Mechanics I: Kinematics

See chapters 3 and 4 of Morin for material on solving differential equations. For general review on kinematics, see chapter 1 of Kleppner and Kolenkow. For fun, see chapters I-1 through I-8 of the Feynman lectures. There is a total of **92** points.

1 Motion in One Dimension

Example 1

When a projectile moves slowly through air, the drag is linear in the velocity, $F = -\alpha m v$. Find the velocity $v(t)$ of a projectile thrown upward at time $t = 0$ with speed v_0.

Solution

We write Newton's second law as

$$\frac{dv}{dt} = -g - \alpha v$$

and multiply through by dt. Integrating both sides from the initial condition to time t_f gives

$$\int_{v_0}^{v(t_f)} \frac{dv}{g + \alpha v} = -\int_0^{t_f} dt.$$

Performing the integrals gives

$$\frac{1}{\alpha} \log(g + \alpha v)\Big|_{v_0}^{v(t_f)} = -t_f.$$

Renaming t_f to t and solving for v yields

$$v(t) = e^{-\alpha t} v_0 + \frac{g}{\alpha}(e^{-\alpha t} - 1).$$

This renaming is necessary because we don't want to confuse t, the dummy variable that we integrating over, with t_f, the time at which we want to evaluate the velocity; t ranges from zero to t_f. Unfortunately, often people just call both of these t, so you need to watch out.

[2] **Problem 1.** Investigating some features of this solution.

(a) By using results from **P1**, verify that $v(t)$ makes sense for both small times and large times.

(b) If the projectile is then caught at the launch point, did it spend more time going up or down?

(c) Do you think the total time is longer or shorter than for a projectile without drag?

[3] **Problem 2.** Now assume quadratic drag, $F = -\alpha m v^2$, which applies for fast-moving projectiles.

(a) Integrate Newton's second law to get an implicit equation for $v(t)$ with the same initial conditions as above. That is, you don't need to solve for $v(t)$, as it'll just make things messy.

(b) Your equation will only be valid when the projectile is going up; explain why.

(c) Find $v(t)$ for an object released from rest at time $t = 0$. (Hint: if needed, look up some standard integrals involving hyperbolic trigonometric functions. But don't worry about memorizing the results, since in competitions, any nontrivial integral needed will usually be given to you.)

(d) Integrate your answer to part (c) with respect to time to find $y(t)$, and verify that the answer makes sense at both small and large times.

Some people only call this quadratic case drag; they call the linear case viscous resistance. This is because they behave fundamentally differently at the microscopic level, as we will explore in **M7**.

[3] **Problem 3.** A projectile of mass m is dropped from a height h above the ground. It falls and bounces elastically, experiencing the same quadratic drag as in problem 2. Find the maximum height to which it subsequently rises. (Hint: don't try to use your results from problem 2.)

Remark

How does the top speed v of a rowboat depend on the number N of rowers? A light, fast-moving rowboat experiences quadratic friction, so that the drag force on it is proportional to $v^2 A$, where A is the submerged cross-sectional area of the boat. A boat designed for N rowers will have a submerged volume $V \propto N$, and a streamlined shape so that $A \propto V^{2/3}$. Thus, the required power input is

$$P = Fv \propto v^3 N^{2/3}.$$

The power output by the rowers scale as N, and combining these results gives the amazingly weak dependence $v \propto N^{1/9}$, which agrees decently with Olympic rowing times.

Idea 1

An ordinary differential equation is any equation involving a quantity $x(t)$ and its derivatives. In introductory physics, we are usually concerned with a few very simple differential equations, with the following nice properties.

- The differential equation is at most second order, meaning it can contain x, its first derivative $\dot{x} = v$, and its second derivative $\ddot{x} = a$, but no higher derivatives. This implies the solution can be determined by an initial position and initial velocity. (We'll focus on second order differential equations for the rest of this section; most first order differential equations can simply be solved by separation and integration, as we've seen above.)

- The differential equation is linear, meaning that terms don't contain products of x, $\dot{x}$, and $\ddot{x}$. For example, a damped driven harmonic oscillator with time-dependent drag,

$$m\ddot{x} = -b(t)\dot{x} - kx + f(t)$$

is a second order linear differential equation. Solutions to such differential equations obey the superposition principle: if $x_1(t)$ and $x_2(t)$ are both solutions, so is $c_1 x_1(t) + c_2 x_2(t)$.

- The differential equation is homogeneous, meaning that each term is proportional to exactly one power of x or its derivatives. The above differential equation is not homogeneous, but it would be if we removed the driving $f(t)$.

- The differential equation is time-translation invariant, meaning that no functions of time appear except for x and its derivatives. The above equation isn't, but it would be if we set $f(t)$ and $b(t)$ to constants.

Idea 2

Linear, homogeneous, time-translation invariant differential equations are very special, and they can all be solved by the exact same method. First, note that we can promote $x(t)$ to a complex variable $\tilde{x}(t)$ and solve the differential equation over the complex numbers. As long as we have a complex solution, we can recover a real solution by taking the real part. Now, the method of solution, which works for *almost* all equations of this form, is to guess a complex exponential solution

$$\tilde{x}(t) = e^{i\omega t}.$$

Plugging this into the differential equation will yield the allowed values of ω, and the general solution can be found by superposing the complex exponentials.

Example 2

Solve the simple harmonic oscillator, $m\ddot{x} + kx = 0$, using the above principles.

Solution

First, we pass to a complex differential equation,

$$m\ddot{\tilde{x}} + k\tilde{x} = 0.$$

We guess $\tilde{x}(t) = e^{i\omega t}$. Plugging this in and using the chain rule gives

$$m(i\omega)^2 e^{i\omega t} + k e^{i\omega t} = 0$$

and canceling $e^{i\omega t}$ and solving gives two solutions,

$$\omega = \pm\omega_0, \quad \omega_0 = \sqrt{k/m}.$$

Since this a second-order linear differential equation, the general solution is given by the superposition of these two complex exponentials,

$$\tilde{x}(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$

where A and B are general complex numbers. The real part of $\tilde{x}(t)$ satisfies the original real differential equation $ma + kx = 0$, and is

$$\operatorname{Re} x(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)$$

where C and D are real numbers.

[1] **Problem 4.** To make sure you know how to go from the complex solution to the real one, write C and D in terms of A and B.

[2] **Problem 5.** Now introduce a damping force and solve the differential equation for the damped harmonic oscillator, $m\ddot{x} + b\dot{x} + kx = 0$, using the same procedure, assuming b is small. (See section 4.3 of Morin if you have trouble with this. We'll consider this system in more detail in **M4**.)

[3] **Problem 6.** USAPhO 2012, problem B1.

[3] **Problem 7.** Above, we mentioned that guessing an exponential works *almost* all the time. The reason is because at the end of the day, the exponential cancels out and we're left with a polynomial in ω, which has just the right number of roots. But if there are repeated roots, there are fewer distinct solutions for ω, and hence not enough solutions.

(a) Consider a second order differential equation with a double root ω. What is the other solution, besides $e^{i\omega t}$? (Hint: to help find a good guess, consider the simple case $ma = 0$, where $\omega = 0$ is the double root. Then generalize your guess to nonzero ω and check that it works.)

(b) This should be setting off alarm bells: the form of the solutions to the equation changes when the two roots are *exactly* equal, while it's just exponentials/sinusoids if the roots are different, no matter how small the difference is. Since no two roots are *ever* exactly equal in practice, it seems the behavior of part (a) can never actually happen in the real world. But it gets taught in applied differential equations courses. Why?

(c) [A] Consider the most general n^{th} order, linear homogeneous time-translation invariant differential equation

$$\left(a_n \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1 \frac{d}{dt} + a_0\right) x = 0.$$

What does the general solution look like?

Remark

You might be wondering how to solve more general differential equations. In **M4**, we will consider three extensions of the above techniques. We'll use the idea of normal modes to solve systems of such differential equations, add driving forces to make the equations inhomogeneous, and use the adiabatic theorem to approximately solve non time-translation invariant equations where the coefficients change slowly in time.

Of course, this just scratches the surface of the subject, and solving more general differential equations can be orders of magnitude harder. We won't try to solve nonlinear differential equations, as there is no general technique for doing so, and the answer is often an obscure special function. (However, such equations will occasionally appear in later problems.) On the other hand, linear differential equations with general time-dependence are more approachable, and the following problem illustrates the most basic method for solving them.

[3] **Problem 8.** [A] Some linear, homogeneous, non time-translation invariant differential equations can be solved by simply guessing a power series. For this problem, don't worry about dimensional analysis; assume all variables have already been redefined to be dimensionless.

(a) As a warmup, consider the differential equation $\dot{x} = kx$ for constant k, which we already know how to solve. By plugging in the ansatz

$$x(t) = \sum_{n=0}^{\infty} a_n t^n$$

with unknown constant coefficients a_n, find the solution with $x(0) = 1$.

(b) Now consider the non time-translation invariant differential equation

$$t^2\ddot{x} + t\dot{x} + t^2 x = 0$$

which is called Bessel's differential equation of order zero. By using the same ansatz, find the unique solution with $x(0) = 1$ and $\dot{x}(0) = 0$.

2 Tricks

In this section we'll consider some kinematics problems that require cleverness, not computation.

Idea 3

Many problems can be solved by a clever choice of reference frame. It is often useful to go to the frame moving with one of the objects in the problem, or to go into a frame that makes the motion in the problem more symmetric. For the purposes of kinematics it can even be useful to use noninertial reference frames, such as a falling frame where projectiles don't accelerate, or a rotating frame, though this will introduce fictitious forces into the dynamics. It is also useful to tilt the coordinate axes to be parallel to various objects.

Example 3: $F = ma$ 2022 B4

A firework explodes, sending shells in all directions. Suppose the shells are all launched with the same speed, and ignore air resistance, but not gravity. What shape do the shells make?

Solution

In the absence of gravity, the shells would always form a sphere. Adding gravity simply shifts all of their locations downward by $gt^2/2$, so the shape is still always a sphere.

[1] **Problem 9** (KoMaL 2019)**.** A cannon A is at the edge of a cliff with a 800 m drop. Cannon B is on the ground below the cliff and 600 m horizontally away from it. Cannon A shoots a cannonball directly towards cannon B at 60 m/s. Cannon B shoots a cannonball directly towards cannon A at 40 m/s. Will the two cannonballs hit each other in midair?

[2] **Problem 10** (Wang)**.** Two particles are released in gravitational acceleration g with leftward and rightward speeds v_1 and v_2. Find the distance between them when their velocities are perpendicular.

[3] **Problem 11** (Kalda)**.** Two intersecting circles of radius r have centers a distance a apart. If one circle moves towards the other with speed v, what is the speed of one of the points of intersection?

[2] **Problem 12** (Kalda)**.** A mirror rotates about its center with angular speed ω. A stationary point source of light sits at a distance a from the rotation axis. What is the speed of its mirror image?

[2] **Problem 13** (Kalda)**.** Two circles of radius r intersect at the point O. One of the circles rotates about the point O with constant angular speed ω. The other point of intersection O' is originally a distance d from O. Find the speed of O' as a function of time.

Idea 4

To find the minimum value of some quantity, it's often useful to think about all possible values of that quantity. This can reveal a solution using geometry or symmetry.

[2] **Problem 14** (PPP 3)**.** A boat can travel a speed of $3\,\text{m/s}$ on still water. A boatman wants to cross a river while covering the shortest possible distance.

(a) In what direction should he row if the speed of the water is $2\,\text{m/s}$?

(b) How about if it is $4\,\text{m/s}$?

Idea 5

In problems with friction, the best reference frame to use is almost always the frame of whatever is causing the friction.

[2] **Problem 15** (Kalda)**.** A block is pushed onto a conveyor belt. The belt is moving with speed $1\,\text{m/s}$, and the block's initial speed is $2\,\text{m/s}$, with initial velocity perpendicular to that of the belt. During the subsequent motion, what is the minimum speed of the block with respect to the ground?

Idea 6

For a variety of kinematics problems, it can be useful to think about the motion from a different perspective. For example, if your problem involves complicated accelerations, it can be useful to think in "velocity space", i.e. directly think about how the velocity vector evolves over time, and deal with the position later. Or, if your problem involves complicated processes occurring in time, it can be useful to think in "spacetime", meaning to visualize the process on a space where time is one of the axes. It can also be useful to parametrize motion in terms of quantities other than the usual Cartesian coordinates.

[2] **Problem 16** (Kalda)**.** A boy enters a patch of ice with a coefficient of friction μ with speed v. By running on the ice, the boy turns his velocity vector by 90° in the minimum possible time, so that his final speed is also v. What is the minimum possible time, and what kind of curve is the trajectory? Assume the normal force with the ice is constant.

[2] **Problem 17** (PPP 5)**.** Four snails travel in uniform, rectilinear motion on a plane. The velocities are chosen so that three snails never meet at once, and no two of the velocities are equal. Since time $t = -\infty$, five of the $\binom{4}{2}$ possible encounters have already occurred. Must the sixth also occur?

[2] **Problem 18.** Six bugs are placed at the vertices of a regular hexagon with side length s. At time $t = 0$ each bug starts moving directly towards the next with speed v. At what time do they collide?

Example 4

A rabbit begins at the origin, and the fox begins at the point $(0, -a)$. The rabbit begins running east, with a constant speed $v\hat{\mathbf{x}}$. At the same time, the fox begins chasing the rabbit, always moving towards it with speed v. After a long time, the rabbit and wolf simply follow each other in a straight line, with a constant separation d. What is d?

Solution

This is the simplest example of a persuit problem. Physicists and mathematicians have been posing them for centuries, though most are too mathematically involved for Olympiads.

Here, the trick to realize that if the displacement between the rabbit and fox is $\mathbf{r}(t) = (x(t), y(t))$, then the quantity $r + x$ is conserved. To see this, let θ be the angle between the rabbit and fox's velocity vectors. Then

$$\frac{dr}{dt} = -v + v\cos\theta$$

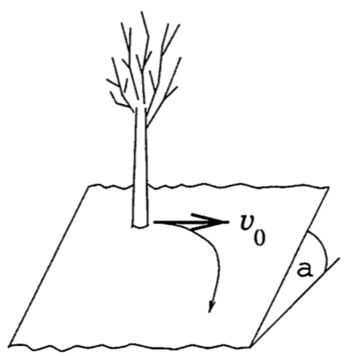
because of the fox's chasing and rabbit's motion, and

$$\frac{dx}{dt} = v - v\cos\theta$$

because of the rabbit's motion and fox's chasing. Then $r + x$ is constant. Initially $r + x = a + 0 = a$, and after a long time $r = x = d$, so the final separation is $d = a/2$.

[2] **Problem 19.** Suppose the fox in the above example instead has speed $u > v$. How long does it take to catch the rabbit?

[2] **Problem 20** (PPP 85)**.** A child is on an icy hill, which may be modeled as an inclined plane.



The coefficient of friction $\mu_k = \mu_s$ is small enough so that, if the child gets the tiniest push, she will begin sliding down the plane. Now suppose the child gets a horizontal push, with initial speed v_0. What is the child's final speed?

[4] **Problem 21.** EuPhO 2023, problem 2. (Warning: the algebra will be a bit messy.)

3 Motion in Two Dimensions

Idea 7

Often, motion in two dimensions can be treated as two independent one-dimensional problems. A change of reference frame may be necessary first.

Idea 8

In problems involving an inclined plane, it helps to draw the inclination angle θ to be less than about 30°. This reduces mistakes, because almost every angle will be either θ or $90^\circ - \theta$, and you can identify which by sight.

Example 5

Consider projectile motion where wind provides a constant horizontal force F. At what angle should a projectile of mass m be launched in order to return to the thrower?

Solution

The key idea is to use tilted coordinate systems. Clearly, when the only force is downward, the projectile must be launched straight upward. Now, the horizontal force acts like an effective horizontal gravitational acceleration of F/m, so that gravity is effectively tilted an angle $\tan^{-1}(F/mg)$ away from the vertical. One must launch the projectile directly "upward" with respect to this effective gravitational field, so the launch angle is an angle $\tan^{-1}(F/mg)$ from the vertical. (For a related problem, see the infamous $F = ma$ 2014 problem 19.)

Example 6: $F = ma$ 2022 A23

For projectiles, the force of air resistance can be modeled as proportional to the speed ("linear drag") or proportional to the square of the speed ("quadratic drag"), depending on the circumstances. Two identical objects, A and B, are dropped from the same height h simultaneously, but object A is given an initial horizontal velocity v. The objects hit the ground at times t_A and t_B. How do these times compare, assuming linear or quadratic drag?

Solution

For linear drag, the horizontal and vertical components of the motion are independent,

$$a_x = -bv_x, \quad a_y = -g - bv_y$$

for some coefficient b. That means the time to hit the ground, which depends on the vertical motion, is independent of the initial horizontal velocity, so $t_A = t_B$. But for quadratic drag,

$$a_y = -g - bv_y|v|$$

which means the upward drag force is larger when the horizontal velocity is larger, so $t_A > t_B$.

Since the components are independent for linear drag, it's not too hard to write down an expression for the trajectory, by recycling the results of example 1. But for quadratic drag, the results of problem 2 won't help much; the two-dimensional problem is much harder.

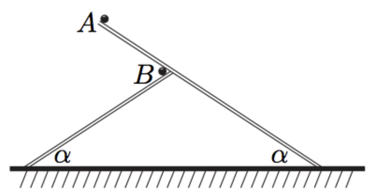
[1] **Problem 22** (Quarterfinal 2002). A cart is rigged with a vertical cannon so that, when the cart is stationary on a horizontal track, the cannonball is fired straight up and lands back in the cannon. In each of the following situations, does the cannonball land back in the cannon, in front of it, or behind it?

(a) The cart is moving on a frictionless horizontal track with speed v.

(b) The cart is accelerating down a frictionless inclined track with angle θ.

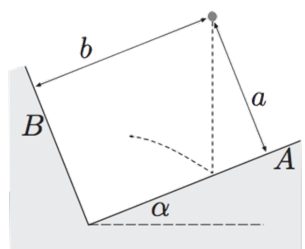
(c) The cart is accelerating down an inclined track with angle θ, and friction slows it down.

[2] **Problem 23** (Kalda). Two balls at points A and B are released from rest at the same moment, from the locations shown below. All surfaces are frictionless.



If it takes time t_A and t_B for the balls to hit the ground, at what time was the distance between the balls the smallest?

[2] **Problem 24** (Kalda). Two planar frictionless walls are placed at right angles, where wall A makes an angle α to the horizontal. A perfectly elastic ball is released from rest at a point a distance a from wall A and b from wall B.



After a long time, what is the ratio of the number of times the ball has bounced against wall B to the number of times it has bounced against wall A?

[2] **Problem 25.** USAPhO 2004, problem A4.

[3] **Problem 26** (EFPhO 2010). A sprinkler can be modeled as a small hemisphere on the ground. Water shoots out from the hemisphere in all directions, with speed v perpendicular to the hemisphere.

(a) Find the total surface area of ground watered by the sprinkler.

(b) At what distance from the sprinkler does the ground get the wettest?

[3] **Problem 27.** USAPhO 2023, problem A1. A neat exercise on collisions and projectile motion.

Example 7

A bug flies towards a light with constant speed v, always making an angle α with the radial direction. If the initial distance to the lamp is L and the radius of the lamp is R, through what total angle does it turn before hitting the lamp?

Solution

In this case we can't avoid solving differential equations, but they're not too hard. It's easiest to work in polar coordinates, with the center of the lamp at the origin. By decomposing the velocity into radial and tangential components, we have

$$\frac{dr}{dt} = -v\cos\alpha, \quad r\frac{d\theta}{dt} = v\sin\alpha.$$

We only care about the path, not the time-dependence, so we divide these equations to get

$$\frac{dr}{d\theta} = -\frac{r}{\tan\alpha}$$

where we manipulated differentials as in **P1**. Separating and integrating,

$$-\int_L^R \frac{dr}{r} = \frac{\Delta\theta}{\tan\alpha}$$

which tells us that

$$\Delta\theta = (\tan\alpha)\log\frac{L}{R}.$$

The shape traced out is a logarithmic spiral.

[2] **Problem 28.** The pilot of a supersonic jet airplane wishes to make a big noise at the origin by flying around it in a path such that all of the noise he makes is heard simultaneously at the origin. The jet travels with Mach number M, meaning that its speed is M times the speed of sound. If the pilot starts at $(r, \theta) = (a, 0)$, find the pilot's path $r(\theta)$.

[4] **Problem 29.** Consider a mass m on a table attached to a spring at the origin with zero relaxed length, which exerts the force

$$\mathbf{F} = -k\mathbf{r}$$

on the mass. We will find the general solution for $\mathbf{r}(t) = (x(t), y(t))$ in two different ways.

(a) Directly write down the answer, using the fact that the x and y coordinates are independent.

(b) Sketch a representative sample of solutions. What kind of curve does the trajectory follow?

(c) $\star$ Here's a more unusual way to arrive at the same answer. Go to a noninertial reference frame rotating with angular velocity ω_0 about the origin, so that the centrifugal force cancels out the spring force. In this frame, the only relevant force is the Coriolis force $-2m\boldsymbol{\omega}_0 \times \mathbf{v}$. Find the general solution in this frame, then transform back to the original frame and show that you get the same answer as in part (a). (This can get a bit messy; the easiest way is to treat the plane as the complex plane, i.e. work in terms of the variable $r = x + iy$.)

4 Optimal Launching

Finally, we'll consider projectile motion questions that involve optimization. These are rare on the USAPhO, but they are quite fun problems, with occasionally very slick solutions.

Example 8

A bug wishes to jump over a cylindrical log of radius R lying on the ground, so that it just grazes the top of the log horizontally as it passes by. What is the minimum launch speed v required to do this?

Solution

Let P be the point at the top of the log. For the bug to be moving horizontally at P, energy conservation applied to the vertical motion gives an initial v_y obeying

$$\frac{1}{2}mv_y^2 = 2mgR, \quad v_y = 2\sqrt{gR}.$$

Thus, we need to find the minimum v_x for the motion to be possible. If v_x is too low, the hypothetical trajectory of the bug will instead pass through the log. At the lowest possible v_x, the bug's trajectory is not just tangent to the log at point P, but also has the same radius of curvature (i.e. the trajectory and the log's shape have the same first and second derivatives).

For uniform motion in a circle of radius r, the acceleration is $a = v^2/r$. Conversely, when an object follows a trajectory of instantaneous radius of curvature r, its acceleration component normal to the path must be $a = v^2/r$. So applying this to the bug at P gives

$$g = \frac{v_x^2}{R}, \quad v_x = \sqrt{gR}.$$

Thus, the minimum initial speed is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{5}\, gR.$$

This radius of curvature trick doesn't come up often, but it's cool when it does.

[2] **Problem 30.** NBPhO 2020, problem 3. A nice warmup for the problems below.

[3] **Problem 31.** An object is launched from the top of a hill, where the ground lies an angle ϕ below the horizontal. Show that the range of a projectile is maximized if it is launched along the angle bisector of the vertical and the ground.

[3] **Problem 32** (PPP 35). A point P is located above an inclined plane with angle α. It is possible to reach the plane by sliding under gravity down a straight frictionless wire, joining P to some point P' on the plane. Geometrically, how should P' be chosen so as to minimize the time taken? (Hint: think about the set of points that can be reached for all possible angles of the wire, after time t.)

Idea 9

Since mechanics is time-reversible, and the speed of a projectile only depends on its height and not the path taken, finding the way to reach point B from point A with the lowest possible initial speed is the same as finding the way to reach point A from point B with the lowest possible initial speed.

[4] **Problem 33.** Two fences of heights h_1 and h_2 are erected on a horizontal plain, so that the *tops* of the fences are separated by a distance d. Show that the minimum speed needed to throw a projectile over both fences is $\sqrt{g(h_1 + h_2 + d)}$.

[4] **Problem 34.** Problems 31 and 33 can be solved with pure geometry. Consider the set of points, in two dimensions, that a projectile can reach with a fixed initial speed v and a fixed launch point. It turns out that the boundary of this set (i.e. the curve of points that a projectile can just barely reach) is a vertical parabola with its focus at the launch point. A parabola is defined as the set of points whose distance to the focus equals the distance to a line, called the directrix.

(a) Show that trajectories that touch the parabola must be tangent to it.

(b) Show that if a point is hit with the smallest possible initial speed, then the initial velocity must be perpendicular to the final velocity.

(c) Using the geometric definition of a parabola, recover the answers to problems 31 and 33.

[3] **Problem 35.** IPhO 2012, problem 1A.

5 Reading Graphs

In some kinematics problems, you'll have to infer what's going on from a diagram. To make progress, you'll have to print out the diagram to make measurements directly on it.

[3] **Problem 36.** EFPhO 2015, problem 6.

[3] **Problem 37.** EFPhO 2008, problem 3.

Remark

For a harder problem from the same genre, see EuPhO 2019, problem 3. Almost all competitors received zero points on it, largely because it relies on a specialized trick introduced earlier in this problem set. You can try it for entertainment if you have time and really like kinematics. The official solutions are here.

Mechanics I: Kinematics

See chapters 3 and 4 of Morin for material on solving differential equations. For general review on kinematics, see chapter 1 of Kleppner and Kolenkow. For fun, see chapters I-1 through I-8 of the Feynman lectures. There is a total of **92** points.

1 Motion in One Dimension

Example 1

When a projectile moves slowly through air, the drag is linear in the velocity, $F = -\alpha m v$. Find the velocity $v(t)$ of a projectile thrown upward at time $t = 0$ with speed v_0.

Solution

We write Newton's second law as

$$\frac{dv}{dt} = -g - \alpha v$$

and multiply through by dt. Integrating both sides from the initial condition to time t_f gives

$$\int_{v_0}^{v(t_f)} \frac{dv}{g + \alpha v} = -\int_0^{t_f} dt.$$

Performing the integrals gives

$$\frac{1}{\alpha} \log(g + \alpha v)\Big|_{v_0}^{v(t_f)} = -t_f.$$

Renaming t_f to t and solving for v yields

$$v(t) = e^{-\alpha t} v_0 + \frac{g}{\alpha}(e^{-\alpha t} - 1).$$

This renaming is necessary because we don't want to confuse t, the dummy variable that we integrating over, with t_f, the time at which we want to evaluate the velocity; t ranges from zero to t_f. Unfortunately, often people just call both of these t, so you need to watch out.

[2] **Problem 1.** Investigating some features of this solution.

(a) By using results from **P1**, verify that $v(t)$ makes sense for both small times and large times.

(b) If the projectile is then caught at the launch point, did it spend more time going up or down?

(c) Do you think the total time is longer or shorter than for a projectile without drag?

Solution. (a) For small times ($\alpha t \ll 1$), we have

$$v(t) \approx (1 - \alpha t) v_0 + \frac{g}{\alpha}(-\alpha t) = v_0 - (g + \alpha v_0) t$$

which makes sense, since it's just the result of uniform acceleration $g + \alpha v_0$, under the initial net force. For large times ($\alpha t \gg 1$), the exponentials decay away and we get $v(t) \approx -g/\alpha$, which is the terminal velocity.

(b) For a fixed height, consider how fast the projectile is moving when it passes that point going up or down. Since the gravitational potential energy is the same, and the drag force does only negative work, it must be going slower on the way down. Since it's going slower at every point going down, the trip down has to take longer.

(c) I'm just asking this so you can exercise your intuition. It's not actually obvious, since the drag force makes the projectile turn around faster, but then slows it on the way down. It turns out that the total time is always shorter with linear drag. More generally, if the drag force is proportional to v^n, then it turns out that the trajectory with drag always takes less time for $n \geq 1$, but for $n < 1$ it depends on the initial speed. (This makes intuitive sense, as when n is high, the drag force rises quickly with speed. The speed will tend to be higher on the way up than the way down, so the effect of the drag force is more important on the upward part, where it points down.) You can find proofs of all these statements here.

[3] **Problem 2.** Now assume quadratic drag, $F = -\alpha m v^2$, which applies for fast-moving projectiles.

(a) Integrate Newton's second law to get an implicit equation for $v(t)$ with the same initial conditions as above. That is, you don't need to solve for $v(t)$, as it'll just make things messy.

(b) Your equation will only be valid when the projectile is going up; explain why.

(c) Find $v(t)$ for an object released from rest at time $t = 0$. (Hint: if needed, look up some standard integrals involving hyperbolic trigonometric functions. But don't worry about memorizing the results, since in competitions, any nontrivial integral needed will usually be given to you.)

(d) Integrate your answer to part (c) with respect to time to find $y(t)$, and verify that the answer makes sense at both small and large times.

Some people only call this quadratic case drag; they call the linear case viscous resistance. This is because they behave fundamentally differently at the microscopic level, as we will explore in **M7**.

Solution. (a) Newton's second law is

$$\frac{dv}{dt} = -g - \alpha v^2.$$

By the same reasoning as before, we find

$$\int_{v_0}^{v(t)} \frac{dv'}{g + \alpha v'^2} = -\int_0^t dt' = -t.$$

By nondimensionalizing the integral as described in **P1**, the left-hand side is

$$-t = \frac{1}{\sqrt{\alpha g}} \int_{v_0\sqrt{\alpha/g}}^{v(t)\sqrt{\alpha/g}} \frac{dx}{1+x^2} = \frac{1}{\sqrt{\alpha g}} \left(\tan^{-1}\left(v(t)\sqrt{\frac{\alpha}{g}} \right) - \tan^{-1}\left(v_0 \sqrt{\frac{\alpha}{g}} \right) \right)$$

where I pulled out a factor of $1/\sqrt{\alpha g}$ to get the right overall dimensions, then used dimensional analysis again to convert the integration bounds to dimensionless numbers. (You can also do this by ordinary u-substitution if you prefer.) This is essentially the final result. It can be solved for $v(t)$, but that just makes it look worse.

(b) The reason the equation only makes sense when the projectile is going up is that the force should always opposite the direction of motion, so we really wanted to solve $F = -m\alpha|v|v$. Equivalently, the sign of α changes when the direction of the velocity changes. This means our solution really should have two separate cases.

(c) By the same reasoning, we have

$$\int_0^{v(t)} \frac{dv'}{g - \alpha v'^2} = -t$$

where the changes are the initial condition and the sign of α. The left-hand side is

$$\frac{1}{\sqrt{\alpha g}} \int_0^{v(t)\sqrt{\alpha/g}} \frac{dx}{1-x^2} = \frac{1}{\sqrt{\alpha g}} \left(\tanh^{-1}\left(v(t)\sqrt{\frac{\alpha}{g}}\right)\right).$$

If you don't know this hyperbolic trig integral, you could also derive it by expanding $1/(1-x^2)$ in partial fractions and integrating each term. You will get a bunch of logarithms, which is equivalent to the hyperbolic tangent. However, if you don't know what the hyperbolic tangent is, you should look it up now, because such functions will be useful later!

Because of the simpler initial condition, we can get an explicit solution,

$$v(t) = -\sqrt{\frac{g}{\alpha}} \tanh(\sqrt{\alpha g}\, t).$$

The speed approaches $\sqrt{g/\alpha}$ with a timescale $1/\sqrt{\alpha g}$, a fact we could also have deduced by physical intuition and dimensional analysis. Actually, another way to arrive at this result is by just substituting $\alpha \to -\alpha$ in the answer for part (a)! This will produce the tangent of an imaginary number, which is in fact how the hyperbolic tangent is defined.

(d) Integrating with respect to time gives

$$y(t) = -\frac{1}{\alpha} \log(\cosh(\sqrt{\alpha g}\, t)).$$

In the large t limit, the cosh grows exponentially, so that

$$y(t) \approx -\frac{1}{\alpha}\sqrt{\alpha g}\, t = -\sqrt{\frac{g}{\alpha}}\, t$$

which is just motion at the terminal velocity. In the small t limit, we can approximate

$$y(t) \approx -\frac{1}{\alpha} \log\left(\frac{e^{\sqrt{\alpha g}\, t} + e^{-\sqrt{\alpha g}\, t}}{2}\right) \approx -\frac{1}{\alpha} \log\left(1 + \alpha g t^2/2\right) \approx -\frac{gt^2}{2}$$

as expected, as drag is negligible in this regime. And of course, when we say that t is large or small, we really mean that $\sqrt{\alpha g}\, t \gg 1$ or $\sqrt{\alpha g}\, t \ll 1$ respectively.

[3] **Problem 3.** A projectile of mass m is dropped from a height h above the ground. It falls and bounces elastically, experiencing the same quadratic drag as in problem 2. Find the maximum height to which it subsequently rises. (Hint: don't try to use your results from problem 2.)

Solution. The reason you shouldn't try to use the results from problem 2 is that they are in terms of time. Given how complicated the implicit expressions for $v(t)$ are, the expressions for $x(t)$ would be extremely clunky. And they're not necessary, because in this problem we don't care about the time-dependence at all; we just want to know the final height.

Another way to say this is that we aren't interested in $v(t)$, we're interested in $v(x)$. While the projectile is moving downward, we can integrate dv/dx to find the speed v_0 at the moment it hits the ground. Then, when it's moving upward, we integrate dv/dx until it has zero speed again, which is its final height. This will be a lot simpler than integrating dv/dt.

For the upward and downward trajectories, Newton's second law says

$$\frac{dv}{dt} = -g \pm \alpha v^2$$

and multiplying both sides by dt/dx gives

$$\frac{dv}{dx} = -\frac{g}{v} \pm \alpha v.$$

Separating and integrating, on the way down we have

$$\int_h^0 dx = \int_0^{-v_0} \frac{dv}{\alpha v - g/v} = \frac{1}{\alpha} \int_0^{-v_0} \frac{v\,dv}{v^2 - g/\alpha}.$$

Carrying out the integral and simplifying,

$$h = -\frac{1}{2\alpha} \log(1 - \alpha v_0^2/g).$$

Now, on the way up, we have

$$\int_0^{h'} dx = \int_{v_0}^0 \frac{dv}{-g/v - \alpha v} = \frac{1}{\alpha} \int_0^{v_0} \frac{v\,dv}{v^2 + g/\alpha}$$

and carrying out the integral gives

$$h' = \frac{1}{2\alpha} \log(1 + \alpha v_0^2/g).$$

Combining the two equations gives

$$h' = \frac{1}{2\alpha} \log(2 - e^{-2\alpha h})$$

which you can check has the right limits. Also note that g drops out, as required by dimensional analysis.

Remark

How does the top speed v of a rowboat depend on the number N of rowers? A light, fast-moving rowboat experiences quadratic friction, so that the drag force on it is proportional to $v^2 A$, where A is the submerged cross-sectional area of the boat. A boat designed for N rowers will have a submerged volume $V \propto N$, and a streamlined shape so that $A \propto V^{2/3}$.

Thus, the required power input is

$$P = Fv \propto v^3 N^{2/3}.$$

The power output by the rowers scale as N, and combining these results gives the amazingly weak dependence $v \propto N^{1/9}$, which agrees decently with Olympic rowing times.

Idea 1

An ordinary differential equation is any equation involving a quantity $x(t)$ and its derivatives. In introductory physics, we are usually concerned with a few very simple differential equations, with the following nice properties.

- The differential equation is at most second order, meaning it can contain x, its first derivative $\dot{x} = v$, and its second derivative $\ddot{x} = a$, but no higher derivatives. This implies the solution can be determined by an initial position and initial velocity. (We'll focus on second order differential equations for the rest of this section; most first order differential equations can simply be solved by separation and integration, as we've seen above.)
- The differential equation is linear, meaning that terms don't contain products of x, $\dot{x}$, and $\ddot{x}$. For example, a damped driven harmonic oscillator with time-dependent drag,

  $$m\ddot{x} = -b(t)\dot{x} - kx + f(t)$$

  is a second order linear differential equation. Solutions to such differential equations obey the superposition principle: if $x_1(t)$ and $x_2(t)$ are both solutions, so is $c_1 x_1(t) + c_2 x_2(t)$.
- The differential equation is homogeneous, meaning that each term is proportional to exactly one power of x or its derivatives. The above differential equation is not homogeneous, but it would be if we removed the driving $f(t)$.
- The differential equation is time-translation invariant, meaning that no functions of time appear except for x and its derivatives. The above equation isn't, but it would be if we set $f(t)$ and $b(t)$ to constants.

Idea 2

Linear, homogeneous, time-translation invariant differential equations are very special, and they can all be solved by the exact same method. First, note that we can promote $x(t)$ to a complex variable $\tilde{x}(t)$ and solve the differential equation over the complex numbers. As long as we have a complex solution, we can recover a real solution by taking the real part. Now, the method of solution, which works for *almost* all equations of this form, is to guess a complex exponential solution

$$\tilde{x}(t) = e^{i\omega t}.$$

Plugging this into the differential equation will yield the allowed values of ω, and the general solution can be found by superposing the complex exponentials.

Example 2

Solve the simple harmonic oscillator, $m\ddot{x} + kx = 0$, using the above principles.

Solution

First, we pass to a complex differential equation,

$$m\ddot{\tilde{x}} + k\tilde{x} = 0.$$

We guess $\tilde{x}(t) = e^{i\omega t}$. Plugging this in and using the chain rule gives

$$m(i\omega)^2 e^{i\omega t} + ke^{i\omega t} = 0$$

and canceling $e^{i\omega t}$ and solving gives two solutions,

$$\omega = \pm\omega_0, \quad \omega_0 = \sqrt{k/m}.$$

Since this a second-order linear differential equation, the general solution is given by the superposition of these two complex exponentials,

$$\tilde{x}(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$

where A and B are general complex numbers. The real part of $\tilde{x}(t)$ satisfies the original real differential equation $ma + kx = 0$, and is

$$\operatorname{Re} x(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)$$

where C and D are real numbers.

[1] **Problem 4.** To make sure you know how to go from the complex solution to the real one, write C and D in terms of A and B.

Solution. Let $A = a_A + b_A i$ and $B = a_B + b_B i$ where a_i, b_i are real. Applying Euler's formula,

$$\operatorname{Re} x(t) = (a_A + a_B)\cos(\omega_0 t) + (-b_A + b_B)\sin(\omega_0 t)$$

from which we read off

$$C = \operatorname{Re}(A + B), \quad D = \operatorname{Im}(B - A).$$

[2] **Problem 5.** Now introduce a damping force and solve the differential equation for the damped harmonic oscillator, $m\ddot{x} + b\dot{x} + kx = 0$, using the same procedure, assuming b is small. (See section 4.3 of Morin if you have trouble with this. We'll consider this system in more detail in **M4**.)

Solution. Guessing an exponential, every time derivative yields a factor of $i\omega$, so

$$m(i\omega)^2 + b(i\omega) + k = 0.$$

Using the quadratic formula,

$$\omega = \frac{-ib \pm \sqrt{4km - b^2}}{-2m}.$$

In other words, we have

$$\omega = \pm\sqrt{k/m - b^2/4m^2} + \frac{ib}{2m}.$$

The oscillation is slightly slowed down, as you might expect, and the frequency has an imaginary part. This corresponds to exponential decay of the solution, by $e^{i(ib)t/2m} = e^{-bt/2m}$.

[3] **Problem 6.** USAPhO 2012, problem B1.

[3] **Problem 7.** Above, we mentioned that guessing an exponential works *almost* all the time. The reason is because at the end of the day, the exponential cancels out and we're left with a polynomial in ω, which has just the right number of roots. But if there are repeated roots, there are fewer distinct solutions for ω, and hence not enough solutions.

(a) Consider a second order differential equation with a double root ω. What is the other solution, besides $e^{i\omega t}$? (Hint: to help find a good guess, consider the simple case $ma = 0$, where $\omega = 0$ is the double root. Then generalize your guess to nonzero ω and check that it works.)

(b) This should be setting off alarm bells: the form of the solutions to the equation changes when the two roots are *exactly* equal, while it's just exponentials/sinusoids if the roots are different, no matter how small the difference is. Since no two roots are *ever* exactly equal in practice, it seems the behavior of part (a) can never actually happen in the real world. But it gets taught in applied differential equations courses. Why?

(c) **[A]** Consider the most general n^{th} order, linear homogeneous time-translation invariant differential equation

$$\left(a_n \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1 \frac{d}{dt} + a_0\right) x = 0.$$

What does the general solution look like?

Solution. (a) In the case of a double root $\omega = 0$, the differential equation is $\ddot{x} = 0$. The solution we get by guessing an exponential is $x(t) = e^{i(0)t} = 1$, which is a constant. The other solution is linear, $x(t) = te^{i(0)t} = t$. This leads us to guess that for a double root ω, the two independent solutions are $e^{i\omega t}$ and $te^{i\omega t}$.

(b) As two roots get closer and closer together, we can get solutions that look more and more like $(A+Bt)e^{i\omega t}$, which is what we would get if they were exactly the same. Of course, it's intuitive that we can get $e^{i\omega t}$. To get a solution that looks like $te^{i\omega t}$, note that for roots $\omega \pm \Delta\omega$,

$$e^{i(\omega+\Delta\omega)t} - e^{i(\omega-\Delta\omega)t} = e^{i\omega t}(2i\sin(\Delta\omega\, t)) \propto \sin(\Delta\omega\, t)e^{i\omega t}.$$

This is an $e^{i\omega t}$ oscillation with a slowly varying envelope $\sin(\Delta\omega\, t)$. For small times, $t \ll 1/\Delta\omega$, the envelope is just proportional to t. As the roots get closer and closer together, this linear behavior persists for longer and longer time, but nothing is ever discontinuous. One can see this kind of envelope behavior in two weakly coupled pendulums, a system which has two nearby oscillation frequencies. You'll investigate this kind of thing in more detail in **M4**.

So the point is that when the roots are to each other, the solutions look like $(A + Bt)e^{i\omega t}$ for times shorter than $1/\Delta\omega$. This is more direct and intuitive than superposing two sinusoids with almost equal frequencies, so we use it in practice.

(c) Guessing $e^{i\omega t}$ gives
$$a_n(i\omega)^n + a_{n-1}(i\omega)^{n-1} + \ldots + a_0 = 0.$$
In the case where the roots are distinct, there are n possible values for ω, and hence n parameters in our trial solution,
$$x(t) = \sum_{i=1}^{n} A_i e^{i\omega_i t}.$$
Since the differential equation has order n, there are n parameters needed to specify the solution, so this is the general solution. If ω_i is a double root, then both $e^{i\omega_i t}$ and $te^{i\omega_i t}$ are solutions. For a triple root, $t^2 e^{i\omega_i t}$ is also a solution, and so on.

Remark

You might be wondering how to solve more general differential equations. In **M4**, we will consider three extensions of the above techniques. We'll use the idea of normal modes to solve systems of such differential equations, add driving forces to make the equations inhomogeneous, and use the adiabatic theorem to approximately solve non time-translation invariant equations where the coefficients change slowly in time.

Of course, this just scratches the surface of the subject, and solving more general differential equations can be orders of magnitude harder. We won't try to solve nonlinear differential equations, as there is no general technique for doing so, and the answer is often an obscure special function. (However, such equations will occasionally appear in later problems.) On the other hand, linear differential equations with general time-dependence are more approachable, and the following problem illustrates the most basic method for solving them.

[3] **Problem 8.** [A] Some linear, homogeneous, non time-translation invariant differential equations can be solved by simply guessing a power series. For this problem, don't worry about dimensional analysis; assume all variables have already been redefined to be dimensionless.

(a) As a warmup, consider the differential equation $\dot{x} = kx$ for constant k, which we already know how to solve. By plugging in the ansatz
$$x(t) = \sum_{n=0}^{\infty} a_n t^n$$
with unknown constant coefficients a_n, find the solution with $x(0) = 1$.

(b) Now consider the non time-translation invariant differential equation
$$t^2\ddot{x} + t\dot{x} + t^2 x = 0$$
which is called Bessel's differential equation of order zero. By using the same ansatz, find the unique solution with $x(0) = 1$ and $\dot{x}(0) = 0$.

Solution. (a) Plugging the ansatz in gives
$$\sum_{n=0}^{\infty} n a_n t^{n-1} = k \sum_{n=0}^{\infty} a_n t^n.$$

Shifting the sum on the left-hand side, we have

$$\sum_{n=0}^{\infty}(ka_n - (n+1)a_{n+1})t^n = 0.$$

For this quantity to be zero for all t, each term in the sum must individually be zero, so

$$a_{n+1} = \frac{k}{n+1}a_n.$$

The initial condition $x(0) = 1$ tells us that $a_0 = 1$, from which we conclude

$$a_1 = k, \quad a_2 = \frac{k^2}{2}, \quad a_3 = \frac{k^3}{6}, \ldots$$

or more generally,

$$x(t) = \sum_{n=0}^{\infty} \frac{k^n}{n!}t^n = e^{kt}$$

which is just as expected.

(b) Plugging the ansatz in gives

$$\sum_{n=0}^{\infty} n(n-1)a_n t^n + na_n t^n + a_n t^{n+2} = 0.$$

Simplifying and shifting the sum as in part (a) gives

$$\sum_{n=0}^{\infty}(n^2 a_n + a_{n-2})t^n = 0.$$

We therefore have the recursion relation $a_n = -a_{n-2}/n^2$. The initial conditions give $a_0 = 1$ and $a_1 = 0$, from which we conclude the a_{2n+1} are all zero. We then have

$$a_2 = -\frac{1}{2^2}, \quad a_4 = \frac{1}{2^2 4^2}, \quad a_6 = -\frac{1}{2^2 4^2 6^2}, \ldots$$

from which we conclude

$$x(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{t}{2}\right)^{2m}.$$

This function is known as the Bessel function of the first kind, of zeroth order, $J_0(t)$.

2 Tricks

In this section we'll consider some kinematics problems that require cleverness, not computation.

Idea 3

Many problems can be solved by a clever choice of reference frame. It is often useful to go to the frame moving with one of the objects in the problem, or to go into a frame that makes the motion in the problem more symmetric. For the purposes of kinematics it can even be useful to use noninertial reference frames, such as a falling frame where projectiles don't accelerate, or a rotating frame, though this will introduce fictitious forces into the dynamics. It is also useful to tilt the coordinate axes to be parallel to various objects.

Example 3: $F = ma$ 2022 B4

A firework explodes, sending shells in all directions. Suppose the shells are all launched with the same speed, and ignore air resistance, but not gravity. What shape do the shells make?

Solution

In the absence of gravity, the shells would always form a sphere. Adding gravity simply shifts all of their locations downward by $gt^2/2$, so the shape is still always a sphere.

[1] **Problem 9** (KoMaL 2019)**.** A cannon A is at the edge of a cliff with a 800 m drop. Cannon B is on the ground below the cliff and 600 m horizontally away from it. Cannon A shoots a cannonball directly towards cannon B at 60 m/s. Cannon B shoots a cannonball directly towards cannon A at 40 m/s. Will the two cannonballs hit each other in midair?

Solution. Work in the frame freely falling with the cannonballs. In this case, the balls have a relative velocity of 100 m/s and initial separation of 1000 m, so it takes 10 s to collide. If there were no gravity, this collision would occur at a point (2/5)(800 m) = 320 m above the ground. However, because of gravity both balls have fallen by an extra $gt^2/2 = 500\,\text{m}$ by this time. Hence the balls hit the ground before they can hit each other in midair.

[2] **Problem 10** (Wang)**.** Two particles are released in gravitational acceleration g with leftward and rightward speeds v_1 and v_2. Find the distance between them when their velocities are perpendicular.

Solution. After time t, the velocity vectors are $(-v_1, -gt)$ and $(v_2, -gt)$. These are perpendicular when the dot product is zero, so $v_1 v_2 = (gt)^2$, which you can also show with basic geometry. Thus,

$$t = \frac{\sqrt{v_1 v_2}}{g}.$$

To compute the distance, we can just work in the frame falling with the masses. Then it's clear that the acceleration g doesn't matter, and the distance is just

$$d = (v_1 + v_2)t = \frac{(v_1 + v_2)\sqrt{v_1 v_2}}{g}.$$

[3] **Problem 11** (Kalda)**.** Two intersecting circles of radius r have centers a distance a apart. If one circle moves towards the other with speed v, what is the speed of one of the points of intersection?

Solution. Work in the frame where the circles are moving towards each other with speed $v/2$. Then by the Pythagorean theorem, the speed of the point of intersection is

$$\frac{d}{dt}\sqrt{r^2 - (a/2)^2} = \frac{av}{4\sqrt{r^2 - a^2/4}}$$

where we used $da/dt = v$. However, we're not done yet, because the speed of the point of intersection depends on the frame; we need to go back to the original frame. Using the Pythagorean theorem again, the answer is

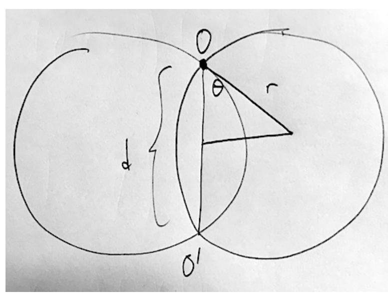
$$\sqrt{\left(\frac{av}{4\sqrt{r^2 - a^2/4}}\right)^2 + \left(\frac{v}{2}\right)^2} = \frac{v}{2}\frac{1}{\sqrt{1-(a/2r)^2}}.$$

[2] **Problem 12** (Kalda). A mirror rotates about its center with angular speed ω. A stationary point source of light sits at a distance a from the rotation axis. What is the speed of its mirror image?

Solution. Work in the frame rotating with the mirror. Because the image is always flipped across the mirror with respect to the source, since the source rotates with angular velocity $-\omega$, the image rotates with angular velocity ω. Then the relative angular velocity of the source and image is 2ω, which holds in all frames. Thus, in the original frame the image has angular velocity 2ω and speed $2\omega a$.

[2] **Problem 13** (Kalda). Two circles of radius r intersect at the point O. One of the circles rotates about the point O with constant angular speed ω. The other point of intersection O' is originally a distance d from O. Find the speed of O' as a function of time.

Solution. Remarkably, the answer does not depend on the time! Let d be the distance between the points of intersection, and work in the rotating frame where the circles rotate with angular velocities $\omega/2$ and $-\omega/2$ about O.



Since $\dot{\theta} = \omega/2$ and $\cos\theta = d/2r$, we have

$$-\frac{\omega}{2}\sin\theta = \frac{\dot{d}}{2r}, \quad \dot{d} = -r\omega\sin\theta.$$

This is the vertical velocity of O'. Now we need to go back to the original frame, which involves rotating with angular velocity $\omega/2$ about O. Then O' picks up a horizontal velocity of $(2r\cos\theta)(\omega/2)$ for a total speed of

$$v = \sqrt{r^2\omega^2\sin^2\theta + r^2\omega^2\cos^2\theta} = r\omega$$

which is constant. The geometrical reason is that the second intersection point rotates around the nonrotating circle with uniform angular velocity ω, as you can show by some angle chasing.

Idea 4

To find the minimum value of some quantity, it's often useful to think about all possible values of that quantity. This can reveal a solution using geometry or symmetry.

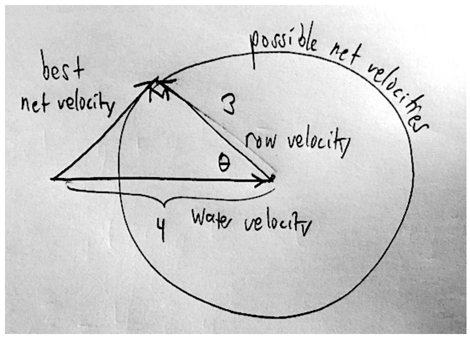
[2] **Problem 14** (PPP 3). A boat can travel a speed of 3 m/s on still water. A boatman wants to cross a river while covering the shortest possible distance.

(a) In what direction should he row if the speed of the water is $2\,\mathrm{m/s}$?

(b) How about if it is $4\,\mathrm{m/s}$?

Solution. (a) The boatman can completely cancel out the horizontal velocity of the water. He should row an angle $\cos^{-1}(2/3)$ from the upstream direction, so that the boat moves directly across the river.

(b) The boatman cannot cancel out the horizontal velocity. Instead, the set of possible velocities forms a circle in velocity space, as shown.



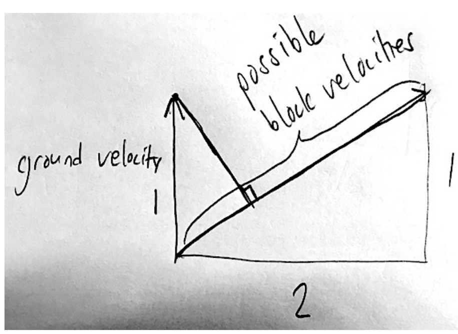
By taking the velocity with the angle closest to directly across the river, we see the boatman should row an angle $\cos^{-1}(3/4)$ from the upstream direction.

Idea 5

In problems with friction, the best reference frame to use is almost always the frame of whatever is causing the friction.

[2] **Problem 15** (Kalda). A block is pushed onto a conveyor belt. The belt is moving with speed $1\,\mathrm{m/s}$, and the block's initial speed is $2\,\mathrm{m/s}$, with initial velocity perpendicular to that of the belt. During the subsequent motion, what is the minimum speed of the block with respect to the ground?

Solution. If the belt were not moving, the block would just decelerate in the direction of its speed, so that's what happens in the reference frame of the belt. The possible block velocities are shown in this frame.



The minimum relative speed with the ground is shown by the altitude, which has length $2/\sqrt{5}\,\mathrm{m/s}$ by similar triangles.

Idea 6

For a variety of kinematics problems, it can be useful to think about the motion from a different perspective. For example, if your problem involves complicated accelerations, it can be useful to think in "velocity space", i.e. directly think about how the velocity vector evolves over time, and deal with the position later. Or, if your problem involves complicated processes occurring in time, it can be useful to think in "spacetime", meaning to visualize the process on a space where time is one of the axes. It can also be useful to parametrize motion in terms of quantities other than the usual Cartesian coordinates.

[2] **Problem 16** (Kalda)**.** A boy enters a patch of ice with a coefficient of friction μ with speed v. By running on the ice, the boy turns his velocity vector by $90°$ in the minimum possible time, so that his final speed is also v. What is the minimum possible time, and what kind of curve is the trajectory? Assume the normal force with the ice is constant.

Solution. The acceleration always has magnitude μg. The velocity needs to change by $v(\hat{\mathbf{x}} - \hat{\mathbf{y}})$ if it starts at $v\hat{\mathbf{y}}$, so $v\sqrt{2} = \mu g t$. Thus, $t = \frac{v}{\mu g}\sqrt{2}$. The acceleration is constant, so the trajectory is a parabola.

There are many other possible "optimal turning" problems. For a much tougher relative of this problem, see Physics Cup 2022, problem 2, and for an explanation of the general theory, see this paper.

[2] **Problem 17** (PPP 5)**.** Four snails travel in uniform, rectilinear motion on a plane. The velocities are chosen so that three snails never meet at once, and no two of the velocities are equal. Since time $t = -\infty$, five of the $\binom{4}{2}$ possible encounters have already occurred. Must the sixth also occur?

Solution. It's hard to visualize what's going on in the plane; instead think about what's going on in *spacetime*. The spacetime here is three-dimensional, and the paths of the worms are lines through it, called worldlines; two worms will encounter each other if their worldlines intersect. For some set of three of the snails, all possible encounters occur, so their worldlines lie in a plane in spacetime. (This means that in space, these three snails move on the same line.)

If the fourth snail's worldline lies in this plane, then it must intersect all three others. If it doesn't, it can intersect at most one. Hence if five encounters have already occurred, the sixth must also occur.

[2] **Problem 18.** Six bugs are placed at the vertices of a regular hexagon with side length s. At time $t = 0$ each bug starts moving directly towards the next with speed v. At what time do they collide?

Solution. By symmetry, the bugs always remain in a hexagon shape, but this hexagon rotates and shrinks. We want to know the time when it collapses completely.

We can first do this by considering how the distance between adjacent bugs changes in an infinitesimal time dt. The first bug moves a distance $v\,dt$ towards the second. The second moves a distance $(\sqrt{3}/2)v\,dt$ to the side, and a distance $(v/2)\,dt$ directly away from the first. The side-to-side motion doesn't contribute to the change in distance (one can use the Pythagorean theorem and binomial theorem to show it is second order, and hence negligible for infinitesimals), so we ignore it. Then the rate of change of distance between the bugs is just $v - v/2 = v/2$, so the bugs meet at $t = 2s/v$.

Another method is to note that all the bugs meet in the center of the original hexagon, so we can consider the component of velocity for each bug directed towards the center. This is always

$v/2$ by the hexagonal symmetry, and the original distance from the center is s, so the bugs again meet in time $t = 2s/v$.

Example 4

A rabbit begins at the origin, and the fox begins at the point $(0, -a)$. The rabbit begins running east, with a constant speed $v\hat{\mathbf{x}}$. At the same time, the fox begins chasing the rabbit, always moving towards it with speed v. After a long time, the rabbit and wolf simply follow each other in a straight line, with a constant separation d. What is d?

Solution

This is the simplest example of a persuit problem. Physicists and mathematicians have been posing them for centuries, though most are too mathematically involved for Olympiads.

Here, the trick to realize that if the displacement between the rabbit and fox is $\mathbf{r}(t) = (x(t), y(t))$, then the quantity $r + x$ is conserved. To see this, let θ be the angle between the rabbit and fox's velocity vectors. Then

$$\frac{dr}{dt} = -v + v\cos\theta$$

because of the fox's chasing and rabbit's motion, and

$$\frac{dx}{dt} = v - v\cos\theta$$

because of the rabbit's motion and fox's chasing. Then $r + x$ is constant. Initially $r + x = a + 0 = a$, and after a long time $r = x = d$, so the final separation is $d = a/2$.

[2] **Problem 19.** Suppose the fox in the above example instead has speed $u > v$. How long does it take to catch the rabbit?

Solution. Solving for the trajectory of the fox is extremely difficult, but we can use an extension of the example. Now the equations of motion are

$$\frac{dr}{dt} = -u + v\cos\theta, \quad \frac{dx}{dt} = v - u\cos\theta.$$

Combining these equations, we can cancel out θ to get

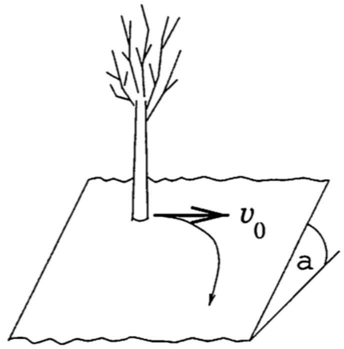
$$u\frac{dr}{dt} + v\frac{dx}{dt} = v^2 - u^2.$$

This can now easily be integrated from between the initial and final time. During this time, the change in r is $-a$, while the change in x is zero, so

$$-au = (v^2 - u^2)t, \quad t = \frac{ua}{u^2 - v^2}.$$

If you're curious what the full trajectory looks like, you can find it in this paper, which was written by a past coach of the U.S. Physics Team.

[2] **Problem 20** (PPP 85)**.** A child is on an icy hill, which may be modeled as an inclined plane.



The coefficient of friction $\mu_k = \mu_s$ is small enough so that, if the child gets the tiniest push, she will begin sliding down the plane. Now suppose the child gets a horizontal push, with initial speed v_0. What is the child's final speed?

Solution. This is easy because you've already solved the problem; it's just the same thing as problem 4. Specifically, the displacement between the rabbit and fox there corresponds to the velocity of the child here. At every increment of time dt, the velocity changes in two ways: it shrinks along its direction by $\mu g \cos\theta \, dt$ due to friction (corresponding to the fox) and it gains a component $g \sin\theta \, dt$ in a fixed direction due to gravity (corresponding to the rabbit). Furthermore, the problem statement implies the coefficient of friction is just enough to prevent sliding from rest, so that $\mu = \tan\theta$ and these two magnitudes are equal. Thus, the problem is exactly analogous to problem 4(c) and the answer is $v_0/2$.

You should definitely not try to solve for the trajectory exactly, since it's very messy, but you can find the gory result in this paper.

[4] **Problem 21.** EuPhO 2023, problem 2. (Warning: the algebra will be a bit messy.)

Solution. See the official solutions here.

3 Motion in Two Dimensions

Idea 7

Often, motion in two dimensions can be treated as two independent one-dimensional problems. A change of reference frame may be necessary first.

Idea 8

In problems involving an inclined plane, it helps to draw the inclination angle θ to be less than about $30°$. This reduces mistakes, because almost every angle will be either θ or $90° - \theta$, and you can identify which by sight.

Example 5

Consider projectile motion where wind provides a constant horizontal force F. At what angle should a projectile of mass m be launched in order to return to the thrower?

Solution

The key idea is to use tilted coordinate systems. Clearly, when the only force is downward, the projectile must be launched straight upward. Now, the horizontal force acts like an effective horizontal gravitational acceleration of F/m, so that gravity is effectively tilted an angle $\tan^{-1}(F/mg)$ away from the vertical. One must launch the projectile directly "upward" with respect to this effective gravitational field, so the launch angle is an angle $\tan^{-1}(F/mg)$ from the vertical. (For a related problem, see the infamous $F = ma$ 2014 problem 19.)

Example 6: $F = ma$ 2022 A23

For projectiles, the force of air resistance can be modeled as proportional to the speed ("linear drag") or proportional to the square of the speed ("quadratic drag"), depending on the circumstances. Two identical objects, A and B, are dropped from the same height h simultaneously, but object A is given an initial horizontal velocity v. The objects hit the ground at times t_A and t_B. How do these times compare, assuming linear or quadratic drag?

Solution

For linear drag, the horizontal and vertical components of the motion are independent,

$$a_x = -bv_x, \quad a_y = -g - bv_y$$

for some coefficient b. That means the time to hit the ground, which depends on the vertical motion, is independent of the initial horizontal velocity, so $t_A = t_B$. But for quadratic drag,

$$a_y = -g - bv_y|v|$$

which means the upward drag force is larger when the horizontal velocity is larger, so $t_A > t_B$.

Since the components are independent for linear drag, it's not too hard to write down an expression for the trajectory, by recycling the results of example 1. But for quadratic drag, the results of problem 2 won't help much; the two-dimensional problem is much harder.

[1] **Problem 22** (Quarterfinal 2002)**.** A cart is rigged with a vertical cannon so that, when the cart is stationary on a horizontal track, the cannonball is fired straight up and lands back in the cannon. In each of the following situations, does the cannonball land back in the cannon, in front of it, or behind it?

(a) The cart is moving on a frictionless horizontal track with speed v.

(b) The cart is accelerating down a frictionless inclined track with angle θ.

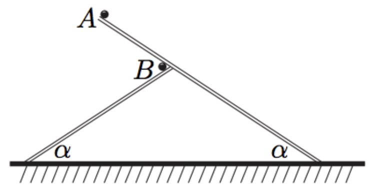
(c) The cart is accelerating down an inclined track with angle θ, and friction slows it down.

Solution. (a) The motion in the x and y directions is independent. In the x direction, both the cannonball and cart just continue moving with speed v, so the cannonball lands right back into the cannon.

(b) Work in the tilted frame where the x axis is parallel to the track. In the x direction, both the cannonball and cart start with the same speed v and accelerate with the same acceleration $g\sin\theta$, so the cannonball lands right back into the cannon, again.

(c) In this case the cart accelerates less, so the cannonball lands in front.

[2] **Problem 23** (Kalda)**.** Two balls at points A and B are released from rest at the same moment, from the locations shown below. All surfaces are frictionless.



If it takes time t_A and t_B for the balls to hit the ground, at what time was the distance between the balls the smallest?

Solution. Both balls have a downward acceleration of $g\sin\alpha$, and they have leftward and rightward accelerations of $g' = g\sin\alpha\cos\alpha$. Since the balls always have the same vertical speed, we can ignore the vertical motion entirely. The distance between the balls is thus smallest when their horizontal separation is zero.

Let the total horizontal distances the balls travel be d_A and d_B. Then

$$d_A = \frac{1}{2}g't_A^2, \quad d_B = \frac{1}{2}g't_B^2$$

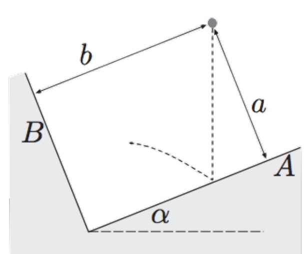
and we are looking for the time t where

$$\frac{d_A - d_B}{2} = \frac{1}{2}g't^2.$$

Solving these equations for t gives

$$t = \sqrt{\frac{t_A^2 - t_B^2}{2}}.$$

[2] **Problem 24** (Kalda)**.** Two planar frictionless walls are placed at right angles, where wall A makes an angle α to the horizontal. A perfectly elastic ball is released from rest at a point a distance a from wall A and b from wall B.



After a long time, what is the ratio of the number of times the ball has bounced against wall B to the number of times it has bounced against wall A?

Solution. In the coordinate system tilted by angle α, the motions in the x and y directions are independent, because collisions with wall A leave v_x unchanged and vice versa. In the y direction, the ball simply bounces up and down with uniform acceleration $g\cos\alpha$ and bounce height a, so

$$\Delta t_A = 2\sqrt{\frac{2a}{g\cos\alpha}}.$$

By similar reasoning, in the x direction

$$\Delta t_B = 2\sqrt{\frac{2b}{g\sin\alpha}}.$$

Thus the answer is

$$\frac{\Delta t_A}{\Delta t_B} = \sqrt{\frac{a\sin\alpha}{b\cos\alpha}}.$$

When this ratio is a rational number, the ball eventually returns to its starting point. If it isn't, it never does; instead it eventually explores all of the space permitted by energy conservation, i.e. it eventually passes arbitrarily close to any point whose height is at most the height of the starting point.

[2] **Problem 25.** USAPhO 2004, problem A4.

[3] **Problem 26** (EFPhO 2010)**.** A sprinkler can be modeled as a small hemisphere on the ground. Water shoots out from the hemisphere in all directions, with speed v perpendicular to the hemisphere.

(a) Find the total surface area of ground watered by the sprinkler.

(b) At what distance from the sprinkler does the ground get the wettest?

Solution. (a) The range of the sprinkler is maximized at 45° and is equal to v^2/g. Then the area is $\pi(v^2/g)^2 = \pi v^4/g^2$.

(b) The outermost circle, at radius v^2/g, gets by far the wettest. This is because a maximum of radius is achieved here, so a large range of launch angles gets to near this radius. (It's the same reason that balls thrown upward spend the most time near the very top of their trajectories.)

This idea is a little tricky, but very general; for instance, it's the principle behind the formation of caustics such as rainbows, as we'll see in **W3**. It is also the way in which classical mechanics emerges from quantum mechanics: classically things follow the trajectory of least action because it's a caustic of the quantum sum over all trajectories. So if you continue in physics, you'll see this beautiful little idea over and over again, in richer and richer settings! For an Olympiad problem that gives a bit more detail about caustics in optics, see here.

[3] **Problem 27.** USAPhO 2023, problem A1. A neat exercise on collisions and projectile motion.

Example 7

A bug flies towards a light with constant speed v, always making an angle α with the radial direction. If the initial distance to the lamp is L and the radius of the lamp is R, through what total angle does it turn before hitting the lamp?

Solution

In this case we can't avoid solving differential equations, but they're not too hard. It's easiest to work in polar coordinates, with the center of the lamp at the origin. By decomposing the velocity into radial and tangential components, we have

$$\frac{dr}{dt} = -v\cos\alpha, \quad r\frac{d\theta}{dt} = v\sin\alpha.$$

We only care about the path, not the time-dependence, so we divide these equations to get

$$\frac{dr}{d\theta} = -\frac{r}{\tan\alpha}$$

where we manipulated differentials as in **P1**. Separating and integrating,

$$-\int_L^R \frac{dr}{r} = \frac{\Delta\theta}{\tan\alpha}$$

which tells us that

$$\Delta\theta = (\tan\alpha)\log\frac{L}{R}.$$

The shape traced out is a logarithmic spiral.

[2] **Problem 28.** The pilot of a supersonic jet airplane wishes to make a big noise at the origin by flying around it in a path such that all of the noise he makes is heard simultaneously at the origin. The jet travels with Mach number M, meaning that its speed is M times the speed of sound. If the pilot starts at $(r, \theta) = (a, 0)$, find the pilot's path $r(\theta)$.

Solution. In order for the sound to reach the origin simultaneously, we must have $r(t) = a - ct$, so that the sound all reaches the origin at time a/c. On the other hand, we have

$$(Mc)^2 = \dot{r}^2 + r^2\dot{\theta}^2 = c^2 + r^2\dot{\theta}^2.$$

This is a bit messy because we have two functions of time, but we can eliminate time by using

$$\dot{\theta} = \frac{d\theta}{dr}\frac{dr}{dt} = c\frac{d\theta}{dr}.$$

Plugging this in above, we have

$$M^2 - 1 = r^2\left(\frac{d\theta}{dr}\right)^2$$

and separating and integrating gives

$$\int \frac{dr}{r} = \int \frac{d\theta}{\sqrt{M^2-1}}, \quad r(\theta) = ae^{-\theta/\sqrt{M^2-1}}.$$

[4] **Problem 29.** Consider a mass m on a table attached to a spring at the origin with zero relaxed length, which exerts the force

$$\mathbf{F} = -k\mathbf{r}$$

on the mass. We will find the general solution for $\mathbf{r}(t) = (x(t), y(t))$ in two different ways.

(a) Directly write down the answer, using the fact that the x and y coordinates are independent.

(b) Sketch a representative sample of solutions. What kind of curve does the trajectory follow?

(c) ⋆ Here's a more unusual way to arrive at the same answer. Go to a noninertial reference frame rotating with angular velocity ω_0 about the origin, so that the centrifugal force cancels out the spring force. In this frame, the only relevant force is the Coriolis force $-2m\boldsymbol{\omega}_0 \times \mathbf{v}$. Find the general solution in this frame, then transform back to the original frame and show that you get the same answer as in part (a). (This can get a bit messy; the easiest way is to treat the plane as the complex plane, i.e. work in terms of the variable $r = x + iy$.)

Solution. (a) We just have two separate equations for each component,

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x, \quad \frac{d^2y}{dt^2} = -\frac{k}{m}y.$$

Both describe a harmonic oscillator with angular frequency $\omega_0 = \sqrt{k/m}$. Then the general solution can be written as

$$x(t) = A\cos(\omega_0 t + \phi_1), \quad y(t) = B\sin(\omega_0 t + \phi_2).$$

In general, it is very rare for the x and y coordinates to be independent. Another example of this type is projectile motion in linear drag, $\mathbf{F} = -k\mathbf{v}$. In these cases the 2D or 3D problem is no harder than the 1D version, but we're rarely so lucky.

(b) In the case where $\phi_1 = \phi_2 = 0$ and $A = B$, the mass moves in a circle centered at the origin. More generally, when the angles ϕ_i are unequal, the mass can move in an ellipse with center at the origin.

(c) The centrifugal force is $m\omega_0^2\mathbf{r}$, so to cancel the spring force we need to choose $\omega_0 = \sqrt{k/m}$. Now, in the rotating frame, the Coriolis force acts just like a magnetic field: it's always perpendicular to the motion, so the solution is circular motion. The angular frequency ω_c of that circular motion satisfies

$$2m\omega_0 v = \frac{mv^2}{r} = m\omega_c v$$

from which we conclude $\omega_c = 2\omega_0$. So in complex notation,

$$r(t) = r_0 + r_1 e^{2i\omega_0 t}$$

in the rotating frame. We can return to the original frame by simply multiplying by $e^{-i\omega_0 t}$, to give

$$r(t) = r_0 e^{-i\omega_0 t} + r_1 e^{i\omega_0 t}.$$

Taking real and imaginary parts and letting $r_i = a_i + ib_i$,

$$x(t) = (a_0 + a_1)\cos(\omega_0 t) + (b_0 - b_1)\sin(\omega_0 t), \quad y(t) = (b_0 + b_1)\cos(\omega_0 t) + (a_1 - a_0)\sin(\omega_0 t).$$

This is the same as our result for part (a), after you use the sine and cosine addition formulas and appropriately redefine the parameters. Evidently, elliptical motion is just the superposition of two opposite circular motions! (In general, complex numbers are a useful way to deal with magnetic or Coriolis forces for motion in a plane, where $\mathbf{B}$ or $\boldsymbol{\omega}$ points perpendicular to the plane. In these cases the force lies in the plane perpendicular to the velocity, so it's just proportional to $i\dot{r}$, which is nice and simple; we'll see this idea again later.)

4 Optimal Launching

Finally, we'll consider projectile motion questions that involve optimization. These are rare on the USAPhO, but they are quite fun problems, with occasionally very slick solutions.

Example 8

A bug wishes to jump over a cylindrical log of radius R lying on the ground, so that it just grazes the top of the log horizontally as it passes by. What is the minimum launch speed v required to do this?

Solution

Let P be the point at the top of the log. For the bug to be moving horizontally at P, energy conservation applied to the vertical motion gives an initial v_y obeying

$$\frac{1}{2}mv_y^2 = 2mgR, \quad v_y = 2\sqrt{gR}.$$

Thus, we need to find the minimum v_x for the motion to be possible. If v_x is too low, the hypothetical trajectory of the bug will instead pass through the log. At the lowest possible v_x, the bug's trajectory is not just tangent to the log at point P, but also has the same radius of curvature (i.e. the trajectory and the log's shape have the same first and second derivatives).

For uniform motion in a circle of radius r, the acceleration is $a = v^2/r$. Conversely, when an object follows a trajectory of instantaneous radius of curvature r, its acceleration component normal to the path must be $a = v^2/r$. So applying this to the bug at P gives

$$g = \frac{v_x^2}{R}, \quad v_x = \sqrt{gR}.$$

Thus, the minimum initial speed is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{5}\, gR.$$

This radius of curvature trick doesn't come up often, but it's cool when it does.

[2] **Problem 30.** NBPhO 2020, problem 3. A nice warmup for the problems below.

Solution. See the official solutions here.

[3] **Problem 31.** An object is launched from the top of a hill, where the ground lies an angle ϕ below the horizontal. Show that the range of a projectile is maximized if it is launched along the angle bisector of the vertical and the ground.

Solution. This is a straightforward if messy problem; we'll show one of many ways to set it up. Setting the origin at the launch point and using ordinary horizontal/vertical coordinates, the object hits the hill when $\tan\phi = -y/x$. Using results for projectile trajectories from the preliminary problem set, we have

$$\frac{y}{x} = -\tan\phi = \tan\theta - \frac{gx}{2v^2\cos^2\theta}$$

where θ is the launch angle from the horizontal. Solving for x,

$$x = \frac{2v^2 \cos^2 \theta}{g}(\tan\theta + \tan\phi) \propto \sin\theta\cos\theta + \cos^2\theta\tan\phi.$$

To maximize the range, we want to maximize x, so setting the derivative to zero gives

$$0 = \cos^2\theta - \sin^2\theta - 2\sin\theta\cos\theta\tan\phi$$

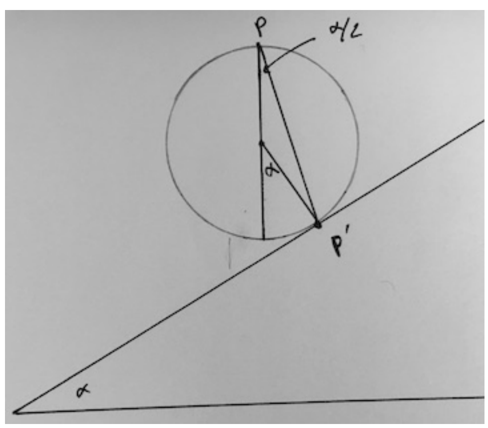
which simplifies to

$$\tan(2\theta) = \frac{1}{\tan\phi} = \tan\left(\frac{\pi}{2} - \phi\right), \quad \theta = \frac{(\pi/2) - \phi}{2}$$

as desired. This famous problem was first posed by Torricelli in the 1640s, and solved by Halley in the 1690s.

[3] **Problem 32** (PPP 35)**.** A point P is located above an inclined plane with angle α. It is possible to reach the plane by sliding under gravity down a straight frictionless wire, joining P to some point P' on the plane. Geometrically, how should P' be chosen so as to minimize the time taken? (Hint: think about the set of points that can be reached for all possible angles of the wire, after time t.)

