 6378\,\mathrm{km}$, Earth's mass $M_\mathrm{Z} = 5.97\cdot 10^{24}\,\mathrm{kg}$. Volume and mass of the bath and Janap will not be disclosed.*

Karel remembered Janap.

The potential energy of a mass m at altitude h is

$$E \approx mgh\,.$$

The heat required to warm up the water to the desired temperature is

$$Q = mc\Delta T\,,$$

where $\Delta T = T_2 - T_1$. This heat should be equal to the difference in potential energy,

$$mgh = mc\Delta T\,.$$

From this equation, we obtain the altitude of the plane

$$h = \frac{c\Delta T}{g} \doteq 7{,}300\,\mathrm{m}\,.$$

At altitude h, the gravitational acceleration is $g' = \frac{GM_\mathrm{Z}}{(R_\mathrm{Z}+h)^2} = 9.77\,\mathrm{kg\cdot m^{-2}} \approx g$, meaning our simplified model is accurate enough. The water has to be released at altitude $h = 7{,}300\,\mathrm{m}$.

Karel Kolář
`karel@fykos.cz`

Problem BG ... mercury column

A thin cylindrical tube with length $l = 1.00\,\mathrm{m}$ is half-way submerged (vertically) in a basin with mercury of density $\varrho = 13.6\cdot 10^3\,\mathrm{kg\cdot m^{-3}}$. We close the top end of the tube and lift it upwards. Part of the mercury spills out of the tube. What will be the height of the column of mercury remaining in the tube? The atmospheric pressure is $p_\mathrm{a} = 101\,\mathrm{kPa}$.
Hint: *Consider an isothermal expansion.*

Vašek couldn't make up an origin for this problem.

The mercury column in the tube is affected by the gravitational force F_g, force F_a induced by atmospheric pressure and by the force F_i induced by air pressure in the tube. At static equilibrium, these forces satisfy the equation

$$F_\mathrm{g} + F_\mathrm{i} = F_\mathrm{a}\,.$$

Force induced by pressure is given by the product of pressure and area, thus we have

$$mg + p_\mathrm{i}S = p_\mathrm{a}S\,,$$

where m is the mass of mercury in the tube, g is the gravitational acceleration, p_i is the pressure of the gas in the tube and S is the inner cross-sectional area of the tube. The mass can be expressed as $m = Sx\varrho$, where x is the equilibrium height of the mercury column. Simple modifications of the force balance equation result in

$$p_\mathrm{a} - p_\mathrm{i} = \varrho x g\,. \tag{1}$$

When the tube is lifted, the gas trapped inside expands isothermally. By Boyle's law, it holds that

$$p_\mathrm{a} S \frac{l}{2} = p_\mathrm{i} S\left(l - x\right)\,.$$

Hence the pressure is

$$p_\mathrm{i} = p_\mathrm{a} \frac{l}{2\left(l - x\right)}\,.$$

This expression can be substituted into equation (1) to obtain a quadratic equation in x

$$2\varrho g x^2 - 2\left(p_\mathrm{a} + l\varrho g\right)x + p_\mathrm{a} l = 0\,.$$

The physically meaningful solution to this equation is

$$x = \frac{1}{2}\left(l + \frac{p_\mathrm{a}}{\varrho g} - \sqrt{\left(\frac{p_\mathrm{a}}{\varrho g}\right)^2 + l^2}\right)\,.$$

For the given values, the height of the mercury column is $x = 0.25\,\mathrm{m}$.

Václav Mikeska
`v.mikeska@fykos.cz`

Problem BH ... sunken

We place an empty cylindrical thin-walled container into a water channel. The height of the container is $h = 15\,\mathrm{cm}$, the radius of its base is $r = 5.0\,\mathrm{cm}$ and its mass is $m = 0.5\,\mathrm{kg}$. The stream in the channel carries the container with constant velocity $v = 2.0\,\mathrm{m{\cdot}s^{-1}}$. There is a hole at the bottom of the container, so $V_0 = 25\,\mathrm{ml}$ of water flows inside it each second. Calculate the length in metres travelled by the container before it sinks under the water surface.

Danka was contemplating the amount of waste in the rivers.

The volumetric flow into the container is $Q_v = V_0\,\mathrm{s^{-1}}$. The container sinks when the buoyant force reaches its maximum value. Up to that moment, buoyancy and weight cancel out, so

$$(m + \varrho Q_v t)g = \pi r^2 h \varrho g$$

where ϱ is the density of water and t is time (measured from the point when the container started leaking water). We can express

$$t = \frac{\pi r^2 h\varrho - m}{Q_v \varrho}\,.$$

Then, the distance travelled by the container is

$$s = vt \doteq 54.25\,\mathrm{m}\,.$$

The container will be carried 54 m away from its starting point before it sinks.

Daniela Pittnerová
`daniela@fykos.cz`

Problem CA ... walk on a boat à la Dan

Daniel, just like a proper rich man, owns a yacht. The yacht isn't the most luxurious and largest, but it's still pretty large. Its displacement is $M = 40\,\mathrm{tons}$ (of water) and it's $D = 20\,\mathrm{m}$ long. Let's imagine that the yacht is in a tank with calm water and at rest with respect to the edge of the water tank; Daniel is standing at rest at its front. By how much does the yacht move with respect to the water tank if Daniel slowly walks from its front to back? Daniel didn't want to tell us his exact weight but assume it's $m = 70\,\mathrm{kg}$. Compute the result to at least two decimal places. *Karel was looking at problems on Brilliant.*

The water displacement of the yacht corresponds to its weight. So the mass of the yacht is M. Because the system was initially at rest, the overall momentum must also be zero during the movement and at the end of the movement. Dan walks from the nose to the stern so the ship will move in the opposite direction. If we assume that Daniel is moving with a velocity v, then the conservation law of momentum ($0 = mv + MV$) means that the ship will move at velocity $V = -mv/M$. I. e. the direction is opposite and the ship's velocity is inversely proportional to the mass of the ship.

We just have to realize that the resulting shift is independent of Daniel's velocity. Daniel walks the same distance in any case. If we think about it, the position of the centre of mass of the ship-Daniel system stays constant and Daniel will move with respect to the tank by $d_1 = MD/(M+m)$. Therefore, the ship moves $d_2 = mD/(M+m) \approx mD/M = 0.035\,\mathrm{m} = 3.5\,\mathrm{cm}$ w.r.t. the tank.

Karel Kolář
`karel@fykos.cz`

Problem CB ... perpetuum mobile of the second kind

Let's assume that the 2nd law of thermodynamics doesn't hold (so heat can flow from a colder body to a warmer one). Consider a boat that, to run, draws water with temperature $T_1 = 3.2\,^\circ\mathrm{C}$ from the sea, cools it down and drops out an ice cube with temperature $T_2 = -5.0\,^\circ\mathrm{C}$ each minute. How long should an edge of the cube be if the power consumption of the ship's motor is $P = 0.8\,\mathrm{MW}$? *Viktor was bored at a thermodynamics lecture.*

Let's denote the edge of the cube as a, specific heat capacity of water as $c_\mathrm{v} = 4{,}200\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$, specific heat capacity of ice as $\varrho_\mathrm{l} = 917\,\mathrm{kg\cdot m^{-3}}$ and specific heat of melting as $l = 334\,\mathrm{kJ\cdot kg^{-1}}$. During each time interval $t = 1\,\mathrm{min} = 60\,\mathrm{s}$, the ship gains energy

$$E = mc_\mathrm{v}(T_1 - T_\mathrm{t}) + ml + mc_\mathrm{l}(T_\mathrm{t} - T_2)$$

from the water of mass $m = \varrho_\mathrm{l} a^3$. This is done by cooling the water to $T_\mathrm{t} = 0^\circ\mathrm{C}$, freezing ice and cooling ice to $T_2 = -5.0\,^\circ\mathrm{C}$.

The energy acquired over time must be equal to the power consumption required, i.e.

$$P = \frac{E}{t} = \frac{\varrho_\mathrm{l} a^3}{t}\left(c_\mathrm{v}(T_1 - T_\mathrm{t}) + l + c_\mathrm{l}(T_\mathrm{t} - T_2)\right)\,,$$

$$a = \sqrt[3]{\frac{Pt}{\varrho_\mathrm{l}}\left(c_\mathrm{v}(T_1 - T_\mathrm{t}) + l + c_\mathrm{l}(T_\mathrm{t} - T_2)\right)^{-1}}\,.$$

By assigning the given values we get $a \doteq 53\,\mathrm{cm}$.

Jakub Šafin
xellos@fykos.cz

Problem CC ... asteroids

There is an asteroid orbiting a star. Far away, there is another asteroid orbiting another star. The orbital radius and orbital period of the second asteroid are three times larger. Determine the ratio of stellar masses (the mass of the first star to the mass of the second star).

Matěj likes ratios.

We start from balance of centrifugal and gravitational forces

$$\frac{GM}{R^2} = m\omega^2 R\,,$$

where M is the mass of the star, m is the mass of the planet, R is the orbital radius, G is the gravitational constant and $\omega = \frac{2\pi}{T}$ is the angular velocity of the orbiting planet. We are able to express the mass of the star

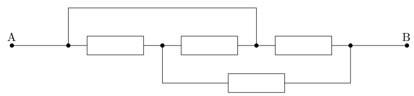
$$M = \frac{4\pi^2 R^3}{GT^2}\,.$$

Although we have derived this relationship "only" for circular orbits, Kepler's third law tells us that it holds even for elliptical orbits. We now know that the mass of each star is directly proportional to the cube of orbital radius and inversely proportional to the square of orbital period. When the radius and period both triple, the mass must triple too, so the ratio of masses is 1:3.

Matěj Mezera
m.mezera@fykos.cz

Problem CD ... a complex network

What will be the total resistance between nodes A and B in the circuit depicted in the figure if each resistor has the same resistance R?



Karel was trying out how good he is at solving physics problems.

It might seem that the circuit contains some loop that would require the use of Kirhoff's laws, or alternatively use some triangle-star transformation, but on the contrary; if the network is correctly redrawn, we find that it is a relatively simple network of resistors connected in parallel and in series, whose resistance we can calculate easily. The original network is the same as the one in figure 1. We continue to modify the network according to basic rules.

As we can see from figure 2, the result is $3R/5 = 0.6\,R$.

Karel Kolář
karel@fykos.cz

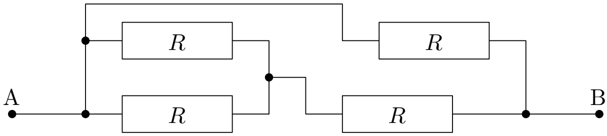


Fig. 1: Modified circuit

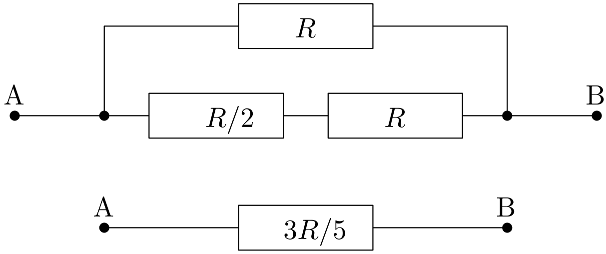


Fig. 2: Modified circuit

Problem CE ... small data

Matěj decided to test a die. He started throwing it and writing down the numbers that came up. After nine throws, he started analysing the gained data. He found out that the minimum number was 1 and the maximum was 6. The median of the set was 4, the (unique) mode was 2 and the arithmetic mean was 3.4 when rounded to one decimal place. What was the geometric mean? *Matěj likes gambling.*

After closer inspection, we find that all 9 values Matěj measured are uniquely determined by the information provided.

We know that each of numbers 1, 4 and 6 occurs at least once. Next, there are at least three 2-s (because 2 is the **unique** mode). Because the median is 4, there cannot be more than four numbers smaller than 4. This means that there are just three 2-s. Now, we don't know three numbers; each of them could be 4, 5 or 6. The value of the arithmetic mean tells us that the sum of all numbers is exactly 31 (30 or 32 would be rounded to 3.3 or 3.6 respectively). The missing numbers could be 4, 4, 6 or 4, 5, 5. We can't have three 4-s because the mode is unique.

The measured values are therefore:

$$1, 2, 2, 2, 4, 4, 5, 5, 6\,.$$

Now, we just calculate the value of the geometric mean $\sqrt[9]{1 \cdot 2 \cdot 2 \cdot 2 \cdot 4 \cdot 4 \cdot 5 \cdot 5 \cdot 6} = 2.99$.

Matěj Mezera
m.mezera@fykos.cz

Problem CF ... hard-working heart

How much work does the human heart do in a lifetime? The pressure amplitude (the difference between systolic and diastolic pressure) is about $\Delta p = 40\,\mathrm{mmHg}$ *(mm of mercury column). Assume that a human lives on average* $T = 82\,\mathrm{years}$ *and the average volumetric flow rate* *through the heart is roughly* $Q = 6\,\mathrm{l{\cdot}min^{-1}}$.

Karel was inspired by Brilliant, where they seemed to get it wrong.

In SI base units, we have $\Delta p = 5.33 \cdot 10^3\,\mathrm{Pa}$, $T = 2.59 \cdot 10^9\,\mathrm{s}$ and $Q = 1 \cdot 10^{-4}\,\mathrm{m^3{\cdot}s^{-1}}$. Let's assume the heart needs to exert a force F over a distance s. The work is given by the formula $W = Fs = S\ \Delta ps\ = V\ \Delta p = Qt\Delta p$, where we consider S to be the effective cross-sectional area of chambers of the heart or the aorta; the total volume of blood it needs to push through is $V = Ss = Qt$. After computing the result for the given values, we get $W \approx 1.4 \cdot 10^9\,\mathrm{J}$.

Karel Kolář
karel@fykos.cz

Problem CG ... I'm just all right

What would be the temperature of the Earth if we considered it a perfect black body which gains energy from solar radiation and doesn't contain any heat source of its own? The power radiated by the Sun is $3.827 \cdot 10^{26}\,\mathrm{W}$, *and its distance from the Earth is* $1.5 \cdot 10^{11}\,\mathrm{m}$. *Assume that Earth is a perfect sphere with constant temperature on its entire surface. Calculate the result in degrees Celsius.*

Štěpán loves classic problems.

The radius of the Earth is r and its distance from the Sun is R. The Earth is always illuminated from one side. The Sun shines just on the (cross-sectional) area πr^2.

We can say that the power of incident solar radiation is $P_S = P\frac{r^2}{4R^2}$, because $r \ll R$.

The power of the Earth's radiation is $P_\mathrm{Z} = \sigma T^4 {\cdot} 4\pi r^2$, where σ denotes the Stefan-Boltzmann constant.

Both of these powers have to be the same because the temperature is constant. From this, we are able to express the thermodynamic temperature T. We can see the radius of the Earth is irrelevant.

$$T = \sqrt[4]{\frac{P}{16\pi R^2\sigma}} = 277.9\,\mathrm{K} = 4.8\,\mathrm{^\circ C}\,.$$

The constant temperature on Earth would be $4.8\,\mathrm{^\circ C}$.

Štěpán Stenchlák
stenchlak@fykos.cz

Problem CH ... lighthouse

There's an isotropic light source with luminous intensity $I_1 = 100\,\mathrm{cd}$ *located in the focal point of a parabolic mirror. The edge of the mirror (a cut through a paraboloid, perpendicular to its axis of symmetry) is circular, contains the focal point in its centre and has radius* $R = 1\,\mathrm{m}$. *At a distance* $L = 1\,\mathrm{km}$ *from the mirror (measured along the axis of the paraboloid), there's a circular target with radius* R. *How many times does the effective brightness of the lighthouse (as measured by the target) increase when the mirror is added?*

Kuba wanted to estimate the importance of the mirror in a lighthouse.

The ratio of fluxes incident on the target surface is equal to the ratio of solid angles formed by rays that reach the target. Without the mirror, we can write (approximately for $R \ll L$)

$$\Omega_1 = \frac{\pi R^2}{L^2} = \frac{\pi R^2}{L^2}\,.$$

The mirror contributes with an additional half-space, i. e. solid angle $\Omega_2 = 2\pi$. For the ratio of incident fluxes, we have

$$\frac{I_2}{I_1} = \frac{\Omega_1 + \Omega_2}{\Omega_1} \approx \frac{\Omega_2}{\Omega_1} = \frac{2L^2}{R^2} \doteq 2.00 \cdot 10^6 \,.$$

The result is a two-million-fold increase in brightness.

Jakub Dolejší
`dolejsi@fykos.cz`

Problem DA ... Bernoulli in real life

Water flows from a tap with a nozzle of radius $R = 0.005\,\mathrm{m}$ at a (volumetric) flow rate $Q_0 = 2\cdot 10^{-5}\,\mathrm{m^3\cdot s^{-1}}$. At what distance below the nozzle does the continuous stream start breaking into droplets? Consider water to be an ideal liquid. Droplets start forming when the stream has radius smaller than $r = 0.003\,\mathrm{m}$. *Matěj almost drowned.*

Thanks to gravity, the stream will accelerate on its way down. The velocity of the stream at the top is

$$v_0 = \frac{Q_0}{\pi R^2} \,.$$

At depth h below the tap, the law of conservation of energy gives

$$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = hg \,,$$
$$v = \sqrt{2gh + v_0^2} \,.$$

We could obtain the same result from the Bernoulli equation, assuming the pressure in the stream is equal to the (constant) atmospheric pressure.

From the continuity equation, we get the depth corresponding to the critical radius

$$\pi r^2 v = Q_0 \,,$$
$$\pi r^2 \sqrt{2gh + \frac{Q_0^2}{\pi^2 R^4}} = Q_0 \,,$$
$$h = \frac{Q_0^2}{2\pi^2 g r^4} - \frac{Q_0^2}{2\pi^2 g R^4} = \frac{Q_0^2}{2\pi^2 g}\left(\frac{1}{r^4} - \frac{1}{R^4}\right) \doteq 0.0222\,\mathrm{m} \,.$$

The stream begins to split apart at about 2 cm below the tap.

Matěj Mezera
`m.mezera@fykos.cz`

Problem DB ... thunderstorm

You're casually running on train tracks in the evening with velocity $v = 15\,\mathrm{km\cdot h^{-1}}$ when you suddenly notice the dark shape of a train on the horizon. The train is moving right towards you with velocity $u = 160\,\mathrm{km\cdot h^{-1}}$. The conductor has already noticed you and sounded the horn, which emits a sound signal with frequency $f = 1{,}000\,\mathrm{Hz}$. What will be the frequency f' you *hear, if you're also running against the wind (the wind velocity is $w = 100\,\mathrm{km\cdot h^{-1}}$)? The speed of sound is $c = 340\,\mathrm{m\cdot s^{-1}}$.* *Tomáš didn't want to jump under a train.*

As a result of the mutual movement of the source and the observer of the wave, the individual waves are densified and the frequency changes. This effect is known as the Doppler phenomenon, and it tells us how frequency depends on the velocity of the observer and the source

$$f' = f\frac{c + v_\mathrm{p}}{c - v_\mathrm{z}}\,,$$

where the original transmitted frequency is f, the velocity of the source is v_z, the velocity of the observer is v_p and the speed of propagation of the waves in the given environment is c. If both observer and source are moving towards each other, their velocities are positive (negative otherwise). In this case, however, we still have to count the wind that blows from the source to the observer at speed w and thus increases the speed of propagation of the waves to $c' = c + w$. The resulting frequency we hear is

$$f' = f\frac{c' + v_\mathrm{p}}{c' - v_\mathrm{z}} = f\frac{c + w + v}{c + w - u}\,.$$

After substituting in the given values, we get the result $f' \doteq 1{,}150\,\mathrm{Hz}$.

Jáchym Bártík
`tuaki@fykos.cz`

Problem DC ... football problem

A football player (European football, soccer in the USA) is standing $d = 15.0\,\mathrm{m}$ in front of a goal with height $h = 2.50\,\mathrm{m}$. He kicks the ball directly towards the target at an angle $\alpha = 30.0^\circ$ with respect to the horizontal plane. How fast should he kick the ball so that it would hit the goal directly without bouncing? (The answer is a range of velocities.) Neglect air resistance and the size of the ball. *Danka is unable to hit the goal.*

We take the motion of the ball as a oblique throw. The centre of the coordinate system is at the point of the kick. We get the motion equations

$$x = vt\cos\alpha\,, \quad y = vt\sin\alpha - \frac{1}{2}gt^2\,.$$

In order to strike the goal, the y component must lie within the range of $0 \le y \le h$ when the x component is equal to d. We get the condition by eliminating time from the equations

$$0 \le d\tan\alpha - \frac{gd^2}{2v^2\cos\alpha^2} \le h\,.$$

The condition for initial velocity is

$$13.04\,\mathrm{m\cdot s^{-1}} \le v \le 15.46\,\mathrm{m\cdot s^{-1}}\,.$$

The football player has to kick the ball out at the speed larger than $13.0\,\mathrm{m\cdot s^{-1}}$ and smaller than $15.5\,\mathrm{m\cdot s^{-1}}$.

Daniela Pittnerová
`daniela@fykos.cz`

Problem DD ... lens and mirror

In a room with height $h = 3\,\mathrm{m}$, we place a mirror on the floor underneath a light source. The light source is exactly at the ceiling level. At the height $5\,\mathrm{cm}$ above the mirror, we place horizontally a thin, positive (converging) lens with focal length $f = 30\,\mathrm{cm}$. How high above the ground will the image of the light source be? *Matěj played with optics.*

Let's denote the distance between the lens and the mirror $d = 5\,\mathrm{cm}$. Firstly, the lens displays the light source to a height h' above the ground (h' can be negative when the image is "under the ground"). Than, this image is reflected by the mirror so it's displayed to a height $-h'$. Finally, the image is displayed again by the lens (but in the other direction) to a height h''.

We can write the thin lens equation for the first image

$$\frac{1}{h-d} + \frac{1}{d-h'} = \frac{1}{f}\,,$$

$$h' = d - \frac{1}{\frac{1}{f} - \frac{1}{h-d}} \doteq -28.4\,\mathrm{cm}\,.$$

Than, the image is reflected by the mirror and we get second thin lens equation

$$\frac{1}{d-(-h')} + \frac{1}{h''-d} = \frac{1}{f}\,,$$

$$h'' = d + \frac{1}{\frac{1}{f} - \frac{1}{d+h'}} \doteq 18.14\,\mathrm{cm}\,.$$

The real image is 18 cm above the ground.

Matěj Mezera
`m.mezera@fykos.cz`

Problem DE ... cleaning glasses

The tap of the kitchen sink has a turnable head, which is currently tilted at the angle $\alpha = 30^\circ$ with respect to the vertical. The head is $h = 30\,\mathrm{cm}$ above the bottom of the sink. The tap is set to form a single continuous stream with the diameter of the opening in the tap $d = 5\,\mathrm{mm}$, the volumetric flow then is $Q = 100\,\mathrm{ml{\cdot}s^{-1}}$. In this setting, the stream from the tap hits the bottom of the sink at horizontal distance l_1 (from the point right below the tap). If we set the tap to "shower" mode, where water exits through ten holes with diameters $d_2 = 1\,\mathrm{mm}$, the streams hit the bottom of the sink at horizontal distance l_2. Determine the ratio l_2/l_1. Assume that the distance between openings in the tap is negligibly small. *Mirek's glass was especially greasy.*

Assuming, with minimal loss of accuracy, that water is incompressible, the volumetric flow rate will remain constant as the area of the openings changes. Outflow velocity in the standard mode is

$$v_1 = \frac{4Q}{\pi d_1^2}\,,$$

after switching to the multiple streams mode, outflow velocity becomes

$$v_2 = \frac{4Q}{10\pi d_2^2}\,.$$

Knowing the initial velocity and height, we can determine the distance at which the stream hits the sink. The motion of the stream is determined

$$x = vt\cos\beta\,, \quad y = vt\sin\beta - \frac{1}{2}gt^2\,,$$

where we defined $\beta = \alpha - \pi/2$. Now setting $y = -h$ we can express x as a function of initial velocity as

$$x(v) = v^2\frac{\cos\beta}{g}\left(\sqrt{\frac{2gh}{v^2} + \sin^2\beta} + \sin\beta\right).$$

Calculating $l_1 = x(v_1) \doteq 16.178\,\mathrm{cm}$, $l_2 = x(v_2) \doteq 17.116\,\mathrm{cm}$ we get a ratio

$$\frac{l_2}{l_1} = 1.058\,.$$

The distance increases by a negligible 6 %. In the limit of decreasing opening sizes, the distance approaches $10\sqrt{3}\,\mathrm{cm}$.

Miroslav Hanzelka
mirek@fykos.cz

Problem DF ... four body problem

Four planets, each with the mass equal to the Moon's mass, orbit their common centre of gravity. The planets move with a stable circular orbit in such a way that they always form the vertices of a square with side length 384,400 km*. Find their orbital period. The Moon has mass* $7.35\cdot10^{22}$ kg. *Three planets aren't enough for Štěpán.*

For the planets to stay in orbit, the centrifugal force F_o and gravitational force F_G must be equal in magnitude.

Distance between the planets and the centre is $\frac{\sqrt{2}}{2}a$, where a is the length of the side of the square. Total centrifugal force at angular velocity $\omega = \frac{2\pi}{T}$, where T is the period, is $F_\mathrm{o} = m\omega^2\cdot\frac{\sqrt{2}}{2}a$.

Gravitational force has to be calculated step by step. The force towards the neighbouring planets at distance a has magnitude $G\frac{m^2}{a^2}$, where G is the gravitational constant. These two forces are equal and perpendicular, so their sum is directed towards the centre and has magnitude $F_{G_1} = \sqrt{2}G\frac{m^2}{a^2}$. The planet opposite, at distance $a\sqrt{2}$, acts with force $F_{G_2} = G\frac{m^2}{2a^2}$ also directed towards the centre. Their sum is $F_G = F_{G_1} + F_{G_2} = G\frac{m^2}{a^2}\cdot\left(\sqrt{2}+\frac{1}{2}\right)$.

Expressing T from the equality $F_\mathrm{o} = F_G$

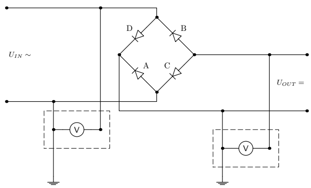
$$T = \sqrt{\frac{8\pi^2a^3}{Gm\left(4+\sqrt{2}\right)}} \doteq 1.30\cdot10^7\,\mathrm{s} \doteq 150.4\,\mathrm{days}\,.$$

The period of one rotation in our system is 150.4 days.

Štěpán Stenchlák
stenchlak@fykos.cz

Problem DG ... Why is it smoking?

Lukáš built a full-wave rectifier and decided to test it (see figure). He took two oscilloscopes (since he knows the negative poles of one oscilloscope are connected...), connected one of them to the AC input and the other to the DC output. Unfortunately, the negative poles of both oscilloscopes were connected to the ground pin in the electrical socket. Which diode started smoking?



Lukáš burned down a germanium diode in the lab and was very surprised, since he used two oscilloscopes.

If we connect grounds of the oscilloscopes together, we see both poles of diode A will always be at the same potential, so we can replace the diode with a conductor. Diode D is connected directly between the terminals of the input voltage, so for one halfway of the input, it shorts the circuit and therefore carries very large current and burns out, so the magic smoke leaves diode D.

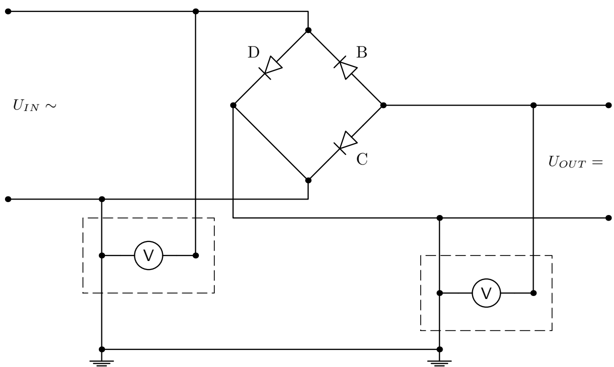


Fig. 3: Connected diodes.

Michal Červeňák
miso@fykos.cz

Problem DH ... solar sniper

Matěj suddenly finds himself on the equator at noon of the equinox, and he's sweating terribly. The sun is making him angry, so he pulls out his house-made weapon, which can shoot lightweight bullets at almost the speed of light, and fires a projectile directly towards the centre of the Sun. However, he forgot about the movement of celestial bodies. Find the minimum *velocity the bullet needs to have so that Matěj could hit the Sun.*

Matěj was thinking about the equator.

Matěj would hit the centre of the Sun if he wasn't in motion relative to it, but he did not consider that the Earth is in orbit around the Sun and rotates.

The velocity of rotation of the earth at the equator can be calculated easily, as we know the radius of the Earth $R_\mathrm{Z} = 6380\,\mathrm{km}$ and the period of rotation $T_\mathrm{Z} = 24\,\mathrm{h}$, being careful to convert the units, we get

$$v_\mathrm{Z} = \frac{2\pi R_\mathrm{Z}}{T_\mathrm{Z}} = 464\,\mathrm{m{\cdot}s^{-1}}\,.$$

The orbital velocity v_S can be calculated similarly, we know the distance between the Earth and the Sun $R_\mathrm{S} = 1\,\mathrm{AU} = 150 \cdot 10^9\,\mathrm{m}$, the orbital period is well known $T_\mathrm{S} = 365.25\,\mathrm{days} = 3.16 \cdot 10^7\,\mathrm{s}$

$$v_\mathrm{S} = \frac{2\pi R_\mathrm{S}}{T_\mathrm{S}} = 29{,}900\,\mathrm{m{\cdot}s^{-1}}\,.$$

We can readily see that the velocity v_Z is negligible, which makes our task much easier as we do not need to consider the direction in which the Earth rotates and the angle between the axis of rotation and the orbital plane.

Denoting the velocity of the projectile v, it will take it approximately $t = \frac{R_\mathrm{S}}{v}$ to reach the Sun. The lateral distance the projectile travels in this time must be less than the radius of the Sun $r = 6.96 \cdot 10^8\,\mathrm{m}$ or it will miss the Sun

$$\begin{aligned} v_\mathrm{S} t &\leq r\,,\\ \frac{2\pi R_\mathrm{Z}}{T_\mathrm{Z}} \frac{R_\mathrm{S}}{v} &\leq r\,,\\ v &\geq \frac{2\pi R_\mathrm{Z} R_\mathrm{S}}{T_\mathrm{Z} r} = \frac{R_\mathrm{S}}{r} v_\mathrm{S} = 6.4 \cdot 10^6\,\mathrm{m{\cdot}s^{-1}}\,. \end{aligned}$$

We get the minimal velocity as approximately one fiftieth of the speed of light.

To get a more accurate result, we need to subtract the velocity due to rotation of the Earth from the orbital velocity, taking into account the tilt of axis of rotation relative to the axis of the orbital plane (approximately $\varphi = 23.4^\circ$). Subtracting the velocity vectors we get the velocity relative to the Sun

$$v' = \sqrt{(v_\mathrm{S} - v_\mathrm{Z} \cos\varphi)^2 + (v_\mathrm{Z} \sin\varphi)^2}\,.$$

Plugging in the numbers we get

$$\begin{aligned} v' &= 29{,}440\,\mathrm{m{\cdot}s^{-1}}\,,\\ v &\geq \frac{R_\mathrm{S}}{r} v' = 6.34 \cdot 10^6\,\mathrm{m{\cdot}s^{-1}}\,, \end{aligned}$$

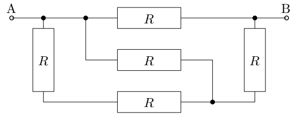
which still is not completely accurate as the orbit of the Earth is elliptical and we do not know whether the problem asks about the spring or autumn equinox, so we cannot solve the problem exactly (but these corrections would be rather small).

Matěj Mezera
`m.mezera@fykos.cz`

Problem EA ... those resistors again

Find the total electrical resistance between nodes A and B of the circuit shown in the figure. Each resistor has resistance R.

Karel was playing with IPE.



Let's mark the vertices in the circuit as C and D (see picture 4).

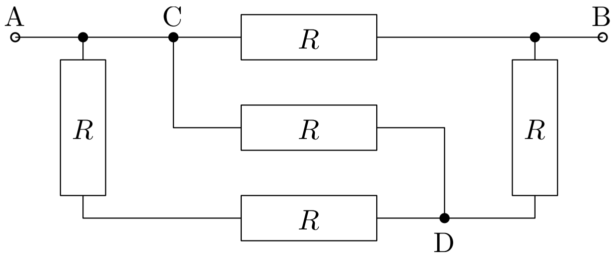


Fig. 4: Sketch of the circuit with the vertices labelled.

We can see that the configuration is quite unsuitable to be solved through adding together of parallel and serial resistors. We could use Kirchhoff's laws but we might get accidentally lost in the equations despite the circuit being relatively simple. One suitable and quite elegant simplification that we can use is the transformation from a triangle to a star. [2]. We modify the triangle BCD in this way. Instead of resistors with resistance R in the triangle, the star will have resistors with resistance $R/3$. You can see the modified circuit in figure 5

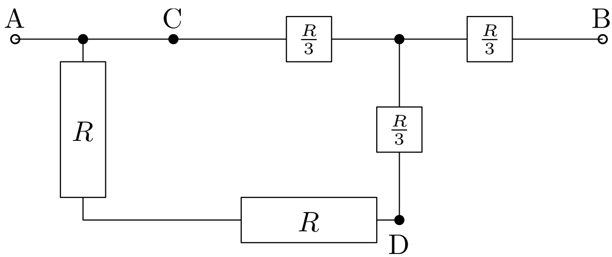


Fig. 5: Modified circuit

Now we can easily add resistors together and get the result

$$R_{\mathrm{tot}} = \frac{\frac{R}{3}\frac{7R}{3}}{\frac{R}{3}+\frac{7R}{3}} + \frac{R}{3} = \frac{5}{8}R = 0.625\,R\,.$$

[2] Derivation and examples of this transformation can be found in many textbooks covering electrical circuits, or on the internet.

The total resistance of the circuit is $0.625\,R$.

Karel Kolář
karel@fykos.cz

Problem EB ... typical problem of a commuter

Matěj is walking to a bus stop along a straight road with velocity $v = 4\,\mathrm{km{\cdot}h^{-1}}$. He sees a bus arriving from the opposite direction to the bus stop with velocity $u = 36\,\mathrm{km{\cdot}h^{-1}}$, but doesn't know if it's bus number 201, which he needs to catch. At the moment when he can read number 201 on the bus, he immediately starts running with velocity $3v$. Will he make it? The resolution of the human eye is one arcminute and the display of the bus is readable if one can distinguish two pixels with distance $2\,\mathrm{cm}$. The deceleration of the bus is constant and it waits at the bus stop for $10\,\mathrm{s}$. Compute the time margin (how much time he can lose and still catch the bus; if he can't make it, this time should be negative). Initially, Matěj and the bus are both $50\,\mathrm{m}$ from the bus stop. *Matěj barely made it.*

Firstly, let's calculate what is the distance Matej can read the number of the bus from. We mark this distance as x. Because 1 arcminute is a very small angle (α), the resolution is very small compared to the distance from the bus and we can write

$$\alpha \approx \frac{y}{x}\,.$$

From this we obtain

$$x \approx \frac{y}{\alpha} \doteq 68.75\,\mathrm{m}\,.$$

Now let's calculate the time at which Matej reads the sign. For this we need to know the constant deceleration of the bus. The initial distance of the bus stop is $s = 50\,\mathrm{m}$. From the basic relations for constant acceleration we get

$$s = \frac{at_1^2}{2} = \frac{ut_1}{2}\,, \quad t_1 = \frac{2s}{u} = 10\,\mathrm{s}\,, \quad a = \frac{u^2}{2s} = 1\,\mathrm{m{\cdot}s^{-2}}\,.$$

As a part of the above calculation we also reached the time the bus needed to stop. The initial distance between Matej and the bus is $2s$. We are looking for a time it takes this distance to become x, i. e. decreases by $2s - x$. Using the same relations

$$vt_\mathrm{x} + ut_\mathrm{x} - \frac{1}{2}at_\mathrm{x}^2 = 2s - x\,, \quad \frac{1}{2}at_\mathrm{x}^2 - (u+v)\,t_\mathrm{x} + (2s - x) = 0\,,$$

$$t_\mathrm{x} = \frac{u + v - \sqrt{(u+v)^2 - 2a\,(2s - x)}}{a} \doteq 3.305\,\mathrm{s}\,.$$

In this time Matej walked $s_1 = vt_\mathrm{x} \doteq 3.67\,\mathrm{m}$.[3] He will run he remaining $s_2 = s - s_1 \doteq 46.33\,\mathrm{m}$ in $t_2 = \frac{s_2}{3v} \doteq 13.90\,\mathrm{s}$. If we add to that the time he walked, we have $t_2 + t_\mathrm{x} \doteq 17.20\,\mathrm{s}$. If we know from before that it took the bus $10\,\mathrm{s}$ to stop, and then another $t_0 = 10\,\mathrm{s}$ to wait on the bus stop, we can deduce Matej's reserve is comfortable $2.8\,\mathrm{s}$ so he manages to board the bus without any issues.

[3]When solving the quadratic equation, we chose the solution with minus sign, because we care about the shortest time, at which bus and Matej get into that distance. The solution with the plus sign would correspond to the situation when the bus slows all the way down to zero and starts accelerating backwards.

In this type of the question, we are forced to make many intermediate steps, and so, for the sake of time, it's useful to look for specific numerical values and not for general formulas, that could be quite lengthy. On the other hand we need to record each value with enough of a precision (3 or 4 significant digits), to stop the inaccuracies from accumulating and propagating to the result. General solution would look like this:

$$
\begin{aligned}
t = t_0 + t_1 - t_2 - t_\mathrm{x} &= t_0 + \frac{2s}{u} - \frac{s - vt_\mathrm{x}}{3v} - t_\mathrm{x} = \\
&= t_0 + \frac{2s}{u} - \frac{s}{3v} - \left(1 - \frac{v}{3v}\right) \frac{u + v - \sqrt{(u+v)^2 - 2a(2s - x)}}{a} = \\
&= t_0 + \frac{2s}{u} - \frac{s}{3v} - \frac{4s}{3} \frac{u + v - \sqrt{(u+v)^2 - \frac{u^2}{s}(2s - \frac{y}{\alpha})}}{u^2}\,,
\end{aligned}
$$

what definitely isn't a beautiful formula.

Matěj Mezera
m.mezera@fykos.cz

Problem EC ... merry-go-flat

Matěj (with mass m) discovered a new equipment at a playground. It is a massive flat merry-go-round in the shape of a homogeneous disc with radius R and mass $2m$, which can rotate around its vertical axis without friction. Matěj stands on its edge and speeds up to velocity v (the velocity of the edge and Matěj with respect to the ground; Matěj is at rest with respect to the merry-go-round). Find the amount of work done by Matěj to move to the centre of the disc. Matěj can be approximated by a point mass. *Matěj likes to turn around.*

Matěj does work against the centrifugal force when he moves towards the centre. The entire merry-go-round is thus accelerated. Using the conservation of angular momentum, we can determine the work done by the difference in energies before and after Matěj moved. The initial kinetic energy is

$$E_0 = \frac{1}{2}mv^2 + \frac{1}{2}J\omega^2\,,$$

where $\omega = \frac{v}{R}$ is the initial angular velocity of the merry-go-round and $J = \frac{1}{2}(2m)R^2$ is the moment of inertia of a disc.The initial angular momentum is the sum of the angular momenta of the disc and of Matěj

$$L = J\omega + mRv\,.$$

When Matěj reaches the centre, his moment of inertia will be zero (his distance to the axis will be zero), but the total angular momentum is conserved, so the merry-go-round will accelerate to $\omega' > \omega$

$$L = J\omega'\,.$$

From the equality of angular momenta ω'

$$\omega' = \omega + \frac{mRv}{J}$$

and calculating the final kinetic energy

$$E_1 = \frac{1}{2}J\omega'^2 = \frac{1}{2}J\omega^2 + \omega mRv + \frac{1}{2}\frac{m^2R^2v^2}{J}\,.$$

The difference in energies before and after the move is equal to the work done

$$W = E_1 - E_0 = mv^2 + \frac{1}{2}\frac{m^2R^2v^2}{J} - \frac{1}{2}mv^2 = \frac{1}{2}mv^2 + \frac{m^2v^2}{2m}\,.$$

The work done is therefore $W = mv^2$.

Matěj Mezera
m.mezera@fykos.cz

Problem ED ... finally a pleasant temperature

A planet orbits around a star with orbital velocity $v = 1.0\,\mathrm{km{\cdot}s^{-1}}$ on a circular trajectory. The mass of the star is $M_{.} = 5.0\cdot10^{30}\,\mathrm{kg}$ and its radius is $R_{.} = 1.0\cdot10^{7}\,\mathrm{km}$. The temperature of the planet is $T = 100\,\mathrm{K}$. Suppose that the planet and the star are both perfect black bodies. What is the temperature of the star in Kelvins? *Danka was cold.*

We can find distance r between the planet and the star from the equilibrium of the gravitational and the centrifugal force .

$$r = \frac{GM_{.}}{v^2}\,.$$

Black body radiates energy

$$I = \sigma T^4\,.$$

so for the thermal equilibrium of the planet, energy received from the star is constantly being irradiated:

$$\sigma T_{.}^4 4\pi R_{.}^2 \frac{\pi R_p^2}{4\pi r^2} = \sigma T_p^4 4\pi R_p^2$$

For the temperature of star we get

$$T_H = T_p\sqrt{\frac{2r}{R_H}} = T_p\sqrt{\frac{2GM_H}{R_H v^2}} \doteq 25{,}800\,\mathrm{K}\,.$$

The temperature of the star is 25,800 K.

Daniela Pittnerová
daniela@fykos.cz

Problem EE ... repulsive

There are 1,000 point masses in an otherwise empty plane. For each $k = 0, \dots, 999$, the coordinates of the kth point are $x_k = (k/1{,}000)^5\,\mathrm{km}$ and $y_k = (k/1{,}000)^3\,\mathrm{m}$. Each point has mass $M = 300\,\mathrm{kg}$. Find the value of the positive *electric charge of every particle (each point should have this charge) required for the system to be in equilibrium. The point masses are initially at rest.* *Lukáš was watching (self)restraining of naked singularities.*

The difficult looking task has a simple solution. Between each pair of particles k and l acts the sum of the gravitational and electrostatic force

$$F_{kl} = -\frac{GM^2}{r_{kl}^2} + \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{r_{kl}^2}\,.$$

Balance means that the sum of forces acting to each particles is zero, that is, $F_{kl} = 0$ for all k, l. Because both forces are decreasing with distance in the same way we meet the required condition by equality of the coefficients of the forces

$$GM^2 = \frac{1}{4\pi\varepsilon_0}Q^2\,,$$
$$Q = M\sqrt{4\pi\varepsilon_0 G} \doteq 2.6\cdot 10^{-8}\,\mathrm{C}\,.$$

Each of the points masses must be charged by the electric charge $Q \doteq 2.6\cdot 10^{-8}$ C.

Lukáš Timko
lukast@fykos.cz

Problem EF ... hang up that pendulum!

Consider a thin, rigid, homogeneous rod. At a certain distance along the length of the rod, we drill in a small hole in such a way that the period of small oscillations of the rod around a perpendicular axis passing through that hole is the minimum possible. The hole separates the rod into two segments. What is the ratio of lengths of these segments?

Matěj would like to hang Jáchym.

We use the formula for the period of a physical pendulum

$$T = 2\pi\sqrt{\frac{J + ml^2}{mgl}}\,,$$

where m is the mass of the pendulum, l is the distance between the center of gravity and the point of suspension and J is moment of inertia to the axis passing through the center of gravity. For homogeneous rod with length L, we have $J = \frac{1}{12}mL^2$. The moment of inertia and the mass is constant. We are looking for l such that the expression inside the square root is minimal.

$$\frac{J + ml^2}{ml} = \frac{\frac{1}{12}L^2 + l^2}{l}$$

We use a first derivative

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{L^2}{12l} + l\right) = -\frac{L^2}{12l^2} + 1\,.$$

By letting this equal to zero we get

$$l = \frac{L}{\sqrt{12}}\,.$$

The required ratio is

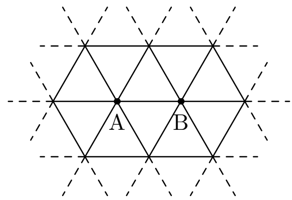
$$\frac{\frac{1}{2} + \frac{1}{\sqrt{12}}}{\frac{1}{2} - \frac{1}{\sqrt{12}}} = \frac{\sqrt{3}+3}{3-\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3} \doteq 3.732\,.$$

Matěj Mezera
m.mezera@fykos.cz

Problem EG ... triangular network

Consider an infinite network of resistive wire depicted in the figure. What will be the resistance between adjacent points A and B if the wire with length |AB| has resistance R_1?

Karel would like to have an infinite network at home.



The problem is best solved by a trick, just as most of those which have an infinite amount of resistors and some type of symmetry.

We use the superposition principle, where a combination of solutions of partial problems is a solution to the whole problem. So although we can't find the current between the two points directly, we can find the current from the first point to infinity and from infinity to the other point. First let us consider there is an electric current I from outside the grid entering at A and exiting at B.

So in the first case, we have current I going to point A, from the source. Because of the 6-fold symmetry, we know that the current through each of the six wires meeting at A must be $I/6$. The same result is when we calcuate the current from infinity to B, where the directions of the currents are pointing inwards this time. The solution for the current from point A to B es equal to the one sending the curren from A to infinity and then from infinity to B. So we see that the total current going through A–B is the sum of the two currents, which is equal to $I/3$.

We don't really care about other wires than the connecting one, so we have no idea about the value of the electric current there, but it's not important for our solution.

To find the resistance between, we use $U_{\mathrm{AB}} = R_1 I/3 = RI$, which solved for R gives $R = R_1/3$.

Karel Kolář
karel@fykos.cz

Problem EH ... zero to hundred for the third time

One of the performance characteristics of a car is the time needed to accelerate from $0\,\mathrm{km\cdot h^{-1}}$ *to* $100\,\mathrm{km\cdot h^{-1}}$. *Consider a car that can do it in* $4.0\,\mathrm{s}$. *Find the distance travelled by the car while accelerating under the assumption of constant jerk of the car. Jerk (often denoted by j) is the rate of change (time derivative) of acceleration, just like acceleration is the rate of change of velocity.*

Karel was wondering about acccelerating.

We start with the basic relationship between time-dependent acceleration and time elapsed from the start

$$a = jt\,.$$

We can derive the formula for time-dependent velocity by integrating previous formula

$$v = \int_0^t jt' \,\mathrm{d}t' = \frac{1}{2} jt^2\,.$$

Now, we are able to calculate the jerk j

$$j = \frac{2v}{t^2} = 3.47\,\mathrm{m\cdot s^{-3}}\,.$$

We need to integrate once again in order to calculate the travelled distance.

$$s = \int_0^t \frac{1}{2} j t'^2 \mathrm{d}t' = \frac{1}{6} j t^3 = \frac{1}{3} v t = 37.0\,\mathrm{m}\,.$$

By the way, this distance is $\frac{1}{3}$ shorter than the distance in the case of constant acceleration.

Matěj Mezera
m.mezera@fykos.cz

Problem FA ... stretched

Consider a rubber band (not a loop, just a single strip) with density ϱ made of a material with Young modulus E. Its rest length (for example when lying freely on a table) is L. We suspend the rubber band from one of its ends. What length will it have now?

Matěj was playing with a rubber band.

Each small part of the rubber band of a length of $\mathrm{d}l$ stretches to a new length $\mathrm{d}x$ given by

$$\mathrm{d}x = \mathrm{d}l \left(1 + \frac{\sigma}{E}\right)\,,$$

where σ is the tension at that point. Let's suppose the cross section of the rubber band is S. The tension is then given by

$$\sigma = \frac{F}{S}\,,$$

where F is the force pulling the rubber band downwards. $F = S l \varrho g$, where l is the distance of the point from the end of the rubber band in the **unstretched** state (otherwise the density wouldn't be constant). Therefore

$$\mathrm{d}x = \left(1 + \frac{l \varrho g}{E}\right) \mathrm{d}l\,.$$

$$L_1 = \int_0^{L_1} \mathrm{d}x = \int_0^{L} \left(1 + \frac{l \varrho g}{E}\right) \mathrm{d}l = L + \frac{L^2 \varrho g}{2E}\,.$$

The solution can be also found noticing that the extension depends linearly on the length of the length under the point, which implies the extension will be the same as if we put a point mass on the end of the rubber band of half of of weight of the rubber band and ignore the weight of the band itself.

Matěj Mezera
m.mezera@fykos.cz

Problem FB ... procrastination

Matěj likes to spend time watching videos on YouTube instead of working. YouTube uses a sophisticated algorithm to suggest videos that you might be interested in after you finish watching the current one. This algorithm is so good that with probability $p = 80\%$, Matěj likes *one of the suggested videos and starts watching it. The average length of one video is* $t = 7\,\mathrm{min}$*. How long will it take on average for Matěj to stop watching videos and start working on this problem?*

The following formulas can be handy: $\sum_{i=1}^{\infty} ix^i = \frac{x}{(x-1)^2}$ *for* $|x| < 1$, $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$.

Mezera: I'll write something here later.

Suggestion are independent, so the probability of watching nth video is

$$p_\mathrm{n} = p^{n-1}\,.$$

Probability, that he stops after nth video is

$$P_\mathrm{n} = p' p_\mathrm{n} = (1-p)\, p^{n-1}\,.$$

and it takes nt of time. For average time, we weight each of the possible watching times with their probability:

$$t_\mathrm{p} = \sum_{\mathrm{n}=1}^{\infty} (1-p)\, p^{n-1} nt = \frac{1-p}{p} t \sum_{\mathrm{n}=1}^{\infty} np^n = \\ = \frac{1-p}{p} t \frac{p}{(1-p)^2} = \frac{t}{1-p} = 35\,\mathrm{min}\,.$$

Matěj stops procrastinating after $35\,\mathrm{min}$ on average.

Matěj Mezera
`m.mezera@fykos.cz`

Problem FC ... a lot of computations

Matěj bought a digital clock which displays time in the 24-hour format. Each digit is formed by up to seven lit diodes (28 diodes in total). Each diode consumes $0.1\,\mathrm{mW}$ *of power. Matěj placed four new AAA batteries in the clock, set the time to 12:00 immediately and hung the clock on the wall. Some time later, he looked at the clock and saw the time for a moment, but suddenly, the clock discharged and switched off. Determine the time Matěj saw. The capacity of one AAA battery is* $2.5\,\mathrm{Wh}$*; energy is only used to power the diodes, there are no other losses.* Note: *Digits six and nine are formed by six diodes each, there's no separator on the clock and all four digits are displayed all the time. All the values are perfectly exact.*

Matěj wanted to make a simple problem which requires computing a lot of stuff.

Clocks can work for months or even years with a single set of batteries. We shall now calculate how much energy this clock uses per day and will then be interested only in the remainder after dividing by this daily power consumption. We can assume that that the change of displayed digits (switching diodes on and off) is instantaneous. We will split the display to specific digits and determine their average daily power usage.

The last digits cycles digits 0 to 9, giving mean power of this digit

$$P_1 = \frac{6+2+5+5+4+5+6+3+7+6}{10} 0.1\,\mathrm{mW} = 0.49\,\mathrm{mW}\,.$$

The ten minutes digit cycles between 0 and 5, so its mean power is

$$P_2 = \frac{6+2+5+5+4+5}{6} 0.1\,\mathrm{mW} = 0.45\,\mathrm{mW}\,.$$

The second digit cycles goes through 0 to 9 twice a day and then goes through 0, 1, 2, 3. This yields a mean of

$$P_3 = \frac{2\,(6+2+5+5+4+5+6+3+7+6)+6+2+5+5}{24} 0.1\,\mathrm{W} = \frac{29}{60}\,\mathrm{mW}\,.$$

The first digits shows 0 for for the first 10 hours, then 1 for 10 hours and finally 2 for the remaining 4 hours

$$P_4 = \frac{10\cdot 6 + 10\cdot 2 + 4\cdot 5}{24} 0.1\,\mathrm{mW} = \frac{25}{60}\,\mathrm{mW}\,.$$

The clock's daily power use is therefore

$$E_1 = (P_1 + P_2 + P_3 + P_4)\cdot 24\,\mathrm{h} = 44.16\,\mathrm{mWh}\,.$$

The remainder after dividing the battery capacity by this value is

$$E_\mathrm{z} = \left(2\cdot 5\cdot 10^3 \mod 44{,}16\right)\,\mathrm{mWh} = 19.84\,\mathrm{mWh}\,.$$

It is now 12:00 on the clock and the remaining battery capacity is E_z. That's less than half of the daily power consumption, so we try to subtract the power used in the next 10 hours:

$$\begin{aligned} E_\mathrm{d} &= (P_1 + P_2)\cdot 10\,\mathrm{h} + P_1\cdot 10\,\mathrm{h} + 8\cdot 2\cdot 0.1\,\mathrm{mWh} + 2\cdot 5\cdot 0.1\,\mathrm{mWh} \\ &= 16.90\,\mathrm{mWh}\,. \end{aligned}$$

The clock is now at 22:00 and we only have $19.84\,\mathrm{mWh} - 16.90\,\mathrm{mWh} = 2.94\,\mathrm{mWh}$ in the batteries left. During the twenty-second hour, the clock uses

$$E_{22} = (P_1 + P_2)\cdot 1\,\mathrm{h} + 5\cdot 0.1\,\mathrm{mWh} + 5\cdot 0.1\,\mathrm{mWh} = 1.94\,\mathrm{mWh}\,.$$

Remaining energy is now $2.94\,\mathrm{mWh} - 1.94\,\mathrm{mWh} = 1.00\,\mathrm{mWh} = 60.0\,\mathrm{mWmin}$. We have now converted the units to the unusual milliwatt-minutes and we will numerically count the minutes. It is 23:00, in the next "30 min" the clock uses

$$\begin{aligned} E_{30\mathrm{min}} &= P_1\cdot 30\,\mathrm{min} + (6+2+5)\cdot 0.1\,\mathrm{mW}\cdot 10\,\mathrm{min} \\ &\quad + 5\cdot 0.1\,\mathrm{mW}\cdot 30\,\mathrm{min} + 5\cdot 0.1\,\mathrm{mW}\cdot 30\,\mathrm{min} \\ &= 57.7\,\mathrm{mWmin}\,. \end{aligned}$$

It is now 23:30 with $60.0\,\mathrm{mWmin} - 57.7\,\mathrm{mWmin} = 2.3\,\mathrm{mWmin}$ left. We find out that during the next minute, the battery loses

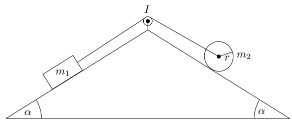
$$\begin{aligned} E_{1\mathrm{min}} &= 6\cdot 0.1\,\mathrm{mWmin} + 5\cdot 0.1\,\mathrm{mWmin} + 5\cdot 0.1\,\mathrm{mWmin} + 5\cdot 0.1\,\mathrm{mWmin} \\ &= 2.1\,\mathrm{mWmin}\,. \end{aligned}$$

leaving us with $0.2\,\mathrm{mWmin}$ and we can easily verify that this will not last for another minute. The clock switches off at 23:31.

This problem does not have a general solution, the only way to solve it is this numerical calculation, but we did not round any intermediate values, so our result is exact. In reality we only know the battery capacity and diode power with a limited precision, which would surely be insufficient to determine the exact time when the clock looses power as we frequently subtracted remaining capacity, which would accumulate the initial uncertainties. We would get a very imprecise result.

Matěj Mezera
m.mezera@fykos.cz

Problem FD ... roof and cylinder



There are two objects placed on a roof, a cuboid and a homogeneous cylinder, connected by a rope. The roof is formed by two inclined planes with inclination angle $\alpha = 30.0^\circ$ with respect to the horizontal plane. The rope passes through a pulley with the moment of inertia $I = 0.100\,\mathrm{kg{\cdot}m^2}$ and radius $r_\mathrm{k} = 0.100\,\mathrm{m}$. The mass of the cuboid is $m_1 = 5.00\,\mathrm{kg}$, the radius of the cylinder's base is $r_\mathrm{v} = 0.30\,\mathrm{m}$ and its mass is $m_2 = 10.0\,\mathrm{kg}$. What will be the acceleration of the cuboid (including its direction)? The coefficient of friction between either object and the roof is $f = 0.50$. Neglect the rolling resistance. *Karel modified another problem.*

First, let's analyse the forces and determine in which direction (if at all), will the system move.

For the bodies to move, the total force acting on the cuboid must exceed the force of frictionThe cuboid is pulled to one side by the parallel component of the force of gravity and pulled the other side by the force due to the cylinder

$$F_2 = m_1 g \sin\alpha - m_2 g \sin\alpha = (m_1 - m_2)\, g \sin\alpha = -24.53\,\mathrm{N}\,.$$

Looking at the numbers we see the cuboid will move upward, as the force of friction is exceeded. The entire system is therefore accelerated by the force

$$F_0 = F_2 - F_\mathrm{t} = -(m_1 - m_2)\, g \sin\alpha - f m_1 g \cos\alpha = 3.29\,\mathrm{N}\,.$$

This force must accelerate the cuboid and the cylinder and at the same time, spin up the cylinder and the pulley. If a force F acts on a body with moment of inertia J at distance r from the axis (perpendicular to the axis and r), from the change of angular momentum we find the expression for angular acceleration $\varepsilon = \frac{a}{r}$

$$F_0 r = J\varepsilon\,, \quad F_0 = \frac{J}{r^2} a\,.$$

Denoting acceleration a and using balance of the forces

$$F_0 = (m_1 + m_2)\, a + \frac{I}{r_\mathrm{k}^2} a + \frac{I_\mathrm{v}}{r_\mathrm{v}^2} a\,,$$

where $I_\mathrm{v} = m_2 r_\mathrm{v}^2/2$ is the moment of inertia of the cylinder.

$$F_0 = \left(m_1 + \frac{3}{2} m_2 + \frac{I}{r_\mathrm{k}^2}\right) a\,,$$

$$a = \frac{(m_2 - m_1)\sin\alpha - f m_1 \cos\alpha}{m_1 + \frac{3}{2} m_2 + \frac{I}{r_\mathrm{k}^2}} g = 0.110\,\mathrm{m{\cdot}s^{-2}}\,.$$

The cuboid will move with acceleration $0.110\,\mathrm{m\cdot s^{-2}}$ upward.

Matěj Mezera
m.mezera@fykos.cz

Problem FE ... zero to hundred for the second time

One of the performance characteristics of a car is the time needed to accelerate from $0\,\mathrm{km\cdot h^{-1}}$ to $100\,\mathrm{km\cdot h^{-1}}$. Consider a car that can do it in $4.0\,\mathrm{s}$. Find the distance travelled by the car while accelerating. Assume that the power of the car's motor is constant.

Karel was wondering about acccelerating.

The power is equal to the change of kinetic energy over time

$$P = \frac{\mathrm{d}E_\mathrm{k}}{\mathrm{d}t} = \frac{\mathrm{d}\frac{1}{2}mv^2}{\mathrm{d}t} = mva = m\dot{x}\ddot{x}\,,$$

where the mass of the car is m, the instantaneous velocity is v and the instantaneous acceleration is $a = \ddot{x}$. In this case, it is easiest to integrate

$$P = \frac{\mathrm{d}\frac{1}{2}m\dot{x}^2}{\mathrm{d}t} \quad \Rightarrow \quad Pt + E_0 = \frac{1}{2}m\dot{x}^2\,,$$

where the integration constant is E_0; it corresponds to initial kinetic energy, so it is equal to 0. Now, we express the velocity before integrating again

$$\dot{x} = \sqrt{\frac{2P}{m}}t^{\frac{1}{2}} \quad \Rightarrow \quad x = \frac{2}{3}\sqrt{\frac{2P}{m}}t^{\frac{3}{2}}\,.$$

We get

$$\sqrt{\frac{2P}{m}} = \frac{\dot{x}}{t^{\frac{1}{2}}}\,.$$

Combining it with the expression for the distance, we get $x = \frac{2}{3}vt \doteq 74.1\,\mathrm{m}$.

Karel Kolář
karel@fykos.cz

Problem FF ... a ring

Consider a non-homogeneous rod with length L and weight M suspended from one of its ends. The distance of the centre of mass of the rod from the point of suspension is l and the corresponding moment of inertia with respect to that point of suspension is J. We place a small ring of negligible dimensions on the rod so that the frequency of oscillations of the rod with the ring is twice the frequency of oscillations of the original rod. Let's denote the weight of the ring by m and its distance from the point of suspension by x. Find the weight m such that the distance x is unambiguous and express this value of x.

Kuba was wondering which solution to choose.

Halved oscillation velocity corresponds to a double period or a half angular frequency. We calculate the angular frequency from the formula

$$\omega = \sqrt{\frac{Mgl}{J}}\,.$$

The total moment of inertia is additive, so after the placement of the ring we have

$$J^* = J + mx^2 \,.$$

Furthermore, we have an equation for the new position of the centre of gravity

$$Ml + mx = (M + m)\, l^* \,.$$

Now, we can write for the new angular frequency

$$\omega^* = \sqrt{\frac{(M + m)\, gl^*}{J^*}} = \sqrt{\frac{g\,(Ml + mx)}{J + mx^2}} \,.$$

By solving the equation $\omega^* = 2\omega$ we get

$$x = \frac{J}{8Ml} \left(1 \pm \sqrt{1 - \frac{48l^2M^2}{Jm}} \right) .$$

We see that both solutions degenerate into one if the square root is zero for which we have

$$m = \frac{48M^2l^2}{J} \,.$$

In this case, we get an equation for the position of the ring

$$x = \frac{J}{8Ml} \,,$$

which is the desired result.

Jakub Dolejší
`dolejsi@fykos.cz`

Problem FG ... Matěj's four spheres

Four identical homogeneous spheres are piled up on a horizontal surface made from the same material as are the spheres. The centres of the spheres form a regular tetrahedron (three spheres are lying on the surface and form an equilateral triangle, the fourth is placed on top of them). Find the minimum possible value of the coefficient of static friction between the surfaces necessary for the spheres to remain at rest. *Matěj was arranging oranges.*

Thanks to the symmetry of the problem, it's sufficient to examine forces acting on only one bottom ball. Top ball exerts force F onto the bottom ball. Let's decompose F in directions parallel (F_t) and perpendicular (F_n) to surface of the balls at the point of contact. From the geometry of a tetrahedron, for an angle φ between F_n and horizontal plane, the following applies:

$$\tan\varphi = \sqrt{2} \,.$$

For the ball to remain in rest in is necessary that:

- Total torque is zero. Therefore friction between the bottom ball and the base has the same magnitude as F_t (in an opposite direction).

- Total force is zero. Equilibrium has to be achieved in both horizontal and vertical direction. We get

$$F_\mathrm{n}\cos\varphi - F_\mathrm{t}\sin\varphi - F_\mathrm{t} = 0\,,$$
$$F_\mathrm{n}\frac{\cos\varphi}{\sin\varphi + 1} = F_\mathrm{t}\,.$$

Balls stay intact, if

$$fF_\mathrm{n} \geq F_\mathrm{t}\,,$$
$$f \geq \frac{\cos\varphi}{\sin\varphi + 1} = \frac{1}{\sqrt{2}+\sqrt{3}} = \sqrt{3}-\sqrt{2} \doteq 0.318\,.$$

In that scenario, the downward force exerted on the base through the single ball is

$$4F_g/3 \geq fF_\mathrm{n} \geq F_\mathrm{t}$$

so the friction between the bottom ball and the base is high enough.

Matěj Mezera
`m.mezera@fykos.cz`

Problem FH ... spinning container

Consider a cylindrical container with base radius $R = 10\,\mathrm{cm}$ containing $V = 2\,\mathrm{l}$ of water. Calculate the minimum height of the container such that we avoid spilling any water when we spin it around its axis for a long time with angular velocity $\omega = 5\,\mathrm{rad{\cdot}s^{-1}}$.

Matěj likes to spin stuff.

After a certain time, the water converges to a state, where it rotates with the same angular velocity as the container. Our task is to find a function describing the height of the surface depending on the distance from the rotational axis. The basic idea is that the water surface is equal to some equipotential, because the water adapts the shape with minimum potential energy. Knowing that in the vertical direction there is a constant (gravitational) acceleration and in the horizontal direction the acceleration is proportional to the distance from the centre, we find the water surface to be a rotational paraboloid.

Let's call the function giving the height of the surface from the radial distance $h(r)$. Because the surface must be perpendicular to the force (therefore also the acceleration from Newtons second law), the slope of the function is equal to the ratio of the horizontal and vertical acceleration.

$$\frac{\mathrm{d}h}{\mathrm{d}r} = \frac{\omega^2 r}{g}\,.$$

Through integration we get $h(r)$

$$h(r) = \int \mathrm{d}h = \int \frac{\omega^2 r}{g}\mathrm{d}r = \frac{\omega^2 r^2}{2g} + C\,,$$

C is an integration constant, the value of which we get from the initial conditions – in this case from the volume V, which remains constant. We split the water into thin cylindrical rings with a volume of $2\pi r h(r)\mathrm{d}r$.

$$V = \int_0^R 2\pi r h(r)\mathrm{d}r = 2\pi \int_0^R \left(\frac{\omega^2 r^3}{2g} + Cr\right)\mathrm{d}r = \frac{\pi\omega^2 R^4}{4g} + C\pi R^2\,,$$

$$C = \frac{V}{\pi R^2} - \frac{\omega^2 R^2}{4g}\,.$$

The value of $h(r)$ in R gives us the lowest height, which is able to keep the water from spilling.

$$h(R) = \frac{\omega^2 R^2}{4g} + \frac{V}{\pi R^2} = 0.070\,\mathrm{m}\,.$$

It should be noted, that our model does not explicitly include the bottom of the container, so we have to be sure it does not intersect with the paraboloid i.e $h(r)$ is non-negative, which would give us incorrect results. This can be shown to hold in this case.

Matěj Mezera
m.mezera@fykos.cz

Problem GA ... rope

There's a coil of rope with linear density ϱ lying on the ground. We pick up one end of the rope and lift it upwards to height h with constant velocity v. What's the difference between the work we need to perform to do this and the sum of potential and kinetic energy of the ordered motion of the rope at the end of the process?

Hint: *It really isn't zero!* *Jáchym felt like he lacked some energy.*

As long as some mass of the rope m is in the air it is affected by the gravitational force $F_G = -mg$. We have to act by force F which we determine from the second Newtonian law as

$$F + F_G = \dot{p} = m\dot{v} + \dot{m}v = \dot{m}v\,. \tag{2}$$

Here we used the fact that $\dot{v} = 0$ because the lifting speed is constant. For some time $\mathrm{d}t$ we lift $\mathrm{d}x = v\mathrm{d}t$ of rope which gives us an equation for the time derivative of velocity

$$\dot{m} = \varrho v\,.$$

Through substituting this equation into the equation (2) we get

$$F = \dot{m}v - F_G = \varrho v^2 + mg = \varrho v^2 + x\varrho g\,,$$

where x means the length of rope that has been already raised. To compute the work, we have to integrate this equation from zero to h

$$W = \int_0^h F\mathrm{d}x = \int_0^h \varrho v^2 + x\varrho g\mathrm{d}x = \left[x\varrho v^2 + \frac{1}{2}x^2\varrho g\right]_0^h = h\varrho v^2 + \frac{1}{2}h^2\varrho g\,.$$

By lifting the end of the rope up to the height h we give it kinetic and potential energy

$$E = \frac{1}{2} h \varrho v^2 + \frac{1}{2} h^2 \varrho g \,,$$

which leads to the energy "loss"

$$\Delta E = W - E = \frac{1}{2} h \varrho v^2 \,.$$

The energy itself is, of course, not lost, only transformed to the energy of the oscillations of the rope. In this way of lifting small parts of the rope of length $\mathrm{d}x$ are accelerated to the speed v in the zero trajectory and for zero time which of course is not possible. Real rope is flexible so the small parts accelerate gradually. If we assumed that the rope is perfectly rigid so that the oscillations and losses of energy can not happen we would realize that the rope could not be lifted in this way.

Jáchym Bártík
`tuaki@fykos.cz`

Problem GB ... solar rod

Consider an infinitely long, thin semiconducting strip with width $1\,\mathrm{cm}$*. The area conductivity of the semiconducting material is directly proportional to illuminance with proportionality constant* $\alpha = 0.03\,\mathrm{S{\cdot}lx^{-1}}$*. We place a point source of light with luminous intensity* $2\,\mathrm{cd}$ *at height* $1\,\mathrm{m}$ *above the axis of the semiconductor. Then, we connect two perfectly conductive infinitely long electrodes to the two infinite edges of the strip. The voltage difference between the electrodes is* $7\,\mathrm{V}$*. What current (in Amperes) will flow between the electrodes?*

Mikuláš likes it when others have to solve unpleasant integrals.

The illumination of the semiconductor decreases both quadratically with distance and with the cosine of the angle of incidence of the rays to the semiconductor so generally according to the formula

$$\frac{hI}{\left(\sqrt{h^2 + x^2}\right)^3} \,,$$

where h is the height of the lamp above the semiconductor, I is luminous intensity of the source and x is the distance from the centre of the semiconductor. The specific resistance is then given by the formula

$$\frac{\left(\sqrt{h^2 + x^2}\right)^3}{hI\alpha}$$

and the resistance of the element of length $\mathrm{d}x$ is

$$\frac{y\left(\sqrt{h^2 + x^2}\right)^3}{hI\alpha \mathrm{d}x} \,,$$

where y is the width of the strip. The inverse quantity of resistance is obtained by integrating the inverse quantity of resistivity because it behaves like a parallel connection of an infinite number of resistors.

$$\frac{1}{R} = \int_{-\infty}^{\infty} \frac{hI\alpha}{y\left(\sqrt{h^2 + x^2}\right)^3} \mathrm{d}x,$$

We solve by hyperbolic substitution

$$\frac{1}{R} = \frac{I\alpha}{hy}\left[\frac{x}{\sqrt{x^2+h^2}}\right]_{-\infty}^{\infty} = \frac{2I\alpha}{hy}\,.$$

The current is then obtained according to the formula

$$I_{\mathrm{El}} = \frac{U}{R} = \frac{2UI\alpha}{hy}\,.$$

By substitution in the equation we get $I_{\mathrm{El}} = 84\,\mathrm{A}$.

Mikuláš Matoušek
`mikulas@fykos.cz`

Problem GC ... fireworks

The organisers forgot to buy fireworks for the New Year's Eve, so they decided to build a copy of the Tsar bomba. Let's assume that $2\,\%$ of the energy released in the reaction ${}^2_1\mathrm{D} + {}^3_1\mathrm{T} \longrightarrow {}^4_2\mathrm{He} + \mathrm{n}^0 +$ is emitted as electromagnetic radiation in the form of one photon per reaction. How fast should the organisers run away in order to be able to watch the "fireworks" as visible light ($550\,\mathrm{nm}$)? Compute the difference $c - v$ between the speed of light and this velocity. The rest masses are $m({}^2_1\mathrm{D}) = 1{,}876.1\,\mathrm{MeV{\cdot}c^{-2}}$, $m({}^3_1\mathrm{T}) = 2{,}809.4\,\mathrm{MeV{\cdot}c^{-2}}$, $m({}^4_2\mathrm{He}) = 3{,}728.4\,\mathrm{MeV{\cdot}c^{-2}}$, $m(\mathrm{n}^0) = 939.6\,\mathrm{MeV{\cdot}c^{-2}}$. *We really forgot.*

The number of photons does not concern us, we can concentrate on just one reaction and the photon generated by it. The total energy ΔE released during this reaction can be calculated from the law of conservation of energy (we can assume that the particles entering the reaction have negligible kinetic energy) in the form

$$m({}^2_1\mathrm{D})c^2 + m({}^3_1\mathrm{T})c^2 = \Delta E + m({}^4_2\mathrm{He})c^2 + m(\mathrm{n}^0)c^2\,.$$

Here, $\Delta E \doteq 17.5\,\mathrm{MeV}$ includes the kinetic energy of all particles produced during the reaction. The energy of the photon (its total energy is the same as its kinetic energy because the rest mass of a photon is 0) according to the problem statement is

$$0.02\Delta E \doteq 0.35\,\mathrm{MeV}\,,$$

the wavelength of the emitted photon (in the centre of mass reference frame of the reaction) is therefore

$$\lambda = \frac{hc}{E} \doteq 3.5\,\mathrm{pm}\,.$$

When escaping from the "fireworks" with velocity v, the Doppler effect occurs and the observed frequency is given by the formula

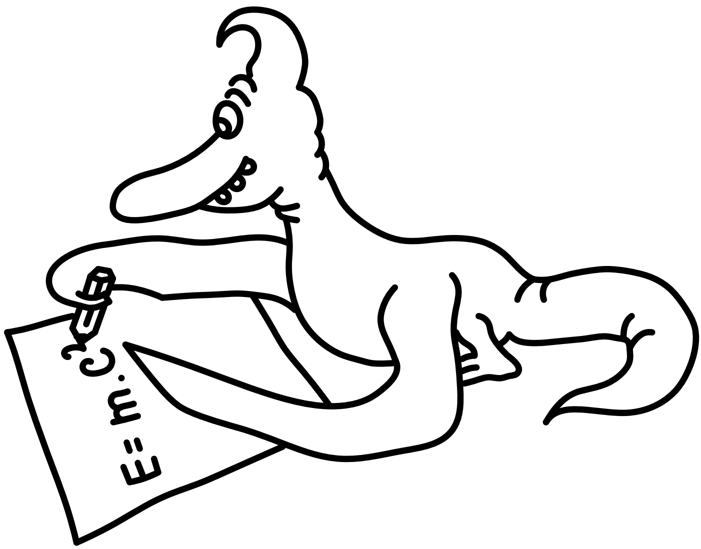
$$\lambda' = \lambda\sqrt{\frac{c+v}{c-v}}\,.$$

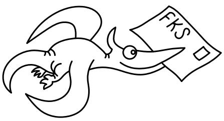
Because the required wavelength λ' is much larger than λ, the velocity must be $v \approx c$ and we can approximate

$$\frac{\lambda'}{\lambda} \approx \sqrt{\frac{2c}{c-v}} \quad \Rightarrow \quad c - v \approx 2c\left(\frac{\lambda}{\lambda'}\right)^2\,.$$

We get $c - v \doteq 8 \cdot 10^{-11} c \doteq 0.024\,\mathrm{m{\cdot}s^{-1}}$ – we need to run away from the Tsar bomba at near-lightspeed.

Jakub Šafin
`xellos@fykos.cz`





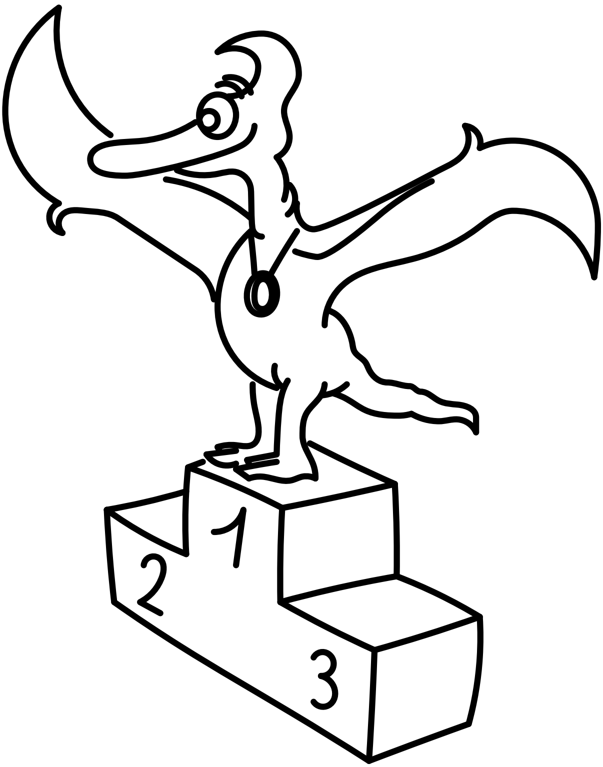
FYKOS
UK, Matematicko-fyzikální fakulta
Ústav teoretické fyziky
V Holešovičkách 2
180 00 Praha 8

www: `http://fykos.cz`
e-mail: `fykos@fykos.cz`

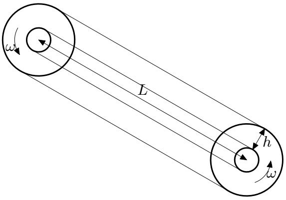
FYKOS is also on Facebook
http://www.facebook.com/Fykos

FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Public Relations Office activities and is supported by Institute of Theoretical Physics, MFF UK, his employees and The Union of Czech Mathematicians and Physicists.

13^{th} *FYKOS Physics Brawl*
Solutions of problems



Problem AA ... escalator strangeness



You may have noticed that the escalator belt for holding frequently moves at a different speed than the stairs themselves. Consider a simplified model where the belt and the stairs are attached to (wrapped around, without slipping) cylinders with a common axis at each end of the escalator. The cylinders are rotating with a constant angular velocity $\omega = 0.36\,\mathrm{rad{\cdot}s^{-1}}$ and the radii of inner cylinders are $r_1 = 1.2\,\mathrm{m}$. Find the ratio of the velocity of the belt to the velocity of the stairs. The escalator in the model has length $l = 32\,\mathrm{m}$ and the distance from the stairs to the belt is $h = 0.6\,\mathrm{m}$.

The radii of the cylinders on which the belt is rotating are $r_2 = r_1 + h$. The relation between linear and angular velocity $v = \omega r$ gives the ratio in question

$$\frac{v_2}{v_1} = \frac{\omega r_2}{\omega r_1} = \frac{r_1 + h}{r_1} = \frac{3}{2}\,.$$

Matěj Mezera
m.mezera@fykos.cz

Problem AB ... far from the planet

How much (in percent) do airliners prolong their path by flying at height $H = 11\,\mathrm{km}$, compared to the distance flown just above the sea level? Neglect the travel distance necessary to reach a particular height. Assume that the radius of Earth is $R_\mathrm{Z} = 6{,}373\,\mathrm{km}$.

Karel was thinking about air traffic.

If we want to fly from point A to point B and compare the distances flown at zero height and at height H, we don't even need to know the ground distance of these points. The aeroplane moves by the same angle ϑ with respect to the centre of the Earth. Therefore, we may write the ratio of the distance s_H flown at height H to s_0 at sea level as

$$K = \frac{s_H}{s_0} = \frac{(R_\mathrm{Z} + H)\,\vartheta}{R_\mathrm{Z}\vartheta} = \frac{R_\mathrm{Z} + H}{R_\mathrm{Z}} = 1 + \frac{H}{R_\mathrm{Z}} \doteq 1.0017\,.$$

Now we have the ratio. The question is about the relative increase in travel distance. We can find this by subtracting 1 from K, since the travel distance is longer by $\Delta = K - 1 = 0.17\,\%$. Compared to other effects that may change fuel consumption or flight length, this is clearly negligible.

Karel Kolář
karel@fykos.cz

Problem AC ... it sparks

Find the distance d between an electric line and a locomotive's pantograph (traction unit) which is needed for electric discharge (a spark) to occur. The electric line uses direct current and its voltage with respect to the rail (the ground) is $U = 3\,\mathrm{kV}$. The dielectric strength of air (the intensity of the electric field between electrodes needed for a spark) is $E = 3.0{\cdot}10^6\,\mathrm{kg{\cdot}m{\cdot}A^{-1}{\cdot}s^{-3}}$. Assume that the relative permittivity and permeability of air are both 1.

Dodo was thinking after a long journey home.

The intensity of a homogeneous field E is related to voltage difference by the formula

$$U = Ed\,,$$

where d is the distance between the points where we measure voltage. In this case, the electric field isn't homogeneous, but we may approximate it as a homogeneous field with the same average intensity E, since d will most likely be very small. Then, we can use this formula to obtain

$$d = \frac{U}{E} = 1\,\mathrm{mm}\,.$$

We can see that d is indeed quite small, so our approximation should be accurate.

As an afternote, consider how much more complicated this problem could get if we wanted an exact answer. First of all, we need to know the exact geometry of the pantograph (and the electric line); of course, a locomotive is moving, which induces additional motion of charge on the electric line and a general electromagnetic field. The electric intensity isn't usually defined as an average either, but for a homogeneous field. It's not even clear what kind of average it should be, precisely because there are so many factors to consider - in this case, we're lucky to have a good approximation.

