ORBITAL MECHANICS

Kien T. Pham

Contents

1 Introduction to Orbital Mechanics

Orbital mechanics or **astrodynamics** is the application of ballistics and celestial mechanics to the practical problems concerning the motion of rockets and other spacecraft. The motion of these objects is usually calculated from Newton's laws of motion and law of universal gravitation. Orbital mechanics is a core discipline within space-mission design and control.

Celestial mechanics treats more broadly the orbital dynamics of systems under the influence of gravity, including both spacecraft and natural astronomical bodies such as star systems, planets, moons, and comets. Orbital mechanics focuses on spacecraft trajectories, including orbital maneuvers, orbital plane changes, and interplanetary transfers, and is used by mission planners to predict the results of propulsive maneuvers.

One of the greatest analytic mathematical achievements that most accurately describe the motion of celestial bodies are the three famous Kepler laws after 20 years of astronomical observations by the Danish astronomer Tycho Brahe.

2 Orbital parameters

2.1 Dynamic parameters

Consider an object with mass m is in motion in the gravitational field of a big celestial body with mass M (assume that $m \ll M$ so that we can assume M is fixed in space).

Both two objects will be pulled centrally by each other according to the Newton's Law of Gravitation. We have the gravitational force and potential which are expressed as

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{r^2}\hat{\mathbf{r}}, \tag{2.1}$$

$$U(r) = -\int_{\infty}^{r} \mathbf{F} \cdot d\mathbf{r} = -\frac{GMm}{r}. \tag{2.2}$$

The angular momentum of the object $\mathbf{L}$ will be conserved and given by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v} = m\mathbf{h} = mr^2\frac{d\theta}{dt}\hat{\mathbf{z}}. \tag{2.3}$$

The total mechanical energy E of the object is also conserved end given by

$$\begin{aligned} E &= \frac{1}{2}mv^2 - \frac{GMm}{r} && (2.4)\\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r} && (2.5)\\ &= \frac{1}{2}m\dot{r}^2 + \left(\frac{L^2}{2mr^2} - \frac{GMm}{r}\right). && (2.6) \end{aligned}$$

In many problems, we can call the term inside the brackets in (Eq.2.5) is the effective potential

$$U_{eff}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{2.7}$$

2.2 Differential equations of motion

From equation (Eq.2.4) we can derive differential equations of motion whose solutions can be very complex:

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - \frac{L^2}{2mr^2} - U(r)}. \tag{2.8}$$

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{L^2} - \frac{1}{r^2} - \frac{2mU(r)}{L^2}. \tag{2.9}$$

We may use this useful integral in computing:

$$\int \frac{dx}{\sqrt{a+bx-cx^2}} = \frac{1}{\sqrt{c}}\arcsin\left[\frac{2c}{\sqrt{4ac+b^2}}\left(x - \frac{b}{2c}\right)\right] + C. \tag{2.10}$$

3 Kepler's laws of celestial motion

3.1 Kepler's laws

Kepler's laws

1. The orbit of every planet is an ellipse with the Sun at one of the two focus.

$$r(\theta) = \frac{p}{1+e\cos\theta}. \tag{3.1}$$

Where $p = \dfrac{L^2}{GMm^2}$ và $e = \sqrt{1 + \dfrac{2EL^2}{G^2M^2m^3}}$.

2. A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

$$\frac{dA}{dt} = \frac{1}{2}h. \tag{3.2}$$

3. The ratio of the square of an object's orbital period with the cube of the semi-major axis of its orbit is the same for all objects orbiting the same primary.