Solution. Suppose the wire is at angle θ with respect to the vertical. Then, the distance traveled in time t is $\frac{1}{2}(g\cos\theta)t^2$. Putting the origin at P, for a fixed t, the locus of all reached points is of the form $r \propto \cos\theta$, which is the polar representation of a circle whose topmost point is P. The diameter of the circle is $\frac{1}{2}gt^2$. Therefore, P' is the point where one of these circles is tangent to the incline, so an $\alpha/2$ angle to the vertical.



Idea 9

Since mechanics is time-reversible, and the speed of a projectile only depends on its height and not the path taken, finding the way to reach point B from point A with the lowest possible initial speed is the same as finding the way to reach point A from point B with the lowest possible initial speed.

[4] **Problem 33.** Two fences of heights h_1 and h_2 are erected on a horizontal plain, so that the *tops* of the fences are separated by a distance d. Show that the minimum speed needed to throw a projectile over both fences is $\sqrt{g(h_1 + h_2 + d)}$.

Solution. It's very confusing to think about how to throw the projectile starting from the ground, because you need to figure out where to launch and at what angle, under the condition that the trajectory just touches the tops of both fences. A much better way is to imagine the projectile *starts* at the top of the higher fence; the goal is then to throw it with minimal energy so that it just touches the top of the lower fence. At some point, this projectile will then reach the ground, though we don't have to worry about where. Since mechanics is time-reversible, its speed at this point (which is found easily by energy conservation) will be the minimal possible speed.

Now there are many ways to do this problem. A very slick solution, which requires no computation at all, is presented in problem 34. However, we'll present a more direct attack for completeness. Note that if you want to hit the top of the lower fence with the minimum velocity, it's equivalent to maximizing your throwing range down an inclined plane, namely the plane that connects the tops of the two fences. Then the optimal launch angle is along the angle bisector, as we found in problem 31. Using the same starting point as the solution to that problem, we have

$$-\frac{h}{\sqrt{d^2-h^2}} = \tan\theta - \frac{g\sqrt{d^2-h^2}}{2v^2\cos^2\theta}$$

where we let $h = h_2 - h_1 > 0$. That solution gives a simple expression for $\tan 2\theta$, so we massage this equation to

$$\frac{g}{v^2} = \frac{\sin 2\theta}{\sqrt{d^2-h^2}} + \frac{h}{d^2-h^2}(1+\cos 2\theta).$$

We then plug in our previous results, which are

$$\sin 2\theta = \frac{\sqrt{d^2-h^2}}{d}, \quad \cos 2\theta = \frac{h}{d}$$

to get the result

$$v^2 = (d-h)g = (d+h_1-h_2)g.$$

By energy conservation, the speed at the ground is

$$v_0^2 = v^2 + 2h_2 g = (d+h_1+h_2)g$$

as desired.

[4] **Problem 34.** Problems 31 and 33 can be solved with pure geometry. Consider the set of points, in two dimensions, that a projectile can reach with a fixed initial speed v and a fixed launch point. It turns out that the boundary of this set (i.e. the curve of points that a projectile can just barely reach) is a vertical parabola with its focus at the launch point. A parabola is defined as the set of points whose distance to the focus equals the distance to a line, called the directrix.

(a) Show that trajectories that touch the parabola must be tangent to it.

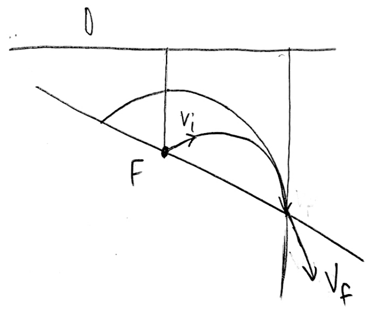
(b) Show that if a point is hit with the smallest possible initial speed, then the initial velocity must be perpendicular to the final velocity.

(c) Using the geometric definition of a parabola, recover the answers to problems 31 and 33.

Solution. (a) This is just because the parabola is defined to be the set of points you can hit. If the trajectory weren't tangent to the parabola, you would be able to hit a point outside the parabola by continuing it.

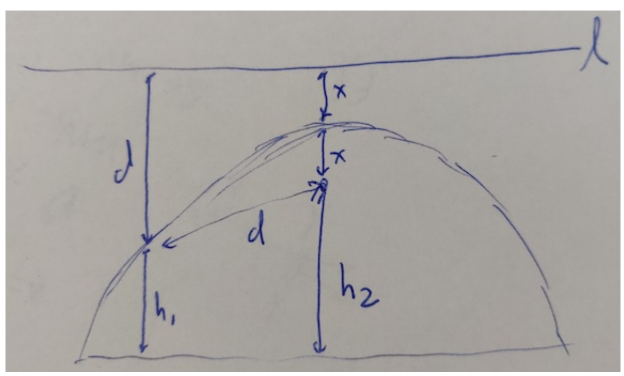
(b) Let $\mathbf{v}_i$ be the initial velocity and $\hat{v}_\perp$ be a unit vector in the perpendicular direction. If we replace the initial velocity by $\mathbf{v}_i + \epsilon\hat{v}_\perp$, where ϵ is infinitesimal, then the speed isn't changed, which implies that the new trajectory should remain inside the parabola. Now suppose the original projectile's velocity is $\mathbf{v}_f$ when it is tangent to the parabola, at position $\mathbf{r}_f$. Then at the same time, the new projectile's position is $\mathbf{r}_f + t\epsilon\hat{v}_\perp$. In order to keep this inside the parabola for all infinitesimal ϵ, both positive and negative, $t\epsilon\hat{v}_\perp$ must be tangent to the parabola at this point. Hence $\hat{v}_\perp$ is parallel to $\mathbf{v}_f$, so $\mathbf{v}_i$ is perpendicular to $\mathbf{v}_f$, as desired.

(c) A parabola is the set of points equidistant from a focus and a line, called the directrix. Refer to the diagram below.



The final velocity $\mathbf{v}_f$ is tangent to the parabola. Therefore, it points along the angle bisector between the vertical and the direction along the plane since the distance from the focus and the directrix will remain equal to each other. Now, $\mathbf{v}_i$ is perpendicular to this, which means it is along the angle bisector between the vertical and the downward direction along the plane, which is precisely the result we found in problem 31.

Assume $h_2 > h_1$. Draw the parabola of the projectile range with its focus on h_2. At the minimum launching velocity, the parabola should just touch the top of the h_1 fence. The (horizontal) directrix will be a distance $2x$ above h_2, where $x = v_2^2/2g$ and v_2 is the launching velocity from h_2. Then use the fact that the distances from a point on a parabola to the focus and directrix are the same.



From the picture, we see that $d + h_1 = 2x + h_2$. Thus the launching velocity at h_2 satisfies $v_2^2/g = d + h_1 - h_2$, and it has a total energy upon launching of

$$\frac{E}{m} = \frac{1}{2}v_2^2 + gh_2 = g\,\frac{d + h_1 - h_2}{2} + gh_2 = \frac{1}{2}mv_0^2.$$

This gives the answer with almost no computation,

$$v_0 = \sqrt{g(d + h_1 + h_2)}.$$

[3] **Problem 35.** IPhO 2012, problem 1A.

5 Reading Graphs

In some kinematics problems, you'll have to infer what's going on from a diagram. To make progress, you'll have to print out the diagram to make measurements directly on it.

[3] **Problem 36.** EFPhO 2015, problem 6.

Solution. See the official solutions here.

[3] **Problem 37.** EFPhO 2008, problem 3.

Solution. See the official solutions here.

Remark

For a harder problem from the same genre, see EuPhO 2019, problem 3. Almost all competitors received zero points on it, largely because it relies on a specialized trick introduced earlier in this problem set. You can try it for entertainment if you have time and really like kinematics. The official solutions are here.

Mechanics II: Statics

For review, read chapter 2 of Morin or chapter 2 of Kleppner and Kolenkow. Statics is covered in more detail in chapter 7 of Wang and Ricardo, volume 1. Surface tension is covered in detail in chapter 5 of *Physics of Continuous Matter* by Lautrup, which is an upper-division level introduction to fluids in general. There is a total of **83** points.

1 Balancing Forces

Idea 1

In principle, you can always solve every statics problem by balancing forces on every individual particle in the setup, but often you can save on effort by considering appropriate systems.

Idea 2

Any problem where everything has a uniform velocity is equivalent to a statics problem, by going to the reference frame moving with that velocity. Any problem where everything has a uniform acceleration $\mathbf{a}$ is also about statics, by going to the noninertial frame with acceleration $\mathbf{a}$, where there is an extra effective gravitational acceleration $-\mathbf{a}$. The same principle applies to uniform rotation, where a centrifugal force appears in the rotating frame, acting like an effective gravitational acceleration $\omega^2\mathbf{r}$.

Example 1

Six blocks are attached in a horizontal line with rigid rods, and placed on a table with coefficient of friction μ. The blocks have mass m and the leftmost block is pulled with a force F so the blocks slide to the left. Find the tension force in the rod in the middle.

Solution

There are six objects here and five rods, each with a different tension, so a direct analysis would involve solving a system of six equations. Instead, first consider the entire set of six blocks as one object; we can do this because the rigid rods force them to move as one. The total mass is $6m$, and applying Newton's second law gives

$$F - 6mg\mu = 6ma, \quad a = \frac{F}{6m} - \mu g.$$

Next, consider the rightmost three blocks as one object. Their total mass is $3m$, and their acceleration is the same acceleration a we computed above. This system experiences two horizontal force: tension and friction. Newton's second law gives

$$T - 3mg\mu = 3ma$$

and solving for T gives

$$T = \frac{F}{2}.$$

This is intuitive, because the differences of any two adjacent tension forces are the same; that's the amount of tension that needs to be spent to accelerate each block. So the middle rod, which has to accelerate only half the blocks, has half the tension.

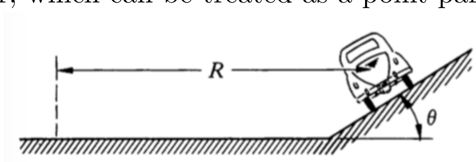
The reason we could ignore the tension forces in the other four rods is that the only thing they do is ensure the blocks move with the same acceleration. Once we assume this is the case, the specific values of the tensions don't matter; we can just zoom out and forget them. It's just like how *within* each block there is also an internal tension which keeps it together, but we rarely need to worry about its details.

Idea 3

To handle a problem where something is just about to slip on something else, set the frictional force to the maximal value μN and assume slipping is not yet occurring, so the two objects move as one. The same idea holds for problems which ask for the minimal force needed to make something move, or the minimal force needed to keep something from moving.

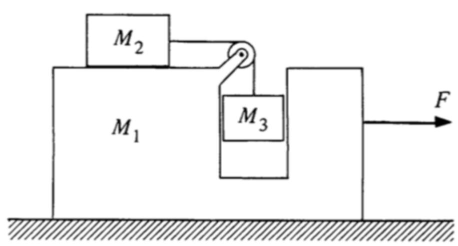
[1] **Problem 1** (KK 2.7)**.** A block of mass M_1 sits on a block of mass M_2 on a frictionless table. The coefficient of friction between the blocks is μ. Find the maximum horizontal force that can be applied to (a) block 1 or (b) block 2 so that the blocks will not slip on each other.

[2] **Problem 2** (KK 2.28)**.** A car, which can be treated as a point particle, enters a turn of radius R.



The road is banked at angle θ, and the coefficient of friction between the wheels and road is μ. Find the maximum and minimum speeds for the car to stay on the road without skidding sideways.

[2] **Problem 3** (KK 2.19)**.** A "pedagogical machine" is illustrated in the sketch below.



All surfaces are frictionless. What force F must be applied to M_1 to keep M_3 from rising or falling?

[3] **Problem 4.** USAPhO 2017, problem A1.

2 Balancing Torques

Idea 4

A static rigid body will remain static as long as the total force on it vanishes, and the total torque vanishes, where the torque about the origin is

$$\boldsymbol{\tau} = \sum_i \mathbf{r}_i \times \mathbf{F}_i$$

where $\mathbf{r}_i$ is the point of application of force $\mathbf{F}_i$. If the total force vanishes, the total torque doesn't depend on where the origin is, because shifting the origin by $\mathbf{a}$ changes the torque by

$$\Delta\boldsymbol{\tau} = \sum_i \mathbf{a} \times \mathbf{F}_i = \mathbf{a} \times \left(\sum_i \mathbf{F}_i \right) = 0.$$

The origin should usually be chosen to set as many torques as possible to zero.

[1] **Problem 5.** The line of a force is defined to the line passing through its point of application parallel to its direction; then the torque of the force about any point on that line vanishes. Suppose a body is static and has three forces acting on it. Show that in two dimensions, the lines of these forces must either be parallel or concurrent. This will be useful for several problems later.

Idea 5

The center of mass $\mathbf{r}_{\text{CM}}$ of a set of masses m_i at locations $\mathbf{r}_i$ with total mass M satisfies

$$M\mathbf{r}_{\text{CM}} = \sum_i m_i \mathbf{r}_i.$$

If a system experiences no external forces, its center of mass moves at constant velocity.

Idea 6

A uniform gravitational field exerts no torque about the center of mass. Thus, for the purposes of applying torque balance on an *entire* object, the gravitational force $M\mathbf{g}$ can be taken to act entirely at its center of mass. (This is a formal substitution; of course, the actual gravitational force remains distributed throughout the object.)

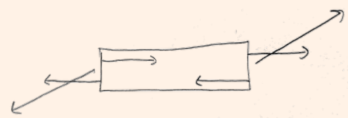
Torque balance works in noninertial frames, as long as one accounts for the torques due to fictitious forces. Thus, for an accelerating frame, the $-M\mathbf{a}$ fictitious force can be taken to act at the center of mass. In a uniformly rotating frame, the total centrifugal force is $M\omega^2\mathbf{r}_{\text{cm}}$, and for the purposes of balancing torques, can be taken to act entirely at the center of mass.

Example 2

Show that the tension in a completely flexible static rope, massive or massless, points along the rope everywhere in the rope.

Solution

Consider a tiny segment $d\ell$ of the rope. Since the rope is static, the tension forces on both ends balance, so they are opposite. Let them both be at an angle θ to the rope direction. Then the net torque on the segment is $(T d\ell) \sin\theta$. Since this must vanish for static equilibrium, we must have $\theta = 0$ and hence the tension is along the rope. In other words, flexible ropes can transmit force, but they can't transmit torque.



It's important to note that the argument above doesn't work for a rigid rod, because the internal forces in a rigid object can look like the picture above. In other words, there can be extra shear forces from the adjacent pieces of the rod that provide the compensating torque. If one tried to set up forces like this in a rope, it would flex instead.

In general, the force distribution within a massless rigid rod can be quite complicated, but if we zoom out, we can replace it with a single tension which does not necessarily point along the rod. This transmits both a force and a torque through the rod, in the sense that a torque is eventually exerted by whatever holds the end of the rod in place. Note that if the rod's supports are free to rotate, then they can't absorb torque, so the rod acts just like a rope, with tension always along it.

Remark

Sometimes, problem writers will intentionally not introduce any variables that are irrelevant to the answer. This can occur in two ways. First, the variables might just cancel out, as one can often see by dimensional analysis. Second, the specific values of the variables might not matter in the limit when they are very large or small. For instance, if a problem simply states a mass is "very heavy" but doesn't give it a name like m, it is asking for the answer in the limit $m \to \infty$.

Idea 7

To handle problems where an object is just about to tip over, note that at this moment, the entire normal force will often be concentrated at a point. (For example, when you're about to fall forward, all your weight goes on your toes.) That often means it's a good idea to take torques about this point.

Example 3: Povey 5.6

In problem 2, we treated the car as a point particle, but in reality it can also tip over. Suppose that on level ground, a car has a distance d between its left and right tires, which are both thin, and its center of mass is a height h above the ground. Now suppose the car turns as in problem 2 on a vertical wall ($\theta = 90°$) with speed v. For what v is this possible?

Solution

Again working in the noninertial frame of the car, force balance gives

$$f_{\text{fric}} = mg, \quad N = \frac{mv^2}{R}$$

where f_{fric} and N are the total friction and normal forces on the four tires. Since $f_{\text{fric}}/N \leq \mu$,

$$v \geq \sqrt{gR/\mu}$$

which matches the general solution to problem 2. But in that problem, we only considered force balance. In this extreme situation, we also have to consider torque balance, i.e. the possibility that the car might topple over. When the car is about to topple over, all the normal and friction force is on the bottom tires. About this point, we have only torques from gravity and the centrifugal force, giving

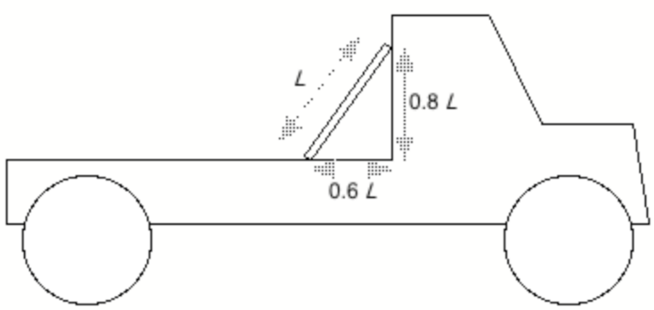
$$mgh = \frac{mv^2}{R}\frac{d}{2}$$

and solving for v gives $v = \sqrt{2gRh/d}$. Toppling is less likely the higher v is, so the answer is

$$v \geq \sqrt{gR}\,\max(1/\sqrt{\mu}, \sqrt{2h/d}).$$

Now here's a puzzle for you. A motorcycle only has one set of wheels, so it is effectively like a car with $d \to 0$. But motorcyclists can perform the motion described here, most famously in the Globe of Death, without toppling over. How is that possible?

[2] **Problem 6** (Quarterfinal 2004.3). A uniform board of length L is placed on the back of a truck.



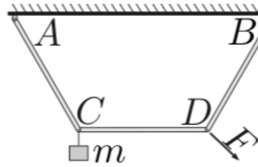
There is no friction between the top of the board and the vertical surface of the truck. The coefficient of static friction between the bottom of the board and the horizontal surface of the truck is $\mu_s = 0.5$. The truck always moves in the forward direction.

(a) What is the maximum starting acceleration the truck can have if the board is not to slip or fall over?

(b) What is the maximum stopping acceleration the truck can have if the board is not to slip or fall over?

(c) For what value of stopping acceleration is the static frictional force equal to zero?

[2] **Problem 7** (Kalda). Three identical massless rods are connected by freely rotating hinges.



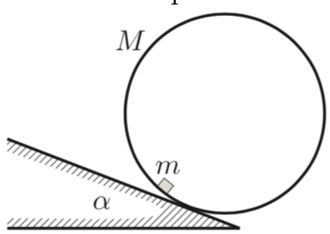
The rods are arranged so that CD is parallel to AB, and $\overline{AB} = 2\overline{CD}$. A mass m is hung on hinge C. What is the minimum force that must be exerted at hinge D to keep the system stationary?

Idea 8

An extended object supported at a point may be static if its center of mass lies directly above or below that point. More generally, if the object is supported at a set of points, it can be static if its center of mass lies above the convex hull of the points.

[2] **Problem 8.** N identical uniform bricks of length L are stacked, one above the other, near the edge of a table. What is the maximum possible length the top brick can protrude over the edge of the table? How does this limit grow as N goes to infinity?

[2] **Problem 9** (Kalda)**.** A cylinder with mass M is placed on an inclined slope with angle α so that its axis is horizontal. A small block of mass m is placed inside it.



The coefficient of friction between the block and cylinder is μ. Find the maximum α so that the cylinder can stay at rest, assuming that the coefficient of friction between the cylinder and slope is high enough to keep the cylinder from slipping.

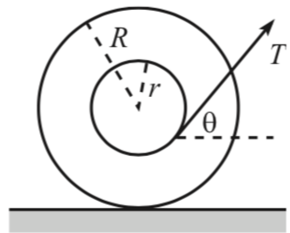
[2] **Problem 10** (PPP 11)**.** A sphere is made of two homogeneous hemispheres stuck together, with different densities. Is it possible to choose the densities so that the sphere can be placed on an inclined plane with incline 30° and remain in equilibrium? Assume the coefficient of friction is sufficiently high so that the sphere cannot slip.

[3] **Problem 11.** An object of mass m lies on a uniform floor, with coefficient of static friction μ.

(a) First, suppose the object is a point mass. What is the minimum force required to make the object start moving, if you can apply the force in any direction?

(b) Now suppose the object is a thin, uniform bar. What is the minimum force required to make the object start moving, if the force can only be applied horizontally? Assume the normal pressure on the floor remains uniform.

[3] **Problem 12** (Morin 2.17)**.** A spool consists of an axle of radius r and an outside circle of radius R which rolls on the ground.



A thread is wrapped around the axle and is pulled with tension T at an angle θ with the horizontal.

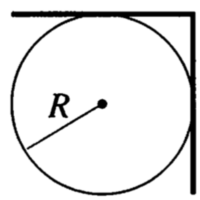
(a) Which way does the spool move if it is pulled with $\theta = 0$?

(b) Given R and r, what should θ be so that the spool doesn't move? Assume that the friction between the spool and the ground is large enough so that the spool doesn't slip.

(c) Given R, r, and the coefficient of friction μ between the spool and the ground, what is the largest value of T for which the spool remains at rest?

(d) Given R and μ, what should r be so that you can make the spool slip from the static position with as small a T as possible? That is, what should r be so that the upper bound on T in part (c) is as small as possible? What is the resulting value of T?

[3] **Problem 13** (PPP 44)**.** A plate, bent at right angles along its center line, is placed on a horizontal fixed cylinder of radius R as shown.



How large does the coefficient of static friction between the cylinder and plate need to be if the plate is not to slip off the cylinder?

3 Trickier Torques

Idea 9

Sometimes, a clever use of torque balance can be used to remove any need to have explicit force equations at all. Rarely, the same situation can occur in reverse.

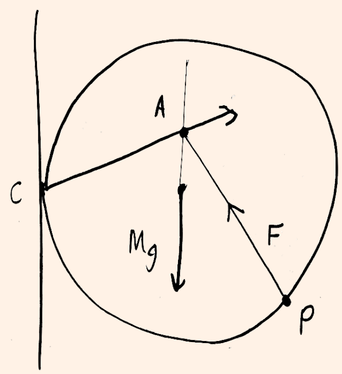
Example 4: EFPhO 2010.4

A spherical ball of mass M is rolled up along a vertical wall, by exerting a force F to some point P on the ball. The coefficient of friction is μ. What is the minimum possible force F, and in this case, where is the point P?

Solution

Following the logic of idea 3, when the minimum possible force is used, the frictional force with the wall must be maximal, $f = \mu N$, and directed upward. (If friction weren't pushing the ball up as hard as possible, we could get by using a smaller force F.) So even though we don't know the magnitude of the normal or the frictional force, we know the direction of the sum of these two forces, so we'll consider them as one combined force.

This reduces the number of independent forces in the problem to three: gravity (acting at the center of mass), the force F (acting at P), and the combined normal and friction forces (acting at the point of contact C with the wall). Therefore, by the result of problem 5, the lines of these forces must all intersect at some point A, as shown.



This ensures that the torques will balance, when taken about point A.

Next, we need to incorporate the information from force balance. Doing this directly will lead us to some nasty trigonometry, but there's a better way. There are in principle two force balance equations, for horizontal and vertical forces. However, one of these equations is just going to tell us the magnitude of the normal/frictional force, which we don't care about. So in reality, we just need one equation, which preferably doesn't involve that force.

The trick is to use torque balance *again*, about the point C, which says that the torques due to gravity and F must cancel. Now you might ask, didn't we already use torque balance? We did, but recall from idea 4 that taking the torque about a different point can give you a different equation if the forces don't balance. So by demanding the torque vanish about two different points, we actually are using force balance! (Specifically, we are using the linear combination of the horizontal and vertical force balance equations that *doesn't* involve the normal/friction force, which we don't need to find anyway.)

When taking the torque about C, we see that F is minimized if P is chosen to maximize the lever arm of the force. This occurs when $CA \perp PA$, in which case the lever arm is $R\sqrt{1+\mu^2}$, where R is the radius of the ball. So we have

$$MgR = FR\sqrt{1+\mu^2}, \quad F = \frac{Mg}{\sqrt{1+\mu^2}}$$

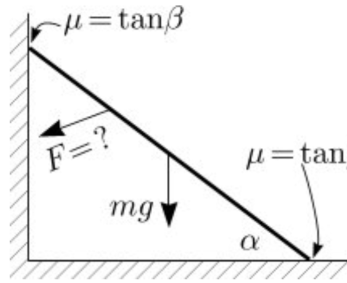
and P is determined as described above.

[2] **Problem 14.** NBPhO 2020, problem 4, parts (i) and (ii).

[3] **Problem 15.** EFPhO 2012, problem 3. The problem statement is missing some information: both the bars and rod have diameter d.

[3] **Problem 16.** EFPhO 2006, problem 6. You will need to print out the problem to make measurements on the provided figure.

[4] **Problem 17** (Physics Cup 2012)**.** A thin rod of mass m is placed in a corner so that the rod forms an angle α with the floor. The gravitational acceleration is g, and the coefficient of friction with the wall and floor is $\mu_s = \tan\beta$, which is not large enough to keep the rod from slipping.

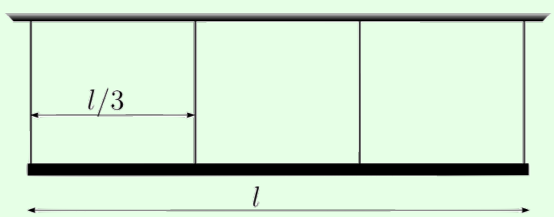


What is the minimum additional force F needed to keep the rod static?

We've now covered some really mathematically elegant problems, but it's important to remember the real-world limitations of this kind of analysis. We discuss two examples below.

Example 5

A uniform bar with mass m and length ℓ hangs on four equally spaced identical light wires. Initially, all four wires have tension $mg/4$.



Find the tensions after the leftmost wire is cut.

Solution

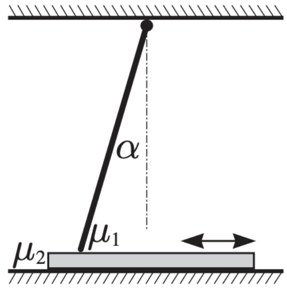
This illustrates a common issue with setups involving rigid supports: there are often more normal forces than independent equations, so there is not a unique solution. In the real world, the result is determined by imperfect characteristics of the wires. For example, if one of the wires was slightly longer than the others, it would go slack, reducing the number of normal forces by one and yielding a solution.

A reasonable assumption, if you aren't given any further information, is to assume that the supports are identical, very stiff springs. In equilibrium, the bar will tilt a tiny bit, so that the length of the middle wire will be the average of the lengths of the other two. By Hooke's law, the force in that wire will than be the average of the other two, so the tensions are

$mg/3 - x$, $mg/3$, and $mg/3 + x$. Applying torque balance yields $7mg/12$, $mg/3$, and $mg/12$.

The general point here is that concepts like rigid bodies or strings characterized by a single tension force are abstractions, made for the idealized problems we study in mechanics classes. A real civil engineer designing a structure would instead use a sophisticated computer program which simulates the complex internal forces, torques, and strains throughout the material.

[2] **Problem 18** (Kalda). A rod is hinged to the ceiling, so that it makes an angle α with the vertical.



Underneath, a thin board is being dragged on the floor. The coefficient of (static and kinetic) friction is μ_1 between the board and rod, and μ_2 between the board and floor. The rod is meant to stop the board from being dragged to the right, no matter how hard or how quickly it is pulled. Is this possible? If so, what are the conditions on the parameters that allow this to occur?

Remark: Subtleties of Friction

Statics problems involving friction can also get quite elegant, but it's important to remember that they're just an imperfect approximation for the real world. Coulomb's law for "dry" friction, $|f| = \mu N$, works for a variety of materials, but for lubricated materials the friction has to be computed using fluid mechanics, as discussed in **M7**. More generally, friction is an entire field of study called tribology, which is essential for engineering. For much more, see *Tribology* by Hutchings and Shipway.

Coulomb's law has other weird features: assuming it can produce mathematical contradictions, in the so-called Painleve paradox! Problem 18 above is the simplest possible example of the paradox. In this problem, you showed that under the right conditions, it is impossible for the board to move to the right. But that means that if you assume the board starts with some rightward velocity, then trying to solve Newton's second law for the acceleration will give a mathematical contradiction.

Of course, in real life there actually would be some finite, albeit large acceleration. To solve for it you would need to adopt a more refined model of the rod and its interaction with the board, e.g. accounting for how the rod and board deform when subject to stress. That requires "contact mechanics", briefly discussed in example 14.

We conclude with some questions that train three-dimensional thinking.

[2] **Problem 19** (PPP 10). In Victor Hugo's novel *les Miserables*, the main character Jean Valjean, an escaped prisoner, was noted for his ability to climb up the corner formed by the intersection of two

vertical perpendicular walls. Suppose for simplicity that Jean has no feet. Let μ be the coefficient of static friction between his hands and the walls. What is the minimum force that Jean had to exert on each hand to climb up the wall? Also, for what values of μ is this feat possible at all?

[3] **Problem 20** (PPP 69)**.** A homogeneous triangular plate has threads of length h_1, h_2, and h_3 fastened to its vertices. The other ends of the string are fastened to a common point on the ceiling. Show that the tension in each thread is proportional to its length. (Hint: with the origin at the point on the ceiling, let the vertices be at positions $\mathbf{r}_i$ and express everything in vector form.)

[4] **Problem 21** (KoMaL 2019, BAUPC 1998)**.** Two identical uniform solid cylinders are placed on a level tabletop next to each other, so that they are touching. A third identical cylinder is placed on top of the other two.

(a) Find the smallest possible values of the coefficients of static friction between the cylinders, and between a cylinder and the table, so that the arrangement can stay at rest.

(b) Repeat part (a) for spheres. That is, put three uniform solid spheres next to each other, with their centers forming an equilateral triangle, and put a fourth sphere on top.

(c) Now return to part (a), and suppose the setup is frictionless. A force is applied directly to the right on the leftmost cylinder, causing the entire setup to accelerate. Find the minimum and maximum accelerations so that all three cylinders remain in contact with each other.

Parts (a) and (b) demonstrate an interesting point: it is possible for a collection of objects to resist some force, even though a single one of those objects would begin moving even with an infinitesimal applied force! This is a simple example of how granular materials, like sand, can give rise to emergent phenomena that are hard to predict from analyzing individual grains alone. Understanding these materials is a whole field of applied research.

4 Extended Bodies

Idea 10: Principle of Virtual Work

Some physical systems have a lot of parts but also a lot of constraints, such as joints, so that they can only move in a few ways. In these cases, it is easiest to determine if the system is static using the principle of virtual work. Suppose that the system moves an infinitesimal amount, in a way allowed by the constraints. If the net work done on the system by this "virtual displacement" is zero, then the system can be in static equilibrium at that position. (Alternatively, if you already know the system can be in static equilibrium, this approach can be used to find the magnitude of one unknown force, such as the tension in a string.)

Example 6: Roberval Balance

Consider the following scale made of rigid bars. The joints ensure that the quadrilateral in the middle always remains a parallelogram, with its left and right sides vertical.

If identical weights are placed on each horizontal arm as shown, can the system remain static?

Solution

There's only one way for the system to move: the rectangle can deform into a parallelogram so that the left horizontal arm moves up, and the right horizontal arm moves down by the same amount. Then the total virtual work done on the scale by the weights is zero, so the system can be in equilibrium no matter where on the arms the weights are placed.

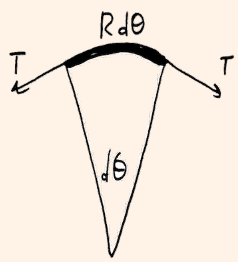
Next, we'll consider problems with continuous bodies, where one often needs to consider forces and torques acting on infinitesimal pieces.

Example 7

Find the tension in a circular rope of radius R spinning with angular velocity ω and mass per length λ.

Solution

Consider an infinitesimal segment of the rope, spanning an angle $d\theta$.



The mass of this segment is $dm = R\lambda\, d\theta$. The total force is downward, with magnitude

$$dF = 2T \sin \frac{d\theta}{2} \approx T\, d\theta$$

where we used the small angle approximation. This is the centripetal force, so

$$dF = (dm)\, \omega^2 R.$$

Combining these results yields $T = R^2\omega^2\lambda$.

Example 8

Find the distance d of the center of mass of a uniform semicircle of radius R to its center. (Note that a semicircle is half of a circle, not half of a disc.)

Solution

This can be done by taking the setup of the previous problem, and taking a subsystem comprising exactly half of the rope. In this case the net tension force is simply

$$F = 2T.$$

The total mass is $m = \pi R\lambda$, and the force must provide the centripetal force, so

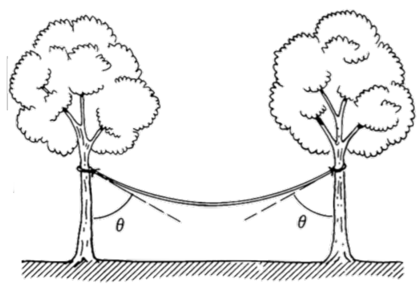
$$F = (\pi R\lambda)(\omega^2 d)$$

But we also know that $T = R^2\omega^2\lambda$ as before, so plugging this in gives

$$d = \frac{2}{\pi}R.$$

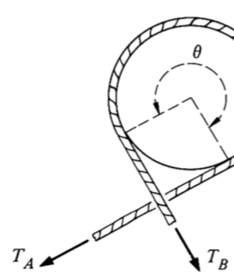
Alternatively, we could have worked in the frame rotating with the rope. The equations would be the same, but instead we would say the tension balances the centrifugal force.

[1] **Problem 22** (KK 2.22)**.** A uniform rope of weight W hangs between two trees. The ends of the rope are the same height, and they each make angle θ with the trees.



Find the tension at either end of the rope, and the tension at the middle of the rope.

[3] **Problem 23** (KK 2.24)**.** A capstan is a device used aboard ships to control a rope which is under great tension.



The rope is wrapped around a fixed drum with coefficient of friction μ, usually for several turns. The load on the rope pulls it with a force T_A. Ignore gravity.

(a) Show that the minimum force T_B needed to hold the other end of the rope in place is $T_A e^{-\mu\theta}$, an exponential decrease.

(b) How does this result depend on the shape of the capstan, if we fix the angle θ between the initial and final tension forces? Would the answer be the same for an oval, or a square?

(c) If $\theta = \pi$, explain why the total normal and friction force of the rope on the drum is $T_A + T_B$.

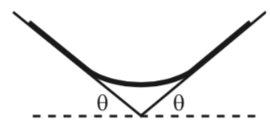
[2] **Problem 24** (***F*** $=$ ***ma*** 2018 B20)**.** A massive, uniform, flexible string of length L is placed on a horizontal table of length $L/3$ that has a coefficient of friction $\mu_s = 1/7$, so equal lengths $L/3$ of string hang freely from both sides of the table. The string passes over the edges of the table, which are smooth frictionless curves, of size much less than L. Now suppose that one of the hanging ends of the string is pulled a distance x downward, then released at rest. Neither end of the string touches the ground.

(a) Find the maximum value of x so that the string does not slip off of the table.

(b) For the case $x = 0$, draw a free body diagram for the string, indicating only the *external* forces on the entire string. Do the forces balance?

(c) Would the answer change significantly if the table's small edges had friction as well?

[3] **Problem 25** (Morin 2.25)**.** A rope rests on two platforms that are both inclined at an angle θ.



The rope has uniform mass density, and the coefficient of friction between it and the platforms is 1. The system has left-right symmetry. What is the largest possible fraction of the rope that does not touch the platforms? What angle θ allows this maximum fraction?

Example 9

A chain is suspended from two points on the ceiling a distance d apart. The chain has a uniform mass density λ, and cannot stretch. Find the shape of the chain.

Solution

First, we note that the horizontal component of the tension T_x is constant throughout the chain; this just follows from balancing horizontal forces on any piece of it. Moreover, by similar triangles, we have $T_y = T_x y'$ everywhere.

Now consider a small segment of chain with horizontal projection Δx. The length of the piece is $\Delta x\sqrt{1+y'^2}$ which determines its weight, and this be balanced by the difference in vertical tensions. Thus

$$\Delta T_y = \lambda g\sqrt{1+y'^2}\,\Delta x.$$

For infinitesimal Δx, we have $\Delta T_y = T_x d(y') = T_x y''\,dx$, so we get the differential equation

$$y'' = \frac{\lambda g}{T_x}\sqrt{1+y'^2}.$$

Usually nonlinear differential equations with second derivatives are very hard to solve, but this one isn't because there is no direct dependence on y, just its derivatives. That means we can treat y' as the independent variable first, and the equation is effectively first order in y'.

Writing $y'' = d(y')/dx$ and separating, we have

$$\int \frac{dy'}{\sqrt{1+y'^2}} = \int \frac{\lambda g}{T_x}\, dx.$$

Integrating both sides gives

$$\sinh^{-1}(y') = \frac{\lambda g x}{T_x} + C.$$

Choosing $x = 0$ to be the lowest point of the chain, the constant C is zero, and

$$y' = \sinh\left(\frac{\lambda g x}{T_x}\right).$$

Integrating both sides again gives the solution for y,

$$y = \frac{T_x}{\lambda g} \cosh\left(\frac{\lambda g x}{T_x}\right)$$

where we suppressed another constant of integration. This curve is called a catenary.

[1] **Problem 26.** To check that you understand the previous example, repeat it for a suspension bridge. In this case the cable is attached by vertical suspenders to a horizontal deck with mass λ per unit length, and supports the weight of the deck. Assume the cable and suspenders have negligible mass.

[4] **Problem 27** (MPPP)**.** A slinky is a uniform spring with negligible relaxed length, with mass m and spring constant k.

(a) Find the shape of a slinky hung from two points on the ceiling separated by distance d. (Hint: to begin, consider the mass and tension of a small piece of the spring with horizontal and vertical extent dx and dy. Don't forget that the slinky's density won't be uniform.)

(b) Suppose a slinky's two ends are fixed, separated by distance d, and rotating uniformly with angular frequency ω like a jump rope in zero gravity. Find the values of ω for which this motion is possible, and the shape of the slinky in this case.

Example 10

A uniform spring of spring constant k, mass m, and relaxed length L is hung from the ceiling. Find its length in equilibrium, as well as its center of mass.

Solution

Problems like this contain subtleties in notation. For example, if you talk about "the piece of the slinky at z", this could either mean the piece that's actually at this position in equilibrium, or the piece that was originally at this place in the absence of gravity. Talking about it the first way automatically tells you where the piece is now, but talking about it the second way makes it easier to keep track of, because then the z of a specific piece of the spring stays the same no matter where it goes.

In fluid dynamics, these are known as the Eulerian and Lagrangian approaches, respectively. If you don't use one consistently, you'll get nonsensical results, and it's easy to mix them up.

There are many ways to solve this problem, but I'll give one that reliably works for me. We're going to use the Lagrangian approach, and avoid confusion with the Eulerian approach by breaking the spring into discrete pieces. Let the spring consist of $N \gg 1$ pieces, of masses m/N, spring constants Nk, and relaxed lengths L/N.

The i^{th} spring from the bottom has tension $(i/N)mg$, and thus is stretched by

$$\Delta L_i = \frac{1}{kN}\frac{i}{N}mg = \frac{mg}{kN^2}\, i.$$

The total stretch is

$$\sum_{i=1}^{N} \Delta L_i = \frac{mg}{kN^2}\int_0^N i\, di = \frac{mg}{2k}.$$

This makes sense, since the average tension is $mg/2$. To find the center of mass, note that the j^{th} spring is displaced downward by a distance

$$\Delta y_j = \sum_{i=j}^{N} \Delta L_i = \frac{mg}{2k}\left(1 - \frac{j^2}{N^2}\right)$$

downward from its position in the absence of gravity. The center of mass displacement is

$$\Delta y_{\text{CM}} = \frac{1}{N}\sum_{j=1}^{N} \Delta y_j \propto \frac{1}{N}\sum_{j=1}^{N}\left(1 - \frac{j^2}{N^2}\right) = \frac{1}{N^3}\int_0^N N^2 - j^2\, dj = \frac{2}{3}$$

so restoring the proportionality constant gives

$$\Delta y_{\text{CM}} = \frac{mg}{3k}.$$

If you want to test your understanding of slinkies, you can also try doing this problem with the Eulerian approach. This would be best done without discretization. The first steps would be finding a relation between the density $\rho(z)$ and tension $T(z)$ from Hooke's law, and finding out how to write down local force balance as a differential equation.

Remark

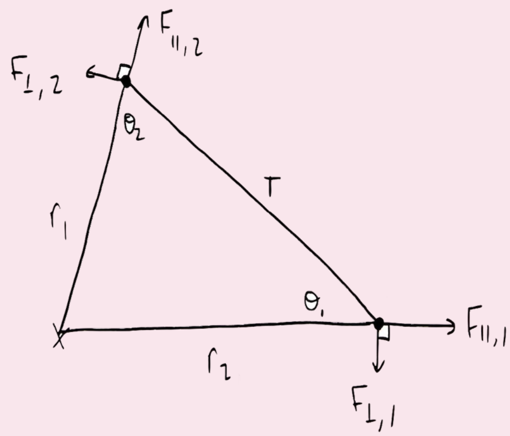
In this problem set, we've given some examples involving static, continuous, one-dimensional objects such as strings and ropes. The general three-dimensional theory of elasticity is mathematically quite complicated, but extremely important in engineering. For more about this subject, which requires comfort with tensors, see chapters 6 through 11 of Lautrup. It is also covered in chapters II-31, II-38, and II-39 of the Feynman lectures.

Remark: Why Use Torque?

Here's a seemingly naive question. Why is the idea of torque so incredibly useful in physics problems, even though in principle, everything can be derived from $F = ma$ alone? Why is it almost impossible to solve any nontrivial problem without referring to torques, and how would a student who's never heard of torque come up with it in the first place?

We don't need torque to analyze the statics of a single, featureless point particle. Torque only became useful in this problem set when we started analyzing rigid bodies with spatial extent. The reason we couldn't reduce torque balance to force balance easily is because the internal forces in these bodies, which maintain their rigidity, are generally very complicated.

To see this, let's consider the simplest possible example of a rigid body: a triangle with masses at the vertices, and sides made of very thin, rigid, massless springs. The triangle is pivoted at one vertex, and experiences external forces $\mathbf{F}_1$ and $\mathbf{F}_2$ at the other two vertices.



Consider force balance on the first marked vertex. The tension in the side of length r_2 takes whatever value is necessary to balance the horizontal force on the vertex, while the tension T in the other side has to balance the vertical force. Thus,

$$F_{\perp,1} = T \sin\theta_1, \qquad F_{\perp,2} = T \sin\theta_2$$

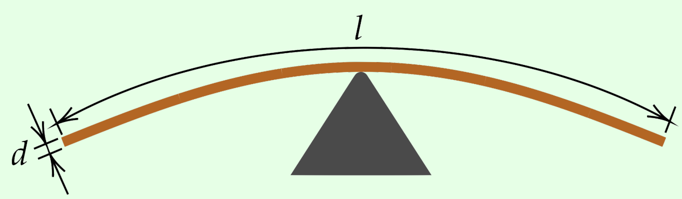
where the second line follows by considering the second marked vertex. Eliminating T and using the law of sines gives $r_2 F_{\perp,1} = r_1 F_{\perp,2}$, which of course is the statement of torque balance. (And if you continue along this line of reasoning, letting the forces be arbitrary, you can also derive the rotational form of Newton's second law, $\tau = I\alpha$.)

However, recovering the results of torque balance is much more complicated in general. For example, consider a rigid bar supported at its ends. The middle of the bar doesn't collapse, despite the force of gravity on it, because the bar contains internal, upward shear forces, which transmit the normal forces applied at its end throughout the rest of the bar. To analyze such systems without using torque, one would have to account for all of these microscopic forces, acting on all of the rod's infinitely many pieces. So for any problem with a continuous object, torque balance is an essential tool.

In fact, now that you're aware of how complicated internal forces can be, you might be wondering why torque balance even works. The simplest explanation is by the principle of virtual work. The torque of a force about a pivot is just the virtual work it does per angle the system rotates about the pivot. If a system is in static equilibrium, it must experience no net virtual work under any displacement, including rotations, so the torques must cancel.

Example 11: IPhO 2022 3A

A thin piece of spaghetti of diameter d is balanced horizontally from its middle.



It can have a length $\ell \gg d$ before it snaps under its own weight. How does ℓ scale with d?

Solution

Let the spaghetti rod have density ρ, and consider its right half. There must be a vertical normal force $F \sim \rho d^2 \ell$ to balance the weight. This vertical force is transmitted through the rod by a shear stress (i.e. an internal force per area, perpendicular to the rod) of order $\sigma_s \sim F/A \sim \rho\ell$. Each piece of the rod exerts such a shear stress on its neighbors, just like how pieces of a string exert tensions on their neighbors.

Now consider torques on the right half of the rod, about the pivot point. The torque $\tau \sim \rho d^2 \ell^2$ of the rod's weight has to be balanced by forces from the other half of the rod. Vertical forces don't work, since they don't provide any torque about the pivot. Instead, the torque is supplied by a horizontal compression force at the bottom, and a horizontal tension force at the top, which cancel out to maintain horizontal force balance. This combination of forces, which produces no net force but does produce a net torque, is a bending moment.

Let the associated normal stresses be of order $\pm\sigma_n$. Then the net compression and tension forces are of order $\pm d^2 \sigma_n$, and the lever arm is of order d, so balancing torques gives

$$\rho d^2 \ell^2 \sim \sigma_n d^3$$

which implies $\sigma_n \sim \rho \ell^2/d$. This is much greater than σ_s, because of the miserably small lever arm, which is why thin rods usually break by snapping, not by shearing or pulling apart. Given a fixed maximum σ_n, we conclude the maximum length scales as $\ell \sim \sqrt{d}$.

[3] **Problem 28.** USAPhO 2022, problem A1. A practical bending moment problem.

5 Pressure and Surface Tension

Example 12

A sphere of radius R contains a gas with a uniform pressure P. Find the total force exerted by the gas on one hemisphere.

Solution

The pressure provides a force per unit area orthogonal to the sphere's surface, so the straightforward way to do this is to integrate the vertical component of the pressure force over a hemisphere. However, there's a neat shortcut in this case.

Momentarily forget about the sphere and just imagine we have a sealed hemisphere of gas at pressure P. The net force of the gas on the hemisphere must be zero, or else it would just begin shooting off in some direction, violating conservation of momentum. So the force on the curved face must balance the force on the flat face, which is $\pi R^2 P$. The same logic must hold for the sphere, since the forces on the curved face are the same, so the answer is $\pi R^2 P$.

This trick will come in handy for several future problems; for example, it's the quick way to do $F = ma$ 2018 B24. It also generalizes to surfaces of arbitrary shape, as discussed in **E1**. Concretely, suppose a surface S has boundary C, and consider any other surface S' with the same boundary. Then by the same logic, the closed surface formed by S and S' together experiences no net pressure force, so the pressure forces on S and S' are equal in magnitude.

Idea 11

The surface of a fluid carries a surface tension γ. If one imagines dividing the surface into two halves, then γ is the tension force of one half on the other per length of the cut. Specifically, for a small segment $d\mathbf{s}$ along the cut, where the normal vector to the surface is $\hat{\mathbf{n}}$, the surface tension force is

$$d\mathbf{F} = \gamma \, d\mathbf{s} \times \hat{\mathbf{n}}$$

which means the force acts along the surface and perpendicular to the cut.

Example 13

A spherical soap bubble of radius R and surface tension γ is in air with pressure P, and contains air with pressure $P + \Delta P$. Compute ΔP.

Solution

We use the result of the previous problem to conclude that the force of one hemisphere on another is $\pi R^2 \Delta P$. This must be balanced by the surface tension force. By imagining cutting the surface of the bubble in half, the surface tension force is γL where L is the total length of the surface connecting the hemispheres.

At this point, we can write $L = 2\pi R$, giving

$$\Delta P = \frac{2\gamma}{R}.$$

This is called the Young–Laplace equation. However, in this particular case, this is not the right answer. The reason is that we should actually take $L = 4\pi R$ because the surface tension is exerted at both the inside and outside surfaces of the bubble wall, and thus the answer is

$$\Delta P = \frac{4\gamma}{R}.$$

The increased pressure inside balances the surface tension, which wants to collapse the bubble.

If you're confused about why $L = 4\pi R$, you can also think about it in terms of energy. Surface tension arises from the fact that it costs energy to take soapy water and stretch it out into a surface, because this breaks some of the attractive intermolecular bonds. The Young–Laplace equation would give the correct answer for a *ball* of soapy water. But for a *bubble* of soapy water, twice as much soapy water/air surface is created. So the energy cost is double, and the force is double.

[2] **Problem 29.** One can also derive the Young–Laplace equation using the principle of virtual work. Suppose the bubble radius changes by dr. The energy of the bubble changes for two reasons: first, there is net $\Delta P\, dV$ work from the two pressure forces, and there is the $\gamma\, dA$ surface tension energy cost. By setting the net virtual work to zero, find ΔP.

[2] **Problem 30** (Kalda)**.** Consider two soap bubbles which have stuck together. The part of the soap film that separates the interior of the first bubble from the outside air has radius of curvature R. The part that separates the interior of the second bubble from the outside air has radius of curvature $2R$. What is the radius of curvature of the part which separates the bubbles from each other?

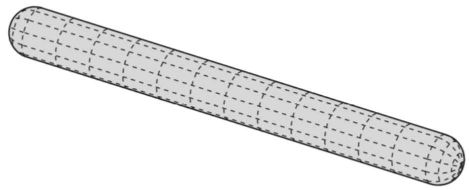
Remark

So far, we've only applied the Young–Laplace equation to spherical surfaces, which are characterized by a single radius of curvature. More generally, a surface has two principle radii of curvature R_1 and R_2 at each point. These are both equal to R for a sphere of radius R, while for a cylinder of radius R, one is equal to R and the other is infinity. For general surfaces, the Young–Laplace equation is

$$\Delta P = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

where the R_i can each be positive or negative, depending on the direction of curvature.

[3] **Problem 31** (MPPP 67)**.** When a pipe bursts under pressure, it often splits "lengthwise" instead of "across". (One familiar example is the process of cooking a long, straight sausage.) The two modes of splitting are shown as dotted lines below.



Explain this observation, assuming the thickness of the sausage skin is uniform, and hence can support a constant surface tension before breaking. (Hint: model the sausage as a cylinder of length L capped by hemispheres of radius $R \ll L$, and consider the surface tension needed to prevent the two modes of splitting mentioned, once an excess pressure P builds up inside the sausage.)

[4] **Problem 32.** Two coaxial rings of radius R are placed a distance L apart from each other in vacuum. A soap film with surface tension γ connects the two rings.

(a) Derive a differential equation for the shape $r(z)$ of the film, and solve it.

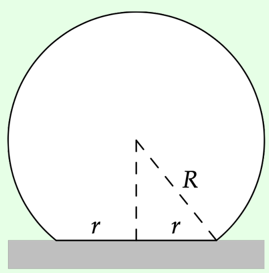
(b) Show that for sufficiently large L, there are no solutions. If L is increased to this value, what happens to the film?

(c) Using a computer or calculator, find the largest possible value of L.

We'll consider surface tension in more detail in **T3**.

Example 14

A solid ball of radius R, density ρ, and Young's modulus Y rests on a hard table. Because of its weight, it deforms slightly, so that the area in contact with the table is a circle of radius r.



Estimate r, assuming that it is much smaller than R.

Solution

Recall from **P1** that the Young's modulus is defined by

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{\text{restoring force/cross-sectional area}}{\text{change in length/length}}$$

and has dimensions of pressure. By dimensional analysis, you can show that

$$r = R\, f(\rho g R/Y)$$

but dimensional analysis alone can't tell us anything more about f. Moreover, an exact analysis using forces would be very difficult, because different parts of the ball are compressed in different amounts, and in different directions; there's little symmetry here.

Instead, we'll roughly estimate the stress and strain near the bottom of the ball. For the part directly in contact with the table, we have

$$\text{stress} \sim F/r^2 \sim \rho g R^3/r^2$$

because the normal pressure has to balance gravity. This is the pressure exactly at the bottom of the ball; at heights much greater than r, the pressure will be smaller because it can spread out over a wider horizontal surface area. Since stress is proportional to strain, that means the part of the ball that is significantly strained has typical height r. (This is an example of Saint–Venant's principle, which states that strain is generally confined near the location that external forces are applied.) So in that region, the strain must be

$$\text{strain} \sim \delta/r \sim r/R.$$

Using the definition of the Young's modulus, we conclude

$$r \propto R\left(\frac{\rho g R}{Y}\right)^{1/3}.$$

We can also phrase this result in terms of force and displacement. The ball's total vertical deformation is $d \sim r^2/R$ and the total force that pushes it into the table is $F \sim \rho g R^3$, so

$$F \propto Y R^{1/2} d^{3/2}.$$

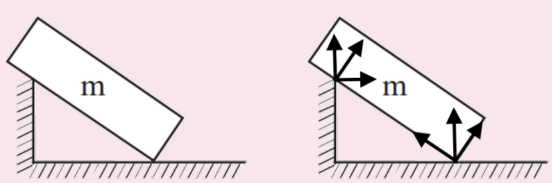
The restoring force is not linear in d, so it doesn't obey Hooke's law.

By the way, there's a whole field of study devoted to figuring out how the normal and other forces behave for realistic, deformable solids, known as contact mechanics, which is essential in engineering. This particular example is about "Hertzian contact". For much more, see *Contact Mechanics* by Johnson, and *Contact Mechanics and Friction* by Popov.

[4] **Problem 33.** EFPhO 2006, problem 5. A tough problem on a deforming object.

Remark: Normal Forces at Corners

Unfortunately, some problem writers don't really understand contact forces, and they'll end up writing questions like the one shown at left below, which is taken from a real book.



Assuming there's no friction, which way do the normal forces on the block point?

Many students have no problem "solving" this, because they've been trained to blindly trust exam questions. They'll just choose some random directions and move on. But the smart student will get seriously confused, because the answer is clearly not well-defined. At the bottom contact point, there are three different possible directions, depending on whether you take the normal to the floor, or either of the two sides of the block. The other contact point is even more ambiguous, because of the wall magically ending. Is the normal force perpendicular to the block, perpendicular to the vertical wall, or something else? In general, there doesn't seem to be any sensible rule to decide which is the "real" normal.

The resolution, of course, is that there *is* no rule. What happens in reality depends on the exact shape of the block and wall, and how deformable it is. For example, suppose the block was perfectly rigid, but actually had slightly rounded corners (not shown in the diagram). Then there's a definite normal direction at the bottom contact point, pointing up. Similarly, we could suppose that at the other contact point, the wall actually ends in a step with a rounded corner, in which case the normal direction points directly into the block.

Alternatively, suppose the block and step weren't rounded, but could deform. Then the answer depends on the relative hardness of the materials, and how they were placed in contact. For instance, if we suppose the block is much softer, then it could squash at the bottom contact point, again leading to a common upward normal direction. But then we would expect the step to dig into the block at the other contact point, which yields two separate normal forces at that point. Or perhaps the step is made of a softer material than the floor, so that it's the step rather than the block that deforms. Or maybe both deform!

To be clear: the issue *isn't* that the problem involved unrealistic idealizations. Physics uses idealizations, like neglecting air resistance and friction, all the time, and they're reasonable within appropriate limits. The issue is that when you apply the idealizations implied by the diagram, the result is mathematically undefined. And you get completely different answers depending on which idealizations you drop, because the true answer depends on the details.

As a result, you certainly won't see this kind of thing on thoroughly vetted competitions, such as the IPhO, APhO, and EuPhO, or large national Olympiads such as those in America or China. I'll never assign such a dysfunctional problem, but they're depressingly common in homework assignments and less carefully written exams, such as the JEE. If you personally encounter such a problem, your best bet is to attempt to read the question writer's mind; that is, simply start guessing and go with whatever gives you tractable results. If you encounter this sort of thing often, in a book or competition, then it's not worth your time. We're in it to learn physics, not to please examiners.

Mechanics II: Statics

For review, read chapter 2 of Morin or chapter 2 of Kleppner and Kolenkow. Statics is covered in more detail in chapter 7 of Wang and Ricardo, volume 1. Surface tension is covered in detail in chapter 5 of *Physics of Continuous Matter* by Lautrup, which is an upper-division level introduction to fluids in general. There is a total of **83** points.

1 Balancing Forces

Idea 1

In principle, you can always solve every statics problem by balancing forces on every individual particle in the setup, but often you can save on effort by considering appropriate systems.

Idea 2

Any problem where everything has a uniform velocity is equivalent to a statics problem, by going to the reference frame moving with that velocity. Any problem where everything has a uniform acceleration $\mathbf{a}$ is also about statics, by going to the noninertial frame with acceleration $\mathbf{a}$, where there is an extra effective gravitational acceleration $-\mathbf{a}$. The same principle applies to uniform rotation, where a centrifugal force appears in the rotating frame, acting like an effective gravitational acceleration $\omega^2\mathbf{r}$.

Example 1

Six blocks are attached in a horizontal line with rigid rods, and placed on a table with coefficient of friction μ. The blocks have mass m and the leftmost block is pulled with a force F so the blocks slide to the left. Find the tension force in the rod in the middle.

Solution

There are six objects here and five rods, each with a different tension, so a direct analysis would involve solving a system of six equations. Instead, first consider the entire set of six blocks as one object; we can do this because the rigid rods force them to move as one. The total mass is $6m$, and applying Newton's second law gives

$$F - 6mg\mu = 6ma, \quad a = \frac{F}{6m} - \mu g.$$

Next, consider the rightmost three blocks as one object. Their total mass is $3m$, and their acceleration is the same acceleration a we computed above. This system experiences two horizontal force: tension and friction. Newton's second law gives

$$T - 3mg\mu = 3ma$$

and solving for T gives

$$T = \frac{F}{2}.$$

This is intuitive, because the differences of any two adjacent tension forces are the same; that's the amount of tension that needs to be spent to accelerate each block. So the middle rod, which has to accelerate only half the blocks, has half the tension.

The reason we could ignore the tension forces in the other four rods is that the only thing they do is ensure the blocks move with the same acceleration. Once we assume this is the case, the specific values of the tensions don't matter; we can just zoom out and forget them. It's just like how *within* each block there is also an internal tension which keeps it together, but we rarely need to worry about its details.

Idea 3

To handle a problem where something is just about to slip on something else, set the frictional force to the maximal value μN and assume slipping is not yet occurring, so the two objects move as one. The same idea holds for problems which ask for the minimal force needed to make something move, or the minimal force needed to keep something from moving.

[1] **Problem 1** (KK 2.7)**.** A block of mass M_1 sits on a block of mass M_2 on a frictionless table. The coefficient of friction between the blocks is μ. Find the maximum horizontal force that can be applied to (a) block 1 or (b) block 2 so that the blocks will not slip on each other.

Solution. Let the horizontal force be F. In both cases the friction is maximal, $f = \mu M_1 g$, and the blocks move together, so $a = F/(M_1 + M_2)$.

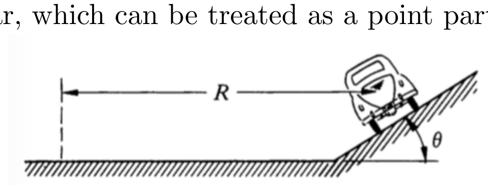
(a) The bottom block experiences only the force $f = M_2 a$, so

$$\mu M_1 g = M_2 a, \quad F = \mu g(M_1 + M_2)\,\frac{M_1}{M_2}$$

(b) The top block experiences only the force $f = M_1 a$, so

$$\mu M_1 g = M_1 a, \quad F = \mu g(M_1 + M_2).$$

[2] **Problem 2** (KK 2.28)**.** A car, which can be treated as a point particle, enters a turn of radius R.



The road is banked at angle θ, and the coefficient of friction between the wheels and road is μ. Find the maximum and minimum speeds for the car to stay on the road without skidding sideways.

Solution. Let N be the normal force, and let f be the friction force (defined to be positive if it's pointing up the hill). We see that $N\cos\theta + f\sin\theta = mg$, and $N\sin\theta - f\cos\theta = mv^2/R$. Therefore,

$$\frac{v^2}{gR} = \frac{N\sin\theta - f\cos\theta}{N\cos\theta + f\sin\theta}.$$

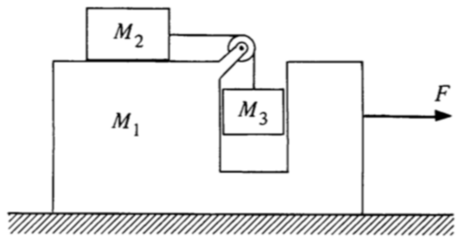
Since $-N\mu \le f \le N\mu$, we have

$$\frac{v_{\text{min}}^2}{gR} = \frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}, \quad \frac{v_{\text{max}}^2}{gR} = \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}.$$

In addition, there are some cases where these formulas break down. If $\mu > \tan\theta$, then the minimum speed is zero. If $\mu > \cot\theta$, then the maximum speed is infinity. (In these cases, the formulas give nonsense, i.e. imaginary numbers for the speeds. That's one way of checking, at the end of a problem, whether there are more special cases that must be accounted for.)

Usually, we are in the regime where $\mu > \tan\theta$, in which case $v_{\text{min}} = 0$ and banking the turn increases v_{max}. Another benefit is that it helps align the direction of the gravitational and centrifugal force with the height of the car, making the turn more comfortable; you get less of a sideways pull along your seat. For this reason, banked turns are very common in highways. In highway engineering, this trick is called superelevation.

[2] **Problem 3** (KK 2.19)**.** A "pedagogical machine" is illustrated in the sketch below.



All surfaces are frictionless. What force F must be applied to M_1 to keep M_3 from rising or falling?

Solution. By considering all the masses as one system, we see that $a = \frac{F}{M_1+M_2+M_3}$. We see that the tension $T = M_3 g$, and $T = M_2 a$, so

$$M_3 g = M_2 a \implies \frac{F}{M_1 + M_2 + M_3} = \frac{M_3}{M_2} g \implies \boxed{F = (M_1 + M_2 + M_3)\frac{M_3}{M_2} g.}$$

[3] **Problem 4.** USAPhO 2017, problem A1.

2 Balancing Torques

Idea 4

A static rigid body will remain static as long as the total force on it vanishes, and the total torque vanishes, where the torque about the origin is

$$\boldsymbol{\tau} = \sum_i \mathbf{r}_i \times \mathbf{F}_i$$

where $\mathbf{r}_i$ is the point of application of force $\mathbf{F}_i$. If the total force vanishes, the total torque doesn't depend on where the origin is, because shifting the origin by $\mathbf{a}$ changes the torque by

$$\Delta\boldsymbol{\tau} = \sum_i \mathbf{a} \times \mathbf{F}_i = \mathbf{a} \times \left(\sum_i \mathbf{F}_i\right) = 0.$$

The origin should usually be chosen to set as many torques as possible to zero.

[1] **Problem 5.** The line of a force is defined to the line passing through its point of application parallel to its direction; then the torque of the force about any point on that line vanishes. Suppose

a body is static and has three forces acting on it. Show that in two dimensions, the lines of these forces must either be parallel or concurrent. This will be useful for several problems later.

Solution. Let $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ be the forces. Suppose two are parallel, then the third must be parallel to the first two to balance forces in the direction perpendicular to the direction of the first two. Now, suppose they are not parallel, and let the origin be at the intersection of the lines of forces of $\mathbf{F}_1$ and $\mathbf{F}_2$. Then, the torque due to these two is zero, so the torque due to $\mathbf{F}_3$ must also be zero, so the line of action of $\mathbf{F}_3$ must also pass through the origin.

Idea 5

The center of mass $\mathbf{r}_{\text{CM}}$ of a set of masses m_i at locations $\mathbf{r}_i$ with total mass M satisfies

$$M\mathbf{r}_{\text{CM}} = \sum_i m_i \mathbf{r}_i.$$

If a system experiences no external forces, its center of mass moves at constant velocity.

Idea 6

A uniform gravitational field exerts no torque about the center of mass. Thus, for the purposes of applying torque balance on an *entire* object, the gravitational force $M\mathbf{g}$ can be taken to act entirely at its center of mass. (This is a formal substitution; of course, the actual gravitational force remains distributed throughout the object.)

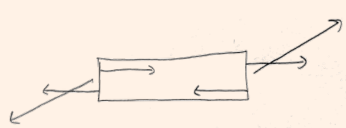
Torque balance works in noninertial frames, as long as one accounts for the torques due to fictitious forces. Thus, for an accelerating frame, the $-M\mathbf{a}$ fictitious force can be taken to act at the center of mass. In a uniformly rotating frame, the total centrifugal force is $M\omega^2\mathbf{r}_{\text{cm}}$, and for the purposes of balancing torques, can be taken to act entirely at the center of mass.

Example 2

Show that the tension in a completely flexible static rope, massive or massless, points along the rope everywhere in the rope.

Solution

Consider a tiny segment $d\ell$ of the rope. Since the rope is static, the tension forces on both ends balance, so they are opposite. Let them both be at an angle θ to the rope direction. Then the net torque on the segment is $(T\,d\ell)\sin\theta$. Since this must vanish for static equilibrium, we must have $\theta = 0$ and hence the tension is along the rope. In other words, flexible ropes can transmit force, but they can't transmit torque.



It's important to note that the argument above doesn't work for a rigid rod, because the internal forces in a rigid object can look like the picture above. In other words, there can be extra shear forces from the adjacent pieces of the rod that provide the compensating torque. If one tried to set up forces like this in a rope, it would flex instead.

In general, the force distribution within a massless rigid rod can be quite complicated, but if we zoom out, we can replace it with a single tension which does not necessarily point along the rod. This transmits both a force and a torque through the rod, in the sense that a torque is eventually exerted by whatever holds the end of the rod in place. Note that if the rod's supports are free to rotate, then they can't absorb torque, so the rod acts just like a rope, with tension always along it.

Remark

Sometimes, problem writers will intentionally not introduce any variables that are irrelevant to the answer. This can occur in two ways. First, the variables might just cancel out, as one can often see by dimensional analysis. Second, the specific values of the variables might not matter in the limit when they are very large or small. For instance, if a problem simply states a mass is "very heavy" but doesn't give it a name like m, it is asking for the answer in the limit $m \to \infty$.

Idea 7

To handle problems where an object is just about to tip over, note that at this moment, the entire normal force will often be concentrated at a point. (For example, when you're about to fall forward, all your weight goes on your toes.) That often means it's a good idea to take torques about this point.

Example 3: Povey 5.6

In problem 2, we treated the car as a point particle, but in reality it can also tip over. Suppose that on level ground, a car has a distance d between its left and right tires, which are both thin, and its center of mass is a height h above the ground. Now suppose the car turns as in problem 2 on a vertical wall ($\theta = 90^\circ$) with speed v. For what v is this possible?

Solution

Again working in the noninertial frame of the car, force balance gives

$$f_{\text{fric}} = mg, \quad N = \frac{mv^2}{R}$$

where f_{fric} and N are the total friction and normal forces on the four tires. Since $f_{\text{fric}}/N \leq \mu$,

$$v \geq \sqrt{gR/\mu}$$

which matches the general solution to problem 2. But in that problem, we only considered force balance. In this extreme situation, we also have to consider torque balance, i.e. the possibility that the car might topple over. When the car is about to topple over, all the normal and friction force is on the bottom tires. About this point, we have only torques from gravity and the centrifugal force, giving

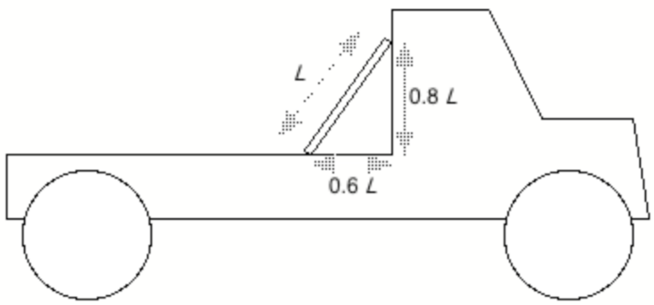
$$mgh = \frac{mv^2}{R}\frac{d}{2}$$

and solving for v gives $v = \sqrt{2gRh/d}$. Toppling is less likely the higher v is, so the answer is

$$v \geq \sqrt{gR}\,\max(1/\sqrt{\mu}, \sqrt{2h/d}).$$

Now here's a puzzle for you. A motorcycle only has one set of wheels, so it is effectively like a car with $d \to 0$. But motorcyclists can perform the motion described here, most famously in the Globe of Death, without toppling over. How is that possible?

[2] **Problem 6** (Quarterfinal 2004.3)**.** A uniform board of length L is placed on the back of a truck.



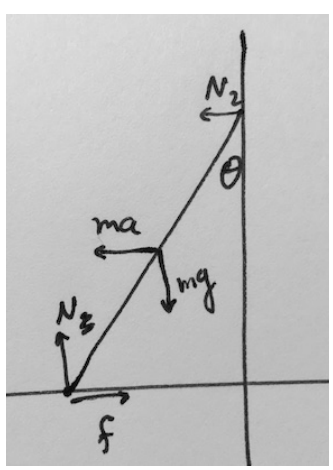
There is no friction between the top of the board and the vertical surface of the truck. The coefficient of static friction between the bottom of the board and the horizontal surface of the truck is $\mu_s = 0.5$. The truck always moves in the forward direction.

(a) What is the maximum starting acceleration the truck can have if the board is not to slip or fall over?

(b) What is the maximum stopping acceleration the truck can have if the board is not to slip or fall over?

(c) For what value of stopping acceleration is the static frictional force equal to zero?

Solution. Let us work in the accelerating frame of the truck.



Force balance gives $mg = N$ and $N_2 + ma = f$, and torque balance gives

$$-mg\frac{L}{2}\sin\theta + ma\frac{L}{2}\cos\theta + N_2 L\cos\theta = 0$$

which implies

$$2N_2 + ma = mg\tan\theta.$$

Thus,

$$N_2 = \frac{m(g\tan\theta - a)}{2}, \quad f = \frac{m(g\tan\theta + a)}{2}.$$

Since $-mg\mu \leq f \leq mg\mu$, to avoid slipping we require

$$-g \leq g\tan\theta + a \leq g \implies -g \leq \frac{3}{4}g + a \leq g \implies -\frac{7}{4}g \leq a \leq \frac{1}{4}g.$$

To avoid falling over, we need $N_2 > 0$, which is equivalent to

$$a \leq g\tan\theta = \frac{3}{4}g.$$

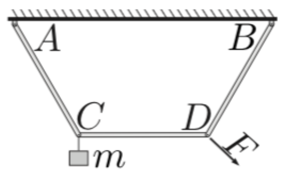
We can now read off the answers.

(a) For starting accelerations above $3g/4$ we would have falling, while for ones above $g/4$ we would have slipping. So slipping kicks in first, and the answer is $g/4$.

(b) Here the only constraint is slipping, and the answer is $7g/4$.

(c) Here $\frac{3}{4}g + a = 0$, so the truck decelerates with acceleration $0.75g$.

[2] **Problem 7** (Kalda)**.** Three identical massless rods are connected by freely rotating hinges.



The rods are arranged so that CD is parallel to AB, and $\overline{AB} = 2\overline{CD}$. A mass m is hung on hinge C. What is the minimum force that must be exerted at hinge D to keep the system stationary?

Solution. Let the rods have length ℓ. There are many ways to solve the problem, but the quickest is to consider the torque on the system of rod CD and its hinges, about the intersection point of AC and BD. About this point, the torque due to the weight of rod CD vanishes. Since the hinges are freely rotating, the force of rod AC on the system is directed along AC, so it also exerts no torque, and the same applies for the force from rod BD.

Thus, the only torque is $mg\ell/2$, from the weight of the mass. The applied force must balance this torque, and by some elementary geometry, we find that its maximum possible lever arm is ℓ, when the force is perpendicular to BD. Therefore, the minimum force is $\boxed{mg/2}$.

Note that it is crucial to assume the rods are massless. If the rods had mass, then the structure can't be supported by freely rotating hinges, even in the absence of the mass m and external force F. (For example, the forces of the hinges on the rod CD would have to be horizontal, which means they can't balance gravity.) Instead, in reality the structure would deform a bit until the hinges were no longer freely rotating, but rather jammed in place.

Idea 8

An extended object supported at a point may be static if its center of mass lies directly above or below that point. More generally, if the object is supported at a set of points, it can be static if its center of mass lies above the convex hull of the points.

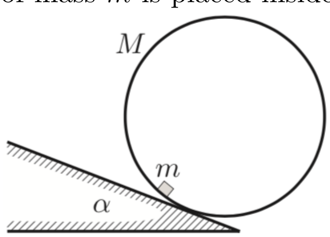
[2] **Problem 8.** N identical uniform bricks of length L are stacked, one above the other, near the edge of a table. What is the maximum possible length the top brick can protrude over the edge of the table? How does this limit grow as N goes to infinity?

Solution. Suppose we begin with all N blocks stacked directly on top of each other and slide them to the right. The maximal extension is reached when the center of mass of the top n blocks lies on the edge of the $(n+1)^{\text{th}}$ block. Let $\ell = L/2$, and suppose we have already adjusted the top $n-1$ blocks to be in the optimal position. Then the center of mass of the top n blocks is a distance ℓ/n from the edge of the $(n+1)^{\text{th}}$ block, so the n^{th} block and everything on top of it may be moved ℓ/n to the right. Hence the total distance is

$$\frac{L}{2}\left(1+\frac{1}{2}+\ldots+\frac{1}{N}\right) \approx \frac{L}{2}\int_1^N \frac{dx}{x} \approx \frac{L}{2}\log N$$

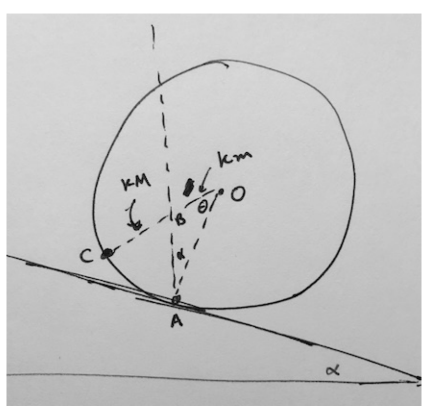
which is unbounded as $N \to \infty$. (By the way, if you allow blocks to be stacked in any combination, not just one on top of the other, then the maximum overhang is much larger. As shown in this neat paper, it grows as $N^{1/3}$.)

[2] **Problem 9** (Kalda)**.** A cylinder with mass M is placed on an inclined slope with angle α so that its axis is horizontal. A small block of mass m is placed inside it.



The coefficient of friction between the block and cylinder is μ. Find the maximum α so that the cylinder can stay at rest, assuming that the coefficient of friction between the cylinder and slope is high enough to keep the cylinder from slipping.

Solution.



We see that the center of mass of the cylinder-block system must be right above the contact point A. Now, we see that the CM is at B where $CB/BO = M/m$. Thus, $OA = OB + OC = k(m+M)$, so by the law of sines on OAB, we have

$$\frac{OB}{\sin\alpha} = \frac{OA}{\sin(\alpha+\theta)} \implies \sin(\alpha+\theta) = (1+M/m)\sin\alpha.$$

We see that m slips when $\tan(\alpha+\theta) = \mu$, or $\sin(\alpha+\theta) = \frac{\mu}{\sqrt{1+\mu^2}}$, so

$$\alpha_{\max} = \sin^{-1}\left(\frac{\mu}{\sqrt{1+\mu^2}}\left(1+\frac{M}{m}\right)^{-1}\right).$$

[2] **Problem 10** (PPP 11)**.** A sphere is made of two homogeneous hemispheres stuck together, with different densities. Is it possible to choose the densities so that the sphere can be placed on an inclined plane with incline 30° and remain in equilibrium? Assume the coefficient of friction is sufficiently high so that the sphere cannot slip.

Solution. By balancing torques around the point of contact, we need the center of mass to be straight above the point of contact. Doing some geometry, we learn that the center of mass be more than a distance $R/2$ away from the center of the sphere.