Jozef Lipták
`liptak.j@fykos.cz`

Problem AD ... beware tram

We are standing next to a straight tram track. We see a tram approaching from the right with a constant velocity v and another tram approaching from the left with a constant velocity u. In the front window of the second tram, we glimpse the reflection of the first tram. Find the velocity with which the reflection is approaching us.

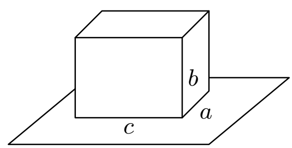
Dodo was bored on the way to climbing class.

The front window of the second tram is approaching us with the velocity u. If we were watching our reflection, it would obviously be approaching with velocity $2u$ and therefore, the velocity of the reflection of the first tram with respect to us is $2u + v$.

Jozef Lipták
`liptak.j@fykos.cz`

Problem AE ... unstable Danka

Consider a cuboid with dimensions $a = 20\,\mathrm{cm}$, $b = 30\,\mathrm{cm}$, $c = 50\,\mathrm{cm}$ and density $\varrho = 620\,\mathrm{kg{\cdot}m^{-3}}$. The cuboid is lying on a horizontal surface in a homogeneous gravitational field in such a way that the faces with dimensions a and c are facing up/down. What is the stability of the cuboid, i.e. the minimum amount of energy necessary to overturn it? *Danka was dropping stuff.*



We can see that it's more efficient to turn the cuboid over one of the edges with length c, because a is shorter, so it's easier to lift the center of mass so that the cuboid would turn over the edge with length c.

In order to turn the cuboid over, we have to move the centre of mass in such a way that it's directly above the edge with height c and push it a bit further. This change in the position of the centre of mass requires increasing the potential energy. We can calculate the stability as the maximum increase in potential energy of the centre of mass. Let's assume that potential is zero on the surface. At the beginning, the cuboid has potential energy

$$E_{p1} = \varrho abcg\frac{b}{2}\,.$$

When the center of mass is right above edge c, its height h is

$$h = \frac{1}{2}\sqrt{a^2 + b^2}$$

and its potential energy is

$$E_{p2} = \varrho abch\,.$$

The stability of the cuboid is the difference between E_{p2} and E_{p1}, which is

$$\Delta E = E_{p2} - E_{p1}\,,$$
$$\Delta E = \frac{1}{2}\varrho gabc(\sqrt{a^2 + b^2} - b)\,,$$
$$\Delta E \doteq 5.52\,\mathrm{J}\,.$$

The stability of the cuboid is 5.52 J.

Daniela Pittnerová
daniela@fykos.cz

Problem AF ... flat-Earth

It is well-known that the Earth is flat and supported by elephants. Above it, there is a dome from which the Sun, the Moon and stars are hanging. If the Moon with mass $7.348 \cdot 10^{22}$ kg *was hanging from a steel rope with a uniform circular cross-section, what would be the minimum diameter of its cross-section if we wanted to keep a* 10 % *reserve (the rope should not break if the mass increased by* 10 %*)? The yield stress of steel is* 700 MPa*. The Moon is placed in uniform gravity* g*. The weight of the rope can be neglected.*

Mikuláš has a weakness for Terry, Terry Pratchett.

The solution can be found by combining formulas for stress $\sigma = F/S$, force due to gravity $F = mg$ and area of a circle $S = \pi d^2/4$. We obtain the formula

$$\sigma_{\max} = \frac{4 \cdot 1.1mg}{\pi d^2}$$

and from it, we can express the diameter d as

$$d = 2\sqrt{\frac{1.1\,mg}{\pi\sigma_{\max}}}\,.$$

Plugging in the numerical values, we get $d = 38{,}000\,\mathrm{km}$. This is several times larger not only than the diameter of the Moon, but also of the Earth, so we can conclude that an explanation of celestial mechanics using massive rocky spheres hurtling across space cannot be completely excluded.

Mikuláš Matoušek
mikulas@fykos.cz

Problem AG ... the Prague Metro

You are running on an escalator. The time it takes you to run from the bottom to the top when the escalator is turned on is t_z. If the escalator is turned off, the time it takes you is t_v ($t_\mathrm{v} < t_\mathrm{z}$). Your running speed is v_c. Find the speed of the escalator.

Legolas was running on escalators in the underground.

We can determine the length of the escalator l_e using the time t_v as

$$l_\mathrm{e} = \frac{1}{2} v_\mathrm{c} t_\mathrm{v} \,,$$

because it's crossed twice (upstairs and downstairs). If t_1 is the time it takes to get upstairs and t_2 the time it takes to get downstairs when the escalator is turned on, we get

$$t_\mathrm{z} = t_1 + t_2 \,,$$

$$t_\mathrm{z} = \frac{l_\mathrm{e}}{v_\mathrm{c} + v_\mathrm{e}} + \frac{l_\mathrm{e}}{v_\mathrm{c} - v_\mathrm{e}} \,,$$

$$\frac{t_\mathrm{z}}{v_\mathrm{c} t_\mathrm{v}} = \frac{v_\mathrm{c}}{v_\mathrm{c}^2 - v_\mathrm{e}^2} \,,$$

from which we obtain the speed of the escalator

$$v_\mathrm{e} = v_\mathrm{c} \sqrt{1 - \frac{t_\mathrm{v}}{t_\mathrm{z}}} \,.$$

Šimon Pajger
legolas@fykos.cz

Problem AH ... prefixious

Sort the first eight letters of the English alphabet according to their use as SI prefixes in descending order of size. Make sure to use correct capitalisation of letters. In case some letter is not used as a prefix, do not use it. In case both forms (lowercase and uppercase) are, use both.

Dodo is studying the alphabet.

From the letters a, b, c, d, e, f, g, h, A, B, C, D, E, F, G, H, only a, c, d, E, f, G, h are SI prefixes, with the following names and meanings: atto = $\cdot 10^{-18}$, centi = $\cdot 10^{-2}$, deci = $\cdot 10^{-1}$, exa = $\cdot 10^{+18}$, femto = $\cdot 10^{-15}$, giga = $\cdot 10^{9}$, hecto = $\cdot 10^{2}$. The correct order is E, G, h, d, c, f, a.

Jozef Lipták
liptak.j@fykos.cz

Problem BA ... landing accident

Some flight navigation instruments are based on air pressure. Let's focus on the altimeter, which displays the altitude of the plane calculated using the static pressure of surrounding air. For this device to work correctly, it is necessary on every airport to recalibrate it with the current air pressure at sea level, calculated from local air pressure. Near the sea level, we can then use the approximation that ascent by $\Delta h = 8.0\,\mathrm{m}$ corresponds to a pressure drop $\Delta p = 1.0\,\mathrm{hPa}$. If a pilot accidentally recalibrates the device with sea level pressure $p_\mathrm{err} = 1{,}021\,\mathrm{hPa}$ instead of the correct $p_\mathrm{r} = 1{,}012\,\mathrm{hPa}$, how high above the sea level would the airplane be when the altimeter shows $450\,\mathrm{m}$?

Karel was learning about the importance of the QHN.

We can write the ratio of altitude change to pressure change as

$$K = -\frac{\Delta h}{\Delta p} = -8.0\,\mathrm{m{\cdot}hPa}^{-1}\,.$$

The difference between the actual pressure and pressure set in the altimeter is $\Delta P = p_\mathrm{r} - p_\mathrm{err} = -9\,\mathrm{hPa}$. This corresponds to the altimeter "thinking" the altitude is greater by

$$H = K\Delta P = \frac{\Delta h}{\Delta p}\left(p_\mathrm{r} - p_\mathrm{err}\right) = 72\,\mathrm{m}\,.$$

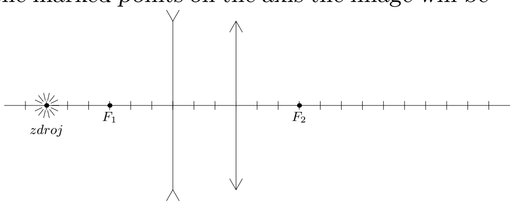
If the instrument shows 450 m, the airplane is actually 378 m above the sea level. This is quite a big difference, for example if the airplane is approaching the Prague - Ruzyne airport, which is at the altitude 380 m. We could say it's a difference between life and death.

If you're confused by the sign convention and whether it determines that the real altitude is higher or lower than the displayed altitude, think about which pressures are used. Pressure always decreases with increasing altitude. If the pressure set in the altimeter is higher than real pressure, it seems that the air column below us is greater. That's because our altimeter measures the surrounding pressure and compares it to the set pressure, which it assumes to be the pressure at sea level. It seems that the pressure drop from sea level is greater than it actually is.

Karel Kolář
`karel@fykos.cz`

Problem BB ... drawing

In the picture, you can see an optical axis, on which a point source of light ("zdroj"), a focus point of the first (diverging) lens, a thin diverging lens, a thin converging lens, and a focus point of the second (converging) lens are marked. Each distance is equal to three units (of distance). Draw into the picture the position of the source's image after it is formed by both lenses. Your task is only to decide between which two of the marked points on the axis the image will be formed. You may solve it by an arbitrary method.

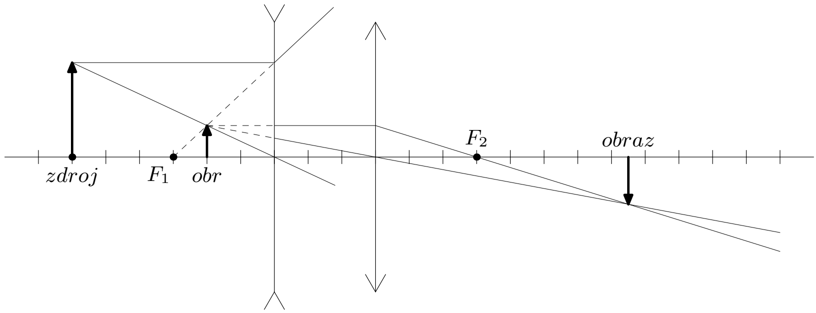


Matěj told himself it would be nice to draw something during the competition.

The following is a solution by image.

We can also measure each distance (the source is placed 6 units of distance before the lens and the focal lengths are -3 and 3) and using the thin lens equation

$$\frac{1}{a} + \frac{1}{a'} = \frac{1}{f}\,,$$



we can find out that the image will be formed 2 units of distance before the first lens ("obr"). This image will be formed 5 units of distance from the second lens, which will form it at the distance 7.5 units before itself ("obraz").

Matěj Mezera
m.mezera@fykos.cz

Problem BC ... acceleration from planets

Consider a system of two homogeneous spherical planets, both with densities $\varrho = 4{,}200\,\mathrm{kg{\cdot}m^{-3}}$. The smaller planet has radius $R = 4.20 \cdot 10^{6}\,\mathrm{m}$ and the larger one has radius $R_2 = 2R$. The distance of their centres is $r = 16R$. What is the gravitational acceleration (including its direction) acting on a dust particle located at a distance $x = 8R$ from the centres of both planets? *Karel likes to play with impossible planetary systems.*

The gravitational acceleration that acts on the dust particle outside a spherically symmetrical body at a distance x from its centre is

$$a_g = G\frac{m}{x^2}\,,$$

where $G = 6.67 \cdot 10^{-11}\,\mathrm{N{\cdot}kg^{-2}{\cdot}m^2}$ is the gravitational constant and m is the mass of the body which imparts this acceleration.

In this problem, there are two spherically symmetrical bodies and the only relevant difference between them is in their masses. The mass of the smaller sphere is

$$m_1 = \varrho V_1 = \frac{4}{3}\pi\varrho R^3\,.$$

Mass is proportional to the cube of radius and the second sphere's radius is twice as large as the first sphere's, so its mass is $m_2 = 8m_1$.

Gravitational acceleration always acts towards the body which causes it. If we place the spheres and dust particle on an axis in such a way that the lighter sphere is on the left of the particle and the heavier one on the right, the principle of superposition gives the total acceleration (pointing to the right)

$$a = a_1 + a_2 = -G\frac{m_1}{x^2} + G\frac{m_2}{x^2} = \frac{7Gm_1}{x^2} = \frac{7\pi G\varrho R}{48} \doteq 0.54\,\mathrm{m{\cdot}s^{-2}}\,.$$

The total gravitational acceleration acting on the dust particle located exactly between the centres of the planets is $0.54\,\mathrm{m\cdot s^{-2}}$. Its direction is away from the centre of the smaller planet, towards the centre of the larger planet.

Karel Kolář
`karel@fykos.cz`

Problem BD ... drinking tea

Consider a cylindrical container with height h and base area S, filled with tea with density ϱ. There is a little hole with area $s \ll S$ at the bottom of the container. This hole is used to attach a horizontal tube with a faucet and a manometer (a vertical tube, open at the top) to the container. Find the formula describing the height of liquid in the manometer $l(t)$ over time after the faucet is opened. Assume that this height adapts instantly to changes in the flow and that tea is an ideal liquid (incompressible, with zero viscosity and surface tension).

Dodo was thinking in the canteen and forgot to eat.

The velocity of the fluid flowing through the hole in the container can be expressed as $v = \sqrt{2gh}$. The overpressure at the bottom of the manometer p (with respect to atmospheric pressure) follows from the Bernoulli equation:

$$p = \varrho gh - \frac{1}{2}\varrho v^2 = 0\,\mathrm{Pa}\,.$$

This conclusion could have also been made after realising that for an ideal liquid, there is no difference between the pressure at the bottom of the manometer and the pressure just above the hole through which the water flows.

Jozef Lipták
`liptak.j@fykos.cz`

Problem BE ... at the bottom of the lake

How much higher is the pressure at the bottom of the Dead Sea as a result of it being salty? We are interested in the difference with respect to a hypothetical situation when the sea would be sweet. Consider a situation where the lake's surface (the Dead Sea is a lake, which is drying out continuously) is $h_0 = 430\,\mathrm{m}$ under the sea level, and the deepest point of the lake is $\Delta h = 298\,\mathrm{m}$ lower. The density of sweetwater is $\varrho_0 = 997\,\mathrm{kg\cdot m^{-3}}$, while the current density of the lake is $\varrho = 1{,}240\,\mathrm{kg\cdot m^{-3}}$. Assume that the depth of the lake after demineralizing would be the same.

Karel was learning something about air pressure.

We do not need data about how deep the water level of The Dead Sea is, because we can consider that above the water level, the atmosphere is the same in both cases and pressure changes only in the water column.

We can consider the depth of the lake after demineralizing to be the same, so the difference of pressures at the bottom of the lake is given by the difference of hydrostatic pressure in saltwater $p_1 = \varrho \Delta h g$ and in sweetwater $p_2 = \varrho_0 \Delta h g$, where g is the acceleration due to gravity. We get the pressure difference

$$\Delta p = p_1 - p_2 = (\varrho - \varrho_0)\,\Delta h g \doteq 710\,\mathrm{kPa}\,.$$

At the bottom of the Dead Sea nowadays, the pressure is about 710 kPa higher than if the water in it was sweet (distilled). The difference is almost the same as 7 atmospheres. It is true that we considered the density of completely clear water, but sweetwater in nature contains some soluble substances and because of that, the real difference should be lower.

Karel Kolář
karel@fykos.cz

Problem BF ... take a guess

By how many orders of magnitude is the volume of the observable universe greater than the volume of a hydrogen atom? In other words, find the order of magnitude of the ratio (volume of observable universe) / (volume of a hydrogen atom). The observable universe has diameter 28.5 Gpc*. The van der Waals radius of hydrogen is* $r = 1.2\,\text{Å}$*.*

Dodo did something wrong once again.

We can convert the diameter of the observable universe to its radius $R = 4.41 \cdot 10^{26}\,\mathrm{m}$. One Angstrom is $1\,\text{Å} = 1 \cdot 10^{-10}\,\mathrm{m}$, so the ratio of radii is $R/r = 3.7 \cdot 10^{36}$. Taking the decimal logarithm, we get $\log \frac{r}{r_0} = 36.56$. The ratio of volumes is

$$\log \frac{V}{V_0} = 3 \log \frac{r}{r_0} = 109.68 \approx 110\,.$$

The volume of the observable universe is bigger by 110 orders of magnitude.

Jozef Lipták
liptak.j@fykos.cz

Problem BG ... exosystem with a planet

Imagine that we discovered another planetary system, which is similar to ours. The local sun has mass $0.900M_\mathrm{S}$ *(a multiple of the mass of our Sun). If the year on a local planet similar to our Earth has the same length as our year, at what distance from the local sun would it have to be? For simplicity, assume that the planet has a circular orbit. Compute the result as a multiple of* au*, i.e. the middle distance between the centers of the Earth and the Sun.*

Karel was thinking about exoplanets.

We assume that the planet orbits on a circle, so the centripetal force F_d has to be equal to gravitational force F_g all the time. Other forces are not appearing in this task (we neglect the effects of other planets because we do not know where they are located) and the planet orbits its sun uniformly (with velocity v). Then we get

$$F_\mathrm{d} = F_\mathrm{g} \qquad \Rightarrow \qquad m\frac{v^2}{r} = G\frac{mM}{r^2} \qquad \Rightarrow \qquad v^2 = G\frac{M}{r}\,,$$

where m is the mass of the planet, M is the mass of the local sun, G is the gravitational constant and r is the distance between the planet and the center of the local sun. Now we use the formula $v = 2\pi r/T$, i.e. the rotational speed of the planet corresponds to the fact that it makes one revolution around its sun in one year (time T). We get a relation between r and other parameters

$$\frac{4\pi^2 r^2}{T^2} = G\frac{M}{r} \qquad \Rightarrow \qquad r = \sqrt[3]{\frac{GM}{4\pi^2}T^2}\,.$$

Now we see that the orbital radius depends on gravitational acceleration, which is constant, other constants and on the mass of the sun and orbital time of the planet. Because we have the same duration of one year and a lower mass of the sun, it is enough to plug these values into the formula for r and we get the orbital radius as a multiple of the Earth's orbital radius r_Z

$$r = \sqrt[3]{G\frac{0.900\,M}{4\pi^2}T^2} \doteq 0.965\,r_Z\,.$$

The radius of the planet's orbit in the discovered solar system has to be 0.965 au. With a decrease of the mass of the sun by about 10 %, the orbital radius has to decrease by about 3.5 %.

Karel Kolář
`karel@fykos.cz`

Problem BH ... price of a flight

We are flying in an aircraft-carrier BAe 146-300/RJ100 and we want to know whether it would be cheaper to fly at low or high velocity, when we want to fly $s = 250\,\mathrm{km}$ at the flight level FL 310. The aircraft at this flight level has fuel consumption $M_\mathrm{h} = 2{,}517\,\mathrm{kg{\cdot}h^{-1}}$ at velocity $v_\mathrm{h} = 429\,\mathrm{kn}$ and $M_\mathrm{l} = 1{,}724\,\mathrm{kg{\cdot}h^{-1}}$ at velocity $v_\mathrm{l} = 377\,\mathrm{kn}$. For which type of flight is the fuel consumption lower and how much fuel (in kg) do we save at the distance s? One knot (kn) is $1{,}852\,\mathrm{m{\cdot}h^{-1}}$. *Karel is at ATCo training.*

We start with converting the units of velocity to the units we need, so $v_\mathrm{h} = 429\,\mathrm{kn} \doteq 794.5\,\mathrm{km{\cdot}h^{-1}}$ and $v_\mathrm{l} = 377\,\mathrm{kn} \doteq 698.2\,\mathrm{km{\cdot}h^{-1}}$. As we are given the fuel consumption in kilograms per hour, we have to determine the flight length for each type of flight,

$$t_\mathrm{h} = \frac{s}{v_\mathrm{h}} \doteq 0.315\,\mathrm{h}\,, \quad t_\mathrm{l} = \frac{s}{v_\mathrm{l}} \doteq 0.358\,\mathrm{h}\,.$$

The total amount of consumed fuel for each type of flight is

$$m_\mathrm{h} = M_\mathrm{h}t_\mathrm{h} = M_\mathrm{h}\frac{s}{v_\mathrm{h}} \doteq 792\,\mathrm{kg}\,, \quad m_\mathrm{l} = M_\mathrm{l}t_\mathrm{l} = M_\mathrm{l}\frac{s}{v_\mathrm{l}} \doteq 617\,\mathrm{kg}\,.$$

Maybe it is suprising, but we found out that if we fly slower, the fuel consumption is lower - at least on flight level 310 (31,000 feet above the sea level, around 9.3 km). The answer is that slower flight is more economical and we save around 175 kg of fuel.

Karel Kolář
`karel@fykos.cz`

Problem CA ... when physicist cooks

In the manual for Legolas' kettle, it is written that $U_0 = 240V$, $P_0 = 2200W$ (i.e. the kettle consumes electrical power P_0 if the voltage is U_0). He poured exactly $V = 1.5\,\mathrm{l}$ of water with temperature $T_1 = 20\,\mathrm{^\circ C}$ into the kettle. Then he switched the kettle on and heated the water to a temperature $T_2 = 80\,\mathrm{^\circ C}$. His stopwatch showed that the process lasted for $t = 4.2\,\mathrm{min}$. What voltage does Legolas have in his power socket? Assume that the resistance of the kettle is independent on temperature and the efficiency of heating water is $\eta = 90\,\%$.

Lego often cooks... tea.

First, we calculate that the water absorbed heat

$$Q = cm\Delta T = cV\varrho\,(T_2 - T_1)\ .$$

From this, we can find the average power consumption of the kettle

$$P_\mathrm{k} = \frac{1}{\eta}\frac{Q}{t} = \frac{cV\varrho\,(T_2 - T_1)}{\eta t}\ .$$

Now we use the information from the manual. We know that the kettle consumes power P_0 at voltage U_0. We can express the resistance of the kettle or we can simply realize that the electrical power consumption is proportional to the square of voltage, which gives

$$\frac{P_\mathrm{k}}{P_0} = \frac{U_\mathrm{k}^2}{U_0^2}\ .$$

Therefore, the result is

$$U_\mathrm{k} = U_0\sqrt{\frac{cV\varrho\,(T_2 - T_1)}{\eta t P_0}}\ .$$

Plugging in the numerical values, we get $U_\mathrm{k} \approx 208\,\mathrm{V}$.

Šimon Pajger
legolas@fykos.cz

Problem CB ... the price of charging a car battery

How much does it cost to fully charge an empty battery ($12\,\mathrm{V}$, $60\,\mathrm{Ah}$)*? We will use a battery charger with parameters* $12\,\mathrm{V}$, $4.2\,\mathrm{A}$ *and effectivness* $72\,\%$. *The price for electricity is* $k = 4.0\,\mathit{Kč}{\cdot}\mathrm{kWh}^{-1}$. *Karel. Don't even ask.*

To obtain the price, we need to specify the energy E_0 that we need to supply the battery. This is given by the product of the battery voltage $U = 12\,\mathrm{V}$ and her capacity $Q = 60\,\mathrm{Ah}$

$$E_0 = QU = 720\,\mathrm{Wh}\,.$$

In order to determine the energy that we need from the power grid, we need to consider the efficiency of the charger

$$E = \frac{E_0}{\eta} = \frac{QU}{\eta} = 1.0\,\mathrm{kWh}\,.$$

The final price x which we have to pay for is then equal to

$$x = Ek = \frac{QU}{\eta}k = 4.0\,\mathrm{Kč}\,.$$

One car battery charging will then cost 4.0 Kč. Note that although it is said that the car battery has a voltage of $12V$, it is not Exactly constant and in fact, the voltage on battery is slightly higher. Especially in a charged state. That would increase the price a little bit.

Karel Kolář
karel@fykos.cz

Problem CC ... mirrors

Consider almost perfectly polished plane mirrors and set them parallel to a distance of 5 m. *When Matej looked in one of them, he did not see the infinity of reflections. How far away is the last visible image if one reflection reflects only* 98.5 % *light? He is able to detect an image that has* 10 % *of the original brightness. Do not consider the dimensions of his head or other loss of brightness.*

The beam of light has to reflect $\log_{0.985}(0.1) = 152$ times, before its intensity drops below 10 %. Each reflection corresponds to five meters of distance and since the number is even, the distance before the first reflection and the distance after the last reflection sum up to 5 m. Thus, the farthest image will be seen at a distance $152 \cdot 5\,\mathrm{m} \doteq 760\,\mathrm{m}$.

Matěj Mezera
m.mezera@fykos.cz

Problem CD ... model of an atom

An electron in the Bohr model of the hydrogen atom is orbiting around fixed proton on a circle with radius r. Its angular momentum is quantized, so it obtains values $L = n\hbar$, where n is a number describing quantum state of the electron. What is the radius r of the circular orbit of the electron, when it is present in the second quantum state?

Danka was reminiscing about the Physics Olympiad.

Let's denote $m = 9.109 \cdot 10^{-31}\,\mathrm{kg}$ as a mass of the electron and v as its velocity. The electron orbiting on a circle around the proton is affected by centrifugal force

$$F_\mathrm{o} = \frac{mv^2}{r}$$

and the electric attractive force

$$F_\mathrm{e} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}\,,$$

where e is elementary charge. These forces are compensated by each other, so we get

$$F_\mathrm{o} = F_\mathrm{e}\,,$$

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}\,.$$

The angular momentum of the electron is

$$L = n\hbar = mvr\,,$$

from where we express the velocity of the electron

$$v = \frac{n\hbar}{mr}\,.$$

We induct this expression to the equation acquired from the equilibrium of forces and we express the radius

$$r = \frac{4\pi\varepsilon_0\hbar^2 n^2}{me^2}\,.$$

In our case $n = 2$, so after evaluating the result numerically we get $r \doteq 2.1 \cdot 10^{-10}\,\mathrm{m}$.

Daniela Pittnerová
daniela@fykos.cz

Problem CE ... degradation of the efficiency of a kettle

Consider a kettle with a power $P = 1{,}800\,\mathrm{W}$ *which heats* $m = 1.2\,\mathrm{kg}$ *of water from temperature* $t_0 = 18\,{}^{\circ}\mathrm{C}$ *to* $t_1 = 100\,{}^{\circ}\mathrm{C}$ *in* $\tau = 4.2\,\mathrm{min}$. *Unfortunately, it takes another* $\Delta\tau = 15\,\mathrm{s}$ *for it to "realise" the water has boiled and to turn off. By how much does the efficiency of the kettle as a heat engine decreases as opposed to it turning off right after the water has boiled? Suppose that the heat capacity of water is constant,* $c = 4{,}200\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$. *Karel was hypnotizing a kettle.*

The heat necessary for the water to boil is given by $Q = mc(t_1 - t_0) \doteq 413\,\mathrm{kJ}$. The kettle's energy consumption for that period τ is $E_0 = P\tau = 454\,\mathrm{kJ}$ and consequentially its efficiency will be

$$\eta_0 = \frac{Q}{E_0} = \frac{mc\,(t_1 - t_0)}{P\tau} \doteq 91.1\,\%\,.$$

With a similar reasoning, we can figure out the energy consumption for the period $\tau + \Delta\tau$ as $E_1 = P\,(\tau + \Delta\tau) \doteq 481\,\mathrm{kJ}$. The corresponding efficiency then follows:

$$\eta_1 = \frac{Q}{E_1} = \frac{mc\,(t_1 - t_0)}{P\,(\tau + \Delta\tau)} \doteq 86.0\,\%\,.$$

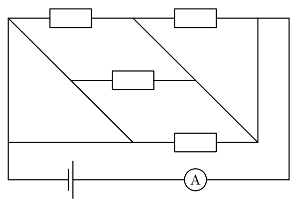
Since the task is to calculate the difference in efficiency, we can easily conclude

$$\Delta\eta = \eta_1 - \eta_0 = \frac{mc\,(t_1 - t_0)\,\Delta\tau}{P\tau\,(\tau + \Delta\tau)} \doteq 5.1\,\%\,.$$

Karel Kolář
`karel@fykos.cz`

Problem CF ... zkkzgt zjazq

A bomb was found at a field. The countdown ends at 13:30, so the poor farmers cannot escape the explosion's deadly radius anymore. At which point(s) should you cut the circuit if you know that the bomb does not explode only when the ammeter shows exactly $15\,\mathrm{mA}$ *at 13:30? The circuit is powered by four* $1.5\,\mathrm{V}$ *batteries and each resistor has resistance* $1\,\mathrm{k\Omega}$. *Mark the point(s) to cut in the figure. Matěj is surely not a terrorist.*

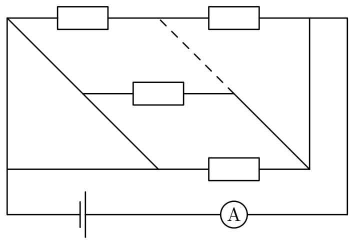


Ohm's law says that the circuit needs to have total resistance $\frac{2}{5}R$, where R is the resistance of one resistor. If there are only four available resistors, the only way to achieve that is by connecting in parallel: one resistor, one resistor and two serially connected resistors. That corresponds to exactly one cut, as shown in the figure.

Matěj Mezera
`m.mezera@fykos.cz`

Problem CG ... playing with marbles

Danka has a marble with radius $R = 1.0\,\mathrm{cm}$ *and mass* $m = 20\,\mathrm{g}$. *She pushes the marble in such a way that a horizontal force* $F = 15\,\mathrm{N}$ *acts on it for a time* $t = 0.2\,\mathrm{s}$ *(we neglect the rolling*



friction during this time period and the marble does not slip). How long does the bead move? The rolling resistance coefficient is $\xi = 0.03m$. *Assume that the bead does not slip.*

Danka found a marble.

The marble gains momentum p,

$$p = Ft = mv_0 \,,$$

where v_0 is the velocity of the marble immediately after it is pushed. There is a constant force of rolling resistance acting on the marble,

$$T = \frac{\xi mg}{R} \,.$$

The marble is moving with uniformly accelerated (decelerated) motion, where the acceleration is given by Newton's 2nd law

$$a = -\frac{T}{m} = -\frac{\xi g}{R} \,.$$

Let's denote the answer by t_v. Then,

$$v_0 - at_\mathrm{v} = 0 \,.$$

Substituting from the previous formulas, we can express t_v

$$0 = \frac{Ft}{m} - \frac{\xi g}{R} t_\mathrm{v} \,,$$

$$t_\mathrm{v} = \frac{FtR}{\xi mg} \doteq 5.1\,\mathrm{s} \,.$$

The marble remains in motion for 5.1 s.

Daniela Pittnerová
daniela@fykos.cz

Problem CH . . . flat-Earth reloaded

It is well-known that the Earth is flat and supported by elephants. Above it, there is a dome from which the Sun, the Moon and stars are hanging. If the Moon with mass $7.348 \cdot 10^{22}$ kg *was hanging from a steel rope with a uniform circular cross-section, what would be the minimum radius of its cross-section? The length of the rope is* 3.2 km, *the yield stress of steel is* 700 MPa *and its density is* 7,850 kg·m^{-3}. *The Moon and the rope are placed in uniform gravity* g. *Neglect the mutual gravitational influence of the rope and the Moon.*

Jáchym has a weakness for reloaded problems.

The stress in the rope is maximum at its top, since the stress in it isn't caused only by gravity acting on the Moon, but also gravity acting on the whole rope. Let's denote the mass of the Moon by m the density of the rope by ϱ, its radius by r and its length by l. The total force acting at the top of the rope due to gravity is

$$F_g = \left(m + \pi r^2 l \varrho\right) g\,.$$

The given yield stress σ, multiplied by the cross-sectional area of the rope πr^2 gives the maximum force which may act on the rope

$$F_{\mathrm{max}} = \sigma \pi r^2\,.$$

Finally, we can just compare the two forces and express the radius

$$F_g = F_{\mathrm{max}}\,,$$
$$r = \sqrt{\frac{mg}{\pi\left(\sigma - l\varrho g\right)}} \doteq 2.25 \cdot 10^7\,\mathrm{m}\,.$$

Note that this value is very unrealistic and construction of such a rope wouldn't make much sense overall.

Jáchym Bártík
tuaki@fykos.cz

Problem DA ... not even an oil stain

There is a standardized light source with luminous intensity $I_0 = 10\,\mathrm{kcd}$ and a light bulb; the distance between them is $d = 3\,\mathrm{m}$. If we place a sheet of paper with an oil stain between the sources at a distance $l = 1\,\mathrm{m}$ from the light bulb, the whole sheet appears equally bright. Find the luminous intensity I of the light bulb.

Dodo was reminiscing about a discontinued lab experiment.

The oil stain becomes invisible when the illuminance is the same on both sides

$$\frac{I}{l^2} = F = \frac{I_0}{(d-l)^2}\,.$$

From this, we obtain the luminous intensity of light bulb $I = I_0\left(d/l - 1\right)^{-2} = 2.5\,\mathrm{kcd}$.

Jozef Lipták
liptak.j@fykos.cz

Problem DB ... rabbit no racist

A small, fat, spherical magic rabbit with diameter $d = 30\,\mathrm{cm}$ is floating in space near Earth. The rabbit is cold, so it decides to paint itself black with some Vanta Black colour, which reflects only $\eta = 0.035$ of incident radiation. What is the power it absorbs from solar radiation? The power radiated by the Sun is $W = 3.8 \cdot 10^{26}\,\mathrm{W}$. *Honza was cold.*

We may consider the distance of the rabbit from the Sun to be $r = 1.50 \cdot 10^{11}\,\mathrm{m}$. Since the rabbit-sphere absorbs basically all sunlight, we don't need to consider angle of incidence. The absorbed power is

$$P = (1-\eta)\,W \frac{\pi\,(d/2)^2}{4\pi r^2} = \frac{(1-\eta)\,W d^2}{16 r^2} \doteq 91.7\,\mathrm{W}\,.$$

where $1-\eta$ is the absorption coefficient.

Jan Střeleček
strelda@fykos.cz

Problem DC ... lumberjack physicist

There is an idealised homogeneous tree with uniform, but negligible thickness and height h. We cut the tree next to the ground in such a way that it starts falling around a fixed point at the bottom. What is the velocity of the top of the tree immediately before it hits the ground?

Lego was trying to make a pen stand on its tip.

The potential energy of the tree is converted to rotational (kinetic) energy:

$$mg\Delta h = \frac{1}{2} J\omega^2$$

The tree is homogeneous, so its center of mass is at its midpoint

$$\Delta h = \frac{1}{2} h\,.$$

The moment of inertia of a rod rotating around an axis passing through its center of mass is

$$J_t = \frac{1}{12} m l^2\,,$$

and from Steiner's theorem, we find that when rotating around an axis passing through its endpoint, the moment of inertia is

$$J = J_t + m\left(\frac{h}{2}\right)^2 = \frac{1}{3} m h^2\,.$$

Finally, we need to realise that the velocity of the top of the tree is $v = \omega h$. Putting everything together, we get

$$v = \sqrt{3gh}\,.$$

Šimon Pajger
legolas@fykos.cz

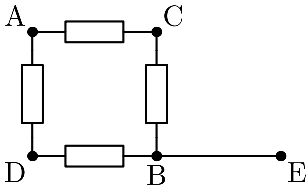
Problem DD ... black box

We have a black box with five terminals labelled A to E. We know that there are only $1\,\Omega$ resistors and wires inside. The table below shows values of resistance measured by a multimeter connected between all pairs of terminals. Draw an electrical diagram of the black box.

Hint *We did not use more than seven resistors.*

	A	*B*	*C*	*D*	*E*
A	*0*	*1*	*3/4*	*3/4*	*1*
B	*1*	*0*	*3/4*	*3/4*	*0*
C	*3/4*	*3/4*	*0*	*1*	*3/4*
D	*3/4*	*3/4*	*1*	*0*	*3/4*
E	*1*	*0*	*3/4*	*3/4*	*0*

First, note that B and E are connected by a wire. Therefore, the problem reduces to a black box with 4 terminals (A to D).



Next, we may notice symmetry between these four terminals. For each terminal, there is one "sister" terminal and the resistance between them is $1\,\Omega$, while the resistance between this terminal and either of the remaining two is $3/4\,\Omega$. Assuming we have ≤ 7 resistors, there are only two possible ways of connecting them symmetrically to all four terminals in such a way that all resistances are non-zero. There may either be a cycle of four resistors or the same cycle with pairs of opposite vertices connected by resistors (6 resistors in total). It's easy to see that only the first configuration fulfills the given conditions.

Matěj Mezera
`m.mezera@fykos.cz`

Problem DE ... two cuboids

One upon a time, there were two cuboids with base areas S_1 and S_2, where $S_1 < S_2$. One day, the first one (the one with a smaller base) fell into the other one, since it was hollow. However, the first cuboid wasn't hollow, it had a height h and density ϱ_1. Therefore, it used that moment to think: what is the volume of a liquid with density $\varrho > \varrho_1$ which needs to be poured inside the second cuboid so that the first one stops sticking to its bottom?

Legolas was picking up a glass from a sink.

For the cuboid to stop sticking to the bottom of the other one, the buoyant force acting on it needs to compensate for the gravitational force

$$\begin{aligned} F_{\mathrm{vz}} &= F_{\mathrm{g}}\,, \\ V_{\mathrm{p}}\varrho g &= S_1 h \varrho_1 g\,, \\ V_{\mathrm{p}} &= S_1 h \frac{\varrho_1}{\varrho}\,. \end{aligned}$$

We know what part of the cuboid is submerged, and from this, we know that water reaches to a height

$$v = \frac{V_\mathrm{p}}{S_1} = h\frac{\varrho_1}{\varrho}\,.$$

Since water fills the space between the cuboids, its volume must be at least

$$V = v\Delta S = h\frac{\varrho_1}{\varrho}\left(S_2 - S_1\right)\,.$$

Šimon Pajger
legolas@fykos.cz

Problem DF ... fast crash

A particle with rest mass $m_1 = 2m_0$ is flying in the positive direction of the x-axis, with velocity $v_1 = \frac{3}{5}c$. A particle with rest mass $m_2 = 3m_0$ is flying in the opposite direction with velocity $v_2 = -\frac{4}{5}c$. These two particles collide (the collision is inelastic), creating a particle with rest mass m_3 flying with velocity v_3. Find its mass and velocity (including the direction).

Danka was reminiscing about a special relativity course.

In the collision, effects of special theory of relativity are important. We can write the law of conservation of relativistic mass (or energy) of the system

$$\frac{m_1}{\sqrt{1-\frac{v_1^2}{c^2}}} + \frac{m_2}{\sqrt{1-\frac{v_2^2}{c^2}}} = \frac{m_3}{\sqrt{1-\frac{v_3^2}{c^2}}}\,,$$

and also the relativistic momentum conservation principle for the whole system

$$\frac{m_1v_1}{\sqrt{1-\frac{v_1^2}{c^2}}} + \frac{m_2v_2}{\sqrt{1-\frac{v_2^2}{c^2}}} = \frac{m_3v_3}{\sqrt{1-\frac{v_3^2}{c^2}}}\,.$$

We express m_3 from the first equation, substitute it into the second equation and then express v_3 as

$$v_3 = \frac{\frac{m_1v_1}{\sqrt{1-\frac{v_1^2}{c^2}}} + \frac{m_2v_2}{\sqrt{1-\frac{v_2^2}{c^2}}}}{\frac{m_1}{\sqrt{1-\frac{v_1^2}{c^2}}} + \frac{m_2}{\sqrt{1-\frac{v_2^2}{c^2}}}}\,.$$

After simplification, we get

$$v_3 = \frac{m_1v_1\sqrt{c^2-v_2^2} + m_2v_2\sqrt{c^2-v_1^2}}{m_1\sqrt{c^2-v_2^2} + m_2\sqrt{c^2-v_1^2}}\,.$$

We plug in the numerical values and get $v_3 = -\frac{1}{3}c$. The minus sign means that the particle will be flying in the negative direction of the x-axis.

Daniela Pittnerová
daniela@fykos.cz

Problem DG ... air goes through

A dormitory room has temperature $t_{\mathrm{in}} = 15\,{}^\circ\mathrm{C}$. Outside, it is freezing, with temperature $t_{\mathrm{out}} = -5\,{}^\circ\mathrm{C}$. In the room, there are two people, each of them generating power $P_0 = 200\,\mathrm{W}$, a radiator with power $P_1 = 1500\,\mathrm{W}$ and a leaking simple glass window with area $S = 5\,\mathrm{m}^2$ and heat transfer coefficient $\Lambda = 0.73\,\mathrm{W{\cdot}m^{-2}{\cdot}K^{-1}}$. Find the volume V of cold air which flows into the room in one minute. Assume that this air is exchanged for an equivalent volume of "hot" air from the room. The room is perfectly thermally insulated otherwise. *Dodo almost froze.*

The heat flowing through the window is

$$P_2 = \Lambda S\ \Delta t\,,$$

where Δt is the difference between inside and outside temperatures. The power P which flows out of the room due to air flow outside is given by $P + P_2 = 2P_0 + P_1$. The air flow is then given by

$$P = c\dot{m}\Delta t = c\varrho Q_V \Delta t\,,$$

where $c = 1.0\,\mathrm{kJ{\cdot}kg^{-1}{\cdot}K^{-1}}$ is the specific heat capacity of air at constant pressure, $\varrho = 1.28\,\mathrm{kg{\cdot}m^{-3}}$ is the density of air and Q_V is the volumetric flow of air through the imperfectly insulating window. The volume of air which flows out of the room during a time τ can be computed as $V = Q_V\tau$. In total, during $\tau = 60\,\mathrm{s}$, the volume of air which flows out of the room is

$$V = \frac{2P_0 + P_1 - \Lambda S\ \Delta t}{c\varrho\Delta t}\tau \doteq 4.3\,\mathrm{m}^3\,.$$

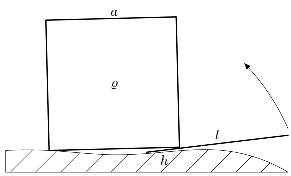
In one minute, approximately $4.3\,\mathrm{m}^3$ of cold air flows into the room.

Jozef Lipták
`liptak.j@fykos.cz`

Problem DH ... lifting by lever

Find the minimum force F necessary to lift a steel cube with edge length $a = 1.5\,\mathrm{m}$ using a crowbar with length $l = 2\,\mathrm{m}$. The crowbar is a straight rod and the part of the crowbar that is inserted under the cube (in the middle of one of the bottom edges of the cube, perpendicularly to this edge) has length $h = 0.1\,\mathrm{m}$. The cube, the crowbar and the floor are rigid.

Dodo likes to save his effort.



From equality of torques at the point where the crowbar touches the cube (the middle of an edge), we can find a formula relating the force F which we're exerting on the crowbar and the force f with which the crowbar is acting on the edge of the cube

$$Fl = fh\,.$$

We're lifting the cube by rotating it around the floor edge opposite to the edge which touches the crowbar. When the torques due to gravity F_g and the force f cancel out, we get

$$F_g\frac{a}{2} = fa\,.$$

The force due to gravity can be expressed using the volume and density of the cube

$$f = \frac{1}{2}\varrho a^3 g\,,$$

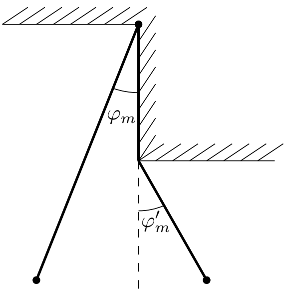
and substituting for the force F, we get

$$F = \frac{1}{2}\varrho a^3 g\frac{h}{l} = 6.5\,\mathrm{kN}\,,$$

so the minimum required force we need to exert on the crowbar is $6.5\,\mathrm{kN}$.

Jozef Lipták
liptak.j@fykos.cz

Problem EA ... asymmetrical pendulum amplitude



There is a pendulum, depicted schematically (with distances and angles not corresponding to this problem) in the figure. When swinging on the left side, the pendulum has length l and angular amplitude φ_m. On the other side, half of the rope is stopped by a wall and the remaining half swings with an angular amplitude φ'_m. If $\varphi_m = 5.0\,^\circ$, what is φ'_m? Assume that it is an ideal mathematical pendulum without energy loss.

Karel likes asymmetrical oscillations.

Since we're implicitly assuming that a pendulum is located in uniform gravity, we know that the point mass at the end of the rope reaches the same height in each extreme point. Let's denote this height by h. Now, we can use two right triangles. From the larger one, we get

$$\cos\varphi_m = \frac{l-h}{l} = 1 - \frac{h}{l}$$

and from the smaller one,

$$\cos\varphi'_m = \frac{\frac{l}{2}-h}{\frac{l}{2}} = 1 - \frac{2h}{l}\,.$$

Expressing h from each equation and putting it together, we get

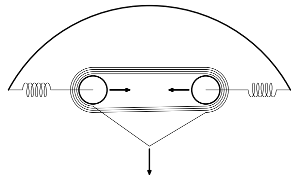
$$\cos\varphi'_m = 2\cos\varphi_m - 1\,, \qquad \Rightarrow \qquad \varphi'_m = \arccos\left(2\cos\varphi_m - 1\right) \doteq 7.07\,^\circ\,.$$

The amplitude on the other side is $7.07\,^\circ$.

Karel Kolář
karel@fykos.cz

Problem EB ... Jachym's bow

Jachym's new compound bow can be approximated as two pulleys with negligible diameters separated by a distance y_0 with a string wound around them in four loops. The string is a closed loop with a constant length $8y_0$. The pulleys can only move along the line passing through their centres and each pulley is connected to the rigid frame of the bow by a spring with a spring constant k. Jachym now pulls one of the threads of the string back, perpendicularly to the line passing through the pulleys. Determine how the force he must exert on the string depends on the drawing distance x. At the beginning, the tension in the springs is zero.



The original version had a longbow. Also, the author was different.

Let's denote the distance between the pulleys by y. We know that initially, $y = y_0$. The total length of the string is $l = 8y_0$. When one of the threads is pulled back by x, the length of this part of the string changes to

$$z = 2\sqrt{x^2 + \left(\frac{y}{2}\right)^2} = \sqrt{4x^2 + y^2}\,.$$

The remaining seven threads remain straight between the pulleys, so the total length of the string is $l = 7y + z$. This length has to remain constant, and from this, we get

$$\begin{aligned} 8y_0 &= 7y + z\,,\\ 0 &= 12y^2 - 28y_0 y + 16y_0^2 - x^2\,,\\ y &= \frac{7y_0 \pm \sqrt{y_0^2 + 3x^2}}{6}\,. \end{aligned}$$

We require $y \leq y_0$, so only the solution with the minus sign is relevant. Each spring exerts a force

$$F_y = \frac{1}{2}\left(y_0 - y\right)k\,.$$

When Jachym exerts a force F_x along a distance $\mathrm{d}x$, the work done is $\mathrm{d}W = F_x \mathrm{d}x$. This is used to change the distance of the pulleys by $\mathrm{d}y$ and therefore to extend the springs by $-\mathrm{d}y$. The change in the energy stored in each spring is

$$\mathrm{d}W_\mathrm{p} = -\frac{1}{2}F_y \mathrm{d}y\,.$$

The total change in the energy of both springs is $\mathrm{d}W = 2\mathrm{d}W_\mathrm{p} = -F_y\mathrm{d}y$. From this, we get

$$F_x = -\frac{\mathrm{d}y}{\mathrm{d}x}F_y = -\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}x}\left(y_0 - y\right)k\,.$$

Now we only need to differentiate y by x, plug this back in and after some manipulation, we get

$$F_x = \frac{1}{24}kx\left(\frac{y_0}{\sqrt{y_0^2 + 3x^2}} - 1\right)\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem EC ... laser

A teacher is illuminating a blackboard with a green laser pointer. The cross-sectional area of the ray is $1cm^2$ and the power of the laser is $1mW$. How many photons are located in one metre of the ray? The wavelength of green light is $530nm$.

Matěj got the idea during a lecture on optics.

Let's denote the time it takes light to travel the distance $l = 1\,\mathrm{m}$ by t. During this time, the laser radiates an energy Pt, where P is its power. This energy must be equal to the energy of all photons NE, where N is the number of photons in this distance and $E = hf = \frac{hc}{\lambda}$ is the energy of one photon. Here, h is Planck's constant, f is the frequency of a photon, λ is its wavelength and c is the speed of light (clearly $t = l/c$). Finally, we obtain

$$\begin{aligned} Pt &= NE\,, \\ P\frac{l}{c} &= N\frac{hc}{\lambda}\,, \\ N &= \frac{P\lambda l}{hc^2} \approx 8{,}89 \cdot 10^6\,. \end{aligned}$$

In one metre, there are almost nine million photons.

Matěj Mezera
m.mezera@fykos.cz

Problem ED ... 1D billiards

Consider an infinite trough and $N = 17$ identical balls which can only move inside the trough. The positions and velocities of the balls may be chosen arbitrarily. What is the maximum possible number of collisions that can occur? All collisions are perfectly elastic and friction inside the trough is negligible.

Matúš was watching a billiards tournament.

Let's take a graph of time-dependent positions of all balls. The movement of each ball forms a line, since they move with uniform velocities. Collisions are represented by intersections of these lines and since the balls are identical, they just swap their velocities during an elastic collision. This means that one ball keeps moving along the trajectory of the other and vice versa, so the lines don't change slopes. The maximum possible number of collisions is the maximum number of intersection points of N lines, which is

$$p = \frac{N\,(N-1)}{2}\,.$$

In our case, $N = 17$, so at most 136 collisions are possible.

Matúš Kopunec
matus.kopunec@fykos.cz

Problem EE ... wailing vibes

Suppose that you look at a particle (a point mass) oscillating at the end of a spring at a random moment in time. What is the probability that the distance of this point from the equilibrium position does not exceed $y_1 = 1.0\,\mathrm{cm}$? The amplitude of the particle's oscillations is $y_\mathrm{m} = 3.0\,\mathrm{cm}$ and there is no damping force.

Karel was staring at springs.

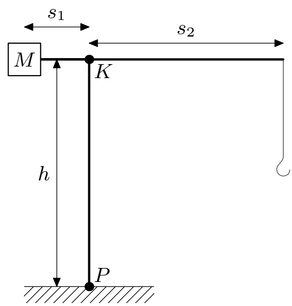
The system we're considering is a linear harmonic oscillator and the position of the point mass can be described as $y(t) = y_\mathrm{m} \sin(\omega t + \varphi_0)$. The phase constant φ_0 has no effect on the probability we are interested in, so we can assume $\varphi_0 = 0$. To further simplify the expression above, let's define $\alpha = \omega t$.

As for the movement itself, now described by $y(t) = y_\mathrm{m} \sin \alpha$, it is 2π-periodic. Its absolute value, though, is π-periodic and of greater interest to us, since we consider only the distance from the equilibrium position. Furthermore, this absolute value is axisymmetric with respect to $x = \pi/2$. This means that we can consider just the interval $[0, \pi/2]$. The total time corresponding to this interval is proportional to the total angle $\pi/2$. The time during which the position of the particle is below y_1 corresponds to the part between time/angle zero and the point where $y_1 = y_\mathrm{m} \sin \alpha$, or $\sin \alpha = 1/3 \quad \Rightarrow \quad \alpha = 0.340$. The probability of the distance from equlibrium being smaller than y_1 is then $P = \alpha/(\pi/2) \doteq 21.6\,\%$. It is more likely for the particle to be at a point more distant from the equilibrium position than the given distance.

Karel Kolář
karel@fykos.cz

Problem EF ... crane driver

A crane driver forgot how much weight his crane can lift up. The height of the crane is $h = 80\,\mathrm{m}$ and a counterweight with mass $M = 10\,\mathrm{t}$ is positioned at a distance $s_1 = 10\,\mathrm{m}$ from the cabin (point K). A hook (ignore its weight) is hanging freely $s_2 = 60\,\mathrm{m}$ from the cabin. The linear density of the whole construction is $\lambda = 10\,\mathrm{kg{\cdot}m^{-1}}$. The crane driver knows that the weakest points of the crane are its foot (point P), where it is anchored to the ground, and its joint (point K). He does not remember the critical magnitude of torque which can act at this point (in either direction), but he knows that after exceeding this value, the crane would collapse. How much weight can the crane driver certainly lift up?



Matěj's dream job.

When there is no additional weight on the crane, a torque τ is acting on the foot. After lifting up a sufficient weight, the final torque starts acting in the other direction (and therefore pulls the crane to the right). Everything is ok while the magnitude of this new torque doesn't exceed τ. Obviously, there is no difference between the torques at the points K and P.

The mass of the whole arm of the crane is $\lambda\,(s_1 + s_2)$ and its center of gravity is lying $(s_2 - s_1)/2$ to the right from the point K. The arm acts with a torque $-\lambda\,(s_1 + s_2)\,(s_2 - s_1)/2$ (against the torque of the weight, Ms_1). We have

$$\tau = Ms_1 - \frac{\lambda\,(s_1 + s_2)\,(S_2 - s_1)}{2}\,.$$

The resulting torque is $-\tau$ when the weight itself causes torque 2τ. Let m be the maximum mass of the weight.

$$\begin{aligned} ms_2 &= 2\tau\,,\\ m &= \frac{2Ms_1}{s_2} - \frac{\lambda}{s_2}\left(s_2^2 - s_1^2\right) = 2{,}750\,\mathrm{kg}\,. \end{aligned}$$

Thus the maximum allowed mass of the weight is 2,750 kg.

Matěj Mezera
m.mezera@fykos.cz

Problem EG ... Christmas snowman reloaded

Consider a large snowball with radius R and acceleration due to gravity at its surface equal to g. Let's use this snowball as the base of a snowman, which is made up of an infinite number of snowballs with the same density, whose radii form an infinite geometric sequence with a coefficient (common ratio, quotient) 1/2. Adjacent snowballs always touch and the centres of all snowballs are collinear. Compute the intensity of gravitational field at the top of this snowman. The snowman is not located in any external gravitational field.

Matúš wanted to build a really large snowman.

From Newton's gravitational law, we know that intensity of gravitational field is proportional to the first power of length, so if all lengths are halved, the intensity is also halved. This halved snowman is identical to the original snowman without the first snowball, which gives us an equation

$$g' = g_1 + \frac{g'}{2}\,;$$

g_1 is the intensity due to the first snowball. Since the radii of snowballs form a geometric sequence, the height of the snowman is

$$h = \frac{2R}{1-\frac{1}{2}} = 4R\,.$$

The distance of the center of the first snowball from the top is $3R$, so the intensity due to this snowball at the top is nine times smaller than at the surface of this snowball, and substituting in the formula above, we get

$$g' = \frac{g}{9} + \frac{g'}{2} = \frac{2}{9}g\,.$$

Matúš Kopunec
matus.kopunec@fykos.cz

Problem EH ... lunar mining

Once upon a time mankind finally managed to build a lunar base. Instead of scientific research, however, they started plundering the Moon when they discovered an invaluable material named Fykosium, with density $\varrho = 5{,}000\,\mathrm{kg\cdot m^{-3}}$, and started mining it homogeneously from the surface of the Moon. The mining operations were so intensive that the radius of the Moon decreased and it wasn't possible to watch a full solar eclipse from the surface of Earth anymore. Calculate the mass of Fykosium that was mined out. The semi-major axis of the Moon's orbit is $a = 384{,}400\,\mathrm{km}$ and its eccentricity is $e = 0,0549$. Assume that the orbit of the Earth around the Sun is a circle with radius $r = 1.496 \cdot 10^{11}\,\mathrm{m}$. The Sun has radius $R_\mathrm{S} = 6.96 \cdot 10^{8}\,\mathrm{m}$, the equatorial radius of Earth is $R_\mathrm{Z} = 6{,}378\,\mathrm{km}$, the radius of the Moon is $R_\mathrm{M} = 1{,}738\,\mathrm{km}$. Assume that $R_\mathrm{Z} \ll r$ and that all the Fykosium that was mined out remains on an orbit around the Moon, so the orbit of the Moon around the Earth has not changed.

Jáchym and Jirka were thinking about the future of mankind.

From similarity of triangles, we get

$$\frac{R_\mathrm{S}}{r} = \frac{R}{a(1-e) - R_\mathrm{Z}}\,,$$

where R is the radius of the Moon after Fykosium is mined out. Therefore,

$$R = \frac{R_\mathrm{S}}{r}\left(a(1-e) - R_\mathrm{Z}\right) \doteq 1660.5\,\mathrm{km}\,.$$

The volume V and mass m that was mined out can then be computed as

$$V = \frac{4}{3}\pi(R_\mathrm{M}^3 - R^3)\,,$$
$$m = V\varrho = \frac{4}{3}\pi\varrho(R_\mathrm{M}^3 - R^3) \doteq 1.41 \cdot 10^{22}\,\mathrm{kg}\,.$$

The mass of Fykosium that was mined out is approximately $1.41 \cdot 10^{22}$ kg.

Jáchym Bareš
`bares@fykos.cz`

Problem FA ... Christmas snowman

Matúš built a snowman in the following way: He made a snowball with radius R. Then, he made a second snowball with radius $R/2$ and placed it on top of the first snowball. On top of the second snowball, he put a third one with radius $R/4$, and so on, placing an infinite number of balls on top of each other, always with a halved radius. All snowballs have the same density. At what height above the ground is the centre of mass of the resulting snowman?

Matúš wanted to build a snowman.

Let's compute the position of the centre of mass as a weighted average of positions of centres of mass of individual balls, where the weights are masses of the balls, i.e. the position of the centre of mass x_T satisfies

$$Mx_T = \sum_{i=1}^{\infty} m_i x_i\,,$$

where M is the mass of the snowman. The origin of the coordinate system can be chosen arbitrarily, so let's choose it in the centre of the first ball, so that the first term in the sum would be zero. The rest of the sum can be rewritten as

$$Mx_T = m_1 \cdot 0 + \sum_{i=2}^{\infty} m_i x_i = M' x'_T\,,$$

where M' and x'_T are the mass and position of the center of mass of the snowman without the first ball, which is identical to the original snowman, but with halved dimensions. Since mass is proportional to the cube of radius and the position of the centre of mass is proportional to radius, we get $M' = M/8$ and $x'_T = x_T/2 + 3R/2$ ($x_T/2$ is the distance from the centre of the second ball). Substituting for M' and x'_T, we get

$$Mx_T = \frac{M}{8} \cdot \frac{3R + x_T}{2}\,,$$
$$x_T = \frac{R}{5}\,,$$

but x_T is the distance of the centre of mass from the centre of the first ball, which has radius R, so the height of the centre of mass above the ground is

$$h_T = \frac{6R}{5}\,.$$

Matúš Kopunec
matus.kopunec@fykos.cz

Problem FB ... slide farther

You are standing on a horizontal plane and throwing an object from zero height. What is the optimal angle of your throw (with respect to the horizontal plane) if you want the object to come to rest after sliding as far as possible? The friction coefficient between the thrown object and the ground is $f = \sqrt{3}/4$, the collision of the thrown object with the ground is perfectly inelastic and it does not rotate.

Matěj was imagining throwing snowballs, since there was no snow :,(.

At first, the object is flying on a parabolic trajectory. Then, it collides with the ground and loses the vertical component of its velocity; the horizontal component remains the same. Then, its velocity decreases due to friction until it comes to rest. Let's assume that we're throwing with a velocity v under an angle φ with respect to the horizontal plane. A well-known formula for the motion of a projectile gives us the distance at which it collides with the ground

$$s_1 = \frac{2v^2}{g}\sin\varphi\cos\varphi\,.$$

After the collision, the object is moving horizontally with velocity $v_x = v\cos\varphi$. There is a uniform frictional deceleration $a_\mathrm{t} = fg$ acting on it. The time of sliding is

$$t = \frac{v_x}{a_\mathrm{t}} = \frac{v_x}{fg}\,.$$

During this time, the object crosses a distance

$$s_2 = v_x t - \frac{1}{2}a_\mathrm{t}t^2 = \frac{v_x^2}{2fg} = \frac{v^2}{2fg}\cos^2\varphi\,.$$

It comes to rest at a distance

$$s = s_1 + s_2 = \frac{2v^2}{g}\sin\varphi\cos\varphi + \frac{v^2}{2fg}\cos^2\varphi\,.$$

We're looking for the maximum of the function $s(\varphi)$. Let's compute the first derivative and solve for φ which makes it zero

$$\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}\varphi} &= \frac{v^2}{g}\left(2\cos^2\varphi - 2\sin^2\varphi - \frac{1}{f}\cos\varphi\sin\varphi\right)\,,\\
0 &= 2f\left(\cos^2\varphi - \sin^2\varphi\right) - \cos\varphi\sin\varphi\,,\\
0 &= 2f\cos 2\varphi - \frac{1}{2}\sin 2\varphi\,,\\
\varphi &= \frac{1}{2}\arctan 4f\,.
\end{aligned}$$

The total distance s is minimum at the endpoints of the interval of φ we consider, i.e. for $\varphi = 0\,^\circ$ and $\varphi = 90\,^\circ$. The extremum we found therefore has to be the maximum we're looking for. Substituting $f = \sqrt{3}/4$, we get $\varphi = 30\,^\circ$.

Matěj Mezera
m.mezera@fykos.cz

Problem FC ... bouncing pendulum

There is a pendulum formed by a steel ball (radius $r = 0.5\,\mathrm{cm}$, specific heat capacity $c = 452\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$, density $\varrho = 7{,}850\,\mathrm{kg{\cdot}m^{-3}}$) attached to a wall by a rope with length $l = 1.0\,\mathrm{m}$, so the ball always bounces off the wall in the bottommost point of its trajectory. During a bounce, the ball loses energy and heats up. The velocity with which the ball bounces off is given by the coefficient of restitution $k = v_1/v_0 = 0.90$ (v_0 and v_1 are the velocities before and after the bounce, respectively). We release the pendulum from an initial position given by a displacement angle $\alpha = 45^\circ$. How much higher is the temperature of the ball after three bounces? Assume that half of the energy lost during bounces is used to heat the ball.

Karel was bashing his head against a wall.

It turns out that the size and mass of the ball aren't relevant parameters, but we'll calculate and use it anyway for clarity.

First, let's compute the initial mechanical energy of the pendulum. This energy corresponds to the potential energy of the pendulum in the initial position, which later transfers into kinetic energy and starts getting dissipated. The initial energy is

$$E_0 = mgl\left(1 - \sin\alpha\right) \doteq 0.0118\,\mathrm{J}\,,$$

where we substituted for the mass $m = \varrho V = 4\pi\varrho r^3/3 \doteq 4.11\,\mathrm{g}$.

The restitution coefficient is defined as a ratio of velocities. Since kinetic energy is proportional to squared velocity,[1] the ratio of energy after the collision to energy before it is

$$K = \frac{E_1}{E_0} = \frac{v_1^2}{v_0^2} = k^2\,.$$

This way, we find out that the mechanical energy of the pendulum after two bounces is $E_2 = KE_1 = k^2E_1 = k^4E_0$ and after three bounces, it's $E_3 = k^6E_0$. The total mechanical energy dissipated into other forms of energy (heat, sound, etc.) is

$$\Delta E = E_0 - E_3 = \left(1 - k^6\right)E_0 = mgl\left(1 - \sin\alpha\right)\left(1 - k^6\right) \doteq 5.53\,\mathrm{mJ}\,.$$

Half of this energy, $\Delta E/2 = Q \doteq 2.77\,\mathrm{mJ}$, is spent on heating the ball. The temperature increase can be found using the formula $Q = mc\Delta T$. Substituting for the already computed heat Q, we get

$$\frac{\Delta E}{2} = mc\Delta T \qquad \Rightarrow \qquad \Delta T = \frac{\left(1 - k^6\right)E_0}{2mc} = \frac{1 - k^6}{2}\frac{gl}{c}\left(1 - \sin\alpha\right) \doteq 1.5\,\mathrm{mK}\,.$$

The ball heats up by a minuscule 1.5 mK.

Karel Kolář
karel@fykos.cz

[1] Indeed if we considered rotational energy, the result is still proportional to squared velocity, because the rotational energy is proportional to the squared angular velocity which is proportional to the velocity in this case.

Problem FD ... capacitors

Initially, there is an LC circuit composed of one coil and one parallel-plate capacitor. The distance between the plates of the capacitor is $d_0 = 4.0\,\mathrm{mm}$*, the surface area is* $S = 500\,\mathrm{cm}^2$ *and the inductance of the coil is* $L = 20\,\mathrm{mH}$*. After some time, we insert a conducting plate with thickness* $r = 0.5\,\mathrm{mm}$ *inside the capacitor exactly in the middle between its plates. How does the resonance frequency of the circuit change? Express the ratio of new to old frequency.*

The initial capacitance of the capacitor is

$$C_0 = \frac{\varepsilon S}{d_0}\,,$$

where ε is the permittivity of the medium inside it. If we insert a conducting plate in the middle, we effectively split it into two smaller capacitors. The distance between plates in each of them is

$$d = \frac{d_0 - r}{2}\,,$$

so each of them has capacitance

$$C' = \frac{2\varepsilon S}{d_0 - r}\,.$$

The new capacitors are connected in series. Their capacitances combine in the same way as resistances of parallel resistors, so the resulting capacitance is

$$C = \frac{C'^2}{C' + C'} = \frac{1}{2}C' = \frac{\varepsilon S}{d_0 - r}\,.$$

In the original LC circuit, the current flowing through both elements is identical, while the voltages on them are opposite to each other. This leads to a differential equation

$$\ddot{I} + \frac{1}{LC_0} I = 0\,.$$

Its solutions are some complex exponentials, in general

$$I = A\mathrm{e}^{\mathrm{i}\omega_0 t} + B\mathrm{e}^{-\mathrm{i}\omega_0 t}\,,$$

where A, B are constants and ω_0 is the angular frequency

$$\omega_0 = \frac{1}{\sqrt{LC_0}}\,.$$

The situation after the plate is inserted gives the same equations, we only need to use the capacitance C computed above instead of C_0. The resulting ratio of frequencies is

$$\frac{f}{f_0} = \frac{\omega}{\omega_0} = \sqrt{\frac{C_0}{C}} = \sqrt{1 - \frac{r}{d_0}} = \sqrt{\frac{7}{8}} \doteq 0.94\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem FE ... Christmas snowman reloaded reloaded

Consider a large snowball with radius R. Let's use this snowball as the base of a snowman, which is made up of an infinite number of snowballs with the same density, whose radii form an infinite geometric sequence with a coefficient (common ratio, quotient) $1/2$. Adjacent snowballs always touch and the centres of all snowballs are collinear. Compute the moment of inertia of the snowman with respect to an axis passing through the top of the snowman and perpendicular to the line passing through the centres of all snowballs. The total mass of the whole snowman is M. *Matúš really really likes snowmen.*

The moment of inertia will be in the form $I = kMR^2$, where k is a dimensionless constant. The mass of the snowman is proportional to the cube of R, so the moment of inertia is proportional to R^5. Let's split the moment of inertia into two parts: the moment of inertia of the largest snowball I_o and the moment of inertia of the remaining snowballs I_z. The resulting moment of inertia is their sum

$$I = I_o + I_z\,.$$

The moment of inertia of the largest snowball can be computed using Steiner's theorem

$$I_o = \frac{2}{5}mR^2 + md^2$$

where d is the distance of center of the largest snowball from the top and m is its mass.

The height of the snowman is the sum of the diameter of the largest snowball and the height of the rest of the snowman. You may notice that the rest of the snowman is identical to the original snowman with all dimensions halved, so its height is also half of the original snowman's height. The diameter of the largest ball is $2R$, which gives us the height of the snowman $4R$ and the distance of the centre of the largest snowball from the top $d = 3R$.

Since the dimensions of the remaining part of the snowman are halved compared to the original and the moment of inertia is proportional to R^5, the moment of inertia of the rest of the snowman is 32 times smaller than the moment of inertia of the original snowman. Since mass is proportional to R^3, the mass of the rest is $1/8$ of the whole snowman's mass, so the mass of the largest snowball is $m = 7/8M$. Substituting everything in the first equation, we get

$$\begin{aligned} I &= \frac{2}{5}\cdot\frac{7}{8}MR^2 + \frac{7}{8}M\left(3R\right)^2 + \frac{I}{32}\,,\\ \frac{31}{32}I &= \frac{329}{40}MR^2\,,\\ I &= \frac{1\,316}{155}MR^2\,. \end{aligned}$$

We may notice that the largest snowball contributes the most by far to the moment of inertia - if we assumed that the whole mass of the snowman is at its center, we'd get $I \approx 9MR^2$, while $k \doteq 8.49$.

Matúš Kopunec
`matus.kopunec@fykos.cz`

Problem FF ... analytical bouncing ball

Vašek likes playing with bouncing balls; specifically, he's interested in the restitution coefficient. He decided to test a bouncing ball on a larger scale, so he threw it from the 15-th floor of the *dorm, at a height $h_1 = 50\,\mathrm{m}$ above the ground, under a randomly chosen angle. At the 20-th floor (height $h_2 = 65\,\mathrm{m}$), however, there is the dorm supervisor, who most definitely would not like to see someone throwing things from windows. Since the dorm supervisor does not lean out of the window, he only sees outside under the angle $\alpha = 60^\circ$ downwards with respect to the horizontal plane. What is the maximum velocity with which Vašek can throw the bouncing ball so that it is guaranteed that the supervisor cannot see it at any point of its trajectory (before it impacts the ground)?*

Consider a coordinate system such that the y-axis corresponds to the wall and the x-axis corresponds to the ground. The supervisor's line of sight is given by the formula $y = h_2 - x\tan\alpha$. Next, we need to determine the trajectory of the ball depending on the initial velocity v and the angle φ under which it's thrown. Let's define the angle φ as the angle between the initial velocity vector and the horizontal plane in the same way as α (positive when throwing down). Then, the ball follows a regular ballistic trajectory with coordinates at time t after the throw given by

$$x = tv\cos\varphi\,,$$
$$y = h_1 - tv\sin\varphi - \frac{1}{2}gt^2\,.$$

We don't need the time dependence at all, only the relation between y and x, so let's express t from the first formula and plug it in the second formula

$$y = h_1 - x\tan\varphi - \frac{gx^2}{2v^2\cos^2\varphi}\,.$$

Now we can find the intersection of the bouncing ball's trajectory and the dorm supervisor's line of sight. Mathematically, we're looking for x such that both curves' y are equal

$$h_2 - x\tan\alpha = h_1 - x\tan\varphi - \frac{gx^2}{2v^2\cos^2\varphi}\,,$$
$$0 = x^2\frac{g}{2v^2\cos^2\varphi} + x\left(\tan\varphi - \tan\alpha\right) + h_2 - h_1\,.$$

This is a quadratic equation. If it has a positive discriminant, it means that the bouncing ball crosses the supervisor's line of sight twice, which we want to avoid. The critical case is when the discriminant is 0, which is when the ball only touches the line of sight.

Let's denote the velocity which leads to discriminant 0 for a fixed φ by $v_0(\varphi)$. We obtain an equation

$$\left(\tan\varphi - \tan\alpha\right)^2 - 4\frac{g}{2v_0(\varphi)^2\cos^2\varphi}\left(h_2 - h_1\right) = 0\,.$$

Next, let's express

$$v_0(\varphi)^2 = \frac{2g\left(h_2 - h_1\right)}{\cos^2\varphi\left(\tan\varphi - \tan\alpha\right)^2}\,. \tag{1}$$

We're supposed to find the maximum v such that for each φ, the ball isn't seen by the supervisor. This velocity is the minimum of $v_0(\varphi)$.

The right hand side of (1) is always non-negative, so if we want to minimise $v_0(\varphi)$, we only need to minimise its square. In addition, the numerator doesn't depend on φ, so we only need to maximise the denominator. If we require its derivative with respect to φ to be 0, we get

$$\left(\tan\varphi - \tan\alpha\right)\left(\left(\tan\varphi - \tan\alpha\right)\cos\varphi\sin\varphi - 1\right) = 0\,.$$

One solution is clearly $\varphi = \alpha$, which means the denominator in (1) is 0. Its physical meaning is throwing in parallel with the dorm supervisor's line of sight (or more vertically), in which case any velocity is allowed. However, this case isn't interesting for us, so the second parenthesised expression must be zero. Further simplifications give us a condition

$$\cos\varphi\,(\tan\alpha\sin\varphi + \cos\varphi) = 0\,.$$

For $\cos\varphi = 0$, we get two possible angles: $90\,^\circ$ (vertically down), which again isn't interesting, and -90° (vertically up). It's easy to compute that $v_0(-90^\circ) = \sqrt{2g\,(h_2 - h_1)}$.

The condition $\tan\alpha\sin\varphi + \cos\varphi = 0$ can be again simplified to

$$\tan\varphi = -\cot\alpha\,,$$

which gives $\varphi = -30\,^\circ$ (throwing upwards under an angle $30\,^\circ$). If we plug this angle into (1), we get

$$v_0(-30^\circ) = \sqrt{\frac{2g\,(h_2 - h_1)}{\tan^2\alpha + 1}} = \sqrt{\frac{1}{2}g\,(h_2 - h_1)}\,.$$

This is less than $v_0(-90^\circ)$, so it's the velocity we were looking for.

For the given numerical values, we get $v_0(-30^\circ) = 8.58\,\mathrm{m{\cdot}s^{-1}}$.

Šimon Pajger
`legolas@fykos.cz`

Problem FG ... ice hockey sticks

Jirka and Jachym wanted to compete in ice hockey puck shooting. Both of them were shooting from the middle of a rink, at a distance $l = 19.50\,\mathrm{m}$ from a hockey goal. Jirka decided to shoot the puck along the surface of the ice into the middle of the goal with velocity $v_0 = 20.00\,\mathrm{m{\cdot}s^{-1}}$, while Jachym was shooting into the left gibbet (= the top left edge of the goal) with velocity u. By total accident, both of them scored a goal at the same time t from shooting. Calculate the velocity u of Jachym's shot. The dimensions of the hockey goal are $1.830\,\mathrm{m} \times 1.220\,\mathrm{m}$ (width $\times$ height), the friction coefficient of a puck on ice is $f = 0.150$ and the mass of the puck is $m = 170.0\,\mathrm{g}$. Calculate the result in $\mathrm{m{\cdot}s^{-1}}$ to 4 significant figures.

Jirka was watching ice hockey.

The friction force acting on a hockey puck with mass m sliding on ice is $F_\mathrm{t} = fmg$. Therefore, it is moving with acceleration $a = -fg$ and if the initial velocity is v_0, we're solving an equation

$$l = v_0 t + \frac{1}{2}at^2 = v_0 t - \frac{1}{2}fgt^2\,.$$

We express the time from shooting the puck to striking a goal from the quadratic equation and get

$$t = \frac{v_0 \pm \sqrt{v_0^2 - 2fgl}}{fg}\,.$$

When we substitute $l = 0$, the time is equal to zero only for the root with the minus sign, so we consider only

$$t = \frac{v_0 - \sqrt{v_0^2 - 2fgl}}{fg} \doteq 1.012{,}7\,\mathrm{s}\,.$$

The trajectory of the puck satisfies

$$x = u\ t\cos\alpha\,,$$
$$y = u\ t\sin\alpha - \frac{1}{2}gt^2\,,$$

where α is an angle between the vector of velocity and the horizontal plane, and y is the height of the hockey goal. The horizontal distance is

$$x = \sqrt{l^2 + \left(\frac{b}{2}\right)^2} \doteq 19.52\,\mathrm{m}\,,$$

where b is the width of the goal. After substituting for u from the first equation, we get

$$\tan\alpha = \frac{y + \frac{1}{2}gt^2}{x}\,,$$
$$\alpha \doteq 17.75\,^\circ\,.$$

Now we can express the velocity from the first equation

$$u = \frac{x}{t\cos\alpha} \doteq 20.24\,\mathrm{m{\cdot}s^{-1}}\,.$$

We may notice that the mass of the puck wasn't needed for solving the problem.

Jiří Vala
j.vala@fykos.cz

Problem FH ... flat-Earth reloaded reloaded

It is well-known that the Earth is flat and supported by elephants. Above it, there is a dome from which the Sun, the Moon and stars are hanging. If the Moon with mass $7.348\cdot 10^{22}$ kg was hanging from a steel rope with a (not necessarily uniform) circular cross-section, what would be the radius of its cross-section at the point of contact with the dome? The length of the rope is $l = 3{,}200\,\mathrm{m}$, the yield stress of steel is $700\,\mathrm{MPa}$ and its density is $7{,}850\,\mathrm{kg{\cdot}m^{-3}}$. The Moon is placed in uniform gravity g. Neglect all other gravitational effects.

Jáchym has a weakness for reloaded reloaded problems.

At the bottommost point, the force acting on the rope is just caused by gravity of the Moon mg, where m is the mass of the Moon. At this point, the radius of the rope is

$$r_0 = \sqrt{\frac{mg}{\pi\sigma}}\,,$$

where σ is the yield stress.

Let's denote the radius of the rope at distance x from this point by $r(x)$. If we move upwards by a small distance $\mathrm{d}x$, the mass of the rope underneath us increases by $\mathrm{d}m = \pi r^2\varrho\mathrm{d}x$. If the radius increases by $\mathrm{d}r$, the cross-sectional area increases by $\mathrm{d}S = 2\pi r\mathrm{d}r$. Now we only need to realise that increasing the mass also increases the force which the rope needs to bear by $g\mathrm{d}m$, while increasing the cross-sectional area increases the force the rope can bear by $\sigma \mathrm{d}S$. These two expressions need to be equal, so we get

$$g\mathrm{d}m = \sigma \mathrm{d}S \ ,$$

$$\frac{\varrho g}{2\sigma}\mathrm{d}x = \frac{\mathrm{d}r}{r} \,.$$

Integrating both sides, we obtain a solution

$$\int_0^x \frac{\varrho g}{2\sigma}\mathrm{d}x = \int_{r_0}^r \frac{\mathrm{d}r}{r} \,,$$

$$\frac{\varrho g}{2\sigma}x = \ln\left(\frac{r}{r_0}\right) = \ln\left(r\sqrt{\frac{\pi\sigma}{mg}}\right) \,.$$

Now it's easy to express

$$r(x) = \sqrt{\frac{mg}{\pi\sigma}}\exp\left(\frac{\varrho g}{2\sigma}x\right) \,.$$

We're interested in radius at height l, which is

$$r(l) = \sqrt{\frac{mg}{\pi\sigma}}\exp\left(\frac{\varrho g}{2\sigma}l\right) \doteq 2.16 \cdot 10^7 \,\mathrm{m} \,.$$

As expected, the radius we got is smaller than in the previous case, but it still doesn't make any sense.

Jáchym Bártík
tuaki@fykos.cz

Problem GA ... centrifuge

We pour water into a hollow cylinder with base radius R to a height h_0. Then, we spin the cylinder up to a constant angular velocity. Determine the angular velocity at which the water level in the centre just touches the bottom of the cylinder.

Matúš wanted to dry the centre of a bucket.

The water level follows an equipotential surface created by the gravitational potential $\varphi_g = gh$ and the potential due to the centrifugal force, which can be determined by integrating the centrifugal acceleration over distance from the centre

$$\varphi_\mathrm{o} = -\int_0^r a_\mathrm{o}\mathrm{d}r = -\int_0^r \omega^2 r\mathrm{d}r = -\frac{1}{2}\omega^2 r^2 \,,$$

where ω is the angular velocity. The sum of both potentials must be constant at the water surface, thus

$$gh - \frac{1}{2}\omega^2 r^2 = k \,.$$

We can use this to express the water level as a function of the distance from the centre r. Water is incompressible, so the volume of water in the container is constant. We can use this to determine the value of k. The initial volume was $V = \pi R^2 h_0$. We are interested in the case where $h \geq 0$ (with the equality for $r = 0$), so we only need to integrate

$$V = \int_0^R \mathrm{d}V = \int_0^R 2\pi r \mathrm{d}rh = \int_0^R \frac{2\pi r}{g}\left(k + \frac{1}{2}\omega^2 r^2\right)\mathrm{d}r = \frac{\pi R^2}{g}\left(k + \frac{1}{4}\omega^2 R^2\right) .$$

Comparing this to the original volume, we get

$$k = gh_0 - \frac{1}{4}\omega^2 R^2 ,$$
$$h = h_0 + \frac{\omega^2}{4g}\left(2r^2 - R^2\right) .$$

Using $r = 0$, $h(r = 0) = 0$, we easily find the required angular velocity

$$\omega = \frac{2}{R}\sqrt{gh_0} .$$

Matúš Kopunec
`matus.kopunec@fykos.cz`

Jáchym Bártík
`tuaki@fykos.cz`

Problem GB ... drinking tea 2.0

Consider a cylindrical container with height H and radius R, filled with tea with density ϱ. There is a little hole with area $s \ll S$ at the bottom of the container. This hole is used to attach a horizontal tube with a faucet and a manometer (a vertical tube, open at the top) to the container. Both tubes have radii r. The distance between the side of the container and the manometer is a, the distance between the manometer and faucet (the end of the horizontal tube) is b. Find the formula describing the height of liquid in the manometer $l(t)$ over time after the faucet is opened. Assume that this height adapts instantly to changes in flow in the horizontal tube. The tea is incompressible with dynamic viscosity η and surface tension σ. Assume laminar flow in the tube and $r \ll a, b \ll H, R$.