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}. \tag{3.3}$$

3.2 Proves

1. First Law

***Solution 1:** We will first show that the angular momentum of an orbit is a conserved quantity.

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{v}) = \mathbf{v}\times\mathbf{v} + \mathbf{v}\times\mathbf{a} = \mathbf{v}\times\mathbf{v} - \frac{GM}{r^3}\mathbf{r}\times\mathbf{r} = 0 - 0 = 0 \tag{3.4}$$

$$\Rightarrow \mathbf{r}\times\mathbf{v} = \mathbf{h} \tag{3.5}$$

where $\mathbf{h}$ is a constant vector. This means that the vector $\mathbf{r}$ will always be perpendicular to the vector $\mathbf{h}$, i.e. the object will always move in a plane perpendicular to $\mathbf{h}$.

To prove Kepler's 1st law we multiply the acceleration times the specific angular momentum:

$$\mathbf{a} \times \mathbf{h} = -\frac{GM}{r^3}\mathbf{r} \times \left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) \tag{3.6}$$

$$= -\frac{GM}{r^3}\left[\left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\right)\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\frac{d\mathbf{r}}{dt}\right] \tag{3.7}$$

$$= \frac{GM}{r}\frac{d\mathbf{r}}{dt} = GM\frac{d\hat{\mathbf{r}}}{dt}. \tag{3.8}$$

Integrating both sides of the equation up, we get:

$$\mathbf{v} \times \mathbf{h} = GM\hat{\mathbf{r}} + \mathbf{c} \tag{3.9}$$

where $\mathbf{c}$ is a constant vector lying on the plane Oxy. Call the angle between $\mathbf{c}$ and $\mathbf{r}$ θ, we have:

$$\begin{aligned} \mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= \mathbf{r} \cdot (GM\hat{\mathbf{r}} + \mathbf{c}) = GMr + rc\cos\theta. \\ &= (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = h^2 \end{aligned} \tag{3.10}$$

$$\Rightarrow r(\theta) = \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{GM + c\cos\theta} = \frac{h^2}{GM(1 + e\cos\theta)}. \tag{3.11}$$

***Solution 2:** For convenience, we will set $u = \frac{1}{r}$ and $\alpha = GMm$. From the equation (2.9) we get:

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2mE}{L^2} + \frac{2GMm^2}{L^2}u - u^2 \tag{3.12}$$

$$= -\left(u - \frac{m\alpha}{L^2}\right)^2 + \frac{2mE}{L^2} + \left(\frac{m\alpha}{L^2}\right)^2. \tag{3.13}$$

It is easy to solve the other differential equation by dissociating the variables and converting it back to get:

$$u = \frac{1}{r} = \frac{m\alpha}{L^2}\left(1 + \sqrt{1 + \frac{2EL^2}{m\alpha^2}}\cos\theta\right) \tag{3.14}$$

$$\Rightarrow r(\theta) = \frac{L^2}{GMm^2(1 + e\cos\theta)}. \tag{3.15}$$

***Solution 3:** We will use Newton's 2nd law for the polar coordinate system:

$$m(\ddot{r} - r\dot{\theta}^2) = -F. \tag{3.16}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0. \tag{3.17}$$

$$\dot{r} = \frac{dr}{dt} = \frac{d\theta}{dt}\frac{dr}{d\theta} = -h\frac{d}{d\theta}\left(\frac{1}{r}\right) = -h\frac{du}{d\theta}. \tag{3.18}$$

$$\ddot{r} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}. \tag{3.19}$$

From that we have **Binet's equation**:

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{mh^2u^2} = \frac{GM}{h^2} \tag{3.20}$$

$$\Rightarrow r(\theta) = \frac{h^2}{GM(1 + e\cos\theta)}. \tag{3.21}$$

As you can see, the function $r(\theta)$ is a conic equation. Here, $e = \sqrt{1 + \frac{2EL^2}{m\alpha^2}} = \sqrt{1 + \frac{2pE}{GMm}}$ is the eccentricity and $p = \ frach^2GM = a(1-e^2)$ is the semi-latus axis of conic.

$$r(\theta) = \frac{h^2}{GM\left(1 + \sqrt{1 + \frac{2Eh^2}{G^2M^2m}}\cos\theta\right)} = \frac{p}{1 + e\cos\theta} = \frac{a(1-e^2)}{1 + e\cos\theta}. \tag{3.22}$$

2. Second Law

The area differential in polar coordinates is: $dA = rdrd\theta$. The area swept through an angle $d\theta$ is:

$$dA = d\theta \int_0^r rdr = \frac{1}{2}r^2 d\theta \tag{3.23}$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}h. \tag{3.24}$$

It can be seen that the sweeping area of the object around the celestial body in a unit of time is constant.