We now show that this is impossible. Consider a homogeneous hemisphere flat on a table. Its center of mass must be at a height lower than $R/2$, since the mass above the plane $z = R/2$ is less than the mass below it, and concentrated closer to the plane. Therefore, the centers of masses of the hemispheres are each within $R/2$ of the center of the sphere. Since the overall center of mass is a convex combination of the two, it is also within $R/2$ of the center, so the sphere cannot be stable.

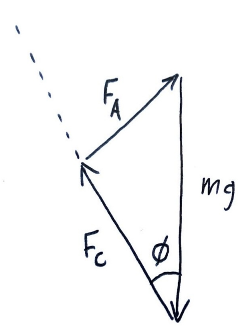
[3] **Problem 11.** An object of mass m lies on a uniform floor, with coefficient of static friction μ.

(a) First, suppose the object is a point mass. What is the minimum force required to make the object start moving, if you can apply the force in any direction?

(b) Now suppose the object is a thin, uniform bar. What is the minimum force required to make the object start moving, if the force can only be applied horizontally? Assume the normal pressure on the floor remains uniform.

Solution. (a) Just before the block slides, the friction force is μ times the normal force, so the sum of the normal force and friction force yields a single force with angle ϕ with respect to the vertical, where $\tan\phi = \mu$. Let's call this sum the "contact force", with magnitude F_C.

The contact force F_C, gravitational force mg, and applied force F_A acting on the object have to sum to zero, so the three force vectors have to form a closed triangle.

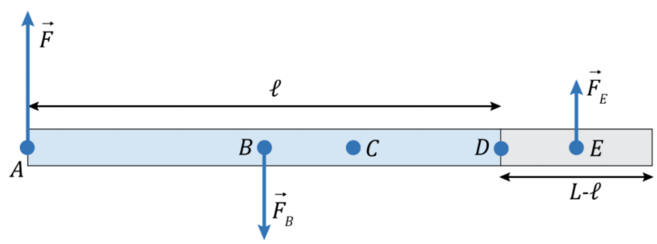


By varying the direction of the applied force, we vary both F_C and F_A. The smallest possible value of F_A occurs when the contact and applied force are perpendicular, so that the force vectors form a right triangle. Then by basic trigonometry, the minimum F_A is

$$F_{\text{min}} = Mg\sin\phi = \boxed{\frac{Mg\mu}{\sqrt{1+\mu^2}}}.$$

As a sidenote, if the block were treated as an extended object, not just a point particle, one would have to worry about whether it's possible to do this without tipping the block over instead. However, by choosing the point of application of the force correctly, it's always possible to make the block slide without tipping. Can you see why?

(b) Naively the answer is μmg, because that's the maximum total friction force. However, we know from everyday experience that it's easier to get the object to start moving if you pull at the edge. That's because the friction forces distributed along the bar also need to balance torque, which means some of them must point *along* the force you exert.



The figure above shows a top-down view of the bar. Just before slipping, friction has the maximum possible magnitude everywhere, and points either directly against or directly along the force you exert. Using the variables defined in the figure, just barely balancing forces and torques simultaneously gives

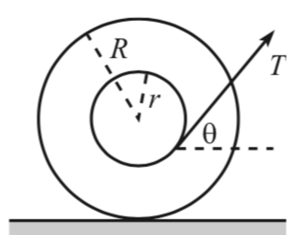
$$F = \mu mg\left(\frac{\ell}{L} - \frac{L-\ell}{L}\right), \quad F\ell = \mu mg\left(\frac{\ell}{L}\frac{\ell}{2} + \frac{L-\ell}{L}\frac{L-\ell}{2}\right).$$

Solving for ℓ gives $\ell = L/\sqrt{2}$, and plugging this in gives

$$F = (\sqrt{2}-1)\mu mg$$

which is less than half the naive answer!

[3] **Problem 12** (Morin 2.17)**.** A spool consists of an axle of radius r and an outside circle of radius R which rolls on the ground.



A thread is wrapped around the axle and is pulled with tension T at an angle θ with the horizontal.

(a) Which way does the spool move if it is pulled with $\theta = 0$?

(b) Given R and r, what should θ be so that the spool doesn't move? Assume that the friction between the spool and the ground is large enough so that the spool doesn't slip.

(c) Given R, r, and the coefficient of friction μ between the spool and the ground, what is the largest value of T for which the spool remains at rest?

(d) Given R and μ, what should r be so that you can make the spool slip from the static position with as small a T as possible? That is, what should r be so that the upper bound on T in part (c) is as small as possible? What is the resulting value of T?

Solution. (a) The torque about the contact point with the ground is clockwise, so the spool rolls to the right. You might think it would roll to the left, by thinking about torque about the center, but one must also account for the torque from friction with the ground; taking torques about the contact point avoids this complication.

(b) Let O be the center of the spool, A the point where the thread leaves the inner circle, and B the point of contact of the outer circle with the floor. We see that $\angle BOA = \theta$. Considering torques about B, we see that gravity provides 0 torque, so the tension must provide 0 torque as well. This means BA is tangent to the inner circle. Since BAO is a right triangle with $\angle BAO = 90°$, we have that $\boxed{\cos\theta = r/R}$.

(c) Let f be the friction force, and N the normal force. We see that $T\cos\theta = f$ and $N = Mg - T\sin\theta$. Since $f \leq \mu N$, we see

$$T\cos\theta \leq \mu(Mg - T\sin\theta) \implies \boxed{T \leq \frac{\mu Mg}{\cos\theta + \mu\sin\theta}},$$

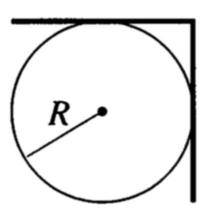
where $\theta = \cos^{-1}(r/R)$.

(d) We see that $\cos\theta + \mu\sin\theta = \frac{1}{\sqrt{1+\mu^2}}\cos(\theta - \beta)$ where $\tan\beta = \mu$. Thus,

$$T = \frac{\mu Mg}{\sqrt{1+\mu^2}\cos(\theta-\beta)},$$

so to minimize T, we want $\theta = \beta$, so $r = R\cos\beta = \boxed{\frac{R}{\sqrt{1+\mu^2}}}$, and the minimum value of T is $\boxed{\frac{\mu Mg}{\sqrt{1+\mu^2}}}$.

[3] **Problem 13** (PPP 44). A plate, bent at right angles along its center line, is placed on a horizontal fixed cylinder of radius R as shown.



How large does the coefficient of static friction between the cylinder and plate need to be if the plate is not to slip off the cylinder?

Solution. Let the normal and friction forces at the top be N_t, f_t and at the right N_r, f_r, and the static coefficient of friction be μ. Balancing forces on the plate gives

$$f_t = N_r, \quad N_t + f_r = mg.$$

Now, it's not obvious whether friction will be maximal at the top or the right contact point, or both, so we define

$$f_t = \mu_t N_t, \quad f_r = \mu_r N_r$$

where $\mu_t, \mu_t \leq \mu$. Eliminating the friction forces and solving the force balance equations gives

$$N_t = \frac{mg}{1 + \mu_r \mu_t}, \quad N_r = \frac{mg\mu_t}{1 + \mu_r \mu_t}.$$

Next, consider torques on the plate about its vertex. (This is an arbitrary choice; taking torques about either of the contact points also works about equally well.) The weight of the vertical of the plate contributes no torque, so the torque balance equation is

$$N_r + mg/2 = N_t.$$

Plugging in our results for N_r and N_t gives

$$\mu_t(2 + \mu_r) = 1.$$

To find the minimum coefficient of friction to avoid slipping, we need to find the solution to this equation where the *larger* of μ_r and μ_t is as small as possible. But it's clear now that increasing one decreases the other, so this is achieved when the two are equal. In other words, at the limit, slipping is just about to occur at both contact points simultaneously. Setting $\mu_r = \mu_t = \mu$ gives

$$\mu^2 + 2\mu - 1 = 0, \qquad \boxed{\mu = \sqrt{2} - 1.}$$

Incidentally, you can also do this problem with the idea of problem 5. At the minimum μ, we assume both friction forces are saturated. The lines of these forces must cross at a point directly above/below the center of mass, where gravity is applied. This quickly yields the same quadratic equation as found above. If you do it this way, though, it's a bit harder to see why both friction forces are saturated simultaneously at the minimum μ. It's usually true, but not guaranteed in general; our more explicit derivation above shows why.

3 Trickier Torques

Idea 9

Sometimes, a clever use of torque balance can be used to remove any need to have explicit force equations at all. Rarely, the same situation can occur in reverse.

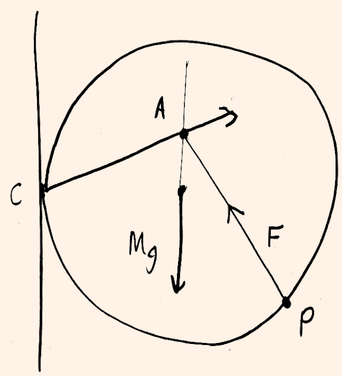
Example 4: EFPhO 2010.4

A spherical ball of mass M is rolled up along a vertical wall, by exerting a force F to some point P on the ball. The coefficient of friction is μ. What is the minimum possible force F, and in this case, where is the point P?

Solution

Following the logic of idea 3, when the minimum possible force is used, the frictional force with the wall must be maximal, $f = \mu N$, and directed upward. (If friction weren't pushing the ball up as hard as possible, we could get by using a smaller force F.) So even though we don't know the magnitude of the normal or the frictional force, we know the direction of the sum of these two forces, so we'll consider them as one combined force.

This reduces the number of independent forces in the problem to three: gravity (acting at the center of mass), the force F (acting at P), and the combined normal and friction forces (acting at the point of contact C with the wall). Therefore, by the result of problem 5, the lines of these forces must all intersect at some point A, as shown.



This ensures that the torques will balance, when taken about point A.

Next, we need to incorporate the information from force balance. Doing this directly will lead us to some nasty trigonometry, but there's a better way. There are in principle two force balance equations, for horizontal and vertical forces. However, one of these equations is just going to tell us the magnitude of the normal/frictional force, which we don't care about. So in reality, we just need one equation, which preferably doesn't involve that force.

The trick is to use torque balance *again*, about the point C, which says that the torques due to gravity and F must cancel. Now you might ask, didn't we already use torque balance? We did, but recall from idea 4 that taking the torque about a different point can give you a different equation if the forces don't balance. So by demanding the torque vanish about two different points, we actually are using force balance! (Specifically, we are using the linear combination of the horizontal and vertical force balance equations that *doesn't* involve the normal/friction force, which we don't need to find anyway.)

When taking the torque about C, we see that F is minimized if P is chosen to maximize the lever arm of the force. This occurs when $CA \perp PA$, in which case the lever arm is $R\sqrt{1+\mu^2}$, where R is the radius of the ball. So we have

$$MgR = FR\sqrt{1+\mu^2}, \quad F = \frac{Mg}{\sqrt{1+\mu^2}}$$

and P is determined as described above.

[2] **Problem 14.** NBPhO 2020, problem 4, parts (i) and (ii).

Solution. See the official solutions here.

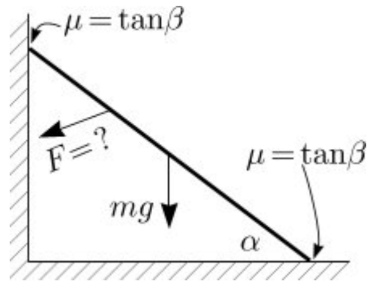
[3] **Problem 15.** EFPhO 2012, problem 3. The problem statement is missing some information: both the bars and rod have diameter d.

Solution. See the official solutions here.

[3] **Problem 16.** EFPhO 2006, problem 6. You will need to print out the problem to make measurements on the provided figure.

Solution. See the official solutions here.

[4] **Problem 17** (Physics Cup 2012)**.** A thin rod of mass m is placed in a corner so that the rod forms an angle α with the floor. The gravitational acceleration is g, and the coefficient of friction with the wall and floor is $\mu_s = \tan\beta$, which is not large enough to keep the rod from slipping.



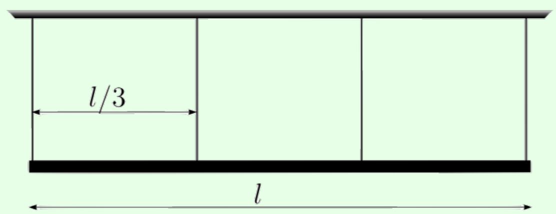
What is the minimum additional force F needed to keep the rod static?

Solution. See the solutions here.

We've now covered some really mathematically elegant problems, but it's important to remember the real-world limitations of this kind of analysis. We discuss two examples below.

Example 5

A uniform bar with mass m and length ℓ hangs on four equally spaced identical light wires. Initially, all four wires have tension $mg/4$.



Find the tensions after the leftmost wire is cut.

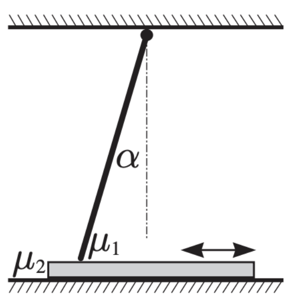
Solution

This illustrates a common issue with setups involving rigid supports: there are often more normal forces than independent equations, so there is not a unique solution. In the real world, the result is determined by imperfect characteristics of the wires. For example, if one of the wires was slightly longer than the others, it would go slack, reducing the number of normal forces by one and yielding a solution.

A reasonable assumption, if you aren't given any further information, is to assume that the supports are identical, very stiff springs. In equilibrium, the bar will tilt a tiny bit, so that the length of the middle wire will be the average of the lengths of the other two. By Hooke's law, the force in that wire will than be the average of the other two, so the tensions are $mg/3 - x$, $mg/3$, and $mg/3 + x$. Applying torque balance yields $7mg/12$, $mg/3$, and $mg/12$.

The general point here is that concepts like rigid bodies or strings characterized by a single tension force are abstractions, made for the idealized problems we study in mechanics classes. A real civil engineer designing a structure would instead use a sophisticated computer program which simulates the complex internal forces, torques, and strains throughout the material.

[2] **Problem 18** (Kalda). A rod is hinged to the ceiling, so that it makes an angle α with the vertical.



Underneath, a thin board is being dragged on the floor. The coefficient of (static and kinetic) friction is μ_1 between the board and rod, and μ_2 between the board and floor. The rod is meant to stop the board from being dragged to the right, no matter how hard or how quickly it is pulled. Is this possible? If so, what are the conditions on the parameters that allow this to occur?

Solution. Let the rod have mass m and length ℓ, and suppose it feels a normal force N and friction force f from the board. Then torque balance on the rod about the hinge gives

$$N\ell \sin\alpha = \frac{\ell}{2} mg \sin\alpha + f\ell\cos\alpha.$$

When friction is maximal and the board is about to move, $f = \mu_1 N$, so

$$N = \frac{mg \sin\alpha}{2(\sin\alpha - \mu_1 \cos\alpha)}.$$

It is impossible to move the board if μ_1 is large enough to make N blow up, so the board is stuck if

$$\mu_1 \geq \tan\alpha.$$

Physically, what's going on is that the harder you pull, the larger the normal force becomes, and so the larger the friction can be; that's how things get jammed.

Remark: Subtleties of Friction

Statics problems involving friction can also get quite elegant, but it's important to remember that they're just an imperfect approximation for the real world. Coulomb's law for "dry" friction, $|f| = \mu N$, works for a variety of materials, but for lubricated materials the friction has to be computed using fluid mechanics, as discussed in **M7**. More generally, friction is an entire field of study called tribology, which is essential for engineering. For much more, see *Tribology* by Hutchings and Shipway.

Coulomb's law has other weird features: assuming it can produce mathematical contradictions, in the so-called Painleve paradox! Problem 18 above is the simplest possible example of the paradox. In this problem, you showed that under the right conditions, it is impossible for the board to move to the right. But that means that if you assume the board starts with some rightward velocity, then trying to solve Newton's second law for the acceleration will give a mathematical contradiction.

Of course, in real life there actually would be some finite, albeit large acceleration. To solve for it you would need to adopt a more refined model of the rod and its interaction with the board, e.g. accounting for how the rod and board deform when subject to stress. That requires "contact mechanics", briefly discussed in example 14.

We conclude with some questions that train three-dimensional thinking.

[2] **Problem 19** (PPP 10)**.** In Victor Hugo's novel *les Miserables*, the main character Jean Valjean, an escaped prisoner, was noted for his ability to climb up the corner formed by the intersection of two vertical perpendicular walls. Suppose for simplicity that Jean has no feet. Let μ be the coefficient of static friction between his hands and the walls. What is the minimum force that Jean had to exert on each hand to climb up the wall? Also, for what values of μ is this feat possible at all?

Solution. Jean Valjean experiences two normal forces and two friction forces, one from each hand. Each friction force must balance the other normal force, plus half the weight, so

$$f_{\text{fric}}^2 = N^2 + (mg/2)^2.$$

Assuming the friction is maximal, $f_{\text{fric}} = \mu N$, we have

$$N = \frac{mg}{2\sqrt{\mu^2 - 1}}$$

and the force Jean Valjean exerts with each hand is

$$F = \sqrt{N^2 + f_{\text{fric}}^2} = \frac{mg}{2}\sqrt{\frac{\mu^2+1}{\mu^2-1}}.$$

The feat is only possible if $\mu > 1$.

[3] **Problem 20** (PPP 69)**.** A homogeneous triangular plate has threads of length h_1, h_2, and h_3 fastened to its vertices. The other ends of the string are fastened to a common point on the ceiling. Show that the tension in each thread is proportional to its length. (Hint: with the origin at the point on the ceiling, let the vertices be at positions $\mathbf{r}_i$ and express everything in vector form.)

Solution. Define the origin to be the attachment point on the ceiling, and let the vertices be at positions $\mathbf{r}_i$. The tensions are along the ropes, so let them be $\mathbf{T}_i = -\eta_i \mathbf{r}_i$. Force balance says

$$\eta_1 \mathbf{r}_1 + \eta_2 \mathbf{r}_2 + \eta_3 \mathbf{r}_3 = m\mathbf{g}.$$

Torque balance tells us that the center of mass of the triangle must lie directly below the attachment point, and the center of mass is at

$$\mathbf{r}_{\text{CM}} = \frac{1}{3}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_2)$$

which means that

$$\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 \propto \mathbf{g}.$$

Thus, we know that the sum of the $\mathbf{r}_i$ is in the vertical direction, and also that the weighted sum of the $\eta_i \mathbf{r}_i$ is in the same vertical direction. This is only possible if all the η_i are equal to each other, which proves the desired result.

In case you're not convinced, we can justify this in more detail. Let $\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = \alpha\mathbf{g}$. Then subtracting this equation from α/m times the force balance equation gives

$$\sum_i \left(1 - \frac{\alpha}{m}\eta_i\right) \mathbf{r}_i = 0.$$

The only way a nontrivial sum of three vectors can vanish is if they lie in a plane, which isn't true here. So each of the coefficients must vanish, so $1 - (\alpha/m)\eta_i = 0$, which means all the η_i are the same, $\eta_i = m/\alpha$.

[4] **Problem 21** (KoMaL 2019, BAUPC 1998)**.** Two identical uniform solid cylinders are placed on a level tabletop next to each other, so that they are touching. A third identical cylinder is placed on top of the other two.

(a) Find the smallest possible values of the coefficients of static friction between the cylinders, and between a cylinder and the table, so that the arrangement can stay at rest.

(b) Repeat part (a) for spheres. That is, put three uniform solid spheres next to each other, with their centers forming an equilateral triangle, and put a fourth sphere on top.

(c) Now return to part (a), and suppose the setup is frictionless. A force is applied directly to the right on the leftmost cylinder, causing the entire setup to accelerate. Find the minimum and maximum accelerations so that all three cylinders remain in contact with each other.

Parts (a) and (b) demonstrate an interesting point: it is possible for a collection of objects to resist some force, even though a single one of those objects would begin moving even with an infinitesimal applied force! This is a simple example of how granular materials, like sand, can give rise to emergent phenomena that are hard to predict from analyzing individual grains alone. Understanding these materials is a whole field of applied research.

Solution. (a) Call the top cylinder A, and the bottom ones B and C. Suppose the normal force between the top cylinder and one of the bottom cylinders is N, and the friction force is f. When the coefficient of static friction is at its smallest between the cylinders, $f = \mu_1 N$. B and C are being pushed apart by A, so only the ground, gravity, and A are exerting forces on B and C.

The total torque about C is 0, so the friction force from the ground is also f (towards B). Then for the net horizontal force on C to be 0, by drawing out the forces and their directions we get

$$f + f\cos(\pi/6) = N\sin(\pi/6) \qquad \mu_1 = \frac{f}{N} = \frac{1}{2+\sqrt{3}} = 0.268$$

The normal force between the cylinder and the ground is $N_2 = N\cos(\pi/6) + f\sin(\pi/6) + mg$. The vertical force exerted on A from C is $N\cos(\pi/6) + f\sin(\pi/6) = mg/2$. Or, you can use the fact that $N_2 = 3mg/2$ by symmetry. Using the latter 2 equations:

$$N_2 = 3\left(\frac{\cos\pi/6}{\mu_1} + \sin\pi/6\right) f$$

$$f = \frac{\mu_1}{3} N_2 = 0.0893 N_2$$

So the coefficient of friction between the cylinders and the ground is 0.0893.

(b) All the spheres are being pushed apart, so the analysis above is the same except now the angle is a bit different and the bottom balls exert a vertical force of $mg/3$ on the top ball since there are 3 supports now.

The lines connecting the centers of the spheres form a tetrahedron by symmetry.

Let the length of the sides of a tetrahedron $ABCD$ be ℓ, and A being the point at the top (center of the top sphere). Then the distance from the centroid of triangle BCD to B is $\ell/\sqrt{3}$ (use the fact that medians intersect in a ratio of 2 to 1 or draw a line from the centroid to a side). Since AB has length ℓ, the angle between the vertical and the lines connecting the centers of the top sphere and a bottom sphere is $\alpha = \arcsin(1/\sqrt{3})$.

We will replace $\sin(\pi/6)$ with $1/\sqrt{3}$ and cos with $\sqrt{2/3}$ in the previous results. Thus with the same analysis on a bottom ball with only the top ball acting on it, the friction coefficient between the balls is:

$$\mu_1 = \frac{\sin\alpha}{1+\cos\alpha} = \sqrt{3} - \sqrt{2} = 0.318$$

And copying the above formula except $N_2 = 4mg/3$ and $N\cos(\pi/6) + f\sin(\pi/6) = mg/3$ will get

$$N_2 = 4\left(\frac{\cos\alpha}{\mu_1} + \sin\alpha\right) f \qquad \mu_2 = \frac{\mu_1}{4} \approx 0.0795.$$

(c) Call the top cylinder A, the left cylinder B, and the right cylinder C, and the normal forces between them N_{ij}. Let $\theta = \pi/6$.

At the minimum acceleration, the weight of cylinder A almost pushes B and C apart, so $N_{BC} = 0$. Under this assumption, considering horizontal forces on cylinders A and C gives

$$N_{AC} \sin\theta = ma, \quad (N_{BA} - N_{AC}) \sin\theta = ma$$

while balancing vertical forces on cylinder C gives

$$(N_{BA} + N_{AC}) \cos\theta = mg$$

Combining these equations and plugging in θ, we find

$$2N_{AC} = 4ma = \frac{mg}{\sqrt{3}/2} - 2ma$$

from which we read off

$$a_{\mathrm{min}} = \frac{g}{3\sqrt{3}}.$$

Now consider the maximum acceleration. In this case, cylinder A will be just about to fly off the top, so that $N_{AC} = 0$. Thus, the only normal force on cylinder A is from cylinder B, and considering horizontal and vertical forces on cylinder A gives

$$N_{BA} \sin\theta = ma, \quad N_{BA} \cos\theta = mg$$

from which we read off

$$a_{\mathrm{max}} = \frac{g}{\sqrt{3}}.$$

4 Extended Bodies

Idea 10: Principle of Virtual Work

Some physical systems have a lot of parts but also a lot of constraints, such as joints, so that they can only move in a few ways. In these cases, it is easiest to determine if the system is static using the principle of virtual work. Suppose that the system moves an infinitesimal amount, in a way allowed by the constraints. If the net work done on the system by this "virtual displacement" is zero, then the system can be in static equilibrium at that position. (Alternatively, if you already know the system can be in static equilibrium, this approach can be used to find the magnitude of one unknown force, such as the tension in a string.)

Example 6: Roberval Balance

Consider the following scale made of rigid bars. The joints ensure that the quadrilateral in the middle always remains a parallelogram, with its left and right sides vertical.

If identical weights are placed on each horizontal arm as shown, can the system remain static?

Solution

There's only one way for the system to move: the rectangle can deform into a parallelogram so that the left horizontal arm moves up, and the right horizontal arm moves down by the same amount. Then the total virtual work done on the scale by the weights is zero, so the system can be in equilibrium no matter where on the arms the weights are placed.

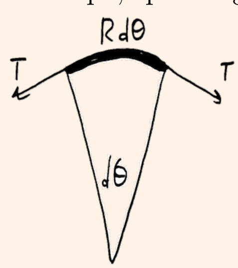
Next, we'll consider problems with continuous bodies, where one often needs to consider forces and torques acting on infinitesimal pieces.

Example 7

Find the tension in a circular rope of radius R spinning with angular velocity ω and mass per length λ.

Solution

Consider an infinitesimal segment of the rope, spanning an angle $d\theta$.



The mass of this segment is $dm = R\lambda\, d\theta$. The total force is downward, with magnitude

$$dF = 2T \sin \frac{d\theta}{2} \approx T\, d\theta$$

where we used the small angle approximation. This is the centripetal force, so

$$dF = (dm)\, \omega^2 R.$$

Combining these results yields $T = R^2 \omega^2 \lambda$.

Example 8

Find the distance d of the center of mass of a uniform semicircle of radius R to its center. (Note that a semicircle is half of a circle, not half of a disc.)

Solution

This can be done by taking the setup of the previous problem, and taking a subsystem comprising exactly half of the rope. In this case the net tension force is simply

$$F = 2T.$$

The total mass is $m = \pi R\lambda$, and the force must provide the centripetal force, so

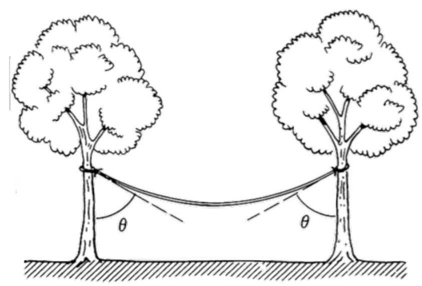
$$F = (\pi R\lambda)(\omega^2 d)$$

But we also know that $T = R^2\omega^2\lambda$ as before, so plugging this in gives

$$d = \frac{2}{\pi}R.$$

Alternatively, we could have worked in the frame rotating with the rope. The equations would be the same, but instead we would say the tension balances the centrifugal force.

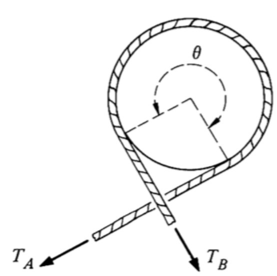
[1] **Problem 22** (KK 2.22)**.** A uniform rope of weight W hangs between two trees. The ends of the rope are the same height, and they each make angle θ with the trees.



Find the tension at either end of the rope, and the tension at the middle of the rope.

Solution. Let the tension at the end be T_0, and T_1 at the center. Considering the entire rope as one system, we see that $2T_0\cos\theta = W$, so $T_0 = \frac{W}{2\cos\theta}$. Considering one half of the rope as a system, we see $T_1 = T_0 \sin\theta = \frac{W}{2}\tan\theta$.

[3] **Problem 23** (KK 2.24)**.** A capstan is a device used aboard ships to control a rope which is under great tension.



The rope is wrapped around a fixed drum with coefficient of friction μ, usually for several turns. The load on the rope pulls it with a force T_A. Ignore gravity.

(a) Show that the minimum force T_B needed to hold the other end of the rope in place is $T_A e^{-\mu\theta}$, an exponential decrease.

(b) How does this result depend on the shape of the capstan, if we fix the angle θ between the initial and final tension forces? Would the answer be the same for an oval, or a square?

(c) If $\theta = \pi$, explain why the total normal and friction force of the rope on the drum is $T_A + T_B$.

Solution. (a) Consider a small piece of the rope that turns through an angle $d\theta$. Using the small angle approximation, the normal force must be $T\,d\theta$, and the friction force must be dT. Setting $f = \mu N$ gives $\mu T\,d\theta = dT$, or $dT/T = \mu\,d\theta$, and integrating gives the desired result.

(b) The infinitesimal reasoning above doesn't care about the shape as long as it's reasonably smooth, so the answer for an oval is the same: just break it into pieces that turn through $d\theta$ again. On the other hand, for a square one has sharp kinks where the normal force is singular, in which case the answer won't be as reliable.

(c) Consider the system consisting of the curved part of the rope. This system experiences a force $T_A + T_B$ from the straight part of the rope. But it is static, which means it must also experience an equal and opposite force from the drum, which comes from integrating the friction and normal forces along the contact surface.

That's all you have to say, but we can also show this more explicitly. For concreteness, let both tensions be vertical. We have a normal force and difference in tension forces

$$dN = T\,d\theta, \quad dT = -df_{\text{fric}}$$

on a small piece $d\theta$ of the rope. The contribution to the vertical force on the drum is

$$dF_y = dN\sin\theta + df_{\text{fric}}\cos\theta = T\sin\theta\,d\theta - dT\cos\theta = -d(T\cos\theta)$$

by the product rule. So the total vertical force is

$$F_y = \int dF_y = -\int_0^{\pi} d(T\cos\theta) = -(T_A + T_B)$$

as expected. A very similar manipulation shows that $F_x = 0$.

[2] **Problem 24 (*F = ma* 2018 B20).** A massive, uniform, flexible string of length L is placed on a horizontal table of length $L/3$ that has a coefficient of friction $\mu_s = 1/7$, so equal lengths $L/3$ of string hang freely from both sides of the table. The string passes over the edges of the table, which are smooth frictionless curves, of size much less than L. Now suppose that one of the hanging ends of the string is pulled a distance x downward, then released at rest. Neither end of the string touches the ground.

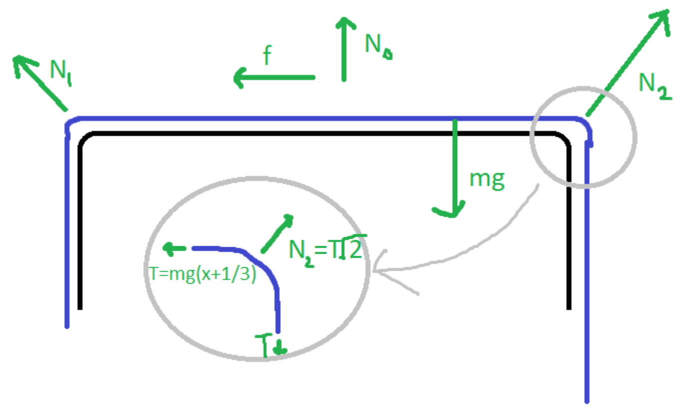
(a) Find the maximum value of x so that the string does not slip off of the table.

(b) For the case $x = 0$, draw a free body diagram for the string, indicating only the *external* forces on the entire string. Do the forces balance?

(c) Would the answer change significantly if the table's small edges had friction as well?

Solution. (a) The difference in weights is $2(Mg/L)x$, and needs to be balanced by the friction force f. At the max value of x, $f = \mu_s N = \mu_s Mg/3$ (the normal force at the top only holds up the top of the string), so $x = (\mu_s/6)L = L/42$.

(b) At first, it may seem that the forces don't balance, because the normal force from the flat part of the table only balances the weight of the string above it, leaving nothing to balance the weight of the vertical parts of the string. But we must recall that there is an enormous normal pressure at the smooth corners. The total normal force there is large enough so that its vertical component holds up *all* of the string underneath it.



(c) Yes, the answer changes sufficiently no matter how small the edges are. This is because, as we saw in part (b), there is a sizable normal force at the edges, since they alone are responsible for holding up a significant part of the rope. So turning on a coefficient of friction at the edges would yield a sizable friction force. (You can calculate it using problem 23.)

[3] **Problem 25** (Morin 2.25)**.** A rope rests on two platforms that are both inclined at an angle θ.

The rope has uniform mass density, and the coefficient of friction between it and the platforms is 1. The system has left-right symmetry. What is the largest possible fraction of the rope that does not touch the platforms? What angle θ allows this maximum fraction?

Solution. Let η be the fraction of the rope that does not touch the platforms. Split the rope into the 3 obvious pieces (the left touching portion, the hanging portion, the right touching portion). Let T be the tension at the boundaries (its the same on both sides by symmetry). Balancing forces on the middle portion tells us

$$2T\sin\theta = \eta mg \implies T = \frac{\eta mg}{2\sin\theta}.$$

We see the friction force on the left piece is $f = T + \frac{1-\eta}{2} mg\sin\theta$, and the normal force is $N = \frac{1-\eta}{2} mg\cos\theta$. We have $f \leq N\mu$, so

$$\frac{\eta mg}{2\sin\theta} + \frac{1-\eta}{2} mg\sin\theta \leq \frac{1-\eta}{2} mg\mu\cos\theta,$$

or

$$\frac{\eta}{\sin\theta} + (1-\eta)\sin\theta \leq (1-\eta)\cos\theta,$$

so some algebra reveals

$$\boxed{\eta \leq \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\tan\theta}.$$

Doing some more algebra turns this into

$$\eta \leq \frac{\sin 2\theta + \cos 2\theta - 1}{\sin 2\theta + \cos 2\theta + 1}.$$

To maximize η, we need to maximize $\sin 2\theta + \cos 2\theta$, which implies $\boxed{\theta = \pi/8}$. The corresponding value of η is $3 - 2\sqrt{2}$.

Example 9

A chain is suspended from two points on the ceiling a distance d apart. The chain has a uniform mass density λ, and cannot stretch. Find the shape of the chain.

Solution

First, we note that the horizontal component of the tension T_x is constant throughout the chain; this just follows from balancing horizontal forces on any piece of it. Moreover, by similar triangles, we have $T_y = T_x y'$ everywhere.

Now consider a small segment of chain with horizontal projection Δx. The length of the piece is $\Delta x\sqrt{1+y'^2}$ which determines its weight, and this be balanced by the difference in vertical tensions. Thus

$$\Delta T_y = \lambda g\sqrt{1+y'^2}\,\Delta x.$$

For infinitesimal Δx, we have $\Delta T_y = T_x d(y') = T_x y''\,dx$, so we get the differential equation

$$y'' = \frac{\lambda g}{T_x}\sqrt{1+y'^2}.$$

Usually nonlinear differential equations with second derivatives are very hard to solve, but this one isn't because there is no direct dependence on y, just its derivatives. That means we can treat y' as the independent variable first, and the equation is effectively first order in y'.

Writing $y'' = d(y')/dx$ and separating, we have

$$\int \frac{dy'}{\sqrt{1+y'^2}} = \int \frac{\lambda g}{T_x}\,dx.$$

Integrating both sides gives

$$\sinh^{-1}(y') = \frac{\lambda g x}{T_x} + C.$$

Choosing $x = 0$ to be the lowest point of the chain, the constant C is zero, and

$$y' = \sinh\left(\frac{\lambda g x}{T_x}\right).$$

Integrating both sides again gives the solution for y,

$$y = \frac{T_x}{\lambda g}\cosh\left(\frac{\lambda g x}{T_x}\right)$$

where we suppressed another constant of integration. This curve is called a catenary.

[1] **Problem 26.** To check that you understand the previous example, repeat it for a suspension bridge. In this case the cable is attached by vertical suspenders to a horizontal deck with mass λ per unit length, and supports the weight of the deck. Assume the cable and suspenders have negligible mass.

Solution. By the same logic as in the example, we have

$$y'' = \frac{\lambda g}{T_x}$$

where there is now no factor of $\sqrt{1 + y'^2}$. Integrating this twice gives

$$y = \frac{\lambda g}{T_x}\frac{x^2}{2}$$

which is a parabola. One result of this analysis is that the required height of the bridge scales as the square of its horizontal span, which is why very long suspension bridges are broken into multiple spans. According to Feynman, engineers were able to watch the shape of the cables of the George Washington bridge turn from a catenary into a parabola as the deck was installed.

By the way, essentially the same calculation can be used to determine the shape of an ideal suspended arch bridge. The main difference is that the arch, being a solid structure, can transmit internal torques (i.e. bending moments, as discussed below) which can result in more general shapes. But in a well-designed arch bridge this internal torque should be negligible, so the analysis is almost identical to the suspended cable bridge, but with an extra minus sign since arches are in compression rather than tension. The shape is an inverted parabola.

[4] **Problem 27** (MPPP)**.** A slinky is a uniform spring with negligible relaxed length, with mass m and spring constant k.

(a) Find the shape of a slinky hung from two points on the ceiling separated by distance d. (Hint: to begin, consider the mass and tension of a small piece of the spring with horizontal and vertical extent dx and dy. Don't forget that the slinky's density won't be uniform.)

(b) Suppose a slinky's two ends are fixed, separated by distance d, and rotating uniformly with angular frequency ω like a jump rope in zero gravity. Find the values of ω for which this motion is possible, and the shape of the slinky in this case.

Solution. (a) Consider a small piece of the spring with mass dm, and horizontal and vertical extent dx and dy. This piece of the spring has spring constant km/dm, which means

$$T_x = km\frac{dx}{dm}, \quad T_y = \frac{dy}{dx}T_x.$$

By horizontal force balance, T_x is a constant, which means dx/dm is a constant; the same amount of mass is contained within each horizontal interval. Thus

$$\frac{dx}{dm} = \frac{d}{m}.$$

Balancing vertical forces on this segment gives

$$dT_y = y'' T_x \, dx = g \, dm$$

and combining this with the previous result gives

$$y'' = \frac{mg}{kd^2}.$$

We thus conclude that the shape is a parabola. Centering it at $x = 0$, we have

$$y = \frac{mgx^2}{2kd^2}.$$

In particular, the lowest point of the parabola is a distance $y(d/2) - y(0) = mg/8k$ below the supports. (This solution is very similar to that of the example; the only difference is that the weight of the segment is proportional to dx instead of $\sqrt{1+y'^2}\,dx$. This is because the slinky's mass per length is not constant, while the chain's was.)

(b) The only difference with respect to part (a) is that now we have a radial "gravity" force of $g_{\text{eff}} = -\omega^2 y$, because of the centrifugal acceleration in the frame rotating with the slinky. Therefore,

$$y'' = -\frac{m\omega^2}{kd^2}\, y$$

The solution is a sinusoid. For concreteness, let's suppose one endpoint is at $x = 0$, imposing $y(0) = 0$. Then

$$y(x) = y_0 \sin\left(\sqrt{\frac{m}{k}}\frac{\omega}{d}\, x\right).$$

For the other endpoint to be fixed, $y(d) = 0$, we must have

$$\sqrt{\frac{m}{k}}\,\omega = n\pi.$$

If ω satisfies this condition for some n, then the slinky can rotate with uniform angular velocity, and its shape is a sinusoid. The value of y_0 is arbitrary.

Another way to say this is that the solutions we have found here are standing waves. The valid values of ω, given the spring parameters, are just the standing wave frequencies. The fact that ω doesn't depend on d follows from dimensional analysis, and reflects the fact that stretching the string further increases the tension and decreases the density, therefore increasing the wave speed. These two effects cancel, keeping the standing wave frequencies the same.

Note that so far we've considered three cases: a hanging rope (in the example), a hanging slinky, and a rotating slinky. So what about a rotating rope? Unfortunately, the differential equation describing it is $y'' \propto y\sqrt{1+y'^2}$, since the centrifugal acceleration is proportional to y. And unlike the example, this is a genuine nonlinear second order differential equation. Mathematica reports that the solution is not an elementary function, but rather an inverse elliptic integral. Unfortunately, that's just what happens most of the time.

Example 10

A uniform spring of spring constant k, mass m, and relaxed length L is hung from the ceiling. Find its length in equilibrium, as well as its center of mass.

Solution

Problems like this contain subtleties in notation. For example, if you talk about "the piece of the slinky at z", this could either mean the piece that's actually at this position in equilibrium, or the piece that was originally at this place in the absence of gravity. Talking about it the first way automatically tells you where the piece is now, but talking about it the second way makes it easier to keep track of, because then the z of a specific piece of the spring stays the same no matter where it goes.

In fluid dynamics, these are known as the Eulerian and Lagrangian approaches, respectively. If you don't use one consistently, you'll get nonsensical results, and it's easy to mix them up.

There are many ways to solve this problem, but I'll give one that reliably works for me. We're going to use the Lagrangian approach, and avoid confusion with the Eulerian approach by breaking the spring into discrete pieces. Let the spring consist of $N \gg 1$ pieces, of masses m/N, spring constants Nk, and relaxed lengths L/N.

The i^{th} spring from the bottom has tension $(i/N)mg$, and thus is stretched by

$$\Delta L_i = \frac{1}{kN}\frac{i}{N}mg = \frac{mg}{kN^2}\,i.$$

The total stretch is

$$\sum_{i=1}^{N} \Delta L_i = \frac{mg}{kN^2}\int_0^N i\,di = \frac{mg}{2k}.$$

This makes sense, since the average tension is $mg/2$. To find the center of mass, note that the j^{th} spring is displaced downward by a distance

$$\Delta y_j = \sum_{i=j}^{N} \Delta L_i = \frac{mg}{2k}\left(1 - \frac{j^2}{N^2}\right)$$

downward from its position in the absence of gravity. The center of mass displacement is

$$\Delta y_{\text{CM}} = \frac{1}{N}\sum_{j=1}^{N} \Delta y_j \propto \frac{1}{N}\sum_{j=1}^{N}\left(1 - \frac{j^2}{N^2}\right) = \frac{1}{N^3}\int_0^N N^2 - j^2\,dj = \frac{2}{3}$$

so restoring the proportionality constant gives

$$\Delta y_{\text{CM}} = \frac{mg}{3k}.$$

If you want to test your understanding of slinkies, you can also try doing this problem with the Eulerian approach. This would be best done without discretization. The first steps would be finding a relation between the density $\rho(z)$ and tension $T(z)$ from Hooke's law, and finding out how to write down local force balance as a differential equation.

Remark

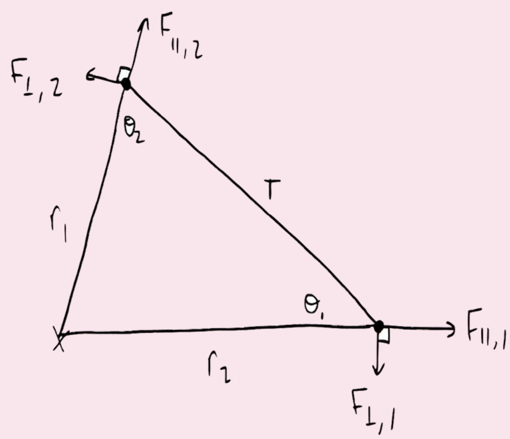
In this problem set, we've given some examples involving static, continuous, one-dimensional objects such as strings and ropes. The general three-dimensional theory of elasticity is mathematically quite complicated, but extremely important in engineering. For more about this subject, which requires comfort with tensors, see chapters 6 through 11 of Lautrup. It is also covered in chapters II-31, II-38, and II-39 of the Feynman lectures.

Remark: Why Use Torque?

Here's a seemingly naive question. Why is the idea of torque so incredibly useful in physics problems, even though in principle, everything can be derived from $F = ma$ alone? Why is it almost impossible to solve any nontrivial problem without referring to torques, and how would a student who's never heard of torque come up with it in the first place?

We don't need torque to analyze the statics of a single, featureless point particle. Torque only became useful in this problem set when we started analyzing rigid bodies with spatial extent. The reason we couldn't reduce torque balance to force balance easily is because the internal forces in these bodies, which maintain their rigidity, are generally very complicated.

To see this, let's consider the simplest possible example of a rigid body: a triangle with masses at the vertices, and sides made of very thin, rigid, massless springs. The triangle is pivoted at one vertex, and experiences external forces $\mathbf{F}_1$ and $\mathbf{F}_2$ at the other two vertices.



Consider force balance on the first marked vertex. The tension in the side of length r_2 takes whatever value is necessary to balance the horizontal force on the vertex, while the tension T in the other side has to balance the vertical force. Thus,

$$F_{\perp,1} = T \sin\theta_1, \qquad F_{\perp,2} = T \sin\theta_2$$

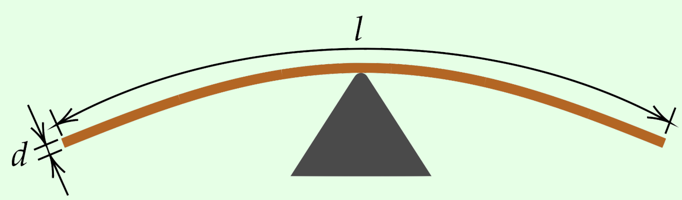
where the second line follows by considering the second marked vertex. Eliminating T and using the law of sines gives $r_2 F_{\perp,1} = r_1 F_{\perp,2}$, which of course is the statement of torque balance. (And if you continue along this line of reasoning, letting the forces be arbitrary, you can also derive the rotational form of Newton's second law, $\tau = I\alpha$.)

However, recovering the results of torque balance is much more complicated in general. For example, consider a rigid bar supported at its ends. The middle of the bar doesn't collapse, despite the force of gravity on it, because the bar contains internal, upward shear forces, which transmit the normal forces applied at its end throughout the rest of the bar. To analyze such systems without using torque, one would have to account for all of these microscopic forces, acting on all of the rod's infinitely many pieces. So for any problem with a continuous object, torque balance is an essential tool.

In fact, now that you're aware of how complicated internal forces can be, you might be wondering why torque balance even works. The simplest explanation is by the principle of virtual work. The torque of a force about a pivot is just the virtual work it does per angle the system rotates about the pivot. If a system is in static equilibrium, it must experience no net virtual work under any displacement, including rotations, so the torques must cancel.

Example 11: IPhO 2022 3A

A thin piece of spaghetti of diameter d is balanced horizontally from its middle.



It can have a length $\ell \gg d$ before it snaps under its own weight. How does ℓ scale with d?

Solution

Let the spaghetti rod have density ρ, and consider its right half. There must be a vertical normal force $F \sim \rho d^2 \ell$ to balance the weight. This vertical force is transmitted through the rod by a shear stress (i.e. an internal force per area, perpendicular to the rod) of order $\sigma_s \sim F/A \sim \rho \ell$. Each piece of the rod exerts such a shear stress on its neighbors, just like how pieces of a string exert tensions on their neighbors.

Now consider torques on the right half of the rod, about the pivot point. The torque $\tau \sim \rho d^2 \ell^2$ of the rod's weight has to be balanced by forces from the other half of the rod. Vertical forces don't work, since they don't provide any torque about the pivot. Instead, the torque is supplied by a horizontal compression force at the bottom, and a horizontal tension force at the top, which cancel out to maintain horizontal force balance. This combination of forces, which produces no net force but does produce a net torque, is a bending moment.

Let the associated normal stresses be of order $\pm\sigma_n$. Then the net compression and tension forces are of order $\pm d^2 \sigma_n$, and the lever arm is of order d, so balancing torques gives

$$\rho d^2 \ell^2 \sim \sigma_n d^3$$

which implies $\sigma_n \sim \rho \ell^2 / d$. This is much greater than σ_s, because of the miserably small lever arm, which is why thin rods usually break by snapping, not by shearing or pulling apart. Given a fixed maximum σ_n, we conclude the maximum length scales as $\ell \sim \sqrt{d}$.

[3] **Problem 28.** USAPhO 2022, problem A1. A practical bending moment problem.

5 Pressure and Surface Tension

Example 12

A sphere of radius R contains a gas with a uniform pressure P. Find the total force exerted by the gas on one hemisphere.

Solution

The pressure provides a force per unit area orthogonal to the sphere's surface, so the straightforward way to do this is to integrate the vertical component of the pressure force over a hemisphere. However, there's a neat shortcut in this case.

Momentarily forget about the sphere and just imagine we have a sealed hemisphere of gas at pressure P. The net force of the gas on the hemisphere must be zero, or else it would just begin shooting off in some direction, violating conservation of momentum. So the force on the curved face must balance the force on the flat face, which is $\pi R^2 P$. The same logic must hold for the sphere, since the forces on the curved face are the same, so the answer is $\pi R^2 P$.

This trick will come in handy for several future problems; for example, it's the quick way to do $F = ma$ 2018 B24. It also generalizes to surfaces of arbitrary shape, as discussed in **E1**. Concretely, suppose a surface S has boundary C, and consider any other surface S' with the same boundary. Then by the same logic, the closed surface formed by S and S' together experiences no net pressure force, so the pressure forces on S and S' are equal in magnitude.

Idea 11

The surface of a fluid carries a surface tension γ. If one imagines dividing the surface into two halves, then γ is the tension force of one half on the other per length of the cut. Specifically, for a small segment $d\mathbf{s}$ along the cut, where the normal vector to the surface is $\hat{\mathbf{n}}$, the surface tension force is

$$d\mathbf{F} = \gamma \, d\mathbf{s} \times \hat{\mathbf{n}}$$

which means the force acts along the surface and perpendicular to the cut.

Example 13

A spherical soap bubble of radius R and surface tension γ is in air with pressure P, and contains air with pressure $P + \Delta P$. Compute ΔP.

Solution

We use the result of the previous problem to conclude that the force of one hemisphere on another is $\pi R^2 \Delta P$. This must be balanced by the surface tension force. By imagining cutting the surface of the bubble in half, the surface tension force is γL where L is the total length of the surface connecting the hemispheres.

At this point, we can write $L = 2\pi R$, giving

$$\Delta P = \frac{2\gamma}{R}.$$

This is called the Young–Laplace equation. However, in this particular case, this is not the right answer. The reason is that we should actually take $L = 4\pi R$ because the surface tension is exerted at both the inside and outside surfaces of the bubble wall, and thus the answer is

$$\Delta P = \frac{4\gamma}{R}.$$

The increased pressure inside balances the surface tension, which wants to collapse the bubble.

If you're confused about why $L = 4\pi R$, you can also think about it in terms of energy. Surface tension arises from the fact that it costs energy to take soapy water and stretch it out into a surface, because this breaks some of the attractive intermolecular bonds. The Young–Laplace equation would give the correct answer for a *ball* of soapy water. But for a *bubble* of soapy water, twice as much soapy water/air surface is created. So the energy cost is double, and the force is double.

[2] **Problem 29.** One can also derive the Young–Laplace equation using the principle of virtual work. Suppose the bubble radius changes by dr. The energy of the bubble changes for two reasons: first, there is net $\Delta P\, dV$ work from the two pressure forces, and there is the $\gamma\, dA$ surface tension energy cost. By setting the net virtual work to zero, find ΔP.

Solution. The work done by the surface tension should be balanced by the work done by the pressure difference. Noting that the total surface area is $8\pi R^2$, we have

$$\Delta P\, dV = \Delta P\, d\left(\frac{4}{3}\pi R^3\right) = \Delta P(4\pi R^2)\, dR = d(8\pi R^2 \gamma) = 16\pi\gamma R\, dR$$

from which we conclude

$$\Delta P = \frac{4\gamma}{R}.$$

Of course, one can generalize this to any other kind of energy. For example, if the bubble was charged, it would grow due to electrostatic repulsion, and the new equilibrium radius could also be found using virtual work.

[2] **Problem 30** (Kalda)**.** Consider two soap bubbles which have stuck together. The part of the soap film that separates the interior of the first bubble from the outside air has radius of curvature R. The part that separates the interior of the second bubble from the outside air has radius of curvature $2R$. What is the radius of curvature of the part which separates the bubbles from each other?

Solution. The key is that the Young–Laplace equation should hold for every point on the surface since the surface tension and pressure should balance for every infinitesimal surface element. The gauge pressure (pressure minus the atmospheric pressure) inside the first bubble is $P_1 = 4\gamma/R$, and for the second $P_2 = 4\gamma/(2R)$. Thus the pressure difference between the two bubbles is $\Delta P = 2\gamma/R = P_2$, giving a radius of curvature of $2R$ for the part separating the bubbles.

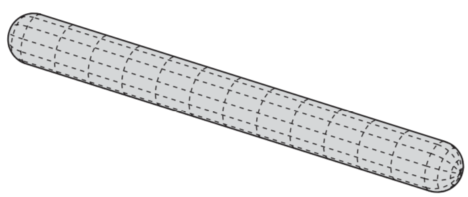
Remark

So far, we've only applied the Young–Laplace equation to spherical surfaces, which are characterized by a single radius of curvature. More generally, a surface has two principle radii of curvature R_1 and R_2 at each point. These are both equal to R for a sphere of radius R, while for a cylinder of radius R, one is equal to R and the other is infinity. For general surfaces, the Young–Laplace equation is

$$\Delta P = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

where the R_i can each be positive or negative, depending on the direction of curvature.

[3] **Problem 31** (MPPP 67)**.** When a pipe bursts under pressure, it often splits "lengthwise" instead of "across". (One familiar example is the process of cooking a long, straight sausage.) The two modes of splitting are shown as dotted lines below.



Explain this observation, assuming the thickness of the sausage skin is uniform, and hence can support a constant surface tension before breaking. (Hint: model the sausage as a cylinder of length L capped by hemispheres of radius $R \ll L$, and consider the surface tension needed to prevent the two modes of splitting mentioned, once an excess pressure P builds up inside the sausage.)

Solution. Let the pressure differential from inside the sausage to outside be P. Cutting it across so the cross section is a circle tells us that the surface tension γ_a will exert a force $F = (2\pi r)\gamma_a$ on each end since $F = \gamma \ell$. Using the trick from example 12, it must balance the force $F = \pi r^2 P$, so $\gamma_a = Pr/2$.

Lengthwise, the cross section has perimeter $2L + 2\pi r \approx 2L$. If we apply the trick to each half-cylinder, we find that the pressure force is $F = (2rL)P$, so balancing forces gives $\gamma_L = Pr$. Since this is a greater requirement on the surface tension, the sausage will break lengthwise, as we regularly observe in the kitchen.

[4] **Problem 32.** Two coaxial rings of radius R are placed a distance L apart from each other in vacuum. A soap film with surface tension γ connects the two rings.

(a) Derive a differential equation for the shape $r(z)$ of the film, and solve it.

(b) Show that for sufficiently large L, there are no solutions. If L is increased to this value, what happens to the film?

(c) Using a computer or calculator, find the largest possible value of L.

We'll consider surface tension in more detail in **T3**.

Solution. (a) Consider a segment of the bubble between z and $z + dz$. The net forces exerted by surface tension on both sides along the z-direction are $2\pi r\gamma/\sqrt{1 + r'^2}$. To balance forces in the z-direction for each segment, the quantity $r/\sqrt{1 + r'^2}$ must be independent of z, so

$$r^2 = A^2(1 + r'^2)$$

for some constant A. Separating and integrating, we have

$$\int dz = \int \frac{A\,dr}{\sqrt{r^2 - A^2}}$$

and substituting $r = A\cosh u$ and integrating yields

$$z + C = A\cosh^{-1}(r/A), \quad r = A\cosh\left(\frac{z + C}{A}\right)$$

for another constant C. Setting the rings to be at $z = \pm L/2$, we have $C = 0$. The quantity A is the minimum radius, which occurs by symmetry at $z = 0$.

You may have noticed that the answer is a catenary, which is the same as the answer to example 9. The reason is that both problems can be solved by minimizing a similar quantity. Here, we want to find the function $r(z)$ that minimizes the area,

$$A = \int 2\pi r\sqrt{1 + r'^2}\,dz$$

where the value of r at two given values of z is fixed. In that example, we wanted to find the shape $y(x)$ of the chain that minimizes the gravitational potential energy,

$$U = \lambda \int y\sqrt{1 + y'^2}\,dx.$$

This function is extremely similar in form, which explains why the form of the solution is similar. But there's an important physical difference: the length of the chain is fixed, and you need to specify it to determine the solution. (To see how this constraint can be imposed with Lagrange multipliers, see here.) By contrast, the soap bubble is more free to vary. That explains why, as we'll see below, you can sometimes have no solution for a soap bubble at all. In those cases, the middle of the film can just get thinner and thinner, always decreasing the area, until it pinches off into two separate pieces.

(b) We introduced the parameter A above, which describes the shape of the solution. It is fixed by R and L by the requirement that the bubble fit the rings,

$$R = A\cosh\frac{L}{2A}.$$

Now, we wish to find the largest L so that there exists some A so that the left-hand side can be R. This is a somewhat annoying optimization problem. It's clearer to note that by dimensional analysis, the only invariant thing is the single dimensionless ratio R/L. (A doesn't count as a dimensionful parameter, because it's fixed by R and L.) So finding the largest L for fixed R is equivalent to finding the smallest R for fixed L.

But this is now easy, because we already have R as a function of L, which is fixed, and A, which can vary. By graphing the function $R(A)$, we see it has a single minimum, so there is indeed a minimum possible R/L and hence a maximum possible L/R.

(c) Setting the derivative dR/dA to zero, the minimum occurs when

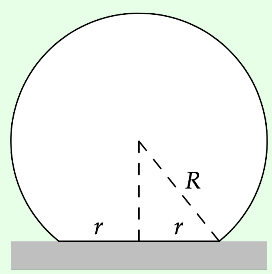
$$\frac{2A}{L} = \tanh \frac{L}{2A}.$$

This equation cannot be solved analytically. Using a calculator and the techniques of **P1**, we find the maximum possible L is about $1.33R$.

By the way, you can also solve this problem by looking at the forces on individual small elements of the bubble. Since the bubble isn't a closed surface, there's no pressure difference across it. Thus, in equilibrium, the Young–Laplace equation implies that the radii of curvature satisfy $R_1 + R_2 = 0$, i.e. the "mean curvature" is zero. This is the condition for the bubble to be a minimal surface. However, actually evaluating this condition in general is somewhat complicated; what we did above is the simplest way, which takes advantage of the axis of symmetry of the setup.

Example 14

A solid ball of radius R, density ρ, and Young's modulus Y rests on a hard table. Because of its weight, it deforms slightly, so that the area in contact with the table is a circle of radius r.



Estimate r, assuming that it is much smaller than R.

Solution

Recall from **P1** that the Young's modulus is defined by

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{\text{restoring force/cross-sectional area}}{\text{change in length/length}}$$

and has dimensions of pressure. By dimensional analysis, you can show that

$$r = R\, f(\rho g R/Y)$$

but dimensional analysis alone can't tell us anything more about f. Moreover, an exact analysis using forces would be very difficult, because different parts of the ball are compressed in different amounts, and in different directions; there's little symmetry here.

Instead, we'll roughly estimate the stress and strain near the bottom of the ball. For the part directly in contact with the table, we have

$$\text{stress} \sim F/r^2 \sim \rho g R^3/r^2$$

because the normal pressure has to balance gravity. This is the pressure exactly at the bottom of the ball; at heights much greater than r, the pressure will be smaller because it can spread out over a wider horizontal surface area. Since stress is proportional to strain, that means the part of the ball that is significantly strained has typical height r. (This is an example of Saint–Venant's principle, which states that strain is generally confined near the location that external forces are applied.) So in that region, the strain must be

$$\text{strain} \sim \delta/r \sim r/R.$$

Using the definition of the Young's modulus, we conclude

$$r \propto R\left(\frac{\rho g R}{Y}\right)^{1/3}.$$

We can also phrase this result in terms of force and displacement. The ball's total vertical deformation is $d \sim r^2/R$ and the total force that pushes it into the table is $F \sim \rho g R^3$, so

$$F \propto Y R^{1/2} d^{3/2}.$$

The restoring force is not linear in d, so it doesn't obey Hooke's law.

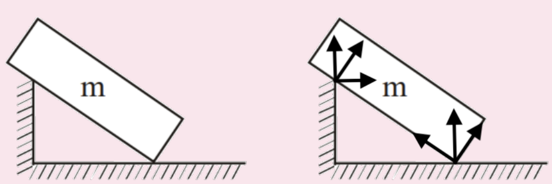
By the way, there's a whole field of study devoted to figuring out how the normal and other forces behave for realistic, deformable solids, known as contact mechanics, which is essential in engineering. This particular example is about "Hertzian contact". For much more, see *Contact Mechanics* by Johnson, and *Contact Mechanics and Friction* by Popov.

[4] **Problem 33.** EFPhO 2006, problem 5. A tough problem on a deforming object.

Solution. See the official solutions here.

Remark: Normal Forces at Corners

Unfortunately, some problem writers don't really understand contact forces, and they'll end up writing questions like the one shown at left below, which is taken from a real book.



Assuming there's no friction, which way do the normal forces on the block point?

Many students have no problem "solving" this, because they've been trained to blindly trust exam questions. They'll just choose some random directions and move on. But the smart student will get seriously confused, because the answer is clearly not well-defined. At the bottom contact point, there are three different possible directions, depending on whether you take the normal to the floor, or either of the two sides of the block. The other contact point is even more ambiguous, because of the wall magically ending. Is the normal force perpendicular to the block, perpendicular to the vertical wall, or something else? In general, there doesn't seem to be any sensible rule to decide which is the "real" normal.

The resolution, of course, is that there *is* no rule. What happens in reality depends on the exact shape of the block and wall, and how deformable it is. For example, suppose the block was perfectly rigid, but actually had slightly rounded corners (not shown in the diagram). Then there's a definite normal direction at the bottom contact point, pointing up. Similarly, we could suppose that at the other contact point, the wall actually ends in a step with a rounded corner, in which case the normal direction points directly into the block.

Alternatively, suppose the block and step weren't rounded, but could deform. Then the answer depends on the relative hardness of the materials, and how they were placed in contact. For instance, if we suppose the block is much softer, then it could squash at the bottom contact point, again leading to a common upward normal direction. But then we would expect the step to dig into the block at the other contact point, which yields two separate normal forces at that point. Or perhaps the step is made of a softer material than the floor, so that it's the step rather than the block that deforms. Or maybe both deform!

To be clear: the issue *isn't* that the problem involved unrealistic idealizations. Physics uses idealizations, like neglecting air resistance and friction, all the time, and they're reasonable within appropriate limits. The issue is that when you apply the idealizations implied by the diagram, the result is mathematically undefined. And you get completely different answers depending on which idealizations you drop, because the true answer depends on the details.

As a result, you certainly won't see this kind of thing on thoroughly vetted competitions, such as the IPhO, APhO, and EuPhO, or large national Olympiads such as those in America or China. I'll never assign such a dysfunctional problem, but they're depressingly common in homework assignments and less carefully written exams, such as the JEE. If you personally encounter such a problem, your best bet is to attempt to read the question writer's mind; that is, simply start guessing and go with whatever gives you tractable results. If you encounter this sort of thing often, in a book or competition, then it's not worth your time. We're in it to learn physics, not to please examiners.

Mechanics III: Dynamics

Chapters 3 and 5 of Morin cover dynamics, energy, and momentum. Alternatively, see chapters 2 and 3 of Kleppner and Kolenkow, or chapters 4 and 6 of Wang and Ricardo, volume 1. For fun, see chapters I-9 through I-14 of the Feynman lectures. There is a total of **82** points.

1 Blocks, Pulleys, and Ramps

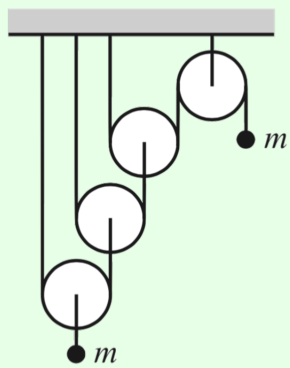
Idea 1

To solve dynamics problems with constraints, it's easiest to first write the constraint in terms of coordinates (e.g. "conservation of string" for pulleys, or stationarity of the CM for an isolated system), then differentiate to get constraints on the velocity and acceleration.

Questions of this type are generally straightforward, as long as you write down the correct equations. The trickiest part is often solving the equations, which can get messy.

Example 1: Morin 3.30

Find the acceleration of the masses in the Atwood's machine shown below.



Neglect friction, and treat all pulleys are massless.

Solution

Let x and x' be the amounts by which the left and right mass have moved down, and number the pulleys 1 through 4 from left to right, and the strings 1 through 3 from left to right. Pulley 4 is stationary, so conservation of string 3 means that pulley 3 moves up by $x'/2$. Next, conservation of string 2 means that pulley 2 moves up by $x'/4$. Finally, conservation of string 1 implies that pulley 1 moves up by $x'/8$, so our final conservation of string constraint is

$$x = -\frac{x'}{8}$$

which upon applying the derivative twice gives

$$a = -\frac{a'}{8}.$$

Now consider the tensions T_i in the strings. We know that

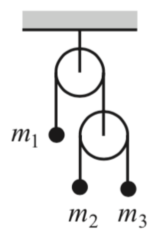
$$a = g - \frac{2T_1}{m}, \quad a' = g - \frac{2T_3}{m}.$$

Since pulley 3 is massless, the forces on it must balance, so $T_2 = 2T_3$. Similarly $T_1 = 2T_2$, so $T_1 = 4T_3$. We hence have a system of three equations in three unknowns (T_1, a, and a'), which can be solved straightforwardly to give

$$a' = \frac{56}{65}\,g, \quad a = -\frac{7}{65}\,g.$$

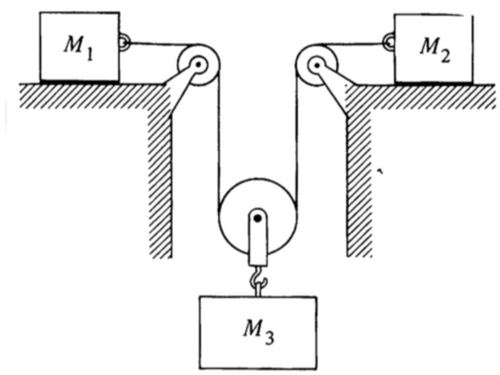
By the way, this arrangement of pulleys is called a Spanish burton. If there are n pulleys chained at the left ($n = 3$ in the above diagram), the mechanical advantage is 2^n, the highest of any possible pulley system. However, in practice such a huge mechanical advantage is rarely useful, since friction would be substantial and the range of motion is small. Instead, people who use pulleys in real life, like sailors, climbers, or auto mechanics, tend to use simpler setups like the block and tackle or chain hoist.

[2] **Problem 1** (Morin 3.2). Consider the double Atwood's machine shown below.



Assuming all pulleys are massless, and neglecting friction, find the acceleration of the mass m_1.

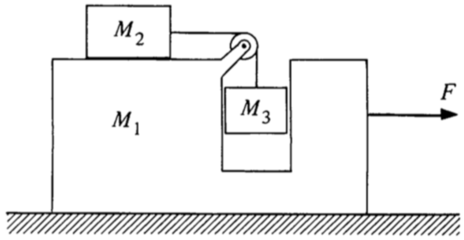
[2] **Problem 2** (KK 2.15). Consider the system of massless pulleys shown below.



The coefficient of friction between the masses and the horizontal surfaces is μ. Show that the tension in the rope is

$$T = \frac{(\mu + 1)g}{2/M_3 + 1/2M_1 + 1/2M_2}.$$

[2] **Problem 3** (KK 2.20). Consider the machine shown below, which we encountered in **M2**.



Show that the acceleration of M_1 when the external force F is zero is

$$a = -\frac{M_2 M_3 g}{M_1 M_2 + M_1 M_3 + 2M_2 M_3 + M_3^2}.$$

[3] **Problem 4.** A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ. The plane rests on a frictionless horizontal table.

(a) When the block is released, what is the horizontal acceleration of the plane?

(b) Assume the block starts a distance d above the table. Using results from part (a), what is the horizontal velocity of the block just before it reaches the floor?

(c) Find the speed of the block after it reaches the floor by applying energy and momentum conservation to the entire process.

(d) Your results for parts (b) and (c) should not match. What's going on?

2 Momentum

Idea 2

The momentum of a system is

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i = M\mathbf{v}_{\text{CM}}.$$

In particular, the total external force on the system is $M\mathbf{a}_{\text{CM}}$, and if there are no external forces, the center of mass moves at constant velocity.

Example 2

A massless rope passes over a frictionless pulley. A monkey hangs on one side, while a bunch of bananas with exactly the same weight hangs from the other side. When the monkey tries to climb up the rope, what happens?

Solution

Remarkably, the answer doesn't depend on how the monkey climbs, whether slowly or quickly, or symmetrically or not! The total vertical force on the monkey is $T - mg$, so the acceleration of the center of mass of the monkey is $T/m - g$. But since the tension is uniform through a massless rope, the acceleration of the bananas is also $T/m - g$. Therefore, the monkey and bananas rise at the same rate, and meet each other at the pulley.

Now here's a question for you: compared to climbing up a rope fixed to the ceiling, climbing up to the pulley takes twice as much work, because the bananas are raised too. But in both cases, isn't the monkey applying the same force through the same distance? Where does the extra work come from? (The answer involves the ideas at the end of this problem set.)

Example 3: KK 3.14 / INPhO 2014.5

Two men, each with mass m, stand on a railway flatcar of mass M initially at rest. They jump off one end of the flatcar with velocity u relative to the car. The car rolls in the opposite direction without friction. Find the final velocities of the flatcar if they jump off at the same time, and if they jump off one at a time. Generalize to the case of $N \gg 1$ men, with a total mass of m_{tot}.

Solution

In the first case, by conservation of momentum, we have

$$Mv + 2m(v - u) = 0$$

where v is the final velocity of the flatcar, so

$$v = \frac{2mu}{M + 2m}.$$

In the second case, by a similar argument, we find that after the first man jumps,

$$v_1 = \frac{mu}{M + 2m}.$$

Now transform to the frame moving with the flatcar. When the second man jumps, he imparts a further velocity $v_2 = mu/(M + m)$ to the flatcar by another similar argument. The final velocity of the flatcar relative to the ground is then

$$v = v_1 + v_2 = mu \left(\frac{1}{M + 2m} + \frac{1}{M + m} \right).$$

It might be a bit disturbing that the final speeds and hence energies of the flatcar are different, even though the men are doing the same thing (i.e. expending the same amount of energy in their legs to jump) in both cases.

The reason for the difference is that in the second case, the second man to jump ends up with less energy, since the velocity he gets from jumping is partially cancelled by the existing velocity v_1. So the extra energy that goes into the flatcar corresponds to less kinetic energy in the men after jumping, which would ultimately have ended up as heat after they slid to a stop. Accounting properly for the kinetic energy of everything in the system solves a lot of paradoxes involving energy, as we'll see below.

In the case of many men, by similar reasoning we have

$$v = \frac{m_{\text{tot}}}{M + m_{\text{tot}}} u$$

in the first case, while in the second case the answer is the sum

$$v = \sum_{i=1}^{N} \frac{m_{\text{tot}} u}{N} \frac{1}{M + (i/N) m_{\text{tot}}}.$$

This can be converted into an integral, by letting $x = i/N$, in which case $\Delta x = 1/N$ and

$$v = \sum_i \Delta x \, \frac{m_{\text{tot}} u}{M + x m_{\text{tot}}} \approx \int_0^1 dx \, \frac{m_{\text{tot}} u}{M + x m_{\text{tot}}} = \log\left(\frac{M + m_{\text{tot}}}{M}\right) u.$$

Note that this is essentially the rocket equation, which we'll derive in a different way in **M6**.

[2] **Problem 5** (KK 4.11)**.** A flexible chain of mass M and length ℓ is suspended vertically with its lowest end touching a scale. The chain is released and falls onto the scale. Find the reading on the scale when a length of chain x has fallen.

[3] **Problem 6.** Some qualitative questions about momentum.

(a) A box containing a vacuum is placed on a frictionless surface. The box is punctured on its right side. How does it move immediately afterward?

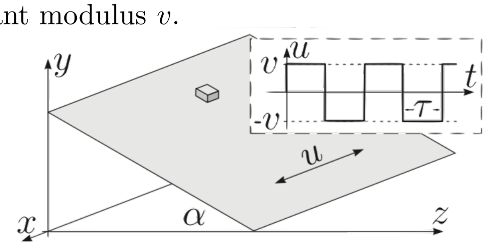
(b) You are riding forward on a sled across frictionless ice. Snow falls vertically (in the frame of the ice) on the sled. Which of the following makes the sled go the fastest or the slowest?

1. You sweep the snow off the sled, directly to the left and right in your frame.
2. You sweep the snow off the sled, directly to the left and right in the ice frame.
3. You do nothing.

(c) An hourglass is made by dividing a cylinder into two identical halves, separated by a small orifice. Initially, the top half is full of sand and the bottom half is empty. The hourglass is placed on a scale, and then the orifice is opened. The total weight of the hourglass and sand is W. How does the scale reading compare to W shortly after the sand starts falling, shortly after it finishing falling, and in between? (For concreteness, assume the surfaces of the sand in the top and bottom halves are always horizontal, and that the sand passes through the orifice at a constant rate.)

[3] **Problem 7.** USAPhO 2018, problem A1.

[3] **Problem 8** (Kalda)**.** A block is on a ramp with angle α and coefficient of friction $\mu > \tan\alpha$. The ramp is rapidly driven back and forth so that its velocity vector $\mathbf{u}$ is parallel to both the slope and the horizontal and has constant modulus v.



The direction of $\mathbf{u}$ reverses abruptly after each time interval τ, where $g\tau \ll v$. Find the average velocity $\mathbf{w}$ of the block. (Hint: as mentioned in **M1**, it's best to work in the frame of the ramp, because it causes the friction, even though this introduces fictitious forces.)

[4] **Problem 9** (Morin 5.21)**.** A sheet of mass M moves with speed V through a region of space that contains particles of mass m and speed v. There are n of these particles per unit volume. The sheet moves in the direction of its normal. Assume $m \ll M$, and assume that the particles do not interact with each other.

(a) If $v \ll V$, what is the drag force per unit area on the sheet?

(b) If $v \gg V$, what is the drag force per unit area on the sheet? Assume for simplicity that the component of every particle's velocity in the direction of the sheet's motion is exactly $\pm v/2$.

(c) Now suppose a cylinder of mass M, radius R, and length L moves through the same region of space with speed V, and assume $v = 0$ and $m \ll M$. The cylinder moves in a direction perpendicular to its axis. What is the drag force on the cylinder?

Parts (a) and (b) are a toy model for the two regimes of drag, mentioned in **M1**. However, it shouldn't be taken too seriously, because as we'll see in **M7**, the typical velocity that separates the two types of behavior doesn't have to be of order v. Instead, it depends on how strongly the particles interact with each other.

3 Energy

Idea 3

The work done on a point particle is

$$W = \int \mathbf{F} \cdot d\mathbf{x}$$

and is equal to the change in kinetic energy, as you showed in **P1**.

Remark: Dot Products

The dot product of two vectors is defined in components as

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z$$

and is equal to $|\mathbf{v}|\,|\mathbf{w}|\cos\theta$ where θ is the angle between them. For example, if $\mathbf{A}$ and $\mathbf{B}$ are the sides of a triangle, the other side is $\mathbf{C} = \mathbf{A} - \mathbf{B}$, and

$$C^2 = |\mathbf{A} - \mathbf{B}|^2 = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = A^2 + B^2 - 2AB\cos\theta$$

which proves the law of cosines. (Or, if you accept the law of cosines, you could regard this as a proof that the dot product depends on $\cos\theta$ as claimed.)

Like the ordinary product, the dot product obeys the product rule. For example,

$$\frac{d}{dt}(\mathbf{v} \cdot \mathbf{w}) = \dot{\mathbf{v}} \cdot \mathbf{w} + \mathbf{v} \cdot \dot{\mathbf{w}}.$$

Using this, it's easy to generalize the derivation of the work-kinetic energy theorem in **P1** to three dimensions; we have

$$\frac{1}{2} d(v^2) = \frac{1}{2} d(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \cdot d\mathbf{v} = \frac{d\mathbf{x}}{dt} \cdot d\mathbf{v} = \frac{d\mathbf{v}}{dt} \cdot d\mathbf{x} = \mathbf{a} \cdot d\mathbf{x}$$

and this is equivalent to the desired theorem. As you can see, it's all basically the same, since the product and chain rule manipulations work the same way for vectors and scalars.

Example 4: IPhO 1996 1(b)

A skier starts from rest at point A and slowly slides down a hill with coefficient of friction μ, without turning or braking, and stops at point B. At this point, his horizontal displacement is s. What is the height difference h between points A and B?