Hint In case of laminar flow, the pressure difference Δp between the ends of a tube satisfies Poiseuille's equation

$$\Delta p = \frac{8\eta l Q}{\pi R^4} ,$$

where R is the diameter of the tube, l its length and Q the volumetric flow rate.

Dodo chronically thinks in the canteen instead of eating.

From the Poiseuille equation, using the hydrostatic pressure at the bottom of the container $H\varrho g$ as the pressure difference between the start and the end of the tube, we get

$$H\varrho g = \frac{8\eta\,(a + b)\,Q}{\pi r^4} .$$

From the continuity equation, we know that this flow rate must correspond to a decrease in the tea level in the container.

$$Q = -\pi R^2 \dot{H} .$$

Plugging this into the previous equation, we get a simple separable differential equation

$$\dot{H} = -\frac{\varrho g r^4}{8\eta\,(a+b)\,R^2} H\,,$$

with a solution

$$H(t) = H_0 \exp\left(-\frac{\varrho g r^4}{8\eta\,(a+b)\,R^2} t\right)\,.$$

The pressure difference (with respect to atmospheric pressure) decreases linearly in the tube from $h\varrho g$ to zero, so at the manometer,

$$\Delta p_b = \frac{b}{a+b}\Delta p\,.$$

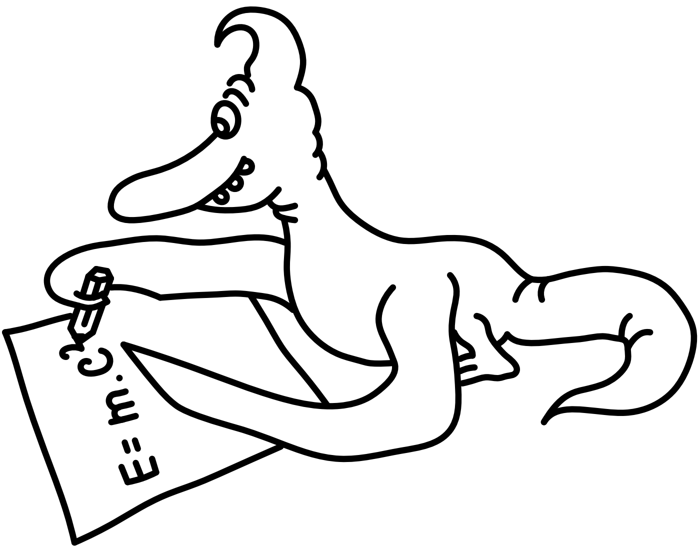
This pressure is further increased by capillary pressure

$$p_\mathrm{k} = \frac{2\sigma}{r}\,.$$

For the level of tea in the tube as a function of time, we get

$$h(t) = \frac{1}{\varrho g}\left(p_\mathrm{k} + \Delta p_b\right) = \frac{2\sigma}{r\varrho g} + \frac{b}{a+b} H_0 \exp\left(-\frac{\varrho g r^4}{8\eta\,(a+b)\,R^2} t\right)\,.$$

Jáchym Bártík
tuaki@fykos.cz



FYKOS
UK, Matematicko-fyzikální fakulta
Ústav teoretické fyziky
V Holešovičkách 2
180 00 Praha 8

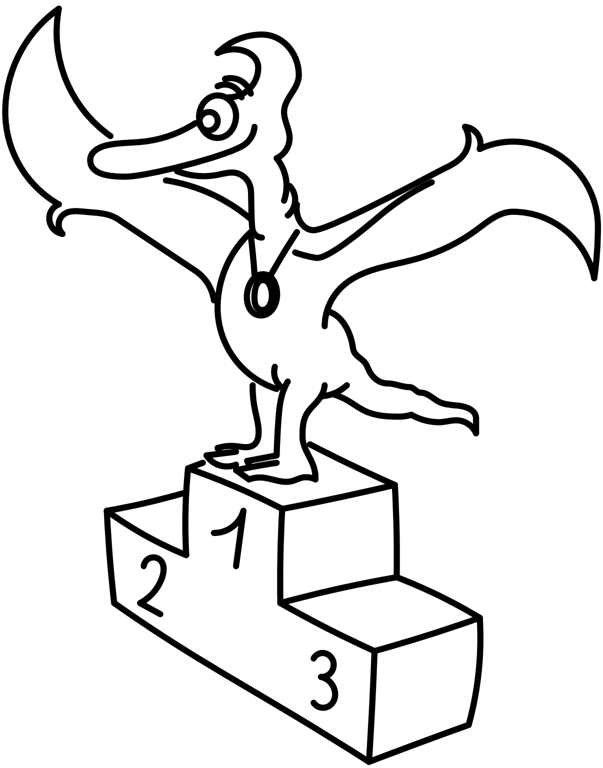
www: `http://fykos.cz`
e-mail: `fykos@fykos.cz`

FYKOS is also on Facebook
http://www.facebook.com/Fykos

FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Public Relations Office activities and is supported by Institute of Theoretical Physics, MFF UK, his employees and The Union of Czech Mathematicians and Physicists.

Solutions of problems

fyziklani2020



Problem AA ... cold lemonade

The FYKOS bird wanted to cool down a bottle of the Bohemsca lemonade, so he dipped it into a well. What volume would he have to drink beforehand in order to make the bottle float? The total external volume of the lemonade bottle is $0.420l$ *and the maximum volume of a liquid that can fit inside it is* $0.336\,\mathrm{l}$*. The original volume of lemonade in a filled bottle is written on the bottle. The density of glass is* 2.5 *times greater than the density of water.*

Jáchym became thirsty while digging a well.

Let's denote the volumes described in the problem statement by V_e, V_i, $V_\mathrm{l} = 330\,\mathrm{ml}$ respectively, and also denote $k = \varrho_\mathrm{g}/\varrho_\mathrm{w} = 2.5$. The bottle floats only if its mean density is equal to that of water. The density of lemonade is almost the same as the density of water, so we only need to consider the rest - the glass and air in the bottle. The volume of glass and air together is

$$V = V_\mathrm{g} + V_\mathrm{i} - (V_\mathrm{l} - \Delta V) \;,$$

where V_g is the volume of glass and ΔV is the volume of lemonade that the FYKOS bird must have drunk. The mass of the air is insignificant, so the total mass of glass and air is $m = m_\mathrm{g} = V_\mathrm{g}\varrho_\mathrm{g}$. The bottle floats if

$$\varrho_\mathrm{w} = \frac{m}{V} = \frac{V_\mathrm{g}\varrho_\mathrm{g}}{V_\mathrm{g} + V_\mathrm{i} - (V_\mathrm{l} - \Delta V)} \,.$$

Finally, after substituting for the volume of the glass $V_\mathrm{g} = V_\mathrm{e} - V_\mathrm{i}$, we get the volume of lemonade the FYKOS bird must drink as

$$\Delta V = k\,(V_\mathrm{e} - V_\mathrm{i}) + V_\mathrm{l} - V_\mathrm{e} = 120\,\mathrm{ml}\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem AB ... ropey

A new rope with a length $l_0 = 60.0\,\mathrm{m}$ *has a diameter* $d_0 = 9.40\,\mathrm{mm}$*. After some use, its diameter changes to* $d = 10.10\,\mathrm{mm}$*. Find out how much the rope shortens, assuming that its volume stays constant.*

Dodo has a new rope.

For the volume of the rope, we can use the formula for a cylinder

$$V = \frac{\pi l_0 d_0^2}{4} = \frac{\pi l d^2}{4} \,,$$

where l is the length of the used rope. Now we express this new length of the rope and calculate the difference

$$\Delta l = l - l_0 = \left(\frac{d_0}{d}\right)^2 l_0 - l_0 = \frac{d_0^2 - d^2}{d^2} l_0 \,.$$

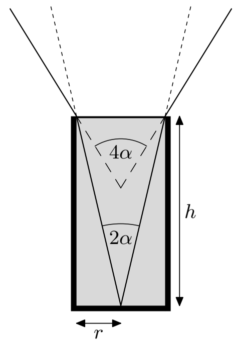
After numerical evaluation, we get that the rope shrank by $8.0\,\mathrm{m}$. That is unexpected.

Jozef Lipták
`liptak.j@fykos.cz`

Problem AC ... at the bottom

Dano has a dry well with constant circular cross-section and depth h. When standing at the bottom, he is able to see an angle $2\alpha \leq 90°$ of the sky above him. We want to increase this angle by pouring some specific liquid into the well (Dano still stays at the bottom). Find the condition for the index of refraction of the liquid if we want the angle of the visible sky to be twice as large.

Jáchym didn't know how it began...



The angle increases the most when we fill the well completely. A light ray from Dano to the top of the well inclined at an angle α from the vertical is bent by the interface in such a way that it forms the angle 2α with the vertical, in order to see double the original angle of the sky. If we denote the refractive index of the liquid by n, Snell's law implies

$$\sin 2\alpha = n \sin \alpha\,.$$

Using the double angle formula $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ for the left hand side,

$$n = 2 \cos \alpha\,.$$

That's the boundary condition, so we need n with greater or equal value,

$$n \geq 2 \cos \alpha\,.$$

The answer is independent of the dimensions of the well.

Jáchym Bártík
`tuaki@fykos.cz`

Problem AD ... snow on the roof

There is a $h = 20\,\mathrm{cm}$ (in vertical direction) thick layer of snow on our roof. The slope of the roof is $\alpha = 55°$, the density of the snow is $\varrho = 0.80\,\mathrm{kg{\cdot}dm^{-3}}$ and the dimensions of the roof are $30\,\mathrm{m} \times 6\,\mathrm{m}$ (when viewed in the direction perpendicular to it). Find the pressure exerted on the roof by the snow. *Dodo was taking a shower.*

The total mass of the snow on the roof is

$$m = Sh\varrho \cos \alpha\,.$$

Pressure is defined as the perpendicular force F_n divided by the surface of the roof S

$$p = \frac{F_\mathrm{n}}{S} = \frac{mg \cos \alpha}{S} = h\varrho g \cos^2 \alpha \doteq 520\,\mathrm{Pa}\,.$$

The snow presses on the roof with $p = 520\,\mathrm{Pa}$ of pressure.

Jozef Lipták
`liptak.j@fykos.cz`

Problem AE ... Danka's glasses

The maximum distance at which Danka can see sharply with a naked eye is 20 cm. *What glasses does Danka need to see properly (i.e. so her far point is located at the correct distance in front of the eye)? Find the type and optical power of the lenses.* *Danka saw nothing.*

Let's use the thin lens equation

$$\frac{1}{a}-\frac{1}{a'}=\frac{1}{f}\,.$$

Danka requires the glasses to project the far point (the point at an infinite distance a) to the distance $a'=20\,\mathrm{cm}$ in front of her eye, where she can see sharply. In our notation, the optical power $\Phi=-\frac{1}{f}$, so we get $\Phi=-5\,\mathrm{D}$. Danka needs concave lenses with optical power $-5\,\mathrm{D}$.

Daniela Pittnerová
`daniela@fykos.cz`

Problem AF ... interdimensional potential

Consider a planet with the same equatorial gravitational acceleration a_g and centrifugal acceleration a_o as the Earth ($a_\mathrm{g}=9.83\,\mathrm{m\cdot s^{-2}}$ and $a_\mathrm{o}=0.034\,\mathrm{m\cdot s^{-2}}$ respectively), but with a radius of only $R=5.00\,\mathrm{km}$. How big is the difference between the gravitational potential energy of a small satellite on the surface of the planet and its potential energy infinitely far from the planet? We are interested in the absolute value of this quantity per one kilogram of the satellite's mass. *Karel was watching where Rick and Morty's family escaped to.*

The gravitational potential energy is defined to be zero at infinite distance from the source if the source is a point mass (or a sphere, which has the same field). In both cases, therefore, the energy at infinity is zero. At the finite distance R, we have the potential energy for our planet

$$E_{p1}=-G\frac{mM}{R}\,.$$

The problem statement asks for the energy per mass of our satelite, which is the gravitational potential

$$U_1=\frac{E_{p1}}{m}=-\frac{GM}{R}=-a_gR\doteq-49\,150\,\mathrm{m^2\cdot s^{-2}}\,,$$

where we have used

$$a_g=\frac{GM}{R^2}$$

obtained from Newton's law of gravity.

In comparison, for the Earth with radius $R_\mathrm{Z}=6\,380\,\mathrm{km}$, we get

$$U_2=-a_gR_\mathrm{Z}\doteq 6.3\cdot 10^7\,\mathrm{m^2\cdot s^{-2}}\,.$$

We can see that for interstellar travel, a planet with a smaller radius is better. On the contrary, for the inhabitants, a bigger planet with smaller tidal forces is better. Anyway, considering the density of this planet, which was computed at the Online Physics Brawl, it's unlikely that such a planet exists. Neutron stars have such density, but they usually have fast periods of rotation and bigger surface gravity.

Karel Kolář
`karel@fykos.cz`

Problem AG ... caffeine

Daniel is watching his intake of caffeine. A standard cup of coffee contains approximately 80.0 mg *of caffeine. Daniel is preparing his coffee in his moka pot, in which case one cup with volume* 1.00 dl *contains three times more caffeine than a standard cup. FYKOS has ordered new cups in the shape of equilateral cylinders – that means their height is the same as the diameter of the base, which is* 8 cm*. How much caffeine (in mg) does Daniel receive by drinking a full new cup of coffee prepared in the moka pot?* *Daniel is drinking too much coffee.*

When 1 dl of coffee from Daniel's moka pot contains three times more caffeine than a standard cup, it contains 240 mg of caffeine. We can calculate the volume of new pot using the formula for the volume of a cylinder

$$V = \frac{1}{4}\pi d^3 \doteq 402\,\mathrm{cm}^3 = 4.02\,\mathrm{dl}\,.$$

In the end, the amount of caffeine is proportional to the volume of coffee, so we get that there are 965 mg of caffeine in a full new cup.

Daniela Pittnerová
daniela@fykos.cz

Problem AH ... non-ideal voltmeter

Consider a resistor with resistance R. We connect a voltmeter in parallel with the resistor and an ammeter in series with them. Then, we connect this circuit to a DC voltage source. The ammeter shows a current I, the voltmeter a voltage U (where $U \neq RI$). Calculate the inner resistance of the voltmeter. *Legolas was measuring resistances as physics lab practice.*

The voltmeter is connected in parallel with the resistor. Therefore, U is the voltage on the resistor. The current which flows through the resistor is

$$I_\mathrm{r} = \frac{U}{R}\,.$$

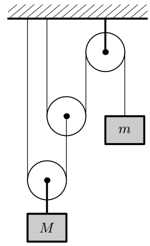
However, the ammeter is connected in series with the parallel combination of the resistor and voltmeter, so the current flowing through it is the sum of currents flowing through each of them, $I = I_\mathrm{r} + I_\mathrm{v}$.

Now, we only need to consider that the voltage shown on the voltmeter is the voltage on its probes. From the ratio of these two quantities, we can calculate the resistance

$$R_\mathrm{v} = \frac{U_\mathrm{v}}{I_\mathrm{v}} = \frac{U}{I - I_\mathrm{r}} = \frac{U}{I - \frac{U}{R}} = \frac{UR}{IR - U} = \left(\frac{I}{U} - \frac{1}{R}\right)^{-1}\,.$$

Šimon Pajger
legolas@fykos.cz

Problem BA ... pulleys again



What is the downward acceleration of the body with mass M? Neglect the masses of all the pulleys and ropes.
We noticed that participants of Online Physics Brawl found pulleys difficult...

Let T be the tension in the rope on which the mass m is hanging. Then (since pulleys are massless and thus the total force acting on them must be zero), the second pulley from the left experiences an upward force $2T$. This force has to be compensated by the other rope, so the tension in it must also be $2T$.

The equations of motion for the masses are

$$Ma_M = Mg - 4T\,,$$
$$ma_m = mg - T\,,$$

where both the accelerations point downwards.

Now we have two equations with three unknowns. We need to find a relation between the accelerations. From the bare fact that the force exerted on the mass M by the rope is 4 times larger, we can assume that it will accelerate 4 times slower.

We can easily prove it geometrically. Imagine that we displace the mass M downwards by x. The second pulley must then move by $2x$ downwards, which moves m by $4x$ upwards. Thus the acceleration of the mass m is 4 times the acceleration of M (in the opposite direction), i.e. $a_m = -4a_M$.

Plugging this into our equations, we get

$$Ma_M = Mg - 4T\,,$$
$$-4ma_M = mg - T\,.$$

After solving the equations, we obtain the result

$$Ma_M + 16ma_M = Mg - 4mg\,,$$
$$a_M = g\frac{M - 4m}{M + 16m}\,.$$

For $M > 4m$, the body with mass M will accelerate downwards.

Šimon Pajger
`legolas@fykos.cz`

Problem BB ... full tank, please

The water level in Matěj's well began to drop, so Matěj started to fill it with a water stream with constant mass flow q. Meanwhile, he noticed that just above the water level (which is $h = 37\,\mathrm{m}$ deep), the cross-section of the stream is 13 times smaller. What is the velocity of the water at the top of the well? Neglect surface tension. *Jáchym likes uncommon wells.*

Let us denote the lower cross-section by S. The upper cross-section is $S_0 = kS$, where $k = 13$. The velocities of water at the bottom and at the top are v and v_0 respectively. The mass flow rate must remain unchanged along the whole stream, therefore

$$vS = v_0S_0\,.$$

From this condition, we get $v = kv_0$. We write the equations of motion

$$v = v_0 + gt\,,$$
$$h = v_0 t + \frac{1}{2} g t^2\,,$$

express time from the first formula as

$$t = \frac{v - v_0}{g} = \frac{(k-1)\,v_0}{g}$$

and substitute it into the second formula, so we get

$$h = v_0 \frac{(k-1)\,v_0}{g} + \frac{1}{2} g \frac{(k-1)^2\,v_0^2}{g^2}\,.$$

The result is

$$v_0 = \sqrt{\frac{2gh}{k^2 - 1}} \doteq 2.1\,\mathrm{m{\cdot}s^{-1}}\,.$$

Jáchym Bártík
tuaki@fykos.cz

Problem BC ... dropped

Štěpán dropped a very heavy cannonball into a well. After the time 3.69 s*, he heard a loud "splash". How deep is the water level in the well? Neglect air resistance, but assume that the speed of sound is finite.* *10/10 made Jáchym happy.*

Let h denote the depth of the water level. The duration of the cannonball's fall is

$$t_1 = \sqrt{\frac{2h}{g}}\,.$$

The time necessary for the signal to propagate back to Štěpán is

$$t_2 = \frac{h}{c}\,,$$

where c is the speed of sound in the air. Obviously, $t_1 + t_2 = t$, and from this, we can express the desired depth

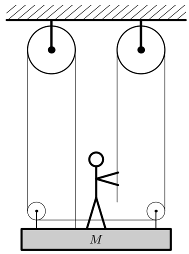
$$0 = h^2 g - 2hc\,(tg + c) + t^2 c^2 g\,,$$
$$h = \frac{c}{g}\left((tg + c) \pm \sqrt{2tgc + c^2}\right)$$

The correct root is the one with the $-$ sign, which gives $h \doteq 60.4\,\mathrm{m}$.

Jáchym Bártík
tuaki@fykos.cz

Problem BD ... phy-lley

A member of FYKOS with mass $m = 50\,\mathrm{kg}$ *is pulling a rope downward with a constant force* $F = 300\,\mathrm{N}$*. Find the magnitude of his acceleration. The mass of the depicted platform is* $M = 50\,\mathrm{kg}$*. Neglect the moments of inertia of all pulleys.* *Matěj likes to dig holes.*



We will assume a simpler situation where the member of FYKOS and the platform form one rigid body. Later, we will explain why this assumption is correct. Between this body and the ceiling, the rope is stretched four times. A typical feature of a rope in a system of ideal pulleys is that the force of tension in it is everywhere the same. The total force that acts on the FYKOS member is $4F$. Furthermore, the force of gravity acting on it is $F_g = (M + m)\,g$. Let us use the standard formula for acceleration

$$a = \frac{4F - F_g}{M + m} = \frac{4F}{M + m} - g = 2.19\,\mathrm{m{\cdot}s^{-2}}\,.$$

The FYKOS member and the surface were assumed to be one body because apart from the surface, only the rope acts on the FYKOS member, upwards with the force $F < mg$. If he wasn't being lifted by the surface, he would start falling.

Matěj Mezera
m.mezera@fykos.cz

Problem BE ... how to move the world

Lego wanted to move the world, so he jumped down from a tree. His centre of mass shifted by h *relative to the Earth. By how much did the Earth move, in the reference frame of the centre of mass of the Lego-Earth system? Lego's mass is* m*, the mass of Earth is* M*. Assume that the radius of Earth* $R_\mathrm{Z} \gg h$*. Do not assume anything about the masses of Lego and the Earth.* *Lego wanted to move the world.*

Since $R_z \gg h$, the forces acting on both Lego and the Earth are constant during the whole fall and both have the same magnitude F. Lego will then fall with the acceleration $a_L = F/m$ towards the Earth and the Earth will fall with the acceleration $a_Z = F/M$ towards Lego. Their relative acceleration is

$$a_v = a_L + a_Z = F\left(\frac{1}{m} + \frac{1}{M}\right) = F\frac{m + M}{mM}\,.$$

We can easily calculate the duration of the fall

$$h = \frac{1}{2}a_v t^2\,,$$
$$t^2 = 2\frac{h}{a_v}\,.$$

We plug this time into the kinematic equation for uniformly accelerated motion and get the resulting displacement

$$s = \frac{1}{2}a_Z t^2 = \frac{1}{2}\frac{F}{M}2\frac{h}{F\frac{m+M}{mM}} = h\frac{m}{m + M}\,.$$

However, the displacement can be found even easier. It is sufficient to find the equation for the relative distance x of the Earth from the centre of mass of the whole system as a function of the distance d between Lego and the Earth. We find the distance from equality of torques

$$xM = (d - x)\,m\,,$$

$$x = \frac{dm}{M + m}\,.$$

During the fall, the distance d changed by h, thus the Earth moved relative to the common centre of mass by $\Delta x = \frac{hm}{M+m}$.

Šimon Pajger
legolas@fykos.cz

Jozef Lipták
liptak.j@fykos.cz

Problem BF ... a bubble in a sea

At the seabed in the depth $h_1 = 130\,\mathrm{m}$ under the sea level, a scuba diver releases an air bubble with temperature $t_1 = 36\,\mathrm{^\circ C}$ and radius $r_1 = 0.50\,\mathrm{cm}$. The bubble moves upwards without dividing into smaller bubbles or changing its shape. What is the radius of the bubble in the depth $h_2 = 5\,\mathrm{m}$ under the sea level? Assume that there's no heat exchange between the bubble and the sea during the ascent of the bubble. The density of the seawater is $\varrho = 1\,020\,\mathrm{kg{\cdot}m^{-3}}$, the atmospheric pressure is $p_\mathrm{a} = 1\,013\,\mathrm{hPa}$. *Danka wants to go diving.*

The bubble has a spherical shape, therefore its volume depends on its radius as $V = \frac{4}{3}\pi r^3$. Assuming no heat exchange, we are dealing with an adiabatic process; therefore, $pV^\kappa = \mathrm{const}$, where $\kappa = 1.4$ is the adiabatic constant (ratio of heat capacities) for air. For volume V_2, we get

$$V_2 = V_1 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\kappa}}\,.$$

Under the sea level, hydrostatic pressure $p = h\varrho g$ affects the bubble, while the total pressure is the sum of hydrostatic and atmospheric pressures. We get

$$\frac{4}{3}\pi r_2^3 = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\kappa}} \frac{4}{3}\pi r_1^3\,,$$

from which we express

$$r_2 = r_1 \left(\frac{h_1\varrho g + p_\mathrm{a}}{h_2\varrho g + p_\mathrm{a}}\right)^{\frac{1}{3\kappa}} \doteq 0.85\,\mathrm{cm}\,.$$

The radius of the bubble in the depth 5 m under the sea level is 0.85 cm.

Daniela Pittnerová
daniela@fykos.cz

Problem BG ... doubly dioptric

We have a thin lens made of flint glass. We manufactured it to have optical power exactly $\varphi = 1.000\,\mathrm{D}$ for the red light. Unfortunately, flint glass has a disadvantage of relatively high dispersion. What optical power does the lens have for blue light? The refractive index of our lens is $n_\mathrm{r} = 1.628$ for the red light and $n_\mathrm{b} = 1.647$ for the blue light.

Karel was wondering about chromatic aberration.

The optical power of a thin lens satisfies the formula

$$\varphi = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (n-1)\,k\,,$$

where the constant k is the difference of multiplicative inverses of curvature radii R_1 and R_2, which remains unchanged with dispersion. Consider the following equation

$$\frac{\varphi_\mathrm{r}}{n_\mathrm{r}-1} = \frac{\varphi_\mathrm{b}}{n_\mathrm{b}-1}\,.$$

From it, we can express the optical power for blue light

$$\varphi_\mathrm{b} = \frac{n_\mathrm{b}-1}{n_\mathrm{r}-1}\varphi_\mathrm{r} = 1.030\,.$$

This calculation would not be suitable for a thick lens due to influence of the refractive index on the shift experienced by a light beam travelling across the lens.

Jáchym Bártík
`tuaki@fykos.cz`

Problem BH ... a molecular rotator

A diatomic molecule of oxygen with a total mass $m = 5.30\cdot 10^{-26}\,\mathrm{kg}$ is rotating around its centre of mass. The bond between the two atoms is elastic with a force constant $k = 180\,\mathrm{N{\cdot}m^{-1}}$ and length $l = 1.21\,\mathrm{A}$. What is the relative change in the length of the bond (the total change in the length divided by the original length) if the molecule starts rotating with an angular speed $\omega = 6.00\cdot 10^{12}\,\mathrm{rad{\cdot}s^{-1}}$? Assume that the atoms are point masses at the ends of the bond.

Danka remembered an exam from Physics 4.

While rotating, the elastic force F_p and centrifugal force F_c are in equilibrium. Then

$$F_\mathrm{p} = k\Delta l\,,$$
$$F_\mathrm{c} = \frac{m}{2}\omega^2\frac{l+\Delta l}{2}\,.$$

Since these forces are equal, we find

$$\frac{\Delta l}{l} = \left(\frac{4k}{m\omega^2} - 1\right)^{-1} \doteq 2.66\cdot 10^{-3}\,.$$

The relative change in length is $2.66\cdot 10^{-3}$.

Daniela Pittnerová
`daniela@fykos.cz`

Problem CA ... danger in class

During Social Sciences lessons, Daniel noticed that most of the neon lamps in the classroom are quite old. These lamps are $l = 1.5\,\mathrm{m}$ long and firmly attached at the endpoints. Sometimes, one of these attachments gets broken and the lamp can spin around the other end. What is the velocity of the free end of the lamp at the lowest point of its trajectory if we assume that there is no resistance? *Daniel is wondering how dangerous school is.*

To solve this problem, we will work with energy. Let's assume that a lamp attached at both ends has zero potential energy. During the fall, its centre of mass is moving down and this potential energy is changing to kinetic rotational energy. We can write

$$-\Delta E_p = \Delta E_k\,,$$
$$mg\frac{l}{2} = \frac{1}{2}I\omega^2\,,$$

where m is the mass of the lamp, $I = \frac{1}{3}ml^2$ is the moment of inertia of a thin rod (which is an approximation for a lamp) spinning around its endpoint and ω is the angular velocity of the lamp at the lowest point. Now, we express the angular welocity as

$$\omega = \sqrt{\frac{3g}{l}}\,.$$

We can express the velocity of the free end as $v = \omega l = \sqrt{3gl} \doteq 6.6\,\mathrm{m\cdot s^{-1}}$.

We can also solve the problem using forces, by considering the force of gravity which acts downwards in the centre of mass (in the middle of the rod). Then, we get the same result by integration with respect to the angle between the force of gravity and the rod.

Daniela Pittnerová
`daniela@fykos.cz`

Problem CB ... make a wish

Danka owns a well with a uniform circular cross-section. The well is magical and makes dreams come true. Danka imagined sunny beaches of the South Pacific and threw a coin into the well. The coin elastically bounced off the walls of the well a few times. The height difference between the first and second points of impact with the wall of the well was $d_1 = 14.6\,\mathrm{m}$. The height difference between the second and third points of impact was $d_2 = 23.7\,\mathrm{m}$. Even before the coin bounced off the wall for the fourth time, Danka knew what the height difference between the third and fourth points of impact was going to be. You should calculate it as well. Assume that the coin is a point mass. *Jáchym is looking forward to holidays.*

The horizontal component v_x of the velocity is constant. The horizontal distance which the coin has to travel is always the same because the well is circular. All the impacts are elastic, so the mechanical energy is conserved and the collisions obey the law of reflection. The time between subsequent impacts is T. Let's say that the first impact occurs at the time $t = 0$ and depth $h = 0$ and the vertical component of the velocity is $v = v_0$ at that moment. Then, the formula for the depth of the i-th collision is

$$h_i = v_0 t_i + \frac{1}{2}g t_i^2\,,$$

where $t_i = iT$. Let's denote $d_3 = h_4 - h_3$, where

$$h_i = \sum_{j=1}^{i} d_j \,.$$

We can express the vertical component of the velocity from the first equation

$$Tv_0 = d_1 - \frac{1}{2} gT^2 \,,$$

and when we substitute it into the second equation, we are able to calculate the time T

$$gT^2 = d_2 - d_1 \,.$$

By substituting into the third equation, we find

$$d_3 = 3v_0T + \frac{9}{2} gT^2 - d_1 - d_2 = 2d_2 - d_1 = 32.8\,\mathrm{m} \,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem CC ... a paintbrush is falling

Find the time t that it takes a paintbrush to fall from the top of a roof to the ground. The roof touches the ground, its height is h, its slope is α and the coefficient of friction between the brush and the roof is f. Assume that the initial velocity of the paintbrush is zero and the slope of the roof is constant. Also, find the conditions that need to be satisfied for the brush to actually fall to the ground. *Dodo was painting his roof.*

In the direction parallel to the plane of the roof, the paintbrush (with a mass m) is affected by the parallel component of its weight, with magnitude

$$F = mg \sin \alpha \,.$$

The magnitude of the friction force F_{t}, acting in the opposite direction, is

$$F_{\mathrm{t}} = fF_{\mathrm{n}} = fmg \cos \alpha \,.$$

From Newton's second law, we obtain the acceleration of the brush as

$$a = \frac{F - F_{\mathrm{t}}}{m} = g \sin \alpha - fg \cos \alpha \,,$$

where the inequality $f < \tan \alpha$ must be satisfied - otherwise, the brush is stopped by friction. If we modify the equation for uniformly accelerated motion

$$s = \frac{1}{2} at^2$$

with $s = h / \sin \alpha$, express the time t and substitute for acceleration, we obtain

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2h}{g\left(\sin \alpha - f \cos \alpha\right) \sin \alpha}} \,.$$

Just out of interest, if we define $\sqrt{2h/g} = T$ (the time of free fall from the roof), we get

$$t = \frac{T}{\sqrt{(\sin\alpha - f\cos\alpha)\sin\alpha}}\,,$$

which gives us t as a function of the coefficient of friction and the slope of the roof. It can be seen that for $\alpha = 90^\circ$, we get free fall, and for less steep roofs, the fall slows down.

Jozef Lipták
`liptak.j@fykos.cz`

Problem CD ... sprinting on a wheelchair

Imagine that we are sitting in an electric wheelchair on an oval athletic track with length $l = 400.0\,\mathrm{m}$. Calculate the shortest time we need to drive around the oval if the inertial acceleration must not exceed $a = 0.1g$ at any time. We start in the curved part of the oval with an arbitrary non-zero velocity (which we also aim to optimise). Both linear parts and both curved parts have lengths $l/4$. *Dodo and his passion for sprinting...*

The radius of each curved part is $r = \frac{l}{4\pi}$. We can travel in these parts only with some maximal speed v_0, at which the centrifugal acceleration is exactly $a = 0.1g$. The time T_z it takes to travel through the curves is then

$$a = 0,1g = \frac{v_0^2}{r} = \frac{v_0^2 4\pi}{l}\,,$$

$$v_0 = \sqrt{\frac{lg}{40\pi}}\,,$$

$$T_\mathrm{z} = \frac{l}{4v_0} = \sqrt{\frac{5\pi l}{2g}}\,.$$

When travelling through the linear parts, the wheelchair accelerates half of the distance and decelerates the other half of the distance, with acceleration of magnitude $a = g/10$. It takes time T_p to travel the distance $l/8$ from the end of a curved part to the middle of the next linear part (or similarly from the middle of a linear part to the start of the next curved part). The equations for uniformly accelerated motion say

$$\frac{l}{8} = v_0 T_\mathrm{p} + \frac{1}{2} a T_\mathrm{p}^2 = \sqrt{\frac{lg}{40\pi}} T_\mathrm{p} + \frac{1}{20} g T_\mathrm{p}^2\,.$$

We got a quadratic equation for the time T_p. Only the positive solution is right, so

$$T_\mathrm{p} = \sqrt{\frac{5}{2\pi}} \sqrt{\frac{l}{g}} \left(\sqrt{1+\pi} - 1\right)\,.$$

The overall time it takes to drive around the oval track is therefore

$$T = 2T_\mathrm{z} + 4T_\mathrm{p} = \sqrt{\frac{l}{g}} \left(2\sqrt{\frac{5\pi}{2}} + 4\sqrt{\frac{5}{2\pi}} \left(\sqrt{1+\pi} - 1\right)\right) \approx 9.30 \sqrt{\frac{l}{g}}\,.$$

After evaluating it numerically, we get $T = 59\,\mathrm{s}$, which is slower than the world records in the 400 m dash for both genders.

Jozef Lipták
`liptak.j@fykos.cz`

Problem CE ... dancing

Even thought it sounds quite unlikely, some Matfyz students like dancing. Daniel is trying to learn advanced steps, for example a simple pirouette. He grabs the hips of his partner and starts to spin around with her. What is the centrifugal force acting on Daniel's partner? Imagine that Daniel is really strong, so he manages to carry his partner just above the floor at the distance of his stretched arms – let's consider it $r = 0.90\,\mathrm{m}$. Assume that the spinning couple makes $f = 0.75$ spins per second, his partner weighs $m_1 = 50\,\mathrm{kg}$ and Daniel weighs $m_2 = 70\,\mathrm{kg}$.

Daniel was dreaming about other forms of procrastination.

Daniel's partner Danka is spinning with an angular velocity $\omega = 0,75 \cdot 2\pi s^{-1} = 1.5\pi s^{-1}$. Daniel and Danka are spinning around their common centre of mass, so Danka is at the distance $R = \frac{m_2 r}{m_1+m_2} = 0.525\,\mathrm{m}$ from the axis of rotation. We can calculate the centrifugal force as

$$F = m_1 a\,,$$

where a is the centrifugal acceleration, which we can express as $a = \omega^2 R$. We get the centrifugal force

$$F = m_1 \omega^2 r \doteq 583\,\mathrm{N}\,.$$

What's interesting is that the centrifugal acceleration acting on Danka is approximately $12\,\mathrm{m\cdot s^{-1}}$, which is more than the acceleration due to gravity. However, Daniel failed and they fell down after a half-spin.

Daniela Pittnerová
`daniela@fykos.cz`

Problem CF ... heat and collisions

Two particles with momenta $m_1\mathbf{v_1}$ and $m_2\mathbf{v_2}$ (these are vector quantities) collided and merged. What was the heat released during the collision?

Jindra was playing with marbles.

Let's start with the laws of conservation of momentum

$$m_1\mathbf{v_1} + m_2\mathbf{v_2} = (m_1 + m_2)\mathbf{u} \tag{1}$$

and conservation of energy

$$\frac{1}{2}(m_1 + m_2)\left|\mathbf{u}\right|^2 + Q = \frac{1}{2}m_1\left|\mathbf{v_1}\right|^2 + \frac{1}{2}m_2\left|\mathbf{v_2}\right|^2\,. \tag{2}$$

We can express the velocity $\mathbf{u}$ from the equation (1) as

$$\mathbf{u} = \frac{m_1\mathbf{v_1} + m_2\mathbf{v_2}}{m_1 + m_2}$$

and substitute into the equation (2)

$$Q = \frac{1}{2} m_1 \left|\mathbf{v_1}\right|^2 + \frac{1}{2} m_2 \left|\mathbf{v_2}\right|^2 - \frac{1}{2}(m_1+m_2)\left|\frac{m_1\mathbf{v_1}+m_2\mathbf{v_2}}{m_1+m_2}\right|^2 .$$

We can simplify this expression to

$$\begin{aligned}
Q &= \frac{1}{2} m_1 \left|\mathbf{v_1}\right|^2 + \frac{1}{2} m_2 \left|\mathbf{v_2}\right|^2 - \frac{1}{2(m_1+m_2)}\left(m_1^2\left|\mathbf{v_1}\right|^2 + 2m_1m_2\mathbf{v_1}\cdot\mathbf{v_2} + m_2^2\left|\mathbf{v_2}\right|^2\right) ,\\
Q &= \frac{m_1m_2}{2(m_1+m_2)}\left|\mathbf{v_1}\right|^2 - \frac{m_1m_2}{m_1+m_2}\mathbf{v_1}\cdot\mathbf{v_2} + \frac{m_1m_2}{2(m_1+m_2)}\left|\mathbf{v_2}\right|^2 ,\\
Q &= \frac{m_1m_2}{2(m_1+m_2)}\left(\mathbf{v_1}-\mathbf{v_2}\right)^2 .
\end{aligned}$$

As expected, the heat depends only on the difference of the velocities and so, it is the same in all inertial reference frames.

Jindřich Jelínek
`jjelinek@fykos.cz`

Problem CG ... cooking

Danka cooks on a hob with input energy consumption (power) P. She pours $2.00\,\mathrm{l}$ of water with temperature $40\,\mathrm{^\circ C}$ into a stock pot with temperature $23\,\mathrm{^\circ C}$. The water starts boiling after a time $t_1 = 6\,\mathrm{min}$. Danka then empties the pot and lets it cool to $70\,\mathrm{^\circ C}$. At this moment, Danka pours $2.00\,\mathrm{l}$ of water with temperature $40\,\mathrm{^\circ C}$ into the pot again. How much time does she save if she wants to wait until the water starts boiling again? The stock pot has temperature $105\,\mathrm{^\circ C}$ when the water is boiling. The hob heats the pot and water with efficiency $\eta = 0.85$. The heat capacity of the pot is $C = 439\,\mathrm{J{\cdot}K^{-1}}$ and the specific heat capacity of water is $c_\mathrm{v} = 4\,180\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$. *Danka uses the electric cooker at the dormitory.*

Let's denote the important temperatures by $T_1 = 23\,\mathrm{^\circ C}$, $T_2 = 40\,\mathrm{^\circ C}$, $T_3 = 70\,\mathrm{^\circ C}$, $T_4 = 105\,\mathrm{^\circ C}$. Water boils at the temperature $T_v = 100\,\mathrm{^\circ C}$. The heat transfer between the hob and the pot with the water is described by the calorimetry equation

$$\eta P t_1 = C\left(T_4 - T_1\right) + c_\mathrm{v} V \varrho \left(T_\mathrm{v} - T_2\right) ,$$

where V is the volume of the water and ϱ is its density. The heat transfer during the second heating process is described by the equation

$$\eta P t_2 = C\left(T_4 - T_3\right) + c_\mathrm{v} V \varrho \left(T_\mathrm{v} - T_2\right) .$$

We can eliminate P and η and write

$$w = \frac{t_2}{t_1} = \frac{C\left(T_4 - T_3\right) + c_\mathrm{v} V \varrho \left(T_\mathrm{v} - T_2\right)}{C\left(T_4 - T_1\right) + c_\mathrm{v} V \varrho \left(T_\mathrm{v} - T_2\right)} .$$

The ratio is $w = 0.961\,6$. We can calculate the time difference

$$\Delta t = (w-1)\,t_1 \doteq -14\,\mathrm{s}.$$

The negative sign means that the water starts boiling $14\,\mathrm{s}$ earlier than in the first case. Danka hasn't saved much time. In reality, the heat capacity of the hob is more important than the heat capacity of the pot. When the hob is hot, the water would boil significantly sooner, in fact.

Daniela Pittnerová
daniela@fykos.cz

Problem CH ... water in an angular tube

We have a tube with water flowing through it, with a mass flow rate Q and a velocity v. At one point, this tube is bent in such a way that it has two arms with an angle α between them (so $\alpha = \pi$ means that there is no bend). What is the magnitute of the force the water is exerting on this bend? *Legolas is glad that he has no water in his knee.*

Through the angular tube, water with mass $\mathrm{d}m = Q\mathrm{d}t$ flows during a small time period $\mathrm{d}t$. This changes its velocity by $\mathbf{u} = \mathbf{v_2} - \mathbf{v_1}$. From Newton's first and third law, we get the force the water exerts on the angular tube

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}m\mathbf{u}}{\mathrm{d}t} = Q\mathbf{u}\,.$$

After applying some geometry, we get

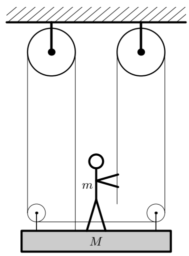
$$F = 2Qv\cos\left(\frac{\alpha}{2}\right)\,.$$

For $\alpha = \pi$, we get $F_\mathrm{v} = 0$, which corresponds to the fact that in such a case, the water would be unaffected.

Šimon Pajger
legolas@fykos.cz

Problem DA ... the floor is lava

A member of FYKOS with mass $m = 50\,\mathrm{kg}$ (depicted in the figure) is pulling a rope with a constant force F. The mass of the platform is $M = 50\,\mathrm{kg}$. Can the FYKOS member lift himself above the platform? If yes, what force does he have to pull with to do so? Neglect the moments of inertia of all pulleys. *Matěj fell into a pit and couldn't get out.*



An important property of every rope in a system of ideal pulleys is that the force of tension in it is everywhere the same. The platform is pulled only by the rope, with an acceleration

$$a_\mathrm{p} = \frac{3F}{M}\,,$$

because there are three parts of the rope and each of them is exerting a force F on the platform. The FYKOS member is pulled upwards by the rope with an acceleration

$$a_\mathrm{F} = \frac{F}{m}\,,$$

since the FYKOS member is pulling only one rope and the force it exerts on him is F. Since after substitution, $a_\mathrm{F} < a_\mathrm{p}$ holds for every positive force F, the upwards acceleration of the platform will always be greater than that of the FYKOS member. Therefore, he can never pull himself above the platform.

Accounting for gravity does not affect the result because g can be subtracted from both accelerations and the inequality remains unchanged.

Matěj Mezera
`m.mezera@fykos.cz`

Problem DB ... lift mechanics

A container with water is placed inside a stationary lift. In the container, there is also a cuboidal weight made of aluminium (with dimensions $x = 3.00\,\mathrm{cm}$, $y = 4.00\,\mathrm{cm}$ and $z = 5.00\,\mathrm{cm}$), which is fully submerged in the water. The weight is hanging on a massless spring with a spring constant $k = 230\,\mathrm{N{\cdot}m^{-1}}$, which is attached to the ceiling and initially stretched. The lift begins to move upwards with a constant acceleration $a = 3.00\,\mathrm{m{\cdot}s^{-2}}$. Find the ratio of the spring's elongation when the lift is moving to its initial elongation. The density of aluminium is $\varrho_{\mathrm{Al}} = 2\,700\,\mathrm{kg{\cdot}m^{-3}}$. Assume that the spring doesn't stretch far enough to reach the floor.

Dodo carried a plate of soup away from Danka.

In the case when the lift is stationary, the force of gravity F_g, which is acting on the weight, is compensated by the buoyant force F_{v} and the tensile force F_{p} of the spring. It satisfies the force balance equation

$$V\varrho_{\mathrm{h}}g = V\varrho_{\mathrm{v}}g + k\delta l_0\,,$$

where $V = xyz$ is the volume of the weight, ϱ_{v} is the density of water and δl_0 is the initial elongation of the spring. The accelerating lift is indistinguishable from a stationary one which is influenced by gravity $a + g$. Therefore, the force balance is

$$V\varrho_{\mathrm{h}}\left(a+g\right) = V\varrho_{\mathrm{v}}\left(a+g\right) + k\delta l\,.$$

Now we simply express the elongations of the spring from both equations and calculate their ratio as

$$\frac{\delta l}{\delta l_0} = 1 + \frac{a}{g} \doteq 1.31\,.$$

The elongation of the spring in the accelerating lift is 1.31 times larger.

Daniela Pittnerová
`daniela@fykos.cz`

Problem DC ... rotation of the velocity vector

Certainly, you have already calculated common projectile motion - for instance, the trajectory of a ball which was kicked under an initial angle α (with respect to the ground) and with an initial speed v_0. Find out how the magnitude of angular velocity ω of its velocity vector $\mathbf{v}$ depends on the initial speed v_0, the initial angle α, the instantaneous speed v and the gravitational acceleration g.

Robo was thinking during a PE class.

The acceleration acting in the direction perpendicular to the trajectory (the radial direction) can be expressed as $a_{\mathrm{n}} = g\cos\vartheta$, where ϑ is angle between the velocity vector and the horizontal direction. Since the instantaneous velocity vector is tangential to the trajectory, the angle satisfies $\cos\vartheta = v_x/v$. No force acts on the ball in the horizontal direction, therefore its horizontal velocity remains constant, $v_x = v_0\cos\alpha$. We also know that the centripetal acceleration satisfies

$$a_\mathrm{c} = \frac{v^2}{R} = \omega v\,,$$

where R is the radius of curvature of the trajectory in the current position. The motion of the ball may be locally approximated by circular motion. For this motion, the angular velocity of the velocity vector is the same as the angular velocity of rotation of the body around the centre of curvature. We write the equation for the accelerations and express the desired angular velocity

$$\omega = \frac{a_\mathrm{c}}{v} = \frac{a_\mathrm{n}}{v} = \frac{gv_0\cos\alpha}{v^2}\,.$$

Róbert Jurčo
robert.jurco@fykos.cz

Problem DD ... heavy can

Imagine a symmetric can. Its mass is m, its height is H and the area of each base is S. There is a liquid with density ϱ in the can.

We want the height of the centre of mass of the system can+liquid to be the smallest possible. For what height h of the liquid does it happen? *Lego loves beer and physics.*

The can is symmetric, so its centre of mass is $H/2$ above the ground. The height of the centre of mass of the liquid is similarly $h/2$. The height of their common centre of mass is

$$x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{mH/2 + Sh\varrho h/2}{m + Sh\varrho}\,.$$

One of the ways to solve this is to differentiate with respect to h and find local extrema. However, there's a quicker (and more elegant) way. The centre of mass is in its lowest position when it is at the same height as the level of the liquid. If we pour more water in such a situation, the centre of mass obviously rises up. If we slop some liquid (which is equivalent to placing a liquid with density $-\varrho$ below the current level of the liquid), the centre of mass rises again. We can therefore conclude that the minimum height of the centre of mass is $x = h$,

$$h = \frac{1}{2}\frac{mH + Sh^2\varrho}{m + Sh\varrho}\,,$$
$$0 = S\varrho h^2 + 2mh - mH\,.$$

The solution is the only positive root of this equation

$$h = \frac{-2m + \sqrt{4m^2 + 4Hms\varrho}}{2S\varrho} = \frac{\sqrt{m^2 + HmS\varrho} - m}{S\varrho}\,.$$

Šimon Pajger
legolas@fykos.cz

Problem DE ... Jindra the tractor driver

Tractor driver Jindra has an uncommon hobby – he likes moving from some point A to B in the shortest possible time. Now he finds himself in a field, at the point A, and the point B is located at the distance d to the east from A. The speed of the tractor depends on the azimuth of its motion as $v = v_0 \left|\cos\alpha\right|$ (so $v = v_0$ if it's moving northwards or southwards). How long does it take Jindra to reach point B? *Jindra still hasn't acquired a driving licence.*

We solve the problem using a trick. Imagine a Cartesian coordinate system in the field such that its x axis points to the east and the y axis points to the north. Jindra has to move by a distance d in the direction of the x axis. The azimuth is measured clockwise from the northward direction. The x-component of the tractor's velocity can be calculated by multiplying the speed by the sine of the azimuth

$$v_x = v_0 \left|\cos\alpha\right| \sin\alpha\,.$$

We are interested in motion in the positive x-direction, i.e. $\alpha \in \langle 0^\circ, 180^\circ\rangle$. Consider that

$$v_x = v_0 \left|\cos\alpha \sin\alpha\right|\,.$$

We find the azimuth corresponding to the maximal velocity by placing the derivative equal to zero. Since we know that in the given range, the velocity is always positive, it's enough to maximise the expression inside the absolute value

$$0 = \frac{\mathrm{d}\cos\alpha\sin\alpha}{\mathrm{d}\alpha} = \cos^2\alpha - \sin^2\alpha\,.$$

The solutions are $\alpha_1 = 45^\circ$ and $\alpha_2 = 135^\circ$. Alternatively, we recall double angle formula $\left|\cos\alpha\sin\alpha\right| = \left|\sin 2\alpha\right|/2$, so we get maxima of 1 for $2\alpha_1 = 90^\circ$ and $2\alpha_2 = 270^\circ$.

Now we are asking: "Can Jindra move in such a direction that his velocity in the positive direction of the x axis would always be equal to this maximum?" Yes, he can. For example, if he moves on the azimuth $\alpha_1 = 45^\circ$ half of the route and on the azimuth $\alpha_2 = 135^\circ$ afterwards. This ensures that his final y coordinate remains the same as at the beginning. Assuming that he always moves with maximal velocity in the direction of the x axis, there is no faster route.

We can calculate the time necessary to move to point B as

$$t = \frac{d}{v_x} = \frac{d}{v_0\left|\cos\alpha_1\sin\alpha_1\right|} = \frac{2d}{v_0}\,.$$

The time the route takes Jindra is $t = 2d/v_0$.

Jindřich Jelínek
`jjelinek@fykos.cz`

Problem DF ... lens-like mirror

We have a thin plano-convex lens with a radius of curvature R and a refractive index n. Onto the convex side of the lens, we place a shiny foil so that it behaves as a convex spherical mirror with a radius R. Find the (positive) focal length of this system.

Matěj is checking himself out in a mirror.

We will find the solution using geometrical methods. A focal point is a place into which light beams parallel to the optical axis are focused. Let us take a beam, parallel to the optical axis, at a small distance x from the axis. Since the lens is thin, we will use the paraxial approximation $\sin x \approx x$, $\tan x \approx x$.

- **passage through the flat side** - The beam doesn't change its direction because it's incident on this side perpendicularly.
- **reflection from the mirror** - The beam, parallel to the axis, is reflected towards the focal point of the mirror, at the distance $f_\mathrm{z} = \frac{R}{2}$. Therefore, the angle between the axis and the reflected beam is $\alpha = \frac{x}{f_\mathrm{z}}$.
- **second passage through the flat side** - The beam refracts in such a way that after passing through the lens, its angle with respect to the axis is β, which is given by Snell's law $n\alpha = \beta$.

Since the lens is thin, the beam exits the lens again at the distance x from the optical axis. The difference is that now, it's at the angle β with respect to it. The focal point we're looking for is the place at which it crosses the optical axis and its distance from the lens is

$$f = \frac{x}{\beta} = \frac{x}{n\alpha} = \frac{x}{n\frac{x}{f_\mathrm{z}}} = \frac{f_\mathrm{z}}{n} = \frac{R}{2n}\,.$$

We can see that the point of intersection doesn't depend on the distance x between the beam and the optical axis and therefore, all beams intersect the axis at the same point. Of course, this only applies for sufficiently small x, when we can use the paraxial approximation. Otherwise, we would find that both the lens and the mirror have optical defects, causing the parallel beams further away from the axis to miss this focal point.

Matěj Mezera
m.mezera@fykos.cz

Problem DG ... at the absolute bottom

Dano has a dry well with constant circular cross-section and depth $h = 23.0\,\mathrm{m}$. When he watches the sky from the centre at the bottom of the well, he sees all the stars which have a smaller zenith distance than $\alpha = 8.00^\circ$. What is the minimum volume of an unknown liquid with a refractive index $n = 2.31$ that Dano has to pour into the well in order to see all the stars with the zenith distance 2α? *... however, Jáchym knows how it ends.*

In both cases, Dano's view is limited by the edge of the well. We can describe the initial situation by the equation $r = h\tan\alpha$, where r is the radius of the well. After the liquid is poured into the well, refraction of light occurs at its surface. The angle of refraction is 2α and we can denote the angle of incidence by β. From Snell's law, we get

$$\sin 2\alpha = n\sin\beta\,,$$

where n is the refractive index of the liquid. Let h_1 be the distance from the bottom to the surface of the liquid and h_2 the distance between the surface and the upper edge of the well, so $h_1 + h_2 = h$. Similarly, let r_1 be the distance of the centre from the point of refraction of a furthest ray, measured along the bottom of the well, and r_2 the distance of the point of refraction from the edge of the well, so $r_1 + r_2 = r$. Now, we can write the equation for the upper triangle

$$\sin 2\alpha = \frac{r_2}{\sqrt{r_2^2 + h_2^2}}\,,$$

from which we get

$$r_2 = \left(\sin^{-2}(2\alpha) - 1\right)^{-\frac{1}{2}} h_2 = \tan(2\alpha)\, h_2 = k_2 h_2\,.$$

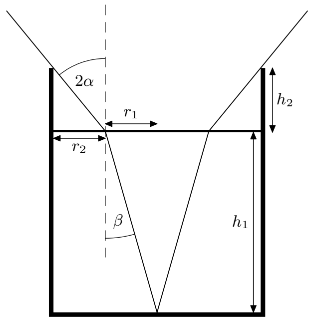
Similarly, for the lower triangle, we can write the equation

$$r_1 = \left(\sin^{-2}\beta - 1\right)^{-\frac{1}{2}} h_1 = \left(n^2 \sin^{-2}(2\alpha) - 1\right)^{-\frac{1}{2}} h_1 = k_1 h_1\,.$$

In these equations, we have used the sine in order to directly use Snell's law. We hid the ugly expressions into the constants k_1 and k_2. Now, we substitute for r_1 and r_2 in the equation $r = r_1 + r_2$, which gives us a system of two equations for two variables

$$h = h_1 + h_2\,,$$
$$r = k_1 h_1 + k_2 h_2\,.$$

Then, the solution for the height h is



$$h_1 = \frac{k_2 h - r}{k_2 - k_1}\,.$$

However, we want to know the total volume of the liquid, so we have to multiply the height by the area of the cross-section πr^2. Now, we can substitute for r and k_2 and get the final result

$$V = \pi r^2 h_1 = \pi h^3 \frac{\tan 2\alpha - \tan\alpha}{\tan 2\alpha - k_1} \tan^2\alpha \doteq 663\,\mathrm{m}^3\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem DH ... retroreflection

Retroreflective elements are called such because they reflect light back to the source. There are two ways of constructing them. We can cover the surface by miniature corner reflectors (like in bike reflectors) or we can cover it by a reflective material and partially embed transparent balls into it (like in white strips on a reflective vest). Calculate an ideal refractive index of the balls in order to reflect back most of the incoming light.

Jindra was solving the competition N-trophy[9]*.*

The material is a retroreflector when the incoming ray is parallel to the outgoing ray. Let's draw a line connecting the source of the light and the centre of a ball and assume that the diameter of the ball is insignificantly small. An incoming ray is parallel to the optical axis, it impacts the surface of the ball at a perpendicular distance h from the optical axis and it forms an angle α with the normal at the point of impact. The radius of the ball is R. The paraxial approximation is valid while $h/R \ll 1$. Rays that are further away from the axis are reflected more to the sides, but in this case, such divergence of the reflected light isn't too significant. We can write

$$\sin\alpha = \frac{h}{R} \approx \alpha\,.$$

From Snell's law, we calculate the angle of refraction β, and we also assume $\beta \ll 1$

$$\sin\alpha = n\sin\beta\,,$$

$$\frac{h}{R} = n\beta\,.$$

Retroreflection occurs when the refracted ray hits the point where the optical axis intersects the opposite surface of the ball. Then, the two rays are symmetrical with respect to the optical axis and the outgoing ray is parallel to the optical axis (like the incoming ray). Under the small angle approximation, we can write $\beta = h/(2R)$, so

$$\frac{h}{R} = n\frac{h}{2R}\,,$$

$$n = 2\,.$$

The ideal refractive index of the ball is $n = 2$.

Jindřich Jelínek
`jjelinek@fykos.cz`

Problem EA ... jerked around

Find the mean square modulus of the change in the vector of velocity of a pollen particle with mass $M = 250\,\mathrm{ng}$ *when this particle collides head on with an atom of argon in air (i.e. in such a way that they are moving directly towards each other). The temperature is* $t = 25.3\,\mathrm{^\circ C}$ *and the pressure is* $p = 1\,003\,\mathrm{hPa}$. *Dodo took a full metro back from school.*

A perfectly elastic collision implies conservation of both momentum and kinetic energy. If the pollen particle remains at rest at the beginning (we can simply choose a coordinate system where it does), then after the collision, it is moving with a velocity u which obeys

$$mv = Mu + mv'\,,$$

$$mv^2 = Mu^2 + mv'^2\,,$$

where m is the mass of an atom of argon and v, v' are the initial and new velocity of the atom respectively. After substituting for v' from the first equation, we obtain

$$u = \frac{2vm}{M+m} \approx \frac{2vm}{M}\,.$$

We're supposed to find the mean square value of this quantity. From the equipartition theorem, we have

$$\frac{1}{2}mv_\mathrm{k}^2 = \frac{s}{2}k_\mathrm{B}T\,,$$

where T is the thermodynamic temperature of the gas and s is the number of active degrees of freedom. After substitution, we get the mean square value of u as

$$u_\mathrm{k} = \frac{2\sqrt{sk_\mathrm{B}Tm}}{M}\,.$$

The mass of an argon atom can be calculated using the Avogadro constant and molar mass as $m = M_\mathrm{m}/N_\mathrm{A} = 6.64 \cdot 10^{-26}\,\mathrm{kg}$. An argon atom has only translational degrees of freedom, so $s = 3$. The numeric value is $u_\mathrm{k} = 2.3 \cdot 10^{-13}\,\mathrm{m{\cdot}s^{-1}}$.

Jozef Lipták
`liptak.j@fykos.cz`

Problem EB ... green shift

Chuck Norris has such a fast car that he sometimes drives through an orange traffic light because he sees it as green. But that's nothing. Paťo (the one who ran a 12-minute run in 6 minutes) runs so fast that he sees red as green. The wavelengths of green, orange and red light are $\lambda_\mathrm{z} = 550\mathrm{nm}$, $\lambda_\mathrm{o} = 600\mathrm{nm}$ and $\lambda_\mathrm{c} = 700\mathrm{nm}$ respectively. What is the difference between the speeds of Paťo and Chuck Norris's car? *Lego was running towards a crosswalk.*

We may assume that Paťo's speed and Chuck Norris's car's speed are both relativistic. At the same time, the waves are light waves, not sound waves, so a relativistic version of the Doppler Law must be used

$$\lambda_\mathrm{observer} = \lambda_\mathrm{source}\sqrt{\frac{1-v/c}{1+v/c}}\,.$$

This formula can either be found in the tables or it can be derived from the original Doppler Law by considering relativistic effects. The ratio v/c is commonly referred to as β. In addition, if we denote $\lambda_\mathrm{observer}/\lambda_\mathrm{source} = \alpha$, we can express

$$\beta = \frac{1-\alpha^2}{1+\alpha^2}\,.$$

Now we get the speed of Chuck Norris's car

$$v_\mathrm{Chuck} = \frac{1-(\lambda_\mathrm{z}/\lambda_\mathrm{o})^2}{1+(\lambda_\mathrm{z}/\lambda_\mathrm{o})^2}c \doteq 2.60 \cdot 10^7\,\mathrm{m{\cdot}s^{-1}}\,.$$

Similarly, we have for Paťo

$$v_\mathrm{Pato} = \frac{1-(\lambda_\mathrm{z}/\lambda_\mathrm{c})^2}{1+(\lambda_\mathrm{z}/\lambda_\mathrm{c})^2}c \doteq 7.10 \cdot 10^7\,\mathrm{m{\cdot}s^{-1}}\,.$$

Therefore, Paťo is faster than Chuck Norris's car by

$$v_{\text{Pato}} - v_{\text{Chuck}} = \frac{2\lambda_{\text{z}}^2\left(\lambda_{\text{c}}^2 - \lambda_{\text{o}}^2\right)}{\left(\lambda_{\text{c}}^2 + \lambda_{\text{z}}^2\right)\left(\lambda_{\text{o}}^2 + \lambda_{\text{z}}^2\right)} c \doteq 0.15c \doteq 4.49 \cdot 10^7 \,\mathrm{m\cdot s^{-1}}$$

Šimon Pajger
legolas@fykos.cz

Problem EC ... Are we climbing?

We are sitting in a train inside a tunnel. The train driver decided to let the train move only by its own inertia. What is the angle α between the water level in a glass and the floor of the train? Assume that when the train is standing on a horizontal surface, the floor is also horizontal. The elevation of the track is $t = 1.2\,\%$ and the rolling resistance coefficient between the track and the wheels of the train is $f = 0.002$. *Dodo was travelling back to school.*

The train is moving uphill. First, let's convert the elevation from percent to an angle. Elevation is the change in altitude per unit horizontal displacement, so the angle is $\varphi = \arctan t$. The motion of the train is directly decelerated by the tangential component of the gravity of Earth and by the friction given by its normal component. The acceleration of the train in the direction of motion is

$$a_{\text{v}} = a_{\text{t}} + f a_{\text{n}} = g\left(\sin\varphi + f\cos\varphi\right) .$$

In the non-inertial reference frame connected with the train, there is the inertial acceleration parallel with the floor, with magnitude a_{v}, but in the opposite direction (in the direction of motion of the train), and the real acceleration - the acceleration due to gravity, with magnitude g and at an angle φ from the normal to the floor, towards the back of the train. To find the angle between the water level and the floor, we need to find the angle between the vertical and the net force because these angles are the same. Decomposing the acceleration due to gravity into the normal and tangential component to the train floor, we get

$$a_{\text{k}} = g\cos\varphi\,,$$
$$a_{\text{r}} = g\sin\varphi\,.$$

Adding the inertial acceleration $a_{\text{z}} = -a_{\text{v}}$ to this tangential component, we get the desired angle as the arctangent of the ratio of magnitudes of accelerations in the "horizontal" and the "vertical" direction

$$\alpha = \arctan\frac{a_{\text{r}} + a_{\text{z}}}{a_{\text{k}}} = \arctan\frac{a_{\text{r}} - a_{\text{v}}}{a_{\text{k}}} = \arctan\frac{\sin\varphi - \left(\sin\varphi + f\cos\varphi\right)}{\cos\varphi} = -\arctan f \doteq -0.11\,^{\circ}\,.$$

We are interested only in its magnitude, the sign gives information about the orientation of the water level. The answer is $\alpha \doteq 0.11\,^{\circ}$. An interesting fact that we can notice from this solution is that we cannot say whether the train goes uphill or not. This follows from the Einstein equivalence principle.

Jozef Lipták
liptak.j@fykos.cz

Problem ED ... hole in a planet

At a large distance from the Earth, there is a rocket which has run out of fuel. The rocket is attracted to the Earth and NASA scientists are trying to figure out how to save the crew. Honza suggests digging a hole through the Earth, so that the rocket would not hit the ground, but fly through the tunnel. For simplicity, assume that the rocket is initially at rest at an infinite distance from the Earth. The tunnel through the Earth is aligned with the direction of the fall of the rocket. Honza wants to know what the velocity of the rocket would be in the middle of the Earth. Assume that the Earth is homogeneous. *Robo found himself inside a planet.*

The mass of the Earth is M and its radius is R. At the beginning, the rocket has both zero potential energy and zero kinetic energy, and the mechanical energy is conserved, so we know that in the centre of the Earth, the sum of the kinetic and potential energies would be zero again.

$$0 = \frac{1}{2}mv^2 + E_p\,.$$

Now, we need to find the gravitational potential in the centre of the Earth φ. From the potential, one can easily find the potential energy as

$$E_p = m\varphi\,.$$

We know that the gravitational potential of the Earth is given by

$$\varphi_R = -\frac{GM}{R}\,.$$

From Gauss's law for gravity, we know the intensity of the gravitational field inside the Earth at a distance r from the centre

$$E = -\frac{GM_r}{r^2}\,,$$

where M_r is the mass below the radius r. We assumed that the Earth is homogeneous, so the mass M_r is directly proportional to the volume, which is proportional to the cube of the radius

$$M_r = \frac{r^3}{R^3}M\,.$$

The potential in the centre of the Earth φ_0 is the sum of the potential on the surface and the integral of $-E$ from the surface to the centre of the Earth

$$\varphi_0 = -\frac{3GM}{2R}\,.$$

Substituting for E_p in the law of energy conservation, we get

$$0 = \frac{1}{2}mv^2 - \frac{3GMm}{2R}$$

This gives the velocity of the rocket in the centre of the Earth $v = \sqrt{\frac{3GM}{R}} = 13.7 \cdot 10^3\,\mathrm{m{\cdot}s^{-1}}$.

Róbert Jurčo
robert.jurco@fykos.cz

Problem EE ... slippery rope

Dodo has a well and a bucket. The mass of the bucket is $m = 1.75\,\mathrm{kg}$ and it can hold at most $V = 15.2\,\mathrm{l}$ of water. The bucket is hanging from a rope, which is thrown over a fixed log with a circular cross-section. When Dodo pulls the bucket full of water upwards, he pulls the rope with a force $F = 237\,\mathrm{N}$. Find the force which Dodo must exert in order to drop the empty bucket down with a constant velocity. *Jáchym prefers aid climbing.*

In this problem, it is essential to know how friction acts on a rope attached around a cylinder with circular cross-section. The exact answer to that can be found e.g. in the solutions of FYKOS problems **27.III.5** and **32.VI.4**. In this case, it is only necessary to know that the ratio between the force exerted on one end of the rope and the force exerted on the other end is something similar to an exponential of an expression involving the coefficient of friction and the total angle of contact between the rope and cylinder. All these values are the same in both situations, so the ratio of the forces on both ends of the rope is also the same.

In the first case, the gravity of the bucket with water is

$$F_\mathrm{s} = (m + \varrho V)\, g\,,$$

and it's the "weaker" force, while on the other end, Dodo pulls with a force F. In the second case, the force F' which Dodo utilises to slow down the rope is the "weaker" force. The force that acts at the other end is the gravity of the empty bucket itself

$$F_\mathrm{b} = mg\,.$$

From the observation above, we get

$$\frac{F}{F_\mathrm{s}} = \frac{F_\mathrm{b}}{F'} = \mathrm{const} > 1\,.$$

From there, we can simply express the desired force

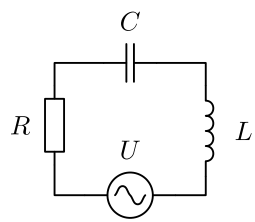
$$F' = \frac{F_\mathrm{b} F_\mathrm{s}}{F} = \frac{m + \varrho V}{F} m g^2 \doteq 12.0\,\mathrm{N}\,.$$

Finally, we can see that it really is less than the approximately $17\,\mathrm{N}$ required to balance an empty bucket without friction.

Jáchym Bártík
`tuaki@fykos.cz`

Problem EF ... RLC for sure this time

Lego took Dodo's circuit from the Online Physics Brawl and slightly changed it. Now it is the circuit shown in the figure, composed of a coil with inductance $L = 10.0\,\mathrm{mH}$, a capacitor with capacitance $C = 4.70\,\mathrm{\mu F}$, a resistor with resistance $R = 1.00\,\mathrm{k\Omega}$, and an AC voltage source with the effective value of voltage $U_\mathrm{ef} = 230\,\mathrm{V}$ and adjustable frequency. Lego set the frequency of the AC source in such a way that the amplitude of the current would be maximised. What is the power drawn by the whole circuit?



Lego felt sorry for FOL participants... so he set a similar troll problem here as well.

We could solve the problem using complex numbers, but we can solve it more easily using some basic knowledge about alternating current.

For example, it is useful to know that for a series RLC circuit (i.e. in which the current can't bypass the resistor through the coil or capacitor), the current (for a given voltage) is maximised when the frequency of the current is equal to the resonant frequency of the given circuit. That's because at the resonant frequency, the impedances of the coil and capacitor cancel out, so $Z = R$.

Therefore, for the effective value of the current, we have $I_{\mathrm{ef}} = U_{\mathrm{ef}}/Z = U_{\mathrm{ef}}/R$. The current is not phase shifted in any way with respect to the voltage, so $\varphi = 0$.

What are the effective values of the voltage and current? Effective value is defined as the amplitude of a given quantity divided by the square root of 2 (so $I_{\mathrm{ef}} = I_{\mathrm{max}}/\sqrt{2}$ and accordingly for the voltage). It's defined that way so that the formula $P = U_{\mathrm{ef}} I_{\mathrm{ef}} \cos\varphi$ would hold. Since $\cos 0 = 1$, we get the power as

$$P = U_{\mathrm{ef}} I_{\mathrm{ef}} = \frac{U_{\mathrm{ef}}^2}{R} = 52.9\,\mathrm{W}\,,$$

which is approximately the power of a standard light bulb and very accurately, the solution of the already mentioned problem from the Online Physics Brawl.

Šimon Pajger
`legolas@fykos.cz`

Problem EG ... rotation of polarisation

We all know that when horizontally polarised light hits an ideal vertical polarisation filter, the light intensity behind the filter is zero - the light cannot pass through the filter. However, if we place another polarisation filter between the two, light can pass through the last filter. Suppose that we have N filters in a series such that each one is rotated by the same angle δ relative to the previous filter. Additionally, the last filter is rotated by 90° relative to the first one. We let light with an original intensity I_0 pass through the filters. For a large value of N, estimate how much the intensity of the light decreases after passing through all the filters.

Štěpán makes photons pass through.

A filter rotated by δ relative to the polarisation decreases the strength of the electric field passing through it from $\mathbf{E}_1$ to $\mathbf{E}_2$, where

$$|\mathbf{E_2}| = |\mathbf{E_1}| \cos\delta\,,$$

Since the intensity is proportional to the square of the field strength, it decreases from I_1 to I_2, where

$$I_2 = I_1 \cos^2\delta\,.$$

Therefore, after passing through N filters,

$$I = I_0 \cos^{2N}\delta = I_0 \cos^{2N}\left(\frac{\pi}{2N}\right)\,,$$

where I_0 is the initial intensity and we plugged in $\delta = \frac{\pi}{2N}$. For large values of N, we can approximate the cosine as

$$\cos\left(\frac{\pi}{2N}\right) \approx 1 - \frac{\pi^2}{2\cdot(2N)^2} = 1 - \frac{\pi^2}{8N^2}$$

and then, we can write

$$I \approx (1 - \frac{\pi^2}{8N^2})^{2N} I_0 \approx \left(1 - \frac{\pi^2}{4N}\right) I_0 \,.$$

The intensity decreased by $\frac{\pi^2}{4N} I_0$.

Štěpán Marek
stepan.marek@fykos.cz

Problem EH ... TNT planet

Consider a sphere of TNT floating freely in space. Assume that firing one kilogram of TNT releases 4.184 MJ *of energy, which is immediately converted into kinetic energy of the reaction products (whose mass is the same as the mass of the original TNT,* 1 kg*). What is the radius of the largest sphere with density* $\varrho = 1\,650\,\mathrm{kg{\cdot}m^{-3}}$ *that can completely scatter through the explosion (i.e. all its mass is thrown to infinity, where it does not gravitationally affect itself anymore)?* *Jáchym joined Karel and decided to also destroy a planet.*

Let our planet have a radius R and a mass

$$M = \frac{4}{3}\pi R^3 \varrho \,.$$

If we denote the calorific value of trinitrotoluene as $H = 4.184\,\mathrm{MJ{\cdot}kg^{-1}}$, the total energy of the explosion is

$$E_\mathrm{v} = HM = \frac{4}{3}\pi R^3 H \varrho \,.$$

Now we take the upper layer of the planet, with width $\mathrm{d}r$, and move it to infinity. The gravitational potential on the surface of a planet with radius r and mass m is

$$\varphi = -\frac{Gm}{r} \,,$$

so we have to add energy

$$\mathrm{d}E = -\varphi \mathrm{d}m \,,$$

where the mass of the layer $\mathrm{d}m$ is calculated as

$$\mathrm{d}m = 4\pi r^2 \varrho \mathrm{d}r \,.$$

By doing so, we get rid of the upper layer of the planet, and thereby reduce its radius and weight

$$m = \frac{4}{3}\pi r^3 \varrho \,.$$

We calculate the total gravitational energy needed to move all parts of the planet to infinity as the integral

$$E_\mathrm{g} = \int_0^R \mathrm{d}E = -\int_0^R \varphi \,\mathrm{d}m = \frac{16\pi^2 G\varrho^2}{3} \int_0^R r^4 \,\mathrm{d}r = \frac{16\pi^2 G\varrho^2}{3} \left[\frac{r^5}{5}\right]_0^R = \frac{16\pi^2 G R^5 \varrho^2}{15} \,.$$

We see that E_g grows with the fifth power of the radius, while E_v only grows with the third power. Therefore, for all R larger than R_0 (for which $E_\mathrm{g} > E_v$), we will not be able to disperse the planet perfectly. We can write the result as

$$R_0 = \sqrt{\frac{5H}{4\pi G\varrho}} \doteq 3\,888\,\mathrm{km}\,.$$

Jáchym Bártík
tuaki@fykos.cz

Problem FA ... Jindra II. the tractor driver

Jindra the tractor driver has an unusual hobby – he likes moving from some point A to B in the shortest possible time. At this moment, Jindra with his tractor is in a field, at the point A, and the point B is located at a distance r from the point A in the same field. At a distance d from the line segment AB, there is a driveway running parallel to it. In the field, the tractor moves with the same velocity u in all directions and on the driveway, it moves with a velocity v, where $v > u$. Jindra found out that it does not matter if he drives from the point A to the point B directly or if he first drives from the point A onto the driveway, then along the driveway and after a while, back into the field towards the point B. In both cases, the time of the journey is t. Express the ratio d/r using u and v.

Jindra wanted to experience total tractor reflection.

First, we must derive the formulas describing the time of driving directly from A to B and the time of travel when using the driveway. Let t_p denote the travel time for the direct case. We can calculate it from the formula for uniform linear motion

$$t_\mathrm{p} = \frac{r}{u}\,.$$

It gets slightly more complicated with the driveway. Consider that there is an infinite number of ways in which the tractor can get on the driveway, as well as an infinite number of ways to get off it. We have to find the most efficient one. The tractor rides from the point A to the point where it gets on the driveway in a straight line [1]. It also rides in a straight line from the point where it leaves the road to the point B. Let's denote the angle between the trajectory of the tractor before it reaches the driveway and a normal to the driveway by α_A. Similarly, let's denote the angle of its trajectory after it leaves the driveway by α_B. The total time of the ride is

$$t_\mathrm{s} = \frac{d}{u\cos\alpha_\mathrm{A}} + \frac{r - d\tan\alpha_\mathrm{A} - d\tan\alpha_\mathrm{B}}{v} + \frac{d}{u\cos\alpha_\mathrm{B}}\,. \tag{3}$$

We want to choose the angles α_A and α_B in such a way that the time t_s is minimal. A local extremum of a function may be found by placing the derivative equal to zero

$$\frac{\mathrm{d}t_\mathrm{s}}{\mathrm{d}\alpha_\mathrm{A}} = \frac{d\sin\alpha_\mathrm{A}}{u\cos^2\alpha_\mathrm{A}} - \frac{d}{v\cos^2\alpha_\mathrm{A}} = 0\,,$$

$$\frac{d(v\sin\alpha_\mathrm{A} - u)}{uv\cos^2\alpha_\mathrm{A}} = 0\,,$$

$$\sin\alpha_\mathrm{A} = \frac{u}{v}\,.$$

[1] Since the velocity of the tractor is constant in all directions, motion in a straight line is the most time-efficient.

$$
\begin{aligned}
\frac{\mathrm{d}t_\mathrm{s}}{\mathrm{d}\alpha_\mathrm{B}} = \frac{d\sin\alpha_\mathrm{B}}{u\cos^2\alpha_\mathrm{B}} - \frac{d}{v\cos^2\alpha_\mathrm{B}} &= 0\,,\\
\frac{d(v\sin\alpha_\mathrm{B} - u)}{uv\cos^2\alpha_\mathrm{B}} &= 0\,,\\
\sin\alpha_\mathrm{B} &= \frac{u}{v}\,.
\end{aligned}
$$

This result may be interpreted with knowledge from optics: the tractor must approach the boundary field-driveway at a critical angle. Since α_A and α_B are equal, from now on, we'll denote $\alpha_\mathrm{A} = \alpha_\mathrm{B} = \alpha$. From relations between goniometric functions $\cos\alpha = \sqrt{1-(u/v)^2}$ and $\tan\alpha = (u/v)/\sqrt{1-(u/v)^2}$, which we substitute into the equation (3),

$$
\begin{aligned}
t_\mathrm{s} &= \frac{2d}{u\cos\alpha} + \frac{r - 2d\tan\alpha}{v} = \frac{r}{v} + 2d\frac{v - u\sin\alpha}{uv\cos\alpha}\,,\\
t_\mathrm{s} &= \frac{r}{v} + 2d\frac{v - \frac{u^2}{v}}{u\sqrt{v^2-u^2}}\,,\\
t_\mathrm{s} &= \frac{r}{v} + \frac{2d}{uv}\sqrt{v^2-u^2}\,.
\end{aligned}
$$

From the condition in the problem statement, we know that $t_\mathrm{p} = t_\mathrm{s} = t$, so

$$
\begin{aligned}
\frac{r}{u} &= \frac{r}{v} + \frac{2d}{uv}\sqrt{v^2-u^2}\,,\\
r\,(v-u) &= 2d\sqrt{v^2-u^2}\,,\\
\frac{d}{r} &= \frac{1}{2}\sqrt{\frac{v-u}{v+u}}\,.
\end{aligned}
$$

If Jindra's ride from A to B takes the same time directly through the field as with the detour on the driveway, then the ratio of distances d and r satisfies $d/r = 1/2\sqrt{(v-u)/(v+u)}$.

Jindřich Jelínek
`jjelinek@fykos.cz`

Problem FB ... dome

The hemispherical dome of an observatory has a diameter $d = 20\,\mathrm{m}$ and mass $m = 200\,\mathrm{t}$, distributed uniformly. Find the minimal power of motors moving the dome which is needed to turn the dome by 180° in $t = 30\,\mathrm{s}$. The dome is sitting frictionlessly on track bearings. At the start and at the end, the dome must be stationary. *Dodo likes to observe the sky.*

Let us take a look at a different problem first. If we are given the maximal power, the fastest way to turn the dome is to accelerate it with this maximal power half of the time and decelerate it in the second half of the time (also with maximal power). That means it rotates with the maximum possible angular velocity at each point in time, so it turns in the shortest possible time. The desired power is the one which turns the dome in the given time t (we are turning it in this fastest way). Therefore, we need to find the power required to turn the dome by the angle $\Phi = \pi/2 = 90^\circ$ in $t/2 = \tau = 15\,\mathrm{s}$ with constant acceleration.

Since we have a constant power P, the kinetic energy satisfies

$$E_\mathrm{k} = \frac{1}{2} I \omega^2 = Pt\,,$$

where I is the moment of inertia of a hemisphere and ω is its instantaneous angular velocity at time t. We express the angular velocity and after integration by time from the beginning to the half-turn, we have

$$\omega = \sqrt{\frac{2Pt}{I}}\,,$$

$$\Phi = \int_0^t \omega \,\mathrm{d}t = \int_0^\tau \sqrt{\frac{2Pt}{I}} \mathrm{d}t = \frac{2}{3}\sqrt{\frac{2P}{I}} \tau^{\frac{3}{2}}\,.$$

From there, we can express the desired power as

$$P = \frac{9\Phi^2 I}{8\tau^3} = \frac{9\Phi^2 I}{t^3}\,.$$

Now we need only to find the moment of inertia of the hemisphere. It can be calculated as half of the moment of inertia of a sphere (not a ball, by sphere we mean only the surface, or rather a thin layer underneath it), since it's cut in half symmetrically (i.e. the upper and lower hemispheres have the same moments of inertia with respect to the given axis). The hemisphere also has half of the weight of a whole sphere. Therefore

$$I = \frac{2}{3} mR^2 = \frac{1}{6} md^2\,,$$

which, after substitution into the expression for power, gives us

$$P = \frac{3\Phi^2 md^2}{2t^3} = \frac{3\pi^2 md^2}{8t^3} \doteq 11.0\,\mathrm{kW}\,.$$

The power required to turn the hemisphere in the given time is $P = 11.0\,\mathrm{kW}$.