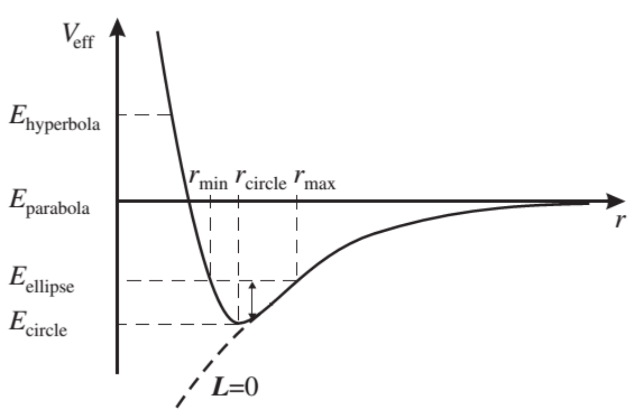
3. Third Law

From equation (Eq.3.24) we have:

$$\frac{\pi ab}{T} = \frac{1}{2}\sqrt{GMa(1-e^2)} \tag{3.25}$$

$$\Rightarrow \frac{T^2}{a^3} = \frac{4\pi^2}{GM} \tag{3.26}$$

3.3 Orbital parameters



Orbit	Eccentricity (e)	$r(\theta)$	Energy (E)	Angular momentum (L)
Circle	0	r_0	$-\dfrac{GMm}{2r_0}$	$m\sqrt{GMr_0}$
Ellipse	$0 \to 1$	$\dfrac{a(1-e^2)}{1+e\cos\theta}$	$-\dfrac{GMm}{2a}$	$m\sqrt{GMa(1-e^2)}$
Parabolar	1	$\dfrac{2a}{1+\cos\theta}$	0	$m\sqrt{2GMa}$
Hyperbolar	$1 \to \infty$	$\dfrac{a(e^2-1)}{1+e\cos\theta}$	$\dfrac{GMm}{2a}$	$m\sqrt{GMa(e^2-1)}$

3.4 Runge-Lenz Vector

A quantity that has a constant value in a stationary orbit is the Runge-Lenz vector, and whose algebraic value is the eccentricity of the orbit:

$$\mathbf{e} = \frac{\mathbf{L} \times \mathbf{v}}{GMm} + \hat{\mathbf{r}}, \quad |\mathbf{e}| = e. \tag{3.27}$$

4 Two-body problem

If m is comparable with M, then the system of two bodies will revolve around its own center of mass, and the problem would be more complicated.

To simplify this, we can consider the two-body system as a fixed body at the center of mass $M + m$ and a body with effective mass $\mu = \dfrac{mM}{M+m}$ revolves around the host with a distance between them $\mathbf{r}$. Kepler's laws still apply this way.

References

[1] David Morin. *Introduction to Classical Mechanics With Problems and Solutions*. Cambridge, 2008.

[2] James Stewart. *Calculus*. Cambridge, 6th edition, 2008.

[3] Wikipedia contributors. Effective potential — Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/Effective_potential, 2022. [Online; accessed 24-August-2022].

[4] Wikipedia contributors. Kepler's laws of planetary motion — Wikipedia, the free encyclopedia, 2022. [Online; accessed 24-August-2022].

[5] Wikipedia contributors. Orbital mechanics — Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/Orbital_mechanics, 2022. [Online; accessed 24-August-2022].