Solution

Since the skier begins and ends at rest, the change in height is the total energy lost to friction,

$$mgh = \int f_{\text{fric}}\, ds$$

where the integral over ds goes over the skier's path. Since the skier is always moving slowly, the normal force is approximately $mg\cos\theta$. (More generally, there would be another contribution to provide the centripetal acceleration.) But then

$$\int f_{\text{fric}}\, ds = \int \mu mg \cos\theta\, ds = \int \mu mg\, dx = \mu mgs$$

which gives an answer of $h = \mu s$. (If the skier's path turned around, then this would still hold as long as s denotes the total horizontal distance traveled.)

[3] **Problem 10** (MPPP 16)**.** On a windless day, a cyclist going "flat out" can ride uphill at a speed of $v_1 = 12\,\text{km/h}$ and downhill at $v_2 = 36\,\text{km/h}$ on the same inclined road. We wish to find the cyclist's top speed on a flat road if their maximal effort is independent of the speed at which the bike is traveling. Note that in this regime, the air drag force is quadratic in the speed.

(a) Solve the problem assuming that "maximal effort" refers to the force exerted on the pedals by the rider, and that the rider never changes gears.

(b) Solve the problem assuming that "maximal effort" refers to the rider's power.

[3] **Problem 11.** USAPhO 2016, problem B1.

[2] **Problem 12.** Alice steps on the gas pedal on her car. Bob, who is standing on the sidewalk, sees Alice's car accelerate from rest to 10 mph. Charlie, who is passing by in another car, sees Alice's car accelerate from 10 mph to 20 mph. Hence Charlie sees the kinetic energy of Alice's car increase by three times as much. How is this compatible with energy conservation, given that the same amount of gas was burned in both frames?

[3] **Problem 13** (KK 4.8)**.** A block of mass M is attached to a spring of spring constant k. It is pulled a distance L from its equilibrium position and released from rest. The block has a small coefficient of friction μ with the ground. Find the number of cycles the mass oscillates before coming to rest.

[3] **Problem 14** (Morin 5.4)**.** A massless string of length 2ℓ connects two hockey pucks that lie on frictionless ice. A constant horizontal force F is applied to the midpoint of the string, perpendicular

to it. The pucks eventually collide and stick together. How much kinetic energy is lost in the collision?

Idea 4

If a problem can be solved using either momentum conservation or energy conservation alone, it usually means one of the two isn't actually conserved. In particular, many processes are inherently inelastic and inevitably dissipate energy. For more about inherently inelastic processes, see section 5.8 of Morin.

[2] **Problem 15** (KK 4.20)**.** Sand falls slowly at a constant rate dm/dt onto a horizontal belt driven at constant speed v.

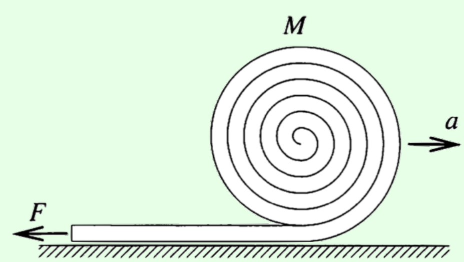
(a) Find the power P needed to drive the belt.

(b) Show that the rate of increase of the kinetic energy of the sand is only $P/2$.

(c) We can explain this discrepancy exactly. Argue that in the reference frame of the belt, the rate of heat dissipation is $P/2$. Since temperature is the same in all frames, the rate of heat dissipation is $P/2$ in the original frame as well, accounting for the missing energy.

Example 5: PPP 108

A fire hose of mass M and length L is coiled into a roll of radius R. The hose is sent rolling along level ground, with its center of mass given initial speed $v_0 \gg \sqrt{gR}$. The free end of the hose is held fixed.



The hose unrolls and becomes straight. How long does this process take to complete?

Solution

First, we need to find what is conserved. The horizontal momentum is not conserved, because there is an external horizontal force needed to keep the end of the hose in place. On the other hand, the energy *is* conserved, even though this process looks inelastic. The hose "sticks" to the floor as it unrolls, but this process dissipates no energy because the circular part of the hose rolls without slipping, so the bottom of this part always has zero velocity.

Once we figure out energy is conserved, the problem is straightforward. The assumption $v_0 \gg \sqrt{gR}$ means we can neglect the change in gravitational potential energy as the hose unrolls. After the hose travels a distance x,

$$\frac{1}{2}\left(1+\frac{1}{2}\right)Mv_0^2 = \frac{1}{2}\left(1+\frac{1}{2}\right)mv^2$$

where the 1/2 terms are from rotational kinetic energy. Since $m(x) = M(1 - x/L)$, we have

$$v(x) = \frac{v_0}{\sqrt{1 - x/L}}$$

which gives a total time

$$T = \int_0^L \frac{dx}{v(x)} = \frac{L}{v_0}\int_0^1 \sqrt{1-u}\,du = \frac{2L}{3v_0}.$$

Evidently, the hose accelerates as it unrolls.

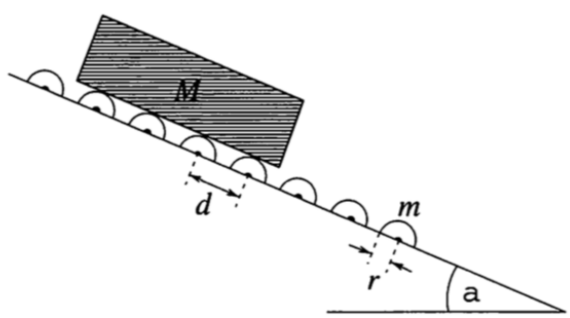
[4] **Problem 16.** Consider the following related problems; in all parts, neglect friction.

(a) A uniform rope of length ℓ lies stretched out flat on a table, with a tiny portion $\ell_0 \ll \ell$ hanging through a hole. The rope is released from rest, and all points on the rope begin to move with the same speed. Since this motion is smooth, energy is conserved. Find the speed of the rope when the end goes through the hole.

(b) $\star$ For practice, repeat part (a) by solving for $x(t)$ explicitly. (Hint: this is best done using the generalized coordinate techniques of **M4**.)

(c) Now suppose a flexible uniform chain of length ℓ is placed loosely coiled close to the hole. Again, a tiny portion $\ell_0 \ll \ell$ hangs through the hole, and the chain is released from rest. In this case, the unraveling of the chain is an inherently inelastic process, because each link of the chain sits still until it is suddenly jerked into motion. Find the speed of the chain when the last link goes through the hole. (Hint: you should get a nonlinear differential equation, which can be solved by guessing $x(t) = At^n$.)

[3] **Problem 17** (PPP 95). A long slipway, inclined at an angle α to the horizontal, is fitted with many identical rollers, consecutive ones being a distance d apart. The rollers have horizontal axles and consist of rubber-covered solid steel cylinders each of mass m and radius r. A plank of mass M, and length much greater than d, is released at the top of the slipway.



Find the terminal speed $v_{\max}$ of the plank. Ignore air drag and friction at the pivots of the rollers.

4 Elastic Collisions

Idea 5

Any temporary interaction between two objects that conserves energy and momentum is a perfectly elastic collision. In one dimension, such collisions are "trivial": their outcome is fully determined by energy and momentum conservation, because there are two final velocities and two conservation laws. In two dimensions, there are four final velocity components and three conservation laws (energy and 2D momentum), so we need one more number to describe what happens, such as the angle of deflection. In a two-dimensional collision, the outcome depends on the details, such as how the objects approach each other, and the force between them. The same holds in three dimensions.

Example 6

Two masses are constrained to a line. The mass m_1 moves with velocity v_1, and the mass m_2 moves with velocity v_2. The masses collide perfectly elastically. Find their speeds afterward.

Solution

The usual method is to directly invoke conservation of energy and momentum, which leads to a quadratic equation. A slicker method is to work in the center of mass frame instead. (This is useful for collision problems in general, and it'll become even more useful for the relativistic collisions covered in **R2**.)

The center of mass of the system has speed

$$v_{\mathrm{CM}} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}.$$

Moreover, by momentum conservation, the center of mass never accelerates. Now we boost into the frame moving with the center of mass. Since the total momentum is by definition zero in the center of mass frame, the momenta of the particles cancel out. The only way for this to remain true after the collision is if we multiply their velocities by the same number. Energy is only conserved if this number is ± 1, with the latter representing no collision at all.

Therefore, during an elastic collision, the velocities in the center of mass frame simply reverse. The initial velocities in that frame are

$$v_{1,\mathrm{CM}} = v_1 - v_{\mathrm{CM}}, \quad v_{2,\mathrm{CM}} = v_2 - v_{\mathrm{CM}}.$$

The final velocities in that frame are

$$v'_{1,\mathrm{CM}} = -v_1 + v_{\mathrm{CM}}, \quad v'_{2,\mathrm{CM}} = -v_2 + v_{\mathrm{CM}}.$$

Finally, going back to the original frame gives the final velocities

$$v'_1 = -v_1 + 2v_{\mathrm{CM}}, \quad v'_2 = -v_2 + 2v_{\mathrm{CM}}.$$

There are many special cases we can check. For example, if $m_1 = m_2$, then the two masses simply swap their velocities, as if they just passed through each other. As another check, consider the case where the second mass is initially at rest, $v_2 = 0$. Then

$$v_1' = v_1 \frac{m_1 - m_2}{m_1 + m_2}, \quad v_2' = v_1 \frac{2m_1}{m_1 + m_2}.$$

When $m_1 = m_2$, the first mass gives all its velocity to the second. When m_2 is large, the first mass just rebounds off with velocity $-v_1$. When m_1 is large, the first mass keeps on going and the second mass picks up velocity $2v_1$. Finally, when $m_1 = m_2/3$, then the final speeds are $v_1' = -v_1/2$ and $v_2' = v_1/2$, a nice result which is worth committing to memory.

Idea 6

The kinetic energy of a set of masses m_i with total mass M can be decomposed as

$$\sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} M v_{\text{CM}}^2 + \sum_i \frac{1}{2} m_i (v_i - v_{\text{CM}})^2$$

where the first term is the "center of mass" contribution, and the second term is the amount of kinetic energy in the center of mass frame. (This statement also holds true for multidimensional collisions, if the squares are replaced with vector magnitudes.) The first term can be rewritten as $P^2/2M$ where P is the total momentum of the masses. Therefore, the kinetic energy of a system of masses with fixed total momentum is minimized when the second term is zero, i.e. when all the masses are traveling with the same velocity. This implies, for instance, that a totally inelastic collision dissipates the highest possible amount of kinetic energy.

Example 7

Three balls of mass M are initially at rest. Then an explosion occurs, giving the system a fixed total kinetic energy. What is the maximum possible fraction of this energy that one ball can carry by itself?

Solution

Suppose we want to maximize the energy of the first ball, and let p_0 be the magnitude of its final momentum. Since the total momentum is zero, the other two balls also have a total momentum of magnitude p_0. As shown in the above idea, the energy of those two balls is minimized if they travel at the same speed. Therefore, the optimal scenario is to have the first ball come out with speed v_0 and have both of the other two come out the other direction with equal speed $v_0/2$. Then the first ball has $2/3$ of the total energy.

This is the simplest possible "optimal collision" problem; we'll see more in **R2**. Many can be solved with the basic idea that some of the outgoing masses should have the same velocity.

[2] **Problem 18** (Morin 5.23)**.** A tennis ball with mass m_2 sits on top of a basketball with a mass $m_1 \gg m_2$. The bottom of the basketball is a height h above the ground. When the balls are dropped, how high does the tennis ball bounce?

[3] **Problem 19** (PPP 46)**.** A Newton's cradle consists of three suspended steel balls of masses m_1, m_2, and m_3 arranged in that order with their centers in a horizontal line. The ball of mass m_1 is drawn aside in their common plane until its center has been raised by h and is then released. If all collisions are elastic, how much m_2 be chosen so that the ball of mass m_3 rises to the greatest possible height, and what is this height? (Neglect all but the first two collisions.)

[3] **Problem 20.** Here's a variety problem involving some "clean" mathematical results. All three parts can be solved without lengthy calculation.

(a) Consider n identical balls confined to a line. Assuming all collisions are perfectly elastic, what is the maximum number of collisions that could happen? Assume no triple collisions happen.

(b) A billiard ball hits an identical billiard ball initially at rest in a perfectly elastic collision. Show that the balls exit at a right angle to each other.

(c) A mass M collides elastically with a stationary mass m. If $M > m$, show that the maximum possible angle of deflection of M is $\sin^{-1}(m/M)$.

[3] **Problem 21** (PPP 72)**.** Beads of equal mass m are strung at equal distances d along a long, horizontal, infinite wire. The beads are initially at rest but can move without friction. The first bead is continuously accelerated towards the right by a constant force F. After some time, a "shock wave" of moving beads will propagate towards the right.

(a) Find the speed of the shock wave, assuming all collisions are completely inelastic.

(b) Do the same, assuming all collisions are completely elastic. What is the average speed of the accelerated bead in this case?

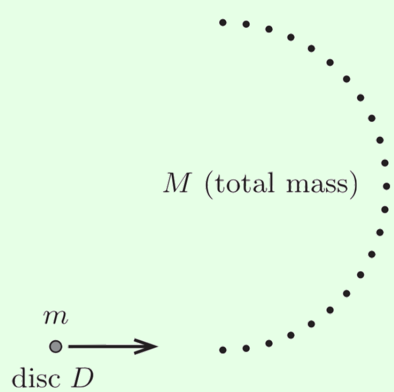
If you're having trouble visualizing this, try plotting all the masses' positions $x(t)$ over time.

[3] **Problem 22.** USAPhO 2019, problem A1.

[3] **Problem 23.** USAPhO 2009, problem B1.

Example 8: MPPP 42

There are N identical tiny discs lying on a table, equally spaced along a semicircle, with total mass M. Another disc D of mass m is very precisely aimed to bounce off all of the discs in turn, then exit opposite the direction it came.



In the limit $N \to \infty$, what is the minimal value of M/m for this to be possible? Given this value, what is the ratio of the final and initial speeds of the disc?

Solution

The reason there is a lower bound on M is that, by problem 20(c), there is a maximal angle that each tiny disc can deflect the disc D. For large N, the deflection is π/N for each disc, so

$$\frac{\pi}{N} = \sin^{-1}\frac{M/N}{m} \approx \frac{M}{Nm}$$

which implies that $M/m \geq \pi$.

To see how much energy is lost in each collision, work in the center of mass frame and consider the first collision. In this frame, the disc D is initially approximately still, and the tiny disc comes in horizontally with speed v. To maximize the deflection angle in the table's frame, the tiny disc should rebound vertically, as this provides the maximal vertical impulse to the disc D.

Thus, going back to the table's frame, where the disc D has speed v, the tiny disc scatters with speed $\sqrt{v^2+v^2} = \sqrt{2}v$. By conservation of energy,

$$\Delta\left(\frac{1}{2}mv^2\right) = -\frac{1}{2}\frac{M}{N}(\sqrt{2}v)^2.$$

This simplifies to

$$\frac{\Delta v}{v} = -\frac{\pi}{N}$$

which means that after N collisions, we have the cute result

$$\frac{v_f}{v_i} = \left(1 - \frac{\pi}{N}\right)^N \approx e^{-\pi}$$

where in the last step we used a result from **P1**.

Example 9: EFPhO 2003.1

A spherical volleyball of radius r and mass m is inflated with excess pressure ΔP. If it is dropped from the ceiling and hits the ground, estimate how long the subsequent elastic collision takes.

Solution

Answering this question requires making a simplified physical model of how the collision occurs. Let's say that when the volleyball hits the ground, it will keep going straight down, deforming the part that touches the ground into a flat circular face. Specifically, when the ball has moved a distance y into the ground, the flat face has area

$$A = \pi\left(\sqrt{r^2-(r-y)^2}\right)^2 = \pi y(2r-y) \approx 2\pi r y$$

where we assumed that $y \ll r$ at all times, which is reasonable as long as the ball's initial speed is not enormous. As a result, the pressure of the volleyball exerts a force

$$F = 2\pi r \, \Delta P \, y$$

on the ground. This assumes the pressure inside the volleyball remains uniform, and that the rest of the volleyball stays approximately spherical, which is again reasonable as long as the initial speed is not huge.

Assuming the initial velocity is not too small, gravity is negligible during the collision, so during the collision the force on the volleyball is effectively that of an ideal spring. The collision lasts for half a period, giving

$$\tau = \pi \sqrt{\frac{m}{k_{\text{eff}}}} = \sqrt{\frac{\pi m}{2r \, \Delta P}}.$$

If we plug in realistic numbers, the result is of order 10 ms, which is plausible.

5 Continuous Systems

Example 10

As shown in **M2**, a hanging chain takes the form of a catenary. Suppose you pull the chain down in the middle. How does the center of mass of the chain move? Does the answer depend on how hard you pull?

Solution

No matter how hard you pull, or in what direction, the height of the center of mass always goes up! This is because this quantity measures the total gravitational potential energy of the chain. If you pull a chain in equilibrium, in any direction whatsoever, you will do work on it. So this raises its potential energy, and hence the center of mass.

Another way of saying this is that the equilibrium position, without the extra pull you supply, is already in the lowest energy state, and hence already has the lowest possible center of mass. Changing this shape in any way raises the center of mass.

[2] **Problem 24.** A uniform half-disc of radius R is nailed to a wall at the center of its circle and allowed to come to equilibrium. The half-disc is then rotated by an angle $d\theta$. By calculating the energy needed to do this in two different ways, find the distance from the pivot point to the center of mass.

[4] **Problem 25** (Morin 5.31)**.** Assume that a cloud consists of tiny water droplets suspended (uniformly distributed, and at rest) in air, and consider a raindrop falling through them. Assume the raindrop is initially of negligible size, remains spherical at all times, and collides perfectly inelastically with the droplets. It turns out that the raindrop accelerates uniformly; assuming this, find the acceleration.

[3] **Problem 26** (Kvant)**.** Half of a flexible pearl necklace lies on a horizontal frictionless table, while the other half hangs down vertically at the edge. If the necklace is released from rest, it will slide off the table. At some point, the hanging part of the necklace will begin to whip back and forth. What fraction of the necklace is on the table when this begins? (Hint: we are considering a pearl necklace with no empty string between adjacent pearls; as a result, all the pearls accelerate smoothly. To solve the problem, think about the vertical forces. There is an important related problem in **M2**.)

[4] **Problem 27** (BAUPC 2002)**.** A small ball is attached to a massless string of length L, the other end of which is attached to a very thin pole. The ball is thrown so that it initially travels in a horizontal circle, with the string making an angle θ_0 with the vertical. As time goes on, the string wraps itself around the pole. Assume that (1) the pole is thin enough so that the length of string in the air decreases very slowly, and (2) the pole has enough friction so that the string does not slide on the pole, once it touches it. Show that the ratio of the ball's final speed (right before it hits the pole) to initial speed is $\sin\theta_0$.

When dealing with an extended system whose parts all move in different ways, conservation of energy is occasionally useless. However, the somewhat obscure idea of "center of mass energy" may become useful instead. For more about this concept, see section 13.5 of Halliday and Resnick.

Idea 7: Center of Mass Energy

The work done on a part of a system is

$$dW = F\,dx$$

where F is the force on that specific part of the system, and dx is its displacement. Then $dW = dE$ where E is the total energy of the system.

Similarly, the "center of mass work" done on a system is

$$dW_{\text{cm}} = F\,dx_{\text{cm}}$$

where F is the total force on the system and dx_{cm} is the displacement of the center of mass. Then $dW_{\text{cm}} = dE_{\text{cm}}$ where the "center of mass energy" is defined as $E_{\text{cm}} = Mv_{\text{cm}}^2/2$.

It should be noted that, like regular energy and work, center of mass energy and work depend on the reference frame you're using.

Example 11

Consider a cyclist who pedals their bike to accelerate. The wheels roll without slipping on the ground. The cyclist moves a distance d, with the bike experiencing a constant friction force f from the ground. Analyze the situation using both energy and center of mass energy.

Solution

Since the wheels roll without slipping, their contact point with the ground is always zero, so the friction force does exactly zero work. Thus the net energy of the cyclist/bike system is conserved. The additional kinetic energy of the cyclist/bike comes from the chemical energy of the cyclist, which ultimately came from what they ate. So conservation of energy is correct, but it doesn't tell us anything useful at all.

Now consider center of mass energy. Considering the cyclist/bike system, the center of mass work is fd, which is the change in $Mv_{\text{cm}}^2/2$. This allows us to compute the change in velocity of the cyclist/bike.

Example 12

Consider the same setup as in the previous example, but now the cyclist brakes hard. The wheels slip on the ground, and experience a friction force $-f$ while the cyclist moves a distance d. Analyze the situation using both energy and center of mass energy.

Solution

The center of mass work equation tells us about the overall deceleration of the cyclist/bike, just as in the previous example.

On the other hand, the work done by the friction force is indeterminate! It can be any quantity between zero and $-fd$. When it is 0, the total energy of the cyclist/bike system is again conserved, which means all the kinetic energy lost is dissipated as heat inside the bike itself. When it is $-fd$, all the kinetic energy lost is dissipated as heat in the *ground*, and hence energy is removed from the cyclist/bike system. In general, the work will be an intermediate value, meaning that both the ground and the bike heat up, but we can't calculate what it is without a microscopic model of how the friction works. It depends on, e.g. how easily the ground and bike tire surface deform.

[1] **Problem 28.** Alice and Bob stand facing each other with their arms bent and hands touching on an ice skating rink. Bob has his back against a wall.

(a) Suppose Bob extends his arms, pushing Alice through a distance d with a force F. Analyze what happens to Alice in terms of both work and center of mass work.

(b) Suppose Alice extends her arms, pushing herself through a distance d with a force F. Repeat the analysis; what is different and what is the same?

(c) Suppose a spherical balloon is compressed uniformly from all sides. Is there work done on the balloon? How about center of mass work?

[4] **Problem 29.** USAPhO 2013, problem B1. This problem is quite tricky! Once you're done, carefully read the official solution, which describes how center of mass work is applied.

Mechanics III: Dynamics

Chapters 3 and 5 of Morin cover dynamics, energy, and momentum. Alternatively, see chapters 2 and 3 of Kleppner and Kolenkow, or chapters 4 and 6 of Wang and Ricardo, volume 1. For fun, see chapters I-9 through I-14 of the Feynman lectures. There is a total of **82** points.

1 Blocks, Pulleys, and Ramps

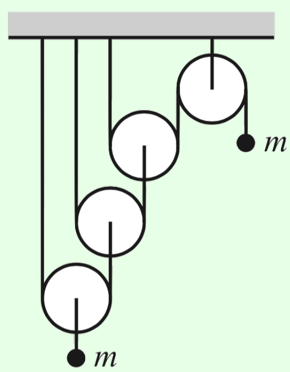
Idea 1

To solve dynamics problems with constraints, it's easiest to first write the constraint in terms of coordinates (e.g. "conservation of string" for pulleys, or stationarity of the CM for an isolated system), then differentiate to get constraints on the velocity and acceleration.

Questions of this type are generally straightforward, as long as you write down the correct equations. The trickiest part is often solving the equations, which can get messy.

Example 1: Morin 3.30

Find the acceleration of the masses in the Atwood's machine shown below.



Neglect friction, and treat all pulleys are massless.

Solution

Let x and x' be the amounts by which the left and right mass have moved down, and number the pulleys 1 through 4 from left to right, and the strings 1 through 3 from left to right. Pulley 4 is stationary, so conservation of string 3 means that pulley 3 moves up by $x'/2$. Next, conservation of string 2 means that pulley 2 moves up by $x'/4$. Finally, conservation of string 1 implies that pulley 1 moves up by $x'/8$, so our final conservation of string constraint is

$$x = -\frac{x'}{8}$$

which upon applying the derivative twice gives

$$a = -\frac{a'}{8}.$$

Now consider the tensions T_i in the strings. We know that

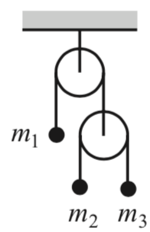
$$a = g - \frac{2T_1}{m}, \quad a' = g - \frac{2T_3}{m}.$$

Since pulley 3 is massless, the forces on it must balance, so $T_2 = 2T_3$. Similarly $T_1 = 2T_2$, so $T_1 = 4T_3$. We hence have a system of three equations in three unknowns (T_1, a, and a'), which can be solved straightforwardly to give

$$a' = \frac{56}{65}\,g, \quad a = -\frac{7}{65}\,g.$$

By the way, this arrangement of pulleys is called a Spanish burton. If there are n pulleys chained at the left ($n = 3$ in the above diagram), the mechanical advantage is 2^n, the highest of any possible pulley system. However, in practice such a huge mechanical advantage is rarely useful, since friction would be substantial and the range of motion is small. Instead, people who use pulleys in real life, like sailors, climbers, or auto mechanics, tend to use simpler setups like the block and tackle or chain hoist.

[2] **Problem 1** (Morin 3.2)**.** Consider the double Atwood's machine shown below.



Assuming all pulleys are massless, and neglecting friction, find the acceleration of the mass m_1.

Solution. Let the downward accelerations be a_1, a_2, and a_3. By conservation of string,

$$2a_1 + a_2 + a_3 = 0.$$

Let T be the tension in the lower pulley. Then we have

$$m_1 g - 2T = m_1 a_1, \quad m_2 g - T = m_2 a_2, \quad m_3 g - T = m_3 a_3.$$

Therefore,

$$0 = 2a_1 + a_2 + a_3 = 2(g - 2T/m_1) + g - T/m_2 + g - T/m_3 = 4g - T(4/m_1 + 1/m_2 + 1/m_3),$$

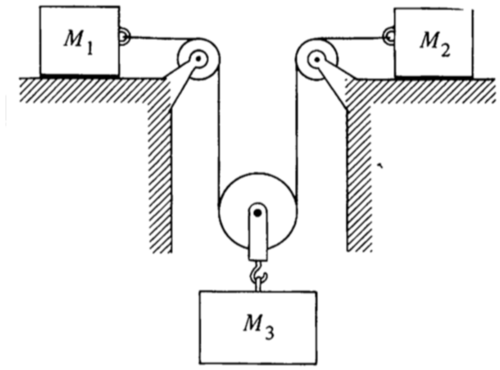
so that

$$T = \frac{g}{1/m_1 + 1/4m_2 + 1/4m_3}.$$

Thus, we conclude

$$a_1 = g - 2T/m_1 = g - \frac{2g}{1 + m_1/4m_2 + m_1/4m_3} = \boxed{g\frac{m_1(m_2 + m_3) - 4m_2m_3}{4m_2m_3 + m_1(m_2 + m_3)}}.$$

[2] **Problem 2** (KK 2.15)**.** Consider the system of massless pulleys shown below.



The coefficient of friction between the masses and the horizontal surfaces is μ. Show that the tension in the rope is

$$T = \frac{(\mu+1)g}{2/M_3 + 1/2M_1 + 1/2M_2}.$$

Solution. Let the acceleration of block 1 be a_1 to the right, and the acceleration of block 2 be a_2 to the left, and the acceleration of block 3 be a_3 down. We see that $2a_3 = a_1 + a_2$ by conservation of string. We also see that $M_3 g - 2T = M_3 a_3$, $T - M_1 g\mu = M_1 a_1$, and $T - M_2 g\mu = M_2 a_2$, or

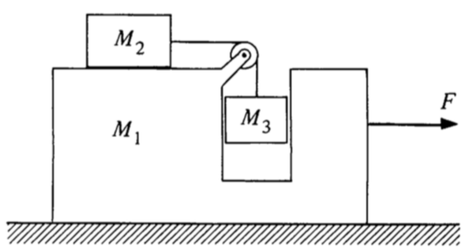
$$2a_3 = 2\frac{M_3 g - 2T}{M_3}, \quad a_1 = \frac{T - M_1 g\mu}{M_1}, \quad a_2 = \frac{T - M_2 g\mu}{M_2}.$$

Thus, we have

$$2(g - 2T/M_3) = (T/M_1 - g\mu) + (T/M_2 - g\mu) \implies 2g(1+\mu) = T(1/M_1 + 1/M_2 + 4/M_3).$$

Solving for T gives the result.

[2] **Problem 3** (KK 2.20)**.** Consider the machine shown below, which we encountered in **M2**.



Show that the acceleration of M_1 when the external force F is zero is

$$a = -\frac{M_2 M_3 g}{M_1 M_2 + M_1 M_3 + 2M_2 M_3 + M_3^2}.$$

Solution. Let the acceleration of M_2 with respect to M_1 be w. Then, we see that $T = M_2(w + a)$, and $M_3 g - T = M_3 w$, so adding gives us $M_3 g = M_2(w + a) + M_3 w$. Since the acceleration of the center of mass is zero,

$$M_1 a + M_2(w + a) + M_3 a = 0 \implies w = -\frac{M_1 + M_3}{M_2} a - a = -a\frac{M_1 + M_2 + M_3}{M_2}.$$

Thus,

$$M_3 g = M_2(-a)\frac{M_1 + M_3}{M_2} - aM_3\frac{M_1 + M_2 + M_3}{M_2},$$

or

$$a = -\frac{M_2 M_3 g}{M_2(M_1 + M_3) + M_3(M_1 + M_2 + M_3)},$$

as desired.

By the way, if you tried to solve the problem by considering just forces, there's a subtlety; it's easy to forget that there must be a force on M_1 due to the normal force of the rope on the pulley. (This force has to be there, or else the forces on the massless rope wouldn't balance.) Indeed, you learned about these forces in the preliminary problem set. The solution above implicitly took this into account, by using the fact that the center of mass doesn't move.

[3] **Problem 4.** A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ. The plane rests on a frictionless horizontal table.

(a) When the block is released, what is the horizontal acceleration of the plane?

(b) Assume the block starts a distance d above the table. Using results from part (a), what is the horizontal velocity of the block just before it reaches the floor?

(c) Find the speed of the block after it reaches the floor by applying energy and momentum conservation to the entire process.

(d) Your results for parts (b) and (c) should not match. What's going on?

Solution. (a) Applying Newton's second law to the plane gives

$$Ma = N\sin\theta.$$

Next, work in the noninertial frame of the plane, where the block only moves parallel to the plane. Balancing the forces perpendicular to the plane gives

$$N = mg\cos\theta - ma\sin\theta.$$

It's now straightforward to eliminate N and thereby solve for a, which gives

$$a = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}.$$

By the way, it's easy to get confused on this problem if you focus too much on the block, because its motion is somewhat confusing; you have to decompose it into motion parallel to the plane, and motion of the plane itself. Once you do that, the problem can be solved straightforwardly. The solution above is especially short because it never considers the acceleration of the block parallel to the plane, which isn't required to get the answer.

(b) Since the only horizontal forces in the problem are between the block and plane, the horizontal acceleration of the block is

$$a_b = \frac{M}{m}a.$$

Thus, the relative acceleration of the two is

$$a_{\text{rel}} = a + a_b = \frac{M + m}{m}a.$$

The block goes off the plane once the two have moved a relative horizontal distance of $d/\tan\theta$, which takes a time $t = \sqrt{2d/a_{\text{rel}} \tan\theta}$. At this point the block has a horizontal velocity

$$v_b = a_b t = \sqrt{2gd}\, \frac{M\cos\theta}{\sqrt{(M + m\sin^2\theta)(M+m)}}.$$

(c) By momentum conservation, the final horizontal speeds obey

$$Mv_p = mv_b$$

while by energy conservation,

$$\frac{1}{2}Mv_p^2 + \frac{1}{2}mv_b^2 = mgd.$$

Combining the two and solving gives

$$v_b = \sqrt{2gd}\sqrt{\frac{M}{M+m}}.$$

(d) It turns out that both results are correct, but they're the answers to different questions. Note that at the instant the block gets to the bottom of the plane, its velocity *isn't* horizontal, but right after it's off the plane, its velocity must be *exactly* horizontal. This requires a rather large vertical impulse. (For an illustration of this, see $F = ma$ 2021, problems 1 and 2.)

Depending on how the plane and block are constructed, there are several possibilities. If the plane abruptly ends, and the block immediately begins moving horizontally, then we have an inherently inelastic process. The vertical kinetic energy $mv_y^2/2$ of the block is lost, so the answer to part (c) doesn't apply, and the answer to part (b) is correct. (Another possibility, if the ball is very bouncy, is that the sign of its v_y will flip, and it'll bounce off the floor. In this case, the answer to part (b) is still correct. Energy is conserved now, but the answer to part (c) is still wrong because it assumes the final velocity of the block is horizontal.)

On the other hand, if the plane ends in a transition region, where θ smoothly goes to zero, then ideally energy remains conserved, and the answer of part (c) applies. In this region, a strong normal force reorients the velocity to be horizontal, supplying both a large horizontal and vertical impulse. As a result, the answer to part (b) doesn't apply.

This subtlety about how inclined planes end applies to lots of Olympiad problems. The usual assumption seems to be that option (c) holds, but in reality it depends sensitively on how the plane and block are made. In fact, in practice you can lose a lot of energy even if there's a smooth curve at the end, if that curve is not gradual enough.

2 Momentum

Idea 2

The momentum of a system is

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i = M\mathbf{v}_{\text{CM}}.$$

In particular, the total external force on the system is $M\mathbf{a}_{\text{CM}}$, and if there are no external forces, the center of mass moves at constant velocity.

Example 2

A massless rope passes over a frictionless pulley. A monkey hangs on one side, while a bunch of bananas with exactly the same weight hangs from the other side. When the monkey tries to climb up the rope, what happens?

Solution

Remarkably, the answer doesn't depend on how the monkey climbs, whether slowly or quickly, or symmetrically or not! The total vertical force on the monkey is $T - mg$, so the acceleration of the center of mass of the monkey is $T/m - g$. But since the tension is uniform through a massless rope, the acceleration of the bananas is also $T/m - g$. Therefore, the monkey and bananas rise at the same rate, and meet each other at the pulley.

Now here's a question for you: compared to climbing up a rope fixed to the ceiling, climbing up to the pulley takes twice as much work, because the bananas are raised too. But in both cases, isn't the monkey applying the same force through the same distance? Where does the extra work come from? (The answer involves the ideas at the end of this problem set.)

Example 3: KK 3.14 / INPhO 2014.5

Two men, each with mass m, stand on a railway flatcar of mass M initially at rest. They jump off one end of the flatcar with velocity u relative to the car. The car rolls in the opposite direction without friction. Find the final velocities of the flatcar if they jump off at the same time, and if they jump off one at a time. Generalize to the case of $N \gg 1$ men, with a total mass of m_{tot}.

Solution

In the first case, by conservation of momentum, we have

$$Mv + 2m(v - u) = 0$$

where v is the final velocity of the flatcar, so

$$v = \frac{2mu}{M + 2m}.$$

In the second case, by a similar argument, we find that after the first man jumps,

$$v_1 = \frac{mu}{M + 2m}.$$

Now transform to the frame moving with the flatcar. When the second man jumps, he imparts a further velocity $v_2 = mu/(M + m)$ to the flatcar by another similar argument. The final velocity of the flatcar relative to the ground is then

$$v = v_1 + v_2 = mu \left(\frac{1}{M + 2m} + \frac{1}{M + m} \right).$$

It might be a bit disturbing that the final speeds and hence energies of the flatcar are different, even though the men are doing the same thing (i.e. expending the same amount of energy in their legs to jump) in both cases.

The reason for the difference is that in the second case, the second man to jump ends up with less energy, since the velocity he gets from jumping is partially cancelled by the existing velocity v_1. So the extra energy that goes into the flatcar corresponds to less kinetic energy in the men after jumping, which would ultimately have ended up as heat after they slid to a stop. Accounting properly for the kinetic energy of everything in the system solves a lot of paradoxes involving energy, as we'll see below.

In the case of many men, by similar reasoning we have

$$v = \frac{m_{\text{tot}}}{M + m_{\text{tot}}}\, u$$

in the first case, while in the second case the answer is the sum

$$v = \sum_{i=1}^{N} \frac{m_{\text{tot}} u}{N} \frac{1}{M + (i/N) m_{\text{tot}}}.$$

This can be converted into an integral, by letting $x = i/N$, in which case $\Delta x = 1/N$ and

$$v = \sum_i \Delta x\, \frac{m_{\text{tot}} u}{M + x m_{\text{tot}}} \approx \int_0^1 dx\, \frac{m_{\text{tot}} u}{M + x m_{\text{tot}}} = \log\left(\frac{M + m_{\text{tot}}}{M}\right) u.$$

Note that this is essentially the rocket equation, which we'll derive in a different way in **M6**.

[2] **Problem 5** (KK 4.11)**.** A flexible chain of mass M and length ℓ is suspended vertically with its lowest end touching a scale. The chain is released and falls onto the scale. Find the reading on the scale when a length of chain x has fallen.

Solution. Because the chain is flexible, each link just crumples when it hits the ground, without pulling the rest of the chain downward. In other words, the assumption of ideal flexibility implies the tension in the chain vanishes, so that the vertical part of the chain is always in free fall.

Now, the lowest end of the chain is moving with velocity $\sqrt{2gx}$, so in time dt, a mass $M\sqrt{2gx}dt/\ell$ falls on to the scale, so the change in momentum of that piece is $(2Mgx/\ell)dt$. Thus, we need a force $2Mgx/\ell$ to stop the links that are falling on the scale. In addition, there must be a force Mgx/ℓ to balance the weight of the chain that's already lying on the scale, for a total of $\boxed{3Mgx/\ell}$.

This is nice and elegant, but is it true? The result is actually pretty accurate, as you can see from experimental data here. The deviation from the expected result is because no chain is perfectly flexible. Since the chain has to bend at the spot it hits the scale, some tension is produced, which pulls down the rest of the chain slightly faster than free fall.

This has a connection to the "inherently inelastic" processes mentioned later in the problem set. The fastest possible fall corresponds to the case where energy is conserved, i.e. when all the kinetic energy of each link hitting the ground is nearly transferred through tension to the still falling part of the chain. The answer we gave above corresponds to the slowest possible fall, where each link collides perfectly inelastically with the ground. For a flexible chain, the latter is closer to reality.

[3] **Problem 6.** Some qualitative questions about momentum.

(a) A box containing a vacuum is placed on a frictionless surface. The box is punctured on its right side. How does it move immediately afterward?

(b) You are riding forward on a sled across frictionless ice. Snow falls vertically (in the frame of the ice) on the sled. Which of the following makes the sled go the fastest or the slowest?

1. You sweep the snow off the sled, directly to the left and right in your frame.
2. You sweep the snow off the sled, directly to the left and right in the ice frame.
3. You do nothing.

(c) An hourglass is made by dividing a cylinder into two identical halves, separated by a small orifice. Initially, the top half is full of sand and the bottom half is empty. The hourglass is placed on a scale, and then the orifice is opened. The total weight of the hourglass and sand is W. How does the scale reading compare to W shortly after the sand starts falling, shortly after it finishing falling, and in between? (For concreteness, assume the surfaces of the sand in the top and bottom halves are always horizontal, and that the sand passes through the orifice at a constant rate.)

Solution. (a) Consider the system of the air plus the box. The air flows to the left, so to keep momentum conserved, the box moves to the right.

(b) It's easiest to think about this using conservation of momentum in the ice frame. Case (2) is clearly the fastest, as the snow steals none of the sled's horizontal momentum.

To decide between (1) and (3), note that in case (3), the snow always has the same speed as the sled. In case (1), the snow that fell and got swept up earlier has a higher speed than the sled, because the sled is constantly slowing down. So in case (1), the snow gets more of the horizontal momentum, so (1) is the slowest and (3) is in the middle.

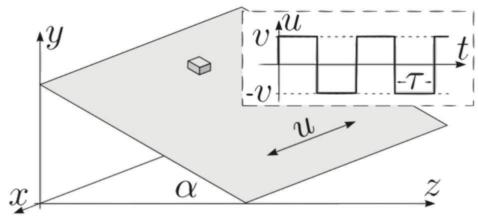
(c) Right before the sand starts falling, and right after it finishes falling, the center of mass is stationary; however, it moves down while the sand is falling. Thus, the center of mass accelerates downward at the beginning, and accelerates upward at the end. So the scale reading is lower than W at the beginning, and higher than W at the end.

This can also be understood directly in terms of forces. Right after the sand starts falling, there's a column of sand that has not yet hit the bottom; the scale reading dips lower because it doesn't have to support this falling sand. And when the last bit of sand arrives, the scale reading jumps higher because the hourglass simultaneously has to support all of the sand, and absorb the impact from the falling sand; this is the effect derived in problem 5.

What about the scale reading in between these two times? In general, it depends on the details of the hourglass's construction. But under the simplifying assumptions made in this problem, the hourglass and sand's center of mass moves downward at constant velocity. So the net external vertical force is zero, and the scale reads just W. You can see an experimental confirmation of these results here.

[3] **Problem 7.** USAPhO 2018, problem A1.

[3] **Problem 8** (Kalda)**.** A block is on a ramp with angle α and coefficient of friction $\mu > \tan\alpha$. The ramp is rapidly driven back and forth so that its velocity vector $\mathbf{u}$ is parallel to both the slope and the horizontal and has constant modulus v.



The direction of $\mathbf{u}$ reverses abruptly after each time interval τ, where $g\tau \ll v$. Find the average velocity $\mathbf{w}$ of the block. (Hint: as mentioned in **M1**, it's best to work in the frame of the ramp, because it causes the friction, even though this introduces fictitious forces.)

Solution. Work in the frame of the ramp and orient the x axis along $\mathbf{u}$ and the y axis along the ramp. At all times, the acceleration due to friction is $\mu g \cos\alpha$ and the acceleration due to gravity is $g \sin\alpha$. Every time period τ, an impulsive fictitious force changes w_x by $\pm 2v$. Since $g\tau \ll v$, the total acceleration during the time period τ due to the friction and gravitational forces is negligible compared to this change. Assuming for now that w_x is symmetric so that $\overline{w_x} = 0$, this means $|w_x| \approx u$ at all times.

Now consider w_y. In the steady state, the acceleration due to friction must be balanced by the acceleration due to gravity, so

$$\frac{\overline{w_y}}{\sqrt{\overline{w_y}^2 + u^2}} \mu g \cos\alpha = g \sin\alpha$$

which yields the answer,

$$\overline{\mathbf{w}} = \overline{w_y}\,\hat{\mathbf{y}}, \quad \overline{w_y} = \frac{u}{\sqrt{\mu^2 \cot^2\alpha - 1}}.$$

Note that this diverges when $\mu = \tan\alpha$, because at that point the friction is not strong enough to prevent the block from accelerating down the ramp indefinitely. For $\mu > \tan\alpha$, we reach a steady state where only a portion of the friction is directed vertically, due to the horizontal speed, and that portion balances gravity.

This also allows us to argue that $\overline{w_x} = 0$. If $\overline{w_x}$ is not zero, $|w_x|$ will be higher during one of the two halves of the cycle. But during that half, a greater share of the frictional acceleration will be directed against the w_x motion, tending to move $\overline{w_x}$ to zero.

This seemingly weird problem actually has real-world applications! The point here is that you can make a block slide down a ramp even if friction would prevent it from doing so, and moreover make it slide at a controlled speed. This technique is used in factories, in the form of vibratory conveyors. In fact, a more complex vibration pattern can even make something slide *up* a ramp!

[4] **Problem 9** (Morin 5.21)**.** A sheet of mass M moves with speed V through a region of space that contains particles of mass m and speed v. There are n of these particles per unit volume. The sheet moves in the direction of its normal. Assume $m \ll M$, and assume that the particles do not interact with each other.

(a) If $v \ll V$, what is the drag force per unit area on the sheet?

(b) If $v \gg V$, what is the drag force per unit area on the sheet? Assume for simplicity that the component of every particle's velocity in the direction of the sheet's motion is exactly $\pm v/2$.

(c) Now suppose a cylinder of mass M, radius R, and length L moves through the same region of space with speed V, and assume $v = 0$ and $m \ll M$. The cylinder moves in a direction perpendicular to its axis. What is the drag force on the cylinder?

Parts (a) and (b) are a toy model for the two regimes of drag, mentioned in **M1**. However, it shouldn't be taken too seriously, because as we'll see in **M7**, the typical velocity that separates the two types of behavior doesn't have to be of order v. Instead, it depends on how strongly the particles interact with each other.

Solution. (a) We can set $v = 0$. In time t, an area A hits $nAVt$ particles, and the total change in momentum of these particles is $(nAVt)m(2V)$, so the pressure is $2nmV^2$.

(b) Let's say the sheet is moving to the right. In the frame of the sheet, the particles are moving at velocity $V \pm v/2$. The particles hitting the sheet from the right will have velocity $v/2 + V$, and from the left $v/2 - V$. From the right in time dt, there will be $\frac{1}{2}nA(v/2 + V)dt$ particles hitting the sheet (with the $\frac{1}{2}$ coming from other particles moving away from the sheet), each with impulse $2m(v/2 + V)$. Thus the pressure will be $nm(v/2 + V)^2$ from the right, and replacing V with $-V$ gives a pressure of $nm(v/2 - V)^2$ from the left. Thus the total pressure on the sheet is $2nmVv$.

(c) Work in cylindrical coordinates, with $\theta = 0$ along the direction of the cylinder's motion. For a segment $d\theta$, we have

$$\frac{\text{collisions}}{\text{time}} = \frac{\text{particles}}{\text{volume}} \frac{\text{volume swept out}}{\text{time}} = nVLR\cos\theta\, d\theta$$

where L is the length of the cylinder. To calculate the rebound velocity, it's best to work in the frame of the cylinder. In this case, the particles come in with vertical velocity $-V$, and then bounce off elastically, ending up with vertical velocity $V\cos 2\theta$. So the impulse per collision is $mV(1 + \cos 2\theta)$. The drag force is

$$F = \int_{-\pi/2}^{\pi/2} mV(1 + \cos 2\theta) nVLR\cos\theta\, d\theta = nmV^2LR \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\cos\theta\, d\theta.$$

The integral can be done straightforwardly using either the cosine double angle identity, or decomposing into complex exponentials, yielding $8/3$, so

$$F = (2nmV^2)(LR)\left(\frac{4}{3}\right).$$

Compare this to the answer to part (a). The force is quadratic in V for the same reason, but now the area is replaced by an effective area $(4/3)LR$. This is slightly less than the actual area $2LR$, since the surface is curved, and hence more aerodynamic.

You can also get a more "realistic" result by averaging over a Maxwell–Boltzmann distribution for the molecular speeds, as introduced in **T1**. But this is a lot more work, and the simpler calculation done in this problem gives all the essential insight.

3 Energy

Idea 3

The work done on a point particle is

$$W = \int \mathbf{F} \cdot d\mathbf{x}$$

and is equal to the change in kinetic energy, as you showed in **P1**.

Remark: Dot Products

The dot product of two vectors is defined in components as

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z$$

and is equal to $|\mathbf{v}|\,|\mathbf{w}|\cos\theta$ where θ is the angle between them. For example, if $\mathbf{A}$ and $\mathbf{B}$ are the sides of a triangle, the other side is $\mathbf{C} = \mathbf{A} - \mathbf{B}$, and

$$C^2 = |\mathbf{A} - \mathbf{B}|^2 = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = A^2 + B^2 - 2AB\cos\theta$$

which proves the law of cosines. (Or, if you accept the law of cosines, you could regard this as a proof that the dot product depends on $\cos\theta$ as claimed.)

Like the ordinary product, the dot product obeys the product rule. For example,

$$\frac{d}{dt}(\mathbf{v} \cdot \mathbf{w}) = \dot{\mathbf{v}} \cdot \mathbf{w} + \mathbf{v} \cdot \dot{\mathbf{w}}.$$

Using this, it's easy to generalize the derivation of the work-kinetic energy theorem in **P1** to three dimensions; we have

$$\frac{1}{2} d(v^2) = \frac{1}{2} d(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \cdot d\mathbf{v} = \frac{d\mathbf{x}}{dt} \cdot d\mathbf{v} = \frac{d\mathbf{v}}{dt} \cdot d\mathbf{x} = \mathbf{a} \cdot d\mathbf{x}$$

and this is equivalent to the desired theorem. As you can see, it's all basically the same, since the product and chain rule manipulations work the same way for vectors and scalars.

Example 4: IPhO 1996 1(b)

A skier starts from rest at point A and slowly slides down a hill with coefficient of friction μ, without turning or braking, and stops at point B. At this point, his horizontal displacement is s. What is the height difference h between points A and B?

Solution

Since the skier begins and ends at rest, the change in height is the total energy lost to friction,

$$mgh = \int f_{\text{fric}}\, ds$$

where the integral over ds goes over the skier's path. Since the skier is always moving slowly, the normal force is approximately $mg\cos\theta$. (More generally, there would be another contribution to provide the centripetal acceleration.) But then

$$\int f_{\text{fric}}\, ds = \int \mu mg \cos\theta\, ds = \int \mu mg\, dx = \mu mgs$$

which gives an answer of $h = \mu s$. (If the skier's path turned around, then this would still hold as long as s denotes the total horizontal distance traveled.)

[3] **Problem 10** (MPPP 16)**.** On a windless day, a cyclist going "flat out" can ride uphill at a speed of $v_1 = 12\,\text{km/h}$ and downhill at $v_2 = 36\,\text{km/h}$ on the same inclined road. We wish to find the cyclist's top speed on a flat road if their maximal effort is independent of the speed at which the bike is traveling. Note that in this regime, the air drag force is quadratic in the speed.

(a) Solve the problem assuming that "maximal effort" refers to the force exerted on the pedals by the rider, and that the rider never changes gears.

(b) Solve the problem assuming that "maximal effort" refers to the rider's power.

Solution. (a) Let F_0 be the force due to gravity along the hill and let kv^2 be the drag force. If the rider exerts force F' on the pedals, then the wheels exert a force F on the ground, but the ratio F/F' is constant if there are no gear switches. Then

$$F - F_0 = kv_1^2, \quad F + F_0 = kv_2^2, \quad F = kv_3^2$$

where v_3 is the answer. Combining these equations gives

$$v_3 = \sqrt{\frac{v_1^2 + v_2^2}{2}} = 27\,\text{km/h}.$$

(b) In this case, the equations are a bit nastier,

$$P/v_1 - F_0 = kv_1^2, \quad P/v_2 + F_0 = kv_2^2, \quad P/v_3 = kv_3^2.$$

Some tedious but straightforward algebra gives

$$v_3 = \sqrt[3]{\frac{v_1 v_2 (v_1^2 + v_2^2)}{v_1 + v_2}} = 23.5\,\text{km/h}.$$

[3] **Problem 11.** USAPhO 2016, problem B1.

[2] **Problem 12.** Alice steps on the gas pedal on her car. Bob, who is standing on the sidewalk, sees Alice's car accelerate from rest to 10 mph. Charlie, who is passing by in another car, sees Alice's car accelerate from 10 mph to 20 mph. Hence Charlie sees the kinetic energy of Alice's car increase by three times as much. How is this compatible with energy conservation, given that the same amount of gas was burned in both frames?

Solution. The difference in energy comes from the change in kinetic energy of the Earth. In Bob's frame, the final kinetic energy of the Earth is $p^2/2M$ where p is the total frictional impulse, and this is negligible since p is moderately sized, while the Earth's mass M is huge. Another way of saying this is that the final kinetic energy of the car is $p^2/2m_{\text{car}}$, which is much larger since $m_{\text{car}} \ll M$.

On the other hand, in Charlie's frame, the Earth has some initial momentum P. The change in kinetic energy of the Earth is

$$\Delta K_E = \frac{(P-p)^2 - P^2}{2M} = -\frac{Pp}{M} + \frac{p^2}{2M}.$$

The last term is again negligible, but now we have a term that is *linear* in p, which isn't negligible. Let $v_0 = 10\,\text{mph}$. We have $P/M = v_0$ and $p = m_{\text{car}} v_0$, so

$$\Delta K_E = -m_{\text{car}} v_0^2.$$

This decrease in Earth's kinetic energy accounts for the extra increase in the car's kinetic energy. The lesson of this problem is that when you go into a different reference frame, kinetic energies and even changes in kinetic energy can differ dramatically. While you can get the right answer either way, it's generally least confusing to work in the rest frame of the largest object in the problem.

When there are multiple large objects, you can get interesting effects. For example, naively a gravitational slingshot can't work, because the gravitational force is conservative. And indeed, a rocket doing a gravitational slingshot off of Jupiter gets no additional energy, *in Jupiter's frame*. However, for rockets that far out, the most important object is the Sun, since it determines, e.g. whether the rocket can escape the solar system. To answer that kind of question we should work in the Sun's frame, and in this frame the rocket does get more energy, as it harvests it from Jupiter's large kinetic energy. You'll investigate this in more detail in **M6**.

[3] **Problem 13** (KK 4.8). A block of mass M is attached to a spring of spring constant k. It is pulled a distance L from its equilibrium position and released from rest. The block has a small coefficient of friction μ with the ground. Find the number of cycles the mass oscillates before coming to rest.

Solution. First let's present a short solution that only works for small μ. Let A be the amplitude, so the energy is $E = \frac{1}{2}kA^2$. Hence in one cycle, the change in energy is related to the change in amplitude by

$$dE = kA\,dA$$

where we can use infinitesimals for one cycle since the friction is assumed small. But the energy loss is also $4\mu mgA$, so plugging this in gives

$$dA = -\frac{4\mu mg}{k}.$$

The oscillation ends when the amplitude drops to zero, so

$$N = \frac{kL}{4\mu mg}.$$

We expect this result to be trustworthy whenever N is large, i.e. when the fractional amplitude change during a cycle is small.

We will now show that, in fact, this result is correct even when N isn't large. Notice that during the left-moving part of a cycle, the friction provides a constant force of μmg to the right. Therefore, just like how gravity shifts the equilibrium position of a vertical spring, the friction shifts the equilibrium position to the right by $\mu mg/k$. The left-moving motion is a perfect sinusoid centered at this position. Similarly, the right-moving part of the oscillation is a perfect sinusoid, but instead centered at $-\mu mg/k$. The net effect of one cycle is thus to decrease the amplitude by exactly $4\mu mg/k$, giving the result.

[3] **Problem 14** (Morin 5.4)**.** A massless string of length 2ℓ connects two hockey pucks that lie on frictionless ice. A constant horizontal force F is applied to the midpoint of the string, perpendicular to it. The pucks eventually collide and stick together. How much kinetic energy is lost in the collision?

Solution. Suppose the bend in the rope is θ, where originally $\theta = 0$. We see that the tension T satisfies $2T \sin\theta = F$, by balancing forces at the midpoint. Thus, the y-component of the force on the top mass is $T\cos\theta$, so the total work done by tension in the y direction is

$$W = -\int_0^{\pi/2} 2(T\cos\theta)\, d(\ell\cos\theta) = \ell F \int_0^{\pi/2} \cos\theta \, d\theta = F\ell.$$

This determines the vertical kinetic energy, $mv_y^2/2$, of each puck. When the pucks collide, all of this energy is lost, giving the answer $F\ell$.

There's also a slick alternate solution using a noninertial reference frame. Now, in general work depends on the reference frame, as we just saw in problem 12, since displacement does, so we always need to be careful calculating energies in other frames. However, the amount of *dissipated* energy determines how much the pucks warm up, which is independent of frame! Therefore, we are free to use any frame we want.

In particular, consider the frame with acceleration $F/2m$ along the force. In this frame, there is a fictitious force $-F/2$ on each puck. The net force on the system is zero, so the pucks move directly towards each other. When the pucks collide, the point of application of the force F has traveled a distance ℓ, doing work $F\ell$. Since the pucks are stationary after collision, all this energy is dissipated, giving the answer $F\ell$ again.

Idea 4

If a problem can be solved using either momentum conservation or energy conservation alone, it usually means one of the two isn't actually conserved. In particular, many processes are inherently inelastic and inevitably dissipate energy. For more about inherently inelastic processes, see section 5.8 of Morin.

[2] **Problem 15** (KK 4.20)**.** Sand falls slowly at a constant rate dm/dt onto a horizontal belt driven at constant speed v.

(a) Find the power P needed to drive the belt.

(b) Show that the rate of increase of the kinetic energy of the sand is only $P/2$.

(c) We can explain this discrepancy exactly. Argue that in the reference frame of the belt, the rate of heat dissipation is $P/2$. Since temperature is the same in all frames, the rate of heat dissipation is $P/2$ in the original frame as well, accounting for the missing energy.

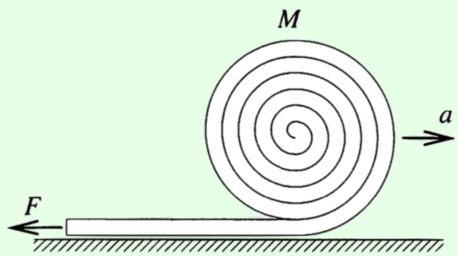
Solution. (a) We have $P = Fv = (dp/dt)\, v = (v\,(dm/dt))v = v^2\, dm/dt$.

(b) It's $\frac{1}{2}(dm/dt)v^2 = P/2$.

(c) In the belt's frame, the sand comes in with a speed of v, and friction slows it down to zero speed. Hence the sand loses all its kinetic energy to heat, at a rate $\frac{1}{2}(dm/dt)v^2 = P/2$.

Example 5: PPP 108

A fire hose of mass M and length L is coiled into a roll of radius R. The hose is sent rolling along level ground, with its center of mass given initial speed $v_0 \gg \sqrt{gR}$. The free end of the hose is held fixed.



The hose unrolls and becomes straight. How long does this process take to complete?

Solution

First, we need to find what is conserved. The horizontal momentum is not conserved, because there is an external horizontal force needed to keep the end of the hose in place. On the other hand, the energy *is* conserved, even though this process looks inelastic. The hose "sticks" to the floor as it unrolls, but this process dissipates no energy because the circular part of the hose rolls without slipping, so the bottom of this part always has zero velocity.

Once we figure out energy is conserved, the problem is straightforward. The assumption $v_0 \gg \sqrt{gR}$ means we can neglect the change in gravitational potential energy as the hose unrolls. After the hose travels a distance x,

$$\frac{1}{2}\left(1+\frac{1}{2}\right)Mv_0^2 = \frac{1}{2}\left(1+\frac{1}{2}\right)mv^2$$

where the 1/2 terms are from rotational kinetic energy. Since $m(x) = M(1-x/L)$, we have

$$v(x) = \frac{v_0}{\sqrt{1-x/L}}$$

which gives a total time

$$T = \int_0^L \frac{dx}{v(x)} = \frac{L}{v_0}\int_0^1 \sqrt{1-u}\,du = \frac{2L}{3v_0}.$$

Evidently, the hose accelerates as it unrolls.

[4] **Problem 16.** Consider the following related problems; in all parts, neglect friction.

(a) A uniform rope of length ℓ lies stretched out flat on a table, with a tiny portion $\ell_0 \ll \ell$ hanging through a hole. The rope is released from rest, and all points on the rope begin to move with the same speed. Since this motion is smooth, energy is conserved. Find the speed of the rope when the end goes through the hole.

(b) $\star$ For practice, repeat part (a) by solving for $x(t)$ explicitly. (Hint: this is best done using the generalized coordinate techniques of **M4**.)

(c) Now suppose a flexible uniform chain of length ℓ is placed loosely coiled close to the hole. Again, a tiny portion $\ell_0 \ll \ell$ hangs through the hole, and the chain is released from rest. In this case, the unraveling of the chain is an inherently inelastic process, because each link of the chain sits still until it is suddenly jerked into motion. Find the speed of the chain when the last link goes through the hole. (Hint: you should get a nonlinear differential equation, which can be solved by guessing $x(t) = At^n$.)

Solution. (a) We use energy conservation. The height of the center of mass falls by $\ell/2$, so $\ell Mg/2 = Mv^2/2$, which gives the answer of $v = \sqrt{\ell g}$.

(b) For convenience, we use the idea of "generalized coordinates", which will be covered in more detail in **M4**. The point is that a *direct* application of Newton's second law would be very tough, because we'd have to solve for the tension everywhere in the rope. But we can instead treat the rope as a single object by parametrizing its motion in terms of "how far it's gone through the hole". The net force "putting the rope into the hole" is just gravity acting on the hanging part of the rope, xMg. Thus,

$$xMg = Ma$$

which implies

$$\ddot{x} = \frac{g}{\ell}\,x.$$

Now, this is a linear differential equation which can be solved with the techniques of **M1**. Guessing exponentials gives growing and decaying solutions $e^{\pm\sqrt{g/\ell}\,t}$. It's most convenient to rewrite these in terms of hyperbolic trig functions,

$$x = A\cosh(\sqrt{g/\ell}\,t) + B\sinh(\sqrt{g/\ell}\,t).$$

The initial conditions then give us $x(t) = \ell_0\cosh(\sqrt{g/\ell}t)$, so in the limit of small ℓ_0, the final time obeys

$$\ell \approx \frac{\ell_0}{2}e^{\sqrt{g/\ell}\,t_f}.$$

The velocity is

$$v(t_f) = \ell_0\sqrt{\frac{g}{\ell}}\sinh(\sqrt{g/\ell}t_f) \approx \frac{\ell_0}{2}\sqrt{\frac{g}{\ell}}e^{\sqrt{g/\ell}t_f} \approx \sqrt{\frac{g}{\ell}}\ell = \sqrt{\ell g}$$

as found by energy conservation.

(c) In this case energy conservation doesn't work, so we need to use momentum/force ideas. Unlike part (a), it's best to use Newton's second law directly, by considering the vertical momentum of the vertical part of the chain. We didn't do this in part (a) because we would have to know the tension at the hole, since this provides an external vertical force, but here it's easy because the chain links on the table are slack, so the tension is zero. On the other hand, the generalized coordinate approach of part (a) wouldn't work here because the chain is not moving as one smooth unit; there's nasty discontinuous stuff going on at the hole, which we don't want to think about.

Now, let m be the time-dependent mass of the vertical part. The only external vertical force is gravity, so applying $F_y = dp_y/dt$ gives

$$mg = m\dot{v} + \dot{m}v = m\dot{v} + (m/x)v^2$$

which implies

$$\ddot{x} = g - \dot{x}^2/x.$$

This is a nonlinear second-order differential equation. There's no general way to solve such equations, so we'll resort to the hint. If we guess a pure power At^n, then all three terms are the same power of t as long as $n = 2$. Plugging in $x(t) = At^2$ gives the solution

$$x(t) = \frac{1}{6}gt^2$$

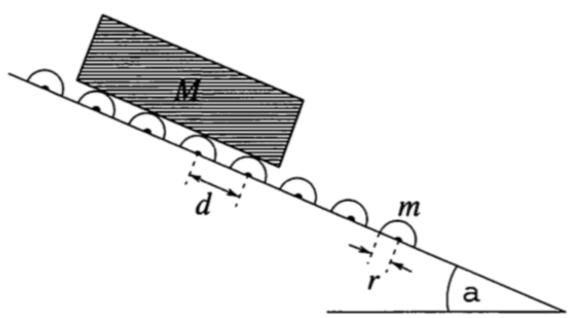
so there is a uniform acceleration of $g/3$. (The 1/6 is *not* an arbitrary constant, if you change it you don't get a solution to the differential equation at all! That's because this equation is nonlinear, so there's no reason to expect that multiplying a solution by a constant gives another solution.)

The amount of time it takes for last link to pass is $t = \sqrt{6\ell/g}$, so the speed there is

$$\boxed{v = (g/3)t = \sqrt{\frac{2\ell g}{3}}.}$$

This is smaller than the answer to part (a) because energy is not conserved.

[3] **Problem 17** (PPP 95). A long slipway, inclined at an angle α to the horizontal, is fitted with many identical rollers, consecutive ones being a distance d apart. The rollers have horizontal axles and consist of rubber-covered solid steel cylinders each of mass m and radius r. A plank of mass M, and length much greater than d, is released at the top of the slipway.



Find the terminal speed $v_{\max}$ of the plank. Ignore air drag and friction at the pivots of the rollers.

Solution. Let the terminal velocity be v, and consider the forces acting on the plank along the plane. There is of course a constant gravitational force $Mg\sin\alpha$. In addition, every time the plank hits a roller, it experiences an impulse as it spin the roller up. The angular impulse on each roller is equal to its angular momentum, so

$$\int f(t) r \, dt = \frac{1}{2}mr^2\omega.$$

This implies the linear impulse on the plank has magnitude

$$J = \int f(t)\, dt = \frac{1}{2} m r \omega = \frac{1}{2} m v.$$

This impulse must be equal to the total gravitational impulse along the plane between rollers,

$$\frac{1}{2} m v = \frac{d}{v} M g \sin\alpha$$

which gives the answer,

$$\boxed{v = \sqrt{\frac{2Mgd\sin\alpha}{m}}.}$$

The subtle thing about this problem is that a similar argument based on energy conservation gives the wrong answer. Equating the gravitational potential energy lost per roller to the rotational kinetic energy given to each roller gives

$$\frac{1}{2} I \omega^2 = \frac{1}{4} m v^2 = M g d \sin\alpha$$

which gives an answer different by a factor of $\sqrt{2}$. The reason is that energy is also dissipated into heat, as the plank and roller initially slip with respect to each other. By an argument extremely similar to that of problem 15, but with angular variables instead of linear ones, you can show that precisely half the gravitational potential energy goes into heat. Accounting for this gives exactly the same answer as momentum conservation.

4 Elastic Collisions

Idea 5

Any temporary interaction between two objects that conserves energy and momentum is a perfectly elastic collision. In one dimension, such collisions are "trivial": their outcome is fully determined by energy and momentum conservation, because there are two final velocities and two conservation laws. In two dimensions, there are four final velocity components and three conservation laws (energy and 2D momentum), so we need one more number to describe what happens, such as the angle of deflection. In a two-dimensional collision, the outcome depends on the details, such as how the objects approach each other, and the force between them. The same holds in three dimensions.

Example 6

Two masses are constrained to a line. The mass m_1 moves with velocity v_1, and the mass m_2 moves with velocity v_2. The masses collide perfectly elastically. Find their speeds afterward.

Solution

The usual method is to directly invoke conservation of energy and momentum, which leads to a quadratic equation. A slicker method is to work in the center of mass frame instead. (This is useful for collision problems in general, and it'll become even more useful for the relativistic collisions covered in **R2**.)

The center of mass of the system has speed

$$v_{\text{CM}} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}.$$

Moreover, by momentum conservation, the center of mass never accelerates. Now we boost into the frame moving with the center of mass. Since the total momentum is by definition zero in the center of mass frame, the momenta of the particles cancel out. The only way for this to remain true after the collision is if we multiply their velocities by the same number. Energy is only conserved if this number is ± 1, with the latter representing no collision at all.

Therefore, during an elastic collision, the velocities in the center of mass frame simply reverse. The initial velocities in that frame are

$$v_{1,\text{CM}} = v_1 - v_{\text{CM}}, \quad v_{2,\text{CM}} = v_2 - v_{\text{CM}}.$$

The final velocities in that frame are

$$v'_{1,\text{CM}} = -v_1 + v_{\text{CM}}, \quad v'_{2,\text{CM}} = -v_2 + v_{\text{CM}}.$$

Finally, going back to the original frame gives the final velocities

$$v'_1 = -v_1 + 2v_{\text{CM}}, \quad v'_2 = -v_2 + 2v_{\text{CM}}.$$

There are many special cases we can check. For example, if $m_1 = m_2$, then the two masses simply swap their velocities, as if they just passed through each other. As another check, consider the case where the second mass is initially at rest, $v_2 = 0$. Then

$$v'_1 = v_1 \, \frac{m_1 - m_2}{m_1 + m_2}, \quad v'_2 = v_1 \, \frac{2m_1}{m_1 + m_2}.$$

When $m_1 = m_2$, the first mass gives all its velocity to the second. When m_2 is large, the first mass just rebounds off with velocity $-v_1$. When m_1 is large, the first mass keeps on going and the second mass picks up velocity $2v_1$. Finally, when $m_1 = m_2/3$, then the final speeds are $v'_1 = -v_1/2$ and $v'_2 = v_1/2$, a nice result which is worth committing to memory.

Idea 6

The kinetic energy of a set of masses m_i with total mass M can be decomposed as

$$\sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} M v_{\text{CM}}^2 + \sum_i \frac{1}{2} m_i (v_i - v_{\text{CM}})^2$$

where the first term is the "center of mass" contribution, and the second term is the amount of kinetic energy in the center of mass frame. (This statement also holds true for multidimensional collisions, if the squares are replaced with vector magnitudes.) The first term can be rewritten as $P^2/2M$ where P is the total momentum of the masses. Therefore, the kinetic energy of a system of masses with fixed total momentum is minimized when the second term is zero, i.e. when all the masses are traveling with the same velocity. This implies, for instance, that a totally inelastic collision dissipates the highest possible amount of kinetic energy.

Example 7

Three balls of mass M are initially at rest. Then an explosion occurs, giving the system a fixed total kinetic energy. What is the maximum possible fraction of this energy that one ball can carry by itself?

Solution

Suppose we want to maximize the energy of the first ball, and let p_0 be the magnitude of its final momentum. Since the total momentum is zero, the other two balls also have a total momentum of magnitude p_0. As shown in the above idea, the energy of those two balls is minimized if they travel at the same speed. Therefore, the optimal scenario is to have the first ball come out with speed v_0 and have both of the other two come out the other direction with equal speed $v_0/2$. Then the first ball has $2/3$ of the total energy.

This is the simplest possible "optimal collision" problem; we'll see more in **R2**. Many can be solved with the basic idea that some of the outgoing masses should have the same velocity.

[2] **Problem 18** (Morin 5.23). A tennis ball with mass m_2 sits on top of a basketball with a mass $m_1 \gg m_2$. The bottom of the basketball is a height h above the ground. When the balls are dropped, how high does the tennis ball bounce?

Solution. Let $v = \sqrt{2gh}$. Once the basketball hits the ground, it has velocity v upwards, whereas the tennis ball has velocity v downwards. Going into CM frame, which coincides with basketball frame since $m_1 \gg m_2$, we see that the new velocity of the tennis ball is $2v$ up, so the velocity in the ground frame is $3v$ up, so the tennis ball bounces to $9h$ since its energy got multiplied by 9.

[3] **Problem 19** (PPP 46). A Newton's cradle consists of three suspended steel balls of masses m_1, m_2, and m_3 arranged in that order with their centers in a horizontal line. The ball of mass m_1 is drawn aside in their common plane until its center has been raised by h and is then released. If all collisions are elastic, how much m_2 be chosen so that the ball of mass m_3 rises to the greatest possible height, and what is this height? (Neglect all but the first two collisions.)

Solution. The ball of mass m_1 has speed $v = \sqrt{2gh}$ once it hits the ball of mass m_2. By applying the result of example 6 twice, the speed of mass m_3 after the first two collisions is

$$v' = \frac{2m_1}{m_1 + m_2} \frac{2m_2}{m_2 + m_3} v$$

which means the final height is

$$h' = \left(\frac{4m_1 m_2}{(m_1 + m_2)(m_2 + m_3)} \right)^2 h.$$

By some basic calculus, this is maximized when

$$m_2 = \sqrt{m_1 m_3}$$

in which case

$$h' = \left(\frac{2}{1 + \sqrt{m_3/m_1}} \right)^4 h.$$

For a wide range of m_3/m_1, this is pretty close to perfect efficiency. (Transferring 100% of the energy would yield $h' = (m_1/m_3)h$.)

[3] **Problem 20.** Here's a variety problem involving some "clean" mathematical results. All three parts can be solved without lengthy calculation.

(a) Consider n identical balls confined to a line. Assuming all collisions are perfectly elastic, what is the maximum number of collisions that could happen? Assume no triple collisions happen.

(b) A billiard ball hits an identical billiard ball initially at rest in a perfectly elastic collision. Show that the balls exit at a right angle to each other.

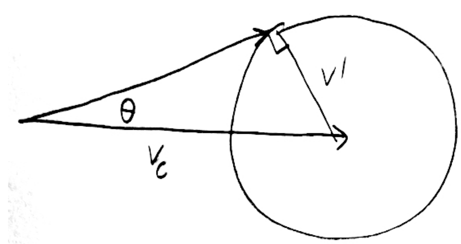
(c) A mass M collides elastically with a stationary mass m. If $M > m$, show that the maximum possible angle of deflection of M is $\sin^{-1}(m/M)$.

Solution. (a) Note when two identical balls collide, they simply swap velocities. Therefore, we can imagine the balls as passing through one another, and we want the maximum number of times two balls pass through each other. All $\boxed{n(n-1)/2}$ pairs can pass through each other, as long as the i^{th} ball on the right has the i^{th} smallest rightward velocity.

(b) Let the initial velocity be $\mathbf{v}$, and the final velocities be $\mathbf{v}_1$ and $\mathbf{v}_2$. By momentum conservation, $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$. By energy conservation, $v_1^2 + v_2^2 = v^2$. By the law of cosines, this is only possible if $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

(c) Let v be the initial speed. Recall that in the center of mass frame, the mass M can only change the direction, but not the magnitude, of its velocity. In this frame, the speed of the mass M is $v' = mv/(m+M)$. Thus, the possible final velocities lie on a circle of radius v'.

Now let's transform back to the lab frame. In this frame, it's still true that the possible final velocities lie on a circle of radius v', but the center of this circle is now at the center of mass velocity $v_c = Mv/(m+M)$.



From the sketch above, the final velocity that maximizes the angle θ of deflection yields

$$\sin\theta = \frac{v'}{v_c} = \frac{m}{M}$$

which is the desired result.

[3] **Problem 21** (PPP 72)**.** Beads of equal mass m are strung at equal distances d along a long, horizontal, infinite wire. The beads are initially at rest but can move without friction. The first bead is continuously accelerated towards the right by a constant force F. After some time, a "shock wave" of moving beads will propagate towards the right.

(a) Find the speed of the shock wave, assuming all collisions are completely inelastic.

(b) Do the same, assuming all collisions are completely elastic. What is the average speed of the accelerated bead in this case?

If you're having trouble visualizing this, try plotting all the masses' positions $x(t)$ over time.

Solution. (a) In the steady state, a large clump of particles will be moving towards the right. If the steady state speed is v_0, then collisions occur at time intervals d/v_0, so the momentum of the blob must grow at rate $(mv_0)(v_0/d)$ as new beads join it. This must be equal to F, and solving gives

$$v_0 = \sqrt{\frac{Fd}{m}}.$$

(b) By basic kinematics, the speed of the first, accelerated bead the moment before it hits the next bead is

$$v_1 = \sqrt{\frac{2Fd}{m}}.$$

At the moment of collision, the first bead loses all its velocity to the second. The second bead moves towards the third with velocity v_1 and gives its velocity to the third, and so on, creating a shock front with velocity v_1.

In the meantime, the first bead is still accelerating. After another time interval, it hits the second bead, which is now where the third bead originally was, and the same phenomenon happens again, creating another bead with velocity v_1 just behind the leading one. So after a long time, we build up a shock front of beads traveling with speed v_1.

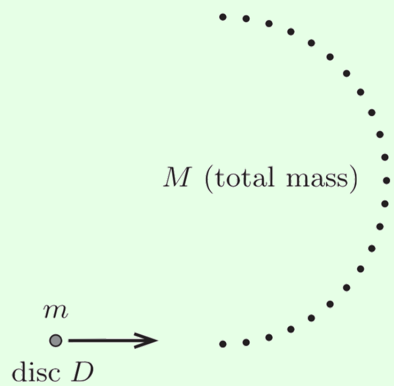
On the other hand, the first bead keeps uniformly accelerating between zero speed and v_1, so its average speed is just $v_1/2$. Evidently, the shock wave separates from the first bead after time.

[3] **Problem 22.** USAPhO 2019, problem A1.

[3] **Problem 23.** USAPhO 2009, problem B1.

Example 8: MPPP 42

There are N identical tiny discs lying on a table, equally spaced along a semicircle, with total mass M. Another disc D of mass m is very precisely aimed to bounce off all of the discs in turn, then exit opposite the direction it came.



In the limit $N \to \infty$, what is the minimal value of M/m for this to be possible? Given this value, what is the ratio of the final and initial speeds of the disc?

Solution

The reason there is a lower bound on M is that, by problem 20(c), there is a maximal angle that each tiny disc can deflect the disc D. For large N, the deflection is π/N for each disc, so

$$\frac{\pi}{N} = \sin^{-1} \frac{M/N}{m} \approx \frac{M}{Nm}$$

which implies that $M/m \geq \pi$.