Jozef Lipták
`liptak.j@fykos.cz`

Problem FC ... light please

On a ceiling at a height $h = 2.5\,\mathrm{m}$, there is a light bulb that shines isotropically into the whole space below it with a luminous flux $\Phi = 1400\,\mathrm{lm}$. How large is the area on the ground where the illuminance is greater than $E_0 = 25\,\mathrm{lx}$? *Danka needs better lights at the dorm.*

Illuminance at a distance r from the source is given by the formula

$$E = \frac{I}{r^2} \cos\alpha\,,$$

where I is the luminous intensity of the source and α is the angle between the incident beam and the normal to the illuminated area. For the luminous intensity, we have

$$I = \frac{\Phi}{\Theta}\,,$$

where Θ is the solid angle at which light propagates from the source. In our case, $\Theta = 2\pi$. The illuminance is then $E = \frac{\Phi\cos\alpha}{2\pi r^2}$. The distance r is simply expressed from a right triangle as $r = \frac{h}{\cos\alpha}$. Then, the condition for illumination can be written as

$$\frac{\Phi\cos^3\alpha}{2\pi h^2} > E_0\,.$$

From there, we get the condition $\cos\alpha > \sqrt[3]{\frac{E_0 h^2 2\pi}{\Phi}}$, so the maximum angle is $\alpha_m \approx 27.3\,^{\circ}$. Then, the area of the circle is $S = \pi x^2$, where $x = h\tan\alpha_m$, so $S = \pi h^2\tan^2\alpha \doteq 5.2\,\mathrm{m}^2$. The area on the ground where the illuminance is greater than $25\,\mathrm{lx}$ is about $5.2\,\mathrm{m}^2$.

Daniela Pittnerová
daniela@fykos.cz

Problem FD ... electron-y

What are the possible values of the total electron spin quantum number of a neutral atom of nitrogen? For each option, it is possible to find the most energetically favourable configuration. State the values of the spin in the order that corresponds to increasing energy of these confugurations. *Dodo is breaking Hund's rules.*

The spin quantum number of an electron is 1/2 and an atom of nitrogen has 7 electrons. The total value of the electron spin of the whole atom can reach only the values which we can get by choosing the signs in the expression

$$\left|\frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2}\right|\,,$$

so after combining the spins, we get values 1/2, 3/2, 5/2 and 7/2. Pauli's principle implies that a full orbital has spin 0. According to Hund's rules, the lowest energy state of the nitrogen atom is $1s^2 2s^2 2p_x^1 2p_y^1 2p_z^1$, where in p orbitals, all of the spins are oriented the same direction (WLOG[2] up). Then, the nitrogen has a total spin 3/2, which we find as the difference between the number of electrons with upward spin and the number of electrons with downward spin. For the other options, we need to increase the energy. For the total spin 1/2, a state with only one unpaired electron works - for instance, $1s^2 2s^2 2p_x^2 2p_y^1 2p_z^0$; another option is to take the ground state and just flip the spin of one of the unpaired electrons. For the other options, we can't use only the orbitals that are full in the ground state, because we need to have at least 5 unpaired electrons. For a spin 5/2, we have to open the orbital $3s$ and use the state $1s^2 2s^1 2p_x^1 2p_y^1 2p_z^1 3s^1$, which will have much higher energy. The energetically worst is the lowest-energy state with the

[2] Without Loss Of Generality

spin 7/2, which requires us to open one more orbital $3p$; all seven lowest-energy orbitals need to have unpaired electrons, so the configuration is $1s^1 2s^1 2p_x^1 2p_y^1 2p_z^1 3s^1 3p^1$.

$$S=\frac{3}{2}:\quad \underset{1s}{\boxed{\upharpoonleft\downharpoonright}}\;\;\underset{2s}{\boxed{\upharpoonleft\downharpoonright}}\;\;\underset{2p}{\boxed{\upharpoonleft}\boxed{\upharpoonleft}\boxed{\upharpoonleft}}$$

$$S=\frac{1}{2}:\quad \underset{1s}{\boxed{\upharpoonleft\downharpoonright}}\;\;\underset{2s}{\boxed{\upharpoonleft\downharpoonright}}\;\;\underset{2p}{\boxed{\upharpoonleft\downharpoonright}\boxed{\upharpoonleft}\boxed{}}$$

$$S=\frac{5}{2}:\quad \underset{1s}{\boxed{\upharpoonleft\downharpoonright}}\;\;\underset{2s}{\boxed{\upharpoonleft}}\;\;\underset{2p}{\boxed{\upharpoonleft}\boxed{\upharpoonleft}\boxed{\upharpoonleft}}\;\;\underset{3s}{\boxed{\upharpoonleft}}$$

$$S=\frac{7}{2}:\quad \underset{1s}{\boxed{\upharpoonleft}}\;\;\underset{2s}{\boxed{\upharpoonleft}}\;\;\underset{2p}{\boxed{\upharpoonleft}\boxed{\upharpoonleft}\boxed{\upharpoonleft}}\;\;\underset{3s}{\boxed{\upharpoonleft}}\;\;\underset{3p}{\boxed{\upharpoonleft}\boxed{}\boxed{}}$$

Jozef Lipták
`liptak.j@fykos.cz`

Problem FE ... thirteen wells

Jáchym usurped all twelve wells from the previous problems and added one more, the thirteenth well. Then, to each well, he poured $V_0 = 169.00\,\mathrm{m}^3$ of one of thirteen different liquids. The last, thirteenth well, was full of blood. The ritual of invoking demons could begin. First, Jáchym transferred a volume $V = 13.00\,\mathrm{m}^3$ from the thirteenth well to the first one. After proper mixing, he transferred a volume V from the first well back to the thirteenth well. Then, he mixed up the liquid in the thirteenth well again and repeated the process with the second well, the third well and so on, up to the 12-th well. Altogether, there were 24 pourings. In the end, Jáchym measured the volume fractions of all liquids in the thirteenth well and multiplied these values. What was the result?

Jáchym was inspired by the problem "thirteen barrels" from the 7th Online Physics Brawl.

Let's substitute $k = \frac{V}{V_0}$ and assume from now on that the volume in each well is 1. The composition of the liquid in the thirteenth well after i steps is described by a vector $\mathbf{x}_i$. The concentration of the j-th liquid is expressed by the number x_i^j, where the blood has the index 0. At the begining (before all pourings), we may write

$$x_0^0 = 1\,,$$
$$x_0^j = 0 \quad \forall j > 0\,.$$

The convention used in this problem is as follows: We have thirteen vectors $\mathbf{x}$, which are indexed from $\mathbf{x}_0$ for the initial state to $\mathbf{x}_{12}$ for the final state. Each of these vectors has thirteen independent components denoted by upper indices (the notation is usually used for powers, but not in this case), so x_i^j is the j-th component of the vector $\mathbf{x}_i$ and it's just a number (a scalar).

Now, we will proceed by induction. After $i-1$ steps, we are in the state described by the vector $\mathbf{x}_{i-1}$. We take the volume $V = kV_0$ from the thirteenth well, represented by the expression $k\mathbf{x}_{i-1}$, and pour it into the i-th well. This well was full of the i-th liquid, so we can describe its initial composition by the vector $\mathbf{e}_i$, whose i-th component is 1 and all other components are 0. After the mixing of the initial liquid and new liquid from the 13-th well, we get the mixture described by $\mathbf{e}_i + k\mathbf{x}_{i-1}$.

The volume of the i-th well is now $V' = V + V_0 = (1+k)\,V_0$. From that, we take the volume

$$V = kV_0 = \frac{k}{1+k}V'\,.$$

Into the thirteenth well, we pour the liquid described by

$$\frac{k}{1+k}\left(\mathbf{e}_i + k\mathbf{x}_{i-1}\right)\,.$$

Before this step, there was the mixture $\mathbf{x}_{i-1}$ in the 13-th well; after taking the volume kV_0, there remained $(1-k)\,\mathbf{x}_{i-1}$, and we pour the mixture described by the expression above into it. The resulting mixture is

$$\mathbf{x}_i = (1-k)\,\mathbf{x}_{i-1} + \frac{k}{1+k}\left(\mathbf{e}_i + k\mathbf{x}_{i-1}\right) = \frac{\mathbf{x}_{i-1} + k\mathbf{e}_i}{1+k}\,.$$

We just expressed the change in the composition of the mixture in the 13-th well for one step. It was more or less a trivial application of the mixing equation, but we worked with 13 components at once. The notation we used may seem a bit complicated, but that is because we worked quite generally. On the other hand, we now have a formula that applies from the beginning to the end of the ritual.

Notice that the i-th step is the only event in which the i-th liquid is added to the thirteenth well. The concentration of the i-th liquid is now just decreasing $(1+k)$ times in each step. Now, we can write the final composition of the mixture in the thirteenth well

$$\mathbf{x}_{12} = \frac{\mathbf{e}_0}{(1+k)^{12}} + \frac{k}{1+k}\sum_{j=1}^{12}\frac{1}{(1+k)^{12-j}}\mathbf{e}_j\,.$$

We want the product of volume fractions, i.e. the product of the expressions before the unit vectors $\mathbf{e}$ in the expression above. Let's denote the result of this problem by P. Then, we may write

$$P = \frac{1}{(1+k)^{12}}\cdot\left(\frac{k}{1+k}\right)^{12}\cdot\prod_{j=1}^{12}\frac{1}{(1+k)^{12-j}} = \frac{k^{12}}{(1+k)^{24}}\cdot\prod_{j=0}^{11}\frac{1}{(1+k)^{j}} =$$
$$= \frac{k^{12}}{(1+k)^{90}} = V^{12}V_0^{78}\left(V+V_0\right)^{-90} \doteq 5.45\cdot 10^{-17}\,.$$

Jáchym Bártík
tuaki@fykos.cz

Problem FF ... boiling

Mikuláš has a well, but he is too lazy to pull water out of it by hand. Therefore, he bought a pump with an electric motor, whose efficiency η does not depend on the voltage or the current. What is the minimal depth of the well such that it would be more efficient to convert water into steam instead of pumping normally? The original temperature of the water is T and it would be boiled by a spiral with resistance R. The resistance per unit length of the power line that connects the spiral with the voltage source on the ground is λ. We expect a general result expressed using the quantities specified in the problem statement and generally known constants. *Jáchym remembered the problem DA from the 11. FYKOSí Fyziklání.*

Let's denote the depth of the well by h. Then the resistance of the power line is $h\lambda$. The power line and the spiral create a series electrical circuit with resistors and a source. As we want to minimise h, we should choose the voltage U on the source in a way that makes the heating the most efficient. The current in the network is

$$I = \frac{U}{R + h\lambda}\,,$$

and the thermal power of the spiral is

$$P_\mathrm{d} = U_R I = RI^2 = \frac{R}{(R + h\lambda)^2} U^2\,.$$

The corresponding electrical power of the engine is

$$P_\mathrm{m} = \eta U I = \eta U I = \eta \frac{1}{R + h\lambda} U^2 = \eta \frac{R + h\lambda}{R} P_\mathrm{d}\,.$$

Now, we have to find the mass of water that we move up per unit time with this power. In the case of the pump, it's quite easy, because the potential energy is mgh, so the mass flow rate is

$$q_\mathrm{m} = \frac{P_\mathrm{m}}{gh}\,.$$

To boil the water, we have to supply the energy $m\left(c\left(T_\mathrm{v} - T\right) + l_\mathrm{v}\right)$, where T_v is the boiling point, c is the specific heat capacity and l_v is the specific latent heat of vaporisation. Altogether, we get

$$q_\mathrm{d} = \frac{P_\mathrm{d}}{c\left(T_\mathrm{v} - T\right) + l_\mathrm{v}}\,.$$

We want to know when the mass flow rate is greater for boiling. This corresponds to the condition $q_\mathrm{d} \geq q_\mathrm{m}$, or

$$\frac{P_\mathrm{d}}{c\left(T_\mathrm{v} - T\right) + l_\mathrm{v}} \geq \frac{P_\mathrm{m}}{gh}\,,$$

$$\frac{Rg}{\eta\left(c\left(T_\mathrm{v} - T\right) + l_\mathrm{v}\right)} \frac{h}{R + h\lambda} \geq 1\,.$$

Now, we can see that the expression on the left is an increasing function that passes through the origin. That implies that equality holds for no more than one value h_0, and for all possible $h > h_0$, the condition above holds. Finally, we express the result

$$h_0 = \frac{R\eta\left(c\left(T_\mathrm{v} - T\right) + l_\mathrm{v}\right)}{Rg - \lambda\eta\left(c\left(T_\mathrm{v} - T\right) + l_\mathrm{v}\right)} = \left(\frac{g}{\eta\left(c\left(T_\mathrm{v} - T\right) + l_\mathrm{v}\right)} - \frac{\lambda}{R}\right)^{-1}\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem FG ... inverse well

Jáchym has an inverse well – water is constantly being replenished into it with a constant mass flow rate $q = 17.0\,\mathrm{kg\cdot min^{-1}}$ by a stream with a circular cross section. At what depth does continuous flow become unstable and break down into droplets with a radius $r = 2.5\,\mathrm{mm}$? The velocity of the water at the top of the well and air resistance are negligible.

Jáchym likes alternative wells.

Nature "tries" to minimise the total energy of any system. The droplets are being formed at a moment when their total energy would be lower than that of the stream. The surface energy is directly proportional to the surface area, so it is sufficient to minimise the area. A droplet with a radius r has surface area $S_\mathrm{k} = 4\pi r^2$. The droplet corresponds to a part of the stream with the same volume, which can be approximated as a cylinder with a height h and cross-sectional area S. Then, since it has the same volume as the droplet,

$$hS = \frac{4}{3}\pi r^3 \,.$$

Both bases of the cylinder are adjacent to other parts of the stream, so the bases don't contribute to the total surface energy. The surface area of the corresponding open cylinder is then

$$S_\mathrm{v} = 2\sqrt{\pi S}h = \frac{8\pi}{3}\sqrt{\frac{\pi}{S}}r^3 \,.$$

For $S_\mathrm{k} < S_\mathrm{v}$, the droplets are energetically more advantageous than the stream. For this case, we get

$$4\pi r^2 = \frac{8\pi}{3}\sqrt{\frac{\pi}{S}}r^3 \,,$$

$$S = \frac{4\pi}{9}r^2 \,.$$

Now we know the cross-sectional area of the stream at the moment of the split and we need the depth at which the split happens. The acceleration of the water is equal to the gravity of Earth, so in a time t, it falls by

$$x = \frac{1}{2}gt^2 \,.$$

The velocity at the time t is $v = gt$. The cross-sectional area S is related to the velocity and the volumetric flow rate as $vS = q_V = q/\varrho$, where ϱ is the density of water. By combining these equations, we get

$$x = \frac{v^2}{2g} = \frac{q^2}{2g\varrho^2 S^2} = \frac{81q^2}{32\pi^2 g\varrho^2 r^4} \doteq 54\,\mathrm{m} \,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem FH ... of course it's drinking water

Jáchym has a well and he constantly draws water from it with a volumetric flow rate $q_V = 0.2\,\mathrm{l{\cdot}s^{-1}}$. Water is being constantly replenished into the well so the total volume stays constant at $V = 68\,\mathrm{m^3}$. However, Jáchym accidentally dropped a piece of radioactive bread into the well, which dissipated perfectly in the water. The bread contained $3.0 \cdot 10^{15}$ radioactive isotopes with a half-life $T = 69\,\mathrm{h}$. Jáchym decided to ignore it and continued to draw water at the original rate. How long does it take until the radioactive activity of the well drops below $A = 1\,900\,\mathrm{s^{-1}}$?

Originally, this should have been about Danka's hair, but Jáchym said it will be about a well.

Radioactive decay obeys the equation

$$N_\mathrm{r} = N_0 \mathrm{e}^{-\lambda_\mathrm{r} t}\,,$$

where N_0 is the initial number of particles and $\lambda_\mathrm{r} = \frac{\ln 2}{T}$ is the decay constant. The number of particles that decay in a time $\mathrm{d}t$ is then

$$-\mathrm{d}N_\mathrm{r} = -\dot{N}_\mathrm{r}\mathrm{d}t = \lambda_\mathrm{r} N_0 \mathrm{e}^{-\lambda_\mathrm{r} t}\mathrm{d}t = \lambda_\mathrm{r} N \mathrm{d}t\,.$$

However, in our case, the radioactive solution is also being diluted by the clean water coming to the well. After the time $\mathrm{d}t$, the volume $q_V \mathrm{d}t$ will flow through the well. The number of radioactive particles leaving the well that way is

$$\frac{q_V \mathrm{d}t}{V} N = -\mathrm{d}N_\mathrm{v}\,.$$

Therefore, the equation describing the change in the number of radioactive particles in the well is

$$\mathrm{d}N = \mathrm{d}N_\mathrm{r} + \mathrm{d}N_\mathrm{v} = -\left(\lambda_\mathrm{r} + \frac{q_V}{V}\right) N \mathrm{d}t = -\lambda N \mathrm{d}t\,.$$

We now see that the number of particles in the well will again be an exponential function, but with a different constant. The activity can be calculated as $A = \lambda_\mathrm{r} N$, or

$$A = \lambda_\mathrm{r} N_0 \mathrm{e}^{-\lambda t}\,.$$

From this, we get the resulting time

$$t = -\frac{1}{\lambda}\ln\left(\frac{A}{\lambda_\mathrm{r} N_0}\right) = \frac{1}{\frac{\ln 2}{T} + \frac{q_V}{V}} \ln\left(\frac{N_0}{A}\frac{\ln 2}{T}\right) \doteq 740\,\mathrm{hod}\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem GA ... hole in a bucket

Dodo has a well with a bucket, which has a cylindrical shape - its height is $h_0 = 32\,\mathrm{cm}$, the radius of its base is $r = 12\,\mathrm{cm}$ and its weight is $m = 2.7\,\mathrm{kg}$. At the bottom of the bucket, there is a hole with a cross-section $S = 1.0\,\mathrm{cm^2}$. Dodo pulls the bucket up from a depth $H = 25\,\mathrm{m}$ at a constant speed $v = 0.40\,\mathrm{m{\cdot}s^{-1}}$. Compared to the situation if the bucket wasn't leaky, how many times less efficient is this procedure? We are asking about the ratio of works which Dodo *has to perform to pull a unit amount of water out of the well in both cases.*

Jáchym was eating soup with a fork.

Water from the bucket flows out with velocity $\sqrt{2hg}$, where h is the current water level. The volumetric flow rate is $q = S\sqrt{2hg}$, which can also be written as $-\dot{V}$, where $V = \pi r^2 h$ is the volume of water in the bucket. Hence, we get the equation

$$\pi r^2 \dot{h} = -S\sqrt{2hg}\,,$$

which has the solution

$$h = \left(\sqrt{h_0} - \frac{S}{\pi r^2}\sqrt{\frac{g}{2}}t\right)^2 .$$

The time Dodo needs to pull the bucket out is simply calculated as

$$\tau = \frac{H}{v}\,.$$

By substituting $t = \tau$ into the previous equation, we verify that all the water does not run out on the way up, and we find out that the water level in the leaky bucket when Dodo pulls it out is $h_1 = 6.75\,\mathrm{cm}$.

The total weight of the bucket with water is $m + \pi r^2 h$. From this, we get the force with which Dodo must pull

$$F = \left(m + \pi r^2 h\varrho\right) g\,.$$

Then, the power is $P = Fv$. Work is the integral of power over time

$$W_1 = \int_0^\tau P\,\mathrm{d}t = mgv\tau + \pi r^2\varrho gv\int_0^\tau h(t)\,\mathrm{d}t = mgv\tau + \pi r^2\varrho gv\left(-\frac{\pi r^2}{S}\sqrt{\frac{2}{g}}\right)\frac{1}{3}\left[h^{\frac{3}{2}}(t)\right]_0^\tau =$$
$$= mgH + \frac{\pi^2 r^4\varrho v\sqrt{2g}}{3S}\left(h_0^{\frac{3}{2}} - h_1^{\frac{3}{2}}\right) = 2.638\,\mathrm{kJ}\,.$$

The resulting volume of water that Dodo pulls up is $V_1 = \pi r^2 h_1$.

In the second case, the situation is much simpler — the force is the same all the time, so we calculate the work as

$$W_2 = \left(m + \pi r^2 h_0\varrho\right) gH = 4.211\,\mathrm{kJ}\,.$$

The volume of water also remains the same, specifically $V_2 = \pi r^2 h_0$.

The solution is the ratio

$$\frac{W_1}{V_1}\frac{V_2}{W_2} = \frac{W_1}{W_2}\frac{h_0}{h_1} \doteq 2.97\,.$$

If Dodo bought a new bucket, he would be nearly three times more efficient in pumping water from the well.

Jáchym Bártík
`tuaki@fykos.cz`

Problem GB ... excess heat

Geothermal energy comes from the decay of radioactive elements, from tidal deformations of the Earth and from the residual heat released by differentiation of the layers of Earth. The Earth has mass $M = 5.97 \cdot 10^{24}\,\mathrm{kg}$ *and radius* $R = 6.38 \cdot 10^{3}\,\mathrm{km}$. *Assume that at the beginning of its existence, it was a homogeneous sphere. After differentiation, a metallic (predominantly iron) core with a radius* $r_\mathrm{j} = 3.50 \cdot 10^{3}\,\mathrm{km}$ *and a density* $\varrho_\mathrm{j} = 13\,000\,\mathrm{kg{\cdot}m^{-3}}$ *formed within the Earth. The remainder of the Earth is the mantle, with a constant density. Find the heat released by this differentiation.* *Jindra felt like his soles were on fire.*

In the problem statement, it was mentioned that we should assume a homogeneous density of the mantle. First of all, though, we have to calculate it. The volume of a spherical shell with an inner radius r_1 and outer radius r_2 is

$$V_\mathrm{sl} = \frac{4}{3}\pi\left(r_2^3 - r_1^3\right).$$

The mass of the whole Earth is $M = 5.97 \cdot 10^{24}\,\mathrm{kg}$. The mass of the Earth's core can be calculated from its radius and density as

$$M_\mathrm{j} = \frac{4}{3}\pi r_j^3 \varrho_\mathrm{j} = 2.335 \cdot 10^{24}\,\mathrm{kg}.$$

Then, the average density of the Earth's mantle ϱ_p can be expressed as

$$\varrho_\mathrm{p} = \frac{M - M_\mathrm{j}}{\frac{4}{3}\pi\left(R^3 - r_\mathrm{j}^3\right)},$$
$$\varrho_\mathrm{p} = 4.003 \cdot 10^{3}\,\mathrm{kg{\cdot}m^{-3}}.$$

Now let's move on to the calculation of energy. At the beginning, the Earth is a homogeneous sphere, so its gravitational potential energy is

$$E_1 = -\frac{3GM^2}{5R},$$
$$E_1 = -2.237 \cdot 10^{32}\,\mathrm{J}.$$

After differentiation, the Earth divides into a core region and a mantle region. Its potential energy is equal to the sum of the potential energy of the core and the potential energy of the mantle. According to the shell theorem, the gravitational forces acting on a mass point within a homogeneous spherical shell cancel each other out. Thus, a mass located at a distance r from the centre of the Earth is affected only by gravitational force from mass below this radius r. In other words, the potential energy of the Earth's core E_j is not affected by the presence of the mantle and is calculated as

$$E_\mathrm{j} = -\frac{3GM_\mathrm{j}^2}{5r_\mathrm{j}} = -6.237 \cdot 10^{31}\,\mathrm{J}.$$

The potential energy of the Earth's mantle can be calculated using an integral. Let's place layers of the mantle on the Earth's core until we build the entire Earth. Suppose that we have already created an "Earth seed" with a radius $r > r_j$. This means that in the middle, there is the core, and above it is part of the shell with width $r - r_\mathrm{j}$. If we bring an infinitesimally thin spherical shell with a mass $\mathrm{d}m$ from infinity, the potential energy changes by

$$\mathrm{d}E = -\frac{G\frac{4}{3}\pi r_\mathrm{j}^3 \varrho_\mathrm{j} \mathrm{d}m}{r} - \frac{G\frac{4}{3}\pi \left(r^3 - r_\mathrm{j}^3\right) \varrho_\mathrm{p} \mathrm{d}m}{r}.$$

We can substitute $\mathrm{d}m = 4\pi \varrho_\mathrm{p} r^2 \mathrm{d}r$ and we get

$$\mathrm{d}E = -\frac{16}{3} G\pi^2 r_\mathrm{j}^3 \varrho_\mathrm{j} \varrho_\mathrm{p} r \mathrm{d}r - \frac{16}{3} G\pi^2 \left(r^3 - r_\mathrm{j}^3\right) \varrho_\mathrm{p}^2 r \mathrm{d}r,$$
$$\mathrm{d}E = \frac{16}{3} G\pi^2 r_\mathrm{j}^3 \varrho_\mathrm{p} \left(\varrho_\mathrm{p} - \varrho_\mathrm{j}\right) r \mathrm{d}r - \frac{16}{3} G\pi^2 \varrho_\mathrm{p}^2 r^4 \mathrm{d}r$$

Since the Earth's mantle extends between the radii r_j and R, its gravitational potential energy is calculated as

$$E_\mathrm{p} = \frac{16}{3} G\pi^2 r_\mathrm{j}^3 \varrho_\mathrm{p} \left(\varrho_\mathrm{p} - \varrho_\mathrm{j}\right) \int_{r_\mathrm{j}}^{R} r \mathrm{d}r - \frac{16}{3} G\pi^2 \varrho_\mathrm{p}^2 \int_{r_\mathrm{j}}^{R} r^4 \mathrm{d}r,$$
$$E_\mathrm{p} = \frac{8}{3} G\pi^2 r_\mathrm{j}^3 \varrho_\mathrm{p} \left(\varrho_\mathrm{p} - \varrho_\mathrm{j}\right) \left(R^2 - r_\mathrm{j}^2\right) - \frac{16}{15} G\pi^2 \varrho_\mathrm{p}^2 \left(R^5 - r_\mathrm{j}^5\right),$$
$$E_\mathrm{p} = -1.903 \cdot 10^{32}\,\mathrm{J}.$$

The total potential energy E_2 of the Earth after differentiation is the sum of the potential energy of the core $E_{\ j}$ and the potential energy of the mantle $E_{\ p}$

$$E_2 = E_\mathrm{j} + E_\mathrm{p} = -2.526 \cdot 10^{32}\,\mathrm{J}.$$

At the beginning, the Earth had more potential energy than after differentiation. The heat Q released by differentiation is their difference

$$Q = E_1 - E_2 = 2.9 \cdot 10^{31}\,\mathrm{J}.$$

The released heat is $2.9 \cdot 10^{31}$ J, which corresponds to "calorific value" 5 MJ·kg−1.

Jindřich Jelínek
`jjelinek@fykos.cz`

Problem GC ... parabolic rays

Your task is to find out how the refractive index of the atmosphere should depend on the height y above the surface, if we want the rays in the atmosphere to travel along parabolic trajectories $y = x^2$. Neglect the curvature of Earth. *Jurčo wants to see sideways.*

Using Snell's law, we should first realise that the refractive index must increase with height. The material (atmosphere) can be divided into thin horizontal layers with a constant refractive index. The beam will tilt by a small angle $\mathrm{d}\vartheta$ when moving up a layer. For two adjacent layers of the atmosphere, we can write Snell's law as

$$(n + \mathrm{d}n)\sin(\vartheta + \mathrm{d}\vartheta) = n \sin\vartheta\,,$$
$$n \sin\vartheta + n\cos\vartheta \mathrm{d}\vartheta + \mathrm{d}n \sin\vartheta = n\sin\vartheta\,,$$

from which we have

$$\frac{\mathrm{d}n}{n} = -\frac{\mathrm{d}\vartheta}{\tan\vartheta}\,.$$

The tangent of the angle between the beam and the vertical is obtained from the tangent of the parabola $y = x^2$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x = \tan(\pi/2 - \vartheta) = \cot\vartheta = \frac{1}{\tan\vartheta}\,.$$

By further differentiating this equation, we get the relationship between $\mathrm{d}\vartheta$ and $\mathrm{d}x$

$$2\mathrm{d}x = -\frac{\mathrm{d}\vartheta}{\sin^2\vartheta}\,,$$

where the sine squared can be expressed as

$$\sin^2\vartheta = \frac{1}{\cot^2\vartheta + 1}$$

$$\sin^2\vartheta = \frac{1}{4x^2 + 1} = \frac{1}{4y + 1}\,.$$

After substituting $\sin^2\vartheta$ and $\mathrm{d}\vartheta$ into our form of Snell's law, we have

$$\frac{\mathrm{d}n}{n} = \frac{1}{2}\frac{4\mathrm{d}y}{4y + 1}\,,$$

and from this, we get the resulting dependence of n on y

$$n = n_0\sqrt{\frac{4y + 1}{4y_0 + 1}}\,.$$

Róbert Jurčo
robert.jurco@fykos.cz

FYKOS
UK, Matematicko-fyzikální fakulta
Ústav teoretické fyziky
V Holešovičkách 2
180 00 Praha 8

www: http://fykos.cz
e-mail: fykos@fykos.cz

FYKOS is also on Facebook
http://www.facebook.com/FYKOS

FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Public Relations Office activities and is supported by Institute of Theoretical Physics, MFF UK, his employees and The Union of Czech Mathematicians and Physicists.

Solutions of the Fyziklani 2021

15^{th} year

Problem M.1 ... a pram 5 points

A pram (baby carriage) weighs $M = 7.0\,\mathrm{kg}$. Its weight is uniformly distributed over four wheels. Find the ratio of the force which a mother must exert on a pram with a baby weighing $m_1 = 7.0\,\mathrm{kg}$ in order to move it, to this force for a pram with a baby weighing $m_2 = 3.5\,\mathrm{kg}$. The main source of resistance is located at the points of contact between the wheels and the ground.

Danka is wondering about maternity.

The rolling resistance is given as

$$F_\mathrm{r} = f F_\mathrm{n}\,,$$

where f is coefficient of friction[1] and F_n is the normal force, that pushes the wheel against the ground, which, in our case, equals $F_\mathrm{n} = F_g = (M + m_{1,2})\,g$. The fact, that the weight is distributed over all four wheels, does not affect the friction force in any way. For ratio of friction forces we get

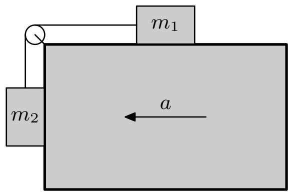
$$\frac{F_{\mathrm{r},1}}{F_{\mathrm{r},2}} = \frac{fg\,(M + m_1)}{fg\,(M + m_2)} \doteq 1.33\,.$$

The mother must exert a force 1.33 times greater while pushing a heavier child.

Daniela Pittnerová
`daniela@fykos.cz`

Problem M.2 ... pulleys on a railway car 5 points

Consider a railway car in the shape of a cuboid which moves with a constant acceleration a. On its upper surface, there is a cuboid (mass $m_1 = 15\,\mathrm{kg}$) attached by a rope to another cuboid (mass $m_2 = 10\,\mathrm{kg}$), which is hanging freely at the front side of the car. Find the value of the acceleration a which prevents those cuboids from accelerating (with respect to the car), assuming that there is zero friction in the system. *Lego likes pulleys.*



To keep the hanging cuboid from accelerating, the vertical component of the net force acting on it must be zero. The horizontal component, which results from the acceleration of the system, is compensated by normal force exerted by the front surface of the car. The gravitational force acting on the cuboid is simply $F_2 = m_2 g$. The only force that could compensate it is the force exerted by the rope. Therefore, this force must pull the cuboid upwards and its magnitude must be $T = F_2$.

The second cuboid is pulled by the same force towards the front of the car. This is the only real force exerted on the cuboid in the horizontal direction. To keep it from accelerating (with respect to the car), this force must cause it to accelerate forward with the acceleration a – therefore, its magnitude is $F_1 = m_1 a$. The gravitational force cancels out with normal force exerted by the roof and the only force left is the force pulling the rope, so $F_1 = T$.

[1] This is often given as the ratio of the arm of rolling friction (also called rolling friction coefficient) and the radius of the wheel, but that is not entirely correct.

Putting it all together, we get the desired acceleration of the car

$$a = \frac{m_2}{m_1} g \doteq 6.5\,\mathrm{m{\cdot}s^{-2}}\,.$$

Šimon Pajger
legolas@fykos.cz

Problem M.3 ... a restless washing machine 5 points

In a washing machine, there is a cylindrical drum with laundry, whose axis is horizontal. We approximate the wet laundry by a rigid body with a mass $m = 3.5\,\mathrm{kg}$ and with the centre of mass located at a distance $r = 4.9\,\mathrm{cm}$ from the axis of rotation of the drum. The drum rotates at 421 rpm. What is the minimum weight of the washing machine needed to ensure that it does not bounce? Assume that the washing machine can only move in the vertical direction.

Jarda was distracted from creating problems by a washing machine.

The laundry is maintained on the circular trajectory by the centripetal force of

$$F_\mathrm{c} = \frac{mv^2}{r} = m\omega^2 r = 4\pi^2 m f^2 r\,,$$

where we express the frequency as $f = \frac{421\,\mathrm{rpm}}{60\,\mathrm{s}} \doteq 7.02\,\mathrm{Hz}$. This centripetal force (in the ground frame of reference) is the sum of the other forces, i.e. the force of gravity and the force imparted on the drum by the washing machine. We can express the latter as a vector $\mathbf{F} = \mathbf{F}_\mathrm{c} - \mathbf{F}_g$. It follows from the law of action-reaction that the same force acts on the washing machine. Its direction changes, sometimes pushing the washing machine to the ground, sometimes lifting it. Just when the laundry is above the drum's axis of rotation, the reaction to the centripetal force is directed upwards. In order for the washing machine to bounce up from the ground, this force has to be larger than its weight. From this we get the minimum weight of the washing machine as

$$M = m\left(\frac{4\pi^2 f^2 r}{g} - 1\right) \doteq 30.5\,\mathrm{kg}\,.$$

Jaroslav Herman
jardah@fykos.cz

Problem M.4 ... negative Moon 5 points

For the purposes of this problem, we can approximate the Earth as a perfect sphere with density $\rho_\mathrm{E} = 5.52\,\mathrm{g{\cdot}cm^{-3}}$. The same applies to the Moon, but its density is $\rho_\mathrm{M} = 3.34\,\mathrm{g{\cdot}cm^{-3}}$. We want to place electric charge uniformly in the whole volume of each body. What should be the value of the charge density (the same for both bodies) if we want the total interaction force between the two bodies to be zero? *Jáchym was walking on the street at night.*

To meet the condition from the problem statement, the size of the attractive gravitational force has to be the same as the size of the repulsive electrostatic force. Marking the distance between the centres of the bodies as r, we get

$$\frac{Gm_\mathrm{E}m_\mathrm{M}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_\mathrm{E}q_\mathrm{M}}{r^2}\,.$$

The mass can be calculated as the product of the density and volume, i.e. $m = \rho V$. If we mark the charge density ρ_c, a similar relation $q = \rho_\mathrm{c} V$ applies for the charge. Substituting it to the previous equation we get

$$G\rho_\mathrm{E} V_\mathrm{E} \rho_\mathrm{M} V_\mathrm{M} = \frac{1}{4\pi\varepsilon_0} \rho_\mathrm{c} V_\mathrm{E} \rho_\mathrm{c} V_\mathrm{M} \quad \Rightarrow \quad \rho_\mathrm{c} = \sqrt{4\pi\varepsilon_0 G \rho_\mathrm{E} \rho_\mathrm{M}} \doteq 3.70 \cdot 10^{-7}\,\mathrm{C{\cdot}m^{-3}}\,.$$

Jáchym Bártík
tuaki@fykos.cz

Problem M.5 ... relaxing at a train station 5 points

Mišo is sitting in a house next to a railway, looking out for his favourite train. He only sees part of the tracks through the window, so he would like to know how much time he has until the train disappears from his view. He measured that the window is $c = 1.5\,\mathrm{m}$ wide and it's $p = 10\,\mathrm{m}$ far from the tracks. The chair he is sitting on is right in front of the center of the window, $l = 2.0\,\mathrm{m}$ far from it. Mišo also knows that the train is $d = 100\,\mathrm{m}$ long and its braking distance is $s = 150\,\mathrm{m}$. How much time does he have to watch the train (i.e. from the moment the locomotive appears to the moment the end of the train disappears from his view), if the train starts to evenly decelerate from a velocity $v = 20\,\mathrm{m{\cdot}s^{-1}}$ right at the time when Mišo first spots it? Neglect the width of the train. *Verča went on an expedition.*

Let us begin by finding the deceleration a of the train. We know the braking distance s of the train and its initial velocity v. Therefore we find the deceleration from the relations for uniformly accelerated motion as

$$s = \frac{1}{2}at^2 = \frac{v^2}{2a} \quad \Rightarrow \quad a = \frac{v^2}{2s}\,.$$

Now we calculate the distance travelled by the train while Mišo can watch it. We will need the length of the rails that are visible from the window; let us denote it b. The triangle formed by Mišo's chair and sides of the window is similar to the one formed by the chair and the furthest points of the rail that is visible from the window. Therefore

$$\frac{c}{b} = \frac{l}{l+p} \quad \Rightarrow \quad b = \left(1 + \frac{p}{l}\right) c\,,$$

holds. The distance the train will travel during Mišo's observation, equals the distance b and the length d of the train. Since we already know the deceleration of the train and the distance that it should travel, we can find the desired time by solving the quadratic equation, which yields

$$b + d = vt - \frac{1}{2}at^2 \quad \Rightarrow \quad t = \frac{2s}{v}\left(1 \pm \sqrt{1 - \left(\left(1 + \frac{p}{l}\right)c + d\right)\frac{1}{s}}\right)\,.$$

Plugging in the numerical values we get two results $t_1 \doteq 22.8\,\mathrm{s}$ and $t_2 \doteq 7.16\,\mathrm{s}$. The desired time is t_2, while t_1 marks the time, after which the train would get into the same position, if it moved backwards with the acceleration $-a$ after stopping.

Veronika Hendrychová
vercah@fykos.cz

Problem M.6 ... the importance of having passengers 5 points

A frontal collision occurred between two cars. They both were the same models with the same manufacturer, each weighing $1\,200\,\mathrm{kg}$*, and both were moving at* $90\,\mathrm{km\cdot h^{-1}}$ *before the impact. One car was occupied only by a driver weighing* $80\,\mathrm{kg}$*, while the other one was occupied by more people (with total weight* $200\,\mathrm{kg}$*). The effects of the impact were mitigated by deformation of the whole engine spaces of the cars and the impact took* $80.0\,\mathrm{ms}$*. Calculate the average decelaration exerted on the crew of the lighter car in the units of the standard acceleration due to gravity* $g = 9.81\,\mathrm{m\cdot s^{-2}}$*.*

Jindra travelled with Dano by car.

In the reference frame connected to the ground, both cars move with the velocity $v = 90\,\mathrm{km\cdot h^{-1}}$, which equals $25\,\mathrm{m\cdot s^{-1}}$, against each other. The masses are different, therefore the centre of mass of the system moves as well. For simplicity, we move into the inertial reference frame connected with the centre of mass (centre of mass reference frame) of the two-car system. The acceleration does not change during transformations between inertial reference frames. Let $m_1 = 1\,280\,\mathrm{kg}$ denote the mass of the car with the driver only and $m_2 = 1\,400\,\mathrm{kg}$ the mass of a car with more passengers. In the centre of mass reference frame the first car moves with the velocity

$$v_1 = v\frac{2m_2}{m_1 + m_2}$$

while the second one with

$$v_2 = v\frac{2m_1}{m_1 + m_2} .$$

After the collision, both cars are at rest in this reference frame. The collision took $t = 80.0\,\mathrm{ms}$, therefore the average deceleration affecting the passengers of the lighter car can be calculated as

$$a_1 = \frac{v_1}{t} = \frac{v}{t}\frac{2m_2}{m_1 + m_2} .$$

Plugging in the numerical values we get $a_1 = 326\,\mathrm{m\cdot s^{-2}} = 33.3\,\mathrm{G}$ (G is the unit used in aerospace – ratio between the acceleration and the gravitational acceleration). Just for your information, the ratio between accelerations of the heavier and the lighter car is $a_1/a_2 = m_2/m_1 \doteq 1.09$.

Jindřich Jelínek
`jjelinek@fykos.cz`

Problem M.7 ... a football problem 5 points

Jarda was watching the football Champions League and wondering how accurate the passes must be. Imagine a player passing the ball to his teammate. The teammate is running directly away from the passing player with a velocity $u = 7.0\,\mathrm{m\cdot s^{-1}}$*. At the time of the pass (when the passing player kicks the ball), he's standing on the offside line, which is at a distance* $L = 20\,\mathrm{m}$ *from the passing player. The passing player decides to kick the ball at an elevation angle* $\alpha = 30^\circ$ *with respect to the ground directly towards the running teammate. Find the velocity of the ball such that it falls just at the feet of the running player.*

Jarda watches football instead of attending lectures.

In order for the running player to get the ball exactly to his feet, he has to be in the place where the ball falls on the ground at the same time as the ball. Let's mark the time that the ball spends in the air as t. The initial vertical velocity is $v_y = v_0 \sin\alpha$ where v_0 is the initial velocity of the ball. The ball stops rising when the vertical velocity drops to zero, which happens in time $t_1 = \frac{v_y}{g}$. It will take the same amount of time for the ball to fall, therefore it spends a total of $t = \frac{2v_y}{g}$ in the air. During this time it flies a horizontal distance of

$$x = v_x t = v_0 \cos\alpha \frac{2v_0 \sin\alpha}{g} = \frac{{v_0}^2 \sin 2\alpha}{g}\,.$$

The running player has to be at this distance from the passer at the time when the ball falls. He had to cover the distance $s = ut = x - L$ during the time t. When substituting t, we get from this and the previous equation a quadratic equation for the initial velocity

$$v_0^2 \sin 2\alpha - 2uv_0 \sin\alpha - Lg = 0\,.$$

The solution is

$$v_0 = \frac{u \sin\alpha + \sqrt{u^2 \sin^2\alpha + Lg \sin 2\alpha}}{\sin 2\alpha} \doteq 19.6\,\mathrm{m{\cdot}s^{-1}}\,,$$

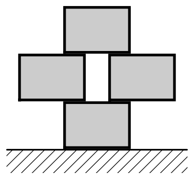
where we chose the positive root, as the negative one has no physical meaning.

Jaroslav Herman
`jardah@fykos.cz`

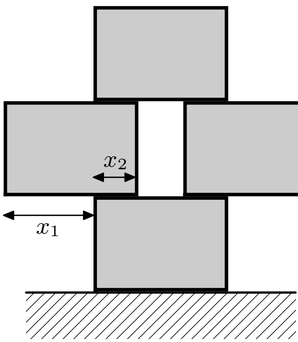
Problem M.8 ... a wall

5 points

We have four identical blocks (cuboids) with edge lengths $a \times b \times b$, where $a = 29.7\,\mathrm{cm}$ and $b = 21.0\,\mathrm{cm}$. We put these blocks on top of each other as shown in the figure. What is the maximum width of the gap between the two middle blocks? *Jáchym wanted to build a wall.*



If the brick was about to fall, it would turn around the edge of the lowest brick. This edge divides the middle brick into two regions $x_1 + x_2 = a$ as you can see in the picture.



The total torque with respect to this axis must be zero. The left part has mass

$$m_1 = \frac{x_1}{a} m\,.$$

The centre of mass of this part of the brick is $x_1/2$ from the edge of the bottom brick. The torque caused by the force of gravity is $x_1 m_1 g/2$.

Similarly, the right part of the brick experiences torque $x_2 m_2 g/2$ and the torque caused by the top brick, too. Its weight is equally spread across the two bricks. This force acts in the upper right edge of the middle brick (imagine that the top brick rotates by a small angle), which means that the perpendicular distance is x_2 and the torque is $x_2 mg/2$.

The torques must be equal

$$\frac{x_1 m_1 g}{2} = \frac{x_2 m_2 g}{2} + \frac{x_2 mg}{2}\,,$$
$$x_1^2 = x_2^2 + a x_2\,.$$

We substitute $x_1 = a - x_2$ and get

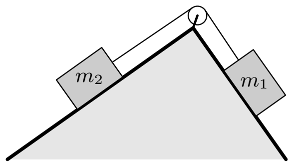
$$x_2 = \frac{1}{3} a\,.$$

The maximum width of the gap is

$$x = a - 2x_2 = \frac{1}{3} a \doteq 9.9\,\mathrm{cm}\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem M.9 ... pulleys on an inclined block — 5 points



Consider a prism whose bases are right triangles. A horizontal view of the prism is shown in the figure (the bases are vertical). On its two perpendicular side faces, there are two blocks with masses $m_1 = 15\,\mathrm{kg}$ and $m_2 = 10\,\mathrm{kg}$ connected by a rope through a pulley. What must be the angle between the side with the block with the mass m_1 and the horizontal plane if we want to keep the blocks at rest, assuming there is no friction in the system?

Besides pulleys, Lego likes problems that are similar.

Three different forces are acting on both blocks:

- gravitational force, acting downwards with magnitude $m_i g$,
- normal force from the face of the prism – this one prevents the blocks from "falling through" by cancelling out the component of the gravitational force, which is perpendicular to the face of the prism – therefore only the component parallel to the wall remains,
- the pull force of the rope, which pulls both blocks upwards, parallel to the wall.

If we want the blocks to remain in rest, then the force, which pulls the rope, must be equal to the magnitude of the component of the gravitational force, which is parallel to the wall. Furthermore, consider that it has the same magnitude for both blocks (otherwise the net force acting on the rope would be non-zero, which will cause the rope and therefore the blocks to accelerate). We conclude that the component of gravitational force parallel to the wall must be equal for both blocks, yielding

$$F_{\mathrm{p},1} = F_{\mathrm{p},2}\,,$$
$$m_1 g \sin\varphi = m_2 g \sin\left(\frac{\pi}{2} - \varphi\right) = m_2 g \cos\varphi\,,$$
$$\varphi = \arctan\frac{m_2}{m_1} \doteq 34^\circ\,.$$

Šimon Pajger
`legolas@fykos.cz`

Problem M.10 ... A

5 points

The letter A consists of two long rods, each with a length $4l$, that are connected by a hinge at the top and by a rope with a length l at their midpoints. If each of these rods has a diameter $r = l/64$ and is made of stiff metal with a density $\rho = 6\,850\,\mathrm{kg\cdot m^{-3}}$, what is the tallest letter A we can build? The rope has the same radius as the rods, but it is made of a material with negligible density, which can handle tensile stress up to $\sigma = 4.50\,\mathrm{MPa}$. Neglect friction.

Jáchym wanted the first problem (at least alphabetically) to be his.

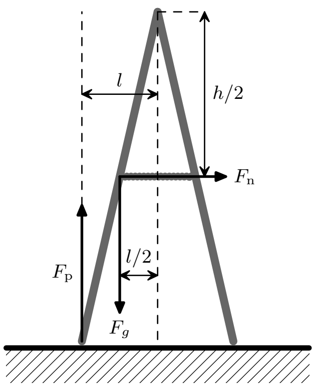
The length of the rod is $4l$, so its mass is

$$m = 4\pi\rho r^2 l\,.$$

The tensile stress on the rope is σ, so the force it imparts on each rod is

$$F_\mathrm{s} = \pi r^2 \sigma\,.$$

Each rod also experiences the reactive force from the ground, the magnitude of which is equal to the force of gravity $F_g = mg$ acting in the centre of mass. The last force is acting at the hinge between the rods, but we won't need its magnitude.



The total torque acting on the rods must be zero to keep the system in balance. Determining the torques relative to the point of the hinge, the moment arm of the reaction force from the ground is l, the momentum arm of the tension force is $h/2$ and the arm of the gravitational force is $l/2$, where h is the height of the letter for which we can write

$$h = \sqrt{(4l)^2 - l^2} = \sqrt{15}l\,.$$

From the condition of zero torque, we obtain

$$\begin{aligned} 0 &= lF_g - \frac{l}{2}F_g - \frac{h}{2}F_\mathrm{s}\,,\\ 0 &= \frac{4\pi\rho r^2 l^2 g}{2} - \frac{\sqrt{15}\pi r^2 l\sigma}{2}\,,\\ l &= \frac{\sqrt{15}\sigma}{4\rho g}\,. \end{aligned}$$

Notice, that the bigger the tension, the bigger l and the bigger h. To determine the maximum height of the letter, we use the maximum tensile stress and obtain

$$h = \sqrt{15}l = \frac{15\sigma}{4\rho g} \doteq 251\,\mathrm{m}\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem M.11 ... falling coins 5 points

We have many identical coins (discs with radii $r = 1\,\mathrm{cm}$) sliding down an inclined plane. The inclination angle of the plane is $\alpha = 45^\circ$ and it is $l = 50\,\mathrm{cm}$ long. At the bottom of the slope, there is a $d = 65\,\mathrm{cm}$ long horizontal plane. Each coin is released from rest at the top of the slope. What is the smallest number of coins (including the first coin) that we have to slide down the plane to ensure that the first coin will fall off the edge of the horizontal plane? Assume perfectly elastic collisions and a constant friction coefficient $f = 0.5$ between the coins and the surface. *Kiko knows his way around money.*

When a coin slides along the inclined plane, a force perpendicular to the plane acts upon it

$$F = mg\sin\alpha - fmg\cos\alpha\,.$$

The resultant acceleration of the coin is

$$a_1 = (\sin\alpha - f\cos\alpha)\,g\,.$$

That is an uniformly accelerated motion. The final velocity after sliding down the plane will be

$$v = \sqrt{2a_1 l}\,.$$

After that, another uniformly accelerated motion follows, this time slowing down due to friction

$$a_2 = fg\,.$$

The center of the first coin will then reach the distance of

$$x = \frac{v^2}{2a_2} = \frac{a_1}{a_2} l = \left(f^{-1} \sin\alpha - \cos\alpha\right) l \doteq 35.4\,\mathrm{cm}\,.$$

Thanks to equal mass of all the coins and the elasticity of the collisions the moving coin will always stop during the collision and transfer all its momentum to the next coin (as it is the case with Newton's pendulum). Thanks to this, the first coin move the distance equal to its diameter with each collision. This result could be calculated, but it is not neccesary since in the end, each collision is essentialy equivalent to teleporting the first coin forward by the distance equal to its diameter, while there are no losses of energy through friction during this "teleportation". As a result, with each successive coin, the same amount of energy must be lost and there is one more coin (and the whole system moves the distance of one diameter of the coin).

This way, we have created a pattern of holes and coins with the same width. The problem would be vastly more complicated if the coins started to accumulate at the spot where the incluned plane meets the horizontal plane. Luckily, that will not happen since the pattern is always symmetrical about the point where the center of the first coin originally stopped. And this point is sufficiently far from the center of the horizontal plane.

For the first coin to fall of the edge, its center must go beyond it. So the total number of coins n is given by the condition

$$2\,(n-1)\,r + x > d\,.$$

Which gives

$$n > \frac{d-x}{2r} + 1\,.$$

Numerically, we obtain $n = 16$.

Kristián Šalata
kristian.salata@fykos.cz

Problem M.12 ... leaping coin 5 points

Kiko placed a thin coin with mass $m = 4\,\mathrm{g}$ on the opening of a bottle of (still) iced tea, which was cooled down to $5\,^\circ\mathrm{C}$. The coin started rattling after a while. Determine how many times the coin lifted from the opening if the ambient temperature of the surrounding air was $25\,^\circ\mathrm{C}$ and the radius of the coin was $r = 0.5\,\mathrm{cm}$. *Kiko was bored while drinking beer.*

After taking the bottle out of a fridge, the air inside it start to heat up and therefore expand. Coin will jump when inside pressure rises enough to overcome coin's weight thus making the inside and outside pressures equal $p_1 = p_\mathrm{a}$. While the bottle is sealed, the gas undergoes an isochoric process

$$\frac{p_1}{T_1} = \frac{p_2}{T_2}\,.$$

Pressure difference required to lift the coin is

$$\Delta p = p_2 - p_\mathrm{a} = \frac{mg}{\pi r^2}\,.$$

Temperature inside the bottle after first jumps becomes

$$T_2 = p_2 \frac{T_1}{p_1} = \left(\frac{mg}{\pi r^2} + p_\mathrm{a}\right) \frac{T_1}{p_\mathrm{a}} = \left(\frac{mg}{\pi r^2 p_\mathrm{a}} + 1\right) T_1 \,.$$

After n^{th} jump the temperature inside the bottle becomes

$$T_n = \left(\frac{mg}{\pi r^2 p_\mathrm{a}} + 1\right)^n T_1 \,,$$

which is bound by temperature of surroundings at 25 °C. The number of jumps can be obtained from inequality

$$T_n < 25\,^\circ\mathrm{C} \,.$$

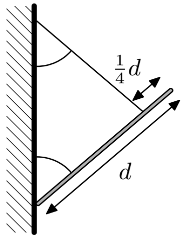
The maximum number of jumps can be computed either by logarithm or íteration. We need to remeber to use the Kelvin scale for temperature, finaly obtaining the result $n = 14$.

Kristián Šalata
`kristian.salata@fykos.cz`

Problem M.13 ... climbing

5 points

A mountain climber with weight $m = 75\,\mathrm{kg}$ *and height* $d = 180\,\mathrm{cm}$ *is climbing up a vertical wall. At some point, he is standing in such a way that the angle between his body and the wall is the same as the angle between the wall and the rope. The climber is connected to the rope via a full-body harness at the distance* $\frac{3}{4}d$ *from the point where his shoes are touching the wall. The current friction force between the mountaineer's shoes and the wall is exactly the threshold static friction force. Find the magnitude of the force acting on (stretching) the rope. The coefficient of static friction between the wall and the shoes is* $f = 0.40$. *Assume that gravity acts on the mountain climber in his centre of mass, located at the midpoint of his body.*



Danka was wondering about climbing.

Let's mark the angle mentioned in the problem statement as α, $k = 3/4$ and F is the force the rope exerts on him. Further, the climber is affected by the normal force from the wall F_n, the frictional force F_f and the gravitational force F_g. The horizontal components of these forces must be in balance, so

$$F_\mathrm{n} - F\sin\alpha = 0\,.$$

Similarly, the sum of the vertical components must be zero

$$F_\mathrm{f} + F\cos\alpha - F_g = 0\,.$$

Applying the condition of zero total torque relative to the point, where the climber touches the rope, we get

$$F_\mathrm{f} kd\sin\alpha - F_\mathrm{n} kd\cos\alpha - F_g\left(k - \frac{1}{2}\right)d\sin\alpha = 0\,.$$

The magnitudes of frictional and gravitational forces are the last two needed relations, written as

$$F_\mathrm{f} = fF_\mathrm{n}\,,$$
$$F_g = mg\,.$$

Now, we have five equations for five variables (four forces and one angle). The first step in the solution is to substitue in the frictional force as above. Next, from the first two equations, we express the required tension of the rope

$$F = \frac{F_\mathrm{n}}{\sin\alpha} = \frac{F_g - fF_\mathrm{n}}{\cos\alpha} \quad\Rightarrow\quad \tan\alpha = \frac{F_\mathrm{n}}{F_g - fF_\mathrm{n}}\,.$$

We got rid of F by this formula and we got tangent of an angle α. This can also be easily expressed from the torques equation, from which we get

$$\tan\alpha = \frac{2F_\mathrm{n}k}{2fF_\mathrm{n}k - F_g\left(2k-1\right)}\,.$$

If we put both formulas for $\tan\alpha$ together, we finally get the relation between normal and gravitational force

$$F_\mathrm{n} = \frac{F_g\left(4k-1\right)}{4fk}\,.$$

Putting this into the previous equation we get

$$\tan\alpha = \frac{4k-1}{f} \quad \Rightarrow \quad \sin\alpha = \frac{4k-1}{\sqrt{f^2+(4k-1)^2}}\,,$$

where we used the mathematical identity

$$\sin\arctan x = \frac{x}{\sqrt{1+x^2}}\,.$$

Now we just plug in one of the previous expressions for F and we get the result

$$F = \frac{F_\mathrm{n}}{\sin\alpha} = \frac{F_g\,(4k-1)}{4fk}\,\frac{\sqrt{f^2+(4k-1)^2}}{4k-1} = \frac{mg}{4k}\sqrt{1+\left(\frac{4k-1}{f}\right)^2} \doteq 1\,250\,\mathrm{N}\,.$$

Jáchym Bártík
tuaki@fykos.cz

Problem M.14 ... an impact into a wall 5 points

Originally, Jindra wanted to let a rolling cylinder impact a wall, but he realised that he cannot calculate how the impact would progress. Therefore, he took a homogeneous cylinder with a mass $m = 12\,\mathrm{kg}$, length $l = 50\,\mathrm{cm}$ and radius $r = 10\,\mathrm{cm}$, let it spin around its axis with an angular velocity $\omega = 50\,\mathrm{rad{\cdot}s^{-1}}$ and placed it on the floor next to the wall in such a way that it was touching both the floor and the wall and slipping occured at both of these points of contact and the cylinder was rolling into the wall (if the wall did not exist, it would roll in its direction). The coefficient of friction between the cylinder and the floor is $\mu_1 = 0.38$, and between the cylinder and the wall it is $\mu_2 = 0.57$. Find the magnitude of the friction force between the cylinder and the wall. *Jindra was demolishing a wall.*

When the cylinder touches the wall, several forces act on it. The force of gravity $\mathbf{F}_g$ acts downwards, the friction force $\mathbf{T}_1$ from the floor decelerates the rotation and pushes the cylinder towards the wall, the reaction force $\mathbf{N}_2$ from the wall counteracts the effects of the friction force from the floor, the friction force $\mathbf{T}_2$ from the wall acts upwards and decelerates the rotation, and the reaction force $\mathbf{N}_1$ from the floor counteracts the effects of the force of gravity and the friction force from the wall. Since the cylinder skids, the magnitudes of the friction forces are maximum possible (i.e. the coefficient of friction multiplied by the magnitude of the normal force). Balancing the forces in both directions

$$\begin{aligned} T_1 &= N_2\,,\\ T_2 &= mg - N_1\,. \end{aligned}$$

We also have the defining relations for the friction force $T_i = \mu_i N_i$. That is a system of linear equations with four variables (N_1, N_2, T_1, T_2) and we want to find T_2. First, we substitute the appropriate $T_i\mu_i^{-1}$ for N_i. Then, we express the force from the first equation

$$T_1 = \mu_2^{-1} T_2\,,$$

and substitute it to the second one, and then we obtain

$$T_2 = mg - (\mu_1\mu_2)^{-1} T_2 \,,$$
$$T_2 = \frac{mg}{(\mu_1\mu_2)^{-1} + 1} \,.$$

After substituting the numbers from the problem statement, we get $T_2 \doteq 21.0\,\mathrm{N}$.

Jindřich Jelínek
`jjelinek@fykos.cz`

Problem M.15 ... non-quantum entanglement 5 points

Assume that we have a point from which a pair of photons originates. Polarisations of these photons are perpendicular to each other. As the name of this problem suggests, we do not claim that they are in superposition – both photons have given polarisations, but we do not know which. Furthermore, for each photon, we place a polariser perpendicularly to its trajectory. Assuming that the planes of polarisation of the polarisers are parallel to each other, what is the probability that both photons from a given pair pass through the polarisers?

Lego was preparing his lecture for a physics camp.

Let's mark the angle between the polarization plane of one photon and the polarization plane of the polarizers φ. The probability that this photon will pass through the polarizer is $p_1 = \cos^2\varphi$. The equally defined angle for the second photon is $\pi/2 - \varphi$ and the probability that it flies through the polarizer is $p_{12} = p_1 p_2 = \cos^2\varphi \sin^2\varphi = 1/4 \sin^2(2\varphi)$. So the probability that both photons fly through, is

$$p_{12} = p_1 p_2 = \cos^2\varphi \sin^2\varphi = 1/4 \sin^2(2\varphi) \;.$$

The angle φ is random and it is needed to define the expected value

$$\overline{p}_{12} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} p_{12}(\varphi)\,\mathrm{d}\varphi = \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \sin^2(2\varphi)\,\mathrm{d}\varphi = \frac{1}{2\pi}\frac{\pi}{4} = \frac{1}{8} \,,$$

where it is enough for the integration to know that the integral of $\sin^2$ is through an integer multiple of its half-period, which is always equal to half of the interval. We could just as well write the average value through the whole 2π, but we could also do it through $\pi/4$. Let us now note that when (for instance) positronium decays into two photons with perpendicular polarisation, and if we put the polarisers with parallel polarisation planes between them, we wouldn't observe any cases where both photons pass. This means that photon polarisation is not determined in the moment of decay, but only in the process of measurement.

Šimon Pajger
`legolas@fykos.cz`

Problem M.16 ... water planet 5 points

Consider a non-rotating planet with a radius $R = 2\,000\,\mathrm{km}$ *formed only by liquid water and an atmosphere. The atmospheric pressure on the surface is* $p_\mathrm{a} = 10\,\mathrm{kPa}$. *What is the pressure in its center? Assume that water is incompressible.*

Honza was wondering what could exist in the universe.

For every part of the planet hydrostatic equilibrium condition holds

$$\nabla p = \rho \mathbf{a}_\mathrm{g}\,,$$

where p is pressure, ρ is density of the water and $\mathbf{a}_\mathrm{g}$ is gravitational acceleration. Due to symmetry we can rewrite the condition in spherical coordinates as

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \rho a_\mathrm{g}\,.$$

As per well known consequence of Gauss's theorem, gravitational acceleration at distance r from the center of a spherically symmetrical mass distribution is exerted only by the mass enveloped in a sphere of diameter r of total mass

$$m = \frac{4}{3}\pi r^3 \rho$$

that acts like a point mass, giving gravitational acceleration at r from the center

$$a_\mathrm{g} = -\frac{Gm}{r^2} = -\frac{4\pi G r\rho}{3}\,,$$

where the sign means that the force acts in direction of decreasing r. Substituting into previous equations we get

$$\mathrm{d}p = -\frac{4\pi G\rho^2}{3} r\,\mathrm{d}r\,.$$

Integrating both sides between boundaries at the surface and the center we get central pressure

$$\int_{p_0}^{p_\mathrm{a}} \mathrm{d}p = -\int_0^R \frac{4\pi G\rho^2}{3} r\,\mathrm{d}r\,,$$

$$p_\mathrm{a} - p_0 = -\frac{2\pi G\rho^2}{3}R^2\,,$$

$$p_0 = p_\mathrm{a} + \frac{2\pi G\rho^2}{3}R^2 \doteq 557\,\mathrm{MPa}\,.$$

Jan Benda
honzab@fykos.cz

Problem M.17 ... slot car racing 5 points

When Jarda was a child, he didn't organise FYKOS, so he had a lot of free time. Once, he got a present from his parents – a slot car track with a shape of a very tall vertical helix, together with slot cars, each with a mass $m = 80\,\mathrm{g}$. The radius of the helix is $r = 10\,\mathrm{cm}$ and a car moves $h = 8\,\mathrm{cm}$ down per one turn. When the car is moving down, it slides on the outer edge of the track where the friction coefficient is $\mu = 0.3$. Jarda launched the first car at a time t_0 (with zero initial velocity) and the second car from the same point five seconds later (also with zero initial velocity). At each point in time afterwards, the first car will be some number of turns ahead of the second car. What is the maximum value of this difference in the travelled number of turns? Assume that each car is a point mass. *Jarda was reminiscing about his childhood.*

We can untwist the helix car track to a usual straight slope. The car will be accelerated by the component of gravitational force parallel to the slope

$$F_g^{\mathrm{t}} = mg\sin\alpha\,,$$

where α is the angle between the slope and a horizontal plane. The car is rolling on wheels with negligible resistance from the surface underneath. By untwisting one story of helix we get a right triangle with sides h and $2\pi r$, therefore we get angle

$$\sin\alpha = \frac{h}{\sqrt{h^2 + (2\pi r)^2}}\,.$$

There is, however, aditional friction force. The car is kept on a circular trajectory of radius r by reaction force from outer side of the track

$$F_{\mathrm{d}} = \frac{mv_{\mathrm{h}}^2}{r}\,,$$

where v_{h} is the horizontal projection of velocity $v_{\mathrm{h}} = v\cos\alpha$. For cosine of angle α we have

$$\cos\alpha = \frac{2\pi r}{\sqrt{h^2 + (2\pi r)^2}}\,.$$

This force is causing friction force from the track side

$$F_{\mathrm{f}} = \mu F_{\mathrm{d}} = \frac{\mu m v_{\mathrm{h}}^2}{r} = \frac{\mu m v^2}{r}\cos^2\alpha\,.$$

Now we can complete the equation of motion

$$F = F_g^{\mathrm{t}} - F_{\mathrm{f}} \quad \Rightarrow \quad \dot{v} = g\sin\alpha - \frac{\mu v^2}{r}\cos^2\alpha\,.$$

This differential equation can be solved, but it is enough to understand that there is certain terminal velocity of cars. We can obtain this velocity by the condition $a = 0$ as

$$v_{\mathrm{t}} = \sqrt{\frac{gr\sin\alpha}{\mu\cos^2\alpha}}\,.$$

This velocity is obtained in infinite time, until then the cars are accelerating more slowly as time goes by. The distance between cars will be increasing up to a maximum value of $\Delta s = v_\mathrm{t} t$, where $t = 5\,\mathrm{s}$ is starting time difference. For height difference we get

$$\Delta h = \Delta s \sin\alpha\,.$$

and for the number of turns one car is preceeding the other as time goes to infinity

$$\frac{\Delta h}{h} = \frac{t}{2\pi}\sqrt{\frac{gh}{\mu r\sqrt{h^2 + (2\pi r)^2}}} \doteq 5.1\,.$$

Jaroslav Herman
`jardah@fykos.cz`

Problem E.1 ... the anti-diet of the Earth 5 points

How would the mass of the Earth increase if its mean radius was increased by $\Delta r = 1.0\,\mathrm{m}$ by adding material with the same density as Earth's mean density? Find the mass of this added material. *Karel was thinking about meteorite impacts.*

First, we determine the density of the Earth. We could simply find this online, but let's calculate it from the quantities we know. We will assume that the Earth is a sphere with radius of $R_\oplus \doteq 6.378 \cdot 10^6\,\mathrm{m}$. In reality, this is its radius on the equator, but the difference is relatively small. The mass of the Earth is $M_\oplus = 5.974 \cdot 10^{24}\,\mathrm{kg}$. Density can then be determined as

$$\rho_\oplus = \frac{M_\oplus}{V_\oplus} = \frac{3M_\oplus}{4\pi R_\oplus^3} \doteq 5\,497\,\mathrm{kg{\cdot}m^{-3}} \doteq 5\,500\,\mathrm{kg{\cdot}m^{-3}}\,.$$

If we compare it with the density found online, for example, on Wikipedia, where $5\,515\,\mathrm{kg{\cdot}m^{-3}}$ is given, our estimation is enough for a result valid to two significant digits.

The whole solution can be very simple if we realize that the radius of the Earth will increase only by a small amount. Therefore, the change to the surface area $S_\oplus$ will be negligible. The change in volume of the Earth is then the product of the Earth's surface area and the change in height

$$\Delta V = S_\oplus \Delta r = 4\pi R_\oplus^2 \Delta r \doteq 5.11 \cdot 10^{14}\,\mathrm{m^3}\,.$$

The mass will change by

$$\Delta m = \rho_\oplus \Delta V = \frac{3M_\oplus}{4\pi R_\oplus^3} 4\pi R_\oplus^2 \Delta r = \frac{3M_\oplus}{R_\oplus} \Delta r \doteq 2.8 \cdot 10^{18}\,\mathrm{kg}\,.$$

The whole mass of the Earth will increase by $2.8 \cdot 10^{18}\,\mathrm{kg}$, what is $0.000\,047\,\%$ of the original mass.

Alternately, we can count the mass of extended Earth and subtract the original mass from it. Due to small increase in radius, this should be a negligible difference. Let's make sure of that

$$\begin{aligned}\Delta m' = \rho_\oplus \Delta V' &= \frac{3M_\oplus}{4\pi R_\oplus^3} \left(\frac{4}{3}\pi \left(R_\oplus + \Delta r\right)^3 - \frac{4}{3}\pi R_\oplus^3\right) \\ &= \frac{M_\oplus}{R_\oplus^3} \left(3R_\oplus^2 \Delta r + 3R_\oplus \Delta r^2 + \Delta r^3\right)\,.\end{aligned}$$

We can see that the result has three terms. The first term is the result that we got in the approximate calculation and the other members are really negligible and after rounding we get the same result. We can use this approximation as long as the increase in radius is significantly smaller than the original radius of the Earth.

Karel Kolář
karel@fykos.cz

Problem E.2 ... Diomede Islands 5 points

Two small islands are located only 3.7 km apart in the Bering strait. The west island belongs to Russia while the east one is part of the USA. The interesting thing about them is that the International Date Line passes just between them. That is why they are sometimes called Tomorrow Island and Yesterday Island). Once in the winter, when the strait was frozen, one traveller *walked from the American island to the Russian island with an average pace of* $4\,\mathrm{km\cdot h^{-1}}$ *(don't try this, it's illegal). However, his GPS showed him a very bizzare average speed since it calculated both the time of departure and the time of arrival using the local time zones. What was this average speed?*

The Russian island is in the time zone +12, the American one in the time zone -9. Be careful – the average speed has a sign, too. *Matěj would like to travel, but can't do it now.*

The trick is to realize the direction in which the Earth rotates. For example, when it is 13 February 12:01pm on the Greenwich meridian, on the Russian island it is 00:01 (12:01am) on the 14th and on the American one it is 3:01am, February 13th. Therefore we have to add a false $12\,\mathrm{h} - (-9\,\mathrm{h}) = 21\,\mathrm{h}$.

The journey actually took him $\frac{3.7\,\mathrm{km}}{4\,\mathrm{km\cdot h^{-1}}} = 0.925\,\mathrm{h}$, and the false average speed is therefore

$$\frac{3.7\,\mathrm{km}}{21.925\,\mathrm{h}} = 0.168\,8\,\mathrm{km\cdot h^{-1}} = 4.69\,\mathrm{cm\cdot s^{-1}}\,.$$

Matěj Mezera
`m.mezera@fykos.cz`

Problem E.3 ... water on Mars 5 points

Mars doesn't have much water. Imagine that we would like to create an ocean on Mars and transport all the water from the Pacific Ocean there, which would create a "new Pacific Ocean" covering almost the entire Mars. What would be the difference in average depth between these two oceans? The answer should be a negative number if the new ocean is shallower. Assume the Pacific Ocean covers a third of the Earth's surface and contains $7.1\cdot 10^8\,\mathrm{km^3}$ *of water. The radius of Mars is* $3\,390\,\mathrm{km}$*. For simplicity, neglect the rotation of Mars.* *Matěj was thirsty.*

The surface area of the Earth is $S_\mathrm{E} = 4\pi R_\mathrm{Z}^2$. Since the depth of the ocean is very small compared to the radius of the Earth, we can calculate the average depth as

$$h_0 = \frac{V}{\frac{1}{3}S_\mathrm{E}} = \frac{3V}{4\pi R_\mathrm{E}^2}\,,$$

where V is the volume of the Pacific and $R_\mathrm{E} = 6\,378\,\mathrm{km}$. The average depth of the Pacific ocean on Mars would be

$$h_1 = \frac{V}{S_\mathrm{M}} = \frac{V}{4\pi R_\mathrm{M}^2}\,,$$

where $R_\mathrm{M} = 3\,390\,\mathrm{km}$ is the radius of Mars. The difference in depths is

$$h_1 - h_0 = \frac{V}{4\pi}\left(\frac{1}{R_M^2} - \frac{3}{R_\mathrm{E}^2}\right) \doteq 750\,\mathrm{m}\,.$$

The Pacific would be almost one kilometer deeper.

Matěj Mezera
`m.mezera@fykos.cz`

Problem E.4 ... rolling in the Neanderthal way 5 points

Ancient civilisations invented the wheel long ago. Determine the steepness of a hill needed for a regular 42-gon to start rolling down the hill on its own.

Kiko was checking out a boneshaker bike in a museum of technology.

In order for the body to tip over, the position of its center of gravity must be directly above the vertex around which is the body turning (actually, it is a bit more to the right but we are solving the critical point). For the angle between the side of the n-gon and the line passing through the center of gravity, and the corresponding vertex of the side we get

$$\varphi = \frac{\pi - \frac{2\pi}{n}}{2}\,.$$

To make the body tip over, the angle must be $\pi/2$. The steepness of the hill adds to this, so it must be

$$\alpha = \pi/2 - \varphi = \frac{\pi}{n}\,.$$

For 42-gon we get approximately $4.3\,^{\circ}$.

Kristián Šalata
kristian.salata@fykos.cz

Problem E.5 ... dry 5 points

The relative humidity in Danka's room is an unpleasant $\Phi_{r,1} = 20\,\%$. How much water does Danka need to evaporate to increase the humidity to $\Phi_{r,2} = 50\,\%$? The room contains $V = 15\,\mathrm{m^3}$ of air with temperature $25^{\circ}\mathrm{C}$ and this air contains $\Phi_1 = 4.6\,\mathrm{g{\cdot}m^{-3}}$ of vaporized water.

Danka's room was too dry.

Absolute humidity of air is defined as

$$\Phi = \frac{m}{V}\,,$$

where m is the mass of the water in the air and V is the total volume of the air. Relative humidity is then

$$\Phi_r = \frac{\Phi}{\Phi_m}\,,$$

where Φ_m is the mass of water vapor per unit of volume, at which the air is saturated with water at a given temperature. From the actual situation, we determine the amount of water in $1\,\mathrm{m^3}$ at saturation Φ_m, as

$$\Phi_m = \Phi_1/\Phi_{r,1}\,.$$

The difference of the mass of the water in the air is expressed from the first equation as

$$\Delta m = m_2 - m_1 = V\left(\Phi_2 - \Phi_1\right) = V\Phi_m\left(\Phi_{r,2} - \Phi_{r,1}\right)\,.$$

Using Φ_m, we finally get

$$\Delta m = V\Phi_1\left(\frac{\Phi_{r,2}}{\Phi_{r,1}} - 1\right) \doteq 104\,\mathrm{g}\,.$$

Danka needs to evaporate approximately $104\,\mathrm{g}$ of water.

Daniela Pittnerová
daniela@fykos.cz

Problem E.6 ... hanging clothes 5 points

Verča hangs towels on the clothes rack in such a way that she throws them over a rod with a square cross-section. One day, she noticed that part of a towel, which does not overlap with the other side, dries faster. Therefore, she decided to hang the towels in such a way that the overlaping part is as small as possible, but the towel still does not slide down. What is the highest ratio of the two hanging parts (the longer part to the shorter part) for Verča's towel such that it remains secured from falling? The coefficient of friction between the rod and the towel is $f = 0.2$ and the mass of the towel is $m = 1\,\mathrm{kg}$.

Verča is looking for physics even in doing household chores.

Let's mark the mass of the longer part as m_1 and the shorter part as m_2, assuming that the mass of the towel is distributed equally. One part will be subjected to the force of gravity $F_{g,1} = m_1 g$, the other to $F_{g,2} = m_2 g$. A force of friction $F_\mathrm{f} = mgf$ will act against the larger of the forces of gravity ($F_{g,1}$) and hold the towel on the rod. We get the equation

$$F_{g,2} + F_\mathrm{f} = F_{g,1}\,,$$

and substituting in the forces

$$m_2 g + mgf = m_1 g\,.$$

We also know that $m_1 + m_2 = m$; therefore after substituting for m we can express the mass ratio as

$$\frac{m_1}{m_2} = \frac{1+f}{1-f}\,.$$

The ratio of the masses is the same as the ratio of the lengths, therefore after substituting numbers we get the result $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{3}{2}$. We can notice that the only thing that affects the result is the coefficient of friction; it does not depend on the mass nor the gravitational acceleration.

Veronika Hendrychová
vercah@fykos.cz

Problem E.7 ... gravitation 5 points

Matěj would love to walk on the Moon, but it's hard to get there. Instead, he used a little trick – he filmed himself jumping on the Earth and then played the video in slow motion. At what frame rate does he need to play the clip to make it look like jumping on the Moon? The camera has a frame rate of 60 fps. Assume that the gravity on the Moon is six times smaller than the gravity on the Earth. *Matěj and film tricks.*

We will consider only the vertical component of the trajectory since the other components are not affected by the gravitational force.

For the movement in a homogenous gravitational field g we have

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0\,,$$

where v_0 is the (vertical) speed at the time $t = 0$ and y_0 is the position at the same time. Now we will transform from unprimed to primed variables representing the altered gravity and slower time. We want the resultant equation to be the same as original since the vertical position y is the same in both the original and the slowed down clip

$$y(t) = y'\left(t'\right) = -\frac{1}{2}g't'^2 + v_0't' + y_0' \,.$$

We also want $g' = g/6$. Comparing the two equations (the equation $y(t) = y'(t')$ must be satisfied for all possible values of v_0 and y_0) we get

$$t' = t\sqrt{6}\,, \quad v_0' = \frac{v_0}{\sqrt{6}}\,, \quad y_0' = y_0\,. \tag{1}$$

We can clearly see that the time must be slowed down by the factor of $\sqrt{6}$, that is while in the original clip 60 frames pass, in the slowed clip, it will only be $\frac{60}{\sqrt{6}} \doteq 24.5$ frames.

Althought that was a clever trick, it has one disadvantage. Matěj's speed during the take-off will, because of (1), appear to be $\sqrt{6}$-times lower. So it will seem Matěj can only jump with a very small speed.

Matěj Mezera
`m.mezera@fykos.cz`

Problem E.8 ... rope on an inclined plane — 5 points

Consider an inclined plane with an angle $\varphi = 50^\circ$ between this plane and the horizontal plane. A rope with a linear density $\lambda = 1.5\,\mathrm{kg{\cdot}m^{-1}}$ is laid on it, along the plane's entire length. The height difference between the top and bottom point of the plane is $h = 1.7\,\mathrm{m}$. What is the force with which we have to hold the rope so it doesn't slip down, if there is zero friction between the rope and the plane? *Lego doesn't feel like coming up with origins anymore, sorry :D*

The mass of the rope is $m = \lambda h/\sin\varphi$. Considering that the angle is constant, we assume that all the mass is located in the center of mass and we will treat it as a point mass. Then, we know that the normal component of the force of gravity is counteracted by normal force from the plane and we only need to couunteract the component parallel with the plane. The magnitude of this component is $F = F_g \sin\varphi = g\lambda h \doteq 25\,\mathrm{N}$.

We can see that the force doesn't depend on the angle, but the only relevant information is the height difference.

Šimon Pajger
`legolas@fykos.cz`

Problem X.1 ... massive crash 5 points

Two point masses $m_1 = 50.0\,\mathrm{kg}$ and $m_2 = 60.0\,\mathrm{kg}$ are $l = 6.00\,\mathrm{m}$ apart at rest. Suddenly, they start attracting each other with a constant force $F = 100\,\mathrm{N}$. What velocity will they collide with? *Verča was reminiscing about skating with her classmates.*

As the first step, let's find the positions of the two point masses at a time t. Suppose that the first point mass is at the origin of our coordinate system and the second point mass is at $x = l$ at the beginning. Because the force is constant, the masses accelerate uniformly. We can calculate each acceleration as the force divided by the mass $a_1 = F/m_1$ and $a_2 = F/m_2$. The positions of the two point masses at a time t are

$$x_1 = \frac{1}{2}a_1 t^2 = \frac{1}{2}\frac{F}{m_1}t^2\,,$$
$$x_2 = l - \frac{1}{2}a_2 t^2 = l - \frac{1}{2}\frac{F}{m_2}t^2\,.$$

They crash when $x_1 = x_2$

$$\frac{1}{2}\frac{F}{m_1}t^2 = l - \frac{1}{2}\frac{F}{m_2}t^2 \quad \Rightarrow \quad t = \sqrt{\frac{2lm_1m_2}{F\left(m_1+m_2\right)}}\,.$$

Now, we have to calculate their velocities using $v = at = \frac{F}{m}t$. The velocity v of one mass relative to the other is simply the sum of the two velocities

$$v = v_1 + v_2 = Ft\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \sqrt{\frac{2lm_1m_2}{F\left(m_1+m_2\right)}}\frac{F\left(m_1+m_2\right)}{m_1m_2} = \sqrt{\frac{2Fl\left(m_1+m_2\right)}{m_1m_2}}\,.$$

The two point masses collide with velocity $v = 6.63\,\mathrm{m\cdot s^{-1}}$.

Veronika Hendrychová
vercah@fykos.cz

Problem X.2 ... Pythagorean theorem not working 5 points

Matěj doesn't like winding paths and he prefers to walk in a straight line, especially when he walks exactly in one of the cardinal directions. He once wondered about the most beautiful way to travel from Prague to Brno. He could either first go straight east and then south, or first go straight south and then east (always making just one 90° turn on the way). What is the difference between the lengths of the first path and the second path? (If the first path is shorter, the answer should be a negative number.) In reality, the specific shape of the terrain would have much greater effect on the result than curvature of the Earth - therefore, consider the Earth to be a perfect sphere. The coordinates of Prague are $50.08^\circ N, 14.44^\circ E$ and the coordinates of Brno are $49.20^\circ N, 16.61^\circ E$. *Matěj was thinking about the type of projection of Google maps.*

We have to realize what Matěj's trajectory looks like. In both cases, a part of the path goes along the parallel of latitude and the other part along the meridian. The paths along the meridian are the same in both cases, because Matěj needs to cross the distance from the parallel going through Prague and the parallel going through Brno. This is not the case of the other part of the path, because the distance between two meridians is not constant. When Matěj goes east, he moves along one parallel. The radius r of the parallel depends on the latitude (we mark it as φ) as

$$r = R\cos\varphi\,,$$

where $R = 6\,378\,\mathrm{km}$ is the radius of the Earth. The distance s Matěj travelled along the parallel is given by the length of the circular arc with the angle $\Delta\lambda = \lambda_\mathrm{B} - \lambda_\mathrm{P}$, therefore

$$s = \frac{\pi}{180^\circ} r\Delta\lambda\,,$$

where we mark the latitude in degrees as λ and the factor $\frac{\pi}{180^\circ}$ converts degrees to radians. We get the difference between the paths as

$$\Delta s = s_1 - s_2 = \frac{\pi}{180^\circ}\left(r_1 - r_2\right)\Delta\lambda = \frac{\pi}{180^\circ} R\Delta\lambda\left(\cos\varphi_\mathrm{P} - \cos\varphi_\mathrm{B}\right) \doteq -2\,830\,\mathrm{m}\,.$$

We can see that even when looking on the flat map of the "small" Czech Republic there is a certain deformation, and the actual distances can differ by up to several kilometers.

Matěj Mezera
m.mezera@fykos.cz

Problem X.3 ... poisoned

5 points

A certain toxic substance got into a pond with $V = 130\,\mathrm{m^3}$ of water. The only tributary of the lake is a stream with a flow rate $Q = 3.8\,\ell{\cdot}\mathrm{s^{-1}}$ and the water flowing in immediately mixes with the contaminated water in the pond. What time does it take to decrease the concentration of the substance to one tenth of the initial concentration, assuming that the water level in the pond doesn't change? *Jarda wanted to go fishing in Bečva.*

Let us denote the concentration of the toxic substance by c. If the water in the pond mixes completely with the water from the stream, the concentration of the toxic substance in the water flowing out of the pond is also c, while the water flowing into the pond has zero concentration of this substance. The constant water level means that the amount of water flowing in is the same as the amount of water flowing out. The difference in the amount of the toxic substance in the pond for a time period $\mathrm{d}t$ is

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\mathrm{d}\left(cV\right)}{\mathrm{d}t} = V\frac{\mathrm{d}c}{\mathrm{d}t}\,.$$

During this time, water with volume $Q\,\mathrm{d}t$ will flow out, therefore the amount of the toxic substance will decrease by $\mathrm{d}n = -cQ\,\mathrm{d}t$. From this, we get a differential equation

$$V\frac{\mathrm{d}c}{\mathrm{d}t} = -cQ\,,$$

and we solve it by separation of variables

$$\frac{\mathrm{d}c}{c} = -\frac{Q}{V}\,\mathrm{d}t\,.$$

By integrating it, we get

$$c(t) = c_0 \mathrm{e}^{-\frac{Q}{V}t}\,.$$

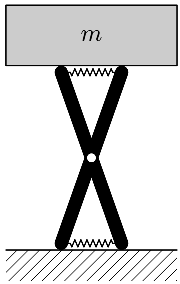
For the concentration to decrease to one tenth, $c(t) = \frac{c_0}{10}$ must apply. From this, we can express the time as

$$t = \frac{V}{Q} \ln \frac{c_0}{c} = \frac{V}{Q} \ln 10 \doteq 21.9\,\mathrm{h}\,.$$

Jaroslav Herman
jardah@fykos.cz

Problem X.4 ... malicious X

5 points



There are two rods in the figure, each with length $l = 1.2\,\mathrm{m}$. They are connected together in the middle in such a way that they can rotate freely. The upper ends and the lower ends of the rods are connected by two springs, each with stiffness $k = 90\,\mathrm{N{\cdot}m^{-1}}$ and rest length $s = 0.4\,\mathrm{m}$. We put a weight with mass m on the top and wait until the system reaches an equilibrium. Find a value of m such that the springs get twice as long. Assume that the system cannot fall over. *Jáchym felt bored.*

Let's denote the length of the spring after elongation by $2x$, its original length by $2x_0 = s$, the final height of the "X" by $2y$ and the length of each rod by $2r = l$. From the Pythagorean theorem, we get

$$r^2 = x^2 + y^2\,.$$

The forces, which the springs exert on each of the rods, have magnitudes $F_x = 2k\,(x - x_0)$. Gravitational (and buoyant) forces are uniformly distributed, so the force on each end of each rod is $F_y = mg/2$. The rods don't move with respect to each other when the net torque on each of them is zero. Therefore, in equilibrium,

$$xF_y = yF_x\,.$$

After we substitute for F_x and F_y, we get

$$m = \frac{4ky\,(x - x_0)}{gx}\,.$$

We are looking for such a mass m that $x = 2x_0 = s$. We can express from the Pythagorean theorem that $y = \sqrt{r^2 - 4x_0^2}$. The result is

$$m = \frac{4k\,(2x_0 - x_0)}{2gx_0}\sqrt{r^2 - 4x_0^2} = \frac{2k}{g}\sqrt{r^2 - 4x_0^2} = \frac{k}{g}\sqrt{l^2 - 4s^2} \doteq 8.21\,\mathrm{kg}\,.$$

Jáchym Bártík
tuaki@fykos.cz

Problem X.5 ... slip 5 points

Jáchym found a stone in the shape of a perfect rotational ellipsoid with semi-axes $a = 5.3\,\mathrm{cm}$ and $b = 3.5\,\mathrm{cm}$, where a is the axis of rotational symetry. He let the stone slip on ice in such a way that the angle φ between the axis of rotational symmetry of the ellipsoid and the vertical axis was constant throughout the whole movement. Calculate the maximum possible magnitude of this angle φ if the mass of the stone is $m = 32\,\mathrm{g}$, the initial speed is $v = 2.79\,\mathrm{m{\cdot}s^{-1}}$ and the coefficient of friction between the stone and ice is $f = 0.30$. *Jáchym was behind with the third series of FYKOS, so he procrastinated by thinking about Fyziklani.*

The key to solving the problem is the stability condition for the stone considering the rotation. From the external point of view, the body is affected by three forces- the weight, reaction force from the ice field F_i and a frictional force $F_\mathrm{f} = fF_\mathrm{i}$. The sum of torques relative to the centre of the stone must be zero, which implies

$$F_\mathrm{f} r \cos\gamma = F_\mathrm{i} r \sin\gamma\,,$$

where r is the length of the line which connects the centre of the stone with the point of contact with the ice, and γ is the angle between this line and the vertical direction (see 1). We'll rewrite the equation as

$$f = \tan\gamma\,. \tag{2}$$

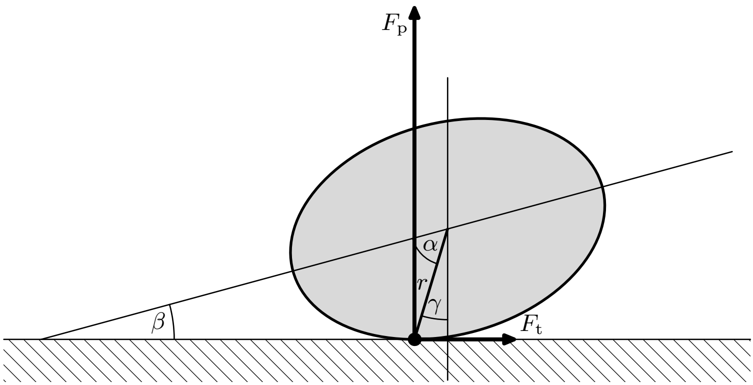


Fig. 1: Drawing of the stone in a slip.

Now let's imagine an ellipsoid rotated in the way that the semi-major axis is on the axis of x and the centre of the stone is at the origin point. The part of its surface lying in the xy plane, can be described by the function

$$y = b\sqrt{1 - \left(\frac{x}{a}\right)^2}\,,$$

as shown in image 2. We will denote the inclination of the tangent line of the function from the horizontal direction as $-\beta$. We chose the minus symbol because we know that this angle is

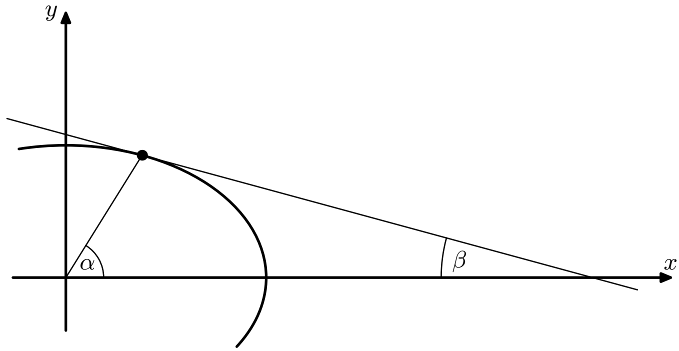


Fig. 2: Part of stone cross-section. The major axis of the ellipse is parallel to axis x, the minor one is parallel to axis y. The magnitudes of the angles α and β corresponds to the image 1.

going to be negative and we're only interested in its magnitude. The tan of this angle is equal to the function's derivative, so we have

$$y' = -b\left(1-\left(\frac{x}{a}\right)^2\right)^{-\frac{1}{2}}\frac{x}{a^2} = -\left(\frac{b}{a}\right)^2\frac{x}{y}\,.$$

For the point on the ellipsoid surface where we calculate the derivative, let's define α as the angle between the line connecting this point with the centre of the ellipsoid and the semi-major axis. Then, the following holds

$$y' = -\left(\frac{b}{a}\right)^2\frac{1}{\tan\alpha} = \tan\left(-\beta\right) = -\tan\beta\,.$$

And so we obtained the second important formula

$$\tan\beta\tan\alpha = \left(\frac{b}{a}\right)^2 = \frac{1}{k}\,, \tag{3}$$

where k isn't a constant. It is a tool of surprisal, which we will use later.

From the geometry of this problem, we realize that $\varphi = \alpha+\gamma$ a $\alpha+\beta+\gamma = \frac{\pi}{2}$. Further, we need the following trigonometric identity

$$\tan\left(\frac{\pi}{2}-x\right) = \frac{1}{\tan x}\,,$$
$$\tan\left(x+y\right) = \frac{\tan x+\tan y}{1-\tan x\tan y}\,.$$

Obviously, it is true that

$$\tan\varphi = \tan\left(\alpha+\gamma\right) = \tan\left(\frac{\pi}{2}-\beta\right) = \frac{1}{\tan\beta} = k\tan\alpha\,.$$

In the last equation, we used the relation (3). The second expression we can rewrite using the formula for the tangent of a sum, and we get

$$\begin{aligned} \tan\alpha + \tan\gamma &= k\tan\alpha\left(1 - \tan\alpha\tan\gamma\right) \,, \\ \tan\alpha + f &= k\tan\alpha\left(1 - f\tan\alpha\right) \,, \\ 0 &= fk\tan^2\alpha + (1-k)\tan\alpha + f \,, \end{aligned}$$

where we have just substituted $\tan\gamma$ by (2). The solution of this quadratic equation is

$$\tan\alpha = \frac{k - 1 \pm \sqrt{(k-1)^2 - 4f^2k}}{2fk} \,,$$

from where we get the result thanks to the formula $\tan\varphi = k\tan\alpha$. We can write

$$\begin{aligned} \varphi &= \arctan\left(\frac{1}{2f}\left(k - 1 \pm \sqrt{(k-1)^2 - 4f^2k}\right)\right) = \\ &= \arctan\left(\frac{1}{2f}\left(\left(\frac{a}{b}\right)^2 - 1 \pm \sqrt{\left(\left(\frac{a}{b}\right)^2 - 1\right)^2 - 4f^2\left(\frac{a}{b}\right)^2}\right)\right) \,. \end{aligned}$$

Because we are looking for the biggest possible angle, we choose the root with the plus sign. After plugging in the numbers, we get $\varphi \doteq 74.8^\circ$. We can see that this solution corresponds to a stable equilibrium position. Choosing the root with the minus sign would result in an unstable equilibrium.

Jáchym Bártík
`tuaki@fykos.cz`

Problem X.6 ... cruel game of fate 5 points

Mr. A and Mr. B were born on the same day in a dystopian future inspired by Orwell's novel 1984. Mr. A is an ordinary citizen, while Mr. B is a member of the Inner Party.

The thought police is making up ficticious charges. Mr. A noticed that the "half-life" of an ordinary citizen is 5 years and he will surely meet the same fate.

The Big Brother organizes a purge in the Party once every 5 years. During that, 60 % of the party members are randomly selected and removed. That may be the fate of Mr. B.

What is the probability that Mr. A will live longer than Mr. B if one purge has just ended? Assume that purges and the thought police are the only causes of death.

Jindra can finally say that mandatory reading in school was useful.

Let's denote the 5-year time interval by T. Next, let $a = 0.5$ and $b = 0.4$ be the fractions of survivors for the two ways to die, respectively.

The probability that Mr. B dies during the n-th purge ($n = 1$ is the first purge from today, in five years) is

$$P_\mathrm{B}(n) = b^{n-1} - b^n \,.$$

The probability that Mr. A dies after the n-th purge is

$$P_\mathrm{A}(n) = a^{\frac{nT}{T}} = a^n \,,$$

so the compound probability that Mr. B dies during the n-th purge and Mr. A dies later is the product of these two probabilities (the two events are independent)

$$P(n) = \left(b^{n-1} - b^n\right) a^n = \frac{1-b}{b} \left(ab\right)^n \ .$$

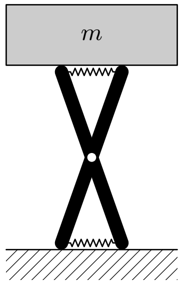
If we want to know the total probability that Mr. A lives longer than Mr. B, we have to calculate a sum over all n

$$P(T_\mathrm{A} > T_\mathrm{B}) = \frac{1-b}{b} \sum_{n=1}^{\infty} (ab)^n = \frac{1-b}{b} \left(\frac{1}{1-ab} - 1\right) = \frac{a\,(1-b)}{1-ab} = \frac{3}{8}\,.$$

The probability that Mr. A lives longer than Mr. B is 3/8.

Jindřich Jelínek
`jjelinek@fykos.cz`

Problem X.7 ... even more malicious X 5 points



There are two rods in the figure, each with length $l = 1.2\,\mathrm{m}$. They are connected together in the middle in such a way that they can rotate freely. The upper ends and the lower ends of the rods are connected by two springs, each with stiffness $k = 90\,\mathrm{N{\cdot}m^{-1}}$ and rest length $s = 0.4\,\mathrm{m}$. We put a weight with mass $m = 7\,\mathrm{kg}$ on the top of this system. What is the difference between the highest and the lowest possible stable equilibrium position of the weight? Do not consider the position where the weight is on the ground. The coefficient of friction between the rods and the weight, and also between the rods and the ground, is $f = 0.6$. Assume that the system cannot fall over, its mass is negligible compared to the mass of the weight and the weight is longer than the rods. *Jáchym felt even more bored.*

In the equilibrium position, the total force moments applied to each of the rods is equal to zero. The range of possible equilibrium positions will be caused by the fact that the frictional force doesn't have any specific magnitude, but it is just big enough for the body not to move.

Mark the spring length after extension $2x$, the initial length $2x_0 = s$, the construction height $2y$ and the rod length $2r = l$. The elastic force acting at the end of the rod will be $F_x = 2k(x - x_0)$, while the weight and pressure force of the pad will be $F_y = mg/2$. The magnitude of force of friction will lie in the interval $\langle -F_\mathrm{t}, F_\mathrm{t}\rangle$, where $F_\mathrm{t} = fF_y$. For the total torque in the maximum and minimum position we can apply the formula

$$xF_y - y\left(F_x + zF_\mathrm{t}\right) = 0\,,$$

where $z \in \langle -1, 1\rangle$. By substituting in the forces

$$xmg - y\left(4k\left(x - x_0\right) + zfmg\right) = 0\,,$$

Trivially, we can write $r^2 = x^2 + y^2$, from which we can express x as a function of y. However, it is immediately apparent that the resulting relation will be a high degree polynomial and so trying to solve it analytically would be impractical. Therefore, we will take a different approach – we will express z as a function of y and we will be interested in what points the condition $z \in \langle -1, 1 \rangle$ holds. We know that

$$z = \frac{1}{f}\left(\sqrt{\frac{r^2}{y^2} - 1} - \frac{4k}{mg}\left(\sqrt{r^2 - y^2} - x_0\right)\right) = \frac{1}{f}\left(\sqrt{\frac{l^2}{h^2} - 1} - \frac{2k}{mg}\left(\sqrt{l^2 - h^2} - s\right)\right),$$

where we have defined the required height as $h = 2y$. The function $z(h)$ is shown in graph 3. The minimum and maximum heights are

$$h_1 \doteq 0.451\,\mathrm{m}\,,$$
$$h_2 \doteq 1.173\,\mathrm{m}\,.$$

The answer is therefore $\Delta h = h_2 - h_1 \doteq 0.722\,\mathrm{m}$.

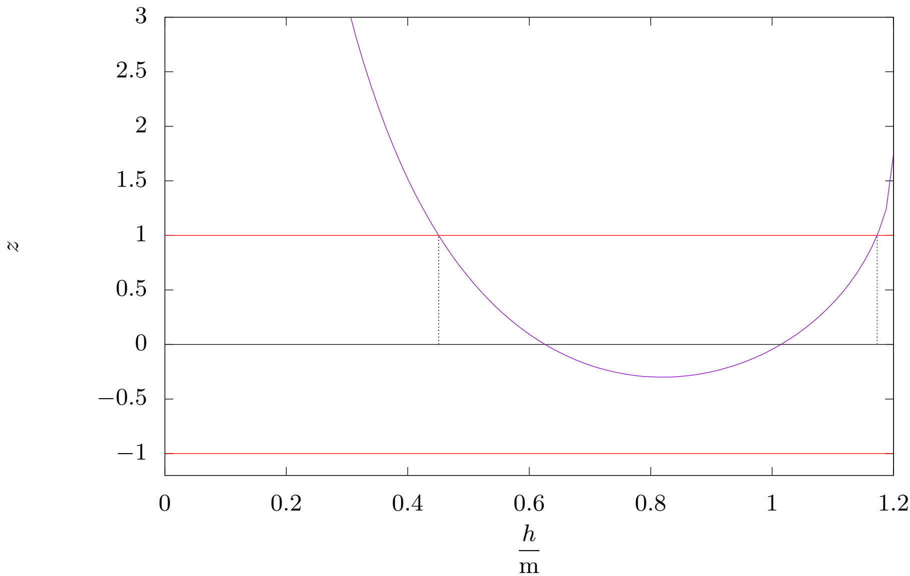


Fig. 3: Function $z(h)$. Horizontal lines indicate the interval $\langle -1, 1 \rangle$.

Jáchym Bártík
tuaki@fykos.cz

Problem A.1 ... flight from the Sun 5 points

At what speed does an airplane need to fly (with respect to the surface of the Earth) at a height $h = 10\,\mathrm{km}$ straight along a 42nd parallel of latitude if it wants to keep flying away from sunrise (i.e. to always be just above the place where the Sun rises)?

Danka was travelling by plane with the Sun rising behind her.

If we think about the situation carefully, we realise that the plane must always be in the same point relative the Earth and the Sun. Therefore the plane moves relative to the surface of the Earth the same way as the surface of the Earth moves relative to the centre of the Earth and to the Sun. We can find such velocity as a ratio between the circumference o of the Earth at the latitude given (42^{nd} parallel) and the period of the Earth rotation $T = 24\,\mathrm{h}$ relative to the Sun. The length of the parallel can be calculated easily as $o = 2\pi r$, where r is the radius of the parallel. We obtain it from the radius of the Earth $R_{\mathrm{E}} = 6\,378\,\mathrm{km}$ as

$$r = R_{\mathrm{E}} \cos\alpha\,,$$

where α is the angle between the equator and the parallel line, which is $42\,^{\circ}$. We obtain, that the velocity of the rotation of the Earth on the 42^{nd} parallel line towards the plane is

$$v = \frac{o}{T} = \frac{2\pi R_{\mathrm{E}} \cos\alpha}{T}\,.$$

Plugging in the numerical values, we get

$$v \doteq 1\,240\,\mathrm{km{\cdot}h^{-1}}\,.$$

Daniela Pittnerová
daniela@fykos.cz

Problem A.2 ... SETI frequency 5 points

SETI is a project trying to find evidence of extraterrestrial civilisations by observing cosmic radio signals. In 1973, Drake and Sagan proposed searching on some "natural frequency" obtained by a combination of the Boltzmann constant k_{B}, Planck constant h and the current temperature of cosmic microwave background $T_0 = 2.725\,\mathrm{K}$. Find a formula for a frequency using these three constants, supposing that the dimensionless constant is 1. Calculate this frequency.

Karel was reading a book about the SETI project.

The solution to this problem is found by dimensional analysis. We can easily guess the formula for the frequency – it is not that hard – but we are going to show a proper solution using a system of linear equations. We are trying to find parameters α, β, γ such that

$$f = k_{\mathrm{B}}^{\alpha} \cdot h^{\beta} \cdot T_0^{\gamma}\,.$$

The unit on the left-hand side must be the same as the unit on the right-hand side. We express the three constants using basic SI units kg, m, s and K

$$\mathrm{Hz} = \left(\mathrm{J{\cdot}K^{-1}}\right)^{\alpha} \cdot \left(\mathrm{J{\cdot}s}\right)^{\beta} \cdot \left(\mathrm{K}\right)^{\gamma}\,,$$
$$\mathrm{s^{-1}} = \left(\mathrm{kg{\cdot}m^2{\cdot}s^{-2}{\cdot}K^{-1}}\right)^{\alpha} \cdot \left(\mathrm{kg{\cdot}m^2{\cdot}s^{-1}}\right)^{\beta} \cdot \left(\mathrm{K}\right)^{\gamma}\,,$$

$$\begin{aligned} 0 &= \alpha + \beta\,, \\ 0 &= 2\alpha + 2\beta\,, \\ -1 &= -2\alpha - \beta\,, \\ 0 &= -\alpha + \gamma\,. \end{aligned}$$

We got a system of four linear equations with three unknowns. However, two of the equations are linearly dependent, so we have effectively three equations with three unknowns. The solution of this system of linear equations is $\alpha = 1$, $\beta = -1$, $\gamma = 1$ (you can check that it is correct). The frequency is

$$f = \frac{k_\mathrm{B} T_0}{h} \doteq 56.78\,\mathrm{GHz}\,.$$

The "natural frequency" is 56.78 GHz, which corresponds to the wavelength 5.3 mm in vacuum.

It is very interesting that this frequency is universal if we assume that the temperature of the universe was homogeneous enough during the recombination epoch and that the expansion of the universe is homogeneous and isotropic. Taking into account the accuracy of our calculation, these assumptions are accurate. Due to the expansion of the universe, wavelengths grow longer with time, but the temperature of cosmic microwave background is decreasing at the same rate. If we sent a signal on this "natural frequency", its wavelength would increase with time; however, possible observers would observe on the "natural frequency" based on their measured temperature of cosmic microwave background, so they could identify our signal.

Karel Kolář
`karel@fykos.cz`

Problem A.3 ... gravitation in The Orwille — 5 points

Isaac, a character in the TV series The Orwille, says that the planet Kaylon 1 has a circumference $o = 57\,583\,\mathrm{km}$, an average density $\rho = 4.42\,\mathrm{g{\cdot}cm^{-3}}$ and "gravity" 1.13 times higher than that of Earth. Assume that by "gravity", he means the gravitational acceleration on the planet's surface and use the value $a_\mathrm{g} = 11.11\,\mathrm{m{\cdot}s^{-2}}$. What is the corresponding value of the gravitational constant, assuming that the planet is a homogenous sphere and the laws of physics work the same way as in reality? *Karel was watching The Orwille.*

The gravitational acceleration at the planet's surface is

$$a_\mathrm{g} = G\frac{M}{R^2}\,, \tag{4}$$

where G is the gravitational constant that we want to find, M is the mass of the planet and R is its diameter. The mass of a homogeneous sphere is simply

$$M = \rho V = \frac{4}{3}\rho\pi R^3\,,$$

where V is its volume. Next, we want to express G as a function of the circumference of the Earth instead of its radius, so we use $o = 2\pi R$. Plugging into (4), we get

$$\begin{aligned} G &= a_\mathrm{g}\frac{R^2}{M} = a_\mathrm{g}\frac{R^2}{\frac{4}{3}\rho\pi R^3} = a_\mathrm{g}\frac{3}{4\pi\rho R} = \\ &= \frac{3}{2}\frac{a_\mathrm{g}}{\rho o} \doteq 6.55\cdot 10^{-11}\,\mathrm{m^3{\cdot}kg^{-1}{\cdot}s^{-2}}\,. \end{aligned}$$

The gravitational constant in such a universe would be $6.55 \cdot 10^{-11}\,\mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}$. Gravitation is then approximately $2\,\%$ weaker than in our universe.

It is interesting that the number π did not show up in the result, so even if it was different in the other universe, but the same laws of physics applied, we would not recognise the difference.

Karel Kolář
karel@fykos.cz

Problem A.4 ... a little crescent moon 5 points

What is the solid angle in the sky occupied by the illuminated part of the Moon if it is visible at an angular distance (elongation) $\alpha = 30^\circ$ from the Sun? The angular diameter of the Moon in the sky is $\theta = 0.52^\circ$. *Matěj was looking dreamily at the Moon.*

Assume that the Moon is far from the Earth and the Sun is far enough from the Moon. Since the angular diameter of the Moon is small, we can easily calculate the solid angle Ω_0 that it occupies in the sky (including its non-illuminated part) just as the surface of a circle of the diameter θ

$$\Omega_0 = \frac{\pi\theta^2}{4}\,.$$

Now we need to find the relative size of the illuminated part of the visible side of the Moon. From the Earth, we observe the circle under an angle and its projection into the plane perpendicular to the observed direction is an ellipse. In the picture we see, that the semi-minor axis of the ellipse is $b = r\sin(\beta - \pi/2) = -r\cos\beta$, where r is the radius of the Moon. The relative share k of the illuminated part of the Moon can be easily expressed as the ratio of the illuminated part (see the picture) and the total projection of the Moon into the observed direction

$$k = \frac{\frac{\pi r^2 - \pi r b}{2}}{\pi r^2} = \frac{r - b}{2r} = \frac{1 + \cos\beta}{2}\,.$$

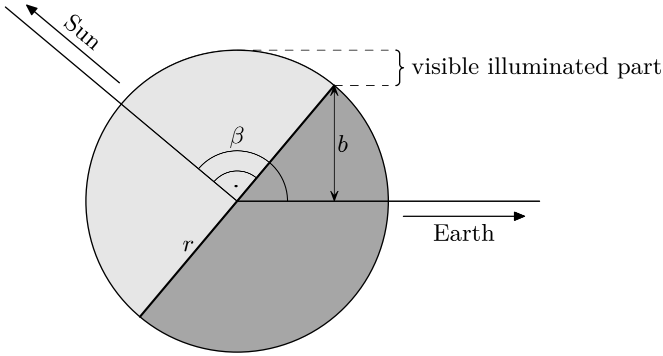
The angle β is the angle Earth-Moon-Sun. It could be easily calculate using the law of cosines, but since the Sun is very far from both the Earth and the Moon, we can approximate

$$\beta \approx \pi - \alpha\,.$$

Finally, the solid angle of the visible part is

$$\Omega = k\Omega_0 = \frac{1 - \cos\alpha}{2}\frac{\pi\theta^2}{4} = \frac{\pi\theta^2}{8}\,(1 - \cos\alpha) = 0.014\,2\,\mathrm{deg}^2 = 4.33 \cdot 10^{-6}\,\mathrm{sr}\,.$$

Matěj Mezera
m.mezera@fykos.cz



Problem A.5 ... a New Year's comet

5 points

Recently, specifically on December 20th 2020, we could see a particular comet in the sky for the first time in 14 years. It appeared at exactly the same spot as on December 20th 2006, the year when the first Fyziklani ever took place. Under what angle does the comet's orbit intersect the orbit of Earth if its eccentricity is 0.9? You may neglect the gravitational influence of Earth on the comet and assume the orbit of Earth to be perfectly circular. The comet's orbit is in the same plane as that of Earth. *Jarda is fascinated by comets.*

According to the assignment, the situation repeats exactly after 14 years, which is also the comet's orbital period. From Kepler's third law, the semi-major axis of the comet's orbit is equal to $a = T^{\frac{2}{3}}$, where the period T is in years and the semi-major axis of the ellipse a in astronomical units, giving us $a \doteq 5.809\,\mathrm{AU}$. The angle is determined by the law of conservation of angular momentum, which holds in a central gravitational field.

When moving on an ellipse, the law of conservation of energy has the form

$$E = -\frac{MmG}{2a}\,,$$

where M is the mass of the central body (in this problem it's the Sun), m is the mass of the comet, G is the gravitational constant and a is the semi-major axis length. The Sun is in one of the focal points of the ellipse. From the ellipse geometry, we are able to find the distance in perihelion (i.e. when the comet is closest to the Sun). This is

$$r_\mathrm{p} = a\left(1 - e\right)\,,$$

where $e = 0.90$ is the eccentricity of the comet's orbit. Using the law of conservation of energy, we can now find the kinetic energy of the comet as

$$E_\mathrm{k} = E - E_\mathrm{p} = -\frac{MmG}{2a} - \left(-\frac{MmG}{r_\mathrm{p}}\right) = (MmG)\left(\frac{1}{r_\mathrm{p}} - \frac{1}{2a}\right)\,.$$

We can express the velocity as

$$v_\mathrm{p} = \sqrt{2MG}\sqrt{\frac{1}{r_\mathrm{p}} - \frac{1}{2a}} = \sqrt{2MG}\sqrt{\frac{1+e}{2a\left(1-e\right)}}\,,$$

where we plugged in the expression for r_p. Angular momentum has magnitude

$$L = mvr\sin\alpha\,,$$

where α is the angle that the line joining the orbiting body and the Sun makes with the velocity vector (which is always a tangent to the trajectory). If the comet is in perihelion, then the velocity vector is perpendicular to this line and the angular momentum is

$$L = mv_\mathrm{p}r_\mathrm{p} = m\sqrt{MGa\left(1-e^2\right)}\,.$$

In a similar way, we calculate the speed of the comet at the moment of intersection with Earth's orbit, at the distance $r = 1\,\mathrm{AU}$ from the Sun. It is

$$v_\mathrm{E} = \sqrt{2MG}\sqrt{\frac{1}{r}-\frac{1}{2a}} = \sqrt{MG}\sqrt{\frac{2a-r}{ar}}\,.$$

From the law of conservation of angular momentum we can find the angle α as

$$\sin\alpha = \frac{v_\mathrm{p}r_\mathrm{p}}{v_\mathrm{E}r_\mathrm{z}} = \frac{\sqrt{MGa\left(1-e^2\right)}}{r\sqrt{MG}\sqrt{\frac{2a-r}{ar}}} = \frac{a\sqrt{\left(1-e^2\right)}}{\sqrt{r\left(2a-r\right)}}\,,$$
$$\alpha \doteq 51.0^\circ.$$

This angle is, therefore, made by the speed and the line connecting the comet and the Sun. But our goal was to find the angle between the velocity vector and the Earth's trajectory. It is

$$\beta = 90^\circ - \alpha \doteq 39.0^\circ\,.$$

The comet's trajectory intersects the Earth's orbit at the angle 39.0°.

Jaroslav Herman
`jardah@fykos.cz`

Problem A.6 ... deadly Sun 5 points

An astronomer noticed a sudden brightening on the surface of the Sun near a sunspot. This coronal mass ejection released a proton with total energy $E = 1\,012\,\mathrm{MeV}$. How long does it take for the proton to fly to Earth (from the time when the astronomer noticed the eruption)?

Dodo wanted to observe, but it was misty.

The required time is given by the difference of flight times of the particle and a photon

$$\Delta T = \frac{l}{v} - \frac{l}{c}\,,$$

where $l = 149.6\cdot 10^9\,\mathrm{m}$ is the distance between the Earth and the Sun. The magnetic fields will have little effect on a proton with such high energy, and so we can assume that it was flying directly to Earth. Further, we can verify that the loss of energy due to gravity is negligible. The velocity of the proton can be determined from the relativistic relation for energy.

$$E = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}\,,$$

where the rest energy m_0c^2 is the rest mass-energy of proton $m_\mathrm{p} = 938.27\,\mathrm{MeV}$. By expressing the velocity and substituting it into the relationship for the time difference we get

$$\Delta T = \frac{l}{c}\left(\frac{c}{v} - 1\right) = \frac{l}{c}\left[\left(1 - \frac{m_\mathrm{p}^2}{E^2}\right)^{-\frac{1}{2}} - 1\right] \doteq 833\,\mathrm{s}\,.$$

From the moment the flash is observed, the particle will reach the Earth in about a quarter of an hour.

Jozef Lipták
`liptak.j@fykos.cz`

Problem C.1 ... non-ideal ammeter 5 points

We have a resistor with resistance $R = 100\,\Omega$. We connect the resistor in series with an ammeter and connect them both in parallel with an ideal voltmeter. Then, we power the circuit by a DC voltage source. The ammeter shows $I = 12\,\mathrm{mA}$ and the voltmeter shows $U = 1.3\,\mathrm{V}$. What is the resistance of the ammeter? *Legolas was measuring resistances as part of lab work.*

The ammeter is connected in series with the resistor, so the value $I = 12\,\mathrm{mA}$ is the true current in the resistor. The voltage on the resistor is therefore $U_\mathrm{r} = IR$.

However, the voltmeter is connected in parallel to both the resistor and the ammeter, so the voltage U measured by the voltmeter is the sum of the voltage on the resistor U_r and the voltage on the ammeter U_a. Remember that the voltmeter is ideal, so no current flows through it. The voltage on the ammeter is $U_\mathrm{a} = U - U_\mathrm{r}$.

The current in the ammeter is the same as the current in the resistor, so the resistance of the ammeter is

$$R_\mathrm{a} = \frac{U_\mathrm{a}}{I} = \frac{U - U_\mathrm{r}}{I} = \frac{U - IR}{I} = \frac{U}{I} - R \doteq 8.3\,\Omega\,.$$

Šimon Pajger
legolas@fykos.cz

Problem C.2 ... cuboids 5 points

In his old box, Kiko discovered five identical conductive blocks (cuboids) with edge lengths $a = 5.0\,\mathrm{cm}$, $b = 4.0\,\mathrm{cm}$ and $c = 3.0\,\mathrm{cm}$. Each of them had resistivity $\rho = 1.21 \cdot 10^{-6}\,\Omega{\cdot}\mathrm{m}$. Kiko connected the blocks a few times and each time, he measured the total resistance of the circuit he made. What is the lowest resistance he could have measured, assuming that he always used all of the blocks? He always connected the blocks to the circuit using two parallel conductive panels that weren't touching each other. *Kiko was playing with bricks.*

We obtain the smallest resistance of a block when the current flows through the widest possible cross-section along the shortest possible path. Comparing the individual combinations of possible rotation of the block we find out that the smallest resistance is

$$R_\mathrm{min} = \rho \frac{c}{ab}\,.$$

The solution for the ideal connection of smallest resistance is now simple. We connect them all in parallel, which gives the equation for the total resistance of the circuit

$$\frac{1}{R_c} = \frac{5}{R_\mathrm{min}}\,.$$

By inverting the equation we get the result

$$R_c = \frac{R_\mathrm{min}}{5} = 3.63 \cdot 10^{-6}\,\Omega\,.$$

Kristián Šalata
kristian.salata@fykos.cz

Problem C.3 ... inductors 5 points

Káťa wanted to make a coil with an inductance that would be as high as possible, but she was also attempting to keep its ohmic resistance low. The old inductor had a radius $r_0 = 2\,\mathrm{cm}$ and $N_0 = 100$ turns which were touching each other but didn't overlap. Káťa used the wire from this coil to make a new coil with a radius $r_1 = 4\,\mathrm{cm}$. The turns on the new coil were also right next to each other and they didn't overlap. By how many percent did the inductance rise? *Káťa was pursuing an internship at a tokamak but she ended up winding coils.*

The inductance L of a long solenoid is

$$L = \frac{\mu N^2 S}{l}\,,$$

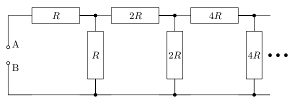
where l is length of the solenoid, S is its cross-sectional area and μ is permeability of the environment. This coil doesn't possess a core, so we can assume that μ is equal to the permeability of vacuum. The ratio of inductances is

$$\frac{L_1}{L_0} = \frac{\mu N_1^2 \pi r_1^2 l_0}{\mu N_0^2 \pi r_0^2 l_1} = \left(\frac{N_1 r_1}{N_0 r_0}\right)^2 \frac{l_0}{l_1} = 2\,,$$

where we noticed that $N_1 r_1 = N_0 r_0$ is proportional to the total length of the wire and the length of the coil l is proportional to the number of turns. The new coil, whose radius is twice as big, has half as many turns compared to the old one. The inductance of the new coil is therefore twice as big – in other words, it rises by $100\,\%$. It seems like we could increase the inductance arbitrarily by using a large radius. However, for large S/l, the relation $L = \mu N^2 S/l$ is not valid anymore.

Kateřina Charvátová
`katerina.charvatova@fykos.cz`

Problem C.4 ... infinite and nasty circuit 5 points



In the figure, we can see part of an infinite circuit of resistors. The resistances of resistors in it are terms of a geometric series. In other words, in the first block, each resistance is R, in the second block, it is $2R$ and in each following block, the resistances are twice as large as for resistors in the previous block. What is the total resistance between points A and B? The answer should be a multiple of R.

Karel was modifying problems on infinite circuits.

In problems with an infinite circuit, the trick is to find the repeating pattern and then to form a suitable equation from which we can (hopefully easily) express the desired total resistance R_∞. In our case, we can notice that by removing the first two resistors, we get a very similar circuit. The only difference is that each resistor in that circuit has twice as large resistance as before. That means the whole circuit has the resistance $2R_\infty$. We can draw the circuit in two equivalent ways, as shown in the figure 4. We get the equation

$$R_\infty = R + \frac{2RR_\infty}{R + 2R_\infty}\,.$$

We can easily convert it to a quadratic equation, which we solve

$$RR_\infty + 2R_\infty^2 = R^2 + 2RR_\infty + 2RR_\infty \,,$$
$$0 = 2R_\infty^2 - 3RR_\infty - R^2 \,,$$
$$R_\infty^\pm = \frac{3R \pm \sqrt{9R^2 + 8R^2}}{4} = \frac{3 \pm \sqrt{17}}{4} R \,.$$

The physically correct solution is the one with the positive sign. The other one would result in negative resistance. The only solution is the total resistance

$$R_\infty = \frac{3 + \sqrt{17}}{4} R \doteq 1.78\,R \,.$$

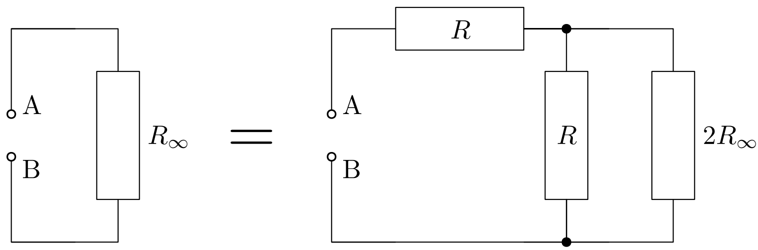
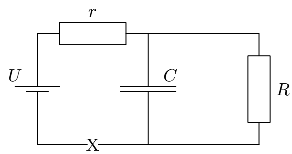


Fig. 4: Two equivalent circuits.

Karel Kolář
`karel@fykos.cz`

Problem C.5 ... long wiring

5 points



Suppose that we have an electric appliance plugged to a $U_0 = 120\,\mathrm{V}$ DC power source via a long cable. Since the conductors are close to each other, we create an electric scheme of the circuit shown in the figure. The resistor R is the appliance, the resistor $r \ll R$ represents the resistance of the cable and the capacitor $C = 420\,\mathrm{nF}$ represents its capacitance. The point denoted by "X" suffers from improper contact, which leads to periodically switching the circuit on and off. Each of these states lasts for the same constant time $T = 0.001\,\mathrm{s}$. Find the power that the power source will supply to the appliance over a very long time after plugging the circuit in. If the connection was stable, the power supplied by the source would be $P_0 = 30\,\mathrm{W}$.

Dodo was fighting with an extension lead.

First, we will solve for the current in the circuit. From Kirchhoff laws we know that current I through the battery and resistor r is equal the sum of currents I_C and I_R through the capacitor and the resistor R. The voltage on the appliance and the capacitor is the same, therefore we have

$$RI_R = U_C \,.$$

Taking the derivative with respect to time using $C\,\mathrm{d}U = \mathrm{d}Q$ we get

$$RC\dot{I}_R = I_C\,,$$

where the dot over I_R means time derivative. Final basic relation we are going to use is for voltage distribution over the resistors

$$U_0 = rI + RI_R\,.$$

Whenever the circuit is interrupted at point X, the capacitor is being discharged through the appliance. For battery curent $I = 0$, therefore $I_R = -I_C$ and

$$-RC\dot{I}_C = I_C\,.$$

The solution is the decresing exponencial relation

$$I_C^{\mathrm{v}}(t) = I_C^{\mathrm{v}}(0)\,\mathrm{e}^{-\frac{t}{RC}}\,.$$

When the battery is connected we have

$$RI_R + rI_C + rI_R = U_0\,,$$

from which, by differentiating, we obtain

$$\frac{R+r}{RC}I_C + r\dot{I}_C = 0\,,$$

with has a solution

$$I_C^{\mathrm{z}} = I_C^{\mathrm{z}}(0)\,\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{t}{RC}}\,.$$

The next step is to make sure these solutions match in the instants of disconnecting and reconnecting the battery. The voltage on the capacitor must always change continuously (otherwise the there would be an infinite flow of energy). This voltage is "measured" on the appliance, therefore the current I_R is continuous. At the moment of disconnecting the battery

$$I_R^{\mathrm{z}}(T) = I_R^{\mathrm{v}}(0)\,,$$

and when reconnecting

$$I_R^{\mathrm{v}}(T) = I_R^{\mathrm{z}}(0)\,.$$

We can determine the current through the resistor from the expressions above

$$I_R^{\mathrm{v}} = -I_C^{\mathrm{v}}\,,$$
$$I_R^{\mathrm{z}} = \frac{U_0 - rI_C^{\mathrm{z}}}{R+r}\,.$$

Substituting the solutions into the preceeding we obtain a system of equations for $I_C^{\mathrm{v}}(0)$ and $I_C^{\mathrm{z}}(0)$

$$\frac{U_0}{r} + \frac{R+r}{r}I_C^{\mathrm{v}}(0) = I_C^{\mathrm{z}}(0)\,\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{T}{RC}}\,,$$
$$I_C^{\mathrm{z}}(0) = \frac{U_0}{r} + \frac{R+r}{r}I_C^{\mathrm{v}}(0)\,\mathrm{e}^{-\frac{T}{RC}}\,.$$

By addition of these equation we get the conservation equation for charge

$$\frac{R+r}{r} I_C^{\mathrm{v}}(0)\left(1-\mathrm{e}^{-\frac{T}{RC}}\right)=-I_C^{\mathrm{z}}(0)\left(1-\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{T}{RC}}\right).$$

Substituting the second equation into the first

$$I_C^{\mathrm{v}}(0)=-\frac{U_0}{R+r}\frac{1-\mathrm{e}^{-\frac{R+r}{r}\frac{T}{RC}}}{1-\mathrm{e}^{-\left(\frac{R}{r}+2\right)\frac{T}{RC}}},$$

and using the preceeding formula

$$I_C^{\mathrm{z}}(0)=\frac{U_0}{r}\frac{1-\mathrm{e}^{-\frac{T}{RC}}}{1-\mathrm{e}^{-\left(\frac{R}{r}+2\right)\frac{T}{RC}}}.$$

We can see that discharging current has negative sign, consistent with our choice of direction of current.

This is the full solution of the circuit, we just need to find the power. Work done by the battery per one cycle is

$$W=\int_0^T U_0 I^{\mathrm{z}}(t)\,\mathrm{d}t+\int_0^T U_0 I^{\mathrm{v}}(t)\,\mathrm{d}t=\int_0^T U_0 I^{\mathrm{z}}(t)\,\mathrm{d}t\,.$$

Battery current is

$$I^{\mathrm{z}}=I_C^{\mathrm{z}}+I_R^{\mathrm{z}}=I_C^{\mathrm{z}}+\frac{U_0-rI_C^{\mathrm{z}}}{R+r}=\frac{U_0+RI_C^{\mathrm{z}}}{R+r},$$

which gives us the work as

$$\begin{aligned}
W&=\frac{U_0}{R+r}\int_0^T\left(U_0+RI_C^{\mathrm{z}}(0)\,\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{t}{RC}}\right)\mathrm{d}t=\\
&=\frac{U_0}{R+r}\left[U_0 t-\left(\left(\frac{R}{r}+1\right)\frac{1}{RC}\right)^{-1}RI_C^{\mathrm{z}}(0)\,\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{t}{RC}}\right]_0^T=\\
&=\frac{U_0}{R+r}\left(U_0T+\frac{rR^2C}{R+r}I_C^{\mathrm{z}}(0)\left(1-\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{T}{RC}}\right)\right)=\\
&=\frac{U_0^2}{R+r}\left(T+\frac{R^2C}{R+r}\frac{\left(1-\mathrm{e}^{-\frac{T}{RC}}\right)\left(1-\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{T}{RC}}\right)}{1-\mathrm{e}^{-\left(\frac{R}{r}+2\right)\frac{T}{RC}}}\right).
\end{aligned}$$

For average power of the circuit using $P_0=\frac{U_0^2}{R+r}$ we get

$$P=\frac{W}{2T}=\frac{P_0}{2}\left(1+\frac{R^2C}{T\,(R+r)}\frac{\left(1-\mathrm{e}^{-\frac{T}{RC}}\right)\left(1-\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{T}{RC}}\right)}{1-\mathrm{e}^{-\left(\frac{R}{r}+2\right)\frac{T}{RC}}}\right).$$

Furthermore, for $R\gg r$ we get a nicer expression

$$P\approx\frac{P_0}{2}\left(1+\frac{RC}{T}\left(1-\mathrm{e}^{-\frac{T}{RC}}\right)\right),$$

where we can see, that it lies in between $P_0/2$ for small RC/T and P_0 for RC/T going to infinity. This makes sense a for a small capacity, the circuit is the same as without the capacitor, and conversely, for high capacity the circuit is always on. This is a common use of capacitors – to overcome short outages in driving voltage. Knowing the power and voltage for the circuit when the battery is always connected, we have the resistance of the appliance

$$R \approx \frac{U_0^2}{P_0}\,,$$

and therefore the power in question

$$P = \frac{P_0}{2} + \frac{U_0^2 C}{2T}\left(1 - \mathrm{e}^{-\frac{P_0 T}{U_0^2 C}}\right) \doteq 18.0\,\mathrm{W}\,.$$

Jozef Lipták
`liptak.j@fykos.cz`

Problem O.1 ... clapping at a concert 5 points

People are watching a concert of a rockstar. The length of the arena, i.e. distance from the closest row of attendees to the most distant row, is $l = 200\,\mathrm{m}$. The singer waves his arm whenever he wants the audience to clap once. Find the highest possible number of arm-waves per minute such that the resulting series of claps do not overlap in the singer's ears.

Dodo was listening to echo at Výstaviště Praha.

All audience members clap at the same time when they see the singer wave his arm (we can neglect the difference caused by a finite speed of light). However, we can't neglect the effect of a finite speed of sound. The time difference t between the singer hearing claps from the first row and claps from the last row is

$$t = \frac{l}{c}\,,$$

where we use the speed of sound $c = 343\,\mathrm{m\cdot s^{-1}}$. If the singer wants to distinguish the series of claps, the highest frequency with which he can wave his arm is

$$f = \frac{1}{t} = \frac{c}{l} \doteq 103\,\mathrm{min^{-1}}\,.$$

Jozef Lipták
`liptak.j@fykos.cz`

Problem O.2 ... a bubble in ice 5 points

Jindra was examining an ice-covered pond. He was intrigued by an air bubble in the ice. In reality, the bubble is $a = 4.0\,\mathrm{cm}$ deep in the ice. However, different refractive indices distort the observed distance. The refractive index of ice is $n_\mathrm{i} = 1.31$, the refractive index of air is $n_\mathrm{a} = 1.00$. What is the apparent depth of the bubble that Jindra observes if he is staring at the bubble straight from above, in the direction perpendicular to the surface of the ice?

Some people ice-skate, Jindra instead observes bubbles.

Rays coming from the bubble are refracted at the ice-air boundary. The refractive index of air is lower than that of ice, so the angle of refraction is higher than the angle of incidence. Therefore, the apparent depth is smaller than the real depth. Because Jindra is looking along the normal to the surface of the ice, we will use the small angle approximation.[2] Let's pick one of the rays and denote the distance between Jindra's line of sight and the point where the ray intersects the ice-air boundary by h. The angle of incidence (inside the ice) is approximately $\varphi_\mathrm{l} = h/a$, whereas the angle of refraction is $\varphi_\mathrm{v} = h/a'$, where a' is the sought apparent depth. The angles φ_l and φ_v are related by Snell's law

$$n_\mathrm{i}\varphi_\mathrm{l} = n_\mathrm{a}\varphi_\mathrm{v}\,.$$

We can express

$$\varphi_\mathrm{v} = \frac{n_\mathrm{i}}{n_\mathrm{a}}\varphi_\mathrm{l} = \frac{n_\mathrm{i}}{n_\mathrm{a}}\frac{h}{a}\,,$$

$$a' = \frac{h}{\varphi_\mathrm{v}} = \frac{n_\mathrm{a}}{n_\mathrm{i}}a \doteq 3.05\,\mathrm{cm}\,.$$

[2] In the limit $\varphi \to 0$, we can write $\sin\varphi = \tan\varphi = \varphi$.

Jindra observes the bubble at the apparent depth $a' \doteq 3.05\,\mathrm{cm}$.

Jindřich Jelínek
`jjelinek@fykos.cz`

Problem O.3 ... at a concert — 5 points

Danka and Dano were standing at a distance $r_1 = 60\,\mathrm{m}$ from a loudspeaker at a rock concert. They heard the music with a sound pressure level $L_1 = 80\,\mathrm{dB}$. What is the difference (in dB) between the sound pressure level heard by them and by the people standing in front of the stage at a distance $r_2 = 10\,\mathrm{m}$ from the loudspeaker? *Danka and Dano went to a concert.*

The physical quantity which describes the power of sound is the sound pressure level L. This quantity is defined as

$$L = 20 \log \frac{p}{p_0}$$

and its unit is decibel. Let p be pressure of the acoustic wave at a given point and $p_0 = 2\cdot 10^{-5}\,\mathrm{Pa}$ be the threshold of hearing. The pressure is proportional to the applied force, which is proportional to the displacement and therefore also to amplitude, which is proportional to the square root of the energy flux density of the waves. Assuming constant power of the speaker, the energy flow through any sphere around it is constant, independent of the radius r. Therefore, the energy flux density is inversely proportional to the surface of the sphere. Mathematically written,

$$p \propto F \propto I \propto \sqrt{\rho_E} \propto \sqrt{S^{-1}} \propto r^{-1} \,.$$

The ratio of the pressures at two different distances satisfies

$$\frac{p_2}{p_1} = \frac{r_1}{r_2} \,.$$

The difference in loudness we're looking for is

$$L_2 - L_1 = 20 \left(\log \frac{p_2}{p_0} - \log \frac{p_1}{p_0} \right) = 20 \log \frac{p_2}{p_1} = 20 \log \frac{r_1}{r_2} \doteq 16\,\mathrm{dB} \,.$$

People standing directly in front of the stage hear the concert approximately 16 dB louder.

Daniela Pittnerová
`daniela@fykos.cz`

Problem O.4 ... bells — 5 points

A church stands at a distance $h = 100\,\mathrm{m}$ from an infinite wall. The church bell rings with a period $T_0 = 1.00\,\mathrm{s}$. Kiko is walking around the church at a constant distance l from it. As he walks, he realises that the time interval between each bell ring and its echo is changing. What is the minimum distance l such that he could hear ringing (without distinguishing between the original sound and the echo) with a period $T = 0.50\,\mathrm{s}$ at some point of his walk. The speed of sound is $v = 310\,\mathrm{m{\cdot}s^{-1}}$. Neglect the height of the church tower with the bell.

Kiko was visiting a midnight mass.

We can substitute the echo by a church which is mirrored by the wall and rings at the same time as the first church. The distance l from the first church is constant - we walk along a circle centered at the church. The minimum time difference between the ringing sound and its echo is

$$\Delta t_{\min} = \frac{2h - 2l}{v} \,.$$

This is the time difference if we stand between the two churches. If we move by 180° to the opposite side, the time difference is the maximum

$$\Delta t_{\max} = \frac{2h}{v} \,.$$

Everywhere else on the circle, the time difference Δt satisfies $\Delta t_{\min} < \Delta t < \Delta t_{\max}$. In order to hear ringing with the period $T = 0.5\,\mathrm{s}$, the time difference must be $\Delta t = 0.5\,\mathrm{s}$. If we stand right next to the church ($l = 0\,\mathrm{m}$), then the time difference is greater than $0.5\,\mathrm{s}$. If we move closer to the wall, the time difference decreases until it finally becomes $0.5\,\mathrm{s}$ at the distance

$$l_{\min} = h - \frac{\Delta t v}{2} = 22.5\,\mathrm{m} \,.$$

Kristián Šalata
kristian.salata@fykos.cz

Problem O.5 ... soliloquy 5 points

Tomáš was in an empty dormitory without any friends to talk with, so he decided to go jogging and talk to himself. He runs with a constant velocity $10\,\mathrm{km\cdot hod^{-1}}$ *along a long wall at a distance* $50\,\mathrm{m}$ *from the wall. He hears his voice not only directly, but also as an echo from the wall. What is the ratio of the frequency of the primary sound to the frequency of the echo?*

Kiko has a friend to talk with.

Since Tomáš runs with constant velocity, the velocities of the transmitter and receiver are the same. Therefore, the frequency of the sound doesn't change and the ratio will be equal to 1.

This result can also be explained when we can replace the wall by a second runner (a mirror image of Tomáš exactly copying his motion) at a distance $100\,\mathrm{m}$ from Tomáš. This situation with two people running in parallel and talking to each other is equivalent to hearing an echo from a wall at the distance $50\,\mathrm{m}$. Relative motion of the air with respect to the runners doesn't change the frequency of the sound. Therefore, the ratio of frequencies is 1.

If you want to see some math, the Doppler shift is

$$f = \frac{c - v_{\mathrm{receiver}}}{c - v_{\mathrm{transmitter}}} f_0 \,.$$

The velocities of the receiver v_{detector} and the transmitter $v_{\mathrm{transmitter}}$ are the same, so $f/f_0 = 1$.

The only condition is that Tomáš must be moving slower than sound.

Kristián Šalata
kristian.salata@fykos.cz

Problem O.6 ... a bubble in a paperweight

5 points

Jindra was intrigued by a spherical glass paperweight with a radius $r = 5.0\,\mathrm{cm}$. There is an air bubble inside the paperweight at a depth $a = 3.0\,\mathrm{cm}$. Jindra observes the bubble along the ray eye-bubble-centre of the sphere. What is the apparent depth of the bubble that Jindra observes? The refractive index of glass is $n_\mathrm{s} = 1.52$, the refractive index of air is $n_\mathrm{v} = 1.00$.

Instead of preparing for exams, Jindra was observing a bubble.

The object (bubble) is at the distance $a = 3.0\,\mathrm{cm}$ from the spherical glass-air boundary, which refracts the light rays coming from it and creates an image of the bubble at a distance a'. In the small angle approximation, we can derive the relation between these distances

$$\frac{n_\mathrm{g}}{a} + \frac{n_\mathrm{a}}{a'} = \frac{n_\mathrm{g} - n_\mathrm{a}}{r}\,.$$

If a' is negative, the image is inside the paperweight and virtual and if a' is positive, the image is outside the paperweight and real. We get

$$\frac{1}{a'} = \left(\frac{n_\mathrm{g}}{n_\mathrm{a}} - 1\right)\frac{1}{r} - \frac{n_\mathrm{g}}{n_\mathrm{a}}\frac{1}{a}\,,$$
$$a' \doteq -2.48\,\mathrm{cm}\,.$$

Jindra sees the bubble at the depth $2.48\,\mathrm{cm}$.

Jindřich Jelínek
jjelinek@fykos.cz

Problem T.1 ... drinking water 5 points

Danka had a glass bottle with a volume $V_\mathrm{b} = 0.500\,\ell$ filled with $V_1 = 4.00\,\mathrm{d}\ell$ of water. She put it to her mouth in such a way that air could not flow inside and drank $V_\mathrm{o} = 2.00\,\mathrm{d}\ell$ of water without blowing any air in. Find the decrease in pressure in the bottle. Assume that the process was fast enough that no heat was exchanged between the water, the bottle and the surrounding environment. At the beginning, the pressure of the air in the bottle is the same as the atmospheric pressure $p_\mathrm{a} = 1\,013\mathrm{hPa}$. Assume that air consists of molecules with $f = 5$ degrees of freedom. *Danka's plastic bottle always shrinks.*

Since there is no heat exchange between the air in the bottle and its surroundings, the process is adiabatic, for which $pV^\kappa = \mathrm{const}$ holds. κ is the heat capacity ratio (typically denoted by γ in some countries), which links to the number of degrees of freedom as

$$\kappa = \frac{f+2}{f}\,.$$

Let p_2 denote the air pressure in the bottle after drinking. Then

$$p_\mathrm{a}\left(V_\mathrm{b} - V_1\right)^{\frac{f+2}{f}} = p_2\left(V_\mathrm{b} - V_1 + V_\mathrm{o}\right)^{\frac{f+2}{f}}\,,$$

holds for the air inside the bottle. The decrease in pressure can be found as the difference between the pressure before and after

$$\Delta p = p_\mathrm{a} - p_2\,.$$

By expressing the pressure p_2 from the penultimate equation and plugging it into the last one we obtain

$$\Delta p = p_\mathrm{a}\left(1 - \left(\frac{V_\mathrm{b} - V_1}{V_\mathrm{b} - V_1 + V_\mathrm{o}}\right)^{\frac{f+2}{f}}\right) \doteq 795\,\mathrm{hPa}\,.$$

We see that if the bottle was ideally sealed, the pressure difference would be more than 75 % of atmospheric pressure, which is an over-estimation. Of course, such negative pressure is not achievable by mouth only. However, even a much smaller and actually achievable pressure difference, is enough to shrink the material of a common plastic bottle.

Daniela Pittnerová
`daniela@fykos.cz`

Problem T.2 ... a little sun 5 points

Determine the temperature of a black body with the same radiated power as the Sun, but with a diameter equal to half of the Sun's diameter $D_\odot = 1\,392\,000\,\mathrm{km}$. The power radiated by the Sun is $P_\odot = 3.826\cdot 10^{26}\,\mathrm{W}$ and the Stefan-Boltzmann constant is $\sigma = 5.67\cdot 10^{-8}\,\mathrm{W{\cdot}m^{-2}{\cdot}K^{-4}}$. *Danka was cold.*

The total radiant intensity of a black body $I(T)$ obeys the Stefan-Boltzmann law

$$I(T) = \sigma T^4\,,$$

where T is the temperature of the black body. In order to get the radiated power, we have to multiply the intensity by the surface area of the body

$$S = \pi D^2 .$$

We start with the equation $P_\odot = P$, which we can rewrite using the previous identities as

$$P_\odot = \pi D^2 \sigma T^4 .$$

Substituting $D = D_\odot/2$ and expressing T, we get

$$T = \sqrt[4]{\frac{4P_\odot}{\pi\sigma D_\odot^2}} \doteq 8\,160\,\mathrm{K} .$$

Daniela Pittnerová
daniela@fykos.cz

Problem T.3 ... a simplified model of a balloon 5 points

Imagine that we have a small deflated balloon with a mass $m = 25\,\mathrm{g}$ made of very thin rubber. When it is inflated, the difference between the pressure inside and the pressure outside is constantly $\Delta p = 10\,\mathrm{kPa}$. We blow $n = 1.0\,\mathrm{mol}$ of air inside it. Assuming normal atmospheric conditions outside the balloon, find the smallest possible temperature inside the balloon such that it would float. Air behaves like an ideal gas and its molar mass is $M = 28.96\,\mathrm{g{\cdot}mol^{-1}}$.

Legolas lives with his head in the clouds.

It follows from the problem statement that inside the balloon, the pressure will be $p = p_\mathrm{a} + \Delta p$. Therefore, we can easily calculate its volume

$$V = \frac{nRT}{p} = \frac{nRT}{p_\mathrm{a} + \Delta p} ,$$

where we neglect the volume of the balloon itself – this is the volume of the air inside it. We are interested in the amount of air inside

$$n_V = \frac{p_\mathrm{a} V}{R T_\mathrm{a}} = n \frac{p_\mathrm{a} T}{(p_\mathrm{a} + \Delta p)\, T_\mathrm{a}} .$$

We must not forget that the air in the balloon also has mass. It remains for us to realise that we get the mass of air by multiplying the amount of air by its molar mass. Then, from equality of the force of gravity and the buoyant force, we get

$$\begin{aligned} F_g &= F_\mathrm{b} , \\ g\,(m + Mn) &= gMn_V , \\ m + Mn &= Mn \frac{p_\mathrm{a} T}{(p_\mathrm{a} + \Delta p)\, T_\mathrm{a}} , \\ T &= T_\mathrm{a} \left(1 + \frac{m}{Mn}\right) \left(1 + \frac{\Delta p}{p_\mathrm{a}}\right) \doteq 600\,\mathrm{K} . \end{aligned}$$

We may notice that this temperature depends on the ratio of the balloon's mass to the mass of air inside it. This ratio decreases with the increasing size of the balloon (the mass of the balloon increases quadratically, the amount of air cubically), which is the reason why it's not necessarry to heat the air in large balloons to very high temperatures.

Šimon Pajger
legolas@fykos.cz

Problem T.4 ... hot ball 5 points

During a sunny day, we place a ball (hollow, filled with air) with a diameter $d = 20.0\,\mathrm{cm}$ in front of a house. The initial temperature of the ball is $t_0 = 20.0\,\mathrm{^\circ C}$, its heat capacity is $C = 500\,\mathrm{J\cdot ^\circ C^{-1}}$ and it absorbs $\eta = 40\,\%$ of the solar energy incident on it. The radiant flux from the Sun at the point in front of the house where the ball is lying is $\Phi = 600\,\mathrm{W\cdot m^{-2}}$. What is the ratio of the pressure in the ball after half of an hour of lying in the sunshine to the initial pressure in it? Assume that the size of the ball remains constant and it is thermally insulated from its surroundings. Neglect any radiative losses. *Danka forgot a ball in the sunshine.*

During the time $\tau = 1\,800\,\mathrm{s}$, the ball absorbs energy

$$Q = \frac{\pi d^2}{4}\eta\Phi\tau\,.$$

The temperature of the ball changes by

$$\Delta T = \frac{Q}{C}\,.$$

From the equation of state of an ideal gas for an isochoric process (with constant volume), we get

$$\frac{p}{T} = \mathrm{const}\,,$$

where T is the thermodynamic temperature. Therefore, we get a formula for the ratio of the pressures

$$\frac{p}{p_0} = \frac{T}{T_0} = \frac{T_0 + \Delta T}{T_0} = 1 + \frac{\pi d^2 \eta\Phi\tau}{4CT_0} \doteq 1,09\,.$$

The pressure in the ball increases 1.09 times.

Daniela Pittnerová
daniela@fykos.cz

Problem T.5 ... a breakfast in Dukovany 5 points

Find the weight of ^{238}Pu (in grams) which we would need to boil $300\,\mathrm{ml}$ of water in just three minutes for a breakfast tea while camping. The initial temperature of the water is $\theta_0 = 15\,\mathrm{^\circ C}$. We assume that we can utilise $80\,\%$ of the energy released by decay. The half-life of ^{238}Pu is 87.7 years and it decays with α decay with mean released energy $5.593\,\mathrm{MeV}$.
Jarda wanted to go camping, but he didn't want to carry a heavy backpack.

The number of radioactive nuclei satisfies

$$N = N_0 \mathrm{e}^{-\lambda t}\,,$$

where N_0 is the initial number of nuclei, $\lambda = \frac{\ln 2}{T}$ is the decay constant and T is the half-life. Activity (the number of decays per second) is given as the absolute value of the time derivative of the formula above, therefore

$$A = N_0 \lambda \mathrm{e}^{-\lambda t} = \frac{N_0 \ln 2}{T} \mathrm{e}^{-\lambda t}\,,$$

where the absolute value was used in order to obtain the "count" – otherwise the number would be negative, because it is a decrease. If we multiply the activity by the mean energy released per decay and the efficiency $\eta = 0.8$, we will obtain the thermal power which can heat the water. In $t = 3\,\mathrm{min} = 180\,\mathrm{s}$ the water receives the heat $Q = mc\Delta T = mc\left(\theta_\mathrm{b} - \theta_0\right)$, where $m = 0.3\,\mathrm{kg}$, $c = 4\,200\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ is the specific heat capacity of water and $\theta_\mathrm{b} = 100\,\mathrm{^\circ C}$ is the boiling point of water. Since $t \ll T$, we assume $\mathrm{e}^{-\lambda t} \approx 1$ and therefore approximate the activity to be constant (otherwise we would have to integrate the activity in time – the integral is not hard, but we like to keep it simple). From the information given we get the equation

$$E\eta N_0 \frac{\ln 2}{T} = \frac{mc\left(\theta_\mathrm{b} - \theta_0\right)}{t}\,,$$

from which we express the number of nuclei

$$N_0 = \frac{mc\left(\theta_\mathrm{b} - \theta_0\right)T}{\eta E t \ln 2} \doteq 3.312 \cdot 10^{24}\,.$$

The molar mass of $^{238}\mathrm{Pu}$ is $M = 238.05\,\mathrm{g}$. Therefore we need

$$m = \frac{N_0 M}{N_\mathrm{A}} = 1\,309\,\mathrm{g}\,.$$

We see that to generate almost 600 W we need only 1.3 kg of the material, which is not that much. It is widely used in so-called radioisotope thermoelectric generators (RTGs). $^{238}\mathrm{Pu}$ is used (or has been used), for example, in Curiosity rover, space probes Cassini, New Horizons or lunar modules from the Apollo programme. RTGs in general need isotopes which emit particles of high enough energy, have an optimal half-life (too short would be less useful due to short lifespan, too high would emit too little radiation and therefore the amount needed would be too high). Furthermore, the radiation has to decelerate rapidly – otherwise the energy will escape the device and the generator will be both inefficient and dangerous for the crew or electric systems. The α-particle is big and heavy, therefore it decelerates very quickly. Therefore, the decay heat remains in the material and a pellet of $^{238}\mathrm{Pu}$ can heat itself to several hundred °C.

In the end, we should note that each radioactive emitter shall be dealt with according to the principles of radiation protection. This, among others, posits that radioisotopes should only be used if they provide sufficient value and there is no safer alternative. Our problem was purely educational – a small camping gas stove is a much better alternative.

Jaroslav Herman
`jardah@fykos.cz`

Problem T.6 ... oscillating piston — 5 points

Suppose that we have an insulated closed cylinder with a length $2l = 2.0\,\mathrm{m}$ and a constant cross-section $S = 100\,\mathrm{cm^2}$ lying on the ground. We add a small partition (wall) with a mass $m = 2.0\,\mathrm{kg}$

inside in such a way that there are $N = 10^{23}$ *molecules of an ideal gas with a heat capacity ratio* $\kappa = 5/3$ *on each side of the partition. The pressure settles on an equilibrium value* $p_0 = 100\,\mathrm{kPa}$. *Find the period of small oscillations of the partition.*

Lego likes small . . . oscillations.

The cylinder is thermally insulated, so the thermodynamic processes inside it will be adiabatic. For such processes, $pV^\kappa = \mathrm{const}$ applies. Since there is the same amount of gas on both sides of the partition, its mechanical equilibrium will be in the middle of the cylinder. Then we can calculate the pressure difference between the two separated chambers when the partition moves by a very small distance x. From the equation $p_0 V_0^\kappa = p_x V_x^\kappa$, we express

$$p_x = p_0 \frac{V_0^\kappa}{V_x^\kappa} = p_0 \frac{S^\kappa l^\kappa}{S^\kappa (l+x)^\kappa} = p_0 \left(1 + \frac{x}{l}\right)^{-\kappa} \approx p_0 \left(1 - \kappa \frac{x}{l}\right) .$$

We must not forget that in the other chamber, the pressure will also change, the partition will just move in the opposite direction. Therefore, we can obtain the pressure in it almost in the same way, we only have to change $x \to -x$. Then we can calculate the resultant force, which acts on the partition when it is displaced from the mechanical equilibrium by x, as

$$F_x = S \Delta p_x = S \left(p_0 \left(1 - \kappa \frac{x}{l}\right) - p_0 \left(1 + \kappa \frac{x}{l}\right)\right) = -\frac{2\kappa S p_0}{l} x .$$

Let us also check the direction. In the chamber towards which the partition moves, the pressure increases (so it pushes harder on the partition), and in contrast, in the other chamber the pressure decreases. Therefore, the force acts against the direction of the partition's displacement and small oscillations will occur.

It only remains to realise that for the stiffness k of a linear harmonic oscillator, $F = -kx$ applies. Therefore we can identify $k = 2\kappa S p_0 / l$. By substituting into the formula for the period of small oscillations, we get

$$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{ml}{2\kappa S\, p_0}} \doteq 0.15\,\mathrm{s} .$$

We may notice that we have not used the amount of the gas in the cylinder nor its temperature.

Šimon Pajger
`legolas@fykos.cz`

Problem T.7 ... a mining poison 5 points

Imagine a cylindrical tank with a height $H = 100\,\mathrm{m}$ *and base radius* $r = 20\,\mathrm{m}$. *It is closed and filled with air (78 % of the volume is nitrogen, 21 % oxygen and 1 % argon) with a temperature* $T = 27\,^\circ\mathrm{C}$ *and pressure* $p = 1\,013\,\mathrm{hPa}$. *We burn some amount of pure carbon inside the tank, which transforms exactly one third of the oxygen molecules into carbon dioxide. Find the ratio of the molar concentration of oxygen to the molar concentration of carbon dioxide at the bottom of the tank when the temperature settles at its original value and the system reaches an equilibrium.*

Dodo sleeps on the upper bunk.

The first important thing is to realise that the gases in the box will be distributed independently of each other.[3] Each of these gases is distributed in the box in equilibrium following the Boltzmann distribution.

$$c(h) = c_0 \mathrm{e}^{-\frac{E_\mathrm{p}}{k_\mathrm{B} T}} = c_0 \mathrm{e}^{-\frac{mgh}{k_\mathrm{B} T}} ,$$

where m is the mass of the gas molecule, c_0 is the molar concentration at the bottom of the box and $c(h)$ is the molar concentration at height h. This relation can be derived, for example, for the pressure evolution with height from the equation of hydrostatic equilibrium

$$\mathrm{d}p = -\rho g \,\mathrm{d}h$$

and from the equation of state of an ideal gas

$$p = \frac{\rho}{M} RT .$$

The value of the molar concentration at the bottom of the box is determined from the condition for the total substance amount of the gases

$$n = \int_0^H Sc(h)\,\mathrm{d}h = Sc_0 \frac{k_\mathrm{B} T}{mg} \left(1 - \mathrm{e}^{-\frac{mgH}{k_\mathrm{B} T}}\right) ,$$

$$c_0 = \frac{nmg}{Sk_\mathrm{B} T} \left(1 - \mathrm{e}^{-\frac{mgH}{k_\mathrm{B} T}}\right)^{-1} \approx \frac{nmg}{Sk_\mathrm{B} T} \left(\frac{mgH}{k_\mathrm{B} T} - \frac{1}{2}\left(\frac{mgH}{k_\mathrm{B} T}\right)^2\right)^{-1} = \frac{n}{SH}\left(1 - \frac{mgH}{2k_\mathrm{B} T}\right)^{-1} ,$$

where we used the approximation $\mathrm{e}^x \approx 1 + x + \frac{1}{2}x^2$. This formula is applicable to each of the gases separately. Let us mark the oxygen with the index c^o, carbon dioxide with c^d and carbon with c^c. We get the required quotient as

$$\frac{c_0^\mathrm{o}}{c_0^\mathrm{d}} = \frac{\frac{n^\mathrm{o}}{SH}\left(1 - \frac{m^\mathrm{o} gH}{2k_\mathrm{B} T}\right)^{-1}}{\frac{n^\mathrm{d}}{SH}\left(1 - \frac{m^\mathrm{d} gH}{2k_\mathrm{B} T}\right)^{-1}} = \frac{n^\mathrm{o}}{n^\mathrm{d}} \frac{1 - \frac{m^\mathrm{d} gH}{2k_\mathrm{B} T}}{1 - \frac{m^\mathrm{o} gH}{2k_\mathrm{B} T}} \approx \frac{n^\mathrm{o}}{n^\mathrm{d}} \left(1 - \frac{m^\mathrm{d} gH}{2k_\mathrm{B} T}\right)\left(1 + \frac{m^\mathrm{o} gH}{2k_\mathrm{B} T}\right) \approx$$

$$\approx \frac{n^\mathrm{o}}{n^\mathrm{d}} \left(1 - \frac{\left(m^\mathrm{d} - m^\mathrm{o}\right) gH}{2k_\mathrm{B} T}\right) = \frac{n^\mathrm{o}}{n^\mathrm{d}} \left(1 - \frac{m^\mathrm{c} gH}{2k_\mathrm{B} T}\right) = \frac{n^\mathrm{o}}{n^\mathrm{d}} \left(1 - \frac{M^\mathrm{c} gH}{2RT}\right) .$$

After substituting in the molar mass of carbon $M^\mathrm{c} = 12\,\mathrm{g{\cdot}mol^{-1}}$ we get

$$\frac{c_0^\mathrm{o}}{c_0^\mathrm{d}} \doteq 2\,(1 - 0.002\,36) \doteq 1.995\,3 .$$

The result may seem surprising - You probably saw experiments where carbon dioxide was poured from one box to another box. However, if we kept this carbon dioxide alone for a sufficiently long time, diffusion would do its work and it would distribute the carbon dioxide to higher levels due to thermal motion.

Jozef Lipták
`liptak.j@fykos.cz`

[3] We consider them to be ideal gases.

www: `https://fyziklani.cz`
e-mail: `fyziklani@fykos.cz`

Fyziklani competition is brought to you by FYKOS.

FYKOS
Institute of Theoretical Physics
Faculty of Mathematics and Physics, Charles University
V Holešovičkách 2
180 00 Praha 8
Czech Republic

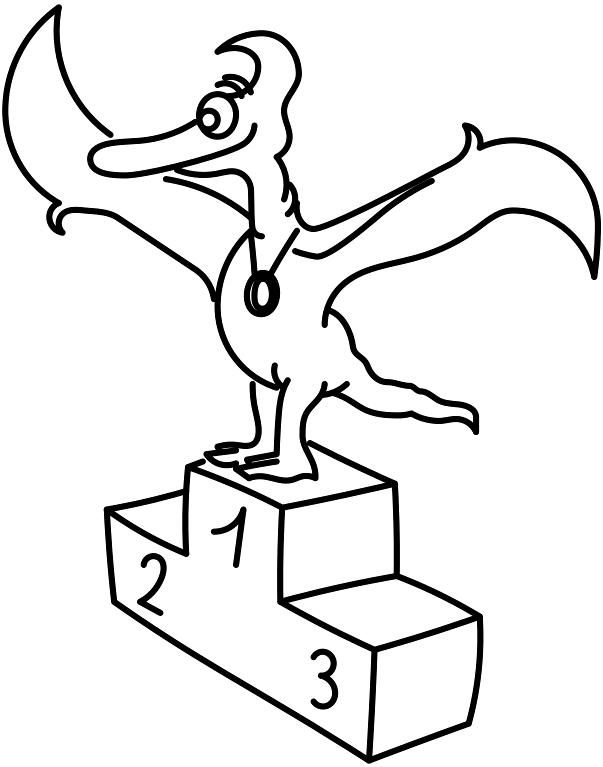
www: `https://fykos.org`
e-mail: `fykos@fykos.cz`

FYKOS is also on Facebook
http://www.facebook.com/Fykos

FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Public Relations Office activities and is supported by Institute of Theoretical Physics, MFF UK, his employees and The Union of Czech Mathematicians and Physicists.

Solutions of problems

Fyziklani



Problem AA ... it flows

A reactor of power $250\,\mathrm{MW}$ *is cooled by a water pump with a volumetric flow rate of* $1\,200\,\mathrm{h\ell}$ *per minute. If the water flows through the reactor only once, how much will its temperature rise? Do not consider heat loss.* *Pepa stole this from a textbook.*

The reactor power tells us that it releases heat $Q = 250\,\mathrm{MJ}$ per $1\,\mathrm{second}$ to its surroundings. During this second, $\Delta V = 20\,\mathrm{h\ell}$ of water of mass $m = \rho \Delta V$ flows through the reactor. We substitute that into the known formula $Q = mc\Delta T$ and express the change in the temperature of the flowed water as

$$\Delta T = \frac{Q}{c\rho\Delta V}\,,$$

which is approximately $30\,\mathrm{K}$ after number substitution.

Josef Trojan
`josef.trojan@fykos.org`

Problem AB ... muddy

The FYKOS-bird went sledging to a hillside. The hillside's perpendicular height is $10\,\mathrm{m}$ *and the path down the hill is* $20\,\mathrm{m}$ *long. However, there is already some grass and mud in the path which slows the sledge down, so the FYKOS-bird stops at the distance* $30\,\mathrm{m}$ *from the bottom of the hill after going downhill. What is the value of the coefficient of friction between the sledge and the ground? Neglect all other resistive forces.*
When Verča was young it was possible to go sledging in winter.

The friction force acting upon the sledge during the ride must do a work equal to the potential energy of the FYKOS-bird at the top of the hillside. During the ride from the hillside, the friction force $F_1 = mgf\cos\alpha$ acts upon the sledge, where f is the friction coefficient and α is the slope of the hill. We obtain the cosine of this angle from the Pythagorean theorem

$$\cos\alpha = \frac{\sqrt{s^2 - h^2}}{s}\,,$$

where h is the hillside's perpendicular height and s is the length of the path. Down below the hill the path is no longer inclined by any angle so the friction force is $F_2 = mgf$. If we denote the distance after which the FYKOS-bird stops as d, we get

$$\begin{aligned} W &= E_\mathrm{p}\,,\\ F_1 s + F_2 d &= mgh\,,\\ mgfs\cos\alpha + mgfd &= mgh\,,\\ f = \frac{h}{s\cos\alpha + d} &= \frac{h}{\sqrt{s^2 - h^2} + d}\,. \end{aligned}$$

The final coefficient equals approximately $f \doteq 0.21$. It is independent of the weight of the FYKOS-bird and the gravitational acceleration.