To see how much energy is lost in each collision, work in the center of mass frame and consider the first collision. In this frame, the disc D is initially approximately still, and the tiny disc comes in horizontally with speed v. To maximize the deflection angle in the table's frame, the tiny disc should rebound vertically, as this provides the maximal vertical impulse to the disc D.

Thus, going back to the table's frame, where the disc D has speed v, the tiny disc scatters with speed $\sqrt{v^2 + v^2} = \sqrt{2}v$. By conservation of energy,

$$\Delta \left(\frac{1}{2} m v^2 \right) = -\frac{1}{2} \frac{M}{N} (\sqrt{2} v)^2.$$

This simplifies to

$$\frac{\Delta v}{v} = -\frac{\pi}{N}$$

which means that after N collisions, we have the cute result

$$\frac{v_f}{v_i} = \left(1 - \frac{\pi}{N}\right)^N \approx e^{-\pi}$$

where in the last step we used a result from **P1**.

Example 9: EFPhO 2003.1

A spherical volleyball of radius r and mass m is inflated with excess pressure ΔP. If it is dropped from the ceiling and hits the ground, estimate how long the subsequent elastic collision takes.

Solution

Answering this question requires making a simplified physical model of how the collision occurs. Let's say that when the volleyball hits the ground, it will keep going straight down, deforming the part that touches the ground into a flat circular face. Specifically, when the ball has moved a distance y into the ground, the flat face has area

$$A = \pi \left(\sqrt{r^2 - (r-y)^2} \right)^2 = \pi y(2r - y) \approx 2\pi r y$$

where we assumed that $y \ll r$ at all times, which is reasonable as long as the ball's initial speed is not enormous. As a result, the pressure of the volleyball exerts a force

$$F = 2\pi r \, \Delta P \, y$$

on the ground. This assumes the pressure inside the volleyball remains uniform, and that the rest of the volleyball stays approximately spherical, which is again reasonable as long as the initial speed is not huge.

Assuming the initial velocity is not too small, gravity is negligible during the collision, so during the collision the force on the volleyball is effectively that of an ideal spring. The collision lasts for half a period, giving

$$\tau = \pi \sqrt{\frac{m}{k_{\text{eff}}}} = \sqrt{\frac{\pi m}{2r \, \Delta P}}.$$

If we plug in realistic numbers, the result is of order 10 ms, which is plausible.

5 Continuous Systems

Example 10

As shown in **M2**, a hanging chain takes the form of a catenary. Suppose you pull the chain down in the middle. How does the center of mass of the chain move? Does the answer depend on how hard you pull?

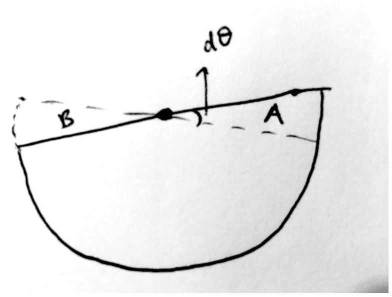
Solution

No matter how hard you pull, or in what direction, the height of the center of mass always goes up! This is because this quantity measures the total gravitational potential energy of the chain. If you pull a chain in equilibrium, in any direction whatsoever, you will do work on it. So this raises its potential energy, and hence the center of mass.

Another way of saying this is that the equilibrium position, without the extra pull you supply, is already in the lowest energy state, and hence already has the lowest possible center of mass. Changing this shape in any way raises the center of mass.

[2] **Problem 24.** A uniform half-disc of radius R is nailed to a wall at the center of its circle and allowed to come to equilibrium. The half-disc is then rotated by an angle $d\theta$. By calculating the energy needed to do this in two different ways, find the distance from the pivot point to the center of mass.

Solution.



Suppose the CM is at radius r. The energy required to turn the disc by $d\theta$ is

$$(1 - \cos(d\theta))\, mgx = \frac{mgx}{2}\, d\theta^2.$$

However, when rotated, all that has changed is that there is a new sector of angle $d\theta$ above (sector A), and one sector is now missing (sector B). The CM of a thin sector is at radius $2R/3$ (it's just an isosceles triangle), so the total extra energy of A is $(dm)g(2R/3)d\theta/2$, where $dm/m = d\theta/\pi$, so it's $mgR(1/3\pi)\, d\theta^2$, and the extra energy due to the absence of B is the same, so we have

$$\frac{2}{3\pi}\, mgR\, d\theta^2 = \frac{mgx}{2}\, d\theta^2$$

which implies

$$x = \frac{4}{3\pi} R.$$

[4] **Problem 25** (Morin 5.31)**.** Assume that a cloud consists of tiny water droplets suspended (uniformly distributed, and at rest) in air, and consider a raindrop falling through them. Assume the raindrop is initially of negligible size, remains spherical at all times, and collides perfectly inelastically with the droplets. It turns out that the raindrop accelerates uniformly; assuming this, find the acceleration.

Solution. Suppose the mass density in the cloud is λ and the mass density of the raindrop is ρ (note $\rho > \lambda$), and suppose r is the radius of the drop, M the mass, and v the velocity. We see that

$$\dot{M} = 4\pi r^2 \dot{r} \rho = 3M \frac{\dot{r}}{r}$$

and

$$\dot{M} = \pi r^2 v \lambda,$$

which combine to give

$$v = \frac{4\rho}{\lambda} \dot{r}.$$

We see that $Mg = \dot{M}v + M\dot{v}$, so Newton's second law is

$$Mg = M\left(\frac{3\dot{r}}{r}\frac{4\rho}{\lambda}\dot{r} + \frac{4\rho}{\lambda}\ddot{r}\right)$$

and writing everything in terms of r gives

$$rg\lambda/\rho = 12\dot{r}^2 + 4r\ddot{r}.$$

This is a nonlinear second-order differential equation; there is no general method to solve these equations. Certainly an exponential won't work, because you won't get the same exponential on the left and right-hand sides. However, we can use the hint, which indicates that v is linear in time. This implies that r is a quadratic, so guessing $r = At^2$ gives

$$At^2 g\lambda/\rho = 4A^2(12t^2 + 2t^2).$$

This implies that we indeed have a solution, as long as

$$A = (g\lambda/\rho)/56.$$

Using our relation between v and $\dot{r}$, we finally have

$$\dot{v} = \frac{4\rho}{\lambda}\ddot{r} = \frac{g}{7}$$

which is the acceleration.

As you can see from this problem and an earlier one, nonlinear second-order differential equations are actually quite common in physics. Trying a pure power At^n is a decent first guess, because monomials remain monomials under both differentiation and multiplication; for the same reason, an exponential Ae^{Bt} can also work. However, in practice, the vast majority of such differential equations don't have analytic solutions at all, or only have solutions in terms of exotic special functions. Problems for Olympiads and undergraduate textbooks are generally chosen precisely to avoid these complications, since they draw attention away from the essential physics.

This raindrop problem is a classic, first invented (though with a slightly different mass accretion rule) for the infamous Cambridge Smith's Prize Examination in 1853. Many papers have been written about it, such as this one by K. S. Krane, who is the K in HRK.

[3] **Problem 26** (Kvant)**.** Half of a flexible pearl necklace lies on a horizontal frictionless table, while the other half hangs down vertically at the edge. If the necklace is released from rest, it will slide off the table. At some point, the hanging part of the necklace will begin to whip back and forth. What fraction of the necklace is on the table when this begins? (Hint: we are considering a pearl necklace with no empty string between adjacent pearls; as a result, all the pearls accelerate smoothly. To solve the problem, think about the vertical forces. There is an important related problem in **M2**.)

Solution. Physically, what happens is that a sizable force is needed to turn the pearls around when they reach the corner, to go from moving horizontally to vertically. At a certain critical velocity, tension will no longer be enough to do this, and the pearl necklace will jump off the corner. This will lead to the hanging part whipping back and forth.

To see when this happens, consider the vertical forces on the pearl necklace. Suppose the necklace has mass M and a fraction x is hanging off the table. As we saw with a similar but static problem in **M2**, the normal force on the horizontal part of the table has to cancel the force of gravity acting on the part of the necklace on the table. This is because otherwise a piece of the horizontal part of the necklace would have an unbalanced vertical force, and would have to go into the table or jump off it, neither of which make sense.

Thus, considering the vertical forces just gives

$$\frac{dp_y}{dt} = xMg - N_{c,y}$$

where we take the downward direction as positive for convenience, and $N_{c,y}$ is the vertical part of the normal force acting at the corner of the table. (Recall from **M2** that this is significant even when the corner is small.) Since the pearl necklace is flexible, the process is elastic, so energy is conserved. Our strategy is to use energy conservation to find dp_y/dt and use that to find $N_{c,y}$. The necklace jumps off the corner when $N_{c,y}$ becomes zero.

Taking the necklace to have length L, energy conservation gives

$$\frac{1}{2}Mv^2 = MxgL\frac{x}{2} - \frac{MgL}{8} \implies v^2 = gL(x^2 - 1/4) \implies \dot{x} = \sqrt{g/L}\sqrt{x^2 - 1/4}.$$

The acceleration is

$$\ddot{x} = \sqrt{g/L}\frac{1}{2\sqrt{x^2 - 1/4}} \cdot 2x\dot{x} = \sqrt{g/L}\frac{x}{\sqrt{x^2 - 1/4}}\sqrt{g/L}\sqrt{x^2 - 1/4} = (g/L)x$$

which we also found using the generalized coordinate approach explained in problem 16. Thus,

$$\frac{dp_y}{dt} = ML\frac{d}{dt}(x\dot{x}) = ML(x\ddot{x} + \dot{x}^2) = ML((g/L)x^2 + (g/L)(x^2 - 1/4)) = Mg(2x^2 - 1/4).$$

Setting $N_{c,y}$ to zero gives

$$2x^2 - 1/4 = x$$

for which the relevant root is $x = (1 + \sqrt{3})/4$. At this point, the fraction on the table is

$$1 - x = \frac{3 - \sqrt{3}}{4}.$$

This is barely less than $1/2$, so the necklace jumps almost immediately!

In general, ideal ropes like pearl necklaces, which aren't inherently inelastic at all, do unintuitive things like this. An even subtler, related phenomenon is the chain fountain, where a chain of beads can seemingly pull itself out of a jar.

[4] **Problem 27** (BAUPC 2002)**.** A small ball is attached to a massless string of length L, the other end of which is attached to a very thin pole. The ball is thrown so that it initially travels in a horizontal circle, with the string making an angle θ_0 with the vertical. As time goes on, the string wraps itself around the pole. Assume that (1) the pole is thin enough so that the length of string in the air decreases very slowly, and (2) the pole has enough friction so that the string does not slide on the pole, once it touches it. Show that the ratio of the ball's final speed (right before it hits the pole) to initial speed is $\sin\theta_0$.

Solution. The official solution is here, but it's a lot longer because it also solves for the evolution of the height of the ball; in this shorter version of the problem, we only want the final speed, which is a lot easier to find.

In this problem, energy is conserved because the string doesn't slide, but angular momentum isn't conserved. Now note that if θ were constant, the ball would move at a constant height. Changes in θ as the string winds up change the ball's height, which then changes its speed by energy conservation. By doing some geometry, you can show

$$mv\,dv = -mg\ell\sin\theta\,d\theta.$$

The radial $F = ma$ equation is

$$mg\tan\theta = \frac{mv^2}{\ell\sin\theta}.$$

Substituting this into the equation above gives

$$-mg\,\frac{mv^2}{mg\tan\theta}\,d\theta = mv\,dv$$

which upon separation gives the simple result

$$-\int_{\theta_0}^{\pi/2}\cot\theta\,d\theta = \int_{v_i}^{v_f}\frac{dv}{v}.$$

Therefore, we have

$$\log\frac{\sin\theta_0}{\sin\pi/2} = \log\frac{v_f}{v_i}$$

which gives the desired result, $v_f/v_i = \sin\theta_0$.

When dealing with an extended system whose parts all move in different ways, conservation of energy is occasionally useless. However, the somewhat obscure idea of "center of mass energy" may become useful instead. For more about this concept, see section 13.5 of Halliday and Resnick.

Idea 7: Center of Mass Energy

The work done on a part of a system is

$$dW = F\,dx$$

where F is the force on that specific part of the system, and dx is its displacement. Then $dW = dE$ where E is the total energy of the system.

Similarly, the "center of mass work" done on a system is

$$dW_{\text{cm}} = F\,dx_{\text{cm}}$$

where F is the total force on the system and dx_{cm} is the displacement of the center of mass. Then $dW_{\text{cm}} = dE_{\text{cm}}$ where the "center of mass energy" is defined as $E_{\text{cm}} = Mv_{\text{cm}}^2/2$.

It should be noted that, like regular energy and work, center of mass energy and work depend on the reference frame you're using.

Example 11

Consider a cyclist who pedals their bike to accelerate. The wheels roll without slipping on the ground. The cyclist moves a distance d, with the bike experiencing a constant friction force f from the ground. Analyze the situation using both energy and center of mass energy.

Solution

Since the wheels roll without slipping, their contact point with the ground is always zero, so the friction force does exactly zero work. Thus the net energy of the cyclist/bike system is conserved. The additional kinetic energy of the cyclist/bike comes from the chemical energy of the cyclist, which ultimately came from what they ate. So conservation of energy is correct, but it doesn't tell us anything useful at all.

Now consider center of mass energy. Considering the cyclist/bike system, the center of mass work is fd, which is the change in $Mv_{\text{cm}}^2/2$. This allows us to compute the change in velocity of the cyclist/bike.

Example 12

Consider the same setup as in the previous example, but now the cyclist brakes hard. The wheels slip on the ground, and experience a friction force $-f$ while the cyclist moves a distance d. Analyze the situation using both energy and center of mass energy.

Solution

The center of mass work equation tells us about the overall deceleration of the cyclist/bike, just as in the previous example.

On the other hand, the work done by the friction force is indeterminate! It can be any quantity between zero and $-fd$. When it is 0, the total energy of the cyclist/bike system is again conserved, which means all the kinetic energy lost is dissipated as heat inside the bike itself. When it is $-fd$, all the kinetic energy lost is dissipated as heat in the *ground*, and hence energy is removed from the cyclist/bike system. In general, the work will be an intermediate value, meaning that both the ground and the bike heat up, but we can't calculate what it is without a microscopic model of how the friction works. It depends on, e.g. how easily the ground and bike tire surface deform.

[1] **Problem 28.** Alice and Bob stand facing each other with their arms bent and hands touching on an ice skating rink. Bob has his back against a wall.

(a) Suppose Bob extends his arms, pushing Alice through a distance d with a force F. Analyze what happens to Alice in terms of both work and center of mass work.

(b) Suppose Alice extends her arms, pushing herself through a distance d with a force F. Repeat the analysis; what is different and what is the same?

(c) Suppose a spherical balloon is compressed uniformly from all sides. Is there work done on the balloon? How about center of mass work?

Solution. (a) The center of mass work and work done on Alice are both Fd, so she moves with speed $v = \sqrt{2Fd/m}$. In this situation Alice effectively behaves like a particle, so the two notions are the same.

(b) The center of mass work on Alice is again Fd, so her final speed is the same. But the work done on her is 0 since the contact point did not move. Accordingly, Alice's total energy did not change; she merely converted some of her internal energy to kinetic energy.

(c) There is no center of mass work on the balloon, but there is work done, at every point on the balloon's surface. This work is just the $P\,dV$ work in thermodynamics.

[4] **Problem 29.** USAPhO 2013, problem B1. This problem is quite tricky! Once you're done, carefully read the official solution, which describes how center of mass work is applied.

Solution. As usual, see the official solutions. Recently, this Veritasium video reignited the debate over the Blackbird, leading to this followup video, which cites the USAPhO solution. If you're still confused about how the Blackbird works, I recommend watching the second video, which shows various arguments and a mechanical model.

Mechanics IV: Oscillations

Chapter 4 of Morin covers oscillations, as does chapter 10 of Kleppner and Kolenkow, and chapter 10 of Wang and Ricardo, volume 1. For a deeper treatment that covers normal modes in more detail, see chapters 1 through 6 of French. Jaan Kalda also has short articles on using Lagrangian-like techniques and the adiabatic theorem. For some fun discussion, see chapters I-21 through I-25, II-19, and II-38 of the Feynman lectures. There is a total of 88 points.

1 Small Oscillations

Idea 1

If an object obeys a linear force law, then its motion is simple harmonic. To compute the frequency, one must the restoring force per unit displacement. More generally, if the force an object experiences can be expanded in a Taylor series with a nonzero linear restoring term, the motion is approximately simple harmonic for small displacements. (However, don't forget that there are also situations where oscillations are not even approximately simple harmonic, no matter how small the displacements are.)

Example 1: KK 4.13

The Lennard–Jones potential

$$U(r) = \epsilon \left(\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^{6} \right)$$

is commonly used to describe the interaction between two atoms. Find the equilibrium radius and the angular frequency of small oscillations about this point for two identical atoms of mass m bound to each other by the Lennard–Jones interaction.

Solution

To keep the notation simple, we'll set $\epsilon = r_0 = 1$ and restore them later. The equilibrium radius is the radius where the derivative of the potential vanishes, and

$$U'(r) = -12r^{-13} + 12r^{-7} = 0$$

implies that that the equilibrium radius is $r = r_0$. Because the force accelerates both of the atoms, the angular frequency is

$$\omega = \sqrt{\frac{U''(r)}{m/2}}$$

where $m/2$ is the so-called reduced mass. At the equilibrium point, we have

$$U''(r_0) = (12)(13)r_0^{-14} - (12)(7)r_0^{-8} = 72.$$

Restoring the dimensionful factors, we have $U''(r_0) = 72\epsilon/r_0^2$, so

$$\omega = \frac{12}{r_0}\sqrt{\frac{\epsilon}{m}}.$$

[3] **Problem 1** (Morin 5.13)**.** A hole of radius R is cut out from an infinite flat sheet with mass per unit area σ. Let L be the line that is perpendicular to the sheet and that passes through the center of the hole.

(a) What is the force on a mass m that is located on L, a distance x from the center of the hole? (Hint: consider the plane to consist of many concentric rings.)

(b) Now suppose the particle is released from rest at this position. If $x \ll R$, find the approximate angular frequency of the subsequent oscillations.

(c) Now suppose that $x \gg R$ instead. Find the period of the resulting oscillations.

(d) Now suppose the mass begins at rest on the plane, but slightly displaced from the center. Do oscillations occur? If so, what is the approximate frequency?

[2] **Problem 2.** Some questions about small oscillations with the buoyant force.

(a) A cubical glacier of side length L has density ρ_i and floats in water with density ρ_w. Find the angular frequency of small oscillations, assuming that a face of the glacier always remains parallel to the water surface, and that the force of the water on the glacier is always given by the hydrostatic buoyant force.

(b) A ball of radius R floats in water with half its volume submerged. Find the angular frequency of small oscillations, making the same assumption.

(c) There are important effects that both of the previous parts neglect. What are some of them? Is the true oscillation frequency higher or lower than the one found here?

[3] **Problem 3.** 🕓 USAPhO 1998, problem A2. To avoid some confusion, skip part (a), since there actually isn't a nice closed-form expression for it.

[3] **Problem 4.** 🕓 USAPhO 2009, problem A3.

[3] **Problem 5.** 🕘 USAPhO 2010, problem B1.

Example 2

Find the acceleration of an Atwood's machine with masses m and M and a massless pulley and string.

Solution

The standard way to do this is to let a_1 and a_2 be the accelerations of the masses, let T be the unknown tension in the string, solve for T by setting a_1 and a_2 to have equal magnitudes, then plug T back in to find the common acceleration. This procedusre is unnecessarily complicated, because we are using two coordinates when the string really ensures the system has only a single degree of freedom.

We can alternatively use energy conservation in terms of "generalized coordinates". Let q be a coordinate which describes "how much the string has moved along itself". In other words,

$q = 0$ initially, and when $q = q_0$, the mass M has moved down by q_0 and the mass m has moved up by q_0. The kinetic and potential energies of the system are simply

$$K = \frac{1}{2}(m+M)\dot{q}^2, \quad V = qg(m-M).$$

To find the acceleration $\ddot{q}$, we differentiate energy conservation with respect to time,

$$0 = \frac{d(K+V)}{dt} = (m+M)\ddot{q}\dot{q} + \dot{q}g(m-M).$$

Solving gives the familiar result

$$\ddot{q} = \frac{M-m}{M+m}\, g.$$

Intuitively, we could say that from the standpoint of this generalized coordinate, the "total force" is $(M-m)g$, and the "total inertia" is $M+m$.

Idea 2

The idea shown in example 2 is very general. Consider *any* system whose configuration can be described by a single "generalized coordinate" q. If its energy can be decomposed into a kinetic energy that depends only on $\dot{q}$ and a potential energy that depends only on q,

$$K = K(\dot{q}), \quad V = V(q)$$

then the energy conservation equation $d(K+V)/dt = 0$ can be used to find the generalized acceleration $\ddot{q}$. Explicitly, the chain rule tells us that

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}} = -\frac{\partial V}{\partial q}.$$

As a simple example, for a particle moving in one dimension, taking $q = x$ gives

$$\frac{d}{dt}(m\dot{x}) = -\frac{dV}{dx}$$

which is the usual Newton's second law. So in general, we will call $\partial K/\partial \dot{q}$ a "generalized momentum", and $-\partial V/\partial q$ a "generalized force". Note that static equilibrium can occur when $\partial V/\partial q = 0$, which is just the principle of virtual work from **M2**.

Remark

The result above is a special case of the Euler–Lagrange equation in Lagrangian mechanics, which states that if a system is described by a Lagrangian L, then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

When $L = K(\dot{q}) - V(q)$, we recover the previous result. But more generally, it might not be possible to meaningfully decompose L into a "kinetic" and "potential" piece at all! We won't use this more general form below. While it is more powerful, it is also more complicated, and if you find yourself using it for an Olympiad problem, there's probably an easier way.

[1] **Problem 6.** A rope is nestled inside a curved frictionless tube. The rope has a total length ℓ and uniform mass per length λ. The shape of the tube can be arbitrarily complicated, but the left end of the rope is higher than the right end by a height h. If the rope is released from rest, find its acceleration. (For a related question, see $F = ma$ 2019 B24.)

Idea 3

Generalized coordinates are really useful for problems that involve complicated objects but only have one relevant degree of freedom, which is especially true for oscillations problems. For instance, if the kinetic and potential energy have the form

$$K = \frac{1}{2} m_{\text{eff}} \dot{q}^2, \quad V = \frac{1}{2} k_{\text{eff}} q^2$$

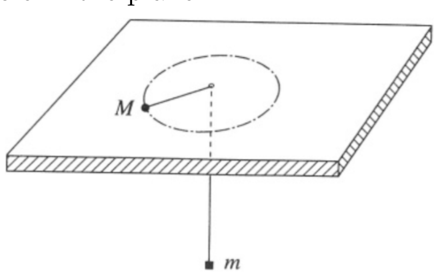
then the oscillation's angular frequency is always

$$\omega = \sqrt{k_{\text{eff}}/m_{\text{eff}}}.$$

Note that q need not have units of position, m_{eff} need not have units of mass, and so on. When $V(q)$ is a more general function, we can expand it about a minimum q_{min}, so that $k_{\text{eff}} = V''(q_{\text{min}})$. This lets us avoid dealing with possibly complicated constraint forces.

[3] **Problem 7.** Suppose a particle is constrained to move on a curve $y(x)$ with a minimum at $x = 0$. We know that if $y(x)$ is a circular arc, then the motion is not exactly simple harmonic, for the same reason that pendulum motion is not. Find a differential equation relating y' and y, so that the motion is exactly simple harmonic for arbitrary amplitudes; you don't have to solve it. (Hint: work in terms of the coordinate s, the arc length along the curve.)

[3] **Problem 8** (Cahn)**.** A particle of mass M is constrained to move on a horizontal plane. A second particle of mass m is constrained to a vertical line. The two particles are connected by a massless string which passes through a hole in the plane.



The motion is frictionless. Show that the motion is stable with respect to small changes in the height of m, and find the angular frequency of small oscillations.

[4] **Problem 9.** IPhO 1984, problem 2. If you use the energy methods above, you won't actually need to know anything about fluid mechanics to do this nice, short problem!

2 Springs and Pendulums

Now we'll consider more general problems involving springs and pendulums, two very common components in mechanics questions. As a first example, we'll use the fictitious forces met in **M2**.

Example 3: PPP 79

A pendulum of length ℓ and mass m initially hangs straight downward in a train. The train begins to move with uniform acceleration a. If a is small, what is the period of small oscillations? If a can be large, is it possible for the pendulum to loop over its pivot?

Solution

The fictitious force in the train's frame due to the acceleration is equivalent to an additional, horizontal gravitational field, so the effective gravity is

$$\mathbf{g}_{\text{eff}} = -a\hat{\mathbf{x}} - g\hat{\mathbf{y}}.$$

For small oscillations, we know the period is $2\pi\sqrt{L/g}$ in ordinary circumstances. By precisely the same logic, it must be replaced with

$$T = 2\pi\sqrt{\frac{L}{g_{\text{eff}}}} = 2\pi\frac{\sqrt{L}}{(g^2 + a^2)^{1/4}}.$$

As a gets larger, the effective gravity points closer to the horizontal. In the limit $g/a \to 0$, the effective gravity is just horizontal, so the pendulum oscillates about the horizontal. Its endpoints are the downward and upward directions, so it never can get past the pivot.

Here's a follow-up question: if the train can decelerate quickly, how should you stop it so that the pendulum doesn't end up swinging at the end? The most efficient way is to first quickly decelerate to half speed, which, in the frame of the train, provides a horizontal impulse to the pendulum. Then wait a half-period $\pi\sqrt{L/g}$, so that the pendulum's momentum turns around, and then quickly stop, providing a second impulse that precisely cancels the pendulum's horizontal motion. Tricks like this are used by crane operators to transport loads, and by physicists to transport clouds of ultracold atoms without warming them up.

Example 4

If a spring with spring constant k_1 and relaxed length ℓ_1 is combined with a spring with spring constant k_2 and relaxed length ℓ_2, find the spring constant and relaxed length of the combined spring, if the combination is in series or in parallel.

Solution

For the series combination, the new relaxed length is clearly $\ell = \ell_1 + \ell_2$. Suppose the first spring is stretched by x_1 and the second by x_2. The tensions in the springs must balance,

$$F = k_1 x_1 = k_2 x_2.$$

Thus, the new spring constant is

$$k = \frac{F}{x_1 + x_2} = \frac{k_2 x_2}{x_2(k_2/k_1 + 1)} = \frac{k_1 k_2}{k_1 + k_2}.$$

For example, if the spring is cut in half, the pieces have spring constant $2k$.

Now consider the parallel combination. In this case it's clear that the new spring constant is $k = k_1 + k_2$, since the tensions of the springs add. The new relaxed length ℓ is when the forces in the springs cancel out, so

$$k_1(\ell - \ell_1) + k_2(\ell - \ell_2) = 0$$

which implies

$$\ell = \frac{k_1 \ell_1 + k_2 \ell_2}{k_1 + k_2}.$$

[2] **Problem 10** (Morin 4.20)**.** A mass m is attached to n springs with relaxed lengths of zero. The spring constants are $k_1, k_2, \ldots, k_n$. The mass initially sits at its equilibrium position and then is given a kick in an arbitrary direction. Describe the resulting motion.

[3] **Problem 11** (Morin 4.22)**.** A spring with relaxed length zero and spring constant k is attached to the ground. A projectile of mass m is attached to the other end of the spring. The projectile is then picked up and thrown with velocity v at an angle θ to the horizontal.

(a) Geometrically, what kind of curve is the resulting trajectory?

(b) Find the value of v so that the projectile hits the ground traveling straight downward.

[5] **Problem 12.** A uniform spring of spring constant k and total mass m is attached to the wall, and the other end is attached to a mass M.

(a) Show that when $m \ll M$, the oscillation's angular frequency is approximately

$$\omega = \sqrt{\frac{k}{M + m/3}}.$$

(b) **[A]** $\star$ Generalize part (a) to arbitrary values of m/M. (Hint: to begin, approximate the massive spring as a finite combination of smaller massless springs and point masses, as in the example in **M2**. It will not be possible to solve for ω in closed form, but you can get a compact implicit expression for it. Check that it reduces to the result of part (a) for small m/M, and interpret the results for large m/M. This is a challenging problem that requires almost all the techniques we've seen so far; you might want to return to it after doing section 4.)

[2] **Problem 13** (PPP 77)**.** A small bob of mass m is attached to two light, unstretched, identical springs. The springs are anchored at their far ends and arranged along a straight line. If the bob is displaced in a direction perpendicular to the line of the springs by a small length ℓ, the period of oscillation of the bob is T. Find the period if the bob is displaced by length 2ℓ.

[3] **Problem 14.** USAPhO 2015, problem A3.

[3] **Problem 15.** USAPhO 2008, problem B1.

Example 5

About how accurately can you measure g with a simple pendulum?

Solution

This simple question illustrates how rich experimental physics can be, even in elementary settings. First, let's think about the uncertainties in the pendulum's length and period.

- Length: a reasonable length for an experiment is $L \sim 1\,\text{m}$. We should use a wire, not a string, to avoid stretching. If you measure the wire with a good ruler, you can get down to $\Delta L \sim 1\,\text{mm}$. If you use calipers, you can get $\Delta L \sim 0.1\,\text{mm}$. Assuming the latter gives a fractional uncertainty $\Delta L/L \sim 10^{-4}$.
- Period: if the length is a meter, the period will be $T \simeq 2\,\text{s}$. (This isn't a coincidence! 17^{th} century scientists defined the meter precisely so this would be true.) One might estimate the timing uncertainty to be given by human reaction speed, $\Delta T \sim 250\,\text{ms}$, but this is too pessimistic, because you can see the pendulum coming. An extensive study of manual timing at swimming competitions found a typical spread $\Delta T \sim 70\,\text{ms}$. Moreover, since a pendulum's motion is regular, you can "lock in" with your sense of rhythm to do even better than this. Finally, we can let the pendulum swing for $N = 100$ consecutive periods and measure the total time. These improvements allow a timing uncertainty $\Delta T/(NT) \sim 10^{-4}$.

Combining these results with the error propagation rules of **P2**, we can estimate $\Delta g/g \sim 10^{-4}$ for a well-performed experiment. But any real experiment also has to contend with systematic effects which can bias the results. Let's consider and estimate a couple of them.

- The bob has finite size, so the pendulum is really a physical pendulum. We can estimate the size of this effect by thinking about how much the bob's size changes the pendulum's moment of inertia. If the bob has radius $r \sim 1\,\text{cm}$, the change is roughly $r^2/L^2 \sim 10^{-4}$.
- The wire isn't massless, so the effective length of the pendulum is less than L. If we use a lead bob whose mass is a few kilograms, and the wire is a thin steel wire whose mass is a few grams, the effect is roughly $m_{\text{wire}}/m_{\text{bob}} \sim 10^{-3}$.
- The motion has finite amplitude θ_0. As we saw in **P1**, this changes the period fractionally by $\theta_0^2/16$, and for an amplitude of a few degrees this is $\sim 10^{-3}$.
- The pendulum oscillates in air. This leads to two distinct effects: the buoyant force on the bob decreases the effective value of g, and the "virtual mass" effect, discussed further in **M7**, increases the bob's effective inertia. These effects shift the period in the same direction, and they are both of order $\rho_{\text{air}}/\rho_{\text{bob}} \sim (1\,\text{kg/m}^3)/(10^4\,\text{kg/m}^3) \sim 10^{-4}$.
- The Earth is rotating, leading to centrifugal and Coriolis forces. The latter turns out to be unimportant; as shown in **M6**, it rotates the pendulum's plane of oscillation, rather than shifting its period. Unless you're conducting the experiment in Greenland or Antarctica, the centrifugal force produces a shift of order $\omega_E^2 R_E/g \sim 10^{-3}$.

- The pendulum's motion is slightly damped, which lengthens the oscillation period. This factor depends on how frictionless the support is. However, if it was set up so that 100 consecutive periods can be measured, one must have quality factor $Q \gtrsim 10^3$. One can show that the fractional shift in frequency is $\sim 1/Q^2 \sim 10^{-6}$.

There are plenty of other factors, but these are the most important ones, and a few of them are larger than the uncertainty from the length and period. But the good thing is that all of them can be calculated, and thereby subtracted out, leading to an ultimate final precision of $\Delta g/g \sim 10^{-4}$. That is indeed the best precision achieved during the 1800s, through extensive effort. For real measurements and further details, see this paper.

3 Damped and Driven Oscillations

We now review damped oscillators, which we saw in **M1**, and consider driven oscillators. For more guidance, see sections 4.3 and 4.4 of Morin.

[2] **Problem 16.** Consider a damped harmonic oscillator, which experiences force $F = -bv - kx$.

(a) As in **M1**, show that the general solution for $x(t)$ is

$$x(t) = A_+ e^{i\omega_+ t} + A_- e^{-i\omega_- t}$$

and solve for the $\omega_\pm$.

(b) For sufficiently small b, the roots are complex. In this limit, show that by taking the real part, one finds an exponentially damped sinusoidal oscillation. Roughly how many oscillation cycles happen when the amplitude damps by a factor of e?

(c) For large b, the roots are pure imaginary, the position simply decays exponentially, and we say the system is overdamped. Find the condition for the system to be overdamped.

[4] **Problem 17.** Analyzing a damped and driven harmonic oscillator.

(a) Consider a damped harmonic oscillator which experiences a driving force $F = F_0 \cos(\omega t)$. Passing to complex variables, Newton's second law is

$$m\ddot{x} + b\dot{x} + kx = F_0 e^{i\omega t}.$$

If $x(t)$ is a complex exponential, then we know that the left-hand side is still a complex exponential, with the same frequency. This motivates us to guess $x(t) = A_0 e^{i\omega t}$. Show that this solves the equation for some A_0.

(b) Of course, the general solution needs to be described by two free parameters, to match the initial position and velocity. Argue that it takes the form

$$x(t) = A_0 e^{i\omega t} + A_+ e^{i\omega_+ t} + A_- e^{-i\omega_- t}$$

where the $\omega_\pm$ are the ones you found in problem 16.

(c) After a long time, the "transient" $A_\pm$ terms will decay away, leaving the steady state solution

$$x(t) \approx A_0 e^{i\omega t}$$

which oscillates at the same frequency as the driving. The actual position is the real part,

$$x(t) \approx |A_0| \cos(\omega t - \phi)$$

where $A_0 = |A_0| e^{-i\phi}$. Evaluate $|A_0|$ and ϕ.

(d) Sketch the amplitude $|A_0|$ and phase shift ϕ as a function of ω. Can you intuitively see they take the values they do, for ω small, $\omega \approx \sqrt{k/m}$, and ω large?

(e) There are several distinct things people mean when they speak of "resonant frequencies". Find the driving angular frequency ω that maximizes (i) the amplitude $|A_0|$, (ii) the amplitude of the velocity, and (iii) the average power absorbed from the driving force. (As you'll see, these are all about the same when the damping is weak, so the distinction between these isn't so important in practice.)

[3] **Problem 18.** The quality factor of a damped oscillator is defined as $Q = m\omega_0/b$, where $\omega_0 = \sqrt{k/m}$. It measures both how weak the damping is, and how sharp the resonance is.

(a) Show that for a lightly damped oscillator,

$$Q \approx \frac{\text{average energy stored in the oscillator}}{\text{average energy dissipated per radian}}.$$

Then estimate Q for a guitar string.

(b) Show that for a lightly damped oscillator,

$$Q \approx \frac{\text{resonant frequency}}{\text{width of resonance curve}}$$

where the width of the resonance curve is defined to be the range of driving frequencies for which the amplitude is at least $1/\sqrt{2}$ the maximum.

For more about Q, see pages 424 through 428 of Kleppner and Kolenkow.

The next two problems explore other ways of driving harmonic oscillators.

[2] **Problem 19.** Consider a pendulum which can perform small-angle oscillations in a plane with natural frequency f. The pendulum bob is attached to a string, and you hold the other end of the string in your hand. There are three simple ways to drive the pendulum:

(a) Move the end of the string horizontally with sinusoidal frequency f'.

(b) Move the end of the string vertically with sinusoidal frequency f'.

(c) Apply a quick rightward impulse to the bob with frequency f'.

In each case, for what value(s) of f' can the amplitude become large? (This question should be done purely conceptually; don't write any equations, just visualize it!)

[5] **Problem 20.** GPhO 2016, problem 1. Record your answers on the official answer sheet.

4 Normal Modes

Idea 4: Normal Modes

A system with N degrees of freedom has N normal modes when displaced from equilibrium. In a normal mode, the positions of the particles are of the form $x_i(t) = A_i \cos(\omega t + \phi_i)$. That is, all particles oscillate with the same frequency. Normal modes can be either guessed physically, or found using linear algebra as explained in section 4.5 of Morin.

The general motion of the system is a superposition of these normal modes. So to compute the time evolution of the system, it's useful to decompose the initial conditions into normal modes, because they all evolve independently by linearity.

Example 6

Two blocks of mass m are connected with a spring of spring constant k and relaxed length L. Initially, the blocks are at rest at positions $x_1(0) = 0$ and $x_2(0) = L$. At time $t = 0$, the block on the right is hit, giving it a velocity v_0. Find $x_1(t)$ and $x_2(t)$.

Solution

The equations of motion are

$$m\ddot{x}_1 = k(x_2 - x_1 - L)$$
$$m\ddot{x}_2 = k(x_1 + L - x_2).$$

The system must have two normal modes. The obvious one is when the two masses oscillate oppositely, $x_1 = -x_2$. The other one is when the two masses move parallel to each other, $x_1 = x_2$, and this normal mode formally has zero frequency. The initial condition is the superposition of these two modes.

We can show this a bit more formally. Define the normal mode amplitudes u and v as

$$x_1 = \frac{u - v}{2}, \quad x_2 = \frac{u + v}{2}.$$

Solving for u and v, we find

$$u = x_1 + x_2, \quad v = x_2 - x_1.$$

Using the equations of motion for x_1 and x_2, we have the equations of motion

$$\ddot{u} = 0, \quad m\ddot{v} = -2k(v - L)$$

which just verifies that the normal modes are independent, with angular frequency zero and $\omega = \sqrt{2k/m}$ respectively. We can fit the initial condition if

$$u(0) = L, \quad v(0) = L, \quad \dot{u}(0) = v_0, \quad \dot{v}(0) = v_0.$$

The normal mode amplitudes are then

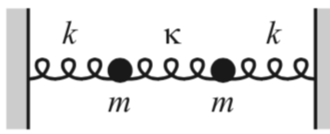
$$u(t) = L + v_0 t, \quad v(t) = L + \frac{v_0}{\omega} \sin \omega t.$$

Plugging this back in gives

$$x_1(t) = \frac{v_0 t}{2} - \frac{v_0}{2\omega}\sin\omega t, \quad x_2(t) = L + \frac{v_0 t}{2} + \frac{v_0}{2\omega}\sin\omega t.$$

Each mass is momentarily stationary at time intervals of $2\pi/\omega$, though neither mass ever moves backwards. If you didn't know about normal modes, you could also arrive at this conclusion by playing around with the equations; you could see that they decouple when you add and subtract them, for instance.

[3] **Problem 21** (Morin 4.10). Three springs and two equal masses lie between two walls, as shown.

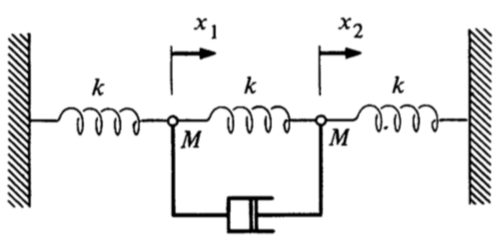


The spring constant k of the two outside springs is much larger than the spring constant $\kappa \ll k$ of the middle spring. Let x_1 and x_2 be the positions of the left and right masses, respectively, relative to their equilibrium positions. If the initial positions are given by $x_1(0) = a$ and $x_2(0) = 0$, and if both masses are released from rest, show that

$$x_1(t) \approx a\cos((\omega+\epsilon)t)\cos(\epsilon t), \quad x_2(t) \approx a\sin((\omega+\epsilon)t)\sin(\epsilon t)$$

where $\omega = \sqrt{k/m}$ and $\epsilon = (\kappa/2k)\omega$. Explain qualitatively what the motion looks like. This is an example of beats, which result from superposition two oscillations of nearly equal frequencies; we will see more about them in **W1**.

[3] **Problem 22** (KK 10.11). Two identical particles are hung between three identical springs.



Neglect gravity. The masses are connected as shown to a dashpot which exerts a force bv, where v is the relative velocity of its two ends, which opposes the motion.

(a) Find the equations of motion for x_1 and x_2.

(b) Show that the equations of motion can be solved in terms of the variables $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$.

(c) Show that if the masses are initially at rest and mass 1 is given an initial velocity v_0, the motion of the masses after a sufficiently long time is

$$x_1(t) = x_2(t) = \frac{v_0}{2\omega}\sin\omega t$$

and evaluate ω.

Example 7

Three identical masses are connected by three identical springs, forming an equilateral triangle in equilibrium. Describe the normal modes of the system.

Solution

Let the system be confined to the xy plane. Then there are three masses that each can move in two dimensions, giving six degrees of freedom. Since we must be able to construct the general solution by superposing normal modes, there should be six normal modes. They are:

- Uniform translation. This yields two independent normal modes, as you can superpose motion in any two distinct directions (e.g. along the x and y axes) to get motion in any direction. These modes have zero frequency, since $\sin(\omega t) \propto t$ in the limit $\omega \to 0$.
- Uniform rotation about the axis of symmetry.
- A "breathing" motion where the whole triangle expands and contracts.
- A "scissoring" motion where one mass moves outward and the other two move inward. You might think there are three scissoring normal modes, but they are redundant: just like how the three sides of the equilateral triangle lie in a plane, these three normal modes formally lie in a plane, in the sense that you can superpose any two of them to get the third. So there are two independent scissoring modes.

Thus we have six normal modes, as expected. If the system can move in three-dimension space, we need three more; they are uniform translation in the z direction, and rotation about the x and y axes.

[5] **Problem 23** (Morin 4.12, IPhO 1986)**.** N identical masses m are constrained to move on a horizontal circular hoop connected by N identical springs with spring constant k. The setup for $N = 3$ is shown below.

(a) Find the normal modes and their angular frequencies for $N = 2$.

(b) Do the same for $N = 3$.

(c) **[A]** $\star$ Do the same for general N. (Hint: consider the normal modes found in (a) and (b), arranged so that in each normal mode, each mass oscillates with unit amplitude but a different phase. Look at the phases and guess a pattern.)

(d) If one of the masses is replaced with a mass $m' \ll m$, qualitatively describe how the set of frequencies changes.

(e) Now suppose the masses alternate between m and $m' \ll m$. Qualitatively describe the set of frequencies.

Part (c) will be useful in **X1**, where we will quantize the normal modes found here.

[4] **Problem 24.** [A] In this problem, you will analyze the normal modes of the double pendulum, which consists of a pendulum of length ℓ and mass m attached to the bottom of another pendulum, of length ℓ and mass m. To solve this problem directly, one has to compute the tension forces in the two strings, which are quite complicated. A much easier method is to use energy.

(a) Parametrize the position of the pendulum in terms of the angle θ_1 the top string makes with the vertical, and the angle θ_2 the bottom string makes with the vertical. Write out the kinetic energy K and the potential energy V to second order in the θ_i and $\dot{\theta}_i$.

(b) The Euler–Lagrange equations for the system are

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\theta}_i} = -\frac{\partial V}{\partial \theta_i}.$$

Using the results of part (a), write these equations in the form

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\frac{g}{L}\, A \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

where A is a 2×2 matrix. This is a generalization of $\ddot{\theta} = -g\theta/L$ for a single pendulum.

(c) Find the normal modes and their angular frequencies, using the general method in section 4.5 of Morin.

Remark

We mostly considered examples with two or three masses, but the techniques above work for systems with arbitrarily many degrees of freedom. However, this quickly becomes intractable unless the setup is highly symmetric, as in problem 23. Without such symmetry, a computer is generally necessary, so this sort of question won't appear on standard Olympiads. However, if you're curious, see ITPO 2016, problem 1 and Physics Cup 2021, problem 3 for examples.

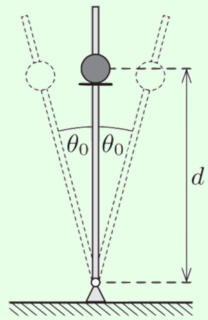
5 [A] Adiabatic Change

Idea 5

When a problem contains two widely separate timescales, such as a fast oscillation superposed on a slow overall motion, one can solve for the fast motion while neglecting the slow motion, then solve for the slow motion by replacing the fast motion with an appropriate average.

Example 8: MPPP 21

A small smooth pearl is threaded onto a rigid, smooth, vertical rod, which is pivoted at its base. Initially, the pearl rests on a small circular disc that is concentric with the rod, and attached to it a distance d from the rotational axis. The rod starts executing simple harmonic motion around its original position with small angular amplitude θ_0.



What angular frequency of oscillation is required for the pearl to leave the rod?

Solution

The reason the pearl leaves the rod is that the normal force rapidly varies in direction, with an average upward component. If this average upward force is greater than gravity, the pearl accelerates upward and leaves the rod.

In this case, the fast motion is the oscillation of the rod, while the slow motion is the rate of change of the pearl's distance from the pivot, which can be neglected during one oscillation. The pearl has horizontal displacement and acceleration

$$x(t) = -d\sin\theta \approx -d\theta(t) = -\theta_0 d\sin\omega t, \quad a_x(t) = \theta\omega^2 d\sin\omega t.$$

This is supplied by the horizontal component of the normal force. The vertical component is

$$N_y = N_x \tan\theta(t) \approx m a_x(t)\theta(t) = m\theta_0^2\omega^2 d\sin^2\omega t.$$

Now we average over the fast motion to understand the slow motion. Since the average value of $\sin^2(\omega t)$ is $1/2$, the condition for the pearl to go up is

$$\frac{1}{2}m\theta_0^2\omega^2 d > mg$$

which gives

$$\omega > \frac{1}{\theta_0}\sqrt{\frac{2g}{d}}.$$

Example 9

A mass m oscillates on a spring with spring constant k_0 with amplitude A_0. Over a very long period of time, the spring smoothly and continuously weakens until its spring constant is $k_0/2$. Find the new amplitude of oscillation.

Solution

In this case the fast motion is the oscillation of the mass, while the slow motion is the weakening of the spring. We can solve the problem by considering how the energy changes in each oscillation, due to the slight decrease in k.

Suppose that the spring constant drops in one instant by a factor of $1-\epsilon$. Then the kinetic energy stays the same, while the potential energy drops by a factor of $1-\epsilon$. Since the kinetic and potential energy are equal on average, this means that if the spring constant gradually decreases by a factor of $1-x$ over a full cycle, with $x \ll 1$, then the energy decreases by a factor of $1-x/2$.

The process finishes after N oscillations, where $(1-x)^N \approx e^{-Nx} = 1/2$. At this point, the energy has dropped by a factor of $(1-x/2)^N \approx e^{-Nx/2} = 1/\sqrt{2}$. Since the energy is also $kA^2/2$, the new amplitude is $\sqrt[4]{2}A_0$.

Amazingly, the question can also be solved in one step using a subtle conserved quantity.

Solution

Sinusoidal motion is just a projection of circular motion. In particular, it's equivalent to think of the mass as being tied to a spring of zero rest length attached to the origin, and performing a circular orbit about the origin, with the "actual" oscillation being the x component. (This is special to zero-length springs obeying Hooke's law, and occurs because the spring force $-k\mathbf{x} = -k(x, y)$ has its x-component independent of y, and vice versa.)

Since the spring constant is changed gradually, the orbit has to remain circular. Then *angular momentum* is conserved, and we have

$$L \propto vr = \omega A^2 \propto \sqrt{k}A^2.$$

Then the final amplitude is $\sqrt[4]{2}A_0$ as before.

Both of these approaches are tricky. The energy argument is very easy to get wrong, while the angular momentum argument seems to come out of nowhere and is inapplicable to other situations. But the formal angular momentum here turns out to be a special case of a more general conserved quantity, which is useful in a wide range of similar problems.

Idea 6: Adiabatic Theorem

If a particle performs a periodic motion in one dimension in a potential that changes very slowly, then the "adiabatic invariant"

$$I = \oint p\,dx$$

is conserved. This integral is the area of the orbit in phase space, an abstract space whose axes are position and momentum.

Solution

By conservation of energy,

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2,$$

the curve $p(x)$ over one oscillation cycle traces out an ellipse in phase space, with semimajor and semiminor axes of $\sqrt{2mE}$ and $\sqrt{2E/k}$. The area of this ellipse is the adiabatic invariant,

$$I = \oint p\,dx = \pi\sqrt{2mE}\sqrt{2E/k} = 2\pi E\sqrt{\frac{m}{k}} \propto A^2\sqrt{km}.$$

Thus, $A \propto k^{-1/4}$ in an adiabatic change of k, recovering the answer found earlier.

Remark

The existence of the adiabatic invariant is hard to see in pure Newtonian mechanics, but it falls naturally out of the framework of Hamiltonian mechanics, which is built on phase space. In fact, Hamiltonian mechanics makes a lot of theoretically useful facts easier to see.

For example, as you will see in **X1** using quantum statistical mechanics, the conservation of the adiabatic invariant for a single classical particle implies the conservation of the entropy for an adiabatic process in thermodynamics! The two meanings of "adiabatic" are actually the same. If you'd like to learn more about Hamiltonian mechanics, see David Tong's lecture notes or chapter 15 of Morin. Tong's notes also contain a proof of the adiabatic theorem.

[3] **Problem 25.** Consider a pendulum whose length adiabatically changes from L to $L/2$.

(a) If the initial (small) amplitude was θ_0, find the final amplitude using the adiabatic theorem.

(b) Give a physical interpretation of the adiabatic invariant.

(c) When quantum mechanics was being invented, it was proposed that the energy in a pendulum's oscillation was always a multiple of $\hbar\omega$, where ω is the angular frequency. At the first Solvay conference of 1911, Lorentz asked whether this condition would be preserved upon slow changes in the length of the pendulum, and Einstein replied in the affirmative. Reproduce Einstein's analysis.

[4] **Problem 26.** A block of mass M and velocity v_0 to the right approaches a stationary puck of mass $m \ll M$. There is a wall a distance L to the right of the puck.

(a) Assuming all collisions are elastic, find the minimum distance between the block and the wall by explicitly analyzing each collision. (Note that it does not suffice to just use the adiabatic theorem, because it applies to slow change, while the collisions are sharp. Nonetheless, you should find a quantity that is approximately conserved after many collisions have occurred.)

(b) Approximately how many collisions occur before the block reaches this minimum distance?

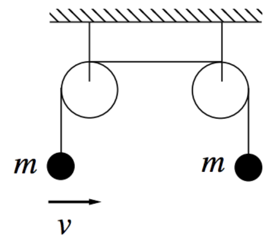
(c) The adiabatic index γ is defined so that PV^γ is conserved during an adiabatic process. In one dimension, the volume V is simply the length, and P is the average force. Using the adiabatic theorem, infer the value of γ for a one-dimensional monatomic gas.

[3] **Problem 27.** A block of mass m is bouncing back and forth in a box spanning $0 < x < L$, with initial speed v_0. At time $t = 0$, the potential energy is slowly raised in part of the box, so that

$$V(x,t) = \begin{cases} V_0 & 0 < x < ut \\ 0 & ut < x < L \end{cases}$$

where $V_0 > mv_0^2/2$, and the speed of the potential u is very small. At time $t = L/u$, when the potential covers the entire box, what is the block's speed?

[4] **Problem 28 ($\boldsymbol{F = ma}$, BAUPC).** Two particles of mass m are connected by pulleys as shown.



The mass on the left is given a small horizontal velocity v, and oscillates back and forth.

(a) Without doing any calculation, which mass is higher after a long time?

(b) Verify your answer is right by computing the average tension in the leftward string, in the case where the other end of the string is fixed, for amplitude $\theta_0 \ll 1$.

(c) Let the masses begin a distance L from the pulleys. Find the speed of the mass which eventually hits the pulley, at the moment it does, in terms of L and the initial amplitude θ_0.

Mechanics IV: Oscillations

Chapter 4 of Morin covers oscillations, as does chapter 10 of Kleppner and Kolenkow, and chapter 10 of Wang and Ricardo, volume 1. For a deeper treatment that covers normal modes in more detail, see chapters 1 through 6 of French. Jaan Kalda also has short articles on using Lagrangian-like techniques and the adiabatic theorem. For some fun discussion, see chapters I-21 through I-25, II-19, and II-38 of the Feynman lectures. There is a total of 88 points.

1 Small Oscillations

Idea 1

If an object obeys a linear force law, then its motion is simple harmonic. To compute the frequency, one must the restoring force per unit displacement. More generally, if the force an object experiences can be expanded in a Taylor series with a nonzero linear restoring term, the motion is approximately simple harmonic for small displacements. (However, don't forget that there are also situations where oscillations are not even approximately simple harmonic, no matter how small the displacements are.)

Example 1: KK 4.13

The Lennard–Jones potential

$$U(r) = \epsilon \left(\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^6 \right)$$

is commonly used to describe the interaction between two atoms. Find the equilibrium radius and the angular frequency of small oscillations about this point for two identical atoms of mass m bound to each other by the Lennard–Jones interaction.

Solution

To keep the notation simple, we'll set $\epsilon = r_0 = 1$ and restore them later. The equilibrium radius is the radius where the derivative of the potential vanishes, and

$$U'(r) = -12r^{-13} + 12r^{-7} = 0$$

implies that that the equilibrium radius is $r = r_0$. Because the force accelerates both of the atoms, the angular frequency is

$$\omega = \sqrt{\frac{U''(r)}{m/2}}$$

where $m/2$ is the so-called reduced mass. At the equilibrium point, we have

$$U''(r_0) = (12)(13)r_0^{-14} - (12)(7)r_0^{-8} = 72.$$

Restoring the dimensionful factors, we have $U''(r_0) = 72\epsilon/r_0^2$, so

$$\omega = \frac{12}{r_0}\sqrt{\frac{\epsilon}{m}}.$$

[3] **Problem 1** (Morin 5.13)**.** A hole of radius R is cut out from an infinite flat sheet with mass per unit area σ. Let L be the line that is perpendicular to the sheet and that passes through the center of the hole.

(a) What is the force on a mass m that is located on L, a distance x from the center of the hole? (Hint: consider the plane to consist of many concentric rings.)

(b) Now suppose the particle is released from rest at this position. If $x \ll R$, find the approximate angular frequency of the subsequent oscillations.

(c) Now suppose that $x \gg R$ instead. Find the period of the resulting oscillations.

(d) Now suppose the mass begins at rest on the plane, but slightly displaced from the center. Do oscillations occur? If so, what is the approximate frequency?

Solution. (a) Consider a ring with radius r and thickness dr. It has mass $dM = 2\pi r\sigma\, dr$. By symmetry, the net force is towards the sheet so we only take that component. Thus we multiply the total force from the ring $dF = Gm\, dM/(x^2 + r^2)$ by $x/\sqrt{x^2 + r^2}$. We integrate from R to infinity due to the hole, giving

$$F = -\int_R^\infty Gm \frac{2\pi r x\sigma\, dr}{(x^2 + r^2)^{3/2}} = -\pi Gmx\sigma \int_{x^2+R^2}^\infty \frac{du}{u^{3/2}} = -2\pi Gm\sigma \frac{x}{\sqrt{x^2 + R^2}}$$

(b) Approximating the force at small x gives

$$F = -2\pi Gm\sigma \frac{x}{R\sqrt{1 + x^2/R^2}} \approx -2\pi Gm\sigma x/R.$$

This implies simple harmonic oscillations,

$$\ddot{x} = -\frac{2\pi G\sigma}{R} x = \omega^2 x, \quad \omega = \sqrt{\frac{2\pi G\sigma}{R}}.$$

(c) In this case, approximating the force at large x gives

$$F = -2\pi Gm\sigma \frac{1}{\sqrt{1 + R^2/x^2}} \approx -2\pi Gm\sigma$$

with the force always directed towards the plane. This corresponds to a uniform acceleration of $g = 2\pi G\sigma$. A ball takes time $t = \sqrt{2x/g}$ to fall from a height x. The period is thus

$$T = 4\sqrt{\frac{x}{\pi G\sigma}}.$$

(d) This situation is unstable; the mass will just accelerate further away from the center, so no oscillations occur. This is related to Earnshaw's theorem, which we cover in **E1**, which tells us that no gravitational field (or electrostatic field) in vacuum can have a point that is stable in all directions.

[2] **Problem 2.** Some questions about small oscillations with the buoyant force.

(a) A cubical glacier of side length L has density ρ_i and floats in water with density ρ_w. Find the angular frequency of small oscillations, assuming that a face of the glacier always remains parallel to the water surface, and that the force of the water on the glacier is always given by the hydrostatic buoyant force.

(b) A ball of radius R floats in water with half its volume submerged. Find the angular frequency of small oscillations, making the same assumption.

(c) There are important effects that both of the previous parts neglect. What are some of them? Is the true oscillation frequency higher or lower than the one found here?

Solution. (a) Let $V = xL^2$ be the submerged volume, and let $V_0 = L^3$. We then have that $F = -\rho_w V g + \rho_i V_0 g = -\rho_w L^2 g x + \text{const}$. Thus, $\omega = \sqrt{\rho_w L^2 g/\rho_i L^3} = \boxed{\sqrt{\frac{\rho_w}{\rho_i}\frac{g}{L}}}$.

(b) The density of the ball is half that of water, so its mass is $(2\pi/3)\rho_w R^3$. The "spring constant" is $\pi R^2 \rho_w g$, so

$$\omega = \sqrt{\frac{\pi R^2 \rho_w g}{(2\pi/3)\rho_w R^3}} = \sqrt{\frac{3g}{2R}}.$$

(c) The most egregious omission is that we have completely neglected the motion of the water. This clearly should add extra inertia, because the water has to move around to accommodate the moving glacier or ball, and it should be a significant change since the water is more dense than these objects. This "virtual mass" effect leads to a decrease in the oscillation frequency. However, treating the motion of the water is usually mathematically involved; we'll see some examples in **M7**.

In fact, the situation is even worse. As we'll also see in **M7**, viscosity between the object and water leads to a boundary layer of water carried along with the object. But since this boundary layer builds up over time, its thickness depends on the *entire history* of the object's motion! This is called the Basset force, and it turns Newton's second law into an "integro-differential equation", one where the second derivative of the position depends on an integral over all the past positions. It has the effect of damping the oscillations. In general, nothing in fluid dynamics is easy.

[3] **Problem 3.** USAPhO 1998, problem A2. To avoid some confusion, skip part (a), since there actually isn't a nice closed-form expression for it.

[3] **Problem 4.** USAPhO 2009, problem A3.

[3] **Problem 5.** USAPhO 2010, problem B1.

Example 2

Find the acceleration of an Atwood's machine with masses m and M and a massless pulley and string.

Solution

The standard way to do this is to let a_1 and a_2 be the accelerations of the masses, let T be the unknown tension in the string, solve for T by setting a_1 and a_2 to have equal magnitudes, then plug T back in to find the common acceleration. This procedusre is unnecessarily complicated, because we are using two coordinates when the string really ensures the system has only a single degree of freedom.

We can alternatively use energy conservation in terms of "generalized coordinates". Let q be a coordinate which describes "how much the string has moved along itself". In other words, $q = 0$ initially, and when $q = q_0$, the mass M has moved down by q_0 and the mass m has moved up by q_0. The kinetic and potential energies of the system are simply

$$K = \frac{1}{2}(m+M)\dot{q}^2, \quad V = qg(m-M).$$

To find the acceleration $\ddot{q}$, we differentiate energy conservation with respect to time,

$$0 = \frac{d(K+V)}{dt} = (m+M)\ddot{q}\dot{q} + \dot{q}g(m-M).$$

Solving gives the familiar result

$$\ddot{q} = \frac{M-m}{M+m}\,g.$$

Intuitively, we could say that from the standpoint of this generalized coordinate, the "total force" is $(M-m)g$, and the "total inertia" is $M+m$.

Idea 2

The idea shown in example 2 is very general. Consider *any* system whose configuration can be described by a single "generalized coordinate" q. If its energy can be decomposed into a kinetic energy that depends only on $\dot{q}$ and a potential energy that depends only on q,

$$K = K(\dot{q}), \quad V = V(q)$$

then the energy conservation equation $d(K+V)/dt = 0$ can be used to find the generalized acceleration $\ddot{q}$. Explicitly, the chain rule tells us that

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}} = -\frac{\partial V}{\partial q}.$$

As a simple example, for a particle moving in one dimension, taking $q = x$ gives

$$\frac{d}{dt}(m\dot{x}) = -\frac{dV}{dx}$$

which is the usual Newton's second law. So in general, we will call $\partial K/\partial \dot{q}$ a "generalized momentum", and $-\partial V/\partial q$ a "generalized force". Note that static equilibrium can occur when $\partial V/\partial q = 0$, which is just the principle of virtual work from **M2**.

Remark

The result above is a special case of the Euler–Lagrange equation in Lagrangian mechanics, which states that if a system is described by a Lagrangian L, then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

When $L = K(\dot{q}) - V(q)$, we recover the previous result. But more generally, it might not be possible to meaningfully decompose L into a "kinetic" and "potential" piece at all! We won't use this more general form below. While it is more powerful, it is also more complicated, and if you find yourself using it for an Olympiad problem, there's probably an easier way.

[1] **Problem 6.** A rope is nestled inside a curved frictionless tube. The rope has a total length ℓ and uniform mass per length λ. The shape of the tube can be arbitrarily complicated, but the left end of the rope is higher than the right end by a height h. If the rope is released from rest, find its acceleration. (For a related question, see $F = ma$ 2019 B24.)

Solution. Let q be the amount the rope has moved. The kinetic energy is $\lambda\ell\dot{q}^2/2$. The "generalized force" is the change in gravitational potential energy per change in q, so $\partial V/\partial q = \lambda g h$. Therefore the acceleration is gh/ℓ. (Of course, you can get the same result by solving for the tension throughout the rope, but that method takes a lot longer.)

Idea 3

Generalized coordinates are really useful for problems that involve complicated objects but only have one relevant degree of freedom, which is especially true for oscillations problems. For instance, if the kinetic and potential energy have the form

$$K = \frac{1}{2} m_{\text{eff}} \dot{q}^2, \quad V = \frac{1}{2} k_{\text{eff}} q^2$$

then the oscillation's angular frequency is always

$$\omega = \sqrt{k_{\text{eff}}/m_{\text{eff}}}.$$

Note that q need not have units of position, m_{eff} need not have units of mass, and so on. When $V(q)$ is a more general function, we can expand it about a minimum q_{min}, so that $k_{\text{eff}} = V''(q_{\text{min}})$. This lets us avoid dealing with possibly complicated constraint forces.

[3] **Problem 7.** Suppose a particle is constrained to move on a curve $y(x)$ with a minimum at $x = 0$. We know that if $y(x)$ is a circular arc, then the motion is not exactly simple harmonic, for the same reason that pendulum motion is not. Find a differential equation relating y' and y, so that the motion is exactly simple harmonic for arbitrary amplitudes; you don't have to solve it. (Hint: work in terms of the coordinate s, the arc length along the curve.)

Solution. The reason s is useful is because

$$K = \frac{1}{2} m \dot{s}^2$$

exactly. Thus, the condition for simple harmonic motion is $V \propto s^2$, which means $y = ks^2$ for some constant k. Taking the derivative with respect to x of both sides, we have

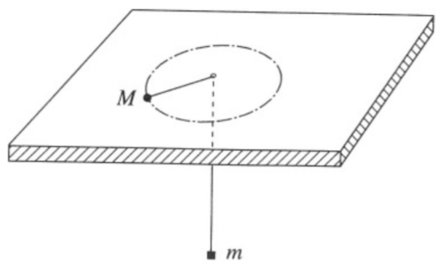
$$y' = 2ks\,\frac{ds}{dx} = 2ks\sqrt{1+y'^2} = 2\sqrt{ky}\sqrt{1+y'^2}.$$

Solving this for y' gives

$$y' = \sqrt{\frac{4ky}{1-4ky}}$$

which is the desired differential equation. Solving this is a bit nasty, but it turns out to be a cycloid. This fact was first discovered by Huygens, who invented the cycloidal pendulum for accurate timekeeping.

[3] **Problem 8** (Cahn)**.** A particle of mass M is constrained to move on a horizontal plane. A second particle of mass m is constrained to a vertical line. The two particles are connected by a massless string which passes through a hole in the plane.



The motion is frictionless. Show that the motion is stable with respect to small changes in the height of m, and find the angular frequency of small oscillations.

Solution. We'll use the energy method for this one. In equilibrium we have

$$mg = \frac{Mv^2}{r} = \frac{L^2}{Mr^3}$$

where r is the radius of the circle, and L is the conserved angular momentum. If the hanging mass goes downward, then r decreases, so the tension in the string increases providing a restoring force; hence the orbit is stable.

To find the angular frequency of small oscillations, we'll use the energy method, with generalized coordinate r. The "kinetic energy", which is the part of the energy dependent on $\dot{r}$, is

$$T = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m\dot{r}^2.$$

The "potential energy", which is the part of the energy dependent on r, is

$$V = mgr + \frac{L^2}{2Mr^2}.$$

Note that $L^2/2Mr^2$ is treated as potential energy here even though it is associated with the motion of the large mass. From the kinetic energy, we see the "effective mass" is $m_{\text{eff}} = M + m$, as one might expect. From the potential energy, we see the "effective spring constant" is

$$k_{\text{eff}} = V'' = \frac{3L^2}{Mr^4} = \frac{3mg}{R}.$$

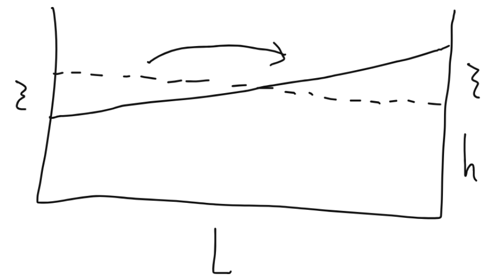
Hence the angular frequency is

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{3g}{R}}\sqrt{\frac{m}{m+M}}.$$

We'll see much more of this technique in **M7**.

[4] **Problem 9.** IPhO 1984, problem 2. If you use the energy methods above, you won't actually need to know anything about fluid mechanics to do this nice, short problem!

Solution. We will find the period of oscillation, we will find expressions for the kinetic energy and potential energy associated with the seiching.



First, to find the potential energy increase when the water is displaced by ξ, note that a triangular prism of water has effectively been moved upward, as shown above. The centers of masses of these triangles are $\xi/3$ from their bases, so the center of mass of the triangle will move up a distance of $2\xi/3$. Let the width of the container by w. Then the potential energy U will be $2mg\xi/3$, where the mass of the triangular prism of water is $m = \frac{1}{2}\frac{L}{2}\xi w\rho$, so

$$U = \frac{1}{6}\rho L w g \xi^2.$$

To find a rough estimate of the kinetic energy, consider the movement of the center of mass alone; this won't get all of the kinetic energy, but it'll get enough to get a reasonable answer. By thinking of the contribution of moving the triangle mentioned above, we have

$$\Delta x_{\text{cm}} = \frac{m(2L/3)}{M} = \frac{\frac{1}{4}L\xi w(2L/3)}{Lwh} = \frac{1}{6}\frac{L\xi}{h}.$$

$$\Delta y_{\text{cm}} = \frac{m(2\xi/3)}{M} = \frac{\xi^2}{6h}.$$

We see that Δx_{cm} dominates since ξ is small, so we focus on it. The total mass of the water is

$$M = \rho L w h$$

and our approximation for the kinetic energy is

$$K \approx \frac{1}{2}M\dot{x}_{\text{cm}}^2 = \frac{1}{2}\rho L w h \frac{L^2\dot{\xi}^2}{36h^2}.$$

Besides the overall side-to-side center of mass motion of the water, the water also has internal motions that can't be described just in terms of the center of mass moving. However, our result is good enough for the purposes of this problem.

Putting this together yields

$$E \approx \frac{\rho w L^3}{72h}\dot{\xi}^2 + \frac{1}{6}\rho L w g \xi^2 = \frac{1}{2}m_{\text{eff}}\dot{\xi}^2 + \frac{1}{2}k_{\text{eff}}\xi^2$$

and thus a period of

$$T \approx 2\pi\sqrt{\frac{L^2}{12gh}}.$$

This is compatible with the data given in the problem statement, up to order-one factors. Your answer may look different, since we've made a lot of approximations throughout the problem; as long as it agrees dimensionally, with the prefactor within an order of magnitude, you can regard it as correct.

This is a very brief taste of the fascinating field of oceanography, which is one of the premier real-world applications of fluid dynamics. For a lot more about seiches and their relatives, see chapter 9 of the *Handbook of Coastal and Ocean Engineering*.

2 Springs and Pendulums

Now we'll consider more general problems involving springs and pendulums, two very common components in mechanics questions. As a first example, we'll use the fictitious forces met in **M2**.

Example 3: PPP 79

A pendulum of length ℓ and mass m initially hangs straight downward in a train. The train begins to move with uniform acceleration a. If a is small, what is the period of small oscillations? If a can be large, is it possible for the pendulum to loop over its pivot?

Solution

The fictitious force in the train's frame due to the acceleration is equivalent to an additional, horizontal gravitational field, so the effective gravity is

$$\mathbf{g}_{\text{eff}} = -a\hat{\mathbf{x}} - g\hat{\mathbf{y}}.$$

For small oscillations, we know the period is $2\pi\sqrt{L/g}$ in ordinary circumstances. By precisely the same logic, it must be replaced with

$$T = 2\pi\sqrt{\frac{L}{g_{\text{eff}}}} = 2\pi\frac{\sqrt{L}}{(g^2+a^2)^{1/4}}.$$

As a gets larger, the effective gravity points closer to the horizontal. In the limit $g/a \to 0$, the effective gravity is just horizontal, so the pendulum oscillates about the horizontal. Its endpoints are the downward and upward directions, so it never can get past the pivot.

Here's a follow-up question: if the train can decelerate quickly, how should you stop it so that the pendulum doesn't end up swinging at the end? The most efficient way is to first quickly decelerate to half speed, which, in the frame of the train, provides a horizontal impulse to the pendulum. Then wait a half-period $\pi\sqrt{L/g}$, so that the pendulum's momentum turns around, and then quickly stop, providing a second impulse that precisely cancels the pendulum's horizontal motion. Tricks like this are used by crane operators to transport loads, and by physicists to transport clouds of ultracold atoms without warming them up.

Example 4

If a spring with spring constant k_1 and relaxed length ℓ_1 is combined with a spring with spring constant k_2 and relaxed length ℓ_2, find the spring constant and relaxed length of the combined spring, if the combination is in series or in parallel.

Solution

For the series combination, the new relaxed length is clearly $\ell = \ell_1 + \ell_2$. Suppose the first spring is stretched by x_1 and the second by x_2. The tensions in the springs must balance,

$$F = k_1 x_1 = k_2 x_2.$$

Thus, the new spring constant is

$$k = \frac{F}{x_1 + x_2} = \frac{k_2 x_2}{x_2(k_2/k_1 + 1)} = \frac{k_1 k_2}{k_1 + k_2}.$$

For example, if the spring is cut in half, the pieces have spring constant $2k$.

Now consider the parallel combination. In this case it's clear that the new spring constant is $k = k_1 + k_2$, since the tensions of the springs add. The new relaxed length ℓ is when the forces in the springs cancel out, so

$$k_1(\ell - \ell_1) + k_2(\ell - \ell_2) = 0$$

which implies

$$\ell = \frac{k_1 \ell_1 + k_2 \ell_2}{k_1 + k_2}.$$

[2] **Problem 10** (Morin 4.20)**.** A mass m is attached to n springs with relaxed lengths of zero. The spring constants are $k_1, k_2, \ldots, k_n$. The mass initially sits at its equilibrium position and then is given a kick in an arbitrary direction. Describe the resulting motion.

Solution. Suppose the anchor of spring i is at $\mathbf{r}_i$. Then the force on the mass is

$$\mathbf{F} = -\sum_i k_i(\mathbf{r} - \mathbf{r}_i) = \left(\sum_i k_i\right)\mathbf{r} - \mathbf{C}$$

where $\mathbf{C}$ is some constant vector. Thus, we see that the mass undergoes simple harmonic motion with angular frequency $\omega = \sqrt{\frac{\sum_i k_i}{m}}$.

[3] **Problem 11** (Morin 4.22)**.** A spring with relaxed length zero and spring constant k is attached to the ground. A projectile of mass m is attached to the other end of the spring. The projectile is then picked up and thrown with velocity v at an angle θ to the horizontal.

(a) Geometrically, what kind of curve is the resulting trajectory?

(b) Find the value of v so that the projectile hits the ground traveling straight downward.

Solution. (a) Let the anchor of the spring be the origin. Then, the force on the particle is $-k\mathbf{r} - mg\hat{\mathbf{y}} = -k(\mathbf{r} - \mathbf{r}_0)$, so it is effectively a single spring force. The motion in 2D due to a spring force is an ellipse (independent x and y oscillations of the same frequency), so the shape is a portion of an ellipse, whose center is a distance mg/k directly below the launch point.

(b) Note that the horizontal velocity takes the form $v_x(t) = (v\cos\theta)\cos(\omega t)$, because the motion in the horizontal direction is just simple harmonic. The horizontal velocity vanishes when the phase is $\pi/2$, a total of a quarter cycle.

At this time, the vertical displacement must vanish. Vertically, the motion is just simple harmonic but with an equilibrium point shifted downward by mg/k. Let the vertical velocity take the form

$$v_y(t) = v_0 \cos(\omega t + \phi).$$

The initial phase is ϕ, and just before the mass hits the ground, its vertical velocity is the opposite of the original one, so the final phase is $\pi - \phi$. So hitting the ground occurs at the same time as having a vertical velocity if the phase difference is $\pi/2$, which implies $\phi = \pi/4$.

Now, by matching the initial velocity and acceleration, we know that

$$v_0 \cos\phi = v\sin\theta, \quad -v_0\omega\sin\phi = -g$$

Dividing these equations gives

$$\tan\phi = \frac{g}{\omega v \sin\theta}$$

and we must have $\tan\phi = 1$, so

$$v = \frac{g}{\omega\sin\theta} = \sqrt{\frac{m}{k}}\frac{g}{\sin\theta}.$$

[5] **Problem 12.** A uniform spring of spring constant k and total mass m is attached to the wall, and the other end is attached to a mass M.

(a) Show that when $m \ll M$, the oscillation's angular frequency is approximately

$$\omega = \sqrt{\frac{k}{M + m/3}}.$$

(b) [A] $\star$ Generalize part (a) to arbitrary values of m/M. (Hint: to begin, approximate the massive spring as a finite combination of smaller massless springs and point masses, as in the example in **M2**. It will not be possible to solve for ω in closed form, but you can get a compact implicit expression for it. Check that it reduces to the result of part (a) for small m/M, and interpret the results for large m/M. This is a challenging problem that requires almost all the techniques we've seen so far; you might want to return to it after doing section 4.)

Solution. In the $m \ll M$ case, we can assume the velocity of a piece of spring that is at position a fraction x of the total length is proportional to x. (More precisely, accounting for nonlinear stretching of the spring would contribute at higher order.) Therefore, the total kinetic energy of the spring is

$$\int_0^1 \frac{1}{2}(xv_0)^2 m\, dx = \frac{1}{6}mv_0^2$$

where v_0 is the velocity of M, and L is the current length of the spring. Therefore, the total kinetic energy is $\frac{1}{2}(M+m/3)v_0^2$, so we have an effective mass of $M+m/3$. The spring is uniformly stretched at this order, so the effective spring constant is still k, giving the desired result.

Part (b) is a nice exercise in dealing with continuum systems, with many possible pitfalls – it is very easy to spend a lot of time attacking extremely complicated equations. We'll present a relatively clean approach.