Veronika Hendrychová
`vercah@fykos.org`

Problem AC ... headphone jack

By how many minutes can a smartphone's battery life be extended by omitting a headphone jack and having a larger battery? Headphone jack needs a volume equivalent to a 1.5 cm *long cylinder with a diameter of* 3.5 mm*. A typical battery has a volume of* 20 cm^3 *and provides* 15 h *of active usage.* *Matěj follows new smartphone trends.*

We assume that a battery capacity is directly proportional to its size. Therefore, using the given values, we obtain battery life per volume as

$$\frac{15\,\mathrm{h}}{20\,\mathrm{cm}^3} = 0.75\,\mathrm{h{\cdot}cm^{-3}}\,.$$

We determine the volume saved by omitting the jack as a volume of cylinder, thus

$$V = \pi \cdot \left(\frac{0.35}{2}\,\mathrm{cm}\right)^2 \cdot 1.5\,\mathrm{cm} \doteq 0.144\,\mathrm{cm}^3\,.$$

To find the upgrade of battery life, we multiply both values we obtained earlier

$$\Delta t \doteq 0.75\,\mathrm{h{\cdot}cm^{-3}} \cdot 0.144\,\mathrm{cm}^3 \doteq 0.11\,\mathrm{h} \doteq 6.5\,\mathrm{min}\,.$$

If a phone has a headphones jack, it loses only several minutes of battery life; therefore, it is not a relevant argument why manufacturers omit it. The more important reasons seem to be improved water resistance of the smartphone or increased profit from selling more wireless headphones.

Matěj Mezera
m.mezera@fykos.org

Problem AD ... Fyziklani Prize

The FYKOS organizers were thinking about the prize for the best teams at Fyziklani. One of the ideas was to issue commemorative coins. It would be 18-karat golden (the rest is silver) coins for the winners, whose value would be $c_1 = 31\,000$ *Kč apiece. The runner-up team would get silver coins, whose value would be* $c_\mathrm{s} = 1\,100$ *Kč apiece. The silver coin is twice as heavy as the golden one. How many times is the unit price of gold greater than the unit price of silver? Assume zero minting costs. One karat is* 1/24 *of the total mass.*

Jarda won a keyboard with lighting effects once.

Let us denote the mass of the golden coin by m. The mass of the silver coins is then $2m$. From the definition of the karat, three quarters $\left(\frac{18}{24}\right)$ of the mass of the golden coins is gold, i.e., $m_\mathrm{z} = \frac{3}{4}m$.

Price of silver per unit mass is $j_\mathrm{s} = \frac{c_2}{2m}$. Price of the gold per unit mass is

$$j_z = \frac{c_\mathrm{z}}{m_z} = \frac{c_1 - j_\mathrm{s}\frac{m}{4}}{\frac{3m}{4}}\,.$$

Hence, the ratio of the unit prices is

$$\frac{j_\mathrm{z}}{j_\mathrm{s}} = \frac{c_1 - \frac{c_2}{8}}{\frac{3c_2}{8}} = \frac{8c_1 - c_2}{3c_2} \doteq 74.8\,.$$

Jaroslav Herman
jardah@fykos.org

Problem AE ... dang reduction

On highway with ongoing road work, the normal three lanes with maximum permitted speed $130\,\mathrm{km\cdot h^{-1}}$ are usually reduced to only two lanes with maximum speed restricted to $80\,\mathrm{km\cdot h^{-1}}$. How much (in percents) does the maximum number of cars that can drive on the highway in hour drop if we assume the traffic is optimal? Cars drive at the speed limit in all lanes and have time spacing $2\,\mathrm{s}$ between their front bumpers. Compare the highway on an ordinary day (no reduction) to a day with restrictions (with reduction).

Karel drove on the D1 (in the Czech Republic and Slovakia).

The solution is a bit surprising because if we have constant time spacing, one car passes one lane in time $\Delta t = 2\,\mathrm{s}$. Hence, if the speed of cars changes, distances between them change, but the total number of cars that pass the highway remains the same. If the allowed time spacing (which we measure between the front bumpers not to make the situation more complicated by the length of cars) is fully utilized, we get that the number of cars depends only on number of lanes. The proportion of cars that pass the highway under road work is therefore $2/3 \doteq 0.667 = 66.7\,\%$. Because we were asked by how much the capacity drops, the answer is $1 - 2/3 = 1/3 \doteq 0.333 = 33.3\,\%$.

If we assumed distance spacing between cars, the solution would be more complicated. However, it is recommended to maintain time spacing mainly due to the driver's reaction time, which remains constant regardless of the car's speed. Indeed, keeping the same space distancing could be seriously dangerous while driving at high speeds while it would be unduly cautious in the cities at lower speeds.

Karel Kolář
`karel@fykos.org`

Problem AF ... oh, those Newton's laws

What is the magnitude of drag force acting on the car, which is going on a straight road at constant velocity $30\,\mathrm{m\cdot s^{-1}}$ if the car has an engine of input power $150\,000\,\mathrm{W}$ that provides car with thrust $3\,000\,\mathrm{N}$? The temperature of surrounding air is $37\,\mathrm{^\circ C}$.

Robert just came up with this problem!

Because the car is driving at a constant speed, all forces acting on it must be in equilibrium (according to Newton's first law of motion). Drag forces are compensated by the force produced by engine. We do not assume any other forces acting on the car. All the other numbers are useless. Thus, the answer is $3\,000\,\mathrm{N}$.

Robert Gemrot
`robert.gemrot@fykos.org`

Problem AG ... toilet vandalism

A toilet paper roll has a cylindrical shape with height $v = 9.4\,\mathrm{cm}$, and outer diameter $D_2 = 12.5\,\mathrm{cm}$. The hollow cardboard cylinder in its center is of diameter $D_1 = 4\,\mathrm{cm}$. Thickness of individual paper rectangles is $h = 0.5\,\mathrm{mm}$. What is the minimum number of toilet paper rolls to cover the entire Earth's surface by a single layer of toilet paper if we assume that the paper is wound up tightly and that we can approximate the spiral winding of paper on a cardboard *cylinder with cylindrical layers? Round the result to two significant digits.*

Robo thinks about problems in unusual places.

By using the provided approximation, we know that a toilet paper roll has

$$n = \frac{\frac{D_2}{2} - \frac{D_1}{2}}{h}$$

paper layers. We want to compute the length of a single paper roll; thus, we have to calculate the sum of circumferences of those layers. The sum has the following form

$$l = \sum_{i=1}^{n} 2\pi \left(\frac{D_1}{2} + ih\right) = 2n\pi\frac{D_1}{2} + 2\pi h \sum_{i=1}^{n} i = n\pi D_1 + 2\pi h \frac{(n+1)\,n}{2}\,,$$

where l is the total length of a unwound single paper roll. If we substitute for n, we get

$$l = \frac{\pi D_1 \left(D_2 - D_1\right)}{2h} + \frac{\pi \left(D_2 - D_1\right)}{2} \left(\frac{D_2 - D_1}{2h} + 1\right)\,,$$

and after the simplification

$$l = \frac{\pi}{4h} \left(D_2 - D_1\right) \left(D_1 + D_2 + 2h\right)\,.$$

Now, get back to the cylindrical shape of the toilet paper roll. The area of one unwound toilet paper is $S_0 = v \cdot l$, while the Earth's surface area is $S = 4\pi R_{\oplus}^2$. Hence, the minimum number of toilet paper rolls needed to cover the entire Earth's surface is

$$N = \frac{S}{S_0} = \frac{4\pi R_{\oplus}^2}{vl} = \frac{16hR_{\oplus}^2}{v\left(D_2 - D_1\right)\left(D_2 + D_1 + 2h\right)} \doteq 2.5 \cdot 10^{14}\ \text{toilet paper rolls}\,.$$

Robert Jurenka
`robert.jurenka@fykos.org`

Problem AH ... time to shine

Neuron Endowment Fund supports talented Czech scientists and their promising projects. Jarda hopes to become a laureate of the prestigious Neuron Award one day for his research in the micro-world. Therefore, he works hard in his laboratory with an electron microscope of accelerating voltage $U = 1.5\,\mathrm{kV}$, with which he can see much more detail than using an optical microscope. Electrons in the microscope have a wavelength λ. Determine the ratio of the energy of photons of the same wavelength to the kinetic energy of electrons in the microscope.

Jarda thinks about problems even on New Year's Eve.

We will use the de Broglie wavelength of the electrons, λ, which is defined as

$$\lambda = \frac{h}{p}\,,$$

where h is Planck's constant and p is the momentum of the particle.

Since the product of the accelerating voltage and the charge of particles is small relative to the rest mass of electrons expressed in an electronvolt ($1.5\,\mathrm{keV} \ll 510\,\mathrm{keV}$), we do not have to use relativistic formulas. Therefore, the momentum is given as $p = mv$, where we obtain speed from the conservation of energy as

$$v = \sqrt{\frac{2Ue}{m_\mathrm{e}}}\,,$$

where m_e is the mass of an electron, and e is its charge.

Putting all of this together and substituing into the first equation, we obtain the wavelength of the electrons as

$$\lambda = \frac{h}{p} = \frac{h}{m_\mathrm{e} v} = \frac{h}{\sqrt{2 m_\mathrm{e} U e}}\,.$$

The energy of a photon of the same wavelength is

$$E_\gamma = hf = \frac{hc}{\lambda} = c\sqrt{2 m_\mathrm{e} U e}\,.$$

Plugging in the kinetic energy of electrons $E_\mathrm{e}^\mathrm{kin} = Ue$ as above, the sought ratio is

$$\frac{E_\gamma}{E_\mathrm{e}^\mathrm{kin}} = c\sqrt{\frac{2m_\mathrm{e}}{Ue}} \doteq 26.1\,.$$

Jaroslav Herman
jardah@fykos.org

Problem BA ... nanoparticle coating

We have a $V_1 = 50\,\mathrm{m\ell}$ solution of silver nanoparticles with molar concentration $c_1 = 9\cdot 10^{-9}\,\mathrm{mol\cdot\ell^{-1}}$. The nanoparticles have a spherical shape with diameter $d = 20\,\mathrm{nm}$ and they are not agglomerated. We add to the beaker $V_2 = 20\,\mathrm{m\ell}$ of glucose solution with concentration $c_2 = 6.5\cdot 10^{-5}\,\mathrm{mol\cdot\ell^{-1}}$. Assume that all added glucose molecules are adsorbed onto the surface of nanoparticles and are evenly distributed on them. What will be the average number of glucose molecules per $1\,\mathrm{nm}^2$ of surface of silver nanoparticles? *Danka was inspired by her research.*

Firstly, we determine the number of silver nanoparticles N_1 in the solution as

$$N_1 = c_1 V_1 N_\mathrm{A}\,,$$

where $N_\mathrm{A} = 6.022\cdot 10^{23}\,\mathrm{mol^{-1}}$ is the Avogadro's constant. Then, using the same formula, we calculate the number of glucose molecules N_2 that were added to the nanoparticle solution as

$$N_2 = c_2 V_2 N_\mathrm{A}\,.$$

Thus, the ratio

$$\frac{N_2}{N_1} \doteq 2\,888\,.$$

yields the average number of molecules that will be adsorbed onto one nanoparticle. In order to compute the number of molecules adsorbed per unit area, we need to calculate the surface area of one nanoparticle. We do this using the formula for the surface of a sphere

$$S_1 = 4\pi\left(\frac{d}{2}\right)^2 = \pi d^2\,.$$

Finally, we can calculate the average number of glucose molecules adsorbed onto $1\,\mathrm{nm}^2$ of a silver nanoparticle surface as

$$\sigma = \frac{N_2}{S_1 N_1} = \frac{c_2 V_2}{c_1 V_1 \pi d^2} \doteq 2.30\,\mathrm{nm}^{-2}\,.$$

Thus, $2.30\,\mathrm{nm}^{-2}$ glucose molecules will be adsorbed on average onto the surface of silver nanoparticles.

Daniela Pittnerová
daniela@fykos.org

Problem BB ... racing

What distance does a car travel when accelerating from 0 *to* $100\,\mathrm{km\cdot h^{-1}}$ *at its maximum possible acceleration? The coefficient of friction between the road and tires is* 0.9*. Do not consider any other drag forces. Assume that the car has a powerful engine enough to maintain the maximum possible acceleration throughout the entire acceleration period.*

Martin was nostalgic about his early years at the university.

The force delivered by the engine that accelerates the car cannot be greater than the frictional force to avoid wheel slip. From Newton's 2nd law of motion, let's express the maximum acceleration a as

$$a = \frac{F}{m} = \frac{fmg}{m} = fg\,,$$

where f is the coefficient of friction. Now from the relations for velocity and the trajectory of uniformly accelerated motion, we can determinate

$$v = at = fgt \quad \Rightarrow \quad t = \frac{v}{fg}\,,$$

$$s = \frac{1}{2}at^2 = \frac{v^2}{2fg} \doteq 43.7\,\mathrm{m}\,.$$

Vojtěch David
vojtech.david@fykos.org

Problem BC ... aerogel on the water

Assume we have an aerogel cuboid with density $2.42\,\mathrm{kg\cdot m^{-3}}$*. We place the aerogel on the water at standard conditions. How much of the aerogel will be under the water surface? For simplicity, assume that the aerogel cannot absorb water and neglect the surface tension of the water.*

Karel wanted a problem with aerogel.

At first, note that the assumptions are not really realistic as the surface tension would be significant in this setup. We could expect that aerogel would not break the water surface, but it would descend by the same height as we determine here.

For simplicity, we can assume a cuboid, which is stable, partially submerged, and its base of area S is parallel to the water surface. The solution for a general shape of aerogel would be the same, but this approach helps us to imagine the problem better. Let us denote the depth of the submerged part of the cuboid as x, and the height of the cuboid in direction perpendicular to the water surface as h. The density of the aerogel will be ρ_1.

The buoyant force (acting on cuboid) exerted by water is $F_1 = \rho_3 Sxg$, where $\rho_3 = 998\,\mathrm{kg{\cdot}m^{-3}}$ is the density of water. We also have to take into account the buoyant force exerted by air, which is $F_2 = \rho_2 S\left(h - x\right) g$, where $\rho_2 = 1.2\,\mathrm{kg{\cdot}m^{-3}}$ is the density of air. Thus, the total buoyant force acting on the aerogel is

$$F = F_1 + F_2 = S\left(\rho_3 x + \rho_2\left(h - x\right)\right) g\,.$$

The magnitude of this force has to be equal to the magnitude of the gravity acting on the aerogel, i.e., $F_g = mg = \rho_1 Shg$, from which we obtain a single equation for one unknown variable x. More precisely, we want to evaluate the ratio x/h, as it is questioned in the problem

$$\begin{aligned} F_g = F \qquad \Rightarrow \qquad \rho Shg &= S\left(\rho_3 x + \rho_2\left(h - x\right)\right) g\,,\\ \rho_1 h &= \rho_3 x + \rho_2 h - \rho_2 x\,,\\ \frac{x}{h} &= \frac{\rho_1 - \rho_2}{\rho_3 - \rho_2} \doteq 0.122\,\%\,. \end{aligned}$$

When we place such an aerogel on water surface without assuming surface tension, it will not descend by more than 0.13 %. In reality, aerogel would lie on the surface without breaking the water surface, and the observer would see almost no change in curvature of the water surface.

Karel Kolář
`karel@fykos.org`

Problem BD ... measuring the Earth radius on budget

Suppose someone lies down on the edge of the beach so that his eyes are exactly at the sea level and that he is looking at the setting sun. When the upper edge of the Sun disappears behind the horizon, he starts his stopwatch and stands up, so that his eyes are suddenly at the height of $h = 164\,\mathrm{cm}$ and he sees the Sun again. When the upper edge of the Sun sets behind the horizon again, he stops the stopwatch. What time will the stopwatch show if it took place at the equator on the day of the equinox? *Lego has actually never been to the seaside.*

In an Earth's reference frame, the upper edge of the Sun disk is a point which retains a circular motion with a large radius (we cannot find our distance from the Sun easily) and an angular velocity $\omega = \frac{2\pi}{24\,\mathrm{h}} \doteq 7.3 \cdot 10^{-5}\,\mathrm{s^{-1}}$. We can determine the elapsed time if we find out the angle of the point's displacement.

If the disk's upper edge disappears behind the horizon, the the line connecting their eyes and the horizon is exactly the tangent line to the Earth's surface. We will focus on two of these. The first one touches the Earth's surface at the observer's location and the second one somewhere in front of him. The angle of Sun's movement matches the included angle of these two lines. Because these tangent lines are always perpendicular to the Earth's radius at the tangent point of tangency, the angle between the lines connecting the center of Earth and the points of tangency is the same. To obtain this angle, let us assume a triangle with vertices in Earth center, the second point of tangency and the standing observer's eyes. It is obvious, that the angle in the second point of tangency is a right angle (one side belonging to this vertex is a part of the tangent line, the second is Earth's radius).

To obtain the angle in the center of Earth, we can use a trigonometric function, because we know the lengths of the sides belonging to this vertex. The adjacent side is the length between the center of Earth to the surface, which is equal to a radius R, and the hypotenuse is the radius plus the height to the observer's eyes $R+h$. The cosine is defined as a ratio of adjacent over hypotenuse, so we get the desired angle by using inverse cosine

$$\varphi = \arccos\left(\frac{R_\oplus}{R_\oplus + h}\right) \doteq 7.2 \cdot 10^{-4}\,\mathrm{rad}\,.$$

Finally, the time measured on the stopwatch is $t = \frac{\varphi}{\omega} \doteq 9.9\,\mathrm{s}$.

Šimon Pajger
legolas@fykos.org

Problem BE ... from extreme to extreme

What would be the length of a cylindrical rod with a radius equal to Planck length l_p and the volume of the observable universe? Assume that the universe's age is $t = 13.8 \cdot 10^9$ years and that the universe is flat. Planck length is only dependent on three fundamental physical constants – the gravitational constant G, the reduced Planck constant $\hbar$, and the speed of light c. Provide the answer in the form of $\log_{10}\left(\frac{l}{1\,\mathrm{m}}\right)$. *Robo was wondering what if...*

Firstly, we compute the Planck length. The problem statement suggests we need to use dimensional analysis to determine coefficients α, β, γ in equation $l_\mathrm{p} = G^\alpha \hbar^\beta c^\gamma$, where all the physical quantities can be expressed in SI units followingly

$$\begin{aligned}[G] &= \mathrm{m^3{\cdot}kg^{-1}{\cdot}s^{-2}}\,, & [c] &= \mathrm{m{\cdot}s^{-1}}\,,\\ [\hbar] &= \mathrm{J{\cdot}s} = \mathrm{m^2{\cdot}kg{\cdot}s^{-1}}\,, & [l_\mathrm{p}] &= \mathrm{m}\,.\end{aligned}$$

Substituting these expressions into the equation for l_p, we get

$$\mathrm{m}^1 = \mathrm{m}^{3\alpha} \cdot \mathrm{kg}^{-\alpha} \cdot \mathrm{s}^{-2\alpha} \cdot \mathrm{m}^{2\beta} \cdot \mathrm{kg}^{\beta} \mathrm{s}^{-\beta} \cdot \mathrm{m}^{\gamma} \cdot \mathrm{s}^{-\gamma}\,.$$

By comparing the bases of powers on the right and left side of the equation, we get three equations with three unknowns

$$\begin{aligned}1 &= 3\alpha + 2\beta + \gamma\,,\\ 0 &= -\alpha + \beta\,,\\ 0 &= -2\alpha - \beta - \gamma\,.\end{aligned}$$

From the second equation, we know that $\beta = \alpha$, consequently from the third equation, we get $\gamma = -3\alpha$, and lastly we substitute into the first equation to obtain $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, and $\gamma = -\frac{3}{2}$. Thus, Planck length can be expressed as

$$l_\mathrm{p} = G^{\frac{1}{2}} \cdot \hbar^{\frac{1}{2}} \cdot c^{-\frac{3}{2}} = \sqrt{\frac{G\hbar}{c^3}} = 1.616 \cdot 10^{-35}\,\mathrm{m}\,.$$

The radius of the observable universe is $R = tc$, expressed numerically (in meters) as

$$R = 13.8 \cdot 10^9 \doteq 1.305\,6 \cdot 10^{26}\,\mathrm{m}\,.$$

We proceed by setting the volumes to be equal,

$$S \cdot l = \frac{4}{3}\pi R^3 \,,$$

where $S = \pi l_\mathrm{p}^2$. The length of the rod is finally

$$l = \frac{4R^3}{3l_\mathrm{p}^2} = \frac{4 \cdot 2.23 \cdot 10^{78}\,\mathrm{m}^3}{3 \cdot 2.61 \cdot 10^{-70}\,\mathrm{m}^2} \doteq 1.1 \cdot 10^{148}\,\mathrm{m}\,.$$

Hence, the correct answer is 148.

Robert Jurenka
robert.jurenka@fykos.org

Problem BF ... little wrecking ball

We are at Jupiter's surface, where the gravity is $a = 24.8\,\mathrm{m{\cdot}s^{-2}}$ and we have two identical pendulums, which oscillate in one plane and share the same hanging point. The length of rigid massless string is $l = 5\,\mathrm{m}$, and the ball made of osmium at its end has diameter $d = 5\,\mathrm{cm}$. Each of pendulums has an initial amplitude of $\theta = 4^\circ$. Consequently, we let them swing towards each other and eventually collide.

Now we assume only one of the pendulums. At what velocity does the ball need to collide with the rigid vertical wall (perpendicular to the plane of oscillations and containing the hanging point and the equilibrium position), so that ball will experience the same force as in the case of the two-pendulum collision mentioned above? All collisions assume to be perfectly elastic.

Delion was thinking 'bout traffic accidents.

We approach the problem using the law of conservation of total energy. At first, we solve the case of two pendulums colliding with each other. Just before the collision, each of the pendulums has its maximum velocity v_max, which they obtained by conversion of potential energy E_p to kinetic energy E_k. The potential energy E_p of one pendulum is

$$E_\mathrm{p} = mah \,,$$

where m is the ball's mass, and h is the height of the ball's initial position with respect to the equilibrium position. Concerning the size of the ball and the characteristic dimensions of the problem, we can approximate the ball by a mass point at its center. Thus, the height of the initial position h is

$$h = \left(l + \frac{d}{2}\right)(1 - \cos\theta) \,.$$

Kinetic energy E_k of the ball is

$$E_\mathrm{k} = \frac{1}{2}mv^2 \,.$$

If we substitute for h in the potential energy formula, and set the potential and kinetic energy equal, we get

$$ma\left(l + \frac{d}{2}\right)(1 - \cos\theta) = \frac{1}{2}mv_\mathrm{max}^2 \,.$$

We simplify the equation for $v_{\max}$ and obtain the formula for ball's velocity just before the collision

$$v_{\max} = \sqrt{2a\left(l+\frac{d}{2}\right)(1-\cos\theta)}\,.$$

The case with two pendulums and the questioned case (a pendulum and a rigid wall) are the equivalent if we look only at the half-plane, where the pendulum is in both cases. Therefore, we only need plug in the numbers to the last equation, obtaining $v_{\max} \doteq 0.78\,\mathrm{m\cdot s^{-1}}$. Concerning the dimensions of the ball at the end of the string, we neglected its moment of inertia.

Daniel Broško
daniel.brosko@fykos.org

Problem BG ... traffic density on a highway

Dano drives on the highway and sees trucks in the opposite direction more often than those he overtakes. He wonders whether this is correct and if there is not just more traffic in the opposite direction than he is driving. What should be the ratio of the number of trucks he passes in the opposite direction to those he overtakes if there is the same number of trucks per unit time in both directions? For simplicity assume that Dano is driving at speed $v_1 = 130\,\mathrm{km\cdot h^{-1}}$, and all trucks drive at speed $v_2 = 90\,\mathrm{km\cdot h^{-1}}$. You can neglect the dimensions of cars, trucks, and highway. *Karel was driving the highway for several hours and has been thinking about it.*

Dano passes trucks in the opposite direction at speed $w_1 = v_1 + v_2$ because the speeds of vehicles going in opposite directions add up. On the contrary, he overtakes the trucks at speed $w_2 = v_1 - v_2$. If we observe a long period, which means many trucks, or if we have no information about spacings, it should be obvious that we must solve directly by using speeds. The faster the trucks move towards Dano's car, the more of them he meets in a certain period of time. The questioned ratio K is thus directly proportional to the ratio of speeds by which he passes the trucks; we can write

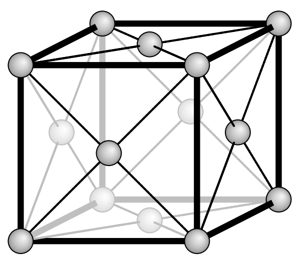
$$K = \frac{w_1}{w_2} = \frac{v_1+v_2}{v_1-v_2} = \frac{220}{40} = 5.5\,,$$

If the truck traffic was the same in both directions, then Dano should pass the oncoming trucks 5.5-times more often than those he overtakes. But if he passes, e.g., only 4-times more, he can assume that the traffic in the opposite direction is lower, although he sees the oncoming trucks more often.

Karel Kolář
karel@fykos.org

Problem BH ... dense lattice

One of the possible arrangements of atoms in the crystal is the face-centered cubic lattice. In this case, the atoms are located at the vertices and centers of the walls of the cube. The crystal is formed by the periodic arrangement of these cubes. We consider atoms to be rigid spheres that come in close contact. What part of the crystal volume is filled with atoms?
Danka studied crystallography.



Let us denote the length of the elementary cell edge as a. The first important thing is to determine the maximum possible radius r of the atom in the lattice. Let us look at how the atoms are arranged in the cube's wall. There is a single atom in each vertex, and another atom lies in the center of the wall. Since the atoms are in close contact, the length of the wall diagonal is four times the atom's radius (two contributions from the center atom and one from each of the two atoms at opposite vertices). The length of the wall diagonal is a hypotenuse of a right triangle with perpendiculars of length a. We find that the wall diagonal has the length $\sqrt{2}a$. It holds

$$\sqrt{2}a = 4r\,,$$

and thus

$$r = \frac{a}{2\sqrt{2}}\,.$$

We also have to determine how many atoms belong to one elementary cell. Each atom at the vertex of this imaginary cube in the crystal lattice belongs to eight adjacent cubes, and thus only the $\frac{1}{8}$ of the vertex atom belongs to our elementary cell. Since the cube has 8 vertices, the total volume of the vertex atoms corresponds to $8 \cdot \frac{1}{8} = 1$ whole atom. Analogously, each atom in the middle of the wall belongs to two elementary cells, so there is only $\frac{1}{2}$ wall atom per cube. The cube has 6 walls, so the total volume of wall atoms in one unit cell is then equal to $6 \cdot \frac{1}{2} = 3$ whole atoms. Thus, a space of four whole atoms is filled in one unit cell. Finally, calculate the ratio p of the total volume of atoms in the unit cell to the volume of the cell itself

$$p = \frac{4\frac{4}{3}\pi r^3}{a^3}\,.$$

This may be recasted using the relation between the radius and the length of the elementary cell edge as

$$p = \frac{\pi}{3\sqrt{2}} \doteq 74\%\,.$$

The filling factor of the face-centered cubic (FCC) grid is, therefore, 74 %.

Daniela Pittnerová
daniela@fykos.org

Problem CA ... arachnophobia

Lying on his bed, Jarda was watching a spider hanging from the ceiling on its spider silk. Jarda thought to himself how lucky he was that spiders are not bigger. However, Jarda got an idea that perhaps, spiders could not be bigger by much. If spider silk can hold a k-multiple of spider's mass, what is the maximum factor by which a spider and spider silk could enlarge (while maintaining all the proportions) so that spider silk does not break?

Jarda was watching a horror movie.

The information that tells us how much spider silk can carry is the ultimate tensile stress that spider silk can withstand. The stress depends on the acting force and the area of the silk's cross-section. If a spider gets bigger by a factor c, its mass increases proportionally to c^3, while the cross-section of silk increases proportionally to c^2. Thus, the maximal acting force increases proportionally to c^2. The ultimate tensile strength of spider silk is kmg; thus, concerning the condition for non-rupture of silk, we get

$$c^3 mg \le c^2 kmg\,,$$

clearly, the maximum factor is $c_{\max} = k$.

Jaroslav Herman
jardah@fykos.org

Problem CB ... eighth-life

What is the time required to decay one-eighth of unstable radioactive particles if we know that half of the particles decays in time T? *Karel wanted to trap naive participants.*

Radioactive decay can be expressed by the following formula

$$N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{T}}\,,$$

where $N(t)$ is the number of particles that have not undergone decay yet at time t, and N_0 is the initial total number of the particles. We express the ratio of number of particles that have not udergone decay yet to the initial number of particles as

$$\frac{N(t)}{N_0} = \left(\frac{1}{2}\right)^{\frac{t}{T}}\,.$$

Now we have to realize that we are looking for the time at which $1 - \frac{1}{8} = \frac{7}{8} = 87.5\,\%$ of the original particles are still left, thus

$$\left(\frac{1}{2}\right)^{\frac{t}{T}} = \frac{7}{8} \quad \Rightarrow \quad \frac{t}{T}\ln\frac{1}{2} = \ln\frac{7}{8} \quad \Rightarrow \quad t = \frac{\ln\frac{7}{8}}{\ln\frac{1}{2}}T \doteq 0.193T\,.$$

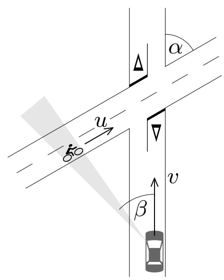
One-eighth of the unstable particles decays in 0.193-multiple of half-life. It is a little quicker than what "a premature estimate," that it is one-quarter of half-life could be. The closer to the beginning, the more original particles we have, and thus the more frequently decay occurs.

Karel Kolář
karel@fykos.org

Problem CC ... attention, cyclist

A car is approaching an intersection on a straight road at speed $v = 50\,\mathrm{km{\cdot}h^{-1}}$. At the intersection, the car should give a right of way to the vehicles going at the other straight road, which crosses the first road from the left at angle $\alpha = 60^\circ$. The driver's view is obstructed by the front left pillar of the car bodywork at angle $\beta = 25^\circ$ (from the direction of the travel). At what speed u must the cyclist approach the intersection in order to collide with the car without being seen by the driver?

Dodo does not ride a bicycle much.



Let t be the time remaining until the collision. Since the car and
cyclist should collide at the intersection, the car must be at distance vt from it, and the cyclist at distance ut, at each time. To prevent the driver from seeing the cyclist, the positions of cyclist C, car A, and intersection K must form a triangle with angles β opposite to side CK, α opposite to side AC, and $\gamma = \pi - \alpha - \beta = 95^\circ$ opposite to side AK. From the knowledge of the angles and the length of side AK, we can use the law of sines to express side KC as

$$|KC| = |AK| \frac{\sin\beta}{\sin\gamma} .$$

Then we obtain the speed of the cyclist as

$$u = v\frac{\sin\beta}{\sin\gamma} = v\frac{\sin\beta}{\sin(\alpha+\beta)} \doteq 21\,\mathrm{km{\cdot}h^{-1}} .$$

Jozef Lipták
`liptak.j@fykos.org`

Problem CD ... neglecting Saturn

The organizers of FYKOS plan to laser evaporate Saturn and then homogenize its rings into a hoop of a radius r and length density τ. Determine the gravitational force that would act on us in the center of the hoop. Consider that the organizers were also careful to evaporate the Saturn moons. Neglect the influences of the surrounding planets. Do not question the motives of the organizers. *Pepa wants to be fired from FYKOS.*

Concerning the problem symmetry, one may intuitively expect that the force in question will be zero – the gravitational effects of the opposite elements of the hoop will cancel each other out. We can verify this assumption by the following calculation.

Working in the plane of the hoop, we introduce the polar coordinates with the origin at its center. The position of any point of the hoop is thus described using a position vector

$$\mathrm{d}\mathbf{r} = (r\cos\varphi, r\sin\varphi) \ .$$

Using the length density, we can express mass of an infinitesimal section of the hoop as $\mathrm{d}m = \tau\,\mathrm{d}l$, or in our coordinate system as $\mathrm{d}m = \tau r\,\mathrm{d}\varphi$. Its contribution to the net force acting on object of a mass M at its center is then calculated using the Newton's law as

$$\mathrm{d}\mathbf{F} = G\frac{M\,\mathrm{d}m}{r^2}\frac{\mathrm{d}\mathbf{r}}{r}.$$

The resulting force is obtained by summing the individual contributions from all "pieces" of the hoop. However, we must not forget that we add up vectors. We can divide the integral into two components

$$\int_{\mathrm{hoop}} \mathrm{d}\mathbf{F}_\mathrm{x} = \frac{GM\tau}{r^2}\int_0^{2\pi} \cos\varphi\,\mathrm{d}\varphi = 0\,,$$
$$\int_{\mathrm{hoop}} \mathrm{d}\mathbf{F}_\mathrm{y} = \frac{GM\tau}{r^2}\int_0^{2\pi} \sin\varphi\,\mathrm{d}\varphi = 0\,.$$

The contributions from the individual sections of the hoop cancel each other out, and thus net force acting on the object in the center of the hoop is indeed zero.

Josef Trojan
josef.trojan@fykos.org

Problem CE ... heating of the synthetic air

Consider an insulated vacuum chamber with dimensions $a = 5\,\mathrm{cm}$, $b = c = 4\,\mathrm{cm}$ filled with nitrogen and oxygen at the temperature $20\,\mathrm{^\circ C}$ in a molar ratio of 85 to 15 so that the pressure in the chamber is $100\,\mathrm{mbar}$. We'll start heating the gas mixture at a rate of $0.8\,\mathrm{^\circ C{\cdot}s^{-1}}$. What will be the power input to the chamber if only $68\,\%$ of used energy is consumed for gas heating? The specific heat capacity of nitrogen is $c_{\mathrm{N_2}} = 743\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ and for oxygen it's $c_{\mathrm{O_2}} = 658\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$. The molar masses of nitrogen and oxygen are $M_{\mathrm{N_2}} = 28\,\mathrm{g{\cdot}mol^{-1}}$ and $M_{\mathrm{O_2}} = 32\,\mathrm{g{\cdot}mol^{-1}}$ respectively. *Danka studied operando methods.*

For the heat Q received by a body, we can write a calorimetric equation in the form

$$Q = mc\Delta T\,,$$

where m is the body's mass, c is its specific heat capacity and ΔT is the change in the body temperature due to received heat. Thus, we can write an equation for a given gas mixture

$$đQ = \left(m_{\mathrm{N_2}}c_{\mathrm{N_2}} + m_{\mathrm{O_2}}c_{\mathrm{O_2}}\right)\mathrm{d}T\,,$$

where $đQ$ is the infinitesimal amount of heat supplied to the mixture, and $\mathrm{d}T$ is the infinitesimal increase in gas temperature. Additionally, the supplied heat is related to the input power P as $đQ = \eta P\,\mathrm{d}t$, where η is the efficiency of the gas heating.

We can determine the amount of substance in the chamber using the equation of the state of an ideal gas

$$pV = nRT\,,$$

where p is the gas pressure, V its volume, n the amount of substance, R the molar gas constant, and T the gas temperature. Since the amount of substance is constant throughout the whole process, we can determine it using the initial values as

$$n = \frac{p_0V}{RT_0}\,.$$

This equation applies to each gas separately. The volume of both gases is equal to the chamber volume, i.e., $V = abc$. According to Dalton's law, the total pressure of a gas mixture is equal to the sum of the partial pressures of the individual components. At the same time, the ratio of the partial pressures of the individual gases is equal to their molar ratio. It follows that the initial partial pressure of nitrogen in the chamber is $p_{\mathrm{N_2}} = 85\,\mathrm{mbar}$ and oxygen $p_{\mathrm{O_2}} = 15\,\mathrm{mbar}$. The amount of substance can be easily converted to the weight of the gas using the relation

$$m = nM\,,$$

where M is the molar mass of the gas. Combining the relations mentioned above we get

$$\eta P\,\mathrm{d}t = \frac{V}{RT_0}\left(p_{\mathrm{N_2}}M_{\mathrm{N_2}}c_{\mathrm{N_2}} + p_{\mathrm{O_2}}M_{\mathrm{O_2}}c_{\mathrm{O_2}}\right)\mathrm{d}T\,,$$

$$P = \frac{abc}{RT_0\eta}\left(p_{\mathrm{N_2}}M_{\mathrm{N_2}}c_{\mathrm{N_2}} + p_{\mathrm{O_2}}M_{\mathrm{O_2}}c_{\mathrm{O_2}}\right)\frac{\mathrm{d}T}{\mathrm{d}t}\,.$$

The expression $\frac{dT}{dt}$ is the rate of change of the mixture temperature. We are left to substitute numerical values into the formula. By doing so we get $P = 8.05 \cdot 10^{-3}$ W. Thus, the power input is $P = 8.05\,\mathrm{mW}$.

We can obtain a similar result in a theoretical way - from the ideal gas model. Since no work is done on the system (its volume does not change, and there is no chemical reaction), the 1st law of thermodynamics takes the form

$$\mathrm{d}U = đQ\,,$$

where U is the internal energy, which in the case of a diatomic gas is $U = \frac{5}{2}nRT$. Pluging this into a aforementioned relationship $\eta P\,\mathrm{d}t = đQ$, one obtains

$$\eta P = \frac{\mathrm{d}U}{\mathrm{d}t} = \frac{5}{2}nR\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{5}{2}\frac{p_0 V}{T_0}\frac{\mathrm{d}T}{\mathrm{d}t}\,.$$

In this way we get the value $P = 8.03\,\mathrm{mW}$. The different result is caused by the difference between the theoretically and experimentally determined value of specific heat capacity. We can determine its value in our microscopic gas model as

$$mc\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{5}{2}nR\frac{\mathrm{d}T}{\mathrm{d}t}\,,$$

from which it follows that

$$c = \frac{5}{2}\frac{nR}{m} = \frac{5}{2}\frac{R}{M}\,.$$

Compared to the exerimental values, we get $c_{\mathrm{N}_2} = 742\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ for nitrogen and $c_{\mathrm{O}_2} = 650\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ for oxygen.

Daniela Pittnerová
daniela@fykos.org

Problem CF ... marble is jumping off the window

A marble (a small hard ball) with mass $1.2\,\mathrm{kg}$ *fastens itself to an elastic rope with stiffness* $6\,\mathrm{N{\cdot}m^{-1}}$ *and free length* $l_0 = 8\,\mathrm{m}$. *The other end of the rope is attached to a pad from which the marble jumps off. The rope unwinds during the fall and starts to extend when the marble is at distance* l_0 *below the pad. The marble is afraid of dizziness, so it wants to know the maximum acceleration (g-force) it faces during its journey? Write your answer a multiple of* g.

Jarda will not try this at the dormitory.

During the fall, until the whole rope unwinds and straightens, the marble moves with acceleration g. When the rope is extended, the marble will start moving like a harmonic oscillator. For the harmonic oscillator, the greatest acceleration is at the maximum amplitude of the motion, and the speed there is zero. Thus, the maximum extension of the rope can be found using the law of conservation of mechanical energy, expressed as

$$mg\,(l_0 + y) = \frac{1}{2}ky^2\,,$$

where m and k are the mass of the marble and the stiffness of the rope. The left-hand side represents the decline of potential energy, the right-hand side is the elasticity energy gain, and y denotes the maximum extension. We express y from the quadratic equation as

$$y = \frac{mg \pm \sqrt{m^2g^2 + 2mgl_0k}}{k}\,,$$

while we seek the positive root. Therefore, the magnitude of the force acting on the marble at the lowest point is

$$F = ky - mg = \sqrt{m^2g^2 + 2mgl_0k}\,,$$

and hence, the acceleration here is

$$a = \frac{F}{m} = \sqrt{g^2 + \frac{2gl_0k}{m}}\,.$$

We find the g-force as the ratio of the final acceleration and the gravitational acceleration

$$\frac{a}{g} = \sqrt{1 + \frac{2l_0k}{mg}} \doteq 3.0\,.$$

Thus, the maximum g-force acting on the marble is three times the gravitational acceleration.

Jaroslav Herman
jardah@fykos.org

Problem CG ... thermos suction cup

Legolas noticed that when rinsing his thermos with cold water, covering the opening with his hand and shaking the thermos, there is a negative pressure forming inside the thermos, sometimes sufficiently large to keep the thermos hanging on his hand. He assumes that this happens because the air in the thermos was initially warm from the tea he had had in the bottle before, and the cold water quickly cools the air inside. So, he decided to make a problem with similar theme.

Consider now a thermos of volume $V = 500\,\mathrm{m}\ell$ *and opening area* $S = 13\,\mathrm{cm}^2$*. Mass of the water and the thermos in total is* $m = 0.35\,\mathrm{kg}$ *and the temperature of the air inside equilibrates to* $T_\mathrm{c} = 20\,^\circ\mathrm{C}$ *after rinsing. What is the (minimal) original temperature of air inside the thermos, so that the created negative pressure keeps the thermos stuck to Lego's hand?* *True story.*

Before covering the opening, the air in the thermos had temperature T_h and was at atmospheric pressure $p_\mathrm{h} = p_\mathrm{a}$. The whole process is isochoric – the volume of the system is constant, hence after cooling of the air to temperature $T_\mathrm{c} = 293.15\,\mathrm{K}$, the equilibrium pressure of the air inside the thermos is $p_\mathrm{c} = p_\mathrm{h}T_\mathrm{c}/T_\mathrm{h}$. The magnitude of the force that pushes the thermos against the hand is given by the product of the opening area and the pressure difference

$$F = S\Delta p = Sp_\mathrm{c}\left(1 - \frac{T_\mathrm{c}}{T_\mathrm{h}}\right) = Sp_\mathrm{c}\frac{T_\mathrm{h} - T_\mathrm{c}}{T_\mathrm{h}}\,.$$

In the opposite direction, the weight of the system pulls on the thermos, leading to an equilibrium condition for minimum initial temperature

$$mg = Sp_\mathrm{c}\frac{T_\mathrm{h} - T_\mathrm{c}}{T_\mathrm{h}} \quad \Rightarrow \quad T_\mathrm{h} = T_\mathrm{c}\frac{Sp_\mathrm{c}}{Sp_\mathrm{c} - mg} \doteq 301\,\mathrm{K} = 28\,^\circ\mathrm{C}\,.$$

This is a relatively low temperature considering that the air was heated by contact with tea. That would suggest that the thermos sticks to Lego's hand quite often, but he denies that. The discrepancy is probably caused by the various approximations we made during the calculation – in a real situation, the air begins to cool down already during the process of pouring the water into the thermos.

Also, Legolas' hand is not perfectly insulating, so some air slips into the thermos even after the "sealing". Furthermore, Legolas' hand is not perfectly stiff, so it changes volume when compressed by the surrounding air, and hence changes the volume of the air inside the system.

For more experimentally inclined problem solvers, Legolas claims absolutely no responsibility for thermos destroyed as a result of this problem.

Šimon Pajger
`legolas@fykos.org`

Problem CH ... Robo shoot!

Robo is at the distance $L = 2\,022\,\mathrm{cm}$ from a soccer goal and is kicking a ball towards it with initial speed $v_0 = 15\,\mathrm{m\cdot s^{-1}}$ at angle $\alpha = 45^\circ$. Robo hits the crossbar, and unexpectedly, the ball bounces off perfectly vertically upwards but loses $10\,\%$ of its velocity due to the bounce. What maximum height above ground does the ball reach after it bounces upwards?

Robo likes to hit the construction of a soccer goal.

We use the motion equations of a upward parabolic throw

$$x(t) = v_0 t\cos\alpha\,,\qquad v_x(t) = v_0\cos\alpha\,,$$
$$y(t) = v_0 t\sin\alpha - \frac{1}{2}gt^2\,,\quad v_y(t) = v_0\sin\alpha - gt\,,$$
$$v = \sqrt{v_x^2 + v_y^2}\,.$$

where $x(t)$ and $y(t)$ are the instantaneous coordinates of the ball in time t, $v_x(t)$ and $v_y(t)$ are components of the velocity v. The ball will be in horizontal distance $x(t) = L$, at time $t_1 = \frac{L}{v_0\cos\alpha}$ and its height above the ground will be the same as the height of the soccer goal $y(t_1) = h_0$. The height of the soccer goal can be obtained by plugging the time t_1 into the second equation of parabolic throw. Consequently we get

$$h_0 = v_0\frac{L}{v_0\cos\alpha}\sin\alpha - \frac{1}{2}g\left(\frac{L}{v_0\cos\alpha}\right)^2 = L\tan\alpha - \frac{gL^2}{2v_0^2\cos^2\alpha}\,.$$

Now, we should take a look on the velocity after bounce. Let us denote the "reduction" factor as $\gamma = 0.9$, then

$$v_2 = \gamma v_1 = \gamma\sqrt{v_0^2\cos^2\alpha + \left(v_0\sin\alpha - \frac{gL}{v_0\cos\alpha}\right)^2} = \gamma\sqrt{v_0^2 - 2g\left(L\tan\alpha - \frac{gL^2}{2v_0^2\cos^2\alpha}\right)}\,,$$

where v_1 is the ball's velocity before the bounce, and v_2 is the ball's velocity after the bounce. Consequently, we can solve the problem using the law of conservation of energy. We take into account two situations – the situation right after the bounce from the crossbar and then the situation when the ball is at its maximum height after the bounce and the ball's velocity is therefore zero. This is represented by the following set of equations

$$\frac{1}{2}mv_2^2 + mgh_0 = mgh_{\max} \quad \Rightarrow \quad h_{\max} = h_0 + \frac{v_2^2}{2g}\,.$$

We substitute for h_0 and v_1 from the equations above, and we finally get the solution

$$h_{\max} = L\tan\alpha - \frac{gL^2}{2v_0^2\cos^2\alpha} + \frac{\gamma^2}{2g}\left(v_0^2 - 2g\left(L\tan\alpha - \frac{gL^2}{2v_0^2\cos^2\alpha}\right)\right)\,,$$

$$h_{\max} = \left(1-\gamma^2\right)\left(L\tan\alpha - \frac{gL^2}{2v_0^2\cos^2\alpha}\right) + \gamma^2\frac{v_0^2}{2g} \doteq 9.74\,\mathrm{m}\,.$$

Robert Jurenka
robert.jurenka@fykos.org

Problem DA ... waterboarding of iron ball

Legolas took his aquarium, filled it with water (but not completely to the edge, not to overfill it), and placed it on a weighing scale. Legolas did not have any fish, so he decided to float his favorite iron ball there. Of course, the ball alone would immediately sink, so he hung it on a gauge meter.

When the ball was floating under the surface (completely submerged, but not touching either the bottom, or the water level), the gauge meter displayed value $\Delta F = 10\,\mathrm{N}$. Nevertheless, Lego noticed that the weighing scale now displayed a greater value than before submerging the iron ball.

By how much did the number displayed on the weighing scale increase? The density of iron is $\rho_{\mathrm{Fe}} = 7\,874\,\mathrm{kg{\cdot}m^{-3}}$. Lego does not know the exact dimensions of his iron ball. Know that no water leaked from the aquarium, and that the weighing scale operates in kilograms.

Legolas has borrowed...

The first and most important is to realize that Newton's third law always applies. Indeed, if water lightens the iron ball by buoyancy force, the ball presses on the water, so its total force has the same magnitude as the buoyancy force and points downwards. The difference in displayed values reflects the buoyancy force acting on the iron ball.

Since the weighing scale converts the difference in forces to weight, we can say that the displayed value will increase by the weight of uplifted water. We only need to find the volume of the iron ball. Furthermore, we know that if the ball is fully submerged, the gauge meter has to compensate for the difference between gravity and buoyancy force by force of magnitude ΔF. This provides us with the equality

$$\Delta F = F_g - F_{\mathrm{b}} = V\rho_{\mathrm{Fe}}g - V\rho_{\mathrm{water}}g \quad \Rightarrow \quad V = \frac{\Delta F}{g\left(\rho_{\mathrm{Fe}} - \rho_{\mathrm{water}}\right)}\,.$$

To get the mass of uplifted water, which makes the difference between displayed values, we only need to multiply obtained volume by water density

$$\Delta m = \frac{\rho_{\mathrm{water}}}{\rho_{\mathrm{Fe}} - \rho_{\mathrm{water}}}\frac{\Delta F}{g} \doteq 0.15\,\mathrm{kg}\,.$$

If the weighing scale operates in kilograms, the displayed value increased by 0.15.

Šimon Pajger
legolas@fykos.org

Problem DB ... repulsive steering wheel

We choose two points on a circumference made of conductive material with specific linear resistance λ. What is the inscribed angle between these two points so that the resistance between them is half the maximum resistance possible (assuming a different position of the points)? Provide the value less than π as the answer. *Jarda comes up with problems when driving.*

We choose an arbitrary point on a circumference, and then we (without loss of generality) search for a second point, which is angularly distant φ ($2\pi - \varphi$, respectively) from the first one. The resistance between these two points is equivalent to the parallel setup of two resistors with resistance $\varphi\lambda r$ and $(2\pi - \varphi)\,\lambda r$, where r is the radius of the circumference. Total resistance is indeed

$$R(\varphi) = \frac{(2\pi - \varphi)\,\varphi\lambda^2 r^2}{2\pi\lambda r} = \frac{(2\pi - \varphi)\,\varphi\lambda r}{2\pi}\,.$$

The denominator is not dependent on φ, to obtain the maximal resistance, we differentiate the numerator

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\,(2\pi - \varphi)\,\varphi = 2\pi - 2\varphi\,.$$

Setting this equal to zero, we find that the angle for which the resistance reaches its maximum (due to the negative second derivative) is $\varphi_{\max} = \pi$. Unsurprisingly, we could have expected this result thanks to the symmetry of the problem. Therefore, the maximum resistance possible is

$$R_{\max} = \frac{(2\pi - \pi)\,\pi\lambda r}{2\pi} = \frac{\pi\lambda r}{2}\,.$$

We are asked to find such an angle φ, for which $R(\varphi) = \frac{R_{\max}}{2}$. To get the answer, we have to solve the following equation

$$\frac{\pi\lambda r}{4} = \frac{(2\pi - \varphi)\,\varphi\lambda r}{2\pi}\,.$$

We simplify it to quadratic equation

$$2\varphi^2 - 4\pi\varphi + \pi^2 = 0\,,$$

whose solution is

$$\varphi_{1,2} = \frac{4\pi \pm \sqrt{16\pi^2 - 8\pi^2}}{4} = \frac{2 \pm \sqrt{2}}{2}\pi\,.$$

Both solutions are valid because their sum is 2π. Nevertheless, in the problem statement, we ask on smaller of both angles; thus, the correct answer is $\frac{2-\sqrt{2}}{2}\pi = 0.92\,\mathrm{rad}$.

Jaroslav Herman
jardah@fykos.org

Problem DC ... current of the metro

Metro train of type M1 is $D = 96.66\,\mathrm{m}$ long, $w = 2.72\,\mathrm{m}$ wide, and its mass is $M = 133\,\mathrm{tons}$. Its capacity is $N = 1\,464$ travelers, the maximum speed is $v_{\mathrm{max}} = 90\,\mathrm{km{\cdot}h^{-1}}$, and the acceleration is $a = 1.4\,\mathrm{m{\cdot}s^{-2}}$. If we assume the metro accelerates with the constant acceleration (from zero to the maximum speed), what is the maximum current that the metro train would need to take from the railways?

Assume the metro train is fully loaded by people of average weight $m = 85\,\mathrm{kg}$, including clothes and luggage. The voltage in the railways is $U = 750\,\mathrm{V}$. Furthermore, neglect resistance forces, the engine efficiency is $80\,\%$. The metro moves prefectly horizontally.

Karel was thinking about power of the metro.

We approach the solution by the output power of the metro train. Since the output power of the engine is $\eta = 80\,\%$ of the input power, the following holds $P = \eta UI$. This is the output power that accelerates the train, it's instantaneous value is

$$P = Fv = M_{\mathrm{tot}}av = (M + mN)\,av\,,$$

where F is the instantaneous force (in the direction of travel) magnitude, v is the train speed, a is the instantaneous (yet constant) acceleration, and M_{tot} is the total mass of the loaded train. Because the mass and the acceleration remain constant, the greatest power output occurs just before reaching the maximum speed (at which the train stops accelerating). If we wanted to be more realistic, we would have to get the data about the particular acceleration of the train at a certain speed. From now on, we use v_{max} instead of v.

Now, we only need to put previous formulae together and compare them

$$P = \eta UI = (M + mN)\,av_{\mathrm{max}} \quad \Rightarrow \quad I = \frac{(M + mN)\,av_{\mathrm{max}}}{\eta U} \doteq 15\,000\,\mathrm{A} = 15.0\,\mathrm{kA}\,.$$

If the train were accelerating by the assumptions, the maximal electric current consumption would occur just before reaching the maximum speed. The electric current value would be $15.0\,\mathrm{kA}$.

Karel Kolář
`karel@fykos.org`

Problem DD ... fun at lecture

Jarda lost his attention at a lecture, so he started to play with his pen, which is of rest length l_0. He put the pen vertically and pressed it by $\Delta l \ll l_0$. Then he released it, the spring inside the pen stretched, the pen jumped vertically upwards, and its tip reached the height h. Jarda had fun for a while, but he got bored quickly.

Consequently, he dismantled the pen, removed the spring, and hung the rest of the pen on it. What was the frequency of the oscillations of a pen on the spring? The mass of the spring is much less than the mass of the rest of the pen.

Jarda is afraid that someone might have a wild imagination...

The elastic energy

$$\frac{1}{2}k\,(\Delta l)^2$$

is used to increase the potential energy by $mg\,(h-l_0)$, where k is the spring stiffness, m is the mass of the pen. The law of conservation of energy can be simplified and written as

$$\frac{k}{m}=\frac{2g\,(h-l_0)}{(\Delta l)^2}\,.$$

The frequency formula for a harmonic oscillator with parameters k and m is

$$f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}\,.$$

We substitute for the fraction in a square-root from the law of conservation of energy, and get

$$f=\frac{1}{2\pi}\frac{\sqrt{2g\,(h-l_0)}}{\Delta l}\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem DE ... hot plates

Danka was heating water on two hot plates simultaneously. The diameter of the large plate is $d_\mathrm{l}=19\,\mathrm{cm}$ and of the small one is $d_\mathrm{s}=15\,\mathrm{cm}$. On the large plate, $V_\mathrm{l}=2\,\ell$ of water is being heated in a pot which weights $m_\mathrm{l}=1\,\mathrm{kg}$. On the small plate, $V_\mathrm{s}=1\,\ell$ of water is being heated in a smaller pot weighting $m_\mathrm{s}=0.5\,\mathrm{kg}$. Both pots are made out of the same material with a specific heat capacity $c=450\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$. Let us assume that the power $P=8\,\mathrm{kW}$ supplied to the two-plate hob is distributed between the hot plates proportionally to their area. The efficiency of each plate is $\eta=80\,\%$. Furthermore, consider that all heat given off by the cooking plates is absorbed by the respective pot and the water inside and is not distributed any further. What is the time delay between reaching the boiling point in the smaller pot and in the large one? In the beginning, the water and the pots have room temperature $T_0=20\,\mathrm{^\circ C}$.

Danka experimentally verified the functioning of her dormitory hob.

For the amount of heat supplied by the plate i in the time t_i to the respective pot filled with water, the following applies

$$P_i t_i = Q_i = (m_i c + V_i \rho c_\mathrm{w})\,(T_1-T_0)\,,$$

where t_i is the necessary time for the water in the pot i to start to boil, $\rho=998\,\mathrm{kg\cdot m^{-3}}$ is the density of water, $c_\mathrm{w}=4\,184\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$ is water's specific heat capacity and $T_1=100\,\mathrm{^\circ C}$ is the boiling point of water. From the problem statement for the effective power of the plate i applies

$$P_i=P\eta\frac{S_i}{S_\mathrm{l}+S_\mathrm{s}}\,.$$

In the previous equation, $S_i=\pi\frac{d_i^2}{4}$ is the area of the hot plate i. From the fist equation, we can express the time it takes for the water to boil, and then we can calculate the difference $\Delta t=t_\mathrm{l}-t_\mathrm{s}$. Substituting all expressions and with appropriate adjustments we get the following equation

$$\Delta t=\frac{T_1-T_0}{P\eta}\left[(m_\mathrm{l}c+V_\mathrm{l}\rho c_\mathrm{w})\left(1+\frac{d_\mathrm{s}^2}{d_\mathrm{l}^2}\right)-(m_\mathrm{s}c+V_\mathrm{s}\rho c_\mathrm{w})\left(1+\frac{d_\mathrm{l}^2}{d_\mathrm{s}^2}\right)\right]\,.$$

After substituting the numerical values, we find that $\Delta t \doteq 35\,\mathrm{s}$. Therefore, the water in the large pot reaches boiling point 35 s later than the water in the smaller one.

Daniela Pittnerová
`daniela@fykos.org`

Problem DF ... fast particles

Very fast particles with mean lifetime $t_0 = 1.3 \cdot 10^{-6}\,\mathrm{s}$ arrive from space. In a balloon at height $h = 2\,\mathrm{km}$ above the ground, the detector measures $N_1 = 1\,100$ of them in some time. At the ground level, the same detector measures $N_2 = 170$ of them in the same time period. What is the speed of these particles? Express the result as a multiple of the speed of light c.

Danka recalled the course on special relativity.

If a particle is relativistic (moves at speed close to the speed of light), a time dilatation occurs, which means that observer at rest measures a longer lifetime of particle t than the proper lifetime of particle t_0. Thus

$$t = t_0\gamma\,,$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\,,$$

is the so-called gamma factor used in relativistic physics. Constant c denotes the speed of light, and v is the speed of a particle with respect to the observer (who is at rest).

Hence, particles decay at slower rate than if at rest. We express the decay law

$$N_2 = N_1 \mathrm{e}^{-\frac{\tau}{t}} = N_1 \mathrm{e}^{-\frac{h}{v t_0 \gamma}}\,,$$

where $\tau = \frac{h}{v}$ represents the time that particle needs to travel from the balloon altitude to reach the ground; thus, the half-life is greater due to multiplication by γ-factor.

By combining these equations, we get

$$-\ln\frac{N_2}{N_1} = \frac{h}{v t_0 \gamma}\,.$$

Now we simplify it to express $\frac{v}{c}$ ratio as

$$\frac{v}{c} = \frac{1}{\sqrt{1 + \left(\frac{c t_0}{h}\right)^2 \ln^2\frac{N_1}{N_2}}} \doteq 0.94\,.$$

Hence, the particles traveled at 0.94 times the speed of light.

Daniela Pittnerová
`daniela@fykos.org`

Problem DG ... marble is falling off the window

A marble (a small hard ball) with a mass 50 g *charged itself by a charge* 50 mC*, and jumped to a narrow vertical tunnel with a height* $h = 50\,\mathrm{dm}$*. At the bottom end of the tunnel, there is a horizontally attached circle (the tunnel and the circle are concentric) with the radius* 50 cm *and the linear charge density* λ*. What is the minimum value of* λ*, if the marble wishes not to get lower than to height* $\frac{h}{5}$ *above the circle?*

Jarda was in the cinema to watch a new Bond movie.

First of all, we compute the electric potential induced by the charged circle. Let us denote the distance between the marble and the center of the circle as x. Then, all points on the circle with the radius R are equidistant from the marble, at the distance $\sqrt{R^2 + x^2}$. Therefore, the electrostatic potential energy of the marble with the charge Q is

$$E_\mathrm{e} = \frac{2\pi\lambda RQ}{4\pi\varepsilon_0\sqrt{R^2 + x^2}}\,.$$

If charges Q and λ are of the same polarity, the force acts upwards and is increasing in x. The first derivative of the potential is negative, but a force is defined as a negative gradient of potential energy; thus, it is positive in this case. The potential energy of the marble due to gravity is simply

$$E_\mathrm{p} = mgx\,,$$

where m is the mass of the marble. At point $x = h$, it's kinetic energy is zero. According to the problem statement, we seek another such point, and we denote its distance from the center as $x_0 = h/5$. The total energy is conserved, so

$$E_\mathrm{e} + E_\mathrm{p} + E_\mathrm{k} = \frac{\lambda RQ}{2\varepsilon_0\sqrt{R^2 + h^2}} + mgh = \frac{\lambda RQ}{2\varepsilon_0\sqrt{R^2 + x_0^2}} + mgx_0\,.$$

We express λ as

$$\lambda = \frac{2\varepsilon_0 mg\,(h - x_0)}{RQ\left(\frac{1}{\sqrt{R^2+x_0^2}} - \frac{1}{\sqrt{R^2+h^2}}\right)} = \frac{8\varepsilon_0 mgh}{5RQ\left(\frac{1}{\sqrt{R^2+\left(\frac{h}{5}\right)^2}} - \frac{1}{\sqrt{R^2+h^2}}\right)} = 2.0\cdot 10^{-9}\,\mathrm{C{\cdot}m^{-1}}\,.$$

If Q and λ were of the opposite polarity, there would be an attractive force between the marble and the circumference. If the marble were above the circle, it would be attracted downwards by both gravity and the electric force. Therefore, it would be impossible for the marble to stop above the circumference (however, the marble could stop below it). Indeed, both the charge-related quantities must be of the same polarity. Consequently, we look for positive λ.

Jaroslav Herman
`jardah@fykos.org`

Problem DH ... attack on the train

One rascal decided to attack a moving train. He stepped in front of it on the rail and threw a bouncy ball against it with horizontal velocity $u = 13\,\mathrm{m{\cdot}s^{-1}}$*. At that time, the train was at a distance* $d = 17\,\mathrm{m}$ *away from him and was approaching at a steady speed. The bouncy ball bounced off the train elastically and reached the rascal after a time* $T = 1.5\,\mathrm{s}$ *from being* *thrown. The surprised rascal understood that the attack was unsuccessful, and it was time to disappear. What time does the rascal have to escape before a train passes through his current position?* *Jarda would like the trains to Prague to go faster.*

Let us denote t_1 the time from throwing the bouncy ball until its collision with the train and t_2 the time from the collision until the return of the bouncy ball to the rascal. Then

$$T = t_1 + t_2 \,.$$

In time t_1 the bouncy ball and the train travel together a distance d, i.e.

$$d = ut_1 + vt_1 \,,$$

where v is the speed of the train. In a system moving at the speed v together with the train, the train is at rest and during the collision, the bouncy ball collides with it at the speed $u + v$. Since the collision is perfectly elastic, the bouncy ball bounces off of it with the same speed. By switching back to the system connected to the ground, we get the speed of the bouncy ball after the bounce as the sum of the speed of the system connected to the train and the speed of the bouncy ball in this system, i.e.

$$u_1 = v + u + v = u + 2v \,.$$

With this speed the bouncy ball flies the time t_2 and between the collision and the rascal is a distance $d - vt_1$. So we have a third equation

$$d - vt_1 = (u + 2v)\, t_2 \,.$$

We modify this equation to the form

$$d = (u + v)\, t_2 + v\,(t_1 + t_2) \ ,$$

to which we substitute from the other two equations

$$d = d\frac{t_2}{t_1} + vT \,.$$

Substituting d at the third equation, we find the ratio of t_1 and t_2, which we insert into the previous expression. We get

$$d = d\frac{u}{u + 2v} + vT \quad \Rightarrow \quad v = \frac{d}{T} - \frac{u}{2} \,.$$

The time the problem statement asks for is

$$t = \frac{d - vT}{v} = \frac{T^2 u}{2d - Tu} = 2.0\,\mathrm{s} \,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem EA ... broken rheostat

Vojta had two light-emitting diodes (LEDs) – a red and a green one. He discovered that by sight, the red diode emits light when the current flowing through the diode is at least $I_1 = 10\,\mathrm{mA}$. Similarly, the green one needs current greater than $I_2 = 20\,\mathrm{mA}$. He was plugging these LEDs in series with a linear rheostat of resistance ranging from $R_1 = 100\,\Omega$ to $R_2 = 400\,\Omega$ into a circuit with source of voltage $U = 5\,\mathrm{V}$. However, Vojta accidentally broke the slider of the rheostat in a random position unknown to him. He thus tried to plug only the red LED and found it emitting light. What is the probability that the green LED will also light up when put instead of the red one? Both LEDs have the same operating voltage $U_\mathrm{p} = 1.7\,\mathrm{V}$.

Vojta played with Christmas lights.

Firstly, we need to determine the current flowing through the LED for a general value of rheostat's resistance R. Apparently

$$U = U_\mathrm{p} + IR \quad \Rightarrow \quad I = \frac{U - U_\mathrm{p}}{R}\,.$$

When calculating conditional probability, we use formula

$$P(I \geq I_2 | I \geq I_1) = \frac{P(I \geq I_2)}{P(I \geq I_1)}\,,$$

that expresses the sought probability in terms of probabilities that the current I is greater than the individual known currents. Let us now modify the expression of interest $P(I \geq I_i)$

$$P(I \geq I_i) = P\left(\frac{U - U_\mathrm{p}}{R} \geq I_i\right) = P\left(R \leq \frac{U - U_\mathrm{p}}{I_i}\right)\,.$$

The probability that the resistance R is greater than value $\frac{U - U_\mathrm{p}}{I_i}$ can be easily determined as

$$P\left(R \leq \frac{U - U_\mathrm{p}}{I_i}\right) = \frac{\frac{U - U_\mathrm{p}}{I_i} - R_1}{R_2 - R_1} = \frac{U - U_\mathrm{p} - R_1 I_i}{(R_2 - R_1)\, I_i}\,.$$

Putting everything together, we get

$$P(I \geq I_2 | I \geq I_1) = \frac{\frac{U - U_\mathrm{p} - R_1 I_2}{(R_2 - R_1) I_2}}{\frac{U - U_\mathrm{p} - R_1 I_1}{(R_2 - R_1) I_1}} = \frac{I_1}{I_2}\left(1 + R_1 \frac{I_1 - I_2}{U - U_\mathrm{p} - I_1 R_1}\right) \doteq 0.283\,.$$

We have also found that the general solution does not depend on R_2, which makes sense since the information about maximal resistance is already contained within the fact that we have managed to light up the red LED.

Vojtěch David
vojtech.david@fykos.org

Problem EB ... blured spectrum of rotations

We can never observe such sharp spectral lines in the star's spectrum as if we were to create the same light in the laboratory. One of the reasons is the rotational motion of the stars. To what width (expressed as the difference of the extreme wavelength $\Delta\lambda$) does this effect broaden the spectral line with the original frequency f_0? We assume that the other parameters are constant and that we could initially consider the line to be sharp. We also consider a star to be a sphere with radius R, which rotates at the equator with an angular velocity ω.

Karel likes astrophysics.

As we are only interested in the width of the spectral line, we don't care how fast a given star moves away from or approaches us. The fact that this value is not given or is to be considered constant is thus not essential.

One edge of the rotating star will move away from us at a speed of $R\omega$, while the other will approach us at the same speed. The wavelengths λ_1 and λ_2 of light incident from these ends can be determined using formula for Doppler effect

$$\lambda_1 = \lambda_0 \frac{c + R\omega}{c} \,,$$
$$\lambda_2 = \lambda_0 \frac{c - R\omega}{c} \,,$$

where λ_0 is the original wavelength of the spectral line and c is the speed of light in a vacuum. Thus, the spectral line is broadened to a width

$$\Delta\lambda = \lambda_1 - \lambda_2 = \frac{2R\omega\lambda_0}{c} \,.$$

We know the frequency f_0, for which $f_0 = c/\lambda_0$ applies. An equation for spectral line width can thus be recast as

$$\Delta\lambda = \frac{2R\omega\lambda_0}{c} = \frac{2R\omega}{f_0} \,.$$

This yields the desired result. In fact, the edges of the spectral line will be relatively dim compared to the line's center. At the same time, many other effects are acting both along the way and during the detection, which causes spectral lines to broaden. However, this formula can be used in practice to approximate the speed of rotation.

Karel Kolář
karel@fykos.org

Problem EC ... watch your head

Jarda hates taking out the garbage at the dormitory, so he usually throws it out of the window. One day, he threw out two balls, both with radius $R = 8\,\mathrm{cm}$. The first was of mass $m_1 = 85\,\mathrm{g}$ and the other was of mass $m_2 = 123\,\mathrm{g}$. How many times will the kinetic energy of the second ball be greater than that of the first one upon impact? Jarda lives on the 16th floor.

Jarda was wondering whether such a ball could be a murderous tool.

Let us denote the quantities from the task (in the same order) as R, m_1 a m_2. Because the balls are falling from an enormous height, we expect their speed to be stable, and the resistance force and gravity to be in equilibrium. Since the flow obviously will not be laminar, we can write this equality as

$$kv_i^2 = F_\mathrm{r} = F_g = m_i g$$

where $i = 1, 2$. We adjust the equation

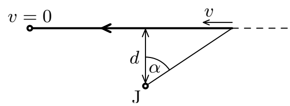
$$\frac{1}{2} m_i v_i^2 = \frac{1}{2} \frac{g}{k} m_i^2 \,.$$

Since g and k are the same for both of the balls, the ratio of balls' kinetic energies is equal to the second power of their masses' ratio, thus

$$\frac{E_{\mathrm{c}2}}{E_{\mathrm{c}1}} = \left(\frac{m_2}{m_1}\right)^2 \doteq 2.09 \,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem ED ... duck



Jarda sits on the shore of a pond with calm surface. Suddenly a duck begins to land on the surface of the pond in a specific way. It moves along a line, which is at distance $d = 2.8\,\mathrm{m}$ from Jarda. After touching the water, the duck slows down uniformly, and her motion creates waves on the surface of the pond. The waves reach Jarda at angle $\alpha = 7.0^\circ$ (see picture), and in time $t_1 = 10.3\,\mathrm{s}$ (since it touched the water). The duck stops moving in time $t_2 = 3.1\,\mathrm{s}$. What distance on a water surface did the duck travel? The speed of waves on the surface is $c = 0.30\,\mathrm{m{\cdot}s^{-1}}$. *Jarda feeds the birds in winter.*

Firstly, suppose that the duck landed (touched the water) on the aforementioned line at a distance $\frac{d}{\cos\alpha} = 2.82\,\mathrm{m}$ from Jarda and that waves propagate by speed c directly towards him. They reach him at time t_1 after landing, and thus they traveled distance $ct_1 = 3.09\,\mathrm{m}$. However, this is not equal to the value we determined earlier.

This means that the duck must have had greater speed than c during the slowing down for some time. Thus, the waves propagate from a different place than where the duck landed. We denote the speed of the duck as v. If $v > c$, then waves are triangle-shaped with vertex angle $\beta = \arcsin\frac{c}{v}$. These waves propagate perpendicular to their wavefront. From the figure, it is clear that $\beta = \alpha$. Therefore, we know the duck's speed at the moment when the waves (that later reached Jarda at angle α) started to propagate from the duck. This speed is

$$v = \frac{c}{\sin\alpha}$$

and waves were formed at

$$t_\mathrm{v} = \frac{d}{\cos\alpha c} \,,$$

prior to reaching Jarda. If we denote time when the duck landed on water as $t_0 = 0\,\mathrm{s}$, the waves were formed at time

$$t_1 - t_\mathrm{v} = t_1 - \frac{d}{\cos\alpha c}$$

when the duck had speed $v = \frac{c}{\sin\alpha}$. The duck has zero speed at time t_2, which means that it decelarated by v in time $t_2 - (t_1 - t_\mathrm{v})$. Hence, its acceleration is

$$a = \frac{v}{t_2 - (t_1 - t_\mathrm{v})} = \frac{\frac{c}{\sin\alpha}}{t_2 - t_1 + \frac{d}{\cos\alpha c}}\,.$$

Therefore, the distance that the duck traveled on the water is

$$s = \frac{1}{2}at_2^2 = \frac{ct_2^2}{2\left((t_2 - t_1)\sin\alpha + \tan\alpha\frac{d}{c}\right)} = 5.4\,\mathrm{m}\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem EE ... we're setting ants on fire

On a beautiful sunny day, at the noon of the summer solstice in Prague, when the air temperature is $35\,^\circ\mathrm{C}$ *we take a convex lens with a diameter* $D = 6\,\mathrm{cm}$, *and we place it perpendicularly to the sun rays. We put a little black ball with high thermal conductivity and radius* $r = 1\mathrm{mm}$ *to the focal point of the magnifying glass behind the lens.*

What will be the equilibrium temperature of the black ball? Heat transfer by convection can be neglected. Only $60\,\%$ *of the solar radiation passes through the atmosphere, and* $65\,\%$ *of the radiation energy passes through the glass lens.*

Jarda wanted to magnify insect, but it turned out badly.

The solar radiation enters the Earth's atmosphere with intensity corresponding to the solar constant $K = 1\,361\,\mathrm{W{\cdot}m^{-2}}$. Only $K_\mathrm{Z} = 0.6K \doteq 817\,\mathrm{W{\cdot}m^{-2}}$ passes through the atmosphere to the Earth's surface. This value is reduced to $K_\mathrm{C} = 0.65K_\mathrm{Z} \doteq 531\,\mathrm{W{\cdot}m^{-2}}$ after passing through the lens. Since the lens is oriented perpendicularly to the rays, the total power passing through the magnifying glass is

$$P = 0.6 \cdot 0.65KS = \frac{0.39\pi D^2 K}{4}\,.$$

The lens directs this power to the focal point, where the little black ball is placed.

In the equilibrium state, this little ball receives the same amount of energy that it radiates. It receives the energy from the Sun through the magnifying glass and also from its surroundings due to the radiation. The equation of equilibrium is

$$4\pi r^2\sigma T^4 = 4\pi r^2\sigma T_\mathrm{o}^4 + \frac{0.39\pi D^2 K}{4}\,,$$

where T is the temperature of the little ball, σ is the Stefan–Boltzmann constant, and $T_\mathrm{o} = 308.15\,\mathrm{K}$ is the temperature of the surroundings. We get

$$T = \sqrt[4]{T_o^4 + \frac{0.39D^2K}{16r^2\sigma}} \doteq 1\,206\,\mathrm{K} = 933\,^\circ\mathrm{C}\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem EF ... charged helicopter

FYKOS organizers decided to go on a helicopter trip, and now they are in the air, above the north magnetic pole, where they just measured the magnitude of a magnetic field $B = 65\,\mathrm{\mu T}$. The length of the rotor blade from the rotor to the end of the blade is $L = 6\,\mathrm{m}$. What voltage arises between the end of the rotor blade and the helicopter main rotor, which rotates with frequency $f = 3\,\mathrm{s^{-1}}$? *Jarda casually takes helicopter rides to the North Pole.*

Let us assume a small rotor blade element of a length $\mathrm{d}r$ and at a distance r from the center of rotation. A magnetic force in the direction along the blade and of magnitude $F_\mathrm{m} = Bvq$ acts on the electrons of this element. In equilibrium, it is compensated by an induced electric force of magnitude $F_\mathrm{e} = Eq$. Speed v depends on the distance from the axis of rotation as $v = \omega r$. From the equality of these forces, we get

$$E = B\omega r\,.$$

We obtain the voltage by integration of electric intensity along the blade

$$U = \int_0^L B\omega r\,\mathrm{d}r = \frac{1}{2}B\omega L^2\,.$$

Therefore, the voltage between the helicopter main rotor and the end of the blade is

$$U = \frac{1}{2}B\omega L^2 = \pi B f L^2 = 22\,\mathrm{mV}\,.$$

We can see that the voltage is negligible.

Jaroslav Herman
`jardah@fykos.org`

Problem EG ... long wait

Lego thinks he waits for the metro pretty often for too long (sometimes even 10 minutes!). So he calculated what percentage of the total waiting time is made up by waits longer than half of the interval (time between the departure of previous and arrival of the following metro, which Lego is waiting for). What result did Lego obtain if he calculated it correctly? By the way, Lego always comes to the metro station at a random moment in the interval. *Lego hates waiting.*

Let us denote the time period between departure of one and arrival of another metro as T. Then the Lego's waiting time is from interval $(0, T)$, and from the problem, we know, that all times from this interval are of equal probability. Precisely, the probability that Lego's waiting time is from interval $(t, t + \mathrm{d}t)$ is for all times $t < T$ equal to $\mathrm{d}t/T$. Average wait will, therefore, last

$$\bar{t} = \int_0^T t\frac{\mathrm{d}t}{T} = \frac{1}{T}\left[\frac{t^2}{2}\right]_0^T = \frac{T}{2}\,.$$

If we think about it, this results seems pretty intuitive. However, what portion of this time is made up by waits longer than $T/2$? We determine it by similar computation

$$\bar{t}_{t>T/2} = \int_{T/2}^T t\frac{\mathrm{d}t}{T} = \frac{1}{T}\left[\frac{t^2}{2}\right]_{T/2}^T = \frac{1}{T}\left(\frac{T^2}{2} - \frac{T^2}{8}\right) = \frac{3}{8}T\,.$$

Hence, despite making only half of the number of all waits, waits longer than one half of the time interval make most of the total waiting time

$$\frac{\frac{3}{8}T}{\frac{1}{2}T} = \frac{3}{4} = 75\,\%\,.$$

This is probably the reason why it seems to Lego that he is always waiting for too long.

Šimon Pajger
legolas@fykos.org

Problem EH ... terrain advantage

Imagine you are standing in the middle of a large inclined plane tilted at angle $\alpha = 25^\circ$. What is the ratio of maximal ranges of the projectile shot directly uphill to projectile shot the opposite way? Assume the maximal projectile velocity does not depend on the shooting angle.

Dodo was trying to shoot uphill.

Let us fix the maximum projectile velocity v. If we shot any slower, the range would be lower. This problem can be solved in different ways. The most straightforward is calculating the range for a given angle of inclination as a function of projectile angle and further maximizing this function by differentiating with respect to the projectile angle. However, this procedure is relatively time-consuming, so in our solution, we will use the fact that all places that can be hit lie below the safety parabola. We set the origin of the coordinate system at the point of firing and the x-axis points in the horizontal plane in uphill orientation. We can describe this in a two-dimensional section with the greatest inclination as

$$y = x\tan\alpha\,.$$

We obtain the equation of safety parabola $y = \alpha x^2 + \beta$ using the position of its vertex at a point at a height $h = \frac{v^2}{2g} = \beta$ above the point of firing (can be found from law of conservation of energy) and the maximum range in the horizontal plane $d = \frac{v^2}{g} = \sqrt{\frac{\beta}{-\alpha}}$ (obtained from vertical and horizontal trajectories). The resulting parabola's equation is

$$y = -\frac{g}{2v^2}x^2 + \frac{v^2}{2g}\,.$$

To find the position of the points of impact, let us set the equality for y-coordinates of both curves, which provides us with a quadratic equation

$$\frac{g}{2v^2}x^2 + x\tan\alpha - \frac{v^2}{2g} = 0\,,$$

whose two roots are the sought ranges measured horizontally. This is suffiecient as we seek only ratio of them so cosines cancel out as the slope is constant

$$x_{+,-} = \frac{v^2}{g}\left(-\tan\alpha \pm \sqrt{\tan^2\alpha + 1}\right) = \frac{v^2}{g}\frac{-\sin\alpha \pm 1}{\cos\alpha}\,.$$

Sign of the result only distinguishes the direction. Hence, we easily obtain the ratio as

$$w = \left|\frac{x_+}{x_-}\right| = \frac{1-\sin\alpha}{1+\sin\alpha} \doteq 0.406\,.$$

Jozef Lipták
`liptak.j@fykos.org`

Problem FA ... cube in slip

An ice cube of mass $m_0 = 7.34\,\mathrm{g}$, and temperature $T = 0\,^\circ\mathrm{C}$ is placed on a long, hot inclined plane tilted at angle $\alpha = 15^\circ$. Since that moment, the cube absorbs heat at rate $P = 1\,\mathrm{kW}$. Determine the work done by the gravitational force on ice acceleration. Assume that the ice cube melts uniformly and only from the bottom base. Neglect all resistance forces.

Vojta was watching old commercials.