First, as usual, we break the spring into pieces. Suppose the spring is made of N masses connected with small springs, and let their displacements from equilibrium be x_i. Each piece has mass m/N and each small spring has spring constant kN, as established in an earlier problem. The equation of motion for each mass is

$$\frac{m}{N}\ddot{x}_i = Nk(x_{i-1} + x_{i+1} - 2x_i).$$

We define $x_0 = 0$ and let x_N be the position of the mass M. Then its equation is different,

$$M\ddot{x}_N = Nk(x_{N-1} - x_N).$$

The spring is really continuous, so we would like to take the limit $N \to \infty$. To this end, define the displacement function $x(s,t)$ to be the continuous function with values

$$x(i/N, t) = x_i(t).$$

The argument s ranges from 0 at the left end of the spring to 1 at the right end. We'll suppress the t argument for brevity. Plugging this into the second equation above gives

$$M\ddot{x}(1) = Nk(x(1-1/N) - x(1)) = k\,\frac{x(1-1/N) - x(1)}{1/N}.$$

Upon taking the limit $N \to \infty$, the fraction on the right becomes a derivative, giving

$$M\ddot{x}(1) = -kx'(1)$$

where a prime denotes a derivative with respect to s. Similarly, in the $N \to \infty$ limit, the quantity $N^2(x_{i-1} + x_{i+1} - 2x_i)$ becomes a second derivative (check this!), so our first equation becomes

$$m\ddot{x}(i/N) = kx''(i/N).$$

Rearranging a bit and defining $\omega_0 = \sqrt{k/M}$, we have shown that

$$\frac{m}{M}\frac{\ddot{x}(s)}{\omega_0^2} = x''(s), \quad \frac{\ddot{x}(1)}{\omega_0^2} = x'(1).$$

Since we are looking for solutions where the whole spring oscillates uniformly with angular frequency ω, we plug in $x(s) = \cos(\omega t)f(s)$ for

$$\frac{m}{M}\frac{\omega^2}{\omega_0^2}f = -f'', \quad \frac{\omega^2}{\omega_0^2}f(1) = f'(1).$$

For simplicity we now define $\alpha = \sqrt{m/M}$ and set $\omega_0 = 1$. Solving the first equation gives

$$f(s) \propto \sin(\alpha\omega s)$$

which yields the expected nonlinear stretching of the spring. The second equation says

$$\omega^2 \sin(\alpha\omega) = \alpha\omega\cos(\alpha\omega)$$

or alternatively

$$\boxed{\tan(\alpha\omega) = \frac{\alpha}{\omega}.}$$

There are generically infinitely many solutions for ω, which correspond to the infinitely many normal modes of the spring. However, we're concerned with the lowest-frequency mode. This is the unique mode with $\alpha\omega < \pi/2$ where all the pieces of the spring are going in the same direction at the same time; it is the fundamental frequency.

The transcendental equation we have here has no closed form solution, but we can approximate it. For small α, if we Taylor expand the tangent to third order we recover the answer to the previous problem. To see this, note that

$$\alpha\omega + \frac{(\alpha\omega)^3}{3} = \frac{\alpha}{\omega}$$

which can be simplified to

$$\frac{\alpha^2}{3}\omega^4 + \omega^2 - 1 = 0.$$

If we parametrize the frequency shift by $\omega^2 = 1 + \epsilon$, then plugging in gives

$$\frac{\alpha^2}{3} + \epsilon + (\text{higher order terms}) = 0$$

which tells us that

$$\epsilon = -\frac{\alpha^2}{3} = -\frac{m}{3M}$$

which is the same result found earlier, to first order.

For large α, the right-hand side is large, so the tangent must be large. The lowest frequency mode has $\alpha\omega \approx \pi/2$. In this case it's also useful to look at all the modes, which have $\alpha\omega \approx (n + 1/2)\pi$. Restoring the units, this means

$$\omega \approx \left(n + \frac{1}{2}\right)\pi\sqrt{k/m}.$$

To understand this, note that in this limit the mass M doesn't matter; the spring acts as if it has a free end. Hence we've just found the standing wave angular frequencies for longitudinal waves with one fixed and one free end! The lowest frequency is the fundamental.

Jumping ahead a bit, we can compare this with some results from **W1**. The wavenumbers for these boundary conditions are

$$k_n = \left(n + \frac{1}{2}\right)\pi$$

and the wave velocity is

$$v = \sqrt{\frac{Y}{\rho}}$$

where Y is the Young's modulus, and ρ is the mass density. (If this isn't familiar, you can also derive it using dimensional analysis.) But this wave velocity can also be written as

$$v = \sqrt{\frac{kL/A}{m/LA}} = L\sqrt{\frac{k}{m}}.$$

Putting these two together using $\omega_n = vk_n$ recovers exactly the angular frequencies we found above! So another way of saying what we've done is that we're *derived* the expression $v = \sqrt{Y/\rho}$ for the speed of sound.

[2] **Problem 13** (PPP 77)**.** A small bob of mass m is attached to two light, unstretched, identical springs. The springs are anchored at their far ends and arranged along a straight line. If the bob is displaced in a direction perpendicular to the line of the springs by a small length ℓ, the period of oscillation of the bob is T. Find the period if the bob is displaced by length 2ℓ.

Solution. Suppose it is stretched x in the perpendicular direction, and let θ be the angle of the spring with respect to the horizontal (the original line of the springs). We see that $\tan\theta = x/L$ where L is the unstretched spring length. Now, the change in the length of one of the springs is $L(1/\cos\theta - 1) \approx L\theta^2/2$. The restoring force is then proportional to $\theta^2 \sin\theta \approx \theta^3 \propto x^3$. Therefore, the potential energy is of the form Cx^4 for some constant C. Thus, we have

$$\frac{1}{2}m\dot{x}^2 + Cx^4 = CA^4$$

where A is the amplitude of oscillation. Thus, $\dot{x} = D\sqrt{A^4 - x^4}$ for some constant D, so $dt \propto \frac{dx}{\sqrt{A^4-x^4}}$. Thus, the period is proportional to $\int_0^A \frac{dx}{\sqrt{A^4-x^4}}$, which by dimensional analysis is proportional to $1/A$. Thus, by doubling the amplitude, we halve the period, so the answer is $\boxed{T/2}$. Note that we saw a similar idea in **P1**.

[3] **Problem 14.** USAPhO 2015, problem A3.

[3] **Problem 15.** USAPhO 2008, problem B1.

Example 5

About how accurately can you measure g with a simple pendulum?

Solution

This simple question illustrates how rich experimental physics can be, even in elementary settings. First, let's think about the uncertainties in the pendulum's length and period.

- Length: a reasonable length for an experiment is $L \sim 1\,\text{m}$. We should use a wire, not a string, to avoid stretching. If you measure the wire with a good ruler, you can get down to $\Delta L \sim 1\,\text{mm}$. If you use calipers, you can get $\Delta L \sim 0.1\,\text{mm}$. Assuming the latter gives a fractional uncertainty $\Delta L/L \sim 10^{-4}$.
- Period: if the length is a meter, the period will be $T \simeq 2\,\text{s}$. (This isn't a coincidence! 17^{th} century scientists defined the meter precisely so this would be true.) One might estimate the timing uncertainty to be given by human reaction speed, $\Delta T \sim 250\,\text{ms}$, but

this is too pessimistic, because you can see the pendulum coming. An extensive study of manual timing at swimming competitions found a typical spread $\Delta T \sim 70\,\text{ms}$. Moreover, since a pendulum's motion is regular, you can "lock in" with your sense of rhythm to do even better than this. Finally, we can let the pendulum swing for $N = 100$ consecutive periods and measure the total time. These improvements allow a timing uncertainty $\Delta T/(NT) \sim 10^{-4}$.

Combining these results with the error propagation rules of **P2**, we can estimate $\Delta g/g \sim 10^{-4}$ for a well-performed experiment. But any real experiment also has to contend with systematic effects which can bias the results. Let's consider and estimate a couple of them.

- The bob has finite size, so the pendulum is really a physical pendulum. We can estimate the size of this effect by thinking about how much the bob's size changes the pendulum's moment of inertia. If the bob has radius $r \sim 1\,\text{cm}$, the change is roughly $r^2/L^2 \sim 10^{-4}$.
- The wire isn't massless, so the effective length of the pendulum is less than L. If we use a lead bob whose mass is a few kilograms, and the wire is a thin steel wire whose mass is a few grams, the effect is roughly $m_{\text{wire}}/m_{\text{bob}} \sim 10^{-3}$.
- The motion has finite amplitude θ_0. As we saw in **P1**, this changes the period fractionally by $\theta_0^2/16$, and for an amplitude of a few degrees this is $\sim 10^{-3}$.
- The pendulum oscillates in air. This leads to two distinct effects: the buoyant force on the bob decreases the effective value of g, and the "virtual mass" effect, discussed further in **M7**, increases the bob's effective inertia. These effects shift the period in the same direction, and they are both of order $\rho_{\text{air}}/\rho_{\text{bob}} \sim (1\,\text{kg/m}^3)/(10^4\,\text{kg/m}^3) \sim 10^{-4}$.
- The Earth is rotating, leading to centrifugal and Coriolis forces. The latter turns out to be unimportant; as shown in **M6**, it rotates the pendulum's plane of oscillation, rather than shifting its period. Unless you're conducting the experiment in Greenland or Antarctica, the centrifugal force produces a shift of order $\omega_E^2 R_E/g \sim 10^{-3}$.
- The pendulum's motion is slightly damped, which lengthens the oscillation period. This factor depends on how frictionless the support is. However, if it was set up so that 100 consecutive periods can be measured, one must have quality factor $Q \gtrsim 10^3$. One can show that the fractional shift in frequency is $\sim 1/Q^2 \sim 10^{-6}$.

There are plenty of other factors, but these are the most important ones, and a few of them are larger than the uncertainty from the length and period. But the good thing is that all of them can be calculated, and thereby subtracted out, leading to an ultimate final precision of $\Delta g/g \sim 10^{-4}$. That is indeed the best precision achieved during the 1800s, through extensive effort. For real measurements and further details, see this paper.

3 Damped and Driven Oscillations

We now review damped oscillators, which we saw in **M1**, and consider driven oscillators. For more guidance, see sections 4.3 and 4.4 of Morin.

[2] **Problem 16.** Consider a damped harmonic oscillator, which experiences force $F = -bv - kx$.

(a) As in **M1**, show that the general solution for $x(t)$ is

$$x(t) = A_+ e^{i\omega_+ t} + A_- e^{-i\omega_- t}$$

and solve for the $\omega_\pm$.

(b) For sufficiently small b, the roots are complex. In this limit, show that by taking the real part, one finds an exponentially damped sinusoidal oscillation. Roughly how many oscillation cycles happen when the amplitude damps by a factor of e?

(c) For large b, the roots are pure imaginary, the position simply decays exponentially, and we say the system is overdamped. Find the condition for the system to be overdamped.

Solution. (a) By setting up and solving a quadratic equation,

$$\omega_\pm = \frac{-ib \pm \sqrt{-b^2 + 4mk}}{-2m} = \frac{ib}{2m} \pm \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}.$$

(b) In this limit, we have

$$\omega_\pm \approx \frac{ib}{2m} \pm \sqrt{\frac{k}{m}}$$

in which case we have

$$e^{i\omega_\pm t} \approx e^{-bt/2m} e^{i\sqrt{k/m}\,t}$$

which is an exponentially damped oscillation. The time for a damping of a factor of e is $2m/b$, which occurs after $\sqrt{km}/\pi b$ cycles.

(c) This occurs if

$$\frac{k}{m} - \left(\frac{b}{2m}\right)^2 < 0$$

which implies $b^2 > 4mk$.

[4] **Problem 17.** Analyzing a damped and driven harmonic oscillator.

(a) Consider a damped harmonic oscillator which experiences a driving force $F = F_0 \cos(\omega t)$. Passing to complex variables, Newton's second law is

$$m\ddot{x} + b\dot{x} + kx = F_0 e^{i\omega t}.$$

If $x(t)$ is a complex exponential, then we know that the left-hand side is still a complex exponential, with the same frequency. This motivates us to guess $x(t) = A_0 e^{i\omega t}$. Show that this solves the equation for some A_0.

(b) Of course, the general solution needs to be described by two free parameters, to match the initial position and velocity. Argue that it takes the form

$$x(t) = A_0 e^{i\omega t} + A_+ e^{i\omega_+ t} + A_- e^{-i\omega_- t}$$

where the $\omega_\pm$ are the ones you found in problem 16.

(c) After a long time, the "transient" $A_\pm$ terms will decay away, leaving the steady state solution

$$x(t) \approx A_0 e^{i\omega t}$$

which oscillates at the same frequency as the driving. The actual position is the real part,

$$x(t) \approx |A_0| \cos(\omega t - \phi)$$

where $A_0 = |A_0| e^{-i\phi}$. Evaluate $|A_0|$ and ϕ.

(d) Sketch the amplitude $|A_0|$ and phase shift ϕ as a function of ω. Can you intuitively see they take the values they do, for ω small, $\omega \approx \sqrt{k/m}$, and ω large?

(e) There are several distinct things people mean when they speak of "resonant frequencies". Find the driving angular frequency ω that maximizes (i) the amplitude $|A_0|$, (ii) the amplitude of the velocity, and (iii) the average power absorbed from the driving force. (As you'll see, these are all about the same when the damping is weak, so the distinction between these isn't so important in practice.)

Solution. (a) If we plug in $x = A_0 e^{i\omega t}$, we find the differential equation is satisfied if

$$(-m\omega^2 + ib\omega + k)A_0 = F_0,$$

which yields

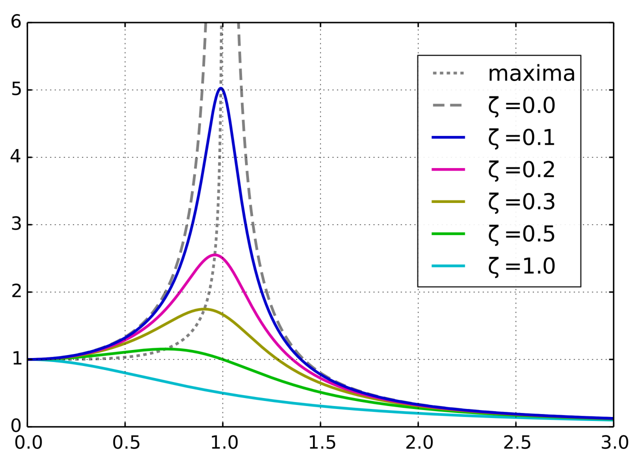
$$A_0 = \frac{F_0}{(k - m\omega^2) + ib\omega}.$$

(b) This follows from linearity. If we plug this solution in, then the first term balances the driving term on the right-hand side. Then the other two terms need to satisfy the damped harmonic oscillator equation with no driving, so they're just the same as in problem 16.

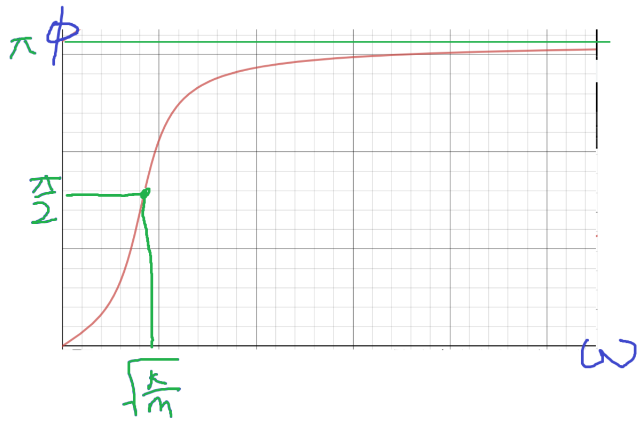
(c) The answers are

$$|A_0| = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}}, \quad \tan\phi = \frac{b\omega}{k - m\omega^2}.$$

(d) The amplitude is shown below, for a few values of b (called ζ here).



The phase shift is shown below.



This all makes physical sense. For very small frequency, we are effectively stretching the spring statically, so the amplitude approaches a constant $|A_0| = F_0/k$, and the phase shift is zero. For $\omega \approx \sqrt{k/m}$, the amplitude is high because we're driving the oscillator at the frequency it wants to oscillate at, in the absence of driving and damping. Here, a large power is absorbed from the driving force, and since $P = Fv$, that means F and v must be approximately in phase, so the phase shift between F and x is $90°$. Finally, for high frequencies, the amplitude goes to zero because the mass doesn't have time to move far before the force turns around. In this case, the driving force is always the largest force acting on the mass, so F and a are in phase, so the phase shift between F and x is $180°$.

(e) First, to find the maximum $|A_0|$, it suffices to minimize the square of its denominator. Setting the derivative of that quantity to zero gives

$$2b^2\omega = 2(k - m\omega^2)(2m\omega)$$

which can be solved to yield

$$\omega = \sqrt{k/m - b^2/2m^2}.$$

The amplitude of the velocity is

$$v_0 = \omega|A_0| = \frac{F_0\omega}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}} = \frac{F_0}{\sqrt{(k/\omega - m\omega)^2 + b^2}}$$

which is clearly minimized when $\omega = \sqrt{k/m}$. Finally, the rate of power dissipation is

$$P = F(t)v(t) = -F_0 v_0 \cos(\omega t)\sin(\omega t - \phi) = F_0 v_0 \cos(\omega t)\cos(\omega t + (\pi/2 - \phi)).$$

As we've just seen, v_0 is maximized at $\omega = \sqrt{k/m}$. In addition, the average value of the product of cosines is maximized when they are in phase with each other, $\phi = \pi/2$, which *also* happens when $\omega = \sqrt{k/m}$. Therefore, the maximum average power dissipation occurs at $\omega = \sqrt{k/m}$.

[3] **Problem 18.** The quality factor of a damped oscillator is defined as $Q = m\omega_0/b$, where $\omega_0 = \sqrt{k/m}$. It measures both how weak the damping is, and how sharp the resonance is.

(a) Show that for a lightly damped oscillator,

$$Q \approx \frac{\text{average energy stored in the oscillator}}{\text{average energy dissipated per radian}}.$$

Then estimate Q for a guitar string.

(b) Show that for a lightly damped oscillator,

$$Q \approx \frac{\text{resonant frequency}}{\text{width of resonance curve}}$$

where the width of the resonance curve is defined to be the range of driving frequencies for which the amplitude is at least $1/\sqrt{2}$ the maximum.

For more about Q, see pages 424 through 428 of Kleppner and Kolenkow.

Solution. (a) Take $x = A\cos\omega_0 t$. In one cycle, the energy dissipated is

$$\int_0^{2\pi/\omega_0} bv \cdot v dt = bA^2\omega_0\pi,$$

so the average energy dissipated per radian is $bA^2\omega_0/2$. The average energy stored is $\frac{1}{2}m\omega_0^2 A^2$, so the ratio is $m\omega_0/b = Q$.

The value of Q depends on the guitar string, but one of the strings in the middle will oscillate for around $\sim 300\,\text{Hz}$ for a few seconds, corresponding to $\sim 10^4$ radians, so we can roughly estimate $Q \sim 10^4$.

(b) We have $|A_0| = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+(b\omega)^2}}$. At the edge of the range that we call the width, we have

$$m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2 = 2(b\omega_0)^2 \implies m(\omega_0^2 - \omega^2) = \pm b\omega_0,$$

so $m(\omega_0 + \omega)(\omega_0 - \omega) = \pm b\omega_0$. We have $\omega \approx \omega_0$ (to first order), so

$$2m\omega_0(\omega_0 - \omega) = \pm b\omega_0 \implies 1 - \omega/\omega_0 = \pm\frac{1}{2Q}.$$

Thus the width is approximately ω_0/Q, as desired.

The next two problems explore other ways of driving harmonic oscillators.

[2] **Problem 19.** Consider a pendulum which can perform small-angle oscillations in a plane with natural frequency f. The pendulum bob is attached to a string, and you hold the other end of the string in your hand. There are three simple ways to drive the pendulum:

(a) Move the end of the string horizontally with sinusoidal frequency f'.

(b) Move the end of the string vertically with sinusoidal frequency f'.

(c) Apply a quick rightward impulse to the bob with frequency f'.

In each case, for what value(s) of f' can the amplitude become large? (This question should be done purely conceptually; don't write any equations, just visualize it!)

Solution. (a) In the frame of the string, this is a sinusoidal horizontal (fictitious) force, so it's just the same kind of sinusoidal driving we saw above. Resonance happens when $f' \approx f$.

(b) In this case, there is a sinusoidal vertical force by the same reasoning. Resonance can happen when $f' \approx 2f$, in which case gravity is weaker whenever the bob is moving up and stronger whenever it is moving down. In fact, one can show that weaker resonance occurs for $f' \approx 2nf$. This is called parametric resonance, and you can find more details here.

(c) This works as long as the impulse always comes when the object is moving to the right, i.e. in the same phase of the object's oscillation. This happens as long as the impulse's period is an integer multiple of the object's period, so $f' \approx f/n$.

[5] **Problem 20.** GPhO 2016, problem 1. Record your answers on the official answer sheet.

Solution. See the official solutions here.

4 Normal Modes

Idea 4: Normal Modes

A system with N degrees of freedom has N normal modes when displaced from equilibrium. In a normal mode, the positions of the particles are of the form $x_i(t) = A_i \cos(\omega t + \phi_i)$. That is, all particles oscillate with the same frequency. Normal modes can be either guessed physically, or found using linear algebra as explained in section 4.5 of Morin.

The general motion of the system is a superposition of these normal modes. So to compute the time evolution of the system, it's useful to decompose the initial conditions into normal modes, because they all evolve independently by linearity.

Example 6

Two blocks of mass m are connected with a spring of spring constant k and relaxed length L. Initially, the blocks are at rest at positions $x_1(0) = 0$ and $x_2(0) = L$. At time $t = 0$, the block on the right is hit, giving it a velocity v_0. Find $x_1(t)$ and $x_2(t)$.

Solution

The equations of motion are

$$\begin{aligned} m\ddot{x}_1 &= k(x_2 - x_1 - L) \\ m\ddot{x}_2 &= k(x_1 + L - x_2). \end{aligned}$$

The system must have two normal modes. The obvious one is when the two masses oscillate oppositely, $x_1 = -x_2$. The other one is when the two masses move parallel to each other, $x_1 = x_2$, and this normal mode formally has zero frequency. The initial condition is the superposition of these two modes.

We can show this a bit more formally. Define the normal mode amplitudes u and v as

$$x_1 = \frac{u-v}{2}, \quad x_2 = \frac{u+v}{2}.$$

Solving for u and v, we find

$$u = x_1 + x_2, \quad v = x_2 - x_1.$$

Using the equations of motion for x_1 and x_2, we have the equations of motion

$$\ddot{u} = 0, \quad m\ddot{v} = -2k(v - L)$$

which just verifies that the normal modes are independent, with angular frequency zero and $\omega = \sqrt{2k/m}$ respectively. We can fit the initial condition if

$$u(0) = L, \quad v(0) = L, \quad \dot{u}(0) = v_0, \quad \dot{v}(0) = v_0.$$

The normal mode amplitudes are then

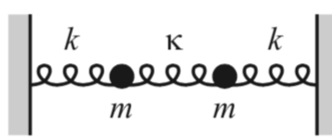
$$u(t) = L + v_0 t, \quad v(t) = L + \frac{v_0}{\omega}\sin\omega t.$$

Plugging this back in gives

$$x_1(t) = \frac{v_0 t}{2} - \frac{v_0}{2\omega}\sin\omega t, \quad x_2(t) = L + \frac{v_0 t}{2} + \frac{v_0}{2\omega}\sin\omega t.$$

Each mass is momentarily stationary at time intervals of $2\pi/\omega$, though neither mass ever moves backwards. If you didn't know about normal modes, you could also arrive at this conclusion by playing around with the equations; you could see that they decouple when you add and subtract them, for instance.

[3] **Problem 21** (Morin 4.10)**.** Three springs and two equal masses lie between two walls, as shown.



The spring constant k of the two outside springs is much larger than the spring constant $\kappa \ll k$ of the middle spring. Let x_1 and x_2 be the positions of the left and right masses, respectively, relative to their equilibrium positions. If the initial positions are given by $x_1(0) = a$ and $x_2(0) = 0$, and if both masses are released from rest, show that

$$x_1(t) \approx a\cos((\omega+\epsilon)t)\cos(\epsilon t), \quad x_2(t) \approx a\sin((\omega+\epsilon)t)\sin(\epsilon t)$$

where $\omega = \sqrt{k/m}$ and $\epsilon = (\kappa/2k)\omega$. Explain qualitatively what the motion looks like. This is an example of beats, which result from superposition two oscillations of nearly equal frequencies; we will see more about them in **W1**.

Solution. The equations of motion are

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - \kappa(x_1 - x_2) \\ m\ddot{x}_2 &= -kx_2 - \kappa(x_2 - x_1). \end{aligned}$$

Again define $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$. Adding and subtracting the two EOMs tells us that

$$\begin{aligned} m\ddot{y}_1 &= -ky_1 \\ m\ddot{y}_2 &= -(k + 2\kappa)y_2. \end{aligned}$$

The initial conditions are $y_1(0) = y_2(0) = a$ and $\dot{y}_1(0) = \dot{y}_2(0) = 0$. The solution is

$$\begin{aligned} y_1(t) &= a\cos(\sqrt{k/m}t) \\ y_2(t) &= a\cos(\sqrt{(k + 2\kappa)/m}t). \end{aligned}$$

Solving for x_1 and x_2, we see that

$$\begin{aligned} x_1(t) &= a\cos\left(\frac{\sqrt{k/m} + \sqrt{(k+2\kappa)/m}}{2}t\right)\cos\left(-\frac{\sqrt{k/m} + \sqrt{(k+2\kappa)/m}}{2}t\right) \\ x_2(t) &= a\sin\left(\frac{\sqrt{k/m} + \sqrt{(k+2\kappa)/m}}{2}t\right)\sin\left(-\frac{\sqrt{k/m} + \sqrt{(k+2\kappa)/m}}{2}t\right). \end{aligned}$$

Note that

$$\frac{\sqrt{k/m} + \sqrt{(k+2\kappa)/m}}{2} \approx \omega$$

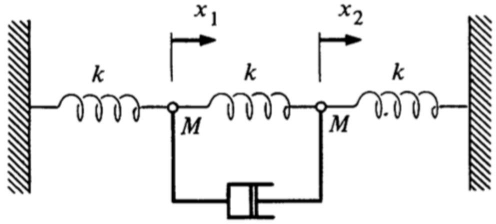
and

$$-\frac{\sqrt{k/m} + \sqrt{(k+2\kappa)/m}}{2} \approx \sqrt{k/m}\frac{\kappa/k}{2} = \epsilon,$$

so the result follows. We have an envelope curve of $a\cos(\epsilon t)$ and $a\sin(\epsilon t)$, and a very high frequency oscillation that matches the envelope. What this looks like is energy gradually sloshing back and forth between the masses. If the second mass begins still, it will gradually pick up energy, until the first mass begins still. Then the process repeats in reverse.

Note that without the weak spring in the middle, we would have two normal modes of equal frequency, while adding the spring causes the frequencies to split apart. This is a very common phenomenon in physics, known as "avoided crossing". For this reason, you will rarely see two acoustic modes of exactly equal frequency in a room, or two electromagnetic modes of equal frequency inside a conducting cavity, or two quantum states of the same energy, unless there's a symmetry at play.

[3] **Problem 22** (KK 10.11)**.** Two identical particles are hung between three identical springs.



Neglect gravity. The masses are connected as shown to a dashpot which exerts a force bv, where v is the relative velocity of its two ends, which opposes the motion.

(a) Find the equations of motion for x_1 and x_2.

(b) Show that the equations of motion can be solved in terms of the variables $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$.

(c) Show that if the masses are initially at rest and mass 1 is given an initial velocity v_0, the motion of the masses after a sufficiently long time is

$$x_1(t) = x_2(t) = \frac{v_0}{2\omega} \sin \omega t$$

and evaluate ω.

Solution. (a) The equations of motion are

$$\begin{aligned} M\ddot{x}_1 &= -kx_1 - k(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2), \\ M\ddot{x}_2 &= -kx_2 - k(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1). \end{aligned}$$

(b) Adding the two tells us that

$$M\ddot{y}_1 = -ky_1$$

and subtracting tells us that

$$M\ddot{y}_2 = -3ky_2 - 2b\dot{y}_2.$$

(c) Let us solve for y_1. The initial condition is $y_1(0) = 0$ and $\dot{y}_1(0) = v_0$. Thus,

$$y_1(t) = \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

where $\omega_0 = \sqrt{k/M}$. After a very long time, y_2 goes to 0, since it is damped. Thus, after a long time we have $x_1 = x_2 = y_1/2$, giving

$$x_1 = x_2 = \frac{v_0}{2\omega_0} \sin(\omega_0 t)$$

for $\omega = \omega_0$.

Example 7

Three identical masses are connected by three identical springs, forming an equilateral triangle in equilibrium. Describe the normal modes of the system.

Solution

Let the system be confined to the xy plane. Then there are three masses that each can move in two dimensions, giving six degrees of freedom. Since we must be able to construct the general solution by superposing normal modes, there should be six normal modes. They are:

- Uniform translation. This yields two independent normal modes, as you can superpose motion in any two distinct directions (e.g. along the x and y axes) to get motion in any direction. These modes have zero frequency, since $\sin(\omega t) \propto t$ in the limit $\omega \to 0$.
- Uniform rotation about the axis of symmetry.

- A "breathing" motion where the whole triangle expands and contracts.
- A "scissoring" motion where one mass moves outward and the other two move inward. You might think there are three scissoring normal modes, but they are redundant: just like how the three sides of the equilateral triangle lie in a plane, these three normal modes formally lie in a plane, in the sense that you can superpose any two of them to get the third. So there are two independent scissoring modes.

Thus we have six normal modes, as expected. If the system can move in three-dimension space, we need three more; they are uniform translation in the z direction, and rotation about the x and y axes.

[5] **Problem 23** (Morin 4.12, IPhO 1986)**.** N identical masses m are constrained to move on a horizontal circular hoop connected by N identical springs with spring constant k. The setup for $N = 3$ is shown below.

(a) Find the normal modes and their angular frequencies for $N = 2$.

(b) Do the same for $N = 3$.

(c) [A] $\star$ Do the same for general N. (Hint: consider the normal modes found in (a) and (b), arranged so that in each normal mode, each mass oscillates with unit amplitude but a different phase. Look at the phases and guess a pattern.)

(d) If one of the masses is replaced with a mass $m' \ll m$, qualitatively describe how the set of frequencies changes.

(e) Now suppose the masses alternate between m and $m' \ll m$. Qualitatively describe the set of frequencies.

Part (c) will be useful in **X1**, where we will quantize the normal modes found here.

Solution. (a) Let the positions of the masses along the circle be x_1 and x_2. Then

$$m\ddot{x}_1 = -k(2x_1 - 2x_2), \quad m\ddot{x}_2 = -k(2x_2 - 2x_1).$$

Adding and subtracting these equations and letting $\omega_0 = \sqrt{k/m}$ gives

$$\ddot{x}_1 + \ddot{x}_2 = 0, \quad \ddot{x}_1 - \ddot{x}_2 = -4\omega_0^2(x_1 - x_2)$$

which tells us the normal mode angular frequencies are zero and $2\omega_0$. These correspond to the masses uniformly rotating around the circle together, and to the two moving oppositely.

(b) Defining quantities analogously to part (a), we have

$$\ddot{x}_1 = -\omega_0^2(2x_1 - x_2 - x_3), \quad \ddot{x}_2 = -\omega_0^2(2x_2 - x_1 - x_3), \quad \ddot{x}_3 = -\omega_0^2(2x_3 - x_1 - x_2).$$

If we subtract the first two equations, we get

$$\ddot{x}_1 - \ddot{x}_2 = -3\omega_0^2(x_1 - x_2)$$

which gives a normal mode with angular frequency $\sqrt{3}\,\omega_0$, where the first two masses move oppositely and the third doesn't move at all. The same happens if we subtract the first and third equation, and second and third equation. Finally, if we add all three equations, we get

$$\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3 = 0$$

which gives a normal mode with zero frequency. Therefore, the normal mode angular frequencies are zero and $\sqrt{3}\,\omega_0$.

Strangely, it seems like we have *four* normal modes even though there are only three masses! The reason is that the first three we found are redundant: if you sum any two of them, you get the third.

(c) Using the same notation as in the previous parts,

$$\ddot{x}_j = -\omega_0^2(2x_j - x_{j-1} - x_{j+1}), \quad j = 1, 2, \ldots N.$$

Solving these equations requires an inspired guess. Notice that in part (a), the zero frequency normal mode has x_1 and x_2 in phase, while the other normal mode has them π out of phase. In part (b), the zero frequency normal mode has x_1, x_2, and x_3 in phase. The other two normal modes can be expressed as having phases

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ e^{2\pi i/3} \\ e^{4\pi i/3} \end{pmatrix} e^{i\omega t}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ e^{-2\pi i/3} \\ e^{-4\pi i/3} \end{pmatrix} e^{i\omega t}.$$

In other words, in each normal mode, the phase differences between adjacent masses are uniform! Therefore, we are inspired to guess

$$x_j = e^{i\omega t}\, e^{ikj}, \quad k = \frac{2\pi n}{N}$$

for integer n. Plugging this in, the differential equations simplify to

$$\omega^2 = \omega_0^2(2 - e^{-ik} - e^{ik})$$

which simplifies to

$$\omega = 2\omega_0 \sin\left(\frac{k}{2}\right) = 2\omega_0 \sin\left(\frac{\pi n}{N}\right).$$

For $n = 0, \ldots, N-1$, these are the normal mode angular frequencies.

As an aside, for $n \ll N$, note that ω_n is proportional to n, which is also proportional to the wavenumber k of the wave. This indicates that waves built out of only normal modes with $n \ll N$ travel with constant velocity, and hence satisfy the ideal wave equation. In general, systems that satisfy the ideal wave equation are often found by taking the low n limit of a system with many discrete parts. We'll see these points in more detail in **W1**.

As another aside, we could have also gotten the solution without "inspiration". As we've discussed above, guessing a complex exponential $e^{i\omega t}$ is the *general* technique when dealing with linear equations with time translation symmetry. Similarly, in this problem we considered linear equations with a discrete spatial translational symmetry, i.e. the equations stay the same upon substituting $j \to j+1$. Thus, the general technique is to guess a complex exponential in j, which is precisely the e^{ikj} factor.

(d) When we add the one light mass, it adds a new normal mode with angular frequency $\sqrt{2k/m'}$, where the light mass oscillates back and forth and nothing else moves. The band of angular frequencies from zero to $2\omega_0$ barely changes.

(e) Naively, if we turn half the masses into light masses, we get $N/2$ modes with angular frequency $\sqrt{2k/m'}$, consisting of each light mass oscillating independently. But this isn't right, because we must take sinusoidal combinations of these modes to get normal modes, by the same logic as we used in the previous parts. This broadens the normal mode angular frequencies into a band centered around $\sqrt{2k/m'}$. Meanwhile, for the low-frequency modes, the heavy masses can't even see the light masses, so it's as if every spring has been doubled in length. We hence have a second band of normal modes with angular frequencies centered on $\sqrt{k/2m}$, which is nonoverlapping if $m' \ll m$.

This idea of normal mode frequencies filling dense but separated bands is crucial in solid state physics. The result of part (d) shows how "defects" in a solid can lead to isolated energy levels, outside the bands. For further discussion, see this paper.

[4] **Problem 24.** [A] In this problem, you will analyze the normal modes of the double pendulum, which consists of a pendulum of length ℓ and mass m attached to the bottom of another pendulum, of length ℓ and mass m. To solve this problem directly, one has to compute the tension forces in the two strings, which are quite complicated. A much easier method is to use energy.

(a) Parametrize the position of the pendulum in terms of the angle θ_1 the top string makes with the vertical, and the angle θ_2 the bottom string makes with the vertical. Write out the kinetic energy K and the potential energy V to second order in the θ_i and $\dot{\theta}_i$.

(b) The Euler–Lagrange equations for the system are

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\theta}_i} = -\frac{\partial V}{\partial \theta_i}.$$

Using the results of part (a), write these equations in the form

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\frac{g}{L}\, A \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

where A is a 2×2 matrix. This is a generalization of $\ddot{\theta} = -g\theta/L$ for a single pendulum.

(c) Find the normal modes and their angular frequencies, using the general method in section 4.5 of Morin.

Solution. (a) To second order, the horizontal displacements of the masses are

$$x_1 = \ell\theta_1, \quad x_2 = \ell(\theta_1 + \theta_2)$$

which gives a kinetic energy of

$$K = \frac{m\ell^2}{2}((\dot{\theta}_1)^2 + (\dot{\theta}_1 + \dot{\theta}_2)^2).$$

The vertical displacements are

$$y_1 = \ell(1 - \cos(\theta_1)), \quad y_2 = \ell(2 - \cos(\theta_1) - \cos(\theta_2))$$

and expanding the cosines to second order gives

$$y_1 = \frac{\ell}{2}\theta_1^2, \quad y_2 = \frac{\ell}{2}(\theta_1^2 + \theta_2^2)$$

which gives a potential energy of

$$V = \frac{mg\ell}{2}(2\theta_1^2 + \theta_2^2).$$

(b) The resulting Euler–Lagrange equations are

$$2\ddot{\theta}_1 + \ddot{\theta}_2 = -\frac{2g}{\ell}\theta_1, \quad \ddot{\theta}_1 + \ddot{\theta}_2 = -\frac{g}{\ell}\theta_2.$$

Solving the system, we find

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

straightforwardly.

(c) We must find the eigenvalues of the matrix, which obey the equation

$$(2 - \lambda)^2 - 2 = 0$$

which implies $\lambda = 2 \pm \sqrt{2}$. The normal mode amplitudes are

$$\text{high frequency} : \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}, \quad \text{low frequency} : \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

and the angular frequencies are $\omega_\pm^2 = (g/L)(2 \pm \sqrt{2})$.

Remark

We mostly considered examples with two or three masses, but the techniques above work for systems with arbitrarily many degrees of freedom. However, this quickly becomes intractable unless the setup is highly symmetric, as in problem 23. Without such symmetry, a computer is generally necessary, so this sort of question won't appear on standard Olympiads. However, if you're curious, see ITPO 2016, problem 1 and Physics Cup 2021, problem 3 for examples.

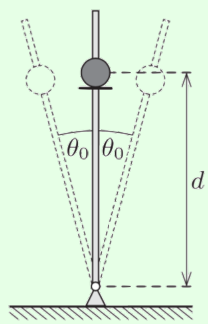
5 [A] Adiabatic Change

Idea 5

When a problem contains two widely separate timescales, such as a fast oscillation superposed on a slow overall motion, one can solve for the fast motion while neglecting the slow motion, then solve for the slow motion by replacing the fast motion with an appropriate average.

Example 8: MPPP 21

A small smooth pearl is threaded onto a rigid, smooth, vertical rod, which is pivoted at its base. Initially, the pearl rests on a small circular disc that is concentric with the rod, and attached to it a distance d from the rotational axis. The rod starts executing simple harmonic motion around its original position with small angular amplitude θ_0.



What angular frequency of oscillation is required for the pearl to leave the rod?

Solution

The reason the pearl leaves the rod is that the normal force rapidly varies in direction, with an average upward component. If this average upward force is greater than gravity, the pearl accelerates upward and leaves the rod.

In this case, the fast motion is the oscillation of the rod, while the slow motion is the rate of change of the pearl's distance from the pivot, which can be neglected during one oscillation. The pearl has horizontal displacement and acceleration

$$x(t) = -d\sin\theta \approx -d\theta(t) = -\theta_0 d \sin\omega t, \quad a_x(t) = \theta\omega^2 d \sin\omega t.$$

This is supplied by the horizontal component of the normal force. The vertical component is

$$N_y = N_x \tan\theta(t) \approx m a_x(t)\theta(t) = m\theta_0^2\omega^2 d \sin^2\omega t.$$

Now we average over the fast motion to understand the slow motion. Since the average value of $\sin^2(\omega t)$ is $1/2$, the condition for the pearl to go up is

$$\frac{1}{2} m\theta_0^2\omega^2 d > mg$$

which gives

$$\omega > \frac{1}{\theta_0}\sqrt{\frac{2g}{d}}.$$

Example 9

A mass m oscillates on a spring with spring constant k_0 with amplitude A_0. Over a very long period of time, the spring smoothly and continuously weakens until its spring constant is $k_0/2$. Find the new amplitude of oscillation.

Solution

In this case the fast motion is the oscillation of the mass, while the slow motion is the weakening of the spring. We can solve the problem by considering how the energy changes in each oscillation, due to the slight decrease in k.

Suppose that the spring constant drops in one instant by a factor of $1-\epsilon$. Then the kinetic energy stays the same, while the potential energy drops by a factor of $1-\epsilon$. Since the kinetic and potential energy are equal on average, this means that if the spring constant gradually decreases by a factor of $1-x$ over a full cycle, with $x \ll 1$, then the energy decreases by a factor of $1-x/2$.

The process finishes after N oscillations, where $(1-x)^N \approx e^{-Nx} = 1/2$. At this point, the energy has dropped by a factor of $(1-x/2)^N \approx e^{-Nx/2} = 1/\sqrt{2}$. Since the energy is also $kA^2/2$, the new amplitude is $\sqrt[4]{2}A_0$.

Amazingly, the question can also be solved in one step using a subtle conserved quantity.

Solution

Sinusoidal motion is just a projection of circular motion. In particular, it's equivalent to think of the mass as being tied to a spring of zero rest length attached to the origin, and performing a circular orbit about the origin, with the "actual" oscillation being the x component. (This is special to zero-length springs obeying Hooke's law, and occurs because the spring force $-k\mathbf{x} = -k(x, y)$ has its x-component independent of y, and vice versa.)

Since the spring constant is changed gradually, the orbit has to remain circular. Then *angular momentum* is conserved, and we have

$$L \propto vr = \omega A^2 \propto \sqrt{k}A^2.$$

Then the final amplitude is $\sqrt[4]{2}A_0$ as before.

Both of these approaches are tricky. The energy argument is very easy to get wrong, while the angular momentum argument seems to come out of nowhere and is inapplicable to other situations. But the formal angular momentum here turns out to be a special case of a more general conserved quantity, which is useful in a wide range of similar problems.

Idea 6: Adiabatic Theorem

If a particle performs a periodic motion in one dimension in a potential that changes very slowly, then the "adiabatic invariant"

$$I = \oint p\, dx$$

is conserved. This integral is the area of the orbit in phase space, an abstract space whose axes are position and momentum.

Solution

By conservation of energy,

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2,$$

the curve $p(x)$ over one oscillation cycle traces out an ellipse in phase space, with semimajor and semiminor axes of $\sqrt{2mE}$ and $\sqrt{2E/k}$. The area of this ellipse is the adiabatic invariant,

$$I = \oint p\, dx = \pi\sqrt{2mE}\sqrt{2E/k} = 2\pi E\sqrt{\frac{m}{k}} \propto A^2\sqrt{km}.$$

Thus, $A \propto k^{-1/4}$ in an adiabatic change of k, recovering the answer found earlier.

Remark

The existence of the adiabatic invariant is hard to see in pure Newtonian mechanics, but it falls naturally out of the framework of Hamiltonian mechanics, which is built on phase space. In fact, Hamiltonian mechanics makes a lot of theoretically useful facts easier to see.

For example, as you will see in **X1** using quantum statistical mechanics, the conservation of the adiabatic invariant for a single classical particle implies the conservation of the entropy for an adiabatic process in thermodynamics! The two meanings of "adiabatic" are actually the same. If you'd like to learn more about Hamiltonian mechanics, see David Tong's lecture notes or chapter 15 of Morin. Tong's notes also contain a proof of the adiabatic theorem.

[3] **Problem 25.** Consider a pendulum whose length adiabatically changes from L to $L/2$.

(a) If the initial (small) amplitude was θ_0, find the final amplitude using the adiabatic theorem.

(b) Give a physical interpretation of the adiabatic invariant.

(c) When quantum mechanics was being invented, it was proposed that the energy in a pendulum's oscillation was always a multiple of $\hbar\omega$, where ω is the angular frequency. At the first Solvay conference of 1911, Lorentz asked whether this condition would be preserved upon slow changes in the length of the pendulum, and Einstein replied in the affirmative. Reproduce Einstein's analysis.

Solution. (a) Using the small angle approximation, we have

$$E = \frac{1}{2}mv^2 + \frac{1}{2}mgL\theta^2$$

and the adiabatic invariant is

$$\oint p\,dx = L\oint p\,d\theta = mL\oint v\,d\theta.$$

On the other hand, from conservation of energy, we know that $v(\theta)$ is an ellipse with semimajor and semiminor axes $\sqrt{2E/m}$ and $\sqrt{2E/mgL}$, so

$$\oint p\,dx \propto mL\sqrt{E/m}\sqrt{E/mgL} = E\sqrt{\frac{L}{g}}.$$

The total energy is $E = mgL\theta_0^2/2$, so

$$\oint p\,dx \propto \theta_0^2 L^{3/2} g^{1/2}$$

which implies that when L halves, the amplitude becomes $2^{3/4}\theta_0$. Since we kept track of factors of g, this derivation also tells us what happens to the amplitude if g is slowly changed.

The most famous literary example of a pendulum with changing length appears in Edgar Allan Poe's short story, *The Pit and the Pendulum.* In the story, the narrator is strapped to a table, and sees a pendulum above him slowly moving and lengthening, bringing its razor edge toward him. Poe describes the pendulum's amplitude as initially small, but "increasing inexorably". As you can see from the solution to this problem, precisely the opposite happens. When the length goes up, the amplitude goes down, so it would just gently land on the narrator's chest.

(b) As for the case of a mass on a spring, we can add a third dimension and let the pendulum oscillate in a horizontal circle. Then the adiabatic invariant is simply

$$\oint L_z\,d\theta = 2\pi L_z \propto L_z$$

which is the angular momentum in the z-direction.

(c) Given the way we did part (a), this is immediate. The adiabatic invariant is

$$E\sqrt{\frac{L}{g}} = \frac{E}{\omega}.$$

Therefore, E/ω remains an integer multiple of $\hbar$ under adiabatic change.

This is admittedly a rather contrived example. However, the conservation of E/ω does have practical applications. For example, in fluid mechanics, a relative of this quantity is called the wave action. The conservation law derived in this problem can be used to see how the amplitudes of water waves change as they travel through the ocean, which has varying depth.

[4] **Problem 26.** A block of mass M and velocity v_0 to the right approaches a stationary puck of mass $m \ll M$. There is a wall a distance L to the right of the puck.

(a) Assuming all collisions are elastic, find the minimum distance between the block and the wall by explicitly analyzing each collision. (Note that it does not suffice to just use the adiabatic theorem, because it applies to slow change, while the collisions are sharp. Nonetheless, you should find a quantity that is approximately conserved after many collisions have occurred.)

(b) Approximately how many collisions occur before the block reaches this minimum distance?

(c) The adiabatic index γ is defined so that PV^γ is conserved during an adiabatic process. In one dimension, the volume V is simply the length, and P is the average force. Using the adiabatic theorem, infer the value of γ for a one-dimensional monatomic gas.

Solution. (a) Let the speeds of the block and puck be v and w. Every collision, w increases by $2v$. If the block is a distance x from the wall, then a collision happens in time $2x/w$. Therefore, we have

$$\frac{\Delta w}{\Delta x} = \frac{2v}{-2xv/w} = -\frac{w}{x}.$$

Because $m \ll M$, many collisions happen. After many collisions have happened, w will be very large, so in the next collision, Δx will be small compared to x, and Δw will be small compared to w. In this case, we can approximate the finite differences with a derivative, giving

$$\frac{dw}{dx} \approx -\frac{w}{x}.$$

Separating and integrating shows that wx is conserved. We could also have arrived at this by the adiabatic theorem,

$$I = \oint p\, dx = mw(2x) \propto wx.$$

However, in the earlier collisions $(\Delta w)/w$ and $(\Delta x)/x$ aren't small, so this reasoning is invalid. For instance, wx is zero before the first collision and nonzero right after it. Thus, we must treat the first few collisions manually. Right before the second collision, we have

$$w \approx 2v_0, \quad x \approx L/3$$

by one-dimensional kinematics. Right before the third collision we have

$$w \approx 4v_0, \quad x \approx L/5$$

where for these early few collisions we are treating v as constant since $m \ll M$. It is not hard to show that right before collision $n+1$, we have $w \approx 2nv_0$ and $x \approx L/(2n+1)$, which means that after many (but not too many collisions) we have $wx \approx Lv_0$. Then, for future collisions, wx stays at this value.

The block turns around when the puck has all its energy, so

$$\frac{1}{2}Mv_0^2 = \frac{1}{2}mw^2.$$

Plugging in $wx = Lv_0$ and solving for x gives the solution, $x = L\sqrt{m/M}$.

(b) At each collision we have $\Delta w = 2v$, and energy conservation gives

$$v^2 + \frac{m}{M}w^2 = v_0^2.$$

Therefore, the number of collisions is approximately

$$n \approx \int \frac{dw}{2v} = \frac{1}{2}\int \frac{dw}{\sqrt{v_0^2 - (m/M)w^2}} = \frac{1}{2}\sqrt{\frac{M}{m}}\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{4}\sqrt{\frac{M}{m}}.$$

Note that we didn't need to separate out the first few collisions here, even though the approximation as an integral technically doesn't work, because they're just that not important for calculating the total number of collisions, which is large. The appearance of π in this result has a nice geometric interpretation, as explained here.

(c) The analogue of pressure in one dimension is just force. The average force exerted by the puck, which we now think of as a gas molecule, is

$$F = \frac{\Delta p}{\Delta t} = \frac{2mw}{2x/w} = \frac{mw^2}{x}.$$

Meanwhile, the analogue of volume is one dimension is simply x. Then the conservation of wx says that Fx^3 is conserved, which means $\gamma = 3$. This is exactly what we would expect for a one-dimensional gas, where $C_v = kT/2$ and $C_p = 3kT/2$.

[3] **Problem 27.** A block of mass m is bouncing back and forth in a box spanning $0 < x < L$, with initial speed v_0. At time $t = 0$, the potential energy is slowly raised in part of the box, so that

$$V(x,t) = \begin{cases} V_0 & 0 < x < ut \\ 0 & ut < x < L \end{cases}$$

where $V_0 > mv_0^2/2$, and the speed of the potential u is very small. At time $t = L/u$, when the potential covers the entire box, what is the block's speed?

Solution. This is a version of Physics Cup 2021, problem 4 with extraneous details removed. The key is to use the adiabatic theorem. Initially, the box's orbit in phase space is a rectangle with width $\Delta x = L$ and $\Delta p_x = 2mv_0$. As the potential barrier enters the box, it effectively makes it shorter. So, just as in problem 26, the rectangle gets narrower while keeping its area the same.

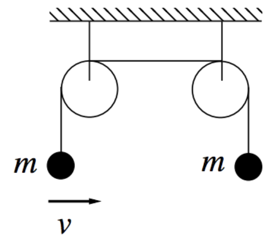
The twist is that eventually, the block will gain enough energy to climb over the potential barrier; at this point, the form of its orbit changes discontinuously, so we have to track exactly what's going on instead of blindly using the adiabatic theorem. Consider the moment when the block's energy becomes just enough to climb the potential barrier, and suppose that at this point, $ut = x_0$. Then the phase space orbit is the union of two rectangles:

- A rectangle with width $\Delta x = x_0$ and height $\Delta p_x = 2mv_0L/x_0$.
- A new rectangle with width $\Delta x = L - x_0$ and negligible height.

The added rectangle has negligible area, so the adiabatic invariant (the total phase space area) doesn't change! After this point, we can continue to apply the adiabatic theorem until the end of the process. The first rectangle shrinks, until it reaches zero width, while the second rectangle grows. At the end of the process, we are back to a single rectangle with width $\Delta x = L$ and the same area as before, so the block ends up with the same speed as before.

In terms of Newton's laws, what's going on is that the moving potential barrier is initially like a piston that does work on the block during each collision, but it also subtracts energy since the block has to eventually climb on top of it. Evidently, these two effects perfectly cancel, but it's much easier to see this with the adiabatic theorem.

[4] **Problem 28 ($\boldsymbol{F = ma}$, BAUPC).** Two particles of mass m are connected by pulleys as shown.



The mass on the left is given a small horizontal velocity v, and oscillates back and forth.

(a) Without doing any calculation, which mass is higher after a long time?

(b) Verify your answer is right by computing the average tension in the leftward string, in the case where the other end of the string is fixed, for amplitude $\theta_0 \ll 1$.

(c) Let the masses begin a distance L from the pulleys. Find the speed of the mass which eventually hits the pulley, at the moment it does, in terms of L and the initial amplitude θ_0.

Solution. (a) The mass on the right will be higher. If the masses didn't move up or down, both would have the same average y-component of tension. But the mass on the left also has an x-component of tension, so its average magnitude of tension would be higher. This is a contradiction; to make the tension constant throughout the rope the mass on the right must rise.

(b) The tension provides the centripetal force, so

$$T = \frac{mv^2}{r} + mg\cos\theta$$

where θ is the angle from the vertical. By energy conservation, the first term is $mgr(\cos\theta - \cos\theta_0)$ where θ_0 is the amplitude, so

$$T = (3\cos\theta - 2\cos\theta_0)mg.$$

Since the amplitude is small, we Taylor expand for

$$T = \left(1 + \theta_0^2 - \frac{3}{2}\theta^2\right) mg.$$

Again using the small angle approximation, so that the motion is approximately simple harmonic, the average value of θ^2 is $\theta_0^2/2$, so

$$\overline{T} = \left(1 + \theta_0^2 - \frac{3}{4}\theta_0^2\right) mg > mg$$

as expected.

(c) As we've seen above,

$$\overline{T} = \left(1 + \frac{1}{4}\theta^2\right) mg$$

where θ is the amplitude. Let x be the distance the right mass has risen. From the standpoint of the left mass, it is simply a pendulum whose length is being adiabatically lengthened, so by the result of problem 25, we have

$$\int \overline{T} - mg\,dx = \frac{mg}{4} \int_L^{2L} L^{3/2} \theta_0^2 \, \frac{dx}{x^{-3/2}} = \frac{mg}{2} L\theta_0^2 \left(1 - \frac{1}{\sqrt{2}}\right).$$

This is the increase in kinetic energy of the right mass, so setting this equal to $mv^2/2$ gives

$$v = \sqrt{gL\left(1 - \frac{1}{\sqrt{2}}\right)}\,\theta_0.$$

Mechanics V: Rotation

Two-dimensional rotation is covered in chapter 6 of Kleppner, chapter 8 of Morin, or chapters 3 and 5 of Wang and Ricardo, volume 1. Further discussion is given in chapters I-18 and I-19 of the Feynman lectures. For more on dot and cross products, see the first two lectures of MIT OCW 18.02. There is a total of **83** points.

1 2D Rotational Kinematics

Idea 1

The instantaneous velocity of a two-dimensional rigid body can always be written as pure rotation about some point $\mathbf{r}_0$, not necessarily in the body, so the velocity of a point in the body at location $\mathbf{r}$ is

$$\mathbf{v} = \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0),$$

as you can check with some geometry. This equation defines $\boldsymbol{\omega}$, the angular velocity vector, which points out of the page. Differentiating gives the acceleration of a point in the body,

$$\begin{aligned}\mathbf{a} &= \boldsymbol{\alpha} \times (\mathbf{r} - \mathbf{r}_0) + \boldsymbol{\omega} \times (\mathbf{v} - \mathbf{v}_0) \\ &= \boldsymbol{\alpha} \times (\mathbf{r} - \mathbf{r}_0) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0)) - \boldsymbol{\omega} \times \mathbf{v}_0\end{aligned}$$

where $\mathbf{v}_0 = d\mathbf{r}_0/dt$ is the rate of change of the location of the pivot point. (Note that this differs from the velocity of the point in the body instantaneously at the pivot, which is always zero.) However, this latter expression is often hard to use, because you usually won't know $\mathbf{r}_0(t)$, or it'll have a complicated form.

Alternatively, we can write the velocity in terms of translation plus pure rotation about any desired point, which is almost always chosen to be the center of mass. This gives

$$\mathbf{v} = \mathbf{v}_{\mathrm{CM}} + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}}).$$

The $\boldsymbol{\omega}$ here is the same as in the previous expression. Differentiating gives the acceleration,

$$\begin{aligned}\mathbf{a} &= \mathbf{a}_{\mathrm{CM}} + \boldsymbol{\alpha} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}}) + \boldsymbol{\omega} \times (\mathbf{v} - \mathbf{v}_{\mathrm{CM}}) \\ &= \mathbf{a}_{\mathrm{CM}} + \boldsymbol{\alpha} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}})).\end{aligned}$$

If you need an acceleration, this form tends to be easiest to use. The three terms represent the acceleration of the center of mass, the angular acceleration, and the centripetal acceleration, written in a slightly fancy way.

Remark 1: Cross Products

The cross product of two vectors is antisymmetric and distributes over addition,

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \quad \mathbf{a} \times (r_1 \mathbf{b}_1 + r_2 \mathbf{b}_2) = r_1 \mathbf{a} \times \mathbf{b}_1 + r_2 \mathbf{a} \times \mathbf{b}_2.$$

The latter means a cross product can be differentiated using the product rule. Moreover,

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}.$$

These are all you need to compute any cross product, but it's also helpful to get geometric intuition. If $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, then the direction of $\mathbf{c}$ is found by applying the right-hand rule to $\mathbf{a}$ and $\mathbf{b}$, and its magnitude is $|\mathbf{a}||\mathbf{b}|\sin\theta$ where θ is the angle between them. Finally, when differentiating a cross product, the ordinary product rule applies (just like for dot products).

Example 1

Describe the velocities of points in a disc rolling without slipping using both methods.

Solution

Let the disc have radius R, lie in the xy plane, and roll along the x axis. Consider the moment where the bottom of the disc touches the origin. At this moment its motion can be thought of as pure rotation about the origin,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = -\omega \hat{\mathbf{z}} \times \mathbf{r}.$$

On the other hand, the motion can also be thought of as simultaneous translation of the center of mass and rotation about the center of mass, so

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_{\text{CM}} + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\text{CM}}) \\ &= \omega R \hat{\mathbf{x}} - \omega \hat{\mathbf{z}} \times (\mathbf{r} - R\hat{\mathbf{y}}) \\ &= \omega R \hat{\mathbf{x}} - \omega \hat{\mathbf{z}} \times \mathbf{r} - \omega R \hat{\mathbf{x}} \\ &= -\omega \hat{\mathbf{z}} \times \mathbf{r}. \end{aligned}$$

As we can see, both decompositions are completely equivalent. Which one you want to use depends on the problem; you might even use both in the same problem.

Example 2

In ancient times, large stone slabs were transported by rolling them on logs. Each log has radius R. How far does a slab of length L move between the time a certain log is at its front and back?

Solution

Since the logs roll without slipping, the tops move twice as fast as the centers, so the slab moves by $2L$.

[4] **Problem 1.** Some brief puzzles about rotation.

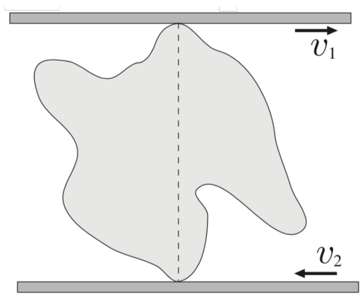
(a) Consider two identical coins laid flat on a table. One is fixed in place, and the second is rolled without slipping around the first. Once the second coin's center has returned to its original position, how many times has it rotated? (Be sure to check your answer experimentally!)

(b) A bicycle wheel is rolling without slipping. When it is photographed, its spokes look blurred, except along a curve of special points, which don't look blurred at all. What is this curve?

(c) Consider a spaceship floating in space, without any thrusters that can expel material. Conservation of momentum implies that it cannot move its center of mass. But is it possible to turn the spaceship around? In other words, is it possible for it to begin stationary in one orientation, and end up stationary in another orientation? If so, why doesn't this violate conservation of angular momentum?

(d) Hold out your arm with your elbow bent at 90° and your palm straight out, facing down. Find a way to end up in the same position but with your palm facing up, without ever bending or rotating your wrist.

[2] **Problem 2** (Kalda). A rigid lump is squeezed between two places, one of which is moving at velocity v_1 and the other at v_2. At some moment, the velocities are horizontal and the two contact points are vertically aligned.



Indicate geometrically all of the points in the body with speed either v_1 or v_2.

[2] **Problem 3.** USAPhO 2010, problem A1.

2 Moments of Inertia

Idea 2

For a two-dimensional object, the moment of inertia

$$I = \int x^2 + y^2 \, dm$$

about the origin obeys the parallel axis theorem

$$I = I_{\text{CM}} + M r_{\text{CM}}^2$$

where I_{CM} is the moment of inertia about the center of mass, and M is the total mass. Defining I_x and I_y to be the moment of inertia about the x and y, we have

$$I = I_x + I_y, \quad I_x = \int y^2 \, dm, \quad I_y = \int x^2 \, dm$$

which is called the perpendicular axis theorem.

[3] **Problem 4.** Basic moment of inertia computations.

(a) Compute the moment of inertia for an $L_x \times L_y$ rectangular plate about an axis passing perpendicular to it through the center.

(b) Compute the moment of inertia for a uniform disc of radius R and mass M, about an axis perpendicular to it through its center. What about an axis lying in the disc, passing through its center?

(c) Compute the moment of inertia of a uniform solid cone of mass M, with height H and a base of radius R, about its symmetry axis.

[2] **Problem 5** (***F* = *ma*** **2016 24).** The moment of inertia of a uniform equilateral triangle with mass m and side length a about an axis through one of its sides and parallel to that side is $ma^2/8$. What is the moment of inertia of a uniform regular hexagon of mass m and side length a about an axis through two opposite vertices?

3 Rotational Dynamics

In this section we'll consider some dynamic problems involving rotation.

Idea 3: Angular Momentum

For a system of particles we define the angular momentum and torque

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i, \quad \boldsymbol{\tau} = \sum_i \mathbf{r}_i \times \mathbf{F}_i, \quad \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}.$$

Using the first part of idea 1, we may write the angular momentum of a rigid body as

$$\mathbf{L} = I\boldsymbol{\omega}, \quad K = \frac{1}{2}I\omega^2$$

where I is the moment of inertia about the instantaneous axis of rotation. Alternatively, using the second part,

$$\mathbf{L} = I_{\text{CM}}\boldsymbol{\omega} + \mathbf{r}_{\text{CM}} \times M\mathbf{v}_{\text{CM}}, \quad K = \frac{1}{2}I_{\text{CM}}\omega^2 + \frac{1}{2}Mv_{\text{CM}}^2$$

where M is the total mass. Both forms are useful in different situations. Systems cannot exert torques on themselves, provided they obey the strong form of Newton's third law: the force between two objects is equal and opposite, and directed along the line joining them.

Idea 4

The idea above refers to taking torques about a fixed point, but often it is easier to consider a moving point P. Let $\mathbf{L}$ be the angular momentum about point P in the frame of P, i.e. the frame whose axes don't rotate, but whose origin follows P around. Working in this frame will produce fictitious forces, since P can accelerate. Such forces act at the center of mass, just like gravity.

The upshot is that if P is the center of mass, then the fictitious force in the frame of P will produce no "fictitious torque". So it's safe to use $\boldsymbol{\tau} = d\mathbf{L}/dt$ about either a fixed point, or in the frame of the center of mass.

Idea 5

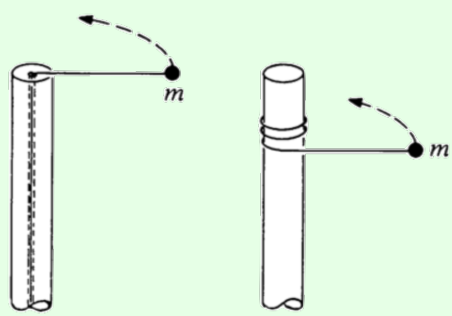
There is a third, more confusing way of applying $\boldsymbol{\tau} = d\mathbf{L}/dt$ that you might rarely see: taking torques about the instantaneous center of rotation. In general, this doesn't work, because the instantaneous center of rotation can accelerate, producing an extra fictitious torque as mentioned above.

However, it turns out this procedure gives the correct answer if the object is instantaneously at rest. That's why taking torques about the contact point for the spool in **M2** to find the initial angular acceleration was valid. It wouldn't have been valid at any instant afterward, after the spool had picked up some velocity.

For more discussion of this subtlety, which isn't mentioned in any textbooks I know of, see the paper *Moments to be cautious of*.

Example 3: KK 6.13

A mass m is attached to a post of radius R by a string. Initially it is a distance r from the center of the post and is moving tangentially with speed v_0. In case (a) the string passes through a hole in the center of the post at the top. The string is gradually shortened by drawing it through the hole. In case (b) the string wraps around the outside of the post. Ignore gravity.



(a) (b)

For each case, find the final speed of the mass when it hits the post.

Solution

In case (a), the energy isn't conserved, since work is done on the mass as it moves inward. (Physically, we can see this by noting there could be a weight slowly descending on the other end of the string.) However, angular momentum conservation says $Rv = rv_0$, so $v = rv_0/R$.

If you don't believe in angular momentum conservation yet, it's not too hard to show this with $F = ma$ as well. Let the tangential and radial speeds of the mass be v_t and v_r, where $v_r \ll v_t$. Since v_r is nonzero, there is a component of acceleration parallel to the velocity,

$$\frac{T}{m} \sin\theta \approx \frac{v_t^2}{r} \frac{v_r}{v_t}$$

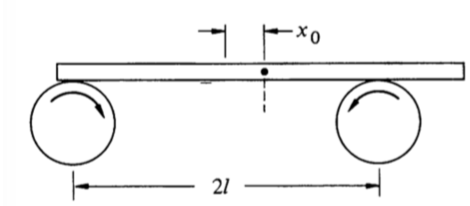
and this is equal to the rate of change of speed, which to first order in v_r/v_t is dv_t/dt. Thus,

$$\frac{dv_t}{dt} = \frac{v_r v_t}{r} = -\frac{v_t}{r}\frac{dr}{dt}$$

from which we conclude rv_t is constant, as expected. (As mentioned in **M2**, you never *need* ideas like torque and angular momentum. Life is just harder without them.)

In case (b), the angular momentum about the axis of the pole isn't conserved, since the tension force has a lever arm about that axis. However, the mass's energy is conserved. A simple physical way to see this is to note that the massless string can't store any energy, and the post doesn't do work on the string, which means the string can't do any work on the mass. Thus, the final speed is just $v = v_0$. (Of course, if you don't believe in energy conservation, you could get the same result by showing that the trajectory of the mass is always perpendicular to the string, though this takes more work.)

[2] **Problem 6** (KK 6.9)**.** A heavy uniform bar of mass M rests on top of two identical rollers which are continuously turned rapidly in opposite directions, as shown.

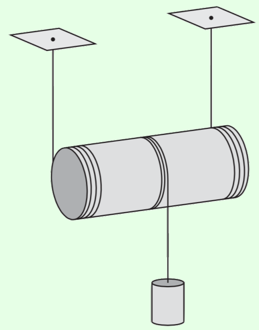


The centers of the rollers are a distance 2ℓ apart. The coefficient of friction between the bar and the roller surfaces is μ, a constant independent of the relative speed of the two surfaces. Initially the bar is held at rest with its center at distance x_0 from the midpoint of the rollers. At time $t = 0$ it is released. Find the subsequent motion of the bar.

[2] **Problem 7** (BAUPC)**.** A mass is connected to one end of a massless string, the other end of which is connected to a very thin frictionless vertical pole. The string is initially wound completely around the pole, in a very large number of small horizontal circles, with the mass touching the pole. The mass is released, and the string gradually unwinds. What angle does the string make with the pole when it becomes completely unwound? (Though the setup is similar to that of example 3, you can't ignore gravity here.)

Example 4: MPPP 49

A uniform rod of mass M and radius R is attached to two identical strings. The strings are wound around the cylinder as shown, and their free ends are fastened to the ceiling.



A third cord is attached to and wound around the middle of the cylinder, and a mass M is attached to the other side. There is sufficient friction so that the strings do not slip. Find the acceleration of the mass immediately after release.

Solution

Let a be the downward acceleration of the center of mass of the rod, let T_1 be the total tension in the first two strings, and let T_2 be the tension in the third. The rod rolls without slipping about its contact axis with the first two strings, which means the downward acceleration of the mass is $a_{\text{mass}} = 2a$.

The Newton's second law equations are thus

$$Ma = T_2 + Mg - T_1, \quad 2Ma = Mg - T_2$$

for the rod and mass. Taking torques about the axis of the rod gives

$$(T_1 + T_2)R = \frac{1}{2}MR^2\,\alpha$$

and using $a = \alpha R$ converts this to

$$Ma = 2T_1 + 2T_2.$$

We now have three equations in three unknowns, so we can straightforwardly solve to find $a = (6/11)g$. This implies that the acceleration of the mass is

$$a_{\text{mass}} = \frac{12}{11}g.$$

Done, right? No, this is the wrong answer! Since the acceleration is greater than free fall, the tension T_2 must be negative. But a string can't support a negative tension, so it instead goes slack. The mass thus free falls, so $a_{\text{mass}} = g$.

In retrospect, we could have seen this conclusion with less work. Suppose the mass were not attached. Then the acceleration of the rod can be computed with the standard rolling without slipping formula,

$$a = \frac{g\sin\theta}{1+\beta} = \frac{g}{1+\beta}, \quad I = \beta MR^2.$$

For *any* (axially symmetric) mass distribution in the rod, we have $0 \leq \beta \leq 1$. The acceleration of the part where the mass would have been attached is hence

$$a_{\text{mass}} = \frac{2g}{1+\beta} \geq g.$$

This implies that any string we attach there must go slack immediately after release.

Example 5

If you're riding a bike and need to stop quickly, what are the advantages and disadvantages of using the front brake versus the rear brake?

Solution

Work in the reference frame moving with the bike. In this frame, the backward friction force is balanced by a forward friction force on the center of mass; the combination of the two produces a torque that tends to lift the rear wheel off the ground. If you use the front brake, you can stop more quickly, because the normal force on the front tire stays higher. But if you brake too hard with the front brake, you could flip yourself over the handlebars. This can't happen when using the rear brake alone, because the brake stops doing anything the moment the rear wheel lifts off the ground.

Idea 6

It is often useful in rotational dynamics to treat the rotational and linear motion of a rigid body conceptually separately.

Example 6: $F = ma$ 2018 B23

Two particles with mass m_1 and m_2 are connected by a massless rigid rod of length L and placed on a horizontal frictionless table. At time $t = 0$, the first mass receives an impulse perpendicular to the rod, giving it speed v. At this moment, the second mass is at rest. When is the next time the second mass is at rest?

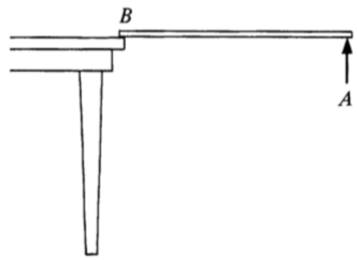
Solution

The motion is the superposition of two motions: uniform translation of both masses with speed $m_1 v/(m_1 + m_2)$ and circular motion about the common center of mass, where the two masses have speeds $m_2 v/(m_1 + m_2)$ and $m_1 v/(m_1 + m_2)$, respectively. This ensures that the second mass begins at rest and the first mass has speed v.

The circular part of the motion determines when the second mass will be at rest again. The radius of the circle the second mass makes is its distance from the center of mass, $Lm_1/(m_1 + m_2)$. This gives a period of

$$t = \frac{2\pi L m_1/(m_1 + m_2)}{m_1 v/(m_1 + m_2)} = \frac{2\pi L}{v}.$$

[2] **Problem 8** (KK 6.14)**.** A uniform stick of mass M and length ℓ is suspended horizontally with end B on the edge of a table, while end A is held by hand.

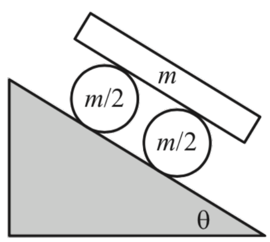


Point A is suddenly released. Right after release, find the vertical force at B, as well as the downward acceleration of point A. You should find a result greater than g. Explain how this can be possible, given that gravity is the only downward external force in the problem.

[2] **Problem 9.** Quarterfinal 2005, problem 4. This is a neat example of separating out rotational and translational motion. For a similar idea, see Morin 8.73.

[2] **Problem 10** (Morin 8.71)**.** A ball sits at rest on a piece of paper on a table. You pull the paper in a straight line out from underneath the ball. You are free to pull the paper in an arbitrary way forward or backwards; you may even jerk it so that the ball starts to slip. After the ball comes off the paper, it will eventually roll without slipping. Show that, in fact, the ball ends up at rest. Is it possible to pull the paper in such a way that the ball ends up exactly where it started?

[2] **Problem 11** (Morin 8.28)**.** Consider the following "car" on an inclined plane.



The system is released from rest, and there is no slipping between any surfaces. Find the acceleration of the board.

[2] **Problem 12.** USAPhO 2006, problem A1.

[2] **Problem 13.** USAPhO 2013, problem A2.

[3] **Problem 14.** USAPhO 2014, problem A1.

[3] **Problem 15.** A uniform stick of length L and mass M begins at rest. A massless rocket is attached to the end of the stick, and provides a constant force F perpendicular to the stick. Find the speed of the center of mass of the stick after a long time. Ignore gravity. You may find the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi},$$

which we first encountered in **P1**, useful.

[4] **Problem 16** (KK 6.41)**.** A plank of length $2L$ leans nearly vertically against a wall. All surfaces are frictionless. The plank starts to slip downward. Find the height of the top of the plank when it loses contact with the wall or floor.

Example 7: EFPhO 2013

A uniform ball and a uniform ring are both released from rest from the same height on an inclined plane with inclination angle θ. They arrive at the bottom of the plane in time T_B and T_R, respectively. The coefficients of friction of both objects with the plane are $\mu_k = 0.3$ and $\mu_s = 0.5$. Find the ratio T_B/T_R as a function of the angle θ.

Solution

When rolling without slipping, the acceleration of an object with moment of inertia $\beta m R^2$ about its center of mass is

$$a = \frac{g \sin\theta}{1+\beta}$$

as mentioned in a previous example. The tangential force from friction is thus

$$f = mg\sin\theta \, \frac{\beta}{1+\beta}$$

which means rolling without slipping occurs when

$$\mu_s mg\cos\theta \geq mg\sin\theta \frac{\beta}{1+\beta}$$

or equivalently

$$\tan\theta \leq \mu_s \, \frac{1+\beta}{\beta}.$$

For the ball, this is when $\theta \leq 60.3^\circ$, and for the ring $\theta \leq 45^\circ$. Whenever either object slips, its acceleration is instead $a = g(\sin\theta - \mu_k \cos\theta)$.

Since the motion is uniformly accelerated, $T_B/T_R = \sqrt{a_R/a_B}$. For $\theta \leq 45^\circ$, both roll without slipping, so the formula above applies, giving a ratio of

$$\frac{T_B}{T_R} = \sqrt{\frac{1+\beta_B}{1+\beta_R}} = \sqrt{\frac{7}{10}}.$$

For $\theta \geq 60.3^\circ$ they both slip, so the ratio is unity. For the angles in between, the ring slips, giving a slightly more complicated expression. At the boundaries between these three regimes, the ratio T_B/T_R jumps discontinuously.

The next two problems require careful thought, and test your understanding of the multiple ways to describe rotational kinematics and dynamics. It will be useful to review idea 1.

[3] **Problem 17.** USAPhO 1999, problem B1.

[3] **Problem 18.** USAPhO 2019, problem B3. It's worth reading the solution carefully afterward.

4 Rotational Collisions

Idea 7: Angular Impulse

During a collision with impulse $\mathbf{J}$, the angular momentum changes by the "angular impulse" $\mathbf{r} \times \mathbf{J}$. In many problems involving collisions which conserve angular momentum, energy is necessarily lost in the collision process. This is another example of an inherently inelastic process, an idea we first encountered in **M3**.

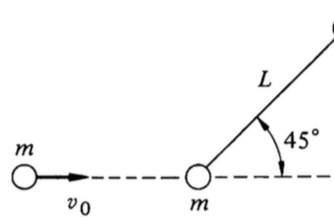
[3] **Problem 19** (Morin 8.22)**.** A uniform ball of radius R and mass m rolls without slipping with speed v_0. It encounters a step of height h and rolls up over it.

(a) Assuming that the ball sticks to the step during this process, show that for the ball to climb over the step,

$$v_0 \geq \sqrt{\frac{10gh}{7}} \left(1 - \frac{5h}{7R}\right)^{-1}.$$

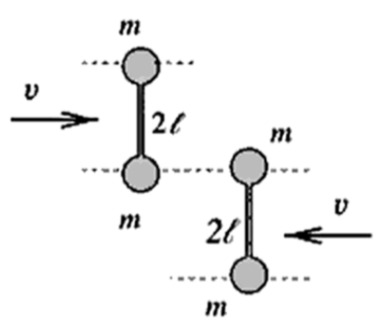
(b) Energy is lost to heat by the inelastic collision of the ball with the step. In the limit of small h, how much heat is produced?

[3] **Problem 20** (KK 6.38)**.** A rigid massless rod of length L joins two particles, each of mass m. The rod lies on a frictionless table, and is struck by a particle of mass m and velocity v_0 as shown.



After an elastic collision, the projectile moves straight back. Find the angular velocity of the rod about its center of mass after the collision.

[3] **Problem 21** (PPP 47)**.** Two identical dumbbells move towards each other on a frictionless table as shown.



Each consists of two point masses m joined by a massless rod of length 2ℓ. The dumbbells collide elastically; describe what happens afterward.

[3] **Problem 22.** USAPhO 2014, problem B1.

[4] **Problem 23.** EuPhO 2018, problem 1. An elegant rotation problem.

5 Rotational Oscillations

In this section we'll consider small oscillations problems involving rotation.

Idea 8

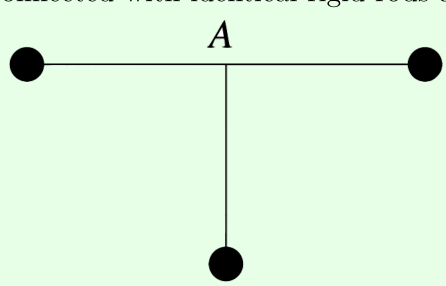
A physical pendulum is a rigid body of mass m pivoted a distance d from its center of mass, with moment of inertia I about the pivot. When considering physical pendulums, we always assume the pivot exerts no torque on the pendulum; that is, it is a "simple support", providing no bending moment, as discussed in **M2**. This is a good approximation if the pivot is smooth and small. In this case, the angular frequency for small oscillations is

$$\omega = \sqrt{\frac{mgd}{I}}.$$

For some neat real-world applications of this formula, see this paper.

Example 8: $F = ma$ 2018 A14

Three identical masses are connected with identical rigid rods and pivoted at point A.



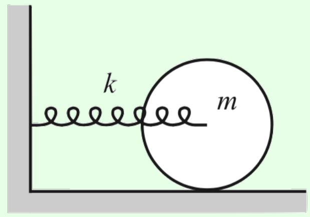
If the lowest mass receives a small horizontal push to the left, it oscillates with period T_1. If it receives a small push into the page, it oscillates with period T_2. Find the ratio T_1/T_2.

Solution

Both modes are physical pendulums, which have period proportional to $\sqrt{I/Mgx}$ where x is the distance from the pivot to the center of mass, and I is the moment of inertia about the pivot. Since x is the same in both cases, $T_1/T_2 \propto \sqrt{I_1/I_2} = \sqrt{3}$, because in the second case only the bottom mass contributes to the moment of inertia.

Example 9: Morin 8.41

The axis of a solid cylinder of mass m and radius r is connected to a spring of spring constant k, as shown.



If the cylinder rolls without slipping, find the angular frequency of the oscillations.

Solution

This is a question best handled using the energy methods of **M4**. The potential energy is $kx^2/2$ as usual, where x describes the position of the cylinder's center of mass. The kinetic energy is $mv^2/2 + I\omega^2/2 = (3/4)mv^2$, since the cylinder is rolling without slipping. Therefore

$$\omega = \sqrt{\frac{k}{m_{\text{eff}}}} = \sqrt{\frac{2k}{3m}}.$$

More complicated variants of this kind of problem can be solved in a similar way.

Example 10: Russia 2011

A uniform ring of mass m and radius r is suspended symmetrically on three inextensible strings of length ℓ. Find the angular frequency of small oscillations.

Solution

The small oscillations are torsional, i.e. the ring rotates about its axis of symmetry. When the ring has twisted by an angle θ, the strings are an angle $\phi \approx (r/\ell)\theta$ from the vertical. Thus, summing over the three strings, the restoring torque is

$$\tau \approx -mgr\phi \approx -\frac{mgr^2}{\ell}\,\theta.$$

Setting this equal to $I\alpha$, we find $\omega = \sqrt{g/\ell}$.

The tricky thing about this problem is that it's harder to solve with the energy method. If you try, you immediately run into the problem that there seems to be no potential energy anywhere, since the strings don't stretch! The source of the potential energy is that the ring moves up a small amount as it oscillates, since the strings are no longer vertical,

$$h = \ell - \sqrt{\ell^2 - r^2\theta^2} \approx \frac{r^2\theta^2}{2\ell}.$$

Therefore we have

$$K = \frac{1}{2}mr^2\dot{\theta}^2, \quad V = \frac{1}{2}\frac{mgr^2}{\ell}\theta^2$$

and the answer follows as usual. (There is also a kinetic energy contribution from the ring's vertical motion, but it's negligible.) The lesson here is that the force/torque and energy approach have different strengths. The energy approach is often easier because it lets you ignore some internal details of the system. But it can be harder because it requires you to understand the kinematics of the system to second order, rather than first order.

[2] **Problem 24.** A circular pendulum consists of a point mass m on a string of length ℓ, which is made to rotate in a horizontal circle. By using only the equation $\boldsymbol{\tau} = d\mathbf{L}/dt$ about an origin of your choice, compute the angular frequency if the string makes a constant angle θ with the horizontal.

[3] **Problem 25.** Using a physical pendulum, one can measure the acceleration due to gravity as

$$g = \frac{4\pi^2}{T^2} \frac{I}{md}.$$

In practice, I is not very precisely known, since it depends on the exact shape of the material. Kater found an ingenious way to circumvent this problem. We pivot the pendulum at an arbitrary point and measure the period T. Next, by trial and error, we find another pivot point which has the same period, which lies at a different distance from the center of mass. Show that

$$g = \frac{4\pi^2 L}{T^2}$$

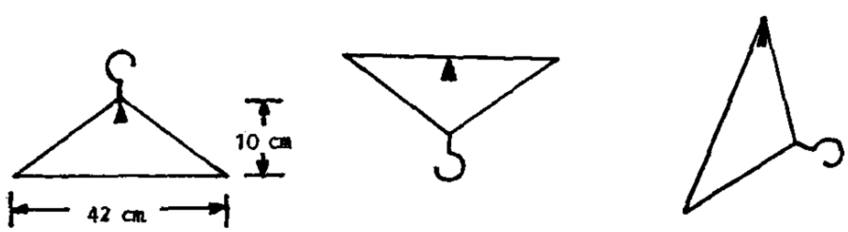
where L is the sum of the lengths from these points to the CM. This allows a measurement of g without knowledge of the moment of inertia about the center of mass. (Kater selected his two pivot points to lie on a line, on opposite sides of the center of mass. This has the additional benefit that L is simply the distance between the pivot points, removing the need to find the center of mass.)

[3] **Problem 26.** USAPhO 1999, problem A4.

[3] **Problem 27.** USAPhO 2011, problem B2.

[3] **Problem 28.** USAPhO 2002, problem B1. This one is trickier than it looks! It can be solved with either a torque or energy analysis, but both require care.

[4] **Problem 29** (IPhO 1982)**.** A coat hanger can perform small oscillations in the plane of the figure about the three equilibrium figures shown.



In the first two, the long side is horizontal. The other two sides have equal length. The period of oscillation is the same in all cases. The coat hanger does *not* necessarily have uniform density. Where is the center of mass, and how long is the period?

[4] **Problem 30** (APhO 2007)**.** A uniform ball of mass M and radius r is encased in a thin spherical shell, also of mass M. The shell is placed inside a fixed spherical bowl of radius R, and performs small oscillations about the bottom. Assume that friction between the bowl and shell is very large, so the shell essentially always rolls without slipping.

The ball is made of an unusual material: it can quickly transition between a liquid and solid state. When the ball is in the liquid state, it has no viscosity, and hence no friction with the shell. When the ball is in the solid state, it rotates with the shell.

(a) Find the period of the oscillations if the ball is always in the solid state.

(b) Find the period of the oscillations if the ball is always in the liquid state.

(c) The ball is now set so that it instantly switches to the liquid state whenever it starts moving downward, and instantly switches to the solid state whenever it starts moving upward. If the initial amplitude of oscillations is θ_0, find the amplitude after n oscillations.

Mechanics V: Rotation

Two-dimensional rotation is covered in chapter 6 of Kleppner, chapter 8 of Morin, or chapters 3 and 5 of Wang and Ricardo, volume 1. Further discussion is given in chapters I-18 and I-19 of the Feynman lectures. For more on dot and cross products, see the first two lectures of MIT OCW 18.02. There is a total of **83** points.

1 2D Rotational Kinematics

Idea 1

The instantaneous velocity of a two-dimensional rigid body can always be written as pure rotation about some point $\mathbf{r}_0$, not necessarily in the body, so the velocity of a point in the body at location $\mathbf{r}$ is

$$\mathbf{v} = \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0),$$

as you can check with some geometry. This equation defines $\boldsymbol{\omega}$, the angular velocity vector, which points out of the page. Differentiating gives the acceleration of a point in the body,

$$\begin{aligned}\mathbf{a} &= \boldsymbol{\alpha} \times (\mathbf{r} - \mathbf{r}_0) + \boldsymbol{\omega} \times (\mathbf{v} - \mathbf{v}_0) \\ &= \boldsymbol{\alpha} \times (\mathbf{r} - \mathbf{r}_0) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0)) - \boldsymbol{\omega} \times \mathbf{v}_0\end{aligned}$$

where $\mathbf{v}_0 = d\mathbf{r}_0/dt$ is the rate of change of the location of the pivot point. (Note that this differs from the velocity of the point in the body instantaneously at the pivot, which is always zero.) However, this latter expression is often hard to use, because you usually won't know $\mathbf{r}_0(t)$, or it'll have a complicated form.

Alternatively, we can write the velocity in terms of translation plus pure rotation about any desired point, which is almost always chosen to be the center of mass. This gives

$$\mathbf{v} = \mathbf{v}_{\mathrm{CM}} + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}}).$$

The $\boldsymbol{\omega}$ here is the same as in the previous expression. Differentiating gives the acceleration,

$$\begin{aligned}\mathbf{a} &= \mathbf{a}_{\mathrm{CM}} + \boldsymbol{\alpha} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}}) + \boldsymbol{\omega} \times (\mathbf{v} - \mathbf{v}_{\mathrm{CM}}) \\ &= \mathbf{a}_{\mathrm{CM}} + \boldsymbol{\alpha} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}})).\end{aligned}$$

If you need an acceleration, this form tends to be easiest to use. The three terms represent the acceleration of the center of mass, the angular acceleration, and the centripetal acceleration, written in a slightly fancy way.

Remark 1: Cross Products

The cross product of two vectors is antisymmetric and distributes over addition,

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \quad \mathbf{a} \times (r_1\mathbf{b}_1 + r_2\mathbf{b}_2) = r_1\mathbf{a} \times \mathbf{b}_1 + r_2\mathbf{a} \times \mathbf{b}_2.$$

The latter means a cross product can be differentiated using the product rule. Moreover,

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}.$$

These are all you need to compute any cross product, but it's also helpful to get geometric intuition. If $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, then the direction of $\mathbf{c}$ is found by applying the right-hand rule to $\mathbf{a}$ and $\mathbf{b}$, and its magnitude is $|\mathbf{a}||\mathbf{b}| \sin\theta$ where θ is the angle between them. Finally, when differentiating a cross product, the ordinary product rule applies (just like for dot products).

Example 1

Describe the velocities of points in a disc rolling without slipping using both methods.

Solution

Let the disc have radius R, lie in the xy plane, and roll along the x axis. Consider the moment where the bottom of the disc touches the origin. At this moment its motion can be thought of as pure rotation about the origin,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = -\omega \hat{\mathbf{z}} \times \mathbf{r}.$$

On the other hand, the motion can also be thought of as simultaneous translation of the center of mass and rotation about the center of mass, so

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_{\mathrm{CM}} + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}}) \\ &= \omega R \hat{\mathbf{x}} - \omega \hat{\mathbf{z}} \times (\mathbf{r} - R\hat{\mathbf{y}}) \\ &= \omega R \hat{\mathbf{x}} - \omega \hat{\mathbf{z}} \times \mathbf{r} - \omega R \hat{\mathbf{x}} \\ &= -\omega \hat{\mathbf{z}} \times \mathbf{r}. \end{aligned}$$

As we can see, both decompositions are completely equivalent. Which one you want to use depends on the problem; you might even use both in the same problem.

Example 2

In ancient times, large stone slabs were transported by rolling them on logs. Each log has radius R. How far does a slab of length L move between the time a certain log is at its front and back?

Solution

Since the logs roll without slipping, the tops move twice as fast as the centers, so the slab moves by $2L$.

[4] **Problem 1.** Some brief puzzles about rotation.

(a) Consider two identical coins laid flat on a table. One is fixed in place, and the second is rolled without slipping around the first. Once the second coin's center has returned to its original position, how many times has it rotated? (Be sure to check your answer experimentally!)

(b) A bicycle wheel is rolling without slipping. When it is photographed, its spokes look blurred, except along a curve of special points, which don't look blurred at all. What is this curve?

(c) Consider a spaceship floating in space, without any thrusters that can expel material. Conservation of momentum implies that it cannot move its center of mass. But is it possible to turn the spaceship around? In other words, is it possible for it to begin stationary in one orientation, and end up stationary in another orientation? If so, why doesn't this violate conservation of angular momentum?

(d) Hold out your arm with your elbow bent at 90° and your palm straight out, facing down. Find a way to end up in the same position but with your palm facing up, without ever bending or rotating your wrist.

Solution. (a) Since the two coins have the same circumference, you might think the answer is 1. However, the answer is 2, as is easily checked experimentally. Rolling around a convex curved surface gives an extra turn, as you can check with limiting cases, such as rolling around a big square.

Another way to think about this is that the center of the coin moves in a circle of radius $2r$, where r is the radius of the coin. Since the coin rolls without slipping, $v_{\text{CM}} = \omega r$ at all times. Integrating this result, $d_{\text{CM}} = \theta r$ where d_{CM} is the distance through which the CM moves, and θ is the total turn angle. Then $2\pi(2r) = \theta r$ which gives $\theta = 4\pi$.

(b) Note that the motion can be described as pure rotation about the contact point C. For the special points P, we want the velocity of that point to be parallel to the spokes, so the line CP to be perpendicular to the spoke OP. It is not hard to see that this locus is the circle with diameter OC.

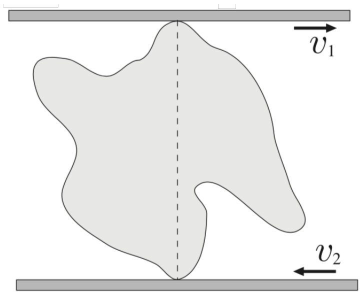
(c) Just rotate a wheel inside the spaceship. If the wheel spins clockwise, then the rest of the spaceship will start spinning counterclockwise, by conservation of angular momentum. Then the wheel can be stopped when the spaceship has the desired final orientation. (This is actually how spaceships turn around: they carry large reaction wheels which are spun up or down as needed. The ability to change orientation is essential for space telescopes, and in practice the wheels are always rotating fairly quickly, because their angular momentum can gyroscopically stabilize the rest of the ship.)

The fundamental reason this works is that rotations are periodic; unlike translations, you can give something a net rotation but also end up back where you started. For a closely related trick, see how a falling cat can turn itself around.

(d) Starting from the original position, bring your forearm horizontally to your chest, then rotate it vertically, then return it to the original position. At this point, your palm should be facing horizontally. Repeat the sequence to get it facing downward.

The basic reason this works is that your wrist and palm are constrained to move along a sphere, and the surface of a sphere is curved. Curvature intrinsically means that this kind of "parallel transport" doesn't necessarily return you to your original configuration. It's an important idea in differential geometry and general relativity. (The detailed math tells us that the angle through which your palm rotates is proportional to the solid angle traced out by the loop. So in theory, you could also achieve the same thing by moving your hand in one giant loop, though this takes some flexibility, or ten times in a small loop. The latter might not work in practice, though, because your brain might unconsciously rotate your wrist a bit to compensate for the effect.)

[2] **Problem 2** (Kalda)**.** A rigid lump is squeezed between two places, one of which is moving at velocity v_1 and the other at v_2. At some moment, the velocities are horizontal and the two contact points are vertically aligned.



Indicate geometrically all of the points in the body with speed either v_1 or v_2.

Solution. Note that the motion of the rigid body can be expressed as rotation about some point. Let this point be O. It must be on the vertical line connecting the two contact points, with distances to those points satisfying $\omega = v_1/r_1 = v_2/r_2$, where $r_1 + r_2$ is the distance between the contact points. Then, all points with speed v_1 lie on the circle centered at O with radius r_1, and radius r_2 for v_2.

[2] **Problem 3.** USAPhO 2010, problem A1.

2 Moments of Inertia

Idea 2

For a two-dimensional object, the moment of inertia

$$I = \int x^2 + y^2 \, dm$$

about the origin obeys the parallel axis theorem

$$I = I_{\text{CM}} + M r_{\text{CM}}^2$$

where I_{CM} is the moment of inertia about the center of mass, and M is the total mass. Defining I_x and I_y to be the moment of inertia about the x and y, we have

$$I = I_x + I_y, \quad I_x = \int y^2 \, dm, \quad I_y = \int x^2 \, dm$$

which is called the perpendicular axis theorem.

[3] **Problem 4.** Basic moment of inertia computations.

(a) Compute the moment of inertia for an $L_x \times L_y$ rectangular plate about an axis passing perpendicular to it through the center.

(b) Compute the moment of inertia for a uniform disc of radius R and mass M, about an axis perpendicular to it through its center. What about an axis lying in the disc, passing through its center?

(c) Compute the moment of inertia of a uniform solid cone of mass M, with height H and a base of radius R, about its symmetry axis.

Solution. (a) For a uniform rod of length L, $I = \int_{-L/2}^{L/2} x^2(M/L)dx = \frac{1}{12}ML^2$. So by the perpendicular axis theorem, the answer is $\boxed{\frac{1}{12}M(L_x^2 + L_y^2)}$.

(b) We see that $I = \int_0^R r^2(M/\pi R^2)2\pi r dr = \boxed{MR^2/2}$. For an axis lying in the disc, the answer is half as much, $\boxed{MR^2/4}$, by the perpendicular axis theorem.

(c) First off, we know the height H doesn't matter, because we can stretch the cone alone its symmetry axis without changing the answer. Letting the density be ρ, we can integrate over the discs making up the cone,

$$I = \int dI = \int \frac{1}{2}(dm)r^2 = \int_0^H \frac{1}{2}(\rho\pi r^2\, dh)r^2$$

where the radius of the disc at height h is, in some set of coordinates, $r(h) = R(h/H)$. Plugging this in, we get

$$I = \int_0^H \frac{\pi}{2}\rho R^4 \frac{h^4}{H^4}\, dh = \frac{\pi\rho R^4 H}{10}.$$

It remains to find ρ, by noting that

$$M = \int dm = \int_0^H \rho\pi r^2\, dh = \rho\pi\frac{R^2}{H^2}\int_0^h h^2\, dh = \frac{\pi\rho R^2 H}{3}.$$

Plugging in the result for ρ gives

$$I = \frac{3}{10}MR^2.$$

This makes sense, as it's somewhat less than the moment of inertia of a uniform disc; a cone has comparatively more of its mass closer to the axis.

[2] **Problem 5** (***F* = *ma*** 2016 24)**.** The moment of inertia of a uniform equilateral triangle with mass m and side length a about an axis through one of its sides and parallel to that side is $ma^2/8$. What is the moment of inertia of a uniform regular hexagon of mass m and side length a about an axis through two opposite vertices?

Solution. We see that 4 of the hexagons contribute $(m/6)a^2/8$, and the other two contribute $(m/6)a^2/8 - (m/6)(a/2\sqrt{3})^2 + (m/6)(a/\sqrt{3})^2 = 3(m/6)a^2/8$ by two applications of the parallel axis theorem. Thus, the total is $(m/6)a^2(4/8 + 3/4) = \boxed{5ma^2/24}$.

3 Rotational Dynamics

In this section we'll consider some dynamic problems involving rotation.

Idea 3: Angular Momentum

For a system of particles we define the angular momentum and torque

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i, \quad \boldsymbol{\tau} = \sum_i \mathbf{r}_i \times \mathbf{F}_i, \quad \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}.$$

Using the first part of idea 1, we may write the angular momentum of a rigid body as

$$\mathbf{L} = I\boldsymbol{\omega}, \quad K = \frac{1}{2}I\omega^2$$

where I is the moment of inertia about the instantaneous axis of rotation. Alternatively, using the second part,

$$\mathbf{L} = I_{\text{CM}}\boldsymbol{\omega} + \mathbf{r}_{\text{CM}} \times M\mathbf{v}_{\text{CM}}, \quad K = \frac{1}{2}I_{\text{CM}}\omega^2 + \frac{1}{2}Mv_{\text{CM}}^2$$

where M is the total mass. Both forms are useful in different situations. Systems cannot exert torques on themselves, provided they obey the strong form of Newton's third law: the force between two objects is equal and opposite, and directed along the line joining them.

Idea 4

The idea above refers to taking torques about a fixed point, but often it is easier to consider a moving point P. Let $\mathbf{L}$ be the angular momentum about point P in the frame of P, i.e. the frame whose axes don't rotate, but whose origin follows P around. Working in this frame will produce fictitious forces, since P can accelerate. Such forces act at the center of mass, just like gravity.

The upshot is that if P is the center of mass, then the fictitious force in the frame of P will produce no "fictitious torque". So it's safe to use $\boldsymbol{\tau} = d\mathbf{L}/dt$ about either a fixed point, or in the frame of the center of mass.

Idea 5

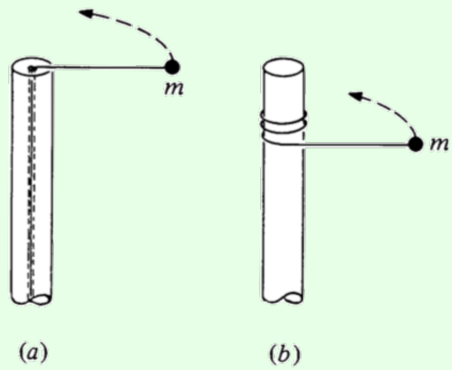
There is a third, more confusing way of applying $\boldsymbol{\tau} = d\mathbf{L}/dt$ that you might rarely see: taking torques about the instantaneous center of rotation. In general, this doesn't work, because the instantaneous center of rotation can accelerate, producing an extra fictitious torque as mentioned above.

However, it turns out this procedure gives the correct answer if the object is instantaneously at rest. That's why taking torques about the contact point for the spool in **M2** to find the initial angular acceleration was valid. It wouldn't have been valid at any instant afterward, after the spool had picked up some velocity.

For more discussion of this subtlety, which isn't mentioned in any textbooks I know of, see the paper *Moments to be cautious of*.

Example 3: KK 6.13

A mass m is attached to a post of radius R by a string. Initially it is a distance r from the center of the post and is moving tangentially with speed v_0. In case (a) the string passes through a hole in the center of the post at the top. The string is gradually shortened by drawing it through the hole. In case (b) the string wraps around the outside of the post. Ignore gravity.



For each case, find the final speed of the mass when it hits the post.

Solution

In case (a), the energy isn't conserved, since work is done on the mass as it moves inward. (Physically, we can see this by noting there could be a weight slowly descending on the other end of the string.) However, angular momentum conservation says $Rv = rv_0$, so $v = rv_0/R$.

If you don't believe in angular momentum conservation yet, it's not too hard to show this with $F = ma$ as well. Let the tangential and radial speeds of the mass be v_t and v_r, where $v_r \ll v_t$. Since v_r is nonzero, there is a component of acceleration parallel to the velocity,

$$\frac{T}{m}\sin\theta \approx \frac{v_t^2}{r}\frac{v_r}{v_t}$$

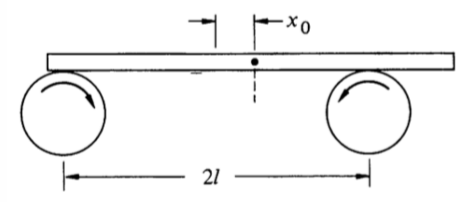
and this is equal to the rate of change of speed, which to first order in v_r/v_t is dv_t/dt. Thus,

$$\frac{dv_t}{dt} = \frac{v_r v_t}{r} = -\frac{v_t}{r}\frac{dr}{dt}$$

from which we conclude rv_t is constant, as expected. (As mentioned in **M2**, you never *need* ideas like torque and angular momentum. Life is just harder without them.)

In case (b), the angular momentum about the axis of the pole isn't conserved, since the tension force has a lever arm about that axis. However, the mass's energy is conserved. A simple physical way to see this is to note that the massless string can't store any energy, and the post doesn't do work on the string, which means the string can't do any work on the mass. Thus, the final speed is just $v = v_0$. (Of course, if you don't believe in energy conservation, you could get the same result by showing that the trajectory of the mass is always perpendicular to the string, though this takes more work.)

[2] **Problem 6** (KK 6.9). A heavy uniform bar of mass M rests on top of two identical rollers which are continuously turned rapidly in opposite directions, as shown.



The centers of the rollers are a distance 2ℓ apart. The coefficient of friction between the bar and the roller surfaces is μ, a constant independent of the relative speed of the two surfaces. Initially the bar is held at rest with its center at distance x_0 from the midpoint of the rollers. At time $t = 0$ it is released. Find the subsequent motion of the bar.

Solution. Let N_1 be the normal force from the right roller, and N_2 be the one of the left roller. Since there is no acceleration in the y-direction, we have that $N_1 + N_2 = Mg$. Also, since the bar is not rotating, we have that the torque about the center is 0, so $N_1(\ell - x_0) = N_2(\ell + x_0)$. One quickly sees that the solution to this system is

$$N_1 = \frac{Mg(\ell + x_0)}{2\ell}, \quad N_2 = \frac{Mg(\ell - x_0)}{2\ell}.$$

Now, the friction force from the right roller points to the left with magnitude $N_1\mu$, and the one from the left roller points to the right with magnitude $N_2\mu$. Therefore, the total net force on this system is

$$N_1\mu - N_2\mu = Mg\mu\frac{x_0}{\ell}$$

to the left. This is simple harmonic motion with angular frequency $\omega = \sqrt{\mu g/\ell}$. This neat system is called a "friction oscillator", or "Timoshenko oscillator".

[2] **Problem 7** (BAUPC). A mass is connected to one end of a massless string, the other end of which is connected to a very thin frictionless vertical pole. The string is initially wound completely around the pole, in a very large number of small horizontal circles, with the mass touching the pole. The mass is released, and the string gradually unwinds. What angle does the string make with the pole when it becomes completely unwound? (Though the setup is similar to that of example 3, you can't ignore gravity here.)

Solution. Let the string have length ℓ, a final angle of θ with the pole, and final angular velocity $\omega = v/\ell \sin\theta$. As it unwinds, there is no source of energy loss so energy is conserved.

$$g\ell\cos\theta = \frac{1}{2}v^2$$

The components of the force on the mass from the string is a horizontal component for a centripetal force, and a vertical component to balance gravity,

$$T\sin\theta = m\omega^2\ell\sin\theta, \quad T\cos\theta = mg.$$

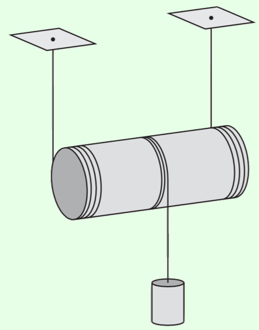
Solving yields

$$\tan\theta = \frac{v^2}{g\ell\sin\theta} = \frac{2}{\tan\theta}.$$

Thus, $\theta = \arctan(\sqrt{2}) \approx 54.74°$.

Example 4: MPPP 49

A uniform rod of mass M and radius R is attached to two identical strings. The strings are wound around the cylinder as shown, and their free ends are fastened to the ceiling.



A third cord is attached to and wound around the middle of the cylinder, and a mass M is attached to the other side. There is sufficient friction so that the strings do not slip. Find the acceleration of the mass immediately after release.

Solution

Let a be the downward acceleration of the center of mass of the rod, let T_1 be the total tension in the first two strings, and let T_2 be the tension in the third. The rod rolls without slipping about its contact axis with the first two strings, which means the downward acceleration of the mass is $a_{\text{mass}} = 2a$.

The Newton's second law equations are thus

$$Ma = T_2 + Mg - T_1, \quad 2Ma = Mg - T_2$$

for the rod and mass. Taking torques about the axis of the rod gives

$$(T_1 + T_2)R = \frac{1}{2}MR^2\,\alpha$$

and using $a = \alpha R$ converts this to

$$Ma = 2T_1 + 2T_2.$$

We now have three equations in three unknowns, so we can straightforwardly solve to find $a = (6/11)g$. This implies that the acceleration of the mass is

$$a_{\text{mass}} = \frac{12}{11}g.$$

Done, right? No, this is the wrong answer! Since the acceleration is greater than free fall, the tension T_2 must be negative. But a string can't support a negative tension, so it instead goes slack. The mass thus free falls, so $a_{\text{mass}} = g$.

In retrospect, we could have seen this conclusion with less work. Suppose the mass were not attached. Then the acceleration of the rod can be computed with the standard rolling without slipping formula,

$$a = \frac{g \sin \theta}{1 + \beta} = \frac{g}{1 + \beta}, \quad I = \beta M R^2.$$

For *any* (axially symmetric) mass distribution in the rod, we have $0 \leq \beta \leq 1$. The acceleration of the part where the mass would have been attached is hence

$$a_{\text{mass}} = \frac{2g}{1 + \beta} \geq g.$$

This implies that any string we attach there must go slack immediately after release.

Example 5

If you're riding a bike and need to stop quickly, what are the advantages and disadvantages of using the front brake versus the rear brake?

Solution

Work in the reference frame moving with the bike. In this frame, the backward friction force is balanced by a forward friction force on the center of mass; the combination of the two produces a torque that tends to lift the rear wheel off the ground. If you use the front brake, you can stop more quickly, because the normal force on the front tire stays higher. But if you brake too hard with the front brake, you could flip yourself over the handlebars. This can't happen when using the rear brake alone, because the brake stops doing anything the moment the rear wheel lifts off the ground.

Idea 6

It is often useful in rotational dynamics to treat the rotational and linear motion of a rigid body conceptually separately.

Example 6: $F = ma$ 2018 B23

Two particles with mass m_1 and m_2 are connected by a massless rigid rod of length L and placed on a horizontal frictionless table. At time $t = 0$, the first mass receives an impulse perpendicular to the rod, giving it speed v. At this moment, the second mass is at rest. When is the next time the second mass is at rest?

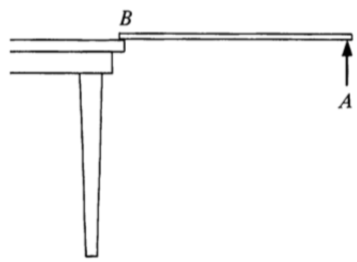
Solution

The motion is the superposition of two motions: uniform translation of both masses with speed $m_1 v/(m_1 + m_2)$ and circular motion about the common center of mass, where the two masses have speeds $m_2 v/(m_1 + m_2)$ and $m_1 v/(m_1 + m_2)$, respectively. This ensures that the second mass begins at rest and the first mass has speed v.

The circular part of the motion determines when the second mass will be at rest again. The radius of the circle the second mass makes is its distance from the center of mass, $Lm_1/(m_1+m_2)$. This gives a period of

$$t = \frac{2\pi L m_1/(m_1+m_2)}{m_1 v/(m_1+m_2)} = \frac{2\pi L}{v}.$$

[2] **Problem 8** (KK 6.14)**.** A uniform stick of mass M and length ℓ is suspended horizontally with end B on the edge of a table, while end A is held by hand.



Point A is suddenly released. Right after release, find the vertical force at B, as well as the downward acceleration of point A. You should find a result greater than g. Explain how this can be possible, given that gravity is the only downward external force in the problem.

Solution. We take torques about B, applying idea 5. Note that $\tau = Mg\ell/2 = I\alpha = \frac{1}{3}M\ell^2\alpha$, so $\alpha = \frac{3}{2}\frac{g}{\ell}$. Thus, the instantaneous acceleration of the CM is $\alpha\ell/2 = \frac{3}{4}g$ down. Therefore, $Mg - F = 3Mg/4$, so $\boxed{F = Mg/4}$. Furthermore, the acceleration of point A is $3g/2$ down. This is indeed greater than g.

The reason this is possible is that the stick is a rigid object, so it supports internal shear stresses, which keep the whole body moving as one piece. If you consider a small piece of the rod near the end, gravity provides a downward acceleration g, while a downward shear stress from the rest of the rod provides the remaining downward acceleration $g/2$.

[2] **Problem 9.** Quarterfinal 2005, problem 4. This is a neat example of separating out rotational and translational motion. For a similar idea, see Morin 8.73.

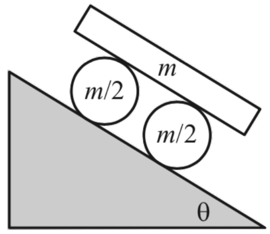
[2] **Problem 10** (Morin 8.71)**.** A ball sits at rest on a piece of paper on a table. You pull the paper in a straight line out from underneath the ball. You are free to pull the paper in an arbitrary way forward or backwards; you may even jerk it so that the ball starts to slip. After the ball comes off the paper, it will eventually roll without slipping. Show that, in fact, the ball ends up at rest. Is it possible to pull the paper in such a way that the ball ends up exactly where it started?

Solution. The normal and gravitational forces cancel, so the only relevant force on the ball is friction, which acts at the bottom. Consider the angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and torques, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, about the point of initial contact. Since $\mathbf{r}$ and $\mathbf{F}$ are always in the same plane, $\boldsymbol{\tau}$ always points perpendicular to the surface, and $\mathbf{L} = \int \boldsymbol{\tau}\, dt$ will likewise be vertical.

During the process, the ball can move, as long as the horizontal components of its spin and orbital angular momentum cancel out. But after the ball comes off the paper, it will eventually roll without slipping, and in this case the spin and orbital angular momenta point in the same direction. So the only way for the sum to be zero is for both to be zero, so the ball stops.

It is possible for the ball to end up where it started. If we just pull the paper out to the right, the ball ends up to the left of where it started. But we can do a little maneuver in the beginning to move the ball right, so that it cancels out the leftward motion in the last step. To do this, just jerk the paper to the right a bit, getting the ball started rolling to the right, then stop it later by jerking the paper to the left. Then pull the paper out to the right.

[2] **Problem 11** (Morin 8.28)**.** Consider the following "car" on an inclined plane.



The system is released from rest, and there is no slipping between any surfaces. Find the acceleration of the board.

Solution. Let the acceleration of the board be a, and the angular accelerations of the cylinders be α. Looking at one cylinder, the motion of the cylinder can be seen as pure rotation about the contact point with the slope (since there's no slipping, that point is stationary). Then the cylinder rotates about the contact point with angular acceleration α, and the top will accelerate at $\alpha(2R)$ where R is the radius of the cylinders. Thus for the board to not slip, $a = 2R\alpha$.

Taking torques about the contact point, with f being the friction force between the cylinders and board,

$$\tau = \left(\frac{m}{2}R^2 + \frac{1}{2}\frac{m}{2}R^2\right)\alpha = \frac{m}{2}gR\sin\theta - 2Rf.$$

For the acceleration of the board,

$$F = ma = 2f + mg\sin\theta.$$

Adding these two equations and substituting $\alpha R = a/2$ yields the answer,

$$a = \frac{12}{11}g\sin\theta.$$

This problem can also be solved using the "Lagrangian"/energy methods of **M4**. Let s be the distance the centers of the wheels have moved. Then by totaling up the kinetic energy,

$$K = \frac{1}{2}m\dot{s}^2 \times \left(1 + \frac{1}{2} + 4\right) \equiv \frac{1}{2}m_{\text{eff}}\dot{s}^2$$

where the $1 + 1/2$ represents the translational and rotational kinetic energy of the wheels, and the 4 represents the kinetic energy of the board, since it travels at twice the speed as the centers of the wheels. On the other hand, the potential energy is

$$V = -mgs\sin\theta(1+2) \equiv -F_{\text{eff}}s$$

where the 1 and 2 represent the potential energies of the wheels and board. Then we have

$$\ddot{s} = \frac{F_{\text{eff}}}{m_{\text{eff}}} = \frac{3mg\sin\theta}{(11/2)m} = \frac{6}{11}g\sin\theta.$$

The acceleration of the board is twice this, giving the same answer,

$$a = 2\ddot{s} = \frac{12}{11} g \sin\theta.$$

[2] **Problem 12.** USAPhO 2006, problem A1.

[2] **Problem 13.** USAPhO 2013, problem A2.

[3] **Problem 14.** USAPhO 2014, problem A1.

Solution. See the official solutions as usual. If you're curious, I also wrote up a solution that doesn't use a rotating frame here. It uses some techniques covered in **M8**.

[3] **Problem 15.** A uniform stick of length L and mass M begins at rest. A massless rocket is attached to the end of the stick, and provides a constant force F perpendicular to the stick. Find the speed of the center of mass of the stick after a long time. Ignore gravity. You may find the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi},$$

which we first encountered in **P1**, useful.

Solution. The uniform stick has moment of inertia $ML^2/12$ about its center, and has a constant torque of $\tau = FL/2$ about its center. Thus if θ is the angular distance the stick has rotated, then

$$\frac{FL}{2} = \frac{1}{12} ML^2 \ddot{\theta}$$

which implies

$$\ddot{\theta} = \frac{6F}{ML}, \quad \theta = \frac{3F}{ML} t^2 = \beta t^2.$$

Let the plane of motion of the stick be the complex plane, let the initial position of the center of mass be the origin, and let the angle between the stick and the real axis be $\theta + \pi/2$ so the force points at an angle θ. Then in the complex plane, the unit vector of the force is just $e^{i\theta}$, so

$$Ma = Fe^{i\beta t^2} \qquad v_f = \frac{F}{M} \int_0^{\infty} e^{i\beta t^2} dt.$$

This trick of using complex numbers allows us to write the two real components of Newton's second law as a single equation.

Now, if you were in a math class, you would be taught that this integral is not convergent (the integrand doesn't even go to zero at infinity), so there is no answer for v_f. But we're in a physics class, so we're allowed to use common sense. As time goes on, the stick will rotate faster and faster, so the acceleration will spin faster, so the endpoint of the velocity vector rotates in tighter and tighter circles. So even though the magnitude of the acceleration never gets smaller, the velocity does approach a limit!

We now compute this limit. Since e^{-x^2} is an even function, $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$. By dimensional analysis or u-sub, as in **P1**, we have

$$\int_0^{\infty} e^{-(ax)^2} dx = \frac{\sqrt{\pi}}{2a}.$$

In order to get $-i = e^{3\pi/2}$ from a^2, we need a factor of $\pm e^{3\pi/4} = \pm(-1+i)/\sqrt{2}$. Thus

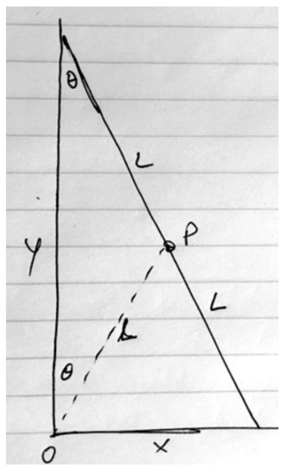
$$\frac{M}{F} v_f = \int_0^\infty e^{i\beta t^2} dt = \int_0^\infty e^{-\left(\pm\frac{-1+i}{\sqrt{2}}\sqrt{\beta}t\right)^2} dt = \pm \frac{\sqrt{\pi}}{2(-1+i)\sqrt{\beta/2}}$$

from which we conclude the final speed is

$$|v_f| = \frac{F\sqrt{\pi}}{2M\sqrt{\beta}} = \sqrt{\frac{\pi F L}{12M}}.$$

[4] **Problem 16** (KK 6.41)**.** A plank of length $2L$ leans nearly vertically against a wall. All surfaces are frictionless. The plank starts to slip downward. Find the height of the top of the plank when it loses contact with the wall or floor.

Solution.



Note that the normal forces at the contact points do no work, since the plank moves in the perpendicular directions at those points. Therefore, mechanical energy is conserved.

The center of mass P moves in a circle of radius L around O, and its speed is $L\dot{\theta}$. Similarly, one also sees that the plank rotates around its CM, P, at angular velocity $\dot{\theta}$ counterclockwise. Therefore, if the plank starts at θ_0, we have by energy conservation that

$$mgL(\cos\theta_0 - \cos\theta) = \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{2}\left(\frac{1}{3}mL^2\right)\dot{\theta}^2 = \frac{2}{3}mL^2\dot{\theta}^2,$$

so

$$\dot{\theta}^2 = \frac{3g}{2L}(\cos\theta_0 - \cos\theta).$$

Taking the time derivative, we obtain

$$2\dot{\theta}\ddot{\theta} = \frac{3g}{2L}\sin\theta\,\dot{\theta} \implies \ddot{\theta} = \frac{3g}{4L}\sin\theta.$$

Now, the plank loses contact when the normal force N_x at the high point of contact is 0. However, by Newton's second law, $N_x = m\ddot{x}$, so $N_x = 0$ when $\ddot{x} = 0$. However, $x = L\sin\theta$, so $\dot{x} = L\cos\theta\,\dot{\theta}$, so $\ddot{x} = L(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2)$. Therefore, we have

$$\cos\theta\,\ddot{\theta} = \sin\theta\,\dot{\theta}^2$$

when contact is lost. Plugging in our earlier results, we find

$$\frac{3g}{4L}\sin\theta\cos\theta = \frac{3g}{2L}(\cos\theta_0 - \cos\theta)\sin\theta,$$

or $\cos\theta = \frac{2}{3}\cos\theta_0$, so $y = \frac{2}{3}y_0$, as desired. (For completeness, we should have checked that the plank actually loses contact with the wall before losing contact with the floor. This is fairly intuitive, but it can be checked explicitly by using the above analysis to compute $\ddot{y}$ and show that N_y is positive until N_x vanishes.)

There is a slick alternative solution using Lagrangian mechanics, though it's subtle enough that I wouldn't recommend trying it in a competition. We note that the center of mass moves on a circle centered at the origin, and that the total kinetic energy of the ladder is proportional to $\dot{\theta}^2$. In particular, we have a Lagrangian of

$$\mathcal{L} = \frac{1}{2}m_{\text{eff}}L^2\dot{\theta}^2 + mgL\cos\theta, \quad m_{\text{eff}} = \frac{4}{3}m$$

where the extra contribution in the first term is due to rotational kinetic energy. Multiplying the Lagrangian by 3/4, which makes no difference to the equations of motion, we get

$$\mathcal{L} = \frac{1}{2}mL^2\dot{\theta}^2 + m\left(\frac{3g}{4}\right)L\cos\theta.$$

However, this is simply the Lagrangian for a mass m sliding on a frictionless hemisphere in gravity $3g/4$. This is a classic, simple problem, and we know in that case that the normal force with the hemisphere vanishes at height $(2/3)L$.

Now, the motion of the mass in this problem is identical to the motion of the center of mass of the ladder in the original problem, so the total external forces are the same. In particular, the horizontal constraint force must vanish when the ladder's center of mass is at height $(2/3)L$, so the ladder loses contact with the wall at this point. On the other hand, the vertical external force must be $3mg/4$, which implies the normal force with the ground is $mg/4$, and hence positive; this shows that the ladder has not lost contact with the ground.

Example 7: EFPhO 2013

A uniform ball and a uniform ring are both released from rest from the same height on an inclined plane with inclination angle θ. They arrive at the bottom of the plane in time T_B and T_R, respectively. The coefficients of friction of both objects with the plane are $\mu_k = 0.3$ and $\mu_s = 0.5$. Find the ratio T_B/T_R as a function of the angle θ.

Solution

When rolling without slipping, the acceleration of an object with moment of inertia βmR^2 about its center of mass is

$$a = \frac{g\sin\theta}{1+\beta}$$

as mentioned in a previous example. The tangential force from friction is thus

$$f = mg\sin\theta\,\frac{\beta}{1+\beta}$$

which means rolling without slipping occurs when

$$\mu_s mg\cos\theta \geq mg\sin\theta\frac{\beta}{1+\beta}$$

or equivalently

$$\tan\theta \leq \mu_s\,\frac{1+\beta}{\beta}.$$

For the ball, this is when $\theta \leq 60.3°$, and for the ring $\theta \leq 45°$. Whenever either object slips, its acceleration is instead $a = g(\sin\theta - \mu_k\cos\theta)$.

Since the motion is uniformly accelerated, $T_B/T_R = \sqrt{a_R/a_B}$. For $\theta \leq 45°$, both roll without slipping, so the formula above applies, giving a ratio of

$$\frac{T_B}{T_R} = \sqrt{\frac{1+\beta_B}{1+\beta_R}} = \sqrt{\frac{7}{10}}.$$

For $\theta \geq 60.3°$ they both slip, so the ratio is unity. For the angles in between, the ring slips, giving a slightly more complicated expression. At the boundaries between these three regimes, the ratio T_B/T_R jumps discontinuously.

The next two problems require careful thought, and test your understanding of the multiple ways to describe rotational kinematics and dynamics. It will be useful to review idea 1.

[3] **Problem 17.** USAPhO 1999, problem B1.

[3] **Problem 18.** USAPhO 2019, problem B3. It's worth reading the solution carefully afterward.

4 Rotational Collisions

Idea 7: Angular Impulse

During a collision with impulse $\mathbf{J}$, the angular momentum changes by the "angular impulse" $\mathbf{r}\times\mathbf{J}$. In many problems involving collisions which conserve angular momentum, energy is necessarily lost in the collision process. This is another example of an inherently inelastic process, an idea we first encountered in **M3**.

[3] **Problem 19** (Morin 8.22)**.** A uniform ball of radius R and mass m rolls without slipping with speed v_0. It encounters a step of height h and rolls up over it.

(a) Assuming that the ball sticks to the step during this process, show that for the ball to climb over the step,

$$v_0 \geq \sqrt{\frac{10gh}{7}}\left(1-\frac{5h}{7R}\right)^{-1}.$$

(b) Energy is lost to heat by the inelastic collision of the ball with the step. In the limit of small h, how much heat is produced?

Solution. (a) Let $\beta = 2/5$. Once the ball collides with the corner, it essentially rotates around that corner, and we will first aim to find the initial angular velocity of the rotation of the ball around the corner. Note that angular momentum about the corner is conserved, since the only relevant force during the very short collision time is the large force applied at the corner, so the net torque is 0. This is an inherently inelastic process; energy is lost during this collision.

Let us compute the angular momentum right before the collision occurs. It is just the sum of the angular momentum of the CM, plus the angular momentum around the CM, so

$$L_i = \beta m R^2 \frac{v_0}{R} + Rmv_0(1 - h/R),$$

since the sine of the angle between $\mathbf{p}$ and $\mathbf{R}$ is $1 - h/R$. Let the angular velocity about the corner be ω. Then, the final angular momentum is

$$L_f = (1+\beta) m R^2 \omega,$$

so equating the two tells us that

$$R\omega = \frac{\beta + 1 - h/R}{\beta + 1} v_0 = \left(1 - \frac{1}{\beta+1}\frac{h}{R}\right) v_0.$$

Now, as the ball rotates about the corner, energy is conserved, so the only way that the ball will make it to the top is if its kinetic energy is at least mgh. Therefore,

$$\frac{1}{2}(\beta+1) m R^2 \omega^2 \geq mgh \implies \frac{1}{2}(\beta+1)\left(1 - \frac{1}{\beta+1}\frac{h}{r}\right)^2 v_0^2 \geq gh,$$

or

$$v_0 \geq \sqrt{\frac{2gh}{\beta+1}} \left(1 - \frac{1}{\beta+1}\frac{h}{r}\right)^{-1},$$

which is exactly what we wanted to show.

(b) The initial kinetic energy of the ball is $\frac{1}{2}(1+\beta) m v_0^2$. We can use the previously found equation

$$v_f = R\omega = \left(1 - \frac{1}{\beta+1}\frac{h}{R}\right) v_0,$$

which helps us find the kinetic energy immediately after the inelastic collision $\frac{1}{2}(1+\beta) m v_f^2$. Thus the kinetic energy dissipated into heat is

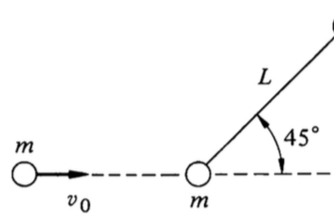
$$\Delta Q = \frac{1}{2}(1+\beta) m (v_0^2 - v_f^2) = \frac{1}{2}(1+\beta) m v_0^2 \left(1 - \left(1 - \frac{1}{1+\beta}\frac{h}{R}\right)^2\right).$$

Using the binomial approximation, we conclude

$$\Delta Q \approx \frac{1}{2}(1+\beta) m v_0^2 \left(\frac{2h}{(1+\beta)R}\right) = \frac{m v_0^2 h}{R}.$$

Interestingly, the ratio of this to the amount of gravitational potential energy needed to climb the step, which is mgh, is independent of h. So even if we turn a big step into many tiny steps, it'll still be substantially less efficient than a smooth slope.

[3] **Problem 20** (KK 6.38). A rigid massless rod of length L joins two particles, each of mass m. The rod lies on a frictionless table, and is struck by a particle of mass m and velocity v_0 as shown.



After an elastic collision, the projectile moves straight back. Find the angular velocity of the rod about its center of mass after the collision.

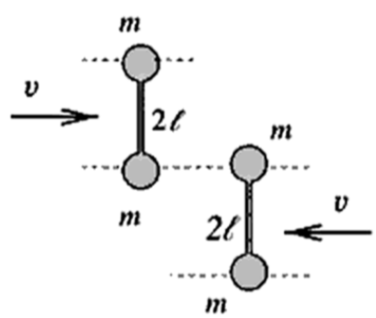
Solution. Suppose the projectile moves back with speed v_1, the CM speed of the dumbbell is v_2, and its angular velocity about its CM is ω. Then, momentum, angular momentum, and energy conservation yield

$$\begin{aligned} mv_0 = -mv_1 + 2mv_2 &\implies v_0 + v_1 = 2v_2 \\ mv_0 L/2\sqrt{2} = (mL^2/2)\omega - mv_1 L/2\sqrt{2} &\implies v_0 + v_1 = \sqrt{2}L\omega \\ mv_0^2 = mv_1^2 + 2mv_2^2 + (mL^2/2)\omega^2 &\implies (v_0 - v_1)(v_0 + v_1) = 3v_2^2. \end{aligned}$$

Combining the first and last equations implies $v_0 - v_1 = (3/2)v_2$, so $2v_0 = (7/2)v_2$, so $v_2 = \frac{4}{7}v_0$. Using the second equation gives

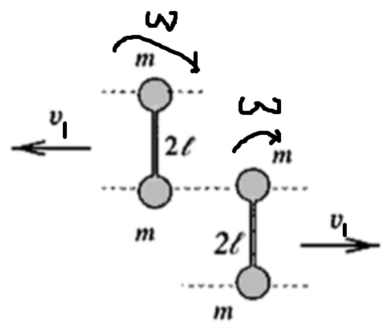
$$L\omega = \sqrt{2}v_2 = \frac{4\sqrt{2}}{7}v_0, \quad \omega = \frac{4\sqrt{2}}{7}\frac{v_0}{L}.$$

[3] **Problem 21** (PPP 47). Two identical dumbbells move towards each other on a frictionless table as shown.



Each consists of two point masses m joined by a massless rod of length 2ℓ. The dumbbells collide elastically; describe what happens afterward.

Solution. First we find out what happens immediately after the collision. Suppose the dumbbells move in the opposite direction at v_1, and have angular velocity $\omega > 0$.



Angular momentum conservation tells us $4m\ell^2\omega - 4mv_1\ell = 4mv\ell$, so $v_1 + v = \ell\omega$. Energy conservation tells us $2mv_1^2 + 2m\ell^2\omega^2 = 2mv^2$, so $(v - v_1)(v + v_1) = \ell^2\omega^2$. Therefore, $v - v_1 = \ell\omega$, so $v_1 = 0$. Thus, the rods both rotate at angular velocity v/ℓ.

Once both rods rotate 180°, they collide again. By using the reasoning of the first collision in reverse, the rods simply lose their angular velocity and regain their original translational velocities. Therefore, the final result is that both rods translate uniformly, as if they passed right through each other, but both rods are flipped upside down.

[3] **Problem 22.** USAPhO 2014, problem B1.

[4] **Problem 23.** EuPhO 2018, problem 1. An elegant rotation problem.

Solution. See the official solutions here.

5 Rotational Oscillations

In this section we'll consider small oscillations problems involving rotation.

Idea 8

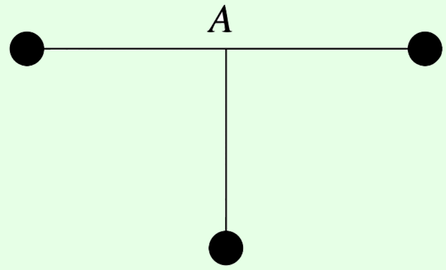
A physical pendulum is a rigid body of mass m pivoted a distance d from its center of mass, with moment of inertia I about the pivot. When considering physical pendulums, we always assume the pivot exerts no torque on the pendulum; that is, it is a "simple support", providing no bending moment, as discussed in **M2**. This is a good approximation if the pivot is smooth and small. In this case, the angular frequency for small oscillations is

$$\omega = \sqrt{\frac{mgd}{I}}.$$

For some neat real-world applications of this formula, see this paper.

Example 8: $F = ma$ 2018 A14

Three identical masses are connected with identical rigid rods and pivoted at point A.



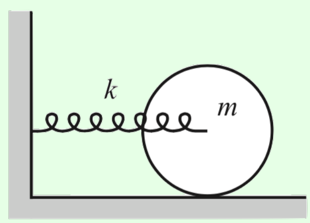
If the lowest mass receives a small horizontal push to the left, it oscillates with period T_1. If it receives a small push into the page, it oscillates with period T_2. Find the ratio T_1/T_2.

Solution

Both modes are physical pendulums, which have period proportional to $\sqrt{I/Mgx}$ where x is the distance from the pivot to the center of mass, and I is the moment of inertia about the pivot. Since x is the same in both cases, $T_1/T_2 \propto \sqrt{I_1/I_2} = \sqrt{3}$, because in the second case only the bottom mass contributes to the moment of inertia.

Example 9: Morin 8.41

The axis of a solid cylinder of mass m and radius r is connected to a spring of spring constant k, as shown.



If the cylinder rolls without slipping, find the angular frequency of the oscillations.

Solution

This is a question best handled using the energy methods of **M4**. The potential energy is $kx^2/2$ as usual, where x describes the position of the cylinder's center of mass. The kinetic energy is $mv^2/2 + I\omega^2/2 = (3/4)mv^2$, since the cylinder is rolling without slipping. Therefore

$$\omega = \sqrt{\frac{k}{m_{\text{eff}}}} = \sqrt{\frac{2k}{3m}}.$$

More complicated variants of this kind of problem can be solved in a similar way.

Example 10: Russia 2011

A uniform ring of mass m and radius r is suspended symmetrically on three inextensible strings of length ℓ. Find the angular frequency of small oscillations.

Solution

The small oscillations are torsional, i.e. the ring rotates about its axis of symmetry. When the ring has twisted by an angle θ, the strings are an angle $\phi \approx (r/\ell)\theta$ from the vertical. Thus, summing over the three strings, the restoring torque is

$$\tau \approx -mgr\phi \approx -\frac{mgr^2}{\ell}\,\theta.$$

Setting this equal to $I\alpha$, we find $\omega = \sqrt{g/\ell}$.

The tricky thing about this problem is that it's harder to solve with the energy method. If you try, you immediately run into the problem that there seems to be no potential energy anywhere, since the strings don't stretch! The source of the potential energy is that the ring moves up a small amount as it oscillates, since the strings are no longer vertical,

$$h = \ell - \sqrt{\ell^2 - r^2\theta^2} \approx \frac{r^2\theta^2}{2\ell}.$$

Therefore we have

$$K = \frac{1}{2}mr^2\dot{\theta}^2, \quad V = \frac{1}{2}\frac{mgr^2}{\ell}\theta^2$$

and the answer follows as usual. (There is also a kinetic energy contribution from the ring's vertical motion, but it's negligible.) The lesson here is that the force/torque and energy approach have different strengths. The energy approach is often easier because it lets you ignore some internal details of the system. But it can be harder because it requires you to understand the kinematics of the system to second order, rather than first order.

[2] **Problem 24.** A circular pendulum consists of a point mass m on a string of length ℓ, which is made to rotate in a horizontal circle. By using only the equation $\boldsymbol{\tau} = d\mathbf{L}/dt$ about an origin of your choice, compute the angular frequency if the string makes a constant angle θ with the horizontal.

Solution. Of course, this would be easier with Newton's second law, but we solve the problem using torques to show the general technique, which will be useful when studying precession in **M8**. We consider the angular momentum about the fixed top end of the string,

$$L = |\mathbf{r} \times \mathbf{p}| = mv\ell = m\ell^2\omega\cos\theta$$

where ω is the angular velocity of the circular motion. The angular momentum points at an angle θ to the vertical. Its vertical component stays the same, while its horizontal component $L\sin\theta$ rotates in a circle, so

$$\left|\frac{d\mathbf{L}}{dt}\right| = \omega L\sin\theta = m\ell^2\omega^2\cos\theta\sin\theta.$$

We equate this to the magnitude of the torque due to gravity,

$$\tau = |\mathbf{r} \times \mathbf{F}| = mg\ell\cos\theta.$$

We thus conclude that

$$\omega = \sqrt{\frac{g}{\ell\sin\theta}}.$$

As a check, in the limit of small oscillations $\theta \to \pi/2$, we get $\omega = \sqrt{g/\ell}$. This makes sense because in this case, we can project in one direction to recover ordinary pendulum motion.

[3] **Problem 25.** Using a physical pendulum, one can measure the acceleration due to gravity as

$$g = \frac{4\pi^2}{T^2}\frac{I}{md}.$$

In practice, I is not very precisely known, since it depends on the exact shape of the material. Kater found an ingenious way to circumvent this problem. We pivot the pendulum at an arbitrary point and measure the period T. Next, by trial and error, we find another pivot point which has the same period, which lies at a different distance from the center of mass. Show that

$$g = \frac{4\pi^2 L}{T^2}$$

where L is the sum of the lengths from these points to the CM. This allows a measurement of g without knowledge of the moment of inertia about the center of mass. (Kater selected his two pivot points to lie on a line, on opposite sides of the center of mass. This has the additional benefit that L is simply the distance between the pivot points, removing the need to find the center of mass.)

Solution. Using the parallel axis theorem where I_c is the moment of inertia about the center of mass and x_1 and x_2 are the distances between the pivots and center of mass,

$$I_1 = I_c + mx_1^2, \qquad I_2 = I_c + mx_2^2.$$

For them to have the same period T, then the ratio $I/x = mgT^2/4\pi^2$ must be the same, so

$$\frac{I_1}{x_1} = \frac{I_2}{x_2}.$$

Combining these equations, we find

$$I_c\left(\frac{1}{x_1} - \frac{1}{x_2}\right) = m(x_2 - x_1).$$

Since $x_1 \neq x_2$, we can divide by $x_2 - x_1$ and find $I_c = mx_1x_2$. Thus,

$$\frac{I_1}{mx_1} = \frac{mx_1x_2 + mx_1^2}{mx_1} = x_1 + x_2 = L.$$

The answer follows straightforwardly,

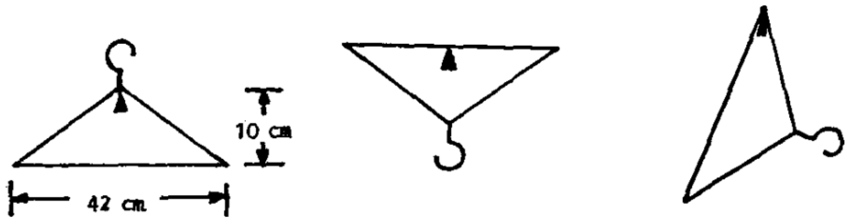
$$g = \frac{4\pi^2}{T^2}\frac{I}{mx_1} = \frac{4\pi^2 L}{T^2}.$$

[3] **Problem 26.** USAPhO 1999, problem A4.

[3] **Problem 27.** USAPhO 2011, problem B2.

[3] **Problem 28.** USAPhO 2002, problem B1. This one is trickier than it looks! It can be solved with either a torque or energy analysis, but both require care.

[4] **Problem 29** (IPhO 1982). A coat hanger can perform small oscillations in the plane of the figure about the three equilibrium figures shown.



In the first two, the long side is horizontal. The other two sides have equal length. The period of oscillation is the same in all cases. The coat hanger does *not* necessarily have uniform density. Where is the center of mass, and how long is the period?

Solution. See the official solutions here.

[4] **Problem 30** (APhO 2007). A uniform ball of mass M and radius r is encased in a thin spherical shell, also of mass M. The shell is placed inside a fixed spherical bowl of radius R, and performs small oscillations about the bottom. Assume that friction between the bowl and shell is very large, so the shell essentially always rolls without slipping.

The ball is made of an unusual material: it can quickly transition between a liquid and solid state. When the ball is in the liquid state, it has no viscosity, and hence no friction with the shell. When the ball is in the solid state, it rotates with the shell.

(a) Find the period of the oscillations if the ball is always in the solid state.

(b) Find the period of the oscillations if the ball is always in the liquid state.

(c) The ball is now set so that it instantly switches to the liquid state whenever it starts moving downward, and instantly switches to the solid state whenever it starts moving upward. If the initial amplitude of oscillations is θ_0, find the amplitude after n oscillations.

Solution. (a) In the solid state, the inside rotates with the shell, so the moment of inertia is

$$I = \frac{2}{5}Mr^2 + \frac{2}{3}Mr^2 = \frac{16}{15}Mr^2.$$

The rolling without slipping condition means $v = \omega r$, so the total kinetic energy is

$$K = \frac{1}{2}(2M)v^2 + \frac{1}{2}I\omega^2 = Mv^2 + \frac{8}{15}Mv^2 = \frac{23}{15}Mv^2.$$

If the angle between the line between the centers of the bowl and ball and the vertical is θ, then $v = (R-r)\dot{\theta}$, and the potential energy is

$$U = 2Mg(R-r)(1-\cos\theta) \approx Mg(R-r)\theta^2.$$

Since both the kinetic and potential energy are quadratic, this is simple harmonic motion. As we saw in **M4**, if we write the total energy as

$$E = \frac{1}{2}m_{\text{eff}}\dot{\theta}^2 + \frac{1}{2}k_{\text{eff}}\theta^2$$

then the period of oscillations is

$$T = 2\pi\sqrt{\frac{m_{\text{eff}}}{k_{\text{eff}}}} = 2\pi\sqrt{\frac{23(R-r)}{15g}}.$$

(b) The only difference here is that the liquid will no longer rotate, so the first term in the moment of inertia above will no longer contribute. Then the kinetic energy is

$$K = \frac{23}{15}Mv^2 - \frac{1}{5}Mv^2 = \frac{4}{3}Mv^2$$

which implies, by the same logic as in part (a), that

$$T = 2\pi\sqrt{\frac{4(R-r)}{3g}}.$$

(c) When the ball goes from solid to liquid, the entire ball is at rest, so no energy is lost. On the other hand, when the ball switches from liquid to solid, the material inside the ball must suddenly start rotating with the shell. This is an angular inelastic collision, where energy is lost, so we expect the amplitude to decay.

Let's suppose that just before the ball switches from liquid to solid, it has an angular velocity ω_i. Then the total energy is

$$E_i = \frac{4}{3}Mr^2\omega_0^2$$

by the work we did in part (b). As the material solidifies, the angular momentum about the ball's contact point with the ground is conserved. Let the final angular velocity be ω_f.

The initial moment of inertia of the shell around the contact point is

$$I_i = \frac{2}{3}Mr^2 + Mr^2 = \frac{5}{3}Mr^2.$$

The shell is instantaneously rotating about the contact point, and so contributes angular momentum $I_i\omega_i$. The liquid is only in translational motion, so it contributes angular momentum $Mv_ir = M\omega_ir^2$. Thus, the total angular momentum is

$$L_i = I_i\omega_i + M\omega_ir^2 = \frac{8}{3}Mr^2\omega_i.$$

After the transition, both the shell and ball will rotate about the contact point, and the moment of inertia is

$$I_f = \frac{5}{3}Mr^2 + \frac{2}{5}Mr^2 + Mr^2 = \frac{46}{15}Mr^2.$$

Conserving the angular momentum gives

$$\frac{8}{3}Mr^2\omega_i = \frac{46}{15}Mr^2\omega_f$$

and therefore

$$\omega_f = \frac{20}{23}\omega_i.$$

Since the energy is $E = \omega L/2$, this means

$$E_f = \frac{20}{23}E_i.$$

The angular amplitude $\theta \propto \sqrt{E}$, so the amplitude decreases by a factor of $\sqrt{20/23}$ after each collision. But there are two collisions per oscillation, so after n oscillations, the amplitude is

$$\theta_n = \theta_0\left(\frac{20}{23}\right)^n.$$

Mechanics VI: Gravitation

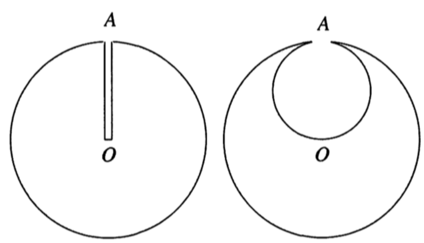
Chapters 8 and 9 of Kleppner and Kolenkow cover orbits and fictitious forces, as do chapters 8 and 11 of Wang and Ricardo, volume 1, and chapters 7 and 10 of Morin. For *much* more, see chapters 2 and 3 of *Galactic Dynamics* by Binney and Tremaine. There is a total of **87** points.

1 Computing Fields

Idea 1

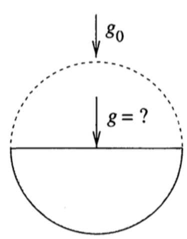
Gravitational fields obey the shell theorem and the superposition principle, which is sufficient to find the field in a variety of setups. One useful trick is to think of objects with holes as superpositions of objects without holes, and holes with negative mass.

[3] **Problem 1** (PPP 110). A spaceship of titanium-devouring little green people has found a perfectly spherical homogeneous asteroid. A narrow trial shaft was bored from point A on its surface to the center O of the asteroid. At that point, one of the little green men fell off the surface of the asteroid into the trial shaft. He fell, without any braking, until he reached O, where he died on impact.



However, work continued and the little green men started secret excavation of the titanium, in the course of which they formed a spherical cavity of diameter AO inside the asteroid. Then a second accident occurred: another little green man similarly fell from point A to point O, and died. Find the ratio of the impact speeds, and total times for impact, of the two little green men.

[3] **Problem 2** (PPP 111). The titanium-devouring little green people of the previous problem continued their excavating. As a result of their environmentally destructive activity, half of the asteroid was soon used up, as shown.



What is the gravitational acceleration at the center of the circular face of the remaining hemisphere if the gravitational acceleration at the surface of the original spherical asteroid was g_0? (This can be done without any integrals.)

[3] **Problem 3** (Morin 5.67). You are given a fixed volume of a moldable material, with a fixed density. Describe the shape it should take to maximize the gravitational field at the origin.

Idea: The Shell Theorems

Newton proved a variety of elegant "shell" theorems, some of which are less well known.

1. Inside a uniform spherical shell, there is no gravitational field. (At the time, this was an important result primarily because it meant that Hell couldn't be at the center of the Earth; if it were, the fire and brimstone would be floating around.)

2. Outside a uniform spherical shell of total mass m, the gravitational field is the same as that of a point mass m at its center. (Of course, this is important because it's required to think about the Earth's gravity at all.)

3. There is no gravitational field inside a uniform homoeoid, i.e. a region defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \in [1, 1+\epsilon]$$

for some constant ϵ, with uniform density. This one is seldom useful. (The gravitational field outside a homoeoid is more complicated, so there's no "fourth" theorem.)

Example 1

Newton was aware that similar shell theorems hold for linear force laws, $F(r) \propto r$. How are his first two theorems modified in this case?

Solution

Consider a spherical shell centered at the origin, and a test mass at $\mathbf{r}_0$. The contribution to the force due to a piece of the shell at $\mathbf{r}$ is $\mathbf{F} \propto \mathbf{r} - \mathbf{r}_0$. When we integrate over the shell, $\mathbf{r}$ averages to zero, giving $\mathbf{F} \propto -\mathbf{r}_0$, which is precisely the result for a mass exactly at the center of the shell. That is, for a linear force law, Newton's first theorem doesn't work; instead the second theorem's result applies both inside and outside the shell.

Example 2

Prove the converse of Newton's second theorem: a spherical shell can be replaced with a point mass at its center only if $F(r)$ is proportional to r, proportional to r^{-2}, or a linear combination of the two.

Solution

It's easiest to consider the potential outside the shell. Let the shell of mass m be centered at the origin with radius R, and consider the potential at a distance $z > R$ from the origin. If a point mass produces a gravitational potential $f(r)\, dm$ at separation r, then integrating over the sphere in spherical coordinates gives

$$V(z) = \frac{m}{4\pi R^2} \int_0^\pi (2\pi R^2 \sin\theta \, d\theta)\, f(\sqrt{z^2 + R^2 - 2zR\cos\theta}).$$

The trick is to switch variables to the separation $r = \sqrt{z^2 + R^2 - 2zR\cos\theta}$, since

$$r\,dr = zR\sin\theta\,d\theta.$$

Plugging this in gives

$$V(z) = \frac{m}{2zR}\int_{z-R}^{z+R} r f(r)\,dr.$$

Newton's second theorem works precisely when dV/dz is independent of R, so that the shell radius can be collapsed to zero without changing the force.

Suppose $f(r)$ is proportional to r^n. Then we have

$$V(z) \propto \frac{(z+R)^{n+2} - (z-R)^{n+2}}{zR}$$

and the force's dependence on R can only drop out in three cases: when $n = -1$ (an inverse square force), $n = 0$ (the trivial case, corresponding to no force), and $n = 2$ (a linear force). The final case is the trickiest; in that case we have

$$V(z) \propto \frac{4z^3R + 4zR^3}{zR} \propto z^2 + R^2$$

so that R drops out of dV/dz, as required. Since any reasonable function can be built by superposing such power laws, this classification is exhaustive.

Incidentally, the same method can be used to prove the converse of Newton's first theorem. The only difference is that $z < R$, so that the lower limit of integration has to be replaced with $|z - R| = R - z$. Then the $n = 2$ case works out the same way, since $z - R$ is squared. By contrast, for $n = -1$ we get no force, since $V(z) \propto ((z+R) - (R-z))/zR = 2/R$ which is constant. Thus, the inverse square force is the only one where Newton's first theorem applies.

2 Central Potentials

Idea 2: Effective Potential

A particle experiencing a central force has a potential energy $V(r)$ which only depends on its radial coordinate, and conserved angular momentum

$$L = |\mathbf{r} \times \mathbf{p}| = mr^2\dot{\theta}.$$

Its kinetic energy can thus be written in terms of the radial velocity $\dot{r}$ and L,

$$E = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_\theta^2 + V(r) = \frac{1}{2}m\dot{r}^2 + \left(V(r) + \frac{L^2}{2mr^2}\right).$$

By setting the time derivative of this expression to zero, we find

$$m\ddot{r} = -\frac{d}{dr}\left(V(r) + \frac{L^2}{2mr^2}\right).$$

Therefore, if we are interested in $r(t)$ alone, we can find it by treating the problem as one-dimensional, where the particle moves in the "effective potential" $V(r) + L^2/2mr^2$. The extra term is called the angular momentum barrier and repels the particle away from the center. Once we know $r(t)$, we can find $\theta(t)$ if desired by using $\dot{\theta} = L/mr^2$.

One way of understanding the effective potential term is to think in terms of the energy methods of **M4**. From the perspective of $r(t)$ alone, any dependence on $\dot{r}^2$ is part of the kinetic energy, and any dependence on r is part of the potential energy. In particular, the kinetic energy of tangential motion depends only on r, because it is fixed by angular momentum conservation, so it appears as part of the potential when considering only radial motion.

Example 3: KK 9.4

For what values of n are circular orbits stable with the potential energy $U(r) = -A/r^n$?

Solution

Note that circular orbits can only possibly exist if the force is attractive. This implies that A must have the same sign as n.

The effective potential is

$$U_{\text{eff}}(r) = -\frac{A}{r^n} + \frac{L^2}{2mr^2}.$$

In a circular orbit, r is constant, so the particle just sits still at a minimum of the effective potential. That is, the circular orbit radius satisfies $U'_{\text{eff}}(r) = 0$, so

$$\frac{An}{r_0^{n+1}} - \frac{L^2}{mr_0^3} = 0, \quad r_0^{2-n} = \frac{L^2}{Anm}.$$

The orbit is stable if $U''_{\text{eff}}(r) > 0$, so

$$-\frac{An(n+1)}{r_0^{n+2}} + \frac{3L^2}{m}\frac{1}{r_0^4} > 0$$

which simplifies to

$$r_0^{n-2} > \frac{m}{3L^2} An(n+1).$$

Plugging in the value of r_0, this becomes simply $n < 2$. As expected, for inverse square forces ($n = 1$) and spring forces ($n = -2$) the orbits are stable, while, e.g. for inverse cube forces, the circular orbits are neutrally stable.

[3] **Problem 4** (Morin 7.4)**.** A particle of mass m moves in a potential $V(r) = \beta r^k$. Let the angular momentum be L.

(a) Find the radius r_0 of the circular orbit.

(b) Find the angular frequency ω_r of small oscillations about this radius.

(c) Now consider a slightly perturbed circular orbit. Explain why the orbit remains a closed curve if the ratio of the time period of small oscillations and the time period of the original circular orbit is rational, and find the integer values of k where this holds.

Remark: Bertrand's Theorem

In problem 4, you showed that for a certain group of potentials, all bound orbits that are nearly circular are closed. Bertrand's theorem states that the *only* central potentials for which *all* bound orbits are closed are $V(r) \propto 1/r$ and $V(r) \propto r^2$.

The idea of the proof is as follows. First, for a general potential $V(r)$, we can compute the ratio of periods of a small radial oscillation and the underlying circular orbit and demand it be rational, just like in part (c) above. However, since this ratio changes continuously as the orbit parameters are varied, it must be a constant if it is to always be rational. Using this condition, you can show that $V(r)$ must be a power law, which we had to assume above.

You found in part (c) that infinitely many power laws give closed nearly circular orbits. To rule out the other ones, we need to expand to higher orders, i.e. account for the fact that the effective potential is not perfectly simple harmonic. A detailed derivation can be found here.

[4] **Problem 5.** In general relativity, the potential describing a black hole of mass M is

$$V(r) = -\frac{GMm}{r} - \frac{GML^2}{mc^2r^3}.$$

The second term is a relativistic effect which strengthens the attraction towards the black hole. (It has nothing to do with the angular momentum barrier; you still have to add that separately.)

(a) Explain why this new term allows particles to fall to the center of the black hole, $r = 0$, and why this is impossible in Newtonian gravity.

(b) For a fixed L, find the values of the circular orbit radii.

(c) Find the radius of the smallest possible stable circular orbit, for any value of L. What happens if you try to orbit the black hole closer than this? (Answer: $6GM/c^2$.)