While moving, cube is subject to a force

$$F(t) = m(t)\,g\sin\alpha\,.$$

We determine work done by gravitational force as

$$W = \int_0^{s_1} m(t)\,g\sin\alpha\,\mathrm{d}s = \int_0^{t_1} m(t)\,v(t)\,g\sin\alpha\,\mathrm{d}t\,.$$

At first, we need to know the total time the ice moves. Let l_t be the specific latent heat of fusion of ice. Let us find the time in which the entire ice cube melts

$$t_1 = \frac{Q}{P} = \frac{m_0 l_\mathrm{t}}{P}\,.$$

Furthermore, we need to express the speed of the ice cube as a function of time. Note that the acceleration remains constant, with magnitude

$$a = g\sin\alpha\,.$$

Thus, it satisfies

$$v = tg\sin\alpha\,.$$

Lastly, we need to express the mass of the ice as a function of time. Note that in time t, the ice cube receives heat Pt and consequently, ice of mass Pt/l_t melts. Now, we have everything prepared and can write integral as

$$W = \int_0^{t_1}\left(m_0 - \frac{Pt}{l_\mathrm{t}}\right) tg^2\sin^2\alpha\,\mathrm{d}t = g^2\sin^2\alpha\int_0^{t_1} m_0 t - \frac{Pt^2}{l_\mathrm{t}}\,\mathrm{d}t = g^2\sin^2\alpha t_1^2\left(\frac{m_0}{2} - \frac{Pt_1}{3l_\mathrm{t}}\right)\,,$$

where after substitution for t_1 we get

$$W = \frac{g^2\sin^2\alpha m_0^3 l_\mathrm{t}^2}{6P^2} \doteq 0.047\,\mathrm{J}\,.$$

Vojtěch David
`vojtech.david@fykos.org`

Problem FB ... double infinity of lenses

Jarda got a very large number of really thin convex lenses for Christmas. He placed them very closely one behind the other so that the first one had a focal length f, the second one $2f$, a third $4f$ and so on. He also found a concave mirror with radius of curvature f, which he placed right behind this row of lenses. He inserted a luminous object into the focus of the first lens. What distance in front of the mirror did its image appear? If it appeared behind the mirror, give a negative answer. *Jarda is bored with problems about infinite circuits, this is a novelty!*

First, let's look at how the beam passes through two lenses placed one behind the other. Let a be the distance of the object in front of the first lens with focal length f_1. Just behind it is a second lens with focal length f_2. Using the Gaussian lens equation, we find the position of the image a' of the object as

$$a' = \frac{af_1}{a - f_1}\,.$$

This distance is positive if the object is imaged behind the first lens. The image will be viewed through the second lens, with the pattern distance now equal to $-a'$. The second lens displays the object as

$$a'' = \frac{-a' f_2}{-a' - f_2} = \frac{-\left(\frac{af_1}{a-f_1}\right) f_2}{-\frac{af_1}{a-f_1} - f_2} = \frac{af_1 f_2}{af_1 + af_2 - f_1 f_2} = \frac{aF}{a - F}\,,$$

where $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$. So we derived an interesting relation that for two lenses in close vicinity to each other, their total optical power is the sum of the optical powers of the two lenses.

The relationship can be iterated and it is evident that the total optical power of all Jarda's lenses in a row is

$$\frac{1}{F} = \frac{1}{f}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\right) = \frac{2}{f}\,,$$

From here $F = \frac{f}{2}$. The focal length of this lens series is thus half that of the first lens. We can calculate the infinite sum above for example from the relation for an infinite geometric series. The set of these lenses will therefore image an object located at a distance f in front of it to a distance

$$x = \frac{f\frac{f}{2}}{f - \frac{f}{2}} = f\,.$$

Now let's move on to the image on the concave mirror. It has a focal length equal to one half of the radius of curvature. For a point with distance $-f$ (the negative sign is due to the fact that the lens system imaged the object behind the mirror) the image will be in distance

$$x' = \frac{-x\frac{f}{2}}{-x - \frac{f}{2}} = \frac{f}{3}\,.$$

Now the object is shown at a distance $x' = \frac{f}{3}$ in front of the mirror.

Further imaging is again performed using a system of lenses. The object is now located $x' = -\frac{f}{3}$ behind the lens system. For the last time we use the imaging equation

$$x'' = \frac{x'\frac{f}{2}}{x' - \frac{f}{2}} = \frac{\frac{-f}{6}}{-\frac{1}{3} - \frac{1}{2}} = \frac{f}{5}\,.$$

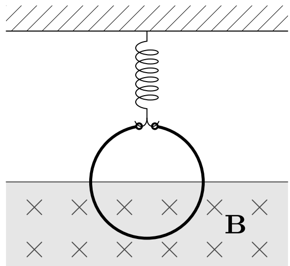
The object was imaged through the entire system to a distance $\frac{f}{5}$ in front of both the mirror and the lenses.

Jaroslav Herman
jardah@fykos.org

Problem FC ... oscillatory voltage

Metal hoop with radius r and mass m is suspended on a spring with stiffness k in such a way, that its lower half is placed in a magnetic field with induction B, which is perpendicular to the hoop, as shown in the figure. What will be the peak voltage on the hoop, if we let it oscillate with the displacement amplitude equal to r? Assume that the voltage is measured at the hinge point in a place where the hoop is disconnected.

Vojta was inventing innovative voltage sources.



Notice that no current will be passing through the hoop, since it is not connected. That means we don't have to deal with damping. From Faraday's law of induction, we can write

$$U = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -B\frac{\mathrm{d}S}{\mathrm{d}t} = -Bl\frac{\mathrm{d}y}{\mathrm{d}t} = -Blv$$

where y is the displacement of the spring, l is the distance between two points on a hoop lying on the border of the magnetic field, and v is the instantaneous velocity of the hoop. Let us now express the displacement as a function of time

$$y = r\cos\omega t\,,$$

from where we can directly determine the instantaneous velocity as

$$v = \frac{\mathrm{d}y}{\mathrm{d}t} = -\omega r\sin\omega t\,.$$

Using the Pythagorean theorem, we can also notice

$$l = 2\sqrt{r^2 - y^2} = 2r\sqrt{1-\cos^2\omega t} = 2r\left|\sin\omega t\right|\,.$$

Substituting these expressions into expression for voltage we get

$$U = 2Br^2\omega\left(\sin\omega t\left|\sin\omega t\right|\right)\,,$$

where the expression in parentheses ranges from -1 to 1. Expressing the spring angular frequency in terms of given quantities, we can express the amplitude of voltage as

$$U_{\mathrm{A}} = 2Br^2\sqrt{\frac{k}{m}}\,.$$

We can also solve the problem by directly expressing the area of a circular segment as a function of time, which yields the same answer. To complement the solution, the graph depicting the voltage as a function of time is included.

Vojtěch David
vojtech.david@fykos.org

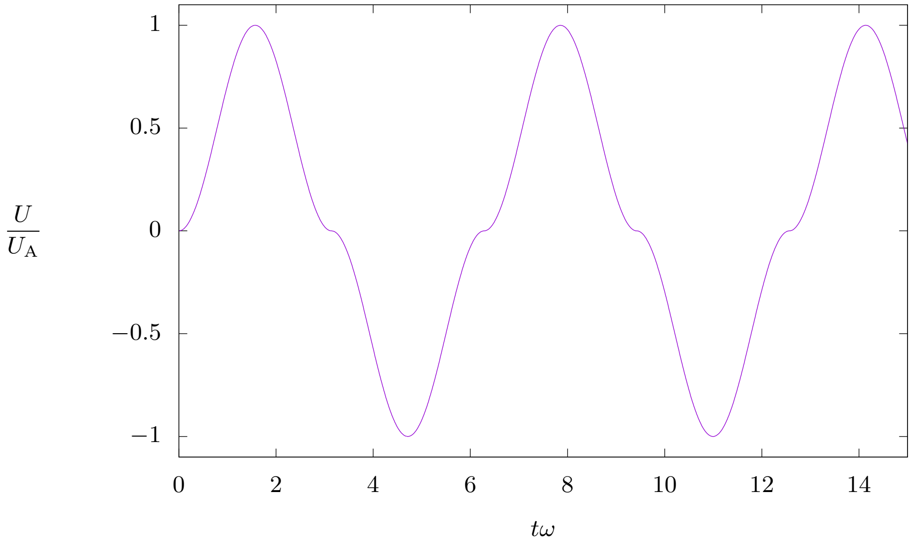


Fig. 1: Voltage as a function of time.

Problem FD ... exhausted Sun

Due to thermonuclear fusion, the Sun loses its mass, which transforms into radiation. Determine the time in which the distance between the Sun and the Earth changes by 1 m *because of this phenomenon. Assume the Earth moves in a circular orbit. Provide answer in years.*

Jarda noticed that the world was not the same anymore.

The Earth moves on a circular orbit thanks to the centripetal force induced by Sun's gravity. We can express this as

$$m\frac{v^2}{r} = \frac{MmG}{r^2}\,,$$

where m is the Earth's mass, M is the mass of the Sun, v is the velocity of Earth's movement around the Sun, and r is the distance between the Earth and the Sun. Now we multiply this equation by r^2 and differentiate it with respect to time

$$2vr\frac{\mathrm{d}v}{\mathrm{d}t} + v^2\frac{\mathrm{d}r}{\mathrm{d}t} = G\frac{\mathrm{d}M}{\mathrm{d}t}\,.$$

In the system, the law of conservation of angular momentum per (Earth's) unit mass holds at each time; we can write it as $l = rv = \text{const}$. Furthermore, we differentiate this equality with respect to time, which provides us with the following formula

$$v\frac{\mathrm{d}r}{\mathrm{d}t} + r\frac{\mathrm{d}v}{\mathrm{d}t} = 0\,.$$

We substitute these expressions into the differentiated equation above and get

$$2vr\left(-\frac{v}{r}\frac{\mathrm{d}r}{\mathrm{d}t}\right)+v^2\frac{\mathrm{d}r}{\mathrm{d}t}=-v^2\frac{\mathrm{d}r}{\mathrm{d}t}=G\frac{\mathrm{d}M}{\mathrm{d}t}\,.$$

The change of the Sun's mass in time is determined by the transformation of its rest mass into radiation. We can write this as

$$\frac{\mathrm{d}M}{\mathrm{d}t}=-\frac{L_\odot}{c^2}\,,$$

where $L_\odot$ is nominal solar luminosity. If we substitute and realize that changes are of small order, we can proceed from differentials to finite changes, obtaining

$$\Delta t=v^2c^2\frac{\Delta r}{GL_\odot}\,.$$

By expressing the square of velocity from the first equation, and number substitution, we get

$$\Delta t=\frac{M_\odot c^2}{L_\odot}\frac{\Delta r}{r}\doteq 99\,\mathrm{years}\,.$$

Due to the change in Sun's mass via radiation, the Earth moves away from it by one meter in almost a hundred years. This effect is truly negligible yet measurable by radar.

Jaroslav Herman
`jardah@fykos.org`

Problem FE ... marble didn't guess

In the previous two problems, the marble (a small hard ball) always stopped somewhere during its journey. Now, the marble wants to float in the air. The marble turned on a monochromatic light source at the bottom, perpendicular to the ground, pointing upwards with intensity P. Then the marble put on a perfectly reflective coat. Afterward, the marble jumps off the window at a height H, directly into the light beam. The marble has a radius R and a mass m. Unfortunately, the marble did not compute the needed intensity correctly, and the light beam cannot keep it floating in the air. At what speed does the marble hit the ground?

Jarda will not try this at the dormitory either.

Firstly, we compute the force by which the light beam acts on the marble. Consider a photon with momentum $p_1=\frac{hf}{c}$ moving from the source directly upwards, which hits the marble. We parametrize the point of interaction by an angle φ formed by the line connecting the marble's center and the point of impact on the cylindrical surface, with the vertical line passing through the marble's center. Satisfying Snell's law, the photon bounces such that its momentum points $\pi-2\varphi$ away from the vertical axis.

Thus, the vertical component of the photon's momentum is $p_2=p_1\cos\left(\pi-2\varphi\right)$. The change in the photon's vertical momentum is therefore,

$$\Delta p=p_1-p_2=p_1\left(1-\cos\left(\pi-2\varphi\right)\right)=p_1\left(1+\cos 2\varphi\right)\,.$$

The total change of marble's horizontal momentum is zero due to the symmetry, so we will not consider it further.

We know the energy of photon satisfies $E = hf$. Therefore, the number of photons emitted per area unit per time unit is

$$n = \frac{P}{hf}\,.$$

Hence, the change in momentum per area unit (actually pressure) is

$$n\Delta p = \frac{P}{c}\left(1 + \cos\left(2\varphi\right)\right)\,.$$

We determine the total force caused by the pressure of the radiation by integrating this change in the momentum per area unit with respect to the area. Regarding the φ, we integrate over the circle element. Note that using simple geometry, we find that it satisfies

$$\mathrm{d}S = 2\pi R\sin\varphi R\,\mathrm{d}\varphi\cos\varphi = \pi R^2\sin 2\varphi\,\mathrm{d}\varphi\,.$$

The magnitude of the force is

$$F_z = \int_0^{\frac{\pi}{2}}\frac{P}{c}\left(1+\cos 2\varphi\right)\pi R^2\sin 2\varphi\,\mathrm{d}\varphi \;= \frac{\pi R^2 P}{c}\int_0^{\frac{\pi}{2}}\left(1+\cos 2\varphi\right)\sin 2\varphi\,\mathrm{d}\varphi\,.$$

We simplify the integrand to

$$\sin 2\varphi + \frac{\sin 4\varphi}{2}\,,$$

which provides us

$$F_z = -\frac{\pi R^2 P}{c}\left[\frac{\cos 2\varphi}{2} + \frac{\cos 4\varphi}{8}\right]_0^{\frac{\pi}{2}} = \frac{\pi R^2 P}{c}\,.$$

Thus, the total force acting on the marble is

$$F = mg - F_z = mg - \frac{\pi R^2 P}{c}$$

and from the law of conservation of energy, we can determine the sought velocity as

$$v = \sqrt{\frac{2FH}{m}} = \sqrt{2H\left(g - \frac{\pi R^2 P}{mc}\right)}\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem FF ... zigzag

Rickon Stark is moving away from Ramsay Bolton at constant speed $v_r = 3.6\,\mathrm{m\cdot s^{-1}}$. Let us imagine that he chooses a better strategy this time than in the source material. Instead of running in a straight line, he will now run zigzag. On top of running forward, he will run at speed $v_\varphi = 2.0\,\mathrm{m\cdot s^{-1}}$ in direction perpendicular to the direction of v_r. Ramsay is standing in the center of a circle sector of angle $\alpha = 20^\circ$. Rickon is running from one edge of the sector to the other (and back) all the time. In the beginning, he is located $r_0 = 8\,\mathrm{m}$ from Ramsay at the edge of the sector. Ramsay knows that after he shoots, the arrow will land at a distance

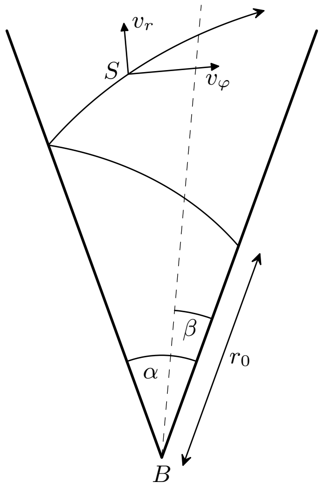


Fig. 2: The circular section in which Rickon moves.

$r_{\mathrm{R}} = 65\,\mathrm{m}$ *from him. At what angle in the segment does he have to aim in order to hit this time as well?* *Jáchym was a Ramsay fan but wanted the story to make sense.*

Let r denote the distance of Rickon from Ramsay. From the problem statement it is apparent, that the distance r as a function of time is given by $r = r_0 + v_r t$. In any given moment Rickon has angular velocity ω, for which $\omega r = v_\varphi$ holds. This means, that angular velocity is not constant (it is actually decreasing). If we want to find out the total angle traveled by Rickon at given time, we need to use integration

$$\varphi = \int_0^t \omega \,\mathrm{d}\tau = \int_0^t \frac{v_\varphi}{r_0 + v_r\tau}\,\mathrm{d}\tau = \left[\frac{v_\varphi}{v_r}\ln\left(r_0 + v_r\tau\right)\right]_0^t = \frac{v_\varphi}{v_r}\ln\left(1 + \frac{v_r t}{r_0}\right) = \frac{v_\varphi}{v_r}\ln\left(\frac{r}{r_0}\right)\,.$$

If we look at φ as a function of distance r, the total angle traveled by Rickon, before the arrow hits him, is given by

$$\varphi = \frac{v_\varphi}{v_r}\ln\frac{r_{\mathrm{R}}}{r_0} = 66.7^\circ\,.$$

Let's denote the angle under which Ramsay has to aim as β. We measure this angle from the edge the Rickon started at. To find out β we simply need to subtract the angle 2α from φ as many times as needed to get the angle $\beta' \in \langle 0, 2\alpha)$. If $\beta' \in \langle 0, \alpha)$, then $\beta = \beta'$, and we have our solution. If $\beta' \in \langle \alpha, 2\alpha)$ then, since we are measuring β from the edge Rickon started at, our solution is $\beta = 2\alpha - \beta'$. With our initial values we get $\varphi = 66.7^\circ$, $\beta' = 26.7^\circ$ and the solution is $\beta = 40.0^\circ - 26.6^\circ = 13.3^\circ$.

Jáchym Bártík
`tuaki@fykos.org`

Problem FG ... magnetic (almost)levitation

On a non-conductive horizontal plane, there lies a point of mass $m = 0.8\,\mathrm{kg}$ and charge $q = 3\,\mathrm{C}$. The coefficient of friction between the point and the plane is $f = 0.4$. At the same time, there is a constant homogeneous horizontal magnetic field $B = 0.5\,\mathrm{T}$. What is the maximum distance this point can travel if its initial horizontal velocity is $v_0 = 4\,\mathrm{m{\cdot}s^{-1}}$?

On his way to Prague, Jarda had a dream about fast trains.

In addition to the constant gravitational force, the magnetic force will also act on the point. This force is perpendicular to the magnetic field and the instantaneous velocity, both of which lie in the horizontal plane. Therefore, the magnetic force will act solely in the vertical direction – just like gravity.

In order to travel the greatest distance possible, the friction force between the point and the plane must be the smallest, i.e., the normal force must be minimal. This occurs when the magnetic field is perpendicular to the velocity vector and the magnetic force points upward. Then it satisfies equation

$$F_\mathrm{n} = mg - Bvq\,,$$

where v is the magnitude of the instantaneous velocity. We can see that point does not detach from the plane by substituting the numbers. Since the friction decelerates the point, we can write Newton's second law as

$$F = ma = -m\frac{\mathrm{d}v}{\mathrm{d}t} = f\left(mg - Bvq\right)\,.$$

This differential equation can be solved by separation of variables (the Fourier method)

$$m\frac{\mathrm{d}v}{Bvq - mg} = f\,\mathrm{d}t \quad \Rightarrow \quad \ln(Bvq - mg) = \frac{Bqf}{m}t + C_0\,,$$

which yields

$$v = C\mathrm{e}^{\frac{Bqf}{m}t} + \frac{mg}{Bq}\,.$$

The integration constant C can be determined using initial condition $v = v_0$ at $t = 0$. Thus,

$$v = \left(v_0 - \frac{mg}{Bq}\right)\mathrm{e}^{\frac{Bqf}{m}t} + \frac{mg}{Bq}\,.$$

The point stops when its velocity is zero, i.e., at a time

$$t_1 = \frac{m}{Bqf}\ln\frac{mg}{(mg - Bqv_0)}\,.$$

By integrating over time we get a covered distance as

$$s = \frac{m}{Bqf}\left(v_0 - \frac{mg}{Bq}\right)\mathrm{e}^{\frac{Bqf}{m}t} + \frac{mg}{Bq}t + C_2\,.$$

A constant C_2 can be also determined using intitial condition $s = s_0 = 0$ at $t = 0$, therefore

$$s = \frac{m}{Bqf}\left(v_0 - \frac{mg}{Bq}\right)\left(\mathrm{e}^{\frac{Bqf}{m}t} - 1\right) + \frac{mg}{Bq}t\,.$$

By substituing $t = t_1$, we obtain the maximum distance this point can travel

$$s = \frac{m^2 g}{B^2 q^2 f} \ln \frac{mg}{(mg - Bqv_0)} - \frac{mv_0}{Bqf} = 4.76\,\mathrm{m}\,.$$

To compare, without a magnetic field, such a point would travel just a little over $2\,\mathrm{m}$.

Jaroslav Herman
`jardah@fykos.org`

Problem FH ... thirteen shots

Consider three cubic water containers, the first of which has a base with an area of $S_a = 72.9\,\mathrm{cm}^2$, the second $S_b = 94.5\,\mathrm{cm}^2$ and the third $S = 3.1\,\mathrm{cm}^2$, while the symbols S_a, S_b and S are also used to label the containers themselves. Initially, the containers S and S_b are connected and the water in them reaches a height of $h_b = 8.1\,\mathrm{cm}$, while the level in the first container is $h_a = 19.3\,\mathrm{cm}$. Then we disconnect the containers S and S_b, join S and S_a, and wait until the levels equalize. Then, we disconnect them, reconnect S and S_b, and wait until the levels equalize again. We will perform the above-described process a total of thirteen times. What will be the final water level in the container S after the last iteration?

Jáchym had to continue the tradition of the famous "thirteen" tasks from previous years.

An important first step in solving this problem is to establish a suitable and clear labeling of all necessary variables. The level in the containers S_a and S_b after the i-th step will be denoted x_i and y_i respectively. The initial values are $x_0 = h_a$ and $y_0 = h_b$.

After we connect the containers S and S_a for the the first time, the levels will equalize to x_1, while initially it was y_0 and x_0 respectively. The conservation of volume implies

$$(S + S_a)\, x_1 = Sy_0 + S_a x_0\,.$$

Then we connect the container S and S_b, in which the levels are x_1 and y_0. Once equalized, the level will be y_1, for which it holds

$$(S + S_b)\, y_1 = Sx_1 + S_b y_0\,.$$

If we define $a = S/S_a$ and $b = S/S_b$, we can generalize these results as

$$x_{i+1} = \frac{x_i + ay_i}{1 + a}\,,$$
$$y_{i+1} = \frac{y_i + bx_{i+1}}{1 + b}\,,$$

where we substitute into the second equation from the first and obtain

$$y_{i+1} = \frac{bx_i + (1 + a + ab)\, y_i}{(1 + a)\,(1 + b)}\,.$$

For the sake of clarity, we redefine the constants as

$$x_{i+1} = \alpha x_i + \beta y_i\,,$$
$$y_{i+1} = \gamma x_i + \delta y_i\,.$$

The transformation relations thus read as follows

$$\alpha = \frac{1}{1+a}\,, \qquad \beta = \frac{a}{1+a}\,,$$
$$\gamma = \frac{b}{(1+a)(1+b)}\,, \quad \delta = \frac{1+a+ab}{(1+a)(1+b)}\,.$$

This system of equations can be solved in several ways. The fastest of them would be a transcription into the language of linear algebra as it holds

$$\mathbf{x}_{i+1} = A\mathbf{x}_i \quad \Rightarrow \quad \mathbf{x}_{13} = A^{13}\mathbf{x}_0\,,$$

where $\mathbf{x}_i = (x_i, y_i)$ and A is a matrix of coefficients α, β, γ, δ. Then we could find a diagonal form of a matrix A, in which it is straightforward to calculate its power and convert it back. Unfortunately, this procedure does not fall under high school mathematics, so we will present a more intuitive solution.

The goal is to separate the equations so that each contains only one variable. We achieve this by a suitable transformation. For the constant k let's define $z_i = x_i + ky_i$, then

$$z_{i+1} = x_{i+1} + ky_{i+1} = (\alpha + k\gamma)\, x_i + (\beta + k\delta)\, y_i = (\alpha + k\gamma)\left(x_i + \frac{\beta + k\delta}{\alpha + k\gamma} y_i\right)\,.$$

Since k is arbitrary, let's choose it such

$$k = \frac{\beta + k\delta}{\alpha + k\gamma} \quad \Rightarrow \quad \gamma k^2 + (\alpha - \delta)\, k - \beta = 0 \quad \Rightarrow \quad k_{1,2} = \frac{\delta - \alpha \pm \sqrt{(\delta - \alpha)^2 + 4\gamma\beta}}{2\gamma}\,.$$

For $K = \alpha + k\gamma$ it holds

$$z_{i+1} = (\alpha + k\gamma)\,(x_i + ky_i) = Kz_i\,,$$

which applies for both possible definitions of z (according to the possible values of the constant k). For clarity, we divide them into separate variables μ_i and ν_i, we get

$$\mu_i = x_i + k_1 y_i \quad \Rightarrow \quad \mu_{i+1} = K_1 \mu_i\,,$$
$$\nu_i = x_i + k_2 y_i \quad \Rightarrow \quad \nu_{i+1} = K_2 \nu_i\,.$$

Note that these new variables are not coupled – μ_{i+1} depends only on μ_i, not on ν_i. This makes it easy to calculate their value for any i as $\mu_i = K_1^i \mu_0$.

Now we need to transform back to the x and y variables. To do this, we simply express x_i and y_i from the definitions of μ_i and ν_i. Substituting $i = 13$ into the result gives us the water levels after thirteen repetitions

$$x_i = \frac{k_2\mu_i - k_1\nu_i}{k_2 - k_1} \quad \Rightarrow \quad x_{13} = \frac{k_2\mu_{13} - k_1\nu_{13}}{k_2 - k_1} = \frac{k_2 K_1^{13}\mu_0 - k_1 K_2^{13}\nu_0}{k_2 - k_1}\,,$$
$$y_i = \frac{\mu_i - \nu_i}{k_1 - k_2} \quad \Rightarrow \quad y_{13} = \frac{\mu_{13} - \nu_{13}}{k_1 - k_2} = \frac{K_1^{13}\mu_0 - K_2^{13}\nu_0}{k_1 - k_2}\,.$$

However, we are only interested in the water level in the container S, for which it holds $h = y_{13}$. All that remains is to use the values given in the problem statement, calculate all constants used and finally plug them into the relation

$$h = \frac{K_1^{13}\,(x_0 + k_1 y_0) - K_2^{13}\,(x_0 + k_2 y_0)}{k_1 - k_2} \doteq 11.1\,\mathrm{cm}\,.$$

It might seem that a general solution has to be lenghty. Surprisingly, this is not the case as

$$k_1 = \frac{a\,(1+b)}{b}\,, \qquad K_1 = 1\,,$$
$$k_2 = -1\,, \qquad K_2 = \frac{1}{(1+a)\,(1+b)}\,.$$

This leads to a relatively simple result

$$h = \frac{bh_a + a\,(1+b)\,h_b - b\,[(1+a)\,(1+b)]^{-13}\,(h_a - h_b)}{b + a\,(1+b)}\,.$$

Jáchym Bártík
`tuaki@fykos.org`

Problem GA ... center of gravity of better snail

How far from the origin of coordinate system lies the center of gravity of a spiral given in polar coordinates as $r = a\mathrm{e}^{b\varphi}$, where $a = 0.1\,\mathrm{m}$, $b = 1/\pi$ and $\varphi \in (-\infty, 4\pi)$ is the polar angle in radians? Assume the spiral has constant linear density λ. *Lego prefers a logarithmic spiral.*

Setting up the Cartesian coordinates in the plane of the spiral, each of its points can be determined as follows

$$x = a\mathrm{e}^{b\varphi}\cos\varphi\,,$$
$$y = a\mathrm{e}^{b\varphi}\sin\varphi\,.$$

From the Pythagorean theorem, we can express the length of the element of the spiral arc $\mathrm{d}l$ corresponding to angle element $\mathrm{d}\varphi$ as

$$\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = a\mathrm{e}^{b\varphi}\sqrt{\left(b\cos\varphi - \sin\varphi\right)^2 + \left(b\sin\varphi + \cos\varphi\right)^2}\,\mathrm{d}\varphi = a\mathrm{e}^{b\varphi}\sqrt{1+b^2}\,\mathrm{d}\varphi\,.$$

The mass of the spiral is thus

$$M = \int_{-\infty}^{4\pi} \lambda a\mathrm{e}^{b\varphi}\sqrt{1+b^2}\,\mathrm{d}\varphi = \lambda\frac{a}{b}\mathrm{e}^4\sqrt{1+b^2}\,.$$

Now, we obtain the coordiantes of the center of the gravity straight from the definiton as

$$x_T = \frac{1}{M}\int_0^L x\lambda\,\mathrm{d}l = \frac{1}{M}a^2\lambda\sqrt{1+b^2}\int_{-\infty}^{4\pi}\mathrm{e}^{2b\varphi}\cos\varphi\,\mathrm{d}\varphi = \mathrm{e}^{-4}ba\int_{-\infty}^{4\pi}\mathrm{e}^{2b\varphi}\cos\varphi\,\mathrm{d}\varphi\,,$$
$$y_T = \mathrm{e}^{-4}ba\int_{-\infty}^{4\pi}\mathrm{e}^{2b\varphi}\sin\varphi\,\mathrm{d}\varphi\,.$$

The integrals can be solved numerically, by twice applied per partes, or by using complex exponential function. Either way we get

$$x_T = \mathrm{e}^{-4}ba\left[\mathrm{e}^{2b\varphi}\frac{2b\cos\varphi + \sin\varphi}{4b^2+1}\right]_{-\infty}^{4\pi} = \frac{2\mathrm{e}^4 b^2 a}{4b^2+1} \doteq 0.79\,\mathrm{m}\,,$$
$$y_T = \mathrm{e}^{-4}ba\left[\mathrm{e}^{2b\varphi}\frac{2b\sin\varphi - \cos\varphi}{4b^2+1}\right]_{-\infty}^{4\pi} = \frac{-\mathrm{e}^4 ba}{4b^2+1} \doteq -1.24\,\mathrm{m}\,.$$

The last thing we need is to find the distance from the origin of coordinates. Clearly from the Pythagorean theorem, it is $\sqrt{x_T^2 + y_T^2} \doteq 1.5\,\mathrm{m}$.

Šimon Pajger
`legolas@fykos.org`

Problem GB ... falling tree

Jáchym cut down a tree, which we can approximate with a thin homogeneous cylinder of height $11\,\mathrm{m}$. *The tree is still partially connected to the stump, so its top describes the arc of the circle. At what time does the top of the tree hit the ground if we start the stopwatch when the tree passes deviation* 1°*?* *Jáchym gives solvers a hard time.*

We will proceed from the law of conservation of energy. Initially, the tree with the length l has the center of gravity at height $\frac{l}{2}$. Therefore, its potential energy is

$$E_{\mathrm{p}0} = mg\frac{l}{2}\,.$$

We denote polar angle (i.e., the angle of tree deviation from vertical axis) during the fall by φ. We can express potential energy of tree as a function of angle φ as

$$E_{\mathrm{p}} = mg\frac{l}{2}\cos\varphi\,.$$

Since the energy is conserved, we can determine the corresponding angular speed. We will approximate the tree by a thin homogeneous rod with the moment of inertia relative to its end being equal to $J = \frac{1}{3}ml^2$. Hence, we can compute the kinetic energy as

$$E_{\mathrm{k}} = \frac{1}{2}J\omega^2 = \frac{1}{2}\frac{1}{3}ml^2\omega^2 = E_{\mathrm{p}0} - E_{\mathrm{p}} = mg\frac{l}{2}\left(1-\cos\varphi\right)\,,$$

where $\omega = \dot{\varphi}$ is the angular speed of the tree.

The usage of the angular speed is convenient as it satisfies expression

$$\omega\,\mathrm{d}t = \mathrm{d}\varphi \quad \Rightarrow \quad \mathrm{d}t = \frac{\mathrm{d}\varphi}{\omega}\,.$$

If we express the angular speed ω as a function of angle φ and integrate it, we obtain the total fall time as a function of φ, i.e.,

$$t = \int_{\varphi_0}^{\varphi_1} \frac{\mathrm{d}\varphi}{\omega}\,,$$

where φ_0 and φ_1 are the initial and final angles, between which we measure the time. After substitution for ω from energy conservation (above), we obtain

$$t = \sqrt{\frac{l}{3g}}\int_{\varphi_0}^{\varphi_1} \frac{\mathrm{d}\varphi}{\sqrt{1-\cos\varphi}}\,.$$

Firstly, we will try to compute the antiderivative when we do not have to consider boundaries.

To get rid of the square root in the denominator, we try substitution $\frac{\varphi}{2} = \psi$, providing

$$2\int\frac{\mathrm{d}\psi}{\sqrt{1-\cos 2\psi}} = 2\int\frac{\mathrm{d}\psi}{\sqrt{1-\cos^2\psi+\sin^2\psi}} = \sqrt{2}\int\frac{\mathrm{d}\psi}{\sin\psi}\,.$$

We will try to halve the angle once again, i.e., $\theta = \frac{\psi}{2}$. Then

$$\sqrt{2} \int \frac{\mathrm{d}\theta}{\sin\theta\cos\theta} \,.$$

Since we obtained the product of sine and cosine of the integrated variable, we will use common trigonometric substitution for this case, $u = \tan\theta$, therefore,

$$\mathrm{d}u = \frac{1}{\cos^2\theta}\,\mathrm{d}\theta \,,$$

from which we get

$$\sqrt{2} \int \frac{\cos\theta}{\sin\theta}\,\mathrm{d}u = \sqrt{2} \int \frac{1}{u}\,\mathrm{d}u \,.$$

Finally, we obtained the expression we can easily integrate, providing

$$\int \frac{\mathrm{d}\varphi}{\sqrt{1-\cos\varphi}} = \sqrt{2} \int \frac{\mathrm{d}u}{u} = \sqrt{2}\ln u + C = \sqrt{2}\ln\tan\theta + C = \sqrt{2}\ln\tan\frac{\psi}{2} + C = \sqrt{2}\ln\tan\frac{\varphi}{4} + C \,.$$

Thus, the fall of the tree with the initial deviation 1° to the horizontal position takes

$$t = \sqrt{\frac{2l}{3g}} \left[\ln\tan\frac{\varphi}{4}\right]_{1^\circ}^{90^\circ} \doteq 6.44\sqrt{\frac{l}{3g}} = 3.9\,\mathrm{s} \,.$$

Jaroslav Herman
jardah@fykos.org

FYKOS
UK, Matematicko-fyzikální fakulta
Ústav teoretické fyziky
V Holešovičkách 2
180 00 Praha 8

www: http://fykos.cz
e-mail: fykos@fykos.cz

FYKOS is also on Facebook
http://www.facebook.com/FYKOS

FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Public Relations Office activities and is supported by Institute of Theoretical Physics, MFF UK, his employees and The Union of Czech Mathematicians and Physicists.

February 10th 2023
PVA EXPO, Prague

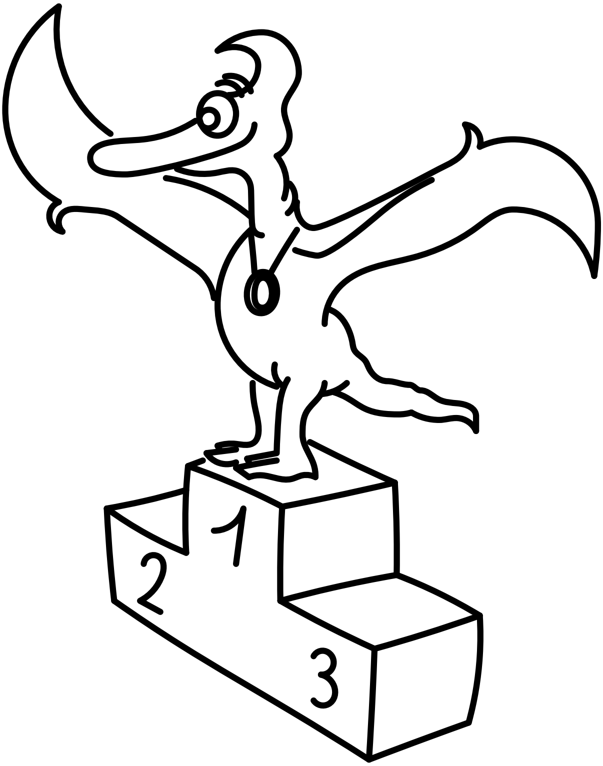
Solutions

fykos.org
fyziklani.org
/fykos
@fykosak

Fyziklani2023

Solutions of problems

Fyziklani



Problem AA ... at the concert

Danka was at the concert at the airport. During one song, they had the drumming and the flashing of the spotlights over the stage synchronized. It was a periodic drumming and flashing with a period of $T = 1.5\,\mathrm{s}$, and the two events always happened simultaneously. However, Danka saw that the spotlights were flashing with a half-period offset from the sound of the drums. What is the smallest possible distance Danka could be from the stage to observe this phenomenon? *Danka and other organizers were at the Rammstein concert.*

When Danka stands at the distance x from the stage, the light from the stage reaches her in $t_1 = x/c$, where c is the speed of light. Similarly, the drumming propagates through the air to Danka at the speed of sound in the air v, so the sound wave reaches her in $t_2 = x/v$. Since Danka sees that the light and sound waves reach her with a $T/2$ offset, the following must hold

$$t_2 - t_1 = \frac{T}{2}\,.$$

We insert the above-mentioned formulas for the times t_1 and t_2 and then express the distance we are looking for

$$\begin{aligned} \frac{x}{v} - \frac{x}{c} &= \frac{T}{2}\,, \\ x\left(\frac{c-v}{cv}\right) &= \frac{T}{2}\,, \\ x &= \frac{T}{2}\left(\frac{cv}{c-v}\right)\,. \end{aligned}$$

Now, we can notice that the second fraction in the last equation can be modified to the form $v/(1 - v/c)$, and since the v/c ratio is several orders of magnitude smaller than 1, the whole fraction is quite exactly equal to v. Then

$$x = \frac{vT}{2} \doteq 257\,\mathrm{m}\,.$$

Hence, Danka had to stand at a distance of 257 m from the stage. In general, Danka can stand at the places that satisfy the condition $t_2 - t_1 = (2n - 1) \cdot T/2$, where n is a natural number. However, since we are interested in the smallest distance, we consider $n = 1$ in the whole solution.

Daniela Dupkalová
daniela@fykos.org

Problem AB ... slow cyclist

Verča is driving her car on the road at speed $v_1 = 82\,\mathrm{km{\cdot}h^{-1}}$. A cyclist is pedaling along the side of the road at speed $v_2 = 16\,\mathrm{km{\cdot}h^{-1}}$ and Verča wants to pass him, so she has to drive into the middle of the road. As she does not want to endanger him, she leaves the lane $d = 20\,\mathrm{m}$ before the cyclist and always returns back at a distance $d = 20\,\mathrm{m}$ after him (in the direction of travel). How long would a stationary obstacle on the side of the road have to be for Verča to spend the same amount of time going around it as she did when passing the cyclist? She only goes around stationary things with a margin of $l = 10\,\mathrm{m}$. Ignore the time required to cross *between the center of the road and the lane. Think of the car and the cyclist as points (do not consider their length).* *Verča doesn't like going around obstacles. So she doesn't drive.*

Let us denote the length of the obstacle we are looking for as S and the distance the cyclist will travel while the car is in the passing lane as s. The key to solving the problem is to express the time t that the car spends here. From the above, we get the equation

$$v_2 \cdot t = s\,.$$

The second equation in the system describes the distance the car travels in time t, i.e.

$$v_1 \cdot t = 2d + s\,,$$

because the car goes around the cyclist with a margin d on both sides.

From this system of equations, we can easily express the time t and distance s as

$$s = \frac{2d}{\frac{v_1}{v_2} - 1}\,, \quad t = \frac{2d}{v_1 - v_2}\,.$$

The distance S we are looking for is this cyclist's path, to which we add the difference in the overtaking margin, so we get

$$S = s + 2(d - l) = \frac{2d}{\frac{v_1}{v_2} - 1} + 2(d - l)\,.$$

After plugging in the numerical values, we find that the obstacle would have to measure approximately $S \doteq 30\,\mathrm{m}$.

Veronika Hendrychová
veronika.hendrychova@fykos.org

Problem AC ... not enough time

Daniel needs to extend the time he has left to write his diploma thesis. Ideally, in a way that three weeks becomes six. The simplest and best solution seems to be to move the Earth by a little. By how many astronomical units does Daniel need to increase the Earth's average distance from the Sun to double the Earth's orbital period? Write the answer with 3 significant digits. *Daniel needs more time to write his diploma thesis.*

We will use the simplified Kepler's Third Law, where $a^3 = P^2$ holds for the orbital period P in years and the average distance from the Sun[1] a in astronomical units. For the new orbital period $P = 2\,\mathrm{years}$, we will get the equation $a^3 = 4$, from which we can take the cube root and get $a = 1.587\,\mathrm{au}$. If the average Earth-Sun distance is $1\,\mathrm{au}$, Daniel will have to move the Earth by approximately 0.587 astronomical units, which is beyond the orbit of Mars.

Daniel Dupkala
daniel.dupkala@fykos.org

[1] more precisely, the length of the orbital semi-major axis

Problem AD ... springs and weights

We have three identical springs of negligible mass with stiffness k and three weights of equal mass m. We attach one spring to the ceiling and hang one of the weights on its other end. To this weight, we add another spring with a weight on its tail and finally a third spring and a third weight. By how much do springs elongate with respect to their rest length?

Karel reminisced about springs.

We solve the elongation of each spring separately and then add them up. If we index springs from the bottom one, the elongation of the first spring is

$$F_1 = mg = ky_1 \quad \Rightarrow \quad y_1 = \frac{mg}{k}\,.$$

Analogically for the second and third one

$$y_2 = \frac{2mg}{k}\,, \quad y_3 = \frac{3mg}{k}\,.$$

Then our result is

$$\Delta y = y_1 + y_2 + y_3 = \frac{6mg}{k}\,.$$

Karel Kolář
karel@fykos.org

Problem AE ... gramophone

The camera records a vinyl record, which is symmetrically crossed by six lines running centrally from one end to the other. The display shows the record starting to spin gradually. At a certain point, it seemed that it had stopped moving. At that moment, the record's perimeter rotates with a speed $v = 3.14\,\mathrm{m\cdot s^{-1}}$, while the record's radius is $r = 10\,\mathrm{cm}$. Determine the frame rate of the camera.

The promising FYKOS-bird forgot to blink.

The angle between two lines on the board is

$$\alpha = \frac{\pi}{6}\,.$$

The record looks as if it stopped on display when the record rotates between two frames by any multiple of this angle. However, the plate is gradually rotating, so we are looking for the smallest rotation, and therefore the frame rate is

$$f = \frac{\omega}{\alpha} = \frac{6v}{\pi r} \doteq 60\,\mathrm{s^{-1}}\,.$$

Jan Pijáček
jan.pijacek@fykos.org

Problem AF ... ball on the boat

Lego and Dodo are sailing down the river on a boat and throwing a ball around on the deck. They both stand exactly parallel to the boat's course and the water in the river. Robo, standing on the shore, watches them from afar. When Lego throws the ball to Dodo, Robo sees that the ball has a horizontal velocity $v_1 = 42\,\mathrm{km{\cdot}h^{-1}}$, when Dodo throws it to Lego, Robo observes a velocity $v_2 = 24\,\mathrm{km{\cdot}h^{-1}}$ in the opposite direction. Lego and Dodo confirm to Robo that they both throw at the same horizontal speed. At what velocity is the ship sailing relative to Robo, and in which direction (i.e., from Dodo to Lego or from Lego to Dodo)?

Karel wanted to trump Nanynka's cabbage problem.

Let's denote the boat's velocity v_L and the speed at which Lego and Dodo throw (i.e., the ball's speed relative to the boat) as v_H. Of the velocities that Robo observes, v_1 is the larger one. He observes this when the ball is thrown in the direction of the boat's motion, so the velocities will add $v_1 = v_\mathrm{L} + v_\mathrm{H}$. From the fact that Robo observes this velocity when Lego throws to Dodo, we can also see that the boat is sailing away from Lego toward Dodo.

The velocity v_2 is observed when the ball is thrown in the opposite direction to the sail. The magnitude of this velocity will be the difference in magnitudes of v_L and v_H, so $v_2 = |v_\mathrm{L} - v_\mathrm{H}|$. We still need to figure out which of the two velocities is larger to eliminate the absolute value. From the problem statement, the velocity v_2 is observed by Robo in the opposite direction to v_1. This is only possible if Lego and Dodo are tossing each other at a velocity greater than the boat's velocity, so $v_2 = v_\mathrm{H} - v_\mathrm{L}$ holds.

In summary, we have a system of equations

$$v_1 = v_\mathrm{H} + v_\mathrm{L}\,,$$
$$v_2 = v_\mathrm{H} - v_\mathrm{L}\,,$$

where the unknowns are v_H and v_L. However, we are only interested in the speed of the boat, so it is sufficient to subtract the second equation from the first to get

$$v_1 - v_2 = 2v_L\,,$$
$$v_L = \frac{v_1 - v_2}{2} = 9\,\mathrm{km{\cdot}h^{-1}}\,,$$

so the answer to the question is that the ship is sailing in the direction from Lego to Dodo at $9\,\mathrm{km{\cdot}h^{-1}}$.

Šimon Pajger
legolas@fykos.org

Problem AG ... weighing a dog

Jarda went with his dog to the vet, where he put him on a scale and weighed him. Once he read his mass, he pulled the leash, but the dog did not move at all. The scale showed a 10 percent lower reading than before. What is the minimal coefficient of friction between the scale and the dog's paws? Jarda pulled the leash at an angle of 40° with respect to the ground.

A final tribute to Dort the dog.

Let us denote the magnitude of force by F, the mass of the dog by m, and the angle by $\alpha = 40^\circ$. When weighing with a taut leash, a normal force acts on the scale

$$F_\mathrm{N} = mg - F\sin\alpha\,,$$

so according to the problem statement $F\sin\alpha = 0.1mg$.

In the horizontal direction, the force $F\cos\alpha$ acts against the friction force, which is directly proportional to the coefficient of friction f as $F_\mathrm{t} = fF_\mathrm{N}$. The dog did not slip on the scale, so it must hold $F_\mathrm{t} > F\cos\alpha$, from which

$$f > \frac{F\cos\alpha}{mg - F\sin\alpha}\,.$$

Now we just substitute for F from the second equation and get

$$f > \frac{\cos\alpha}{9\sin\alpha} = 0.13\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem AH ... filling a bucket

A garden hose has a length of $15\,\mathrm{m}$ *and an inner diameter of* $1.5\,\mathrm{cm}$. *The* $11\,\mathrm{m}$ *long first part of the hose lies in direct sunlight; the rest leads through the shade to the tap. The sun beams warmed the water in the first part to* $35\,\mathrm{^\circ C}$, *while a* $15\,\mathrm{^\circ C}$ *water flows from the tap. Assume that between the warmed-up part of the hose and the tap, the water temperature changes linearly. Now, we begin to fill up the bucket from the hose. What temperature will the water have at the end if we fill the bucket with* $5.5\,\ell$ *of water?*

Jarda reminisces his garden and warm summer days.

A volume of water $V_1 = l_1 S \doteq 1.9\,\ell$ lies in the hose in the direct sunlight, where $l_1 = 11\,\mathrm{m}$ a $S = \pi d^2/4$ is the cross-sectional area of the hose, with $d = 1.5\,\mathrm{cm}$. This water has a temperature of $t_1 = 35\,\mathrm{^\circ C}$. Let us calculate the heat stored in it. Since heat is an additive quantity, let us set the zero heat level of water at $0\,\mathrm{^\circ C}$. Thus, after the subsequent calculation, we get the temperature in degrees Celsius.

$$Q_1 = l_1 S\rho c t_1\,,$$

where c is the specific heat capacity of water, and ρ is its density.

In the second part of the hose, the temperature changes linearly between $35\,\mathrm{^\circ C}$ and $t_2 = 15\,\mathrm{^\circ C}$, which corresponds to an average temperature $t_\mathrm{p} = (t_1 + t_2)/2$. Thus, the heat of this part is

$$Q_2 = l_2 S\rho c\frac{t_1 + t_2}{2}\,,$$

where $l_2 = l - l_1 = 11\,\mathrm{m}$.

Since $Sl \doteq 2.6\,\ell$ is still less than $V = 5.5\,\ell$, we also have to fill the bucket with water that has not gone through the tap yet. We need $V - lS$ of this water. Its heat is

$$Q_3 = (V - lS)\,\rho c t_2\,.$$

Since the fluid in the bucket is homogeneous, we obtain its temperature in degrees Celsius as the ratio of total heat to total mass and specific heat capacity as

$$t = \frac{1}{\rho c}\frac{Q_1 + Q_2 + Q_3}{V_1 + V_2 + V_3} = \frac{l_1 S t_1 + (l - l_1)\,S\,\frac{t_1 + t_2}{2} + (V - lS)\,t_2}{V} = 23.4\,\mathrm{^\circ C}\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem BA ... CR7

Perhaps the best footballer in the world, Cristiano Ronaldo, is 187 cm *tall. However, in header duel, he can jump so high that the top of his head is at a height of* 268 cm*. The timing of the jump is very important to score a header. What percentage of the total airtime is a part of his head above the* 250 cm *level, where he can hit a flying ball?* *Jarda still thinks Messi is better.*

Let us denote Ronaldo's height by h_R, the height he can jump to h_v, and the height where he needs to hit the ball by $h_\mathrm{b} = 250\,\mathrm{cm}$. Ronaldo lifts by unbelievable $h_\mathrm{v} - h_\mathrm{R} = 81\,\mathrm{cm}$ during his jump. According to laws of motion in a homogeneous gravitational field, which holds on a pitch, he spends the time

$$t_\mathrm{v} = 2\sqrt{\frac{2\left(h_\mathrm{v} - h_\mathrm{R}\right)}{g}}$$

in the air, where g is the gravity of Earth.

Similarly, we can express a condition when he is higher than h_b. That gives us time

$$t_\mathrm{p} = 2\sqrt{\frac{2\left(h_\mathrm{v} - h_\mathrm{b}\right)}{g}}\,.$$

We are interested in the ratio

$$\frac{t_\mathrm{p}}{t_\mathrm{v}} = \sqrt{\frac{h_\mathrm{v} - h_\mathrm{b}}{h_\mathrm{v} - h_\mathrm{R}}} = 0.47\,,$$

which is 47 %. Thus, for almost half the time he is in the air, Ronaldo can hit a ball flying at 250 cm.

Jaroslav Herman
`jardah@fykos.org`

Problem BB ... chopping a parsley

Jarda chopped the parsley, which had the shape of a perfect cone with an apex angle of $2\alpha = 10\,^\circ$*. He got bored of cutting all the pieces with the same width, so he started chopping them into pieces with constant volume* $V = 0.9\,\mathrm{cm^3}$*. What is the width of the seventh piece if he starts slicing from the tip?* *A large piece of vegetable was floating in Jarda's soup.*

Let us denote the distance from the tip to the nearer plane of the n-th circle as a h_{n-1}, and the distance of the tip from the second slice as a h_n. Therefore h_0 is equal to zero.

The volume of the first piece is V, which shape is a cone. Its volume is

$$V = \frac{1}{3}\pi r^2 h_1 = \frac{1}{3}\pi h_1^3 \tan^2\alpha\,,$$

where we expressed the radius of the base using the apex angle and height.

The volume of the first n pieces is nV, so the total height is

$$h_n = \sqrt[3]{\frac{3nV}{\pi \tan^2\alpha}}\,.$$

Subtracting the h_6 from the h_7 will give us the requested result (remember that the $\alpha = 5\,^\circ$)

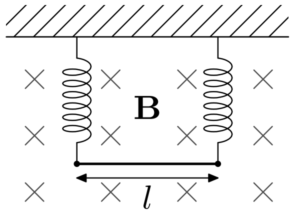
$$t_n = \sqrt[3]{\frac{3V}{\pi \tan^2 \alpha}} \left(\sqrt[3]{n} - \sqrt[3]{n-1}\right) = 0.46\,\mathrm{cm}\,.$$

Obviously, the width of the discs is decreasing.

Jaroslav Herman
`jardah@fykos.org`

Problem BC ... a rod on springs

A metal rod with a length $l = 34\,\mathrm{cm}$ and a mass $m = 85\,\mathrm{g}$ is suspended by its ends from the ceiling using two conductive springs. We create a homogeneous magnetic field of size $B = 0.440\,\mathrm{T}$ oriented as is shown in the figure. How much current does it need to be running through the rod in the right direction to have the springs not stretched at all?
Danka remembered the time she was studying electromagnetism.



In order for the springs not to be stretched at all, the gravitational force acting on the rod $F_\mathrm{G} = mg$ in the downward direction must be offset by another force acting upward. When we insert a conductor through which an electric current is flowing into a magnetic field, a magnetic force will start to act on it. Its magnitude is described by Ampere's law

$$F_\mathrm{m} = IlB \sin \alpha\,,$$

where α is the angle between the direction of the magnetic field and the direction of the current. In our case, the rod is perpendicular to the magnetic field, so $\alpha = 90\,^\circ$, and therefore $\sin \alpha = 1$. When the magnetic and gravitational forces are equal, the following equation holds

$$mg = IlB \sin \alpha\,.$$

From here, we express the magnitude of the current we are looking for and plug in the numerical values

$$I = \frac{mg}{lB \sin \alpha} = \frac{mg}{lB} \doteq 5.6\,\mathrm{A}\,.$$

Therefore a current of $5.6\,\mathrm{A}$ has to be flowing through the rod.

Daniela Dupkalová
`daniela@fykos.org`

Problem BD ... a swing

The swing chain deviates from the vertical line at the highest point by $\alpha = 70\,^\circ$. When the seat of the swing passes through the lowest point, the chain suddenly breaks. The chain is $l = 2\,\mathrm{m}$ long, and the seat is at rest at the height $h_0 = 1\,\mathrm{m}$ above the ground. How far does the seat fall from the vertical line of the swing? *A promising FYKOS-bird is thinking about the future.*

First, we determine the maximum height that the swing can reach

$$h = h_0 + l\left(1 - \cos\alpha\right) .$$

Since we assume that mechanical energy is being conserved ($E_\mathrm{k} = E_\mathrm{p}$), we can determine the horizontal component of the velocity as

$$\frac{1}{2}mv_x^2 = mg\left(h - h_0\right) ,$$

from which we express the horizontal velocity

$$v_x = \sqrt{2gl\left(1 - \cos\alpha\right)} .$$

We write the height of the vertical throw in a homogeneous gravitational field as $h_0 = gt^2/2$. From this, we find the time

$$t = \sqrt{\frac{2h_0}{g}} ,$$

which, when multiplied by the formula for v_x, gives us the horizontal distance we are looking for

$$s_x = v_x t = \sqrt{4h_0 l\left(1 - \cos\alpha\right)} = 2.3\,\mathrm{m} .$$

Jan Pijáček
jan.pijacek@fykos.org

Problem BE ... dense energy

Nowadays, we are looking for ways to get energy. One way is to generate electricity from renewable sources such as the wind or the sun. However, these do not produce electricity under all conditions, so the energy needs to be stored. Let us compare two options – pumped hydroelectric energy storage and compressed hydrogen. The Dlouhé stráně power plant has a working volume of water in the upper reservoir of $2\,580\,000\,\mathrm{m}^3$*, a water gradient of* $510\,\mathrm{m}$ *and an efficiency of* $90\,\%$*. Hydrogen gas* H_2 *can be stored compressed in* $50\,\ell$ *cylinders at a pressure of* $70\,\mathrm{MPa}$*. The heating value of hydrogen is* $120\,\mathrm{MJ{\cdot}kg^{-1}}$ *and the conversion efficiency to electricity is* $50\,\%$*. Consider hydrogen an ideal gas with a temperature of* $25\,\mathrm{^\circ C}$*. How many of these filled hydrogen cylinders are equivalent to the available electricity of the Dlouhé stráně power plant?* *Jarda wanted to stock up electricity before it gets more expensive.*

The available energy at a pumped storage power plant is the potential energy of a homogeneous gravity field

$$E_\mathrm{e} = \eta_\mathrm{e} V_\mathrm{e} \rho_\mathrm{v} g h = 11\,600\,\mathrm{GJ} ,$$

where $\eta_\mathrm{e} = 0.9$, V_e is the volume of the upper reservoir, ρ_v is the density of the water, and $h = 510\,\mathrm{m}$ is the gradient of the water.

We calculate the energy in one bottle of hydrogen from its mass and gravimetric energy (energy stored per unit mass). Considering hydrogen is an ideal gas, its density at pressure $p = 70\,\mathrm{MPa}$ and temperature $T = 298\,\mathrm{K}$ is

$$\rho_\mathrm{h} = \frac{M_\mathrm{m} p}{TR} = 56.5\,\mathrm{kg{\cdot}m^{-3}} ,$$

where $M_\mathrm{m} \doteq 2\,\mathrm{g{\cdot}mol^{-1}}$ is the molar mass of the hydrogen molecule.

The available energy from a bottle of hydrogen is thus

$$E_\mathrm{h} = \eta_\mathrm{h} l V_\mathrm{l} \rho_\mathrm{h} = 170\,\mathrm{MJ}\,,$$

where $\eta_\mathrm{h} = 0.5$, l is the gravimetric energy and V_l is the volume of the bottle.

By dividing the two energies, we get the number of bottles $N = 68\,000$. However, the total volume of hydrogen in all the bottles is several orders of magnitude smaller than the volume of the hydropower plant reservoir.

Jaroslav Herman
`jardah@fykos.org`

Problem BF ... a pulley that shifts

We have a massless movable pulley. One end of a massless rope that goes through the pulley is attached directly to the ceiling; the other end is connected to a spring with stiffness $k = 80\,\mathrm{N{\cdot}m^{-1}}$, which is attached to the same ceiling. We will hang a weight with mass $m = 1.0\,\mathrm{kg}$ onto the pulley. How far will the pulley shift downwards?

This crossed Lego's mind when he was writing down a different problem...

The whole system will be at equilibrium when both sides of the rope pull the pulley with force $mg/2$. The spring will be strained by $mg/(2k)$ with respect to its initial length. The extensional strain is distributed equally between the two halves of the rope coming out of the pulley; thus, the pulley will only move by $mg/(4k) = 3.1\,\mathrm{cm}$.

Šimon Pajger
`legolas@fykos.org`

Problem BG ... does is float?

We have a cube, and the only thing you know about it is that its sides are shorter than $10\,\mathrm{cm}$. Furthermore, we have a container with a square base of side $a = 10\,\mathrm{cm}$, which we fill with water of volume $V = 1\,\ell$. When we drop our cube into the container, the level of water rises by $\Delta h_1 = 3.5\,\mathrm{cm}$. We then remove the cube, pour the water out, and replace it with methanol of volume V and density $\rho_\mathrm{M} = 792\,\mathrm{kg{\cdot}m^{-3}}$. When we drop the cube into the container with methanol, the level rises by $\Delta h_2 = 4.2\,\mathrm{cm}$. What is the density of our cube? Yes, you do have all the data needed.

Lego wanted to assign an interesting Archimedes problem.

The key observation in this problem is that $\Delta h_1 \rho_\mathrm{V} \neq \Delta h_2 \rho_\mathrm{M}$, i.e. in at least one liquid the cube lies at the bottom of the container. If that wasn't the case, and the cube was indeed floating in both liquids, the buoyant force would have to be equal in both cases, thus $a^2 \Delta h_1 \rho_\mathrm{V} g = a^2 \Delta h_2 \rho_\mathrm{M} g$. Divide the previous expression by $a^2 g$ and we get a condition on the product of the rise of the level and density, which is not satisfied, therefore it cannot be true that the cube floats in both liquids.

Moreover, it cannot lie on the bottom of the container in both liquids, since $\Delta h_1 \neq \Delta h_2$. Thus, it obviously floats in one liquid and lies on the bottom in the other. Logically, the cube will float in the liquid with higher density and lie on the bottom in the one with the lower density.

From the rise of the methanol level, we can easily calculate the volume of the cube as $V_\mathrm{k} = a^2\Delta h_2$. We can also calculate the mass of the cube from the rise of the water level since its weight must be the same as the weight of a liquid whose volume is equal to the volume of the submerged part of the cube, i.e. $m_\mathrm{k} = a^2\Delta h_1\rho_\mathrm{V}$.

The density of the cube will therefore be

$$\rho_\mathrm{k} = \frac{m_\mathrm{k}}{V_\mathrm{k}} = \frac{a^2\Delta h_1\rho_\mathrm{V}}{a^2\Delta h_2} = \frac{\Delta h_1}{\Delta h_2}\rho_\mathrm{V} = 832\,\mathrm{kg{\cdot}m^{-3}}\,.$$

Šimon Pajger
legolas@fykos.org

Problem BH ... restless coin

When riding up the inclined moving walkway of inclination $\alpha = 10^\circ$ and length $l = 30\,\mathrm{m}$, a coin drops out of Verča's pocket when she is exactly in the middle of it. It falls into one of the grooves on the walkway and starts rolling down without slipping. How much time does Verča have to catch the coin before it falls under the bottom edge of the walkway? The velocity of the moving walkway is $v = 0.9\,\mathrm{m{\cdot}s^{-1}}$.

Verča is constantly losing the content of her pockets and fears the escalators.

First, we express the distance of the coin from the lower end of the walkway in terms of time. We determine it from the total energy balance for a decrease in height h. Potential energy converts to translational energy

$$E_\mathrm{k,t} = \frac{1}{2}mv^2$$

and rotational energy

$$E_\mathrm{k,r} = \frac{1}{2}J\omega^2\,.$$

Letter J denotes the moment of inertia. In our particular case, for a coin shaped like a cylinder, $J = mR^2/2$, where m is the mass of the coin and R is its radius. Angular velocity ω is bound to the velocity of the coin due to the no-slip condition as $v = R\omega$. Thus, adding the two contributions to the kinetic energy, we get the following from the law of conservation of energy

$$mg\Delta h = E_\mathrm{k,t} + E_\mathrm{k,r} = \frac{1}{2}mv^2 + \frac{1}{2}\frac{1}{2}m\left(R\omega\right)^2 = \frac{1}{2}\frac{3}{2}mv^2\,.$$

That is the same formula as for uniformly accelerated motion, except that the mass of the coin on the right is multiplied by a factor of 3/2. Thus, the coin will have a constant acceleration of magnitude

$$a = \frac{2}{3}g\sin\alpha\,.$$

The coin's motion down the walkway is uniformly accelerated, but its initial velocity v relative to the ground is in the opposite direction. So we get the equation

$$x = \frac{l}{2} + vt - \frac{1}{2}\frac{2}{3}gt^2\sin\alpha\,.$$

Putting $x = 0$ gives us the time in which the coin will reach the bottom edge. So, we need to solve the quadratic equation for t to get the roots

$$t_{1,2} = \frac{-v \pm \sqrt{v^2 + \frac{2}{3} lg \sin\alpha}}{-\frac{2}{3} g \sin\alpha}\,.$$

Putting in the numerical values, we get $t_1 \doteq -4.4\,\mathrm{s}$ and $t_2 \doteq 6.0\,\mathrm{s}$. We will not consider negative time, so Verča has roughly 6.0 seconds to get the coin.

Veronika Hendrychová
veronika.hendrychova@fykos.org

Problem CA ... oscillating pulley

Let us have a massless movable pulley. One end of a massless rope that goes through the pulley is attached directly to the ceiling, while the other end is attached to a spring of stiffness $k = 80\,\mathrm{N{\cdot}m^{-1}}$, *which is attached to the same ceiling. We hang a weight of mass* $m = 1.0\,\mathrm{kg}$ *on the pulley. What is the period of small oscillations of the weight?*

Lego loves pulleys and oscillators, so why not to combine them?!

When we deviate the weight from the equilibrium position lowering it by $\mathrm{d}x$, the pulley also moves by $\mathrm{d}x$. Thus, the total length of the rope and the spring must increase by $2\,\mathrm{d}x$. As the rope does not elongate, it must be the spring that is extended by this length. Thus, the spring will pull with a force $2k\,\mathrm{d}x$ greater than it was pulling while in the equilibrium position. Since the rope and the pulley are both massless, the force exerted by the spring is equal to the tension along the entire length of the rope. As the pulley is pulled upward by the rope on both sides (i.e. twice), the pulling force is $4k\,\mathrm{d}x$ greater than in the equilibrium position. The pulley is weightless, so this is also the contribution to the force, which is pulling our weight upward. When we divide the force contribution by the displacement that caused it, we get the effective stiffness that the weight experiences as $k_{\mathrm{ef}} = 4k$. (We reached the same conclusion in the problem "a pulley that shifts", where we found that, when a weight is hung on the pulley, the pulley moves down by $mg/(4k) = mg/k_{\mathrm{ef}}$.)

All that remains is to substitute into the formula for the period of a linear harmonic oscillator

$$T = 2\pi\sqrt{\frac{m}{k_{\mathrm{ef}}}} = \pi\sqrt{\frac{m}{k}} = 0.35\,\mathrm{s}\,.$$

Šimon Pajger
legolas@fykos.org

Problem CB ... a crystal of beryl

Beryl crystallizes in a hexagonal crystal system, i.e., with the unit cell in the shape of a perpendicular prism, whose base is a rhombus. If we correctly join 3 unit cells, the resulting perpendicular prism has the base of a regular hexagon, with the side a. The height of the prism, c, is approximately equal to a. In the unit cell, there are atoms corresponding to two molecules of beryl, whose chemical formula is $\mathrm{Be_3Al_2(SiO_3)_6}$. *The relative atomic mass of beryllium is* $A_{\mathrm{Be}} = 9.01$, *aluminium* $A_{\mathrm{Al}} = 27.0$, *silicon* $A_{\mathrm{Si}} = 28.1$ *and oxygen* $A_{\mathrm{O}} = 16.0$. *What is* *the length of the hexagonal side a, if the density of beryl is $\rho = 2\,760\,\mathrm{kg{\cdot}m^{-3}}$?*

Karel made a simple high school problem more complicated.

Let us first calculate the mass of the beryllium molecule. We are given the relative atomic masses of all the atoms in the molecule. These give us the masses of the atoms as multiples of the atomic mass unit, which is defined as one-twelfth the mass of a carbon atom ^{12}C, which is $m_{\mathrm{u}} = 1.66 \cdot 10^{-27}\mathrm{kg}$.

One molecule of beryl contains 3 atoms of Be, 2 Al, 6 Si, and $6 \cdot 3 = 18$ O, then the total mass of the molecule is

$$m_{\mathrm{molecule}} = (3A_{\mathrm{Be}} + 2A_{\mathrm{Al}} + 6A_{\mathrm{Si}} + 18A_{\mathrm{O}})m_u = 537,6m_u = 8,92 \cdot 10^{-25}\,\mathrm{kg}\,.$$

Next, we need to calculate the volume of one unit. The volume of a prism is the height times the area of the base; the height is simply a in our case. The base is a regular hexagon, so we calculate its area as 6 times the area of an equilateral triangle with side a

$$S_p = 6S_t = 3a\ v_a = 3a\sqrt{a^2 - \left(\frac{a}{2}\right)^2} = \frac{3\sqrt{3}}{2}a^2\,,$$

and to get the volume, we need to multiply the calculated area by a.

All that is left to do is plug the obtained result in the formula for density, not forgetting that there are $2 \cdot 3$ molecules per one hexagonal base, and express a

$$\rho = \frac{m}{V} = \frac{6m_{\mathrm{molecule}}}{\frac{3\sqrt{3}}{2}a^3}\,,$$

$$a = \left(\frac{4m_{\mathrm{molecule}}}{\sqrt{3}\rho}\right)^{\frac{1}{3}} \approx 0.91\,\mathrm{nm}\,.$$

Šimon Pajger
legolas@fykos.org

Problem CC ... lasagna

Consider a layering lasagna. The first layer with everything in its place has a mass $m_0 = 100\,\mathrm{g}$ and has a temperature $T_0 = 300\,\mathrm{K}$. Each additional layer weighs $p = 1/3$ times more than the previous layer; however, it is fresher. That means its absolute temperature is $q = 2$ times higher than the previous layer. Since Pepa was very hungry, he stacked an infinite number of these layers on each other. At what temperature will the lasagna in its final form stabilize if the system reaches thermal equilibrium? Do not consider heat loss. Bon appétit.

Pepa was hungry.

First, let us calculate the weight of the whole lasagne. The weight of the ith layer will be $m_i = m_0p^i$. Therefore the mass of the whole lasagne will be

$$M = \sum_{i=0}^{\infty} m_0p^i\,.$$

That is a geometric series; those of you who know the formula for its sum, go ahead and use it; however, for those that do not know the formula, we will show you how to get it. The trick is to multiply both sides of the equation by p, and we will also substitute $j = i+1$ on the right-hand side

$$pM = \sum_{i=0}^{\infty} m_0 p^{i+1} = \sum_{j=1}^{\infty} m_0 p^j \,.$$

Notice that on the right-hand side, we have the same sum, which was equal to M, with the difference that it starts from the first term, not the zeroth term. Thus at the right-hand side, we have $M - m_0$, from which we get the linear equation

$$pM = M - m_0$$
$$M = \frac{m_0}{1-p} \,.$$

We still need to calculate the total heat contained in the whole lasagna and divide it by the total mass. The heat contained in one layer can be calculated as $Q_i = m_i T_i c = m_0 p^i T_0 q^i c = Q_0 p^i q^i$, where c is the specific heat capacity of the lasagna and $Q_0 = m_0 T_0 c$ is the heat in the zeroth layer. Therefore, we will get the total heat using the same procedure we used to get the total mass

$$Q = \frac{Q_0}{1-pq} = \frac{m_0 T_0 c}{1-pq} \,.$$

If the whole lasagna settles at the thermal equilibrium, the following must be true for its temperature

$$T_\mathrm{v} c M = Q$$
$$T_\mathrm{v} = \frac{\frac{m_0 T_0 c}{1-pq}}{c \frac{m_0}{1-p}}$$
$$T_\mathrm{v} = T_0 \frac{1-p}{1-pq} = 600\,\mathrm{K} \,.$$

Šimon Pajger
`legolas@fykos.org`

Problem CD ... lenses for nothing

Consider two lenses at a mutual distance $d = 25\,\mathrm{cm}$. The first is a biconvex lens with focal length $f = 10\,\mathrm{cm}$, and the second is a biconcave lens with focal length $-f$. How far from the first lens do we need to place the object for the system to project it on itself?

Jarda likes to project himself on himself.

Let us denote the object distance from the connecting lens as a. According to the Gaussian lens formula, the object will appear through the lens at a distance

$$a' = \frac{af}{a-f} \,.$$

The distance of this image from the second lens is $a_2 = d - a'$. We will use the Gaussian lens formula for the second time and get

$$a_2' = \frac{-a_2 f}{a_2 + f} = \frac{-\left(d - a'\right) f}{d - a' + f} = \frac{-\left(d - \frac{af}{a-f}\right) f}{d - \frac{af}{a-f} + f} \,.$$

According to the problem statement, the position of the image is equal to the position of the object, therefore $a'_2 = -d - a$. So we are trying to solve the equation

$$\frac{-\left(da - df - af\right)f}{da - df - f^2} = -d - a\,,$$

from which we get

$$a^2 + \left(d - 2f\right)a - df = 0\,.$$

The solution to this equation is

$$a = \frac{-d + 2f \pm \sqrt{d^2 + 4f^2}}{2}\,.$$

Since we place the object in front of the first lens, the solution will be a root with a positive sign. The answer is $a = 13.5\,\mathrm{cm}$

Jaroslav Herman
`jardah@fykos.org`

Problem CE ... heating with hydrogen

Electrolysis of water produces oxygen and hydrogen, the latter of which we capture and store in an inflatable balloon. The process takes place at a voltage of $1.48\,\mathrm{V}$, *electric current of* $15\,\mathrm{A}$ *and produces* $0.16\,\mathrm{g}$ *of hydrogen molecules* $\mathrm{H_2}$. *Next, we will let the hydrogen burn in the air to heat a water container. The container has a heat capacity* $22\,\mathrm{J{\cdot}K^{-1}}$ *and an inner volume of* $55\,\mathrm{ml}$ *filled with water with initial temperature* $23\,\mathrm{^\circ C}$. *After the hydrogen gets burned, the temperature inside the container rises to* $94\,\mathrm{^\circ C}$. *What percentage of energy was used to heat the container and water?* *Jarda wanted to warm himself up with water.*

The molar mass of hydrogen approximately equals $M_{\mathrm{H_2}} = 2.0\,\mathrm{g{\cdot}mol^{-1}}$ because the hydrogen molecule is composed of two protons (as a matter of fact, it is $M_{\mathrm{H_2}} = 2.016\,\mathrm{g{\cdot}mol^{-1}}$, but in this case, less precision is sufficient). Mass $m = 0.16\,\mathrm{g}$ corresponds to the number of molecules by $n = N_\mathrm{A} m / M_{\mathrm{H_2}}$. The number of hydrogen atoms is double that of molecules, thus the total electric charge in the reactions is

$$Q = 2\frac{N_\mathrm{A} m}{M_{\mathrm{H_2}}} e\,.$$

The work done by the electrolyzer is simply

$$W = 2U\frac{N_\mathrm{A} m}{M_{\mathrm{H_2}}} e\,.$$

We got this energy, which stored itself in the hydrogen molecules by breaking up the water molecules. On the other hand, if we burn the hydrogen in the air, the energy will be released as heat.

By heating the container and the water, we have added heat

$$Q = \left(C + V_\mathrm{w}\rho_\mathrm{w}c_\mathrm{w}\right)\left(t_2 - t_1\right)\,,$$

where C is the heat capacity of the container, V_w, ρ_w and c_w are the volume, density and Specific heat capacity of the water in the container and $t_2 - t_1$ is the difference between the initial and final temperature.

The desired efficiency is

$$\eta = \frac{Q}{W} = \frac{M_{\mathrm{H_2}}\left(C + V_{\mathrm{w}}\rho_{\mathrm{w}}c_{\mathrm{w}}\right)\left(t_2 - t_1\right)}{2UeN_{\mathrm{A}}m} = 78\,\%\,.$$

Note that in order to heat this amount of water to nearly the boiling point, we used only 0.16 g of hydrogen! Hydrogen has the highest calorific value per mass unit of all chemical fuels.

Jaroslav Herman
jardah@fykos.org

Problem CF ... a cart with a plumb bob

Let us consider a hill with a slope at an angle α. We place a wagon with a mass M on top of the hill. There is a string in the wagon of a length l hanging from the roof with a point mass m at its end (this mass is not included in the cart's mass). We then release the wagon down the hill. What angle relative to the vertical direction does the string stabilize at? Submit a positive result if the string is deflected in the direction of travel or negative if it is deflected in the opposite direction. The wagon moves without friction. *Lego has had this idea for a long time and he has no clue why he hadn't written this problem earlier.*

We are interested in the steady state situation, i.e., when the rope and the point mass stop moving with respect to the wagon. Then, they appear as if they formed one perfectly rigid body together with the wagon. We can calculate the acceleration of this body moving down the hill.

Its total mass is $M + m$, and the component of the gravitational force in the direction parallel to the hill is $(M + m)\, g \sin\alpha$. Thus the acceleration is $a = g \sin\alpha$.

Let us transfer to a system accelerating with the wagon. If the point mass suspended on the rope in this system is supposed not to move, there must be a zero resultant force. So now, we need to discuss the forces acting on it. The gravitational force mg acts vertically downward; the inertial force ma acts parallel to the roof toward the rear; and finally, there is a force exerted by the rope on which it hangs. The magnitude and direction of the force from the rope is (in a steady situation precisely such that this force compensates the two remaining forces. It is essential to realize that the direction of the force from the rope is in the same direction as the rope. Thus, we must find the direction of the vector sum of the remaining two forces.

The vertical component of the inertial force has magnitude $ma \sin\alpha = mg \sin^2\alpha$, and it is directed upward. The horizontal component has magnitude $ma \cos\alpha = mg \sin\alpha \cos\alpha$, which is directed backward. The combined force of gravity and inertia thus has a component in the vertical direction $mg\left(1 - \sin^2\alpha\right) = mg \cos^2\alpha$ pointing downward and a horizontal component $mg \sin\alpha \cos\alpha$ pointing backward. Note that for the limiting case of a vertical hill ($\alpha = \pi/2$), the mass point does not experience any force in the system associated with the wagon. This is due to the fact that in this situation, the system free falls, and the point mass is in a weightless state from the viewpoint of this system.

Now back to the angle of deflection of the rope. We are interested in its deflection with respect to the vertical direction. We obtain this angle as the inverse tangent of the ratio of the horizontal and the vertical component

$$\beta = \arctan\frac{mg \sin\alpha \cos\alpha}{mg \cos^2\alpha} = \arctan\frac{\sin\alpha}{\cos\alpha} = \arctan(\tan\alpha) = \alpha\,.$$

The string will be deflected backward at calculated angle α; therefore, the desired result is $-\alpha$.

Šimon Pajger
legolas@fykos.org

Problem CG ... hot air balloon

A hot air balloon with volume $V = 3\,000\,\mathrm{m^3}$ and mass $M = 724\,\mathrm{kg}$ (without air) ascends to the sky. The temperature of the air inside the balloon is $T_\mathrm{b} = 120\,\mathrm{^\circ C}$, while the temperature of the surrounding air is constant $T_0 = 20\,\mathrm{^\circ C}$. However, as the balloon goes up, the ambient pressure drops. Determine at what value of ambient pressure does the weight of the balloon equal the buoyant force acting on it? The volume of the rest of the balloon is negligible compared to the volume of air in it. *Lego recalled the Physics Olympiad.*

We know that at temperature T_0 and pressure p_0 the density of air is ρ_0. The quantities in the equation of state imply, that the density is proportional to N/V, hence we can write

$$\begin{aligned} pV &= NkT \\ \frac{p}{kT} &= \frac{N}{V} \sim \rho \\ \frac{p}{T}C &= \rho\,, \end{aligned}$$

where C is a constant whose value can be determined from the density under normal conditions.

$$C = \frac{T_0\rho_0}{p_0}$$

By back-substitution, we get the air density under general temperature and pressure.

$$\rho = \rho_0\frac{T_0 p}{T p_0}\,.$$

Thus the mass of air in the balloon is at pressure p

$$m = V\rho_0\frac{T_0 p}{T_\mathrm{b} p_0}\,,$$

and the mass of air displaced by the balloon

$$m_\mathrm{vyt} = V\rho_0\frac{p}{p_0}\,.$$

The task asks at what pressure p the gravitational and buoyant forces are balanced

$$\begin{aligned} F_\mathrm{g} &= F_\mathrm{vz} \\ (M+m)g &= m_\mathrm{vyt}g \\ M + V\rho_0\frac{T_0 p}{T_\mathrm{b} p_0} &= V\rho_0\frac{p}{p_0} \\ M &= V\rho_0\frac{p}{p_0}\left(1-\frac{T_0}{T_\mathrm{b}}\right) \\ p &= p_0\frac{1}{1-\frac{T_0}{T_b}}\frac{M}{V\rho_0} = 80\,\mathrm{kPa}\,. \end{aligned}$$

Šimon Pajger
legolas@fykos.org

Problem CH ... heating

In scientific applications, it is sometimes necessary to heat a sample suspended in a vacuum. This is done using a stream of accelerated electrons, whose kinetic energy is converted to thermal energy upon impact. Consider a cathode with an emission stream of electrons I_e, which are accelerated by a voltage U. These electrons hit a sample with a total surface area S. Assume that the surface is small enough that the temperature is the same everywhere on it, and all of the electron energy is converted into heat. At what temperature T, will the sample stabilize? Consider that it behaves like a perfect black body, i.e., heat loss is only in the form of radiation, the ambient temperature is much lower than T, and neglect the initial velocity of the electrons upon emission.

The instrument measures so often that some elementary functions are memorized at last...

We assume a constant emission current I_e, which is defined as usual

$$I_e = \frac{Q}{t}\,,$$

where Q is the charge of electrons transferred in time t.

Using the equation for the energy of the charge accelerated by the voltage U, we can express the power P heating the sample

$$E = QU = UI_e t \quad \Rightarrow \quad P = UI_e\,.$$

The magnitude of the power radiated to the surroundings from our sample is governed by the Stefan-Boltzmann law

$$P_{ok} = \sigma S T^4\,.$$

The temperature of the sample stabilizes when $P = P_{ok}$, thus we have

$$UI_e = \sigma S T^4 \quad \Rightarrow \quad T = \sqrt[4]{\frac{UI_e}{\sigma S}}\,.$$

According to the problem statement, we could neglect the incoming power of thermal radiation from the surrounding environment. Therefore, the last equation gives us the desired result.

Jiří Blaha
jirka.b@fykos.org

Problem DA ... cutting an apple

We want to cut a perfectly spherical apple with a radius $R = 3.6\,\mathrm{cm}$ so that only the core is left. At first, we cut off a section of the apple with a straight cut at a distance $a = 0.7\,\mathrm{cm}$ from the center of the apple so that the plane of the cut is parallel to the core's axis of symmetry. Perpendicularly to this plane, we make two other cuts at a distance a from the center. Again at a distance a from the center, we make the last cut parallel with the first one such that the core is a square with a side $2a$ when viewed from above. Assume that the force needed to make the cut is proportional to the length of the knife in the apple. Determine the ratio of the work done during the last cut and the first cut.

Jarda was sitting under a tree, and an apple fell on his head.

According to the problem statement, the force is proportional to the length of the cut, which we denote by l. We can write

$$F = kl\,,$$

where k is a constant of proportionality. If we make the cut $\mathrm{d}x$ deeper, we do the work $\mathrm{d}W$. When cutting through the entire apple, we perform the work

$$W = \int kl\,\mathrm{d}x = \int k\,\mathrm{d}S = kS\,,$$

so the work is directly proportional to the area of the cut.

The area of the first cut is in the shape of a circle. Its radius is according to the Pythagorean theorem $r = \sqrt{R^2 - a^2}$. Thus, the area of the first cut is $S_1 = \pi\left(R^2 - a^2\right)$.

The plane of the last cut is also at a distance a from the center and the shape of the cut is a subset of a circle, whose radius is also r. A strip of width $2a$ is cut out from this circle, and we need to calculate its area.

The chord intersecting the circle has a length $2t = 2\sqrt{r^2 - a^2}$ according to the Pythagorean theorem. From the circle's center, we see it at an angle $2\alpha = 2\arctan(t/a)$. The area of the circular sector with this central angle is $S_\mathrm{v} = (2\alpha/2\pi)\pi r^2$.

Now, let us calculate the area of the triangle ABS. It is $S_\mathrm{t} = ta$. The area of the segment is, therefore, $S_\mathrm{u} = S_\mathrm{v} - S_\mathrm{t} = \alpha r^2 - ta$. We subtract the area of this segment two times (for each side) from the area of the circle with a radius r, so the area of the last cut is

$$S_2 = (\pi - 2\alpha)\,r^2 + 2ta = \left(\pi - 2\arctan\left(\frac{\sqrt{R^2 - 2a^2}}{a}\right)\right)\left(R^2 - a^2\right) + 2a\sqrt{R^2 - 2a^2}\,.$$

The ratio of work we are looking for is equal to the ratio of the areas, which is

$$\frac{S_2}{S_1} = \frac{\left(\pi - 2\arctan\left(\frac{\sqrt{R^2-2a^2}}{a}\right)\right)\left(R^2 - a^2\right) + 2a\sqrt{R^2 - 2a^2}}{\pi\left(R^2 - a^2\right)} \doteq 0.251\,.$$

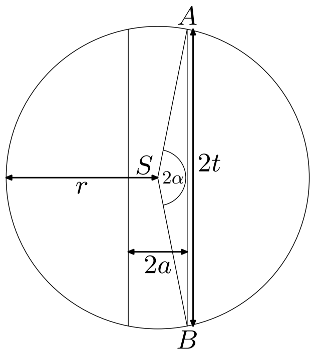


Fig. 1: The cut of the apple.

Jaroslav Herman
`jardah@fykos.org`

Problem DB ... emergency and problematic

"Fireman" jumps onto the pole, but suddenly the pole breaks off the ground, and he begins reassessing the situation. He will either fall with the pole to the floor, holding onto the top of the rod throughout the whole fall. Or he may let go of the bar and free fall from the initial height. What is the ratio of the impact velocities from the first and second situations? The length of the pole is l, its mass is M, and the mass of the fireman is m.

Consider zero initial velocity in both cases, and the pole rotates about an axis passing through the broken end just off the ground. The size of the "fireman" is negligible compared to the length of the pole. *Pepa wanted to call "fireman".*

Let us look at the first case, where the fireman holds onto the pole and falls with it. In this case, the whole system is in rotational motion around the base of the pole. We will calculate the impact velocity using the law of conservation of energy. In the beginning, the rod is perpendicular to the ground, so its center of gravity is at height $l/2$; and the firefighter's center of gravity is at height l. The potential energy of the whole system in the beginning is

$$E_0 = mgl + Mg\frac{l}{2}\,.$$

At the end of the fall, the total kinetic energy of the system is equal to

$$E = \frac{1}{2}I\omega^2\,,$$

where I is the moment of inertia of the entire system. We can calculate it as the sum of moments of inertia, both the fireman: $I_h = ml^2$ and the pole, which in case of rotation about one end equals $I_t = Ml^2/3$. Thus we get the equation

$$mgl + Mg\frac{l}{2} = \frac{1}{2}\left(ml^2 + \frac{1}{3}Ml^2\right)\omega^2\,.$$

Now, we can express the fall velocity of the fireman $v_1 = \omega l$:

$$2\frac{mgl + Mg\frac{l}{2}}{m + \frac{1}{3}M} = v_1^2\,,$$

$$v_1^2 = 2gl\frac{m + \frac{M}{2}}{m + \frac{M}{3}}\,,$$

$$v_1 = \sqrt{2gl\frac{m + \frac{M}{2}}{m + \frac{M}{3}}}\,.$$

In the second case, the fireman will let go of the rod immediately, i.e., he will experience free fall. We can once again use the law of conservation of energy. In the beginning, he has only potential energy mgh; and in the end, only kinetic energy

$$mgl = \frac{1}{2}mv_2^2\,,$$

$$v_2^2 = 2gl\,,$$

$$v_2 = \sqrt{2gl}\,.$$

So the ratio of the fall velocities in both cases is

$$\frac{v_1}{v_2} = \sqrt{\frac{m + \frac{M}{2}}{m + \frac{M}{3}}}\,.$$

Note that for a non-zero mass of the rod, we get that the fireman holding onto the pole will fall with greater impact velocity compared to the case when he lets go of it immediately.

Kateřina Rosická
kacka@fykos.org

Problem DC ... floating

Let us have a buoy made of a hollow sphere of outer radius $r = 51\,\mathrm{cm}$ and thickness $t = 1\,\mathrm{cm}$ whose material has density $\rho = 854\,\mathrm{kg{\cdot}m^{-3}}$. A rod of linear density $\lambda = 0.74\,\mathrm{kg{\cdot}m^{-1}}$ is attached to it on the outside perpendicular to the sphere's surface. In the buoy, there is a weight of mass $m_\mathrm{z} = 1.12\,\mathrm{kg}$ (of negligible dimensions), which is fixed inside the sphere on the opposite side as the rod.

The buoy floats on the water; the rod is initially in a vertical position above the sphere. What can be the maximum length of the rod l so that the buoy is stable, i.e. the buoy returns to its original position after a small deflection? *Martin and the mechanics exam.*

The position where the rod is vertically above or under the buoy is the equilibrium position. That follows from symmetry. However, one of these vertical positions is stable, while the other is unstable. A stable equilibrium is one, where there is (locally) a minimum of potential energy. In an unstable equilibrium, there is a maximum of potential energy. We need to express the potential energy as a function of the deflection of the rod with respect to the vertical direction above the buoy.

The mass of the whole buoy is constant, so the same volume of the buoy will be submerged at all times. Moreover, the rod will not go under the water when deflected by a bit, so the submerged part will always have the shape of a spherical cap of radius r and with a given volume. Therefore, even though the hollow sphere may shift a bit in the horizontal direction by tilting the buoy in the vertical direction, the sphere will remain at the same height. That implies two things – the potential energy of the sphere itself does not change because its center (which is also the center of gravity) does not change its height; the potential energy of the water in which the buoy floats does not change either.

It remains for us to calculate how the potential energy of the weight and the rod changes. The weight is fixed on the inside of the sphere at a distance $r - t$ from the sphere's center. If we put the zero point of the height to the center of the sphere, the potential energy of this weight will be $m_\mathrm{z} g(r - t)(-\cos\varphi)$, where φ is the angle that the rod makes with the vertically upward direction.

The mass of the rod will be $m_\mathrm{t} = l\lambda$, and the distance and its center of gravity will be in the center of the rod. The center of gravity will be at a distance $r + l/2$ from the center of the sphere, and its potential energy will be $l\lambda g(r + l/2)\cos\varphi$. The sum of the potential energy of the weight and the rod will be

$$E_\mathrm{p}(\varphi) = g(l\lambda(r + l/2) - m_\mathrm{z}(r - t))\cos\varphi\,.$$

The cosine has a local maximum in $\varphi = 0$; if we want a local minimum for the position when the rod direction is vertically upwards, the expression in the brackets must be negative. We get a condition

$$0 > \frac{\lambda}{2}l^2 + r\lambda l - m_z(r-t)\,.$$

The discriminant is $D = r^2\lambda^2 + 2\lambda m_z(r-t)$, therefore the roots are

$$l_{1,2} = \frac{-r\lambda \pm \sqrt{r^2\lambda^2 + 2\lambda m_z(r-t)}}{\lambda}\,,$$

where the negative root does not interest us and looking at the initial inequality, we see that the positive root gives us an upper bound for the length of the rod at which the vertically upward position will still be stable. This maximum length is

$$l_{\max} = -r + \sqrt{r^2 + 2\frac{m_z(r-t)}{\lambda}} = 82\,\mathrm{cm}\,.$$

Notes: Looking again at the inequality we derived, we see that this condition is equivalent to the condition that the resulting center of gravity is below the center of the sphere. That makes sense because when the buoy rotates, the center of gravity goes up, and its potential energy therefore increases.

How would we solve this through the forces? The important points would again be the center of gravity of the buoy and the center of buoyancy, which is the point at which we could place the center of buoyant force. This point is obviously (by symmetry) somewhere below the center of the sphere. So, when we deflect the buoy, the center of gravity and the center of buoyancy will not be directly under each other, and thus the forces acting in the two centers there will be spinning the buoy with their torque. In case the center of gravity is lower than the center of the sphere, this torque will counteract the deflection, and therefore it will be in a stable position and vice versa.

Šimon Pajger
legolas@fykos.org

Problem DD ... snowfall

Overnight, 6 cm *of snow fell, so in the morning, Jarda went to shovel it out of the sidewalk. He took a big shovel with a width of* $d = 60\,\mathrm{cm}$*, put it on the ground, and started pushing it in front of him with a velocity of* $v = 0.6\,\mathrm{m\cdot s^{-1}}$*. In this process, the snow with a density of* $\rho = 120\,\mathrm{kg\cdot m^{-3}}$ *accumulates on the shovel. The coefficient of friction between the shovel and the ground is* $f = 0.6$*; do not account for the shovel's mass. How long will it take Jarda to stop if he can exert a maximum force of* $F = 60\,\mathrm{N}$*? Consider that snow on the shovel moves with the shovel.* *Jarda's greenhouse is covered in snow…*

The snow shovel is subject to two forces. The first is a frictional force of magnitude $F_t = fF_N = fmg$, where m is the mass of snow on the shovel and g is the gravitational acceleration. In addition, Jarda must accelerate the accumulated snow to a velocity v, which requires another part of the force F_p. The mass m of the accumulated snow increases with time as $m = \rho h d v t$, so the corresponding force is

$$F_p = \frac{\mathrm{d}m}{\mathrm{d}t}v = \rho d h v^2\,,$$

where h is the height of the snowfall, d is the width of the shovel. We see that the force is constant over time.

Thus, the total force exerted by the Jarda is

$$F_\mathrm{p} + F_\mathrm{t} = \rho dhv\left(v + tgf\right) \,.$$

Putting $F = F_\mathrm{p} + F_\mathrm{t}$ gives us the time

$$t = \frac{F}{\rho dhvgf} - \frac{v}{gf} = 3.8\,\mathrm{s}\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem DE ... USA annexes

The border between Canada and the USA is said to contain the longest straight line, which length is $l = 2\,057\,\mathrm{km}$. However, this is not true because this part of the border runs along the forty-ninth circle of latitude. Calculate by how many kilometers the length of this "straight" part of the border would be reduced if it were a truly straight line (i.e., the shortest line) running along the surface of a sphere whose endpoints remained the same.

Matěj is fascinated by strange borders `https://www.youtube.com/shorts/caJeL1sjqJQ`.

It is not a straight line because the circles of latitude (except the equator) are not straight. If we wanted to walk along a circle of latitude, we would have to keep turning slightly. The forty-ninth circle of latitude is a circle with radius $r = R_\oplus \cos 49\,^\circ$. The current border between Canada and the USA is a circular arc with an angle of $\alpha = l/r$. The actual distance of the endpoints in 3D space is $d = 2r\sin(\alpha/2)$.

A straight line running along the surface of a sphere is always a circle centered at the center of the sphere and with radius $R_\oplus$. This is because it is the shortest path on the surface that connects the two points. To calculate the magnitude of the angle β of the arc of a truly straight border, we use the same formula as in the paragraph above, but in the inverse form $\beta = 2\arcsin\left(d/(2R_\oplus)\right)$. The length of this arc is, therefore, $\beta R_\oplus$. Putting it together, we get

$$\Delta l = l - 2R_\oplus \arcsin\left(\cos 49^\circ \sin\frac{l}{2R_\oplus \cos 49^\circ}\right) = 12\,\mathrm{km}\,.$$

Note: the notion of a straight line on a curved surface (e.g., the surface of a sphere) has a precise definition in differential geometry. These lines are called geodesics and have a special meaning in general relativity. Here, geodesics describe the trajectories of objects in 4D space that is curved by the presence of physical bodies (to give an idea, e.g., the motion of things falling into a black hole).

Matěj Mezera
`m.mezera@fykos.org`

Problem DF ... wind velocity

On a warm summer afternoon, Robo hides from a storm under a shelter where he stands at a distance $R = 1.2\,\mathrm{m}$ from the edge. At what velocity is the wind blowing against him if it is pouring down on him from his feet to his shoulders? The shelter's roof is at height $H = 2.3\,\mathrm{m}$, and Robo's shoulders are at height $h = 1.5\,\mathrm{m}$. Assume that the wind blows only horizontally and that the droplets are subject to air resistance (drag). Assume the droplets to be small spheres with radius $r = 0.8\,\mathrm{mm}$. The drag coefficient for a sphere is $C = 0.5$.

Rain is fine if you have a place to hide.

First, let us note that droplets move at a constant velocity. In the vertical direction, air resistance acts on them, and in the horizontal direction, they are carried by the wind.

When we draw a picture, we see that we get a right triangle with legs R and $H - h$ and a hypotenuse that characterizes the trajectory of the droplet. We'll denote the angle between the hypotenuse and leg R θ and note that the same angle is formed by the velocity components v_x in the horizontal direction and v in the direction of the droplet's trajectory. Then

$$\tan\theta = \frac{H-h}{R} = \frac{v_y}{v_x}\,,$$

where v_y is the steady velocity component in a vertical direction. This velocity is determined from the equilibrium of the gravitational force F_g of the droplet and the drag force F_o. Thus

$$\begin{aligned} F_g &= F_o\,,\\ mg &= \frac{1}{2}CS\rho v_y^2\,,\\ \frac{4}{3}\pi r^3\rho_\mathrm{w} g &= \frac{1}{2}C\pi r^2\rho v_y^2\,,\\ v_y &= \sqrt{\frac{8g\rho_\mathrm{w} r}{3C\rho}}\,, \end{aligned}$$

where we used the spherical shape of the droplet and its cross-section $S = \pi r^2$. The density of water is ρ_w, and the air density is ρ. We noted that the droplet's component v_x is purely made up of the wind velocity, and so we get a final expression for the wind velocity

$$v_x = \frac{R}{H-h}\sqrt{\frac{8g\rho_\mathrm{w} r}{3C\rho}} = 8.9\,\mathrm{m{\cdot}s^{-1}}\,.$$

Robert Jurenka
`robert.jurenka@fykos.org`

Problem DG ... race of shadows

Legolas was walking at night along a sidewalk parallel to street lights, observing the movement of his shadows on the ground. The lights, however, were of varying heights, so the shadows moved strangely. Lego wants to calculate the speed at which the two shadows cast by his head move relative to one another. Assume Legolas' head is a point at height $h_\mathrm{h} = 1.7\,\mathrm{m}$, moving at speed $v = 1.5\,\mathrm{m{\cdot}s^{-1}}$. Lamps are placed on a straight line that is $d = 2.1\,\mathrm{m}$ away from the line that Lego is walking on, and the bases of the lamps are $l = 13\,\mathrm{m}$ apart. Their heights *are* $h_1 = 3.1\,\mathrm{m}$ *and* $h_2 = 3.6\,\mathrm{m}$ *and think of them as point sources. Legolas is interested in the magnitude of the relative speed of the shadows when he is separated from the bases of both lamps equally.*

Legolas was walking down the sidewalk.

The triangle given by the head's shadow, Lego's head, and Lego's feet is similar to the triangle given by the head's shadow, the base of the lamp, and the point source itself. Hence their angles are identical.

If we project the whole situation into a plane perpendicular to Lego's direction, we see Lego's projection will not move at all; namely, Lego's feet will be away from the base of the lamp by d. Thus, the shadow in this projection will not move either and will be at distance $t_{1,2}$ from the lamp's base, where (from the similarity triangles)

$$\frac{t}{h_{1,2}} = \frac{t-d}{h_\mathrm{h}}\,,$$

$$d\frac{h_{1,2}}{h_{1,2}-h_\mathrm{h}} = t_{1,2}\,.$$

Thus, the velocities of the shadows are parallel to the direction in which Legolas is walking. Since the velocity of one shadow with respect to the other is given by the difference between these two vectors, we see that it is sufficient to subtract their magnitudes.

So what are those magnitudes? Again we use the similarity of triangles, this time as seen from the top view. We remember Lego's original position and the original position of the shadow and let Lego move. Then the triangle formed by the lamp and the Lego's original and new position is similar to the triangle formed by the lamp and the original and new position of the shadow. Since we also know that the ratio of the distance between the shadow and the lamp to the distance of the Lego and the lamp is $t_{1,2}/d$, we know that the ratio of shadow displacement to Lego's displacement is the same. Thus, the shadow of Lego's head moves $t_{1,2}/d$-times faster than Lego's, that is $v_{1,2} = vt_{1,2}/d$.

It only remains, to plug in the numbers and subtract these two magnitudes of velocity

$$\Delta v = v_1 - v_2 = \frac{v}{d}\left(t_1 - t_2\right) = \,,$$

$$\Delta v = \frac{v}{d}\left(d\frac{h_1}{h_1-h_\mathrm{h}} - d\frac{h_2}{h_2-h_\mathrm{h}}\right)\,,$$

$$\Delta v = v\frac{h_\mathrm{h}(h_2-h_1)}{(h_2-h_\mathrm{h})(h_1-h_\mathrm{h})}\,,$$

$$\Delta v = 0.48\,\mathrm{m\cdot s^{-1}}\,.$$

Šimon Pajger
`legolas@fykos.org`

Problem DH ... termoanemometer

To measure the speed of water flow in a tube, we can use a wire, which we place into the tube. We run a constant current $I = 200\,\mathrm{mA}$ *through the wire and measure the voltage on it. For volumetric flow* $Q = 70\,\mathrm{ml\cdot s^{-1}}$ *we measured the voltage* $U = 522\,\mathrm{mV}$. *We then increased the flow rate by* $\Delta Q = 8\,\mathrm{ml\cdot s^{-1}}$ *while keeping the same current, and the voltage dropped by* $\Delta U = 25\,\mathrm{mV}$ *due to the temperature change of the wire resistance. How did the average temperature* *of the fluid change after passing through the device compared to the first measurement? Also state whether the temperature is now higher or lower.*

Jarda has heard of a device with a strange name.

Let us denote the temperature of the water before entering the device by T_0. In the first case, the temperature of the water after exiting the device is

$$T_1 = T_0 + \frac{IU}{Q\rho c}\,,$$

where c is specific heat capacity of water and ρ its density. In the second case, it holds

$$T_2 = T_0 + \frac{I\,(U - \Delta U)}{(Q + \Delta Q)\,\rho c}\,.$$

We then subtract these two equations and obtain the result

$$T_2 - T_1 = \frac{I}{\rho c}\left(\frac{U - \Delta U}{Q + \Delta Q} - \frac{U}{Q}\right) = -5.2 \cdot 10^{-5}\,\mathrm{K}\,.$$

Jaroslav Herman
jardah@fykos.org

Problem EA ... clumsy Danka

Danka dropped a small ball with a mass $m = 10\,\mathrm{g}$ from a height $h = 1.3\,\mathrm{m}$ above the floor so that it landed on the "bosu" at a horizontal distance $x_0 = 5\,\mathrm{cm}$ from its axis of symmetry and bounced off it elastically. How far from the axis of the bosu did the ball fall after the bounce? The bosu is a rubber exercise mat shaped like a hemisphere with a radius of the base equal to $29\,\mathrm{cm}$. Assume that the bosu does not move on impact. *Danka's brother owns bosu.*

The ball's movement consists of a free fall, a perfectly elastic bounce from the bosu, and a parabolic projectile motion. Let's look at them one at a time. The height at which the ball bounces off the bosu indicates how much of its potential energy is converted into kinetic energy, giving us the initial velocity of the projectile motion. We shall introduce a coordinate system with the origin at the center of the base of the bosu. The x-coordinate defines the horizontal distance from the center of the bosu, and the y-coordinate determines the vertical distance. Let us denote by y_0 the height at which the ball bounces from the bosu. We can calculate it by applying Pythagorean theorem to the triangle in the figure as

$$y_0 = \sqrt{r^2 - x^2} \doteq 28.565\,7\,\mathrm{cm}\,.$$

Using the law of conservation of energy, we then calculate the velocity v_0 of the ball after bouncing as

$$mg(h - y_0) = \frac{1}{2} m v_0^2\,,$$
$$v_0 = \sqrt{2g(h - y_0)} \doteq 4.461\,1\,\mathrm{m{\cdot}s^{-1}}\,.$$

Next, we need to calculate the angle of the ball's velocity after bouncing off the bosu with respect to the horizontal plane. Let us denote the angle by θ. When we consider a perfectly elastic collision, the angles of incidence and ricochet are identical. We see in the figure that

$$90^\circ = 2\alpha + \theta\,.$$

We calculate the angle α using the aforementioned triangle as

$$\alpha = \arcsin\frac{x}{r}\,.$$

We get

$$\theta = 90^\circ - 2\arcsin\frac{x}{r} \doteq 70.14^\circ$$

Then we compute the components v_{0x} and v_{0y} of the velocity at the beginning of the projectile motion as

$$v_{0x} = v_0\cos\theta\,, \qquad v_{0y} = v_0\sin\theta\,.$$

Now we know both the initial coordinates and the initial velocity components. We can therefore write the equations of motion

$$x = x_0 + v_{0x}t\,, \qquad y = y_0 + v_{0y}t - \frac{1}{2}gt^2\,.$$

Time t starts when the ball bounces off the bosu. When the ball hits the ground, its y-coordinate is zero. From there, we get the quadratic equation

$$0 = y_0 + v_0\sin\theta t - \frac{1}{2}gt^2\,,$$

and consequently, the variable t

$$t = \frac{v_0\sin\theta \pm \sqrt{(v_0\sin\theta)^2 + 2gy_0}}{g} \doteq 0.918\,8\,\mathrm{s}\,.$$

Since we are looking for a positive value of time, the only real solution is the one with a positive sign in front of the square root. The final step is to plug this expression into the equation for x, and get the numerical result.

Ideally, the resulting equation for x should contain only the quantities defined in the problem statement. In this case, however, the relationship would be too intricate, and it is very easy to make a mistake when substituting numerical values in. Therefore, it is better to calculate the partial results of the crucial variables with sufficient precision, as we have indeed done while solving this problem, and plug these into the final formula. We arrive at the result that the ball hits the ground at distance $x = 1.44\,\mathrm{m}$ from the bosu's axis.

Daniela Dupkalová
`daniela@fykos.org`

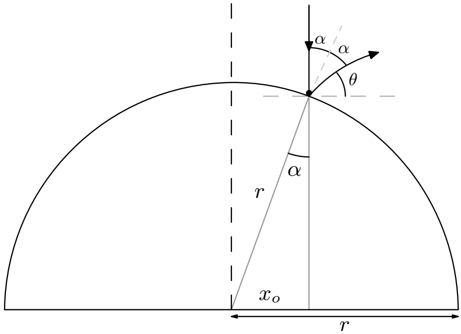
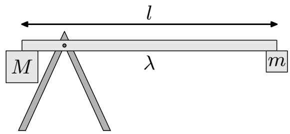


Fig. 2: A sketch of the situation.

Problem EB ... simplified trebuchet



The arm of a trebuchet has a length of $l = 9.14\,\mathrm{m}$, with the axis of rotation dividing it in a ratio of 1 : 5. The linear density of the arm is $\lambda = 10\,\mathrm{kg{\cdot}m^{-1}}$. For simplicity, assume that the weight and the projectile are mass points attached at the ends of the arm. The weight has a mass of $M = 15\,\mathrm{t}$ and is attached at the shorter end. The projectile has a mass $m = 60\,\mathrm{kg}$ and is attached at the longer end. What is the angular acceleration when the arm is horizontal?

Legolas does not need just pulleys...

Probably the most straightforward way to get the angular acceleration is to divide the total torque of the system by the total moment of inertia, all of which must be calculated with respect to the axis of rotation. Since both quantities are additive, we can calculate them by summing contributions from the weight, projectile, and arm.

The torque of the weight is simply $M_z = Mgl/6$. Similarly, the torque of the projectile is $M_\mathrm{p} = -5mlg/6$, where we used the minus sign to reflect that this torque tends to rotate the arm in the opposite direction as the weight. The torque of the arm itself also acts in the opposite direction. Its mass is $l\lambda$ and its center of gravity is at a distance $2l/6 = l/3$ from the axis of rotation. The torque of the arm is, therefore $M_\mathrm{r} = -g\lambda l^2/3$. The resulting torque acting on the arm with respect to the axis of rotation is

$$M_\mathrm{v} = M_\mathrm{z} + M_\mathrm{p} + M_\mathrm{r} = gl\left(\frac{M}{6} - \frac{5m}{6} - \frac{l\lambda}{3}\right) .$$

Moment of inertia of a point mass m, with respect to the rotational axis at distance r is mr^2, thus the moment of inertia of the weight is $J_z = Ml^2/36$ and the moment of inertia for the projectile is $J_p = 25ml^2/36$.

The moment of inertia of the arm about its center is $ml^2/12$. But we need the moment of inertia about the axis of rotation. We can get this using Steiner's theorem, i.e., by adding mr^2, where r is the distance between the center of gravity of the arm and the axis of rotation, in our case $l/3$. All in all, we get

$$J_r = J_0 + J_S = \frac{1}{12}\lambda l^3 + \frac{1}{9}\lambda l^3 = \frac{7}{36}\lambda l^3 .$$

The resulting moment of inertia is therefore calculated as

$$J = J_{\mathrm{z}} + J_{\mathrm{p}} + J_{\mathrm{r}} = \left(\frac{M}{36} + \frac{25m}{36} + \frac{7}{36}\lambda l\right) l^2 \,.$$

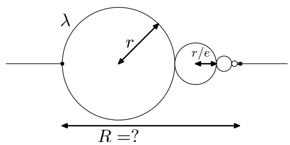
Now, we just need to substitute obtained results into the formula for angular acceleration

$$\varepsilon = \frac{M_{\mathrm{v}}}{J} = \frac{gl\left(\frac{M}{6} - \frac{5m}{6} - \frac{l\lambda}{3}\right)}{\left(\frac{M}{36} + \frac{25m}{36} + \frac{7}{36}\lambda l\right) l^2}$$

$$\varepsilon = 6\frac{g}{l}\,\frac{M - 5m - 2l\lambda}{M + 25m + 7\lambda l} = 5.5\,\mathrm{s}^{-2} \,.$$

Šimon Pajger
`legolas@fykos.org`

Problem EC ... repulsive 2D snowman



We have a conductive ring with linear resistance λ and radius r. We also have a ring of the same material but with a radius e*-times smaller. Surprisingly, we gradually found infinitely many more such rings, each with a radius of* $1/\mathrm{e}$ *compared to the previous one. We arrange them side by side in the shape of a snowman and connect them conductively. What will be the total resistance of this shape between the endpoints on the axis of symmetry?* *Do not disturb Jarda's circles.*

First, we determine the ratio between the two opposite sides of a circle of radius R. These are actually two resistors in parallel, each with a resistance $\pi r\lambda$. The total resistance of the ring is thus $R_0 = \pi r\lambda/2$.

This relationship holds for each of the rings. Once we connect them conductively in series, their total resistance is given by the simple sum. The only thing that changes for each ring is its radius. We get

$$R_{\mathrm{c}} = \frac{\pi r\lambda}{2}\sum_{n=0}^{\infty}\frac{1}{\mathrm{e}^n} = \frac{\pi r\lambda}{2}\,\frac{\mathrm{e}}{\mathrm{e}-1} \,,$$

where we used the formula for the sum of an infinite series.

Jaroslav Herman
`jardah@fykos.org`

Problem ED ... searching for a type II civilization

Organizers of FYKOS are flying through space, looking for new civilizations to where they could expand. So far, they have found only one ring that could be a remnant of a Dyson sphere. It has a radius of $R = 1.2 \cdot 10^8\,\mathrm{m}$, but the organizers are also interested in its mass. Therefore, they flew to the ring's center with their rocket and then moved perpendicularly to its cross-section. When they waited long enough, they discovered that their oscillatory period was $T = 60\,\mathrm{h}$. What is the total ring's mass if we assume that the mass is uniformly distributed and the amplitude of the oscillations is an order of magnitude smaller than R? *Pepa likes to swing.*

Firstly, we will calculate the force F exerted on the organizers when they are displaced by a small distance z in the direction of the ring's axis. A ring's element of mass $\mathrm{d}M$ will then exert a force with a magnitude $\mathrm{d}F = Gm\,\mathrm{d}M/(R^2+z^2)$ on the organizers. When summing these forces, the components in the ring's cross-section cancel out. Only the components perpendicular to the plane of the ring will remain, and for them, we can write

$$\frac{\mathrm{d}F_z}{\mathrm{d}F} = \sin\varphi = \frac{z}{\sqrt{(R^2+z^2)}}\,,$$

where φ is the angle, which is given by the line connecting the organizers to the element $\mathrm{d}M$ and the ring's cross-section. For $\mathrm{d}F_z$ the following holds

$$\mathrm{d}F_z = Gm\,\mathrm{d}M\frac{z}{(R^2+z^2)^{\frac{3}{2}}}\,.$$

In order to get the total force in the perpendicular direction to the ring's cross-section, we need to sum all these infinitesimal contributions. But notice that the force component $\mathrm{d}F$ is the same for all elements $\mathrm{d}M$. The sum(integral) is then drastically simplified, and we get the total force as

$$F_z = GmM\frac{z}{(R^2+z^2)^{\frac{3}{2}}}\,.$$

Now recall that the displacement z is small so we can neglect its second power in the denominator, and hence $F_z = GmMz/R^3$. The stiffness therefore is

$$k = \frac{GmM}{R^3}\,.$$

The final thing we need to do is to plug this formula into the equation for the period of a linear harmonic oscillator

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{R^3}{GM}}\,,$$

from where we can express the total mass of the ring as

$$M = 4\pi^2\frac{R^3}{GT^2} = 2.2\cdot10^{25}\mathrm{kg}\,.$$

Šimon Pajger
legolas@fykos.org

Problem EE ... Doppler on a stroll

Christian Doppler took a stroll. After a while, he noticed that in both directions (in his direction and the opposite direction), people are walking at velocity $v = 1.0\,\mathrm{m\cdot s^{-1}}$ and with $l = 4.0\,\mathrm{m}$ spaces between them. He decided to take advantage of this to find the speed at which he must move to meet as few people as possible. How fast should he walk to meet the least number of people in time $T \gg l/v$? Find all solutions. *Legolas is very neuroatypical.*

Let us denote v_D the speed of Doppler. The time between the meeting of the two bypassers will be

$$t_\mathrm{p} = \frac{l}{v+v_\mathrm{D}}\,.$$

Therefore, the frequency of the meeting of the bypassing people is

$$f_\mathrm{p} = \frac{1}{t_\mathrm{p}} = \frac{v + v_\mathrm{D}}{l}\,.$$

Similarly, the frequency of meeting people going in the same direction will be

$$f_\mathrm{r} = \frac{1}{t_\mathrm{r}} = \frac{|v - v_\mathrm{D}|}{l}\,.$$

The absolute value then divides the solution into two cases where the passers-by bypass[2] Doppler ($v_\mathrm{D} \leq v$), and the case where the Doppler bypasses the passers-by ($v_\mathrm{D} \geq v$).

The number of people the Doppler meets is the product of the total frequency of meetings and the time T. Total frequency means the sum of the frequencies of meeting people in one direction and the other. Actually, it is more like an average frequency, but since we only need the total number of people Doppler meets and we know he meets a lot of people (because $T \gg l/v$), this figure is perfectly enough.

To get rid of the absolute value, let us split the problem into the cases discussed above.

Passer-by bypass Doppler ($v_\mathrm{D} \leq v$)

The final frequency will be

$$f_\mathrm{v1} = \frac{v + v_\mathrm{D}}{l} + \frac{v - v_\mathrm{D}}{l} = 2\frac{v}{l}\,,$$

which is a constant independent of the speed of Doppler. Therefore, the total number of people he meets will be

$$N_1 = f_\mathrm{v1} T = 2\frac{v}{l} T\,.$$

Doppler bypasses the passers-by ($v_\mathrm{D} \geq v$)

The final frequency will be

$$f_\mathrm{v2} = \frac{v + v_\mathrm{D}}{l} + \frac{v_\mathrm{D} - v}{l} = 2\frac{v_\mathrm{D}}{l}\,.$$

Therefore, the total number of people he meets will be

$$N_2 = f_\mathrm{v2} T = 2\frac{v_\mathrm{D}}{l} T\,,$$

so if the speed is greater than v Doppler will always meet more than N_1 people. Thus, the minimum number of people he will meet in a given time is N_1, and he will meet that many if he walks at a speed less than or equal to v, so our solution (since we had to find all the solutions) is $[0, v]$ or $[-v, v]$ if we assume he can also go in the opposite direction.

It is worth noting that if we interpret the crowd of passers-by as a wave with wavelength l and wave speed v, i.e. with frequency $f_0 = v/l$, we get by expanding the expression for the frequency of meeting

$$f_\mathrm{p} = \frac{v + v_\mathrm{D}}{l}\frac{v}{v} = \frac{v + v_\mathrm{D}}{v} f_0\,,$$

which is the relation for the Doppler effect with a moving observer, as well as by analogy for f_r.

Šimon Pajger
`legolas@fykos.org`

[2] even if they travel at the speed

Problem EF ... pigeon doesn't fall far from the stone

A pigeon with a mass $M = 300\,\mathrm{g}$ is flying $h = 30\,\mathrm{m}$ above the ground at velocity $V = 20\,\mathrm{km{\cdot}h^{-1}}$. Suddenly, he is hit from behind (i.e., in the direction of his velocity) by a stone thrown from the ground with a mass $m = 100\,\mathrm{g}$, and the pigeon blacks out. The stone, which was at the highest point of its trajectory at that moment, flew with velocity $v = 30\,\mathrm{km{\cdot}h^{-1}}$ and lost half of its kinetic energy. Determine how far the pigeon's lifeless body falls from the spot where the rock was thrown.

Coo.

First, we determine the horizontal distance from the point from which the stone was thrown to the collision point. We know that at that moment, the stone was at the highest point of its trajectory, so the vertical component of its velocity was zero. From this, we can calculate the time from the moment the rock was thrown as

$$h = \frac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{2h}{g}}\,.$$

The horizontal component of the velocity will remain constant throughout this entire period, so the stone will travel the distance

$$d = v\sqrt{\frac{2h}{g}}\,.$$

in the horizontal direction. Now we focus on the collision. If the stone had lost half of its kinetic energy, it moved after the collision with a velocity v', which we determine as

$$\frac{1}{2}mv'^2 = E_\mathrm{k}' = \frac{1}{2}E_\mathrm{k} = \frac{1}{2}m\left(\frac{v}{\sqrt{2}}\right)^2 \quad \Rightarrow \quad v' = \frac{v}{\sqrt{2}}\,.$$

Therefore, from the law of conservation of momentum for the velocity of a lifeless body of the pigeon V, the following will hold

$$MV + mv = MV' + m\frac{v}{\sqrt{2}} \quad \Rightarrow \quad V' = V + \frac{mv}{M}\frac{2-\sqrt{2}}{2}\,.$$

At this velocity, the pigeon will fly for $\sqrt{2h/g}$, for a total horizontal distance

$$s = \left(V + \frac{mv}{M}\frac{2-\sqrt{2}}{2}\right)\sqrt{\frac{2h}{g}}\,.$$

So the answer to the question in the problem statement is

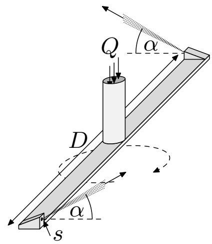
$$s + d = \left(V + \frac{mv}{M}\frac{2-\sqrt{2}}{2} + v\right)\sqrt{\frac{2h}{g}} = 36\,\mathrm{m}\,.$$

Vojtěch David
vojtech.david@fykos.org

Problem EG ... dishwasher

Inside the dishwasher is a rotating propeller, into which water flows with the volumetric flow rate Q through its center. The water causes the propeller to rotate in a horizontal plane and flows radially inside it in tubes with cross-sectional area S on both sides. At the ends of the propeller are holes with cross-sectional area s, through which the water sprays and washes the dishes. The direction of the water is tangential to the rotation and forms an angle α with the horizontal plane. What is the angular frequency of the propeller if its diameter is D?

Jarda cleaned the dishes after party, and his head was still spinning.



Let us first consider the system associated with the propeller. Water flows into the propeller with a flow rate of Q, and as usual, all water that flows into the propeller along the axis of symmetry must also flow out through the holes at the ends. Let us denote the speed at which water flows out as v and the cross-sectional area of one hole as s. Thus, the flow rate from each hole at the end is

$$v_{\text{out}} = \frac{Q}{2s}\,,$$

since Q is the flow rate for both halves of the propeller.

By changing the direction of the water from radial to tangential at the ends of the propeller, the water changes its angular momentum. And consequently changing the angular momentum of the propeller. The torque, i.e., the change in the angular momentum of the whole propeller, is

$$M_1 = \frac{\mathrm{d}L_1}{\mathrm{d}t} = 2\frac{RQ\cos\alpha}{2s}\frac{Q\rho}{2}\frac{\mathrm{d}t}{\mathrm{d}t} = \frac{Q^2\rho R\cos\alpha}{2s}\,.$$

However, in our reference frame, the Coriolis force acts on the water flowing in the tube. It acts perpendicular to the direction of the water flow, thus creating an angular momentum. On the element of the water with a length $\mathrm{d}x$, which is at a distance x from the center, acts a torque with magnitude $\mathrm{d}M_2 = x2\omega\frac{Q}{2S}S\rho\,\mathrm{d}x$. Here $Q/2S$ denotes the radial velocity of the water in the propeller. The total torque due to the Coriolis force is

$$M_2 = 2\int_0^R x\omega Q\rho\,\mathrm{d}x = R^2\omega Q\rho\rho,\,.$$

Since we are interested in a steady state of the system and these two torques act against each other, by equating them: $M_1 = M_2$, we can obtain the rotation frequency as

$$\omega = \frac{Q\cos\alpha}{Ds}\,,$$

where we substituted for $R = D/2$.

Jaroslav Herman
`jardah@fykos.org`

Problem EH ... watering the flower bed

Jarda's garden hose ends with a "gun" that divides the water beam into many smaller ones. Suppose that these small beams originate from a hemispherical cap at the end of this gun and that water gushes from all its points at the same velocity in a direction perpendicular to the tangent plane at that point. When Jarda lets the water squirt out of the gun pointing straight up, holding it at a height h, the water hits the surface A. How large of an area will the water fall on if he points the gun horizontally? The radius of the cap is small compared to the other sizes.

The garden is a highly inspiring environment for Jarda.

When Jarda holds the hose upwards, the water falls on the surface A, which is a circle of radius r. We can define this radius as the maximum possible distance that water can reach for each nozzle of a single hose with the same velocity at the same height h. Assume that this distance is reached for some inclination angle φ_0. The points with this angle form a circle on the hemispherical cap. When Jarda turns the hose such that it points horizontally, only half the number of points of this circle will be at the current position of the cap. At the same time, the water in that semi-plane will splash above and below the gun, but no longer backward. The resulting area on which the water falls will therefore be half of a circle with the same radius r, so the area will be $A/2$.

Kateřina Rosická
kacka@fykos.org

Problem FA ... too fast electron

What atomic number would an element have to have for the ground state speed of an electron in a shell in order to exceed the speed of light, if we do not consider relativistic corrections? Use the Bohr model of the atom and assume that the atom is ionized to exactly one electron.

Jarda can't keep track of how much work there is in FYKOS.

In the Bohr model of the atom, we assume that the electron orbits the nucleus of the atom along a path that is curved by the electrostatic interaction between the nucleus and the electron. This results in a circular motion similar to the orbit of the planets around the Sun.

However, as observed, atoms can emit light only at certain wavelengths. The energy of this emitted light depends on the change in the radius of the trajectory. Since the spectrum of these radiated energies is discrete, electrons can orbit the nucleus only at certain discrete distances.

So much for the introduction. In the Bohr model of the atom, it is postulated that electrons orbit only on orbits on which the magnitude of the angular momentum is $L = n\hbar$, where n is the number of the shell on which the electron orbits, and $\hbar = h/2\pi$ is the reduced Planck constant. For the ground state of the electron we have $n = 1$.

In a force interaction, the magnitude of the centrifugal force is equal to the attractive electrostatic interaction

$$F_\mathrm{d} = m_\mathrm{e} \frac{v^2}{r} = F_\mathrm{e} = \frac{1}{4\pi\varepsilon_0} \frac{Ze^2}{r^2}\,,$$

where m_e is the mass of the orbiting electron, v is the orbital velocity, r is the distance from the nucleus, Z is the number of protons in the nucleus, ε_0 is vacuum permitivitty, and e is the elementary charge.

Now, we use the aforementioned condition for the magnitude of the angular momentum, which is

$$L = m_\mathrm{e} r v = n\hbar\,.$$

From this equation, we express r and insert it into the previous equation. This gives us

$$v = \frac{Ze^2}{4\pi\varepsilon_0 n\hbar}\,.$$

After substituting for $v = c$, $n = 1$ and expressing Z. We obtain

$$Z = \frac{4\pi\varepsilon_0\hbar c}{e^2} \doteq 137.065\,.$$

The inverse of this result is called the fine-structure constant. Now we need to proceed carefully – the charge of the nucleus has to be an integer. But if we substitute $Z = 137$ into the formula for the speed, we get the orbital velocity $v = 2.997 \cdot 10^8\,\mathrm{m{\cdot}s^{-1}}$ which is, however, a lower value than $c = 2.998 \cdot 10^8\,\mathrm{m{\cdot}s^{-1}}$. Therefore, the charge we are looking for has to be even higher, hence the answer is $Z = 138$.

An element with so many protons has not been observed yet. Moreover, according to the assumptions made in the problem statement, it would have to be 137 times ionized, which is a lot. Furthermore notice that the result is not far away from the atomic numbers of the heavy elements, for which special relativity has to be accounted for.

Finally, let us note that the Bohr model of the atom gives, in first approximation, the same structure of the electron shell as more advanced quantum models, but it does not describe the physical nature correctly.

Jaroslav Herman
jardah@fykos.org

Problem FB ... annihilation peak

Far from a material with a thickness t, we will place a source of gamma rays with an energy of approximately $2\,\mathrm{MeV}$. The radiation has an absorption coefficient of mu_1 in the material. For radiation of this energy can happen that a photon converts into an electron-positron pair inside the material (the loss in intensity due to this effect is already included in the μ_1 coefficient). However, the positron immediately annihilates with an electron and produces 2 gamma flashes, each with energy $E_\gamma = 511\,\mathrm{keV}$. The absorption coefficient of the material for the radiation with this energy is μ_2. We place a detector far behind the material and observe the intensity of radiation of the energy E_γ. For what thickness of material t will this intensity be the largest?

Hint: The infinitesimal loss of gamma-ray intensity in a material is proportional to the magnitude of the intensity, the absorption coefficient, and the unit of distance.

Jarda keeps losing photons somewhere.

From the hint, we derive the relation according to which the intensity I (i.e., the number of photons in the radiation beam) decreases as a function of the distance in the material. Namely, $-\,\mathrm{d}I = \mu_1 I\,\mathrm{d}x$, which is a differential equation that we solve by separation of variables to get the expected relation

$$I(x) = I_0 \mathrm{e}^{-\mu_1 x}\,,$$

where I_0 is the intensity before entering the material and x is the depth at which the intensity is calculated.

Some of this radiation is converted into an electron-positron pair. We assume that the probability of this effect does not depend on the position of the material, so the intensity of the E radiation is proportional only to $I(x)$. Thus, most of this radiation is produced at the beginning of the material, and the least at the end.

Once a gamma ray of energy E is produced and begins propagating through the material, it is also attenuated, now with a coefficient of μ_2. Let the intensity of this radiation $\mathrm{d}J_0$ occur in the $\mathrm{d}x$-neighbourhood of the x coordinate. According to the formula above, this intensity drops after exiting the material to

$$\mathrm{d}J = \mathrm{d}J_0 \mathrm{e}^{-\mu_2(t-x)}\,,$$

since the radiation has now passed a distance $t - x$ in the material.

But the resulting intensity $\mathrm{d}J_0$ is proportional to $I(x)$ and the length $\mathrm{d}x$ through some proportionality coefficient α, which we write as

$$\mathrm{d}J_0 = \alpha I(x)\,\mathrm{d}x = \alpha I_0 \mathrm{e}^{-\mu_1 x}\,\mathrm{d}x\,.$$

Substituting into the previous equation and integrating over the entire thickness of the material gives the total radiation intensity with energy E as

$$J = \alpha I_0 \int_0^t \mathrm{e}^{-\mu_1 x} \mathrm{e}^{-\mu_2(t-x)}\,\mathrm{d}x = \alpha I_0 \mathrm{e}^{-\mu_2 t} \int_0^t \mathrm{e}^{(\mu_2-\mu_1)x}\,\mathrm{d}x = \frac{\alpha I_0 \mathrm{e}^{-\mu_2 t}}{\mu_2 - \mu_1}\left(\mathrm{e}^{(\mu_2-\mu_1)t} - 1\right)\,.$$

The maximum intensity is found by a derivative with respect to t

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \frac{\alpha I_0}{\mu_2 - \mu_1}\left(-\mu_2 \mathrm{e}^{-\mu_2 t}\left(\mathrm{e}^{(\mu_2-\mu_1)t} - 1\right) + \mathrm{e}^{-\mu_2 t}\left(\mu_2 - \mu_1\right)\left(\mathrm{e}^{(\mu_2-\mu_1)t}\right)\right)\,,$$

which we set equal to zero. We get

$$0 = -\mu_2 \mathrm{e}^{(\mu_2-\mu_1)t_\mathrm{m}} + \mu_2 + \left(\mu_2 - \mu_1\right)\mathrm{e}^{(\mu_2-\mu_1)t_\mathrm{m}}\,,$$

from which

$$\mu_1 \mathrm{e}^{(\mu_2-\mu_1)t_\mathrm{m}} = \mu_2\,,$$

and now we can simply express the thickness for which the intensity is extremal as

$$t_\mathrm{m} = \frac{1}{\mu_2 - \mu_1}\ln\left(\frac{\mu_2}{\mu_1}\right)\,.$$

Looking at the relation $J(t)$, it can be seen that J is positive for all (positive) t, and that for points $t = 0$ and $t = \infty$, J decreases to zero. Thus, in computing the zero derivative of $J(t)$, we have indeed found the maximum of this function.

Regardless of which of the absorption coefficients is larger, it is clear that the thickness t_m is positive. If μ_2 is close to μ_1, our relation also holds in this limit.

Jaroslav Herman
`jardah@fykos.org`

Problem FC ... evil chandelier

When Jarda is at home, he often happens to bump his head against the chandelier in the kitchen. The chandelier has the shape of a cylindrical shell without bases, a height of $h = 27\,\mathrm{cm}$, and a radius of $R = 12\,\mathrm{cm}$. In the upper part of the chandelier, a massless bar runs across its diameter. At the center of this bar, a rope by which the chandelier hangs from the ceiling is attached. The length of the rope is $l = 42\,\mathrm{cm}$. What is the period of the small oscillations of the chandelier when Jarda deflects it?

To find the period of the small oscillations we use the relation for a physical pendulum

$$T = 2\pi\sqrt{\frac{J}{mgd}}\,,$$

where J is the moment of inertia of the oscillating body with respect to the axis of rotation, m is the mass of the body, g is the gravitational acceleration and d is the distance of the center of gravity from the axis of rotation. The distance of the center of gravity from the axis of rotation is clearly $d = l + h/2$.

The moment of inertia with respect to the axis of rotation is given by Steiner's theorem as

$$J = J_T + md^2\,,$$

where J_T is the moment of inertia with respect to the axis running through the center of gravity of the body and has to be determined by integration.

Consider a thin circular element of the chandelier of thickness $\mathrm{d}x$ at a distance of x from the center of the chandelier and mass $\mathrm{d}m = m\,\mathrm{d}x/h$, where m is the mass of the whole chandelier. Relative to the center and the axis passing perpendicularly to the axis of symmetry, this element has a moment of inertia

$$\mathrm{d}J_T = \frac{1}{2}R^2\,\mathrm{d}m + x^2\,\mathrm{d}m\,,$$

where we used again Steiner's theorem. The factor $1/2$ appears due to the moment of inertia of the thin ring with respect to the axis perpendicular to the axis of symmetry.

By integration we obtain

$$J_T = \frac{m}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{1}{2}R^2 + x^2\right)\mathrm{d}x = \frac{m}{h}\left[\frac{1}{2}R^2x + \frac{1}{3}x^3\right]_{-\frac{h}{2}}^{\frac{h}{2}} = m\left(\frac{1}{2}R^2 + \frac{1}{12}h^2\right)\,.$$

Thus, the solution to the problem is

$$T = 2\pi\sqrt{\frac{\frac{1}{2}R^2 + \frac{1}{12}h^2 + \left(l + \frac{h}{2}\right)^2}{g\left(l + \frac{h}{2}\right)}} = 1.53\,\mathrm{s}\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem FD ... sphere in the road curve

Let us have a horizontal circular surface with the radius R. An edge of height h is raised around this circle. A sphere of mass m and radius r rolls along the edge while touching it. The friction between the sphere and all other surfaces is large enough not to make the sphere slip. Determine the kinetic energy of the sphere if one revolution in this enclosure takes time T. It holds that $r < h$ and $R > r$. *You do not need a steering wheel to turn.*

We have described the kinematics of the sphere. Let us denote the angular velocity of the center of gravity's movement as $\omega = 2\pi/T$. The center of gravity moves on a circle with a radius $R-r$; thus, its velocity is $v = \omega\,(R-r)$.

Now let us switch to a system that moves with velocity v in the direction of motion of the sphere. Let us draw the axis z perpendicular to the bottom plane, the axis x from the circle's center towards the sphere.

Consider rotation along the bottom plane. The sphere is stationary, and the bottom plane moves with velocity v. Since the sphere is moving without slipping, the velocity of the points on edge must be also v. Let us denote the angular velocity of rotation along the x axis as ω_x, then the following is true

$$\omega_x r = v = \omega\,(R-r)\ .$$

Alternatively, let us just consider rotation relative to the edge. The sphere is stationary, and the edge moves with a velocity ωR. Using the same argument, we get that the angular velocity of rotation with respect to the z axis is

$$\omega_z r = \omega R\,.$$

Therefore, the magnitude of the angular velocity vector will be $\sqrt{\omega_x^2+\omega_z^2}$. Since the sphere has the same moment of inertia with respect to all its axes, we can express the rotational kinetic energy as

$$E_{\mathrm{k,r}} = \frac{1}{2}J\left(\omega_x^2+\omega_z^2\right) = \frac{1}{2}\frac{2}{5}mr^2\omega^2\left(\left(\frac{R-r}{r}\right)^2+\left(\frac{R}{r}\right)^2\right) = \frac{1}{2}\frac{2}{5}m\omega^2\left(2R^2-2Rr+r^2\right)\ .$$

This solves the rotation of the sphere, and we only need to add the kinetic energy of the translation to it, from which we get

$$E_{\mathrm{k}} = \frac{1}{2}mv^2 + E_{\mathrm{k,r}} = \frac{2\pi^2 m}{5T^2}\left(9R^2-14Rr+7r^2\right)\ .$$

Jaroslav Herman
`jardah@fykos.org`

Problem FE ... we will prevent armageddon

In August 2022, NASA conducted a test to see if a probe impacting an asteroid could affect its orbit. The chosen object was Dimorphos, a small asteroid orbiting its larger colleague called Didymos. The probe with a mass of 570 kg *shortened Dimorphos' orbital period around Didymos by 32 minutes. What's the maximum change in Didymos' orbital period if this probe crashed directly into Didymos? This asteroid orbits the Sun with a semi-major axis* 1.64 au *and* *eccentricity* 0.384*, has a mass of* $5.2 \cdot 10^{11}$ kg*, and the probe would hit the surface with a velocity of* $22\,000\,\mathrm{km{\cdot}h^{-1}}$*. The mass of Dimorphos (which was, by the way, discovered at Ondřejov Observatory here in the Czech Republic) is neglected in this problem. Assume no material was ejected from Didymos due to the collision.* *Jarda wants to move something heavy.*

We first state Kepler's third law, from which we get the period of Didymos T as

$$T = T_{\oplus}\sqrt{\frac{a^3}{a_{\oplus}^3}}\,,$$

where $T_{\oplus} = 1\,\mathrm{year}$ is the period of the Earth, a the lengh of Didymos' semi-major axis and $a_{\oplus} = 1\,\mathrm{au}$ the semi-major axis of the Earth's orbit. Thus we see that the period is proportional to the semi-major axis. Since the probe is much lighter than the asteroid, we can assume that the impact will only slightly change the asteroid's orbit. We, therefore, differentiate this relation to obtain the dependence of the change in the period on the change in the semi-major axis of Didymos' orbit

$$\Delta T = \frac{3}{2}T_{\oplus}\sqrt{\frac{a^3}{a_{\oplus}^3}}\frac{\Delta a}{a}\,.$$

As we know, the conservation of energy law applies for motion in a gravitational field. The total energy of an object (neglecting its mass relative to the central body) per unit of its mass is $E = -GM_{\odot}/(2a)$, where G is the gravitational constant, $M_{\odot}$ is in our case the mass of the Sun and a is the semi-major axis. Thus we see there is a simple relation between energy per unit of mass and the semi-major axis. We can also calculate this energy as the sum of a potential and a kinetic energy per unit of mass

$$E = \frac{1}{2}v^2 - \frac{M_{\odot}G}{r}\,.$$

By comparing these two parts we get

$$-\frac{GM_{\odot}}{2a} = \frac{1}{2}v^2 - \frac{M_{\odot}G}{r}\,.$$

This equation has to hold before, as well as after the collision but with different a and v, because r is not changed by the collision. Therefore, we can express the change in the semi-major axis using the change in the asteroid's speed by differentiating this equation

$$\frac{GM_{\odot}}{2a^2}\Delta a = \mathbf{v}\cdot\Delta\mathbf{v}\,.$$

To maximize the change in the semi-major axis, we require $\mathbf{v}$ and $\Delta\mathbf{v}$ to point in the same or the opposite direction.

Let us now consider for a moment the change in Didymos' velocity $\Delta\mathbf{v}$ due to the collision. We can find that using the law of conservation of momentum. This change in velocity will be the same as in the reference frame of the Sun and the whole planetary system, as well as in the frame of reference where Didymos is originally at rest. There the following holds

$$m\mathbf{u} = (m + M)\,\Delta\mathbf{v}\,.$$

Now we can insert into the previous relation

$$\frac{GM_{\odot}}{2a^2}\Delta a = v\frac{mu}{M + m}\,.$$

At the right-hand side of the equation, we can neglect m in the denominator relative to M. The change of the semi-major axis is therefore

$$\Delta a = \frac{2muva^2}{GM_{\odot}M}\,.$$

Now we want to maximize v so that Δa is as large as possible. The asteroid has a maximum magnitude of velocity when it is closest to the Sun, which occurs at a distance $a\,(1-e)$. There, the magnitude of its velocity is

$$v = \sqrt{\frac{M_{\odot}G}{a}\left(\frac{2}{1-e}-1\right)} \doteq 35\,\mathrm{km{\cdot}s^{-1}}\,.$$

By substituting for all unknown parameters we get

$$\Delta T = \frac{3}{2}T_{\oplus}\sqrt{\frac{a^3}{a_{\oplus}^3}}\frac{2mavu}{GM_{\odot}M} = 0.086\,\mathrm{s}\,.$$

The orbital period of Didymos around the Sun would change by 0.086 s. For simplicity, we did not consider the effect of material ejection due to the probe hitting the surface of the body in the problem statement. However, the fragments produced by the impact would also have some momentum, which on average would be in the opposite direction to the probe. Thus, the change in momentum (and, therefore, as a consequence, the orbital period) would be higher.

Jaroslav Herman
jardah@fykos.org

Problem FF ... two rings

Two rings made out of a thin conductive wire lie on a common axis at a distance of $z = 15\,\mathrm{cm}$ from each other. One ring has a radius of $a = 5\,\mathrm{cm}$, while the other has a radius of $b = 2\,\mathrm{mm}$, you can therefore consider that $b \ll a$. Of course, both rings can act as coils. Determine the mutual inductance of the rings. *Jindra wears two rings on one finger.*

The mutual inductance is the same whether the first ring acts on the second one or vice versa. The magnetic induction on the axis of the bigger ring at a distance of z is

$$B = \frac{\mu_0 I_1 a^2}{2(z^2+a^2)^{3/2}}\,,$$

where I_1 is the current in the bigger ring. Since the second ring has a significantly smaller radius than the first ring, the magnetic flux through the second ring can be approximated as

$$\Phi_{12} = B\pi b^2 = \frac{\pi\mu_0 I_1 a^2 b^2}{2(z^2+a^2)^{3/2}}\,.$$

The mutual induction of the two coils is then

$$M = \frac{\Phi_{12}}{I_1} = \frac{\pi\mu_0 a^2 b^2}{2(z^2+a^2)^{3/2}} = 5\cdot 10^{-12}\,\mathrm{H}\,.$$

Jindřich Jelínek
jjelinek@fykos.org

Problem FG ... stone doesn't fall far from pigeon

A pigeon with a head at height $h = 20\,\mathrm{cm}$ is eating bread crumbs from the ground. Suddenly, a rock hits the ground at a distance r from its head, and the pigeon flies away with a velocity $v = k/r^2$ away from the rock (the direction is taken from the point of the impact to the pigeon's head), where k is a constant. Determine the value of r for the pigeon to fly as far as possible from its initial position if it exerts no additional force to stay in the air as it moves.

Vojta really doesn't mind pigeons.

The pigeon's flight is nothing but an oblique throw. A distance reached by a mass point thrown at an angle α is given by

$$d = \frac{v^2}{g} 2 \sin\alpha \cos\alpha$$

for initial velocity $v = k/r^2$. The sine of the angle α is determined from simple goniometry as h/r and the cosine as $\sqrt{r^2 - h^2}/r$. Plugging everything into the equation, we obtain

$$d = 2\frac{k^2 h}{g} \frac{\sqrt{r^2 - h^2}}{r^6} = 2\frac{k^2 h}{g} \sqrt{\frac{r^2 - h^2}{r^{12}}} \,.$$

So, we just need to find the maximum (which, by physics intuition, should exist) of the expression

$$\mathcal{D} = \frac{r^2 - h^2}{r^{12}} \,.$$

We set the derivative of the expression equal to zero, resulting in

$$\mathcal{D}' = \frac{12h^2 - 10r^2}{r^{13}} = 0 \quad \Rightarrow \quad r_{\max} = \frac{\sqrt{30}}{5} h \,,$$

which gives us that the pigeon flies the farthest when the rock hits at a distance $r_{\max} \doteq 21.9\,\mathrm{cm}$.

Vojtěch David
`vojtech.david@fykos.org`

Problem FH ... love and truth will overcome lies and hatred

Freedom of speech has a speed $v = 1.00\,\mathrm{m\cdot s^{-1}}$ and flies through space toward a better society. However, it hits hard against the immovable wall the local regime has erected against freedom and elastically reflects. In the opposite direction, however, it travels only the distance $l_1 = 1.0\,\mathrm{m}$ before it hits a second wall, which is approaching the first wall at speed $u = 0.01\,\mathrm{m\cdot s^{-1}}$. The regime began to repress freedom of speech. Free speech is elastically reflecting from the walls and trying to break free from this clench. Since the world is supposed to end well, it will break its prison if it starts to apply an average force $F = 10\,\mathrm{N}$ to the moving wall. The question, however, is how long it will take to break out after the first hit. Think of the force F as the average change in momentum for two successive impacts over the time elapsed between them. The mass of freedom is $m = 1.0\,\mathrm{g}$.

Jarda feels under pressure.

Let us denote by Δp_n the change in momentum of the free speech at the nth impact to the moving wall, and t_n the time that elapses between nth and $n+1$st impact into the moving wall. Then according to the problem statement, the average force is

$$F_n = \frac{\Delta p_n + \Delta p_{n+1}}{2t_n} \,.$$

Now we have to calculate p_n and t_n.

We solve the whole problem in a system where the speed of freedom was v. After the first impact into the immutable wall, the freedom of speech moves with the same speed in the opposite direction. After each reflection from the moving wall, the speed increases by $2u$. We easily justify that by moving to the system where this wall is at rest. Here, the speed of freedom of speech is $v+u$, after the elastic reflection is the speed just reversed. This system, however, was moving at a speed u compared to the original one, so after the reflection is the speed of freedom of speech in the original system $v+2u$. This is true for every reflection on this wall, therefore the speed after the nth reflection from the moving wall is $v+2nu$.

We calculate the time elapsed between two reflections on this moving wall. We denote the distance between walls after the nth reflection by l_n (we know the distance l_1 after the first impact from the problem statement). For the $n+1$st reflection the following condition must hold

$$(v+2nu)\,t_n + u\ t_n = 2l_n \quad \Rightarrow \quad t_n = \frac{2l_n}{v+(2n+1)\,u}\,,$$

which determines when the next reflection occurs.

We see that t_n depends on l_n, so we have to find it. Between the nth and the following reflection the distance between the two walls decreases by ut_n, so

$$l_{n+1} = l_n - u\ t_n = l_n \frac{v+(2n-1)\,u}{v+(2n+1)\,u}\,.$$

Let us write this expression also for l_n as

$$l_n = l_{n-1}\frac{v+(2n-3)\,u}{v+(2n-1)\,u}$$

and let us insert it into the equation above. We get

$$l_{n+1} = l_{n-1}\frac{v+(2n-3)\,u}{v+(2n-1)\,u}\frac{v+(2n-1)\,u}{v+(2n+1)\,u} = l_{n-1}\frac{v+(2n-3)\,u}{v+(2n+1)\,u}\,.$$

Several elements have been eliminated. We can iterate this procedure and we get

$$l_n = l_1\frac{v+u}{v+(2n-1)\,u}\,.$$

Now we know how t_n depends on n. Let us now express Δp_n as

$$\Delta p_n = m\left(v+2nu-\left(-v-2\left(n-1\right)u\right)\right) = m\left(2v+2u\left(2n-1\right)\right)\,.$$

Then we get the final relation for the force F_n as

$$F_n = 2m\frac{v+2nu}{t_n} = m\frac{\left(v+(2n-1)\,u\right)\left(v+2nu\right)\left(v+(2n+1)\,u\right)}{l_1\left(u+v\right)}\,.$$

Now we check the values given to us. For small n we can neglect the elements with u compared to v and we get a force on the order of 10^{-3} N. Presumably then, the n needs to be very large for the freedom of speech to exert such a large average force. But then we can put $2n-1 \approx 2n+1 \approx 2n$ and express

$$F_n \doteq m\frac{\left(v+2nu\right)^3}{l_1\left(u+v\right)} \quad \Rightarrow \quad n \doteq \frac{\sqrt[3]{\frac{Fl_1(u+v)}{m}} - v}{2u} = 1\ 031\,.$$

If we insert $n = 1\ 030$, we get $F_{1\ 030} \doteq 9.98\,\mathrm{N}$, for $n = 1\ 031$ it is already $F_{1\ 031} \doteq 10.01\,\mathrm{N}$. Therefore, there must be $n = 1\ 031$ reflections on the moving wall.

The time we are looking for is then obtained from the knowledge of l_n and the motion of the wall as

$$t = \frac{l_1 - l_{1031}}{u} + \frac{l_1}{v} = \frac{l_1}{u} \frac{2\,(1\ 031 - 1)\,u}{v + (2 \cdot 1\ 031 - 1)\,u} + \frac{l_1}{v} \doteq 96\,\mathrm{s}\,.$$

Note also that for $nu \gg v$ is $ml'v' = ml_1 v$, where $v' = 2nu$ and l' is the distance of the walls at the speed of freedom of speech v'. Thus, the law of conservation of some kind of angular momentum holds.

Jaroslav Herman
jardah@fykos.org

Problem GA ... running princess

The princess was walking around the castle when she suddenly noticed a spider crawling up her veil at $u = 1.00\,\mathrm{m\cdot s^{-1}}$ towards her. She screamed terribly (which of course didn't help her at all) and started running away at $v = 3.00\,\mathrm{m\cdot s^{-1}}$ (which also didn't help her at all, as she was dragging the veil behind her...). She ran through the door like that, which slammed shut just behind the spider. At that point, $l_0 = 3.00\,\mathrm{m}$ of the veil remained between the spider and the princess.

Both the princess and the spider continued to move, and the slammed veil began to stretch perfectly. The princess thought she could escape the spider because she was running faster than it, but she was wrong. In what time t_f will the spider catch up with the princess? Think of the spider as a point. *Legolas likes spiders.*

With this problem, the hardest part is finding a way to approach it mathematically. In my opinion, the most efficient way is to introduce the quantity p, which we will use to denote the ratio of the section of the veil that the spider has already passed and the current length of the veil, where we denote the current length of the veil l and $l = l_0 + vt$ will hold, where t is the time since the door was slammed.

In the beginning, $p = 0$. The spider catches up with the princess when $p = 1$.

This approach is advantageous because p is not directly affected by the princess's running since the veil extends the same along its entire length, and thus if the spider had remained stationary, p would not have changed at all.

However, when the spider moves, it travels a distance $u\,\mathrm{d}t$ in a small instant $\mathrm{d}t$. So in this instant p increases by an element $\mathrm{d}p = u\,\mathrm{d}t/l$.

We get a differential equation

$$\mathrm{d}p = \frac{u\,\mathrm{d}t}{l_0 + vt}\,,$$

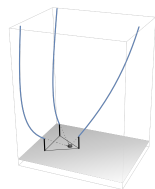
which is already in the form of separated variables, i.e., we only need to integrate and express t_f

$$\begin{aligned}\int_0^1 \mathrm{d}p &= \frac{u}{v}\int_0^{t_\mathrm{f}} \frac{\mathrm{d}t}{l_0/v+t}\,,\\ [p]_0^1 &= \frac{u}{v}\left[\ln\left(\frac{l_0}{v}+t\right)\right]_0^{t_\mathrm{f}} = \frac{u}{v}\left(\ln\left(\frac{l_0}{v}+t_\mathrm{f}\right)-\ln\left(\frac{l_0}{v}\right)\right)\,,\\ \frac{v}{u} &= \ln\left(1+\frac{t_\mathrm{f}v}{l_0}\right)\,,\\ \mathrm{e}^{\frac{v}{u}} &= 1+\frac{t_\mathrm{f}v}{l_0}\,,\\ \left(\mathrm{e}^{\frac{v}{u}}-1\right)\frac{l_0}{v} &= t_\mathrm{f}\,.\end{aligned}$$

We can observe that for v approaching 0 in the limit, we can write $\mathrm{e}^{v/u} = 1 + v/u$, and then we get $t_\mathrm{f} = l_0/u$. Substituting the values from the problem statement, we get $t_\mathrm{f} = 19.1\,\mathrm{s}$.

Šimon Pajger
`legolas@fykos.org`

Problem GB ... stone is falling among pigeons



Three pigeons of height $h = 20\,\mathrm{cm}$ are standing at the vertices of an equilateral triangle with side length $a = 50\,\mathrm{cm}$ and eating bread crumbs. Suddenly, a stone falls among them on one of the triangle's medians such that the point of impact divides the median in the ratio $2:1$, and it is not the triangle's centroid. All the pigeons fly away with an initial velocity $v = k/r^2$ away from the stone (the direction is taken from the point of impact to the bird's head), where k is a constant and r is the initial distance of the bird's head from the point of impact. At this point, however, group behavior kicks in for the pigeons. At each moment, each bird instinctively averages the velocity vectors of its two fellow pigeons and chooses an acceleration such that after a period $T = 3\,\mathrm{s}$ of uniformly accelerated motion, it moves at that average velocity. After some time, the movement of the feathered friends becomes steady, and they all fly in the same direction at the same speed. Determine the angle that this direction forms with the ground. *Vojta wonders how a pigeon works.*

Let us denote the velocities of the pigeons $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$ and their accelerations correspondingly. The heuristic that the pigeons follow can be mathematically rewritten as

$$\begin{aligned}\mathbf{v_1}+\mathbf{a_1}T &= \frac{1}{2}\left(\mathbf{v_2}+\mathbf{v_3}\right)\,,\\ \mathbf{v_2}+\mathbf{a_2}T &= \frac{1}{2}\left(\mathbf{v_1}+\mathbf{v_3}\right)\,,\\ \mathbf{v_3}+\mathbf{a_3}T &= \frac{1}{2}\left(\mathbf{v_1}+\mathbf{v_2}\right)\,.\end{aligned}$$

If we add these three equations, we get an interesting equation

$$\left(\mathbf{a_1}+\mathbf{a_2}+\mathbf{a_3}\right)T = 0\,.$$

This tells us that the total instantaneous acceleration of the system of three pigeons is zero. This means that the whole system is either at rest or moving in a uniform linear motion – that is, the sum of the velocities of the three birds is constant throughout the motion. This means that the total velocity of the flock of pigeons after the direction is steady is determined as the average of the velocity vectors at the beginning of the movement, i.e.,

$$\mathbf{V} = \frac{1}{3}\left(\mathbf{v_1}(0) + \mathbf{v_2}(0) + \mathbf{v_3}(0)\right) ,$$

Now we only need to solve the initial conditions. Let's introduce a Cartesian coordinate system with the origin at the center of the triangle, the z-axis perpendicular to the ground, and the first pigeon standing on the x-axis. The positions of the heads of the pigeons $\mathbf{h_i}$ and the position of the stone $\mathbf{s}$ can then be determined from simple geometry as

$$\begin{aligned}
\mathbf{h_1} &= \left[\frac{\sqrt{3}}{3}a, 0, h\right] ,\\
\mathbf{h_2} &= \left[-\frac{\sqrt{3}}{6}a, \frac{1}{2}a, h\right] ,\\
\mathbf{h_3} &= \left[-\frac{\sqrt{3}}{6}a, -\frac{1}{2}a, h\right] ,\\
\mathbf{s} &= \left[\frac{\sqrt{3}}{6}a, 0, 0\right] ,
\end{aligned}$$

from which we can easily determine the directions and magnitudes of the velocities $\mathbf{v_i}$ at time zero as

$$\begin{aligned}
\mathbf{v_1}(0) &= \frac{k}{||\mathbf{h_1} - \mathbf{s}||^3}\left(\mathbf{h_1} - \mathbf{s}\right) \doteq k \cdot \begin{pmatrix} 9.620 \\ 0.000 \\ 13.33 \end{pmatrix} \mathrm{m}^{-2} ,\\
\mathbf{v_2}(0) &= \frac{k}{||\mathbf{h_2} - \mathbf{s}||^3}\left(\mathbf{h_2} - \mathbf{s}\right) \doteq k \cdot \begin{pmatrix} -3.604 \\ 3.121 \\ 2.497 \end{pmatrix} \mathrm{m}^{-2} ,\\
\mathbf{v_3}(0) &= \frac{k}{||\mathbf{h_3} - \mathbf{s}||^3}\left(\mathbf{h_3} - \mathbf{s}\right) \doteq k \cdot \begin{pmatrix} -3.604 \\ -3.121 \\ 2.497 \end{pmatrix} \mathrm{m}^{-2} .
\end{aligned}$$

The sought vector is, therefore, equal to

$$\mathbf{v} = k' \begin{pmatrix} 2.413 \\ 0.000 \\ 18.32 \end{pmatrix} ,$$

where k' is a certain constant. Now we need to determine the angle that this vector makes with the ground, which we do simply by using goniometry because

$$\tan\varphi = \frac{v_z}{v_x} \doteq \frac{18.32}{2.413} \doteq 7.59 .$$

From this, we determine the angle $\varphi \doteq 82.5^\circ$.

Note that the entire situation can also be resolved in an exact way, which implies, among other things, the validity of the assumption that the speed of the feathered friends will stabilize. One of the methods to resolve this situation is given below.

The original triple of equations of motion can also be rewritten in matrix form, but we must remember that all the elements are vectors themselves.

$$\begin{pmatrix}\mathbf{a_1}\\\mathbf{a_2}\\\mathbf{a_3}\end{pmatrix} = \begin{pmatrix}\mathbf{v_1}\\\mathbf{v_2}\\\mathbf{v_3}\end{pmatrix}' = \frac{1}{2T}\begin{pmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{pmatrix}\begin{pmatrix}\mathbf{v_1}\\\mathbf{v_2}\\\mathbf{v_3}\end{pmatrix}$$

At this point, our goal is to solve a system of linear differential equations. For this purpose, we need to diagonalize the matrix shown above. This matrix has eigenvalues 0, -3 and -3, which are associated with eigenvectors

$$M_0 = \operatorname{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}, \qquad M_{-3} = \operatorname{span}\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}.$$

So we can rewrite our set in the form

$$\begin{pmatrix}\mathbf{v_1}\\\mathbf{v_2}\\\mathbf{v_3}\end{pmatrix}' = \frac{1}{2T}\begin{pmatrix}1 & -1 & -1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -3\end{pmatrix}\begin{pmatrix}1 & -1 & -1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{v_1}\\\mathbf{v_2}\\\mathbf{v_3}\end{pmatrix}.$$

Now, if we introduce substitution

$$\begin{pmatrix}\mathbf{u_1}\\\mathbf{u_2}\\\mathbf{u_3}\end{pmatrix} = \begin{pmatrix}1 & -1 & -1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{v_1}\\\mathbf{v_2}\\\mathbf{v_3}\end{pmatrix},$$

we can further simplify our set of equations to

$$\begin{pmatrix}\mathbf{u_1}\\\mathbf{u_2}\\\mathbf{u_3}\end{pmatrix}' = \frac{1}{2T}\begin{pmatrix}0 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -3\end{pmatrix}\begin{pmatrix}\mathbf{u_1}\\\mathbf{u_2}\\\mathbf{u_3}\end{pmatrix},$$

which are already three simple separable differential equations that we can solve. We can directly write

$$\begin{pmatrix}\mathbf{u_1}\\\mathbf{u_2}\\\mathbf{u_3}\end{pmatrix} = \begin{pmatrix}\mathbf{c_1}\\ \mathbf{c_2}\mathrm{e}^{-\frac{3}{2T}t}\\ \mathbf{c_3}\mathrm{e}^{-\frac{-3}{2T}t}\end{pmatrix},$$

from where, by reverse substitution

$$\begin{pmatrix}\mathbf{v_1}\\\mathbf{v_2}\\\mathbf{v_3}\end{pmatrix} = \begin{pmatrix}\mathbf{c_1} - (\mathbf{c_2} + \mathbf{c_3})\,\mathrm{e}^{-\frac{3}{2T}t}\\ \mathbf{c_1} + \mathbf{c_2}\mathrm{e}^{-\frac{3}{2T}t}\\ \mathbf{c_1} + \mathbf{c_3}\mathrm{e}^{-\frac{3}{2T}t}\end{pmatrix}, \tag{1}$$

where the vectors $\mathbf{c_1}$, $\mathbf{c_2}$, and $\mathbf{c_3}$ are vector integration constants that we determine from the initial conditions. Before we find these initial conditions, note that after a sufficiently long time, all components that do not depend on $\mathbf{c_1}$ disappear, since

$$\lim_{t\to\infty}\begin{pmatrix}\mathbf{v_1}\\ \mathbf{v_2}\\ \mathbf{v_3}\end{pmatrix}=\begin{pmatrix}\mathbf{c_1}\\ \mathbf{c_1}\\ \mathbf{c_1}\end{pmatrix}\,,$$

which indeed corresponds to the assumption that pigeons "converge" to the same velocity, and we can consider our intuition correct. So we only need to find the vector $\mathbf{c_1}$. At time $t = 0\,\mathrm{s}$, from equation (1), the following holds

$$\mathbf{c_1}=\frac{1}{3}\left(\mathbf{v_1}(0)+\mathbf{v_2}(0)+\mathbf{v_3}(0)\right)\,,$$

thus obtaining purely mathematically the same result as using the physical intuition above.

Vojtěch David
vojtech.david@fykos.org

Problem GC ... crashed charges

Two small particles, each of mass $m = 0.1\,\mathrm{g}$, are at rest in a vacuum at a distance $L = 1.0\,\mathrm{m}$ apart. One of them has charge $Q = 0.1\,\mathrm{\mu C}$, the other the same charge of the opposite sign. How long will it take for them to reach each other? You can neglect the loss of energy due to bremsstrahlung. *Robo wanted to annihilate everything, but all he got was a collision.*

First of all, we need to realize that the sizes of the particles are much smaller than their mutual distances; thus, we can say that they will reach each other when their distance equals 0. The two particles are the same, which means they will move with the same acceleration, which implies that each particle must travel a distance $L/2$. Using Kepler's third law, we can find the time it takes the particles to travel the distance $L/2$ to the system's center of gravity. The law describes that if two bodies orbit around the same point of mass on conic sections (ellipse, circle, ...) with semi-major and semi-minor axis a_1, a_2 and their orbital periods T_1, T_2, then the following is true

$$\left(\frac{T_1}{T_2}\right)^2=\left(\frac{a_1}{a_2}\right)^3\,.$$

The two particles will travel in straight lines to their mutual center of mass, which is $L/2$ far from each. The line segment is actually an ellipse with minor semi-axes equal to 0, with its focuses at the ends of the ellipse, and the major semi-axes have a length $L/4$. Half the orbital period along such an ellipse (segment) equals the time it takes a body to travel from one end of the segment to the other. From Kepler's third law, we know that its orbital period must be equal to the orbit of particles moving along a circle with a radius $L/4$ with the center at a mutual center of gravity. Therefore, we only need to calculate half of its orbital period along this circle. From the equilibrium of the electric and centrifugal forces, we will get

$$\frac{mv_0^2}{L/4}=\frac{Q^2}{4\pi\varepsilon_0(2\cdot L/4)^2}\,.$$

For the orbital period along the circle, the following holds

$$T = \frac{2\pi L/4}{v_0}\,,$$

from which we get

$$T^2 = \left(\frac{2\pi L/4}{v_0}\right)^2 = \frac{\varepsilon_0 m \left(\pi L\right)^3}{Q^2}\,.$$

Halving it gives us

$$t = \frac{T}{2} = \frac{\sqrt{\varepsilon_0 m \left(\pi L\right)^3}}{2Q}\,.$$

What numerically equals $t = 0.83\,\mathrm{s}$.

Róbert Jurčo
robert.jurco@fykos.org

FYKOS
UK, Matematicko-fyzikální fakulta
Ústav teoretické fyziky
V Holešovičkách 2
180 00 Praha 8

www: http://fykos.cz
e-mail: fykos@fykos.cz

FYKOS is also on Facebook
http://www.facebook.com/FYKOS

FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Public Relations Office activities and is supported by Institute of Theoretical Physics, MFF UK, his employees and The Union of Czech Mathematicians and Physicists.

Organizers

FYKOS

matfyz

General partner

ESPR

Platinum partners

NADACE ČEZ

Neuron

Silver partners

KARÁSKOVY LIMONÁDY A SIRUPY

Quark Magazín o vede a technike

FACULTY OF SCIENCE Charles University

Partners

PROMETHEUS®

HUMUSOFT® | MathWorks®

MINDOK

PETR HOBŽA SNACK GROUP

CASIO

VIDA! science centrum

escape point

MERKUR

DOLLER

Efektivní altruismus