(d) Find the closest possible approach radius of an unbound object. That is, among the set of all trajectories that start and end far away from the black hole (i.e. without falling into it), find the smallest possible minimum value of r. (Answer: $3GM/c^2$.)

For all parts, assume the particle is moving nonrelativistically.

Example 4: Binney 5.1

For over 150 years, most astronomers believed that Saturn's rings were rigid bodies, until Laplace showed that a solid ring would be unstable. The same instability plagues Larry Niven's *Ringworld*, a science fiction novel once popular among boomer nerds. Following Laplace, consider a rigid, circular ring of radius R and mass m, centered on a planet of mass $M \gg m$. The ring rotates around the planet with the Keplerian angular velocity $\omega = (GM^2/R^3)^{1/3}$. Show that this configuration is linearly unstable.

Solution

One way to understand the stability of an ordinary planetary orbit is angular momentum conservation: if you displace a planet radially inward, then it'll start moving faster tangentially, which will tend to make it go back out, even though the inward gravitational force gets stronger too. This tendency is absent for a rigid ring, because the entire ring always rotates with the same angular velocity $\omega = L/mR^2$.

The simplest way to see that this configuration is unstable is to calculate the gravitational potential ϕ due to the ring at the planet's position. If the planet starts at the center of the ring, then displacing it along the axis of the ring increases ϕ. But since $\nabla^2\phi = 0$, displacing it towards the ring must decrease ϕ, so the system is unstable. (This is just the gravitational analogue of Earnshaw's theorem from **E1**.)

To make this more concrete, fix the planet at the origin, and parametrize the ring by the angle θ along it. If the whole ring is shifted by a small distance a in the plane of the ring, the elements of the ring are at

$$r^2 = (R\cos\theta + a)^2 + (R\sin\theta)^2.$$

The total gravitational potential energy is

$$U = -GMm\int_0^{2\pi} \frac{d\theta}{2\pi}\frac{1}{r} = -\frac{GMm}{R}\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{1}{\sqrt{1 + (2a/R)\cos\theta + a^2/R^2}}.$$

We have to be a bit careful here, remembering some lessons from **P1**. The first order term in a is going to vanish, because we started at an equilibrium point, which means we need to expand everything to second order in a. Using the Taylor series

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + O(x^3)$$

we conclude

$$U \approx -\frac{GMm}{R}\int_0^{2\pi}\frac{d\theta}{2\pi}\left(1 - \frac{a}{R}\cos\theta + \frac{a^2}{R^2}\frac{3\cos^2\theta - 1}{2}\right) = -\frac{GMm}{R}\left(1 + \frac{a^2}{4R^2}\right).$$

The energy goes down upon a small displacement, so the configuration is unstable. The ring will soon crash into the planet.

3 Kepler's Laws

Idea 3

Kepler's laws for a general orbit are:

1. The trajectories of planets are conic sections, with a focus at the Sun. Bound orbits are ellipses, which contain circles as a special case. Unbound orbits are hyperbolas, which contain parabolas as a special case.

2. The trajectories sweep out equal areas in equal times.
3. When the orbit is bound, the period T and semimajor axis a obey $T^2 \propto a^3$.

Unlike the other laws, the second is valid for any central force, because the rate of area sweeping is $rv_\perp/2 \propto |\mathbf{r} \times \mathbf{v}| \propto |\mathbf{L}|$.

Idea 4

For a general orbit with semimajor axis a, the total energy is

$$E = -\frac{GMm}{2a}.$$

This identity also applies to hyperbolas, where a is negative, and the parabola in the limit of infinite a, where the total energy vanishes.

Remark: Virial Theorem

For bound orbits, the time-averaged values of the kinetic and potential energy are related by

$$\langle K \rangle = -\frac{1}{2}\langle V \rangle.$$

In fact, the virial theorem holds for more complicated bound systems of particles as well, as long as they interact by a power law potential $V(r) \propto r^n$. In this case, we have

$$\langle K \rangle = \frac{n}{2}\langle V \rangle$$

where gravity corresponds to the case $n = -1$.

You can easily check that the virial theorem works in one dimension for a particle bouncing in a uniform gravitational field ($n = 1$), or a particle on a spring ($n = 2$). It's also easy to check for a planet in a circular orbit ($n = -1$). With some more work, you can check that it also holds for arbitrary elliptical orbits – to do this most efficiently, convert the time integral to an integral over angle θ, and use the form of an ellipse in polar coordinates.

In astrophysics, the virial theorem is useful because it allows us to estimate V, which can be hard to measure, given K. For discussion of the virial theorem along with applications to dark matter, see section 1.4.3 of these notes. We will return to these subjects in **X3**.

[3] **Problem 6.** In this problem we'll verify some of the basic facts stated above.

(a) Prove the statement of idea 4 for the case of elliptical orbits.

(b) Using this result, prove the vis-viva equation

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

which is often used in rocketry.

(c) Prove Kepler's third law. (Hint: see the area of an ellipse below.)

Remark: Scaling Symmetry

There's a variant of Kepler's third law for unbound orbits. Suppose a planet is right next to the Sun at time $t = 0$, but has a large initial radial velocity, so that it has zero total energy. Then its distance to the Sun evolves as $r(t) \propto t^{2/3}$, like how $a \propto T^{2/3}$ for bound orbits.

Both of these results come from the scaling symmetry of inverse square force laws: any solution to Newton's second law remains a solution if you multiply all distances by 4 and all times by 8. The widest-reaching application of this idea is to the whole universe itself. If it contains only matter, which started at the origin at time $t = 0$, and it expands under gravity with zero total energy, then its "scale factor" evolves as $a(t) \propto t^{2/3}$. This was a good description of our universe for most of its lifetime, but in the past few billion years the effects of dark energy took over, accelerating the expansion. We'll revisit cosmology in **X3**.

[3] **Problem 7.** [A] A simple derivation of Kepler's first law is given in section 7.4 of Morin, and revolves around solving a differential equation for $1/r(\theta)$. (You can motivate this by noting that the polar form of an ellipse is quite simple, $1/r = (1 + e\cos\theta)/p$, where p is the semilatus rectum and e is the eccentricity.) However, in this problem, we'll consider an alternative approach that uses a subtle conserved quantity, which is also important in more advanced physics.

(a) Show that the Laplace–Runge–Lenz vector

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - GMm^2\hat{\mathbf{r}}$$

is conserved, where the star is at the origin and $\hat{\mathbf{r}}$ is the radial unit vector at the planet's position $\mathbf{r}$. (Hint: use the fact that $\mathbf{L} = mr^2\boldsymbol{\omega}$ to evaluate the time derivative.)

(b) We have $\mathbf{A} \cdot \mathbf{r} = Ar\cos\theta$, where θ is the angle between $\mathbf{A}$ and $\mathbf{r}$. Evaluate $\mathbf{A} \cdot \mathbf{r}$ using the definition of $\mathbf{A}$, and the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, in order to derive an expression for r in terms of θ and constants. Then use this to show that the orbit is a conic section.

(c) As another simple application of the conservation of $\mathbf{A}$, show that the set of velocities during an elliptical orbit traces out a *circle* in velocity space.

The ideas discussed in this problem are almost never required to solve Olympiad problems, but they can dramatically simplify very tough orbital mechanics problems. For two examples, see Physics Cup 2021, problem 2 and Physics Cup 2024, problem 4.

Now we'll consider some really slick problems that can be solved with pure geometry.

Idea 5

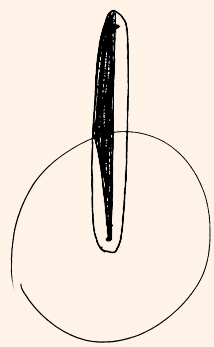
An ellipse is defined by two foci F_1 and F_2 separated by a distance $2d$. It consists of the set of points P so that $PF_1 + PF_2 = 2\ell$ is a constant. The semimajor and semiminor axes a and b of the ellipse are related by $a = \sqrt{b^2 + d^2} = \ell$, and the area is πab.

Example 5

An object is dropped from rest at a distance R above the Earth's surface, where R is the radius of the Earth. How long does it take to hit the Earth's surface?

Solution

The answer doesn't change much if we give the object a tiny horizontal velocity. In this case, the orbit becomes a part of a very thin ellipse, where $a \approx d \approx \ell$, with one focus at the center of the Earth (by the shell theorem) and the other near the starting point.



If the Earth were replaced by a point mass at its center, then the object could perform a full orbit, with total period T. The time until the object actually hits the Earth's surface is determined by the fraction of the orbit's area swept out. Referring to the diagram, this is

$$t = T\,\frac{\pi ab/4 + ab/2}{\pi ab} = T\left(\frac{1}{4} + \frac{1}{2\pi}\right)$$

by summing a quarter of an ellipse and a triangle. All that's left is to solve for T. Note that the semimajor axis is R. Another orbit with the same semimajor axis is simply a circular orbit around the Earth, just above its surface. This orbit has

$$\frac{v^2}{R} = \frac{GM}{R^2}$$

so $v = \sqrt{GM/R}$. Using $T = 2\pi R/v$ gives the answer,

$$t = \left(\frac{\pi}{2} + 1\right)\sqrt{\frac{R^3}{GM}}.$$

Example 6: MPPP 39

An astronaut jumps out of the international space station directly towards the Earth. What happens afterward? In particular, will the astronaut survive?

Solution

If you've seen certain movies, you might get the impression that the astronaut spirals into the Earth, and so will surely die. But that isn't what Kepler's laws say! After the jump, the astronaut simply performs a Keplerian orbit. Since the change in energy is negligible, so is the change in semimajor axis and hence the change in period. The astronaut simply orbits in a nearly circular ellipse, with the same period as the space station.

After one rotation period of the space station, which takes time $T = 92\,\text{min}$, the astronaut arrives back. They are unharmed as long as their oxygen and cooling supply lasts this long. (If you draw some pictures of the orbits, you may think the answer is $T/2$, because the orbits intersect twice. This is incorrect because while the orbits do intersect geometrically halfway through, the space station and the astronaut won't arrive at that point at the same time.)

Example 7: Wang and Ricardo 8.4

A particle moves in a circle of radius R, under the influence of a central force. If its minimum and maximum speeds are v_1 and v_2, what is the period T?

Solution

At first the problem statement might sound confusing, until you realize that the origin need not be at the center of the circle; it must be off-center. Now, it would be intractable to find the trajectory for a general central force law, but we can infer T by thinking about how quickly area is swept out, as in Kepler's second law. This works because conservation of angular momentum holds for all central force laws, not just the inverse square.

At the furthest and closest points, the distances from the origin must be r_1 and r_2, and by conservation of angular momentum, the speeds v_1 and v_2 are achieved at these points, so

$$r_1 v_1 = r_2 v_2, \quad r_1 + r_2 = 2R, \quad \frac{dA}{dt} = \frac{1}{2} r_1 v_1.$$

Using the first two equations, we can solve for r_1 and plug it into the third, for

$$r_1 = \frac{2R}{1 + v_1/v_2}, \quad \frac{dA}{dt} = \frac{R}{1/v_1 + 1/v_2}.$$

Since $dA/dt = \pi R^2/T$, we have

$$T = \pi R \left(\frac{1}{v_1} + \frac{1}{v_2} \right).$$

[3] **Problem 8** (PPP 88)**.** A rocket is launched from and returns to a spherical planet of radius R so that its velocity vector on return is parallel to its velocity vector at launch. The angular separation at the center of the planet between the launch and arrival points is θ. How long does the flight take, if the period of a satellite flying around the planet just above its surface is T_0?

[4] **Problem 9** (Physics Cup 2012)**.** A cannon at the equator fires a cannonball, which hits the North pole. Neglecting air resistance and the Earth's rotation, at what angle to the horizontal should the

cannonball be fired to minimize the required speed?

[4] **Problem 10** (EFPhO 2015)**.** An asteroid is initially stationary, a distance R from a star of mass M. The asteroid suddenly explodes into many pieces, with speed ranging from zero to v_0. What is the set of all points that can be hit by a piece of the asteroid? (Hint: this problem requires more geometry than the rest. For simplicity, you can begin by treating the problem as two-dimensional, but the solution you find will work just as well for three.)

Idea 6: Reduced Mass

Consider two objects of mass m_1 and m_2 with positions $\mathbf{r}_1$ and $\mathbf{r}_2$ with relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, interacting by a central potential $V(r)$. For the purposes of computing $\mathbf{r}$ alone, we may replace this system with a single mass μ in the same central potential $V(r)$, where μ is the reduced mass, obeying

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

Both systems have the same solutions for $\mathbf{r}(t)$.

Example 8

Consider two planets of mass m. If one planet is somehow fixed in place, the other can perform a circular orbit of radius R with period T. If both planets are allowed to move, they can simultaneously perform circular orbits of radius $R/2$ about their center of mass. What is the period of this motion?

Solution

First let's try an explicit solution. In the first case,

$$\frac{mv^2}{R} = \frac{Gm^2}{R^2}, \quad v = \sqrt{\frac{Gm}{R}}.$$

In the second case, we have

$$\frac{mv^2}{R/2} = \frac{Gm^2}{R^2}, \quad v = \frac{1}{\sqrt{2}}\sqrt{\frac{Gm}{R}}.$$

The velocity in this case is a factor of $1/\sqrt{2}$ smaller, but the arc length of the orbit is a factor of 2 smaller, so the period is $T/\sqrt{2}$.

We can also handle the problem with reduced mass. Consider the relative position $\mathbf{r}_1 - \mathbf{r}_2$ in the second case, which orbits in a circle of radius R. Applying the above idea, we can work in the reduced system. In this system, there is a single mass $\mu = (1/m + 1/m)^{-1} = m/2$ in a circular orbit of radius R, experiencing the same force Gm^2/R^2, so

$$\frac{\mu v^2}{R} = \frac{Gm^2}{R^2}, \quad v = \sqrt{\frac{2Gm}{R}}.$$

The speed is $\sqrt{2}$ bigger than in the first case, but the arc length of the orbit is the same, so the period is $T/\sqrt{2}$.

Reduced mass is a bit unintuitive, since you need to work in two very different pictures. On the other hand, some people like it because it's mathematically concrete, and can reduce problems to one-liners. Whether you use it is up to you.

[2] **Problem 11** (MPPP 27)**.** Two permanent magnets are aligned on a horizontal frictionless table, separated by a distance d. The magnets are held in such a way so that the net force between them is attractive, and there are no torques generated.

If one of the magnets is held and the other is released, the two collide after time t_1. If instead the roles are reversed, the two collide after time t_2. If instead both magnets are released from rest, how long does it take for them to collide? (For a simpler related problem, see $F = ma$ 2019 B4.)

[3] **Problem 12.** USAPhO 2012, problem A4.

4 Rocket Science

So far you've done some challenging problems, but they haven't exactly been rocket science. These questions literally *are* rocket science. Since almost all of these kinds of questions rely on the same few essential ideas, I've grouped them into a few longer questions. The 4 and 5 point problems below are representative of older IPhO problems in length.

[2] **Problem 13.** A rocket begins at rest in empty space. The engine is turned on and exerts a constant thrust, so $P = Fv$ increases over time. After a long time, the power of the engine can become arbitrarily high, even though it's doing the same thing at all times. This is called the Oberth effect, and has real practical consequences; all else equal, a rocket should be preferentially used when the velocity is high, since it will produce extra power. Where does the extra power come from?

[4] **Problem 14.** A rocket with a full fuel tank has a mass M and is initially stationary. The fuel is ejected at a rate σ, where σ has units of kg/s, at a relative velocity of u.

(a) If the rocket begins in space, show that the velocity of the rocket when its total mass is M' is

$$v = u \log \frac{M}{M'}.$$

This is the Tsiolkovsky rocket equation.

(b) Repeat part (a) for a rocket in a uniform gravitational field g. Do you get the best final velocity if σ is high or low? (Ignore gravity for the rest of this problem.)

(c) In a multi-stage rocket, an empty fuel tank detaches from the rocket once it is used up, after which a second engine starts up. Explain why this can achieve a much higher final velocity than just firing both engines at once. (If you want a quantitative treatment of this, you can see INPhO 2016, problem 3.)

(d) It is desired for a rocket to begin at zero speed and accelerate to speed v, to deliver a given payload. If the exhaust comes out with a relative velocity of u, how should u be chosen to minimize the fuel energy that must be spent to perform this maneuver? (Hint: let the *final* mass of the rocket be fixed, since that's the mass of the payload we want to transport. You will have to solve an equation numerically.)

(e) If u has this value, what fraction η of the spent fuel's energy ends up in the rocket's final kinetic energy?

(f) Now suppose u can be freely varied over time. Qualitatively, how should it be chosen to maximize η, and what is the maximum possible value of η?

[3] **Problem 15.** USAPhO 2015, problem B1. A basic, two-step rocket maneuver.

[5] **Problem 16.** The classic cosmic speeds. For each part, evaluate your answers numerically, using

$$M_{\text{Earth}} = 5.97 \times 10^{24}\,\text{kg}, \quad M_{\text{Sun}} = 1.99 \times 10^{30}\,\text{kg}, \quad R_{\text{Earth}} = 6.37 \times 10^{6}\,\text{m}, \quad d_{\text{Sun}} = 1.50 \times 10^{11}\,\text{m}.$$

Neglect the rotation of the Earth about its own axis for all parts except for part (b).

(a) What is the minimum launch speed required to put a satellite into orbit around the Earth? This is the first cosmic speed. (It's useful to think in terms of speeds because the Tsiolkovsky rocket equation tells us that directly determines the amount of fuel needed. Multistage rocket maneuvers are often described in terms of their "total Δv".)

(b) If you account for the rotation of the Earth, what is the new minimum speed and how should the satellite be launched?

(c) What is the minimum launch speed required for a rocket to escape the gravitational field of the Earth? This is the second cosmic speed.

(d) What is the minimum launch speed required for a rocket to leave the solar system? This is the third cosmic speed. How should the satellite be launched? (Hint: doing this exactly is very hard; instead use the approximation $R_{\text{Earth}} \ll d_{\text{Sun}}$. To check, the answer is 16.7 km/s.)

(e) What is the minimum launch speed required for a rocket to hit the Sun? Assume you cannot make any adjustments to the rocket's path after launch. (To check, the answer is 31.8 km/s.)

(f) If subsequent adjustments are allowed, the minimum launch speed to hit the Sun can be dramatically reduced. Find the minimum launch speed required to hit the Sun if an infinitesimal adjustment later is allowed.

Remark

There's a whole science of multi-stage rocket maneuvers. For example, suppose your goal is to quickly escape the solar system. As you found in part (d) of problem 16, the minimum launch speed necessary is the third cosmic speed. However, you can also start by doing the maneuver of part (e). Once the rocket is very close to the Sun, it'll be moving extremely quickly, which means that a second impulse can provide a huge amount of energy. This is called the Oberth maneuver, as it uses the Oberth effect. Doing it this way costs more fuel, in terms of total Δv, but can allow the rocket to leave much faster.

In practice, you can only get within some distance $r_{\min}$ of the Sun without the rocket burning up, so there's a limit to how much you can employ the Oberth effect. Thus, in some cases a three-impulse maneuver, called the Edelbaum maneuver, can be even better. In the Edelbaum maneuver, you begin with a forward impulse to get to a higher elliptical orbit, then perform a backward impulse to drop to $r_{\min}$. This gives a higher speed at $r_{\min}$, since the rocket is on an elliptical orbit with higher total energy. Then a final forward impulse can be used to escape the solar system. You can read more about these maneuvers here. However, neither the Oberth or Edelbaum maneuvers have ever been used, because the Δv requirement is too high for them to be feasible. For an authoritative reference on rocket maneuvers, see *An Introduction to the Mathematics and Methods of Astrodynamics* by Battin.

[4] **Problem 17** (MPPP 36). Consider a solar system with two planets, in circular orbits with radii R_1 and $R_2 = xR_1$, where $x > 1$. A space probe is planned to be launched from the first planet, which we will call the Earth, and use a gravitational slingshot from the second planet to exit the solar system. The goal is to do this with the smallest fuel energy expenditure possible.

(a) The space probe is launched so that, after it has exited the gravitational field of the Earth, but before it has moved very far, it has speed v_0 in the Sun's frame. Furthermore, its velocity is parallel to the Earth's velocity in the Sun's frame. Explain why this direction of launch minimizes the energy needed.

(b) Assume the space probe arrives near the second planet, with radial and tangential speeds v_r and v_t with respect to the Sun. Find v_r and v_t.

(c) Suppose the planet have speed v_p. In terms of v_p, v_r, and v_t, what is the largest possible speed v_f of the space probe (relative to the Sun) after the gravitational slingshot ends?

(d) Find the value of x that minimizes the required initial launch speed v_0, for the probe to be able to escape the solar system. (Hint: to save space, consider nondimensionalizing variables. Unless you are very good at algebra, you will have to optimize something numerically.)

(e) Which real solar system planet is closest to this ideal planet?

Remark

Above we discussed the Oberth and Edelbaum maneuvers, which use two and three impulses, respectively. In general, if you only deal with the gravity of the Sun, optimal maneuvers never require more than three impulses, so they can't get *too* complicated. But in reality, it would be impractical to exit the solar system or reach the Sun without also using gravitational slingshots. The Voyager probes used multiple slingshots off the gas giants to do the former, while the Parker Solar Probe did an incredible seven gravitational slingshots off Venus in a row to do the latter! These kinds of trajectories need to be calculated years in advance.

Still, you might be thinking, is this really the hardest stuff in the world, when it just boils down to Newtonian mechanics? Well, as Lee DuBridge, the president of Caltech once said:

> I [like] to talk about space to nonscientific audiences. In the first place, they can't check up on whether what you are saying is right or not. And in the second place, they can't make head or tail out of what you are telling them anyway—-so they just gasp with surprise and wonderment, and give you a big hand for being smart enough to say such incomprehensible things. And I never let on that all you have to do to work the whole thing out is to set the centrifugal force equal to the gravitational force and solve for the velocity. That's all there is to it!

I'm just being glib here – the moon landing is unquestionably one of the greatest engineering feats in history. The physical laws at play are elementary, but their application is subtle, and the engineering required getting thousands of tricky real-world details right.

Example 9

An object quickly flies past a star of mass M, with nearly constant speed v, so that its distance of closest approach is R. Estimate the angle by which the object is deflected.

Solution

To solve this exactly, we would write down Newton's second law in polar coordinates and solve a differential equation. However, we can get a rough estimate a simpler way. Since the object is flying quickly, its path is approximately a straight line. Most of the transverse impulse it experiences occurs when it is at a distance of order R from the star, and we can approximate this as

$$\Delta p_\perp = \int F_\perp \, dt \sim F_\perp \, \Delta t \sim \frac{GMm}{R^2} \frac{R}{v}.$$

The small angle of deflection is

$$\Delta\theta \approx \frac{\Delta p_\perp}{mv} \sim \frac{GM}{Rv^2}.$$

This is a decent approximation for the true answer, which turns out to be $2GM/Rv^2$.

In Newtonian gravity, we can think of light as consisting of massless particles moving at speed c, so we can find the deflection of light by setting $v = c$. However, in general relativity the bending of light is actually twice as large, $\Delta\theta = 4GM/Rc^2$. The observation of this factor of 2 by Eddington during a solar eclipse was one of the first tests of general relativity, but it's pretty tricky; Einstein himself missed it in his original paper of the subject!

Roughly speaking, the source of the factor of 2 is that general relativity is a theory of spacetime curvature. The "temporal" curvature corresponds, in the Newtonian limit, to motion under a potential. But the star also sources "spatial" curvature. As explained here, it corresponds to an "angular defect", so that moving tangentially by $2\pi r$ doesn't bring you back where you started. We could neglect this effect in problem 5 because we were considering nonrelativistic particles, which move much faster through time than space. But for light, the two effects are comparable and contribute equally to the deflection.

Remark

Another famous prediction of general relativity is the perihelion precession of Mercury. As you will show in **R3**, the orbit of Mercury advances by a tiny angle $\Delta\theta$ on each cycle. However, knowing that general relativity is a relativistic theory of gravity, we can estimate this angle by dimensional analysis. The only dimensionful parameters are the strength of gravity GM, the radius R of Mercury's orbit, and the speed of light c. (Other parameters we might care about can be expressed in terms of these; for instance, the speed of Mercury is $v = \sqrt{GM/R}$.)

By similar logic to the above problem, the only possible expression is

$$\Delta\theta \sim \frac{GM}{Rc^2} \sim 10^{-8}$$

which is incredibly tiny. The true answer is larger by a factor of $6\pi/(1-e^2)$, where $e \approx 0.2$ is the eccentricity.

5 Fictitious Forces

Idea 7

Consider an inertial frame and a rotating frame with angular velocity $\boldsymbol{\omega}$. For any vector $\mathbf{V}$, the time derivatives of $\mathbf{V}$ in these two frames are related by

$$\left(\frac{d\mathbf{V}}{dt}\right)_{\text{in}} = \left(\frac{d\mathbf{V}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{V}.$$

For example, when $\mathbf{V}$ is the position $\mathbf{r}$, we have the familiar result

$$\mathbf{v}_{\text{in}} = \mathbf{v}_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{r}.$$

Applying this equation to the velocity $\mathbf{v}$, we find

$$\mathbf{a}_{\text{in}} = \mathbf{a}_{\text{rot}} + 2\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

The two terms on the right correspond to the Coriolis and centrifugal forces,

$$\mathbf{F}_{\text{rot}} = \mathbf{F} - 2m\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

In the case where $\boldsymbol{\omega}$ can change, we also have the azimuthal force $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$. (If you prefer, these forces can also be derived by working in components in polar coordinates, as shown in chapter 11 of Wang and Ricardo, volume 1.)

Idea 8

Sometimes, the best way to deal with fictitious forces is to just avoid them by using an inertial frame instead. This is especially true when the Coriolis force is not small; it's straightforward to treat it approximately if it's small, but otherwise it's quite complicated. If a problem presents a situation in a rotating frame, there's no reason you have to stay in that frame! (One example of this is $F = ma$ 2018, problem B12.)

Example 10

Angular momentum conservation tells us that an ice skater increases their angular velocity as they pull their arms inward. Derive this result by working in the frame that always rotates with the skater, as the skater pulls their arms in radially. Specifically, model the skater as two point masses m a distance r from the axis. Show that balancing the Coriolis and azimuthal forces yields a result equivalent to using angular momentum conservation in an inertial frame.

Solution

Let ω be the (time-dependent) angular velocity of the skater's frame. Balancing the forces on one arm,

$$2m\omega\dot{r} = -m\dot{\omega}r$$

which is equivalent, by the product rule, to the statement that ωr^2 is constant. Then $mr^2\omega$ is constant, which is exactly the angular momentum in an inertial frame.

Example 11

A projectile is dropped from height h at the equator. Let the Earth have angular velocity ω, and let the local gravitational acceleration be g. Counting only the Coriolis force, which direction is it deflected when it hits the ground, and by about how far? Is the correction due to the centrifugal force significant?

Solution

The earth rotates from west to east, so the angular velocity points from the south pole to the north pole. The velocity of the falling ball points radially inward, so the Coriolis force points east. We naturally assume the height h is much less than the radius of the Earth, so the inward gravitational acceleration is constant. The Coriolis acceleration is thus

$$a_c = 2\omega v = 2\omega g t$$

in the eastward direction, and integrating this twice gives a deflection

$$d(t) = \frac{1}{3}\omega g t^3.$$

The projectile hits the ground at $t = \sqrt{2h/g}$, giving a final eastward deflection of

$$d = \frac{\omega}{3}\sqrt{\frac{(2h)^3}{g}}.$$

This is the right answer to first order in ω. For a neat, geometric method that arrives at the same result, see the solutions to NBPhO 2016, problem 9.

Remark

What if you wanted to refine the answer above? It turns out this is quite subtle, because a slew of other effects appear at higher order in ω and h, including the centrifugal force (which affects both the mass and the shape of the Earth), and the variation in g with height. If you want to explore this in detail, see problems 10.12 and 10.13 of Morin.

Example 12

The Eotvos effect is the fact that the apparent weight of an object on Earth depends on its motion. How large is this effect at latitude ϕ, and what directions of motion have an effect?

Solution

The Eotvos effect is due to the Coriolis force. As we mentioned in the previous example, the angular velocity of the Earth points out of the north pole. For concreteness, let's suppose we're in the northern hemisphere, $\phi > 0$. Then a object moving east with speed v will yield an outward Coriolis force $2m\omega v$, making the apparent weight lighter, while a westward velocity will make the apparent weight heavier. (Moving north or south, or up and down, just deflects the object east or west.) Eotvos first measured this effect in the lab in 1915, by rotating a balance. It must be accounted for by surveys of g, which are used to find oil deposits.

Example 13

Explain where the factor of 2 in the Coriolis force comes from, working in an inertial frame.

Solution

For concreteness, consider a rotating cylindrical space station of radius R with angular velocity ω. An astronaut initially stands on its rim, then jumps upward, picking up an inward radial velocity u in the space station's rotating frame. The Coriolis force implies that the astronaut will have tangential acceleration $2\omega u$.

In an inertial frame, this 2 comes from the combination of two effects of equal magnitude. Let $v = \omega R$ be the initial tangential velocity of the astronaut in this frame. As the astronaut moves radially inward, angular momentum conservation implies that their tangential velocity increases, so that after a time dt it is

$$\omega R \frac{R}{R - u\,dt} = \omega R + \omega u\,dt.$$

At the same time, the tangential speed of the rotating frame decreases, to

$$\omega(R - u\,dt) = \omega R - \omega u\,dt.$$

The relative tangential acceleration is thus $2\omega u$, giving the desired result.

[2] **Problem 18.** A cylindrical space station of radius R can create artificial gravity by rotating with angular velocity ω about its axis.

(a) For an observer rotating along with the spaceship on the rim, what gravitational acceleration g do they perceive?

(b) The observer throws a ball parallel to the floor. For some launch speed v, the observer will see the ball perform a circular orbit along the spaceship, always parallel to the floor. Find v.

(c) What does the motion of part (b) look like, in a frame that isn't rotating with the ship?

[2] **Problem 19.** Every satellite in orbit around the Earth is slowly falling due to drag. Consider a satellite steadily falling, with a large tangential velocity and small inward radial velocity.

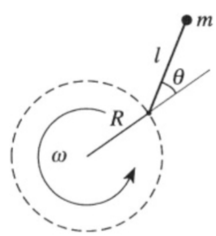
(a) Show that for a satellite initially in a circular orbit, losing energy U to drag *increases* the kinetic energy of the satellite. By how much is it increased?

(b) The result of part (a) seems almost paradoxical. How can it be explained in an inertial frame, given that the drag force always acts to slow down the satellite?

(c) Now consider a uniformly rotating frame, whose angular velocity is equal to the initial angular velocity of the satellite. In this frame, the drag force always points tangentially backwards, but the satellite ends up going tangentially forward. What force is responsible?

[2] **Problem 20.** A frictionless tube of length R is rotated with angular velocity ω about one of its ends. A package is placed in the tube at a distance r_0 from the axis of rotation, with no initial radial velocity. What the package flies out the other end of the tube, what is its speed?

[2] **Problem 21** (Cahn)**.** A pendulum is designed for use on a gravity-free spacecraft. The pendulum consists of a mass at the end of a rod of length ℓ. The pivot at the other end of the rod is forced to move in a circle of radius R with angular frequency ω. Let θ be the angle the rod makes with the radial direction.



Show this system behaves exactly like a pendulum of length ℓ in a uniform gravitational field $g = \omega^2 R$. That is, show that $\theta(t)$ is a solution for one system if and only if it is for the other.

[4] **Problem 22.** IPhO 2016, problem 1B. A useful set of Coriolis force exercises.

[3] **Problem 23.** USAPhO 2020, problem A2. A tricky question on the Foucault pendulum. For an algebraic derivation of the final result, see section 9.9 of Taylor; it uses the complex number method introduced for a problem in **M1**. For a beautiful but more abstract geometric derivation, see section 11.5.1 of Griffiths' *Introduction to Quantum Mechanics* (3rd edition).

As a warning, this problem and its solution are a lot rougher than in a typical USAPhO, making the question more confusing than it should be. That's because the 2020 USAPhO was cancelled for the pandemic, and AAPT released a rough draft to give people something to practice on. So if you have unexpected trouble on this problem, or find the solution puzzling, don't worry about it. I only include the 2020 problems in these problem sets because they illustrate some new ideas.

[4] **Problem 24** (Morin 10.26)**.** A coin stands upright on a turntable rotating with angular frequency ω, and rolls without slipping so that its center is motionless in the lab frame. Thus, in the frame of the turntable, the coin rolls without slipping in a large circle with angular frequency ω.

(a) In the lab frame, explain how $\mathbf{F} = d\mathbf{p}/dt$ and $\boldsymbol{\tau} = d\mathbf{L}/dt$ are satisfied. (This is the easy part.)

(b) In the frame of the turntable, verify that $\mathbf{F} = d\mathbf{p}/dt$.

(c) In the frame of the turntable, verify that $\boldsymbol{\tau} = d\mathbf{L}/dt$. (This is the hard part.)

If you slogged through part (c), you'll understand why we rarely want to think about torques for extended bodies in rotating frames.

Idea 9

An object freely falling in a gravitational field will experience tidal forces, due to the spatial variation of gravitational fields. Specifically, suppose a mass M is placed at $(R, 0)$. Then the gravitational acceleration near the origin is

$$\mathbf{g} \approx \frac{GM}{R^2}\hat{\mathbf{x}} + \mathbf{g}_{\text{tidal}}, \quad \mathbf{g}_{\text{tidal}} = \frac{GM}{R^3}\left(2x\hat{\mathbf{x}} - y\hat{\mathbf{y}}\right).$$

In particular, if a small rigid object with center of mass at the origin is released, it will have acceleration $(GM/R^2)\hat{\mathbf{x}}$. In the frame of the center of mass, the rest of the object will experience the residual gravitational acceleration $\mathbf{g}_{\text{tidal}}$. Intuitively $\mathbf{g}_{\text{tidal}}$ tends to stretch the object in the longitudinal direction and squeeze it in the transverse direction.

Example 14

Estimate the height of the tides on Earth induced by the Moon.

Solution

We will simplify the problem by treating the Earth and Moon as stationary, and the Earth as a ball of fluid. The result above tells us that the tidal acceleration due to the Moon is

$$\mathbf{g} = \frac{GM_M}{D^3}(2x\hat{\mathbf{x}} - y\hat{\mathbf{y}})$$

where D is the distance to the Moon.

Suppose that without the Moon's presence, the Earth is a sphere of radius R_E. By integrating this, the tidal gravitational potential induced on the Earth's surface is

$$\Delta\Phi = -\frac{GM_M R_E^2}{D^3}\frac{3\cos^2\theta - 1}{2}.$$

Changing the gravitational potential on the surface by $\Delta\Phi$ means the equilibrium height of the water shifts by $\Delta h = \Delta\Phi/g$, where g is the gravitational acceleration of the Earth. Thus,

$$\Delta h \sim \frac{GM_M R_E^2/D^3}{GM_E/R_E^2} = \frac{M_M}{M_E}\left(\frac{R_E}{D}\right)^3 R_E.$$

This can also be written in terms of the densities,

$$\Delta h \sim \frac{\rho_M}{\rho_E}\left(\frac{R_M}{D}\right)^3 R_E.$$

The first term is of order one. The second term is the angular radius of the Moon in the sky, which is about a quarter of a degree, giving

$$\Delta h \sim \left(\frac{2\pi}{360 \times 4}\right)^3 R_E \sim 0.5\,\mathrm{m}$$

which is of the right order of magnitude.

Remark

The tidal effects of the Sun and Moon are comparable, but the Moon's are somewhat larger. This can be seen by the final equation above: the Sun and Moon have comparable angular diameter, since eclipses can just barely happen, but the Moon is 2.4 times denser.

When the Moon and Sun's tidal effects reinforce, one has a larger tidal effect, called a spring tide; otherwise, one has a neap tide. Given what's been said so far, you should be able to figure out what phase(s) of the Moon and time(s) of day correspond to each (at least for an idealized "spherical cow" Earth), and also roughly how much stronger tides are at spring tides than at neap tides.

It's worth noting that the *actual* tides are far more complicated than just computing the tidal force, because they depend on the dynamical response of the water, which in turn depends on the detailed lay of the land. For some more details, see here.

One should be careful when applying tidal forces, because many astronomical objects are also spinning or orbiting, which provides an additional centrifugal force. For a good discussion of tides, see section 10.3 of Morin.

[3] **Problem 25** (Morin 10.31)**.** A small spherical rock covered with sand falls in radially toward a planet. Let the planet have radius R and density ρ_p and let the rock have density ρ_r. When the rock gets close enough to the planet, the tidal force will pull the sand off the rock. The cutoff distance is called the Roche limit; it gives the radial distance below which loose objects can't coalesce into larger ones.

(a) Show that the Roche limit is

$$d = R\left(\frac{2\rho_p}{\rho_r}\right)^{1/3}.$$

(b) Now suppose the rock is orbiting the planet and rotating so that the same side always faces the planet. Show that the Roche limit is

$$d = R\left(\frac{3\rho_p}{\rho_r}\right)^{1/3}.$$

[4] **Problem 26.** IPhO 2009, problem 1. A neat problem about how the Moon has slowed down the Earth's rotation. This is a fascinating subject; in the deep past, not only were days shorter, but the weather on Earth was completely different, owing to the much higher Coriolis forces.

[5] **Problem 27.** IPhO 1992, problem 1. A problem on a strange propulsion mechanism, which is in some sense the reverse of the previous problem. Fill in your answers on the attached answer sheet, along with your solutions.

Remark: Discovering Gravity

In elementary school, we hear that Newton understood gravity in a flash of inspiration, after being hit on the head with an apple. Later, you might learn that it didn't quite work that way: there was an apple tree in Newton's childhood home, but an apple didn't hit him, and Newton didn't publish his ideas on gravity until decades afterward.

However, the story is an oversimplification in a much more significant way. Newton's law of gravity actually contains many independent insights. For example, you need to realize that gravitational forces occur between *pairs* of objects, rather than emanating from an object, or reflecting an object's desire to move towards its "natural" place of being. To explain the orbits, you need to understand that the force is radial, not tangential, and moreover that it is *not* balanced by any other radial force. You need to see that the force acts between *all* pairs of objects, and not just certain pairs of objects with the right qualities, like iron and magnets, that the force is proportional to mass and falls off with distance, and that it occurs "at a distance" with nothing in between.

All of these insights, which we think of as obvious today, were viewed as unintuitive or downright occult by intelligent thinkers of the time. For example, you probably think the astrological idea that Jupiter governs blood and Venus governs phlegm is laughable, as did many 17$^{\text{th}}$ century astronomers, but would the idea that the Moon governs the rise and fall of water on Earth sound any more plausible, if you hadn't been told early on that it's true by people you trust? (If you flip this logic around, you can understand why so many people believe in astrology.) Or, going further back to antiquity, if you claimed then that everything is affected by gravity, how could you explain why flames go up? (Before you embark on an explanation of buoyancy, you would first have to explain how air exerts a massive yet somehow unobservable pressure on everything, why air has mass but doesn't fall, and that buoyant forces for air exist at all. The helium balloon wouldn't be around for thousands of years, and without modern tools, it's surprisingly tricky to make a stable vacuum.)

Between Galileo and Newton, there were many incremental steps towards the development of universal gravitation. For instance, Borelli proposed that Jupiter's moons obeyed Kepler's laws, and Horrocks found that Jupiter and Saturn slightly deviated from Kepler's laws because of their mutual attraction. Newton played an important role by putting everything on a solid foundation, such as by deriving Kepler's first law and the shell theorems. But as you can see from Newton's notebooks, these insights came from years of experience tinkering with concrete calculations. Indeed, think about the massive distance between the law of gravity and Aristotle's idea that things fall because they seek their "natural" place. To get from Aristotle's "rocks want to go home" theory to Newtonian mechanics requires not just genius, but many geniuses. And of course, there were just as many steps needed to get from noticing static electricity existed to writing down Coulomb's law, including a bewildering array of homemade experiments with medieval technology. Nothing is trivial in physics.

Remark: Negative Mass

One intriguing difference between electrostatics and gravity is that charge can be positive or negative, while mass is apparently only positive. Could there be a "negative mass", to complete the analogy, which falls upward in a gravitational field? If general relativity is true, the answer is no: it respects the equivalence principle, which implies that particles all accelerate the same way in a gravitational field. (Every month or so, people will write clickbait articles saying that negative mass has been discovered, but most of those examples are like balloons. They don't go up in a gravitational field because they actually have negative mass, they go up because they have less mass than the air they're displacing.)

However, there's an alternative formulation of negative mass that does respect the equivalence principle: let $\mathbf{F} = m\mathbf{g}$ and $\mathbf{F} = m\mathbf{a}$ with the same negative m in both equations. Then the force on the mass points upward, but it accelerates downward anyway.

This leads to some seriously strange consequences. For example, suppose we had a positive and a negative mass interacting. Then the two would repel each other, which implies that both will start zooming off in the *same* direction! This is completely compatible with momentum and energy conservation, since the negative mass has flipped momentum and negative energy, but it's quite strange. In a universe with an equal amount of both kinds of masses, they'll both eventually end up zooming around at the speed of light, but the energy will be zero. Anyway, at the moment we don't know of any ordinary matter with these weird properties.

Remark: Lagrangian Points

Earnshaw's theorem states that a point charge in an electrostatic field cannot be in stable equilibrium, i.e. it is impossible for the charge to be in equilibrium and feel a restoring force when displaced in an arbitrary direction. The proof is simply Gauss's law: for some candidate equilibrium point P, draw a small Gaussian surface around it. Since there is no charge inside (the test charge itself doesn't count, since we're only caring about the fields of *other* charges, which act on it), the flux must be zero. But a stable equilibrium requires the field to have an inward-pointing component in all directions, which implies nonzero flux.

Earnshaw's theorem also applies to gravity; by the same logic, in space there are no gravitational potential extrema. However, you may have heard that there are stable Lagrangian points in the Earth-Sun system, where satellites can be stored. How is this possible, given that stability usually requires a potential minimum?

The first subtlety is that Lagrangian points rotate with the Earth about the Sun, so we really should be working in a rotating frame. In this frame, there is an additional centrifugal force. However, this doesn't help, because you can check that the divergence of the centrifugal force is *positive* everywhere, i.e. it causes instability rather than stability.

As usual for subtleties involving fictitious forces, the resolution comes from the Coriolis force. Near the stable Lagrangian points, the Coriolis force deflects satellites sideways. The stability of the points is not because there is an ordinary restoring force, but because of this continuous sideways deflection, which keeps particles from moving far away.

How can this can be analyzed mathematically? For a typical restoring force, we would expand the force linearly in $\Delta \mathbf{r}$ and determine stability using the methods of **M4**. Here that doesn't work, because we have a velocity-dependent force, but we can still write the force as linear in the *four* coordinates $(\Delta \mathbf{r}, \mathbf{v})$. This gives a set of four linear differential equations, which can be solved in terms of normal modes as in **M4**. The motion is stable if no modes are exponentially growing. For a more complete analysis, see here.

Mechanics VI: Gravitation

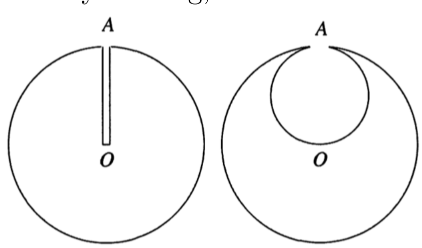
Chapters 8 and 9 of Kleppner and Kolenkow cover orbits and fictitious forces, as do chapters 8 and 11 of Wang and Ricardo, volume 1, and chapters 7 and 10 of Morin. For *much* more, see chapters 2 and 3 of *Galactic Dynamics* by Binney and Tremaine. There is a total of **87** points.

1 Computing Fields

Idea 1

Gravitational fields obey the shell theorem and the superposition principle, which is sufficient to find the field in a variety of setups. One useful trick is to think of objects with holes as superpositions of objects without holes, and holes with negative mass.

[3] **Problem 1** (PPP 110). A spaceship of titanium-devouring little green people has found a perfectly spherical homogeneous asteroid. A narrow trial shaft was bored from point A on its surface to the center O of the asteroid. At that point, one of the little green men fell off the surface of the asteroid into the trial shaft. He fell, without any braking, until he reached O, where he died on impact.



However, work continued and the little green men started secret excavation of the titanium, in the course of which they formed a spherical cavity of diameter AO inside the asteroid. Then a second accident occurred: another little green man similarly fell from point A to point O, and died. Find the ratio of the impact speeds, and total times for impact, of the two little green men.

Solution. Say we have a sphere with center $\vec{c}$, and some point $\vec{r}$ is inside the sphere. Then, we know from Gauss's law that the acceleration of a mass at $\vec{r}$ due to the sphere is $-k\rho(\vec{r}-\vec{c})$ for some constant k.

Let O be the origin, and let $\vec{m}$ be the midpoint of AO. Then, the acceleration in case 1 is

$$-k\rho\vec{r}$$

and in case 2 is

$$-k\rho\vec{r}-k(-\rho)(\vec{r}-\vec{m})=-k\rho\vec{m}.$$

Thus, the impact speed in case 1 where R is AO is

$$\frac{1}{2}v_1^2=\frac{1}{2}k\rho R^2 \implies v_1=R\sqrt{k\rho}$$

where we are thinking of the force as coming from a spring. For case 2, the speed is

$$v_2^2=2k\rho(R/2)R \implies v_2=R\sqrt{k\rho}.$$

Thus, the ratio of the speeds is $v_1/v_2=1$.

Now for times. We know that t_1 is 1/4 the period of the full oscillation, or

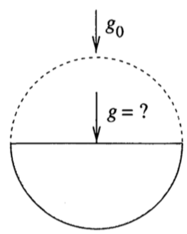
$$t_1 = \frac{\pi}{2}\sqrt{\frac{1}{k\rho}}.$$

Now, for t_2, we have that

$$v_2 = at_2 \implies R\sqrt{k\rho} = (k\rho R/2)t_2 \implies t_2 = 2\sqrt{\frac{1}{k\rho}},$$

so $t_1/t_2 = \boxed{\pi/4}$.

[3] **Problem 2** (PPP 111)**.** The titanium-devouring little green people of the previous problem continued their excavating. As a result of their environmentally destructive activity, half of the asteroid was soon used up, as shown.



What is the gravitational acceleration at the center of the circular face of the remaining hemisphere if the gravitational acceleration at the surface of the original spherical asteroid was g_0? (This can be done without any integrals.)

Solution. Set up coordinates such that the hemisphere is described by $0 \leq r \leq R$ and $0 \leq \theta \leq \pi/2$. If the mass density is ρ, the total field at the origin is

$$\vec{g} = \int_0^R \int_0^{\pi/2} \int_0^{2\pi} \frac{G\rho}{r^2}\hat{\mathbf{r}}r^2 \sin\theta \, d\phi \, d\theta \, dr.$$

Due to symmetry reasons, only the $\hat{\mathbf{z}}$ component of $\hat{\mathbf{r}}$ will survive, so we can replace $\hat{\mathbf{r}}$ with $\hat{\mathbf{z}}\cos\theta$. Thus,

$$\begin{aligned}\vec{g} &= \int_0^R \int_0^{\pi/2} \int_0^{2\pi} \frac{G\rho}{r^2}\hat{\mathbf{z}}(\cos\theta)r^2 \sin\theta \, d\phi \, d\theta \, dr \\ &= 2\pi G\rho R\hat{\mathbf{z}} \int_0^{\pi/2} \sin\theta\cos\theta \, d\theta \\ &= \pi G\rho R.\end{aligned}$$

Here's another, slick way to do this: note that every hemisphere contributes the same amount, since its mass increases as the square of its radius. The entire solid hemisphere can hence be replaced with a hemispherical shell at its surface, with the same total mass. The resulting gravitational field is then computed using the hemisphere trick of **M2**.

Originally we had

$$g_0 = G\frac{4}{3}\pi R^3\rho/R^2 = \frac{4}{3}\pi G\rho R.$$

Thus, $g/g_0 = 3/4$.

[3] **Problem 3** (Morin 5.67)**.** You are given a fixed volume of a moldable material, with a fixed density. Describe the shape it should take to maximize the gravitational field at the origin.

Solution. Set up spherical coordinates (r, θ, ϕ) so that the field points in the $\hat{\mathbf{z}}$ direction in the optimal case. We will actually maximize g_z, but since we chose $\hat{\mathbf{z}}$ such that it is anti-parallel to $\mathbf{g}$ in the optimal case, this is actually good enough. Firstly, we should clearly have everything above the plane $z = 0$, else the masses will contribute negative g_z.

Note that the surfaces given by $\cos\theta/r^2 = A$ for some constant A provide equal z-field at the origin. Note that A is proportional to the field. Pick B such that the volume $\cos\theta/r^2 \geq B$ is equal to the fixed volume we have. If all the charge is not in here, then some is outside, with lower $\cos\theta/r^2$, so by moving it into this region, we strictly increase g_z. Therefore, the optimal case is when the region is of the form $\cos\theta/r^2 \geq B$.

Idea: The Shell Theorems

Newton proved a variety of elegant "shell" theorems, some of which are less well known.

1. Inside a uniform spherical shell, there is no gravitational field. (At the time, this was an important result primarily because it meant that Hell couldn't be at the center of the Earth; if it were, the fire and brimstone would be floating around.)
2. Outside a uniform spherical shell of total mass m, the gravitational field is the same as that of a point mass m at its center. (Of course, this is important because it's required to think about the Earth's gravity at all.)
3. There is no gravitational field inside a uniform homoeoid, i.e. a region defined by
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \in [1, 1+\epsilon]$$
for some constant ϵ, with uniform density. This one is seldom useful. (The gravitational field outside a homoeoid is more complicated, so there's no "fourth" theorem.)

Example 1

Newton was aware that similar shell theorems hold for linear force laws, $F(r) \propto r$. How are his first two theorems modified in this case?

Solution

Consider a spherical shell centered at the origin, and a test mass at $\mathbf{r}_0$. The contribution to the force due to a piece of the shell at $\mathbf{r}$ is $\mathbf{F} \propto \mathbf{r} - \mathbf{r}_0$. When we integrate over the shell, $\mathbf{r}$ averages to zero, giving $\mathbf{F} \propto -\mathbf{r}_0$, which is precisely the result for a mass exactly at the center of the shell. That is, for a linear force law, Newton's first theorem doesn't work; instead the second theorem's result applies both inside and outside the shell.

Example 2

Prove the converse of Newton's second theorem: a spherical shell can be replaced with a point mass at its center only if $F(r)$ is proportional to r, proportional to r^{-2}, or a linear combination of the two.

Solution

It's easiest to consider the potential outside the shell. Let the shell of mass m be centered at the origin with radius R, and consider the potential at a distance $z > R$ from the origin. If a point mass produces a gravitational potential $f(r)\,dm$ at separation r, then integrating over the sphere in spherical coordinates gives

$$V(z) = \frac{m}{4\pi R^2} \int_0^\pi (2\pi R^2 \sin\theta\, d\theta)\, f(\sqrt{z^2 + R^2 - 2zR\cos\theta}).$$

The trick is to switch variables to the separation $r = \sqrt{z^2 + R^2 - 2zR\cos\theta}$, since

$$r\, dr = zR \sin\theta\, d\theta.$$

Plugging this in gives

$$V(z) = \frac{m}{2zR} \int_{z-R}^{z+R} r f(r)\, dr.$$

Newton's second theorem works precisely when dV/dz is independent of R, so that the shell radius can be collapsed to zero without changing the force.

Suppose $f(r)$ is proportional to r^n. Then we have

$$V(z) \propto \frac{(z+R)^{n+2} - (z-R)^{n+2}}{zR}$$

and the force's dependence on R can only drop out in three cases: when $n = -1$ (an inverse square force), $n = 0$ (the trivial case, corresponding to no force), and $n = 2$ (a linear force). The final case is the trickiest; in that case we have

$$V(z) \propto \frac{4z^3R + 4zR^3}{zR} \propto z^2 + R^2$$

so that R drops out of dV/dz, as required. Since any reasonable function can be built by superposing such power laws, this classification is exhaustive.

Incidentally, the same method can be used to prove the converse of Newton's first theorem. The only difference is that $z < R$, so that the lower limit of integration has to be replaced with $|z - R| = R - z$. Then the $n = 2$ case works out the same way, since $z - R$ is squared. By contrast, for $n = -1$ we get no force, since $V(z) \propto ((z+R) - (R-z))/zR = 2/R$ which is constant. Thus, the inverse square force is the only one where Newton's first theorem applies.

2 Central Potentials

Idea 2: Effective Potential

A particle experiencing a central force has a potential energy $V(r)$ which only depends on its radial coordinate, and conserved angular momentum

$$L = |\mathbf{r} \times \mathbf{p}| = mr^2\dot{\theta}.$$

Its kinetic energy can thus be written in terms of the radial velocity $\dot{r}$ and L,

$$E = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_\theta^2 + V(r) = \frac{1}{2}m\dot{r}^2 + \left(V(r) + \frac{L^2}{2mr^2}\right).$$

By setting the time derivative of this expression to zero, we find

$$m\ddot{r} = -\frac{d}{dr}\left(V(r) + \frac{L^2}{2mr^2}\right).$$

Therefore, if we are interested in $r(t)$ alone, we can find it by treating the problem as one-dimensional, where the particle moves in the "effective potential" $V(r) + L^2/2mr^2$. The extra term is called the angular momentum barrier and repels the particle away from the center. Once we know $r(t)$, we can find $\theta(t)$ if desired by using $\dot{\theta} = L/mr^2$.

One way of understanding the effective potential term is to think in terms of the energy methods of **M4**. From the perspective of $r(t)$ alone, any dependence on $\dot{r}^2$ is part of the kinetic energy, and any dependence on r is part of the potential energy. In particular, the kinetic energy of tangential motion depends only on r, because it is fixed by angular momentum conservation, so it appears as part of the potential when considering only radial motion.

Example 3: KK 9.4

For what values of n are circular orbits stable with the potential energy $U(r) = -A/r^n$?

Solution

Note that circular orbits can only possibly exist if the force is attractive. This implies that A must have the same sign as n.

The effective potential is

$$U_{\text{eff}}(r) = -\frac{A}{r^n} + \frac{L^2}{2mr^2}.$$

In a circular orbit, r is constant, so the particle just sits still at a minimum of the effective potential. That is, the circular orbit radius satisfies $U'_{\text{eff}}(r) = 0$, so

$$\frac{An}{r_0^{n+1}} - \frac{L^2}{mr_0^3} = 0, \quad r_0^{2-n} = \frac{L^2}{Anm}.$$

The orbit is stable if $U''_{\text{eff}}(r) > 0$, so

$$-\frac{An(n+1)}{r_0^{n+2}} + \frac{3L^2}{m}\frac{1}{r_0^4} > 0$$

which simplifies to

$$r_0^{n-2} > \frac{m}{3L^2}An(n+1).$$

Plugging in the value of r_0, this becomes simply $n < 2$. As expected, for inverse square forces ($n = 1$) and spring forces ($n = -2$) the orbits are stable, while, e.g. for inverse cube forces, the circular orbits are neutrally stable.

[3] **Problem 4** (Morin 7.4)**.** A particle of mass m moves in a potential $V(r) = \beta r^k$. Let the angular momentum be L.

(a) Find the radius r_0 of the circular orbit.

(b) Find the angular frequency ω_r of small oscillations about this radius.

(c) Now consider a slightly perturbed circular orbit. Explain why the orbit remains a closed curve if the ratio of the time period of small oscillations and the time period of the original circular orbit is rational, and find the integer values of k where this holds.

Solution. In this problem, the effective potential is

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + \beta r^k.$$

(a) We have circular orbits when the effective potential is minimized, or $V'_{\text{eff}}(r) = 0$, or

$$\frac{L^2}{2m}(-2)r_0^{-3} + k\beta r_0^{k-1} = 0,$$

or

$$r_0 = \left(\frac{L^2}{mk\beta}\right)^{\frac{1}{k+2}}.$$

(b) Note that r_0 is a minimum for $U \equiv V_{\text{eff}}$, so $U'(r_0) = 0$. Thus, for small $|r - r_0|$, we have that

$$U(r) \approx U(r_0) + \frac{1}{2}U''(r_0)(r - r_0)^2,$$

so $\omega_r = \sqrt{U''(r_0)/m}$. Thus, we must compute $U''(r_0)$. Note that

$$U''(r) = \frac{3L^2}{m}r^{-4} + k(k-1)\beta r^{k-2} = r^{-4}\left(\frac{3L^2}{m} + k(k-1)\beta r^{k+2}\right),$$

so $U''(r_0) = \frac{1}{r_0^4}\frac{L^2}{m}(k+2)$, so $\omega_r = \frac{L}{mr_0^2}\sqrt{k+2}$.

(c) This is true because, if the ratio of periods is rational, there is a "least common multiple" at which point an integer number of both cycles (both radial oscillation and the overall orbit) have completed. At this point we are back to the original starting point, so the orbit is closed.

To do the calculation, note that it is equivalent for $\omega_r/\dot{\theta}$ to be zero, where $\dot{\theta}$ is the angular velocity of a circular orbit. We have $mr^2\dot{\theta} = L$, so $\omega_\theta = \frac{L}{mr_0^2}$. Thus, $\omega_r/\omega_\theta = \sqrt{k+2}$. Given that k is an integer, this is rational when $k+2$ is a perfect square, so $k = n^2 - 2$. Examples are $k = -1, 2, 7, 14, \ldots$.

Remark: Bertrand's Theorem

In problem 4, you showed that for a certain group of potentials, all bound orbits that are nearly circular are closed. Bertrand's theorem states that the *only* central potentials for which *all* bound orbits are closed are $V(r) \propto 1/r$ and $V(r) \propto r^2$.

The idea of the proof is as follows. First, for a general potential $V(r)$, we can compute the ratio of periods of a small radial oscillation and the underlying circular orbit and demand it be rational, just like in part (c) above. However, since this ratio changes continuously as the orbit parameters are varied, it must be a constant if it is to always be rational. Using this condition, you can show that $V(r)$ must be a power law, which we had to assume above.

You found in part (c) that infinitely many power laws give closed nearly circular orbits. To rule out the other ones, we need to expand to higher orders, i.e. account for the fact that the effective potential is not perfectly simple harmonic. A detailed derivation can be found here.

[4] **Problem 5.** In general relativity, the potential describing a black hole of mass M is

$$V(r) = -\frac{GMm}{r} - \frac{GML^2}{mc^2r^3}.$$

The second term is a relativistic effect which strengthens the attraction towards the black hole. (It has nothing to do with the angular momentum barrier; you still have to add that separately.)

(a) Explain why this new term allows particles to fall to the center of the black hole, $r = 0$, and why this is impossible in Newtonian gravity.

(b) For a fixed L, find the values of the circular orbit radii.

(c) Find the radius of the smallest possible stable circular orbit, for any value of L. What happens if you try to orbit the black hole closer than this? (Answer: $6GM/c^2$.)

(d) Find the closest possible approach radius of an unbound object. That is, among the set of all trajectories that start and end far away from the black hole (i.e. without falling into it), find the smallest possible minimum value of r. (Answer: $3GM/c^2$.)

For all parts, assume the particle is moving nonrelativistically.

Solution. (a) The effective potential adds a term of $L^2/2mr^2$ to the potential energy term, and that makes the effective potential go to $+\infty$ as $r \to 0$. Thus it would take an infinite amount of energy to reach $r = 0$ in Newtonian gravity (unless $L = 0$). However, the addition of the $-GML^2/mc^2r^3$ term makes the effective potential go to $-\infty$ as $r \to 0$, thus particles can fall to the center.

(b) Circular orbits occur when $V'_{\text{eff}}(r) = 0$:

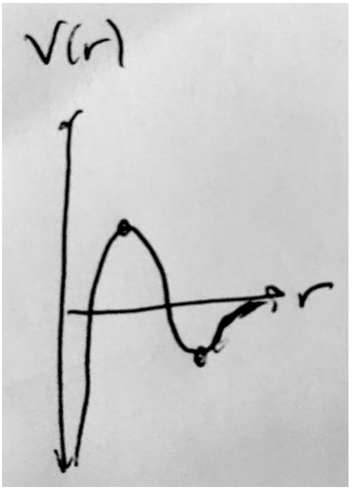
$$\frac{L^2}{mr^3} = \frac{GMm}{r^2} + \frac{3GML^2}{mc^2r^4}$$

$$GMmr^2 - \frac{L^2}{m}r + \frac{3GML^2}{mc^2} = 0$$

$$r = \frac{L^2 \pm \sqrt{L^4 - 12(GMmL/c)^2}}{2GMm^2}$$

Note that there are no extrema at all when the discriminant is negative. That is, circular orbits only exist when $L > \sqrt{12}\,GMm/c$.

(c) From part a), we know that $\lim_{r\to 0} V_{\text{eff}}(r) = -\infty$ and from part b), we know that there's two 1st order extremes. Thus the graph of $V_{\text{eff}}(r)$ should look like this, when L is large:



The particle is trying to minimize its effective potential, so the qualitiative behavior can be easily understood from the graph. Since $V''_{\text{eff}}(r) > 0$ indicates a stable equilibrium, the larger radius $r_2 = (L^2 + \sqrt{L^4 - 12(GMmL/c)^2})/(2GMm^2)$ is stable while the smaller one r_1 is not.

As the angular momentum is decreased, r_1 and r_2 get closer together. When L reaches the critical value $\sqrt{12}\,GMm/c$, the two radii coincide. For smaller L, the curve $V_{\text{eff}}(r)$ has no extrema, so there are no circular orbits at all.

Therefore, the radius of the smallest stable circular orbit is the minimum possible value of r_2, which is achieved when $L = \sqrt{12}\,GMm/c$, giving

$$r_{\text{min}} = \frac{6GM}{c^2}.$$

If you orbit in a circular orbit with a smaller radius, it is necessarily unstable, which means that under any perturbation, you will either drift into the black hole, *or* outward away from it. If you have rockets, this can be prevented by continual orbital adjustment. (Of course, if you try to orbit closer than the Schwarzschild radius $2GM/c^2$, you don't have any choice; you must fall into the black hole. But this effect isn't visible in our simple Newtonian analysis.)

(d) For this to happen, the effective potential needs a maximum, so the particle can "bounce" off it and get back to $r \to \infty$. Thus we need $L > \sqrt{12}\, GMm/c$. For each value of L, the closest we can get while still bouncing off corresponds to the lower critical value of $V_{\text{eff}}(r)$, which was shown in part (b) to be

$$r_{\text{min}}(L) = \frac{L^2 - \sqrt{L^4 - 12(GMmL/c)^2}}{2GMm^2}.$$

This function is minimized when $L \to \infty$ (i.e. when the particle is launched from a very large impact parameter), so the closest possible approach radius is

$$\lim_{L\to\infty} r_{\text{min}}(L) = \frac{6(GMm/c)^2}{2GMm^2} = \frac{3GM}{c^2}$$

where we used the binomial theorem in the first step. (Again, you could get slightly closer, but you would necessarily have to fire a rocket to make it back out.)

Example 4: Binney 5.1

For over 150 years, most astronomers believed that Saturn's rings were rigid bodies, until Laplace showed that a solid ring would be unstable. The same instability plagues Larry Niven's *Ringworld*, a science fiction novel once popular among boomer nerds. Following Laplace, consider a rigid, circular ring of radius R and mass m, centered on a planet of mass $M \gg m$. The ring rotates around the planet with the Keplerian angular velocity $\omega = (GM^2/R^3)^{1/3}$. Show that this configuration is linearly unstable.

Solution

One way to understand the stability of an ordinary planetary orbit is angular momentum conservation: if you displace a planet radially inward, then it'll start moving faster tangentially, which will tend to make it go back out, even though the inward gravitational force gets stronger too. This tendency is absent for a rigid ring, because the entire ring always rotates with the same angular velocity $\omega = L/mR^2$.

The simplest way to see that this configuration is unstable is to calculate the gravitational potential ϕ due to the ring at the planet's position. If the planet starts at the center of the ring, then displacing it along the axis of the ring increases ϕ. But since $\nabla^2\phi = 0$, displacing it towards the ring must decrease ϕ, so the system is unstable. (This is just the gravitational analogue of Earnshaw's theorem from **E1**.)

To make this more concrete, fix the planet at the origin, and parametrize the ring by the angle θ along it. If the whole ring is shifted by a small distance a in the plane of the ring, the elements of the ring are at

$$r^2 = (R\cos\theta + a)^2 + (R\sin\theta)^2.$$

The total gravitational potential energy is

$$U = -GMm \int_0^{2\pi} \frac{d\theta}{2\pi}\frac{1}{r} = -\frac{GMm}{R}\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{1}{\sqrt{1 + (2a/R)\cos\theta + a^2/R^2}}.$$

We have to be a bit careful here, remembering some lessons from **P1**. The first order term in a is going to vanish, because we started at an equilibrium point, which means we need to expand everything to second order in a. Using the Taylor series

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + O(x^3)$$

we conclude

$$U \approx -\frac{GMm}{R} \int_0^{2\pi} \frac{d\theta}{2\pi} \left(1 - \frac{a}{R}\cos\theta + \frac{a^2}{R^2}\frac{3\cos^2\theta - 1}{2}\right) = -\frac{GMm}{R}\left(1 + \frac{a^2}{4R^2}\right).$$

The energy goes down upon a small displacement, so the configuration is unstable. The ring will soon crash into the planet.

3 Kepler's Laws

Idea 3

Kepler's laws for a general orbit are:

1. The trajectories of planets are conic sections, with a focus at the Sun. Bound orbits are ellipses, which contain circles as a special case. Unbound orbits are hyperbolas, which contain parabolas as a special case.

2. The trajectories sweep out equal areas in equal times.

3. When the orbit is bound, the period T and semimajor axis a obey $T^2 \propto a^3$.

Unlike the other laws, the second is valid for any central force, because the rate of area sweeping is $rv_\perp/2 \propto |\mathbf{r} \times \mathbf{v}| \propto |\mathbf{L}|$.

Idea 4

For a general orbit with semimajor axis a, the total energy is

$$E = -\frac{GMm}{2a}.$$

This identity also applies to hyperbolas, where a is negative, and the parabola in the limit of infinite a, where the total energy vanishes.

Remark: Virial Theorem

For bound orbits, the time-averaged values of the kinetic and potential energy are related by

$$\langle K \rangle = -\frac{1}{2}\langle V \rangle.$$

In fact, the virial theorem holds for more complicated bound systems of particles as well, as long as they interact by a power law potential $V(r) \propto r^n$. In this case, we have

$$\langle K \rangle = \frac{n}{2} \langle V \rangle$$

where gravity corresponds to the case $n = -1$.

You can easily check that the virial theorem works in one dimension for a particle bouncing in a uniform gravitational field ($n = 1$), or a particle on a spring ($n = 2$). It's also easy to check for a planet in a circular orbit ($n = -1$). With some more work, you can check that it also holds for arbitrary elliptical orbits – to do this most efficiently, convert the time integral to an integral over angle θ, and use the form of an ellipse in polar coordinates.

In astrophysics, the virial theorem is useful because it allows us to estimate V, which can be hard to measure, given K. For discussion of the virial theorem along with applications to dark matter, see section 1.4.3 of these notes. We will return to these subjects in **X3**.

[3] **Problem 6.** In this problem we'll verify some of the basic facts stated above.

(a) Prove the statement of idea 4 for the case of elliptical orbits.

(b) Using this result, prove the vis-viva equation

$$v^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right)$$

which is often used in rocketry.

(c) Prove Kepler's third law. (Hint: see the area of an ellipse below.)

Solution. (a) Let the closest approach distance be r_1, farthest be r_2. We have $L/m = J = r_1 v_1 = r_2 v_2 = J$, and we also have

$$2E/m = \epsilon = v_1^2 - \frac{2GM}{r_1} = v_2^2 - \frac{2GM}{r_2}.$$

Therefore,

$$\epsilon = J^2/r^2 - 2GM/r$$

is satisfied by r_1 and r_2, so $\epsilon r^2 + 2GMr - J^2$ is satisfied by r_1 and r_2. Thus, by Vieta, we have that $2a = r_1 + r_2 = -2GM/\epsilon = -GMm/E$, or that $E = -GMm/2a$.

(b) This follows from energy conservation, or

$$-GMm/2a = \frac{1}{2} m v^2 - GMm/r.$$

(c) Note that the rate of area being swept per unit time is $L/2m$, so $\frac{L}{2m} T = \pi ab$. Ellipse geometry tells us that

$$b = \sqrt{a^2 - (r_2 - r_1)^2/4} = \sqrt{a^2 - \frac{(r_1 + r_2)^2 - 4 r_1 r_2}{4}} = \sqrt{r_1 r_2},$$

so the Vieta again says that $b = \sqrt{-J^2/\epsilon} = \frac{L/m}{\sqrt{-2E/m}}$. Therefore,

$$\frac{L}{2m}T = \pi a \frac{L/m}{\sqrt{GM/a}},$$

which implies $T^2 \propto a^3$.

Remark: Scaling Symmetry

There's a variant of Kepler's third law for unbound orbits. Suppose a planet is right next to the Sun at time $t = 0$, but has a large initial radial velocity, so that it has zero total energy. Then its distance to the Sun evolves as $r(t) \propto t^{2/3}$, like how $a \propto T^{2/3}$ for bound orbits.

Both of these results come from the scaling symmetry of inverse square force laws: any solution to Newton's second law remains a solution if you multiply all distances by 4 and all times by 8. The widest-reaching application of this idea is to the whole universe itself. If it contains only matter, which started at the origin at time $t = 0$, and it expands under gravity with zero total energy, then its "scale factor" evolves as $a(t) \propto t^{2/3}$. This was a good description of our universe for most of its lifetime, but in the past few billion years the effects of dark energy took over, accelerating the expansion. We'll revisit cosmology in **X3**.

[3] **Problem 7.** [A] A simple derivation of Kepler's first law is given in section 7.4 of Morin, and revolves around solving a differential equation for $1/r(\theta)$. (You can motivate this by noting that the polar form of an ellipse is quite simple, $1/r = (1 + e\cos\theta)/p$, where p is the semilatus rectum and e is the eccentricity.) However, in this problem, we'll consider an alternative approach that uses a subtle conserved quantity, which is also important in more advanced physics.

(a) Show that the Laplace–Runge–Lenz vector

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - GMm^2\hat{\mathbf{r}}$$

is conserved, where the star is at the origin and $\hat{\mathbf{r}}$ is the radial unit vector at the planet's position $\mathbf{r}$. (Hint: use the fact that $\mathbf{L} = mr^2\boldsymbol{\omega}$ to evaluate the time derivative.)

(b) We have $\mathbf{A} \cdot \mathbf{r} = Ar\cos\theta$, where θ is the angle between $\mathbf{A}$ and $\mathbf{r}$. Evaluate $\mathbf{A} \cdot \mathbf{r}$ using the definition of $\mathbf{A}$, and the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, in order to derive an expression for r in terms of θ and constants. Then use this to show that the orbit is a conic section.

(c) As another simple application of the conservation of $\mathbf{A}$, show that the set of velocities during an elliptical orbit traces out a *circle* in velocity space.

The ideas discussed in this problem are almost never required to solve Olympiad problems, but they can dramatically simplify very tough orbital mechanics problems. For two examples, see Physics Cup 2021, problem 2 and Physics Cup 2024, problem 4.

Solution. (a) Since the angular momentum is conserved,

$$\dot{\mathbf{A}} = \mathbf{F} \times \mathbf{L} - GMm^2\frac{d\hat{\mathbf{r}}}{dt} = \frac{GMm}{r^2}(\omega m r^2)(-\hat{\mathbf{r}} \times \hat{\mathbf{z}}) - GMm^2(\omega\hat{\mathbf{z}} \times \hat{\mathbf{r}}) = 0$$

as desired.

(b) We have

$$\mathbf{A} \cdot \mathbf{r} = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) - GMm^2 r = (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{L} - GMm^2 = L^2 - GMm^2 r$$

which tells us that

$$Ar \cos\theta = L^2 - GMm^2 r.$$

But now this can be solved for r to give the trajectory,

$$r = \frac{L^2}{GMm^2 + A\cos\theta}.$$

This is precisely the form of a conic section. Specifically, the general form is

$$r = \frac{p}{1 + e\cos\theta}$$

and we can identify

$$p = \frac{L^2}{GMm^2}, \quad e = \frac{A}{GMm^2}.$$

As a check, note that A indeed vanishes for circular motion, where

$$A = (mv)(mvr) - GMm^2 = mr^2 \left(\frac{mv^2}{r} - \frac{GMm}{r^2} \right) = 0.$$

For an elliptical orbit, $\mathbf{A}$ lies in the plane of the orbit and points along the major axis.

(c) Take the cross product of the vector with $\mathbf{L}$, which is always conserved, for

$$(\mathbf{A} + GMm^2\hat{\mathbf{r}}) \times \mathbf{L} = (\mathbf{p} \times \mathbf{L}) \times \mathbf{L} = -mL^2\mathbf{v}$$

since $\mathbf{p}$ is always perpendicular to $\mathbf{L}$. Now, during an elliptical orbit, the values of $\mathbf{A} + GMm^2\hat{\mathbf{r}}$ trace out a circle because $\mathbf{A}$ is conserved and $\hat{\mathbf{r}}$ has constant magnitude. Since $\mathbf{A}$ and $\hat{\mathbf{r}}$ are perpendicular to $\mathbf{L}$, taking the cross product with $\mathbf{L}$ just scales the circle and rotates it by 90° in the orbit plane, so the set of $\mathbf{v}$ lies on a circle.

Now we'll consider some really slick problems that can be solved with pure geometry.

Idea 5

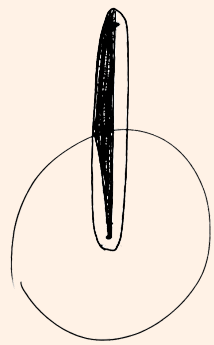
An ellipse is defined by two foci F_1 and F_2 separated by a distance $2d$. It consists of the set of points P so that $PF_1 + PF_2 = 2\ell$ is a constant. The semimajor and semiminor axes a and b of the ellipse are related by $a = \sqrt{b^2 + d^2} = \ell$, and the area is πab.

Example 5

An object is dropped from rest at a distance R above the Earth's surface, where R is the radius of the Earth. How long does it take to hit the Earth's surface?

Solution

The answer doesn't change much if we give the object a tiny horizontal velocity. In this case, the orbit becomes a part of a very thin ellipse, where $a \approx d \approx \ell$, with one focus at the center of the Earth (by the shell theorem) and the other near the starting point.



If the Earth were replaced by a point mass at its center, then the object could perform a full orbit, with total period T. The time until the object actually hits the Earth's surface is determined by the fraction of the orbit's area swept out. Referring to the diagram, this is

$$t = T\,\frac{\pi ab/4 + ab/2}{\pi ab} = T\left(\frac{1}{4} + \frac{1}{2\pi}\right)$$

by summing a quarter of an ellipse and a triangle. All that's left is to solve for T. Note that the semimajor axis is R. Another orbit with the same semimajor axis is simply a circular orbit around the Earth, just above its surface. This orbit has

$$\frac{v^2}{R} = \frac{GM}{R^2}$$

so $v = \sqrt{GM/R}$. Using $T = 2\pi R/v$ gives the answer,

$$t = \left(\frac{\pi}{2} + 1\right)\sqrt{\frac{R^3}{GM}}.$$

Example 6: MPPP 39

An astronaut jumps out of the international space station directly towards the Earth. What happens afterward? In particular, will the astronaut survive?

Solution

If you've seen certain movies, you might get the impression that the astronaut spirals into the Earth, and so will surely die. But that isn't what Kepler's laws say! After the jump, the astronaut simply performs a Keplerian orbit. Since the change in energy is negligible, so is the change in semimajor axis and hence the change in period. The astronaut simply orbits in a nearly circular ellipse, with the same period as the space station.

After one rotation period of the space station, which takes time $T = 92\,\text{min}$, the astronaut arrives back. They are unharmed as long as their oxygen and cooling supply lasts this long. (If you draw some pictures of the orbits, you may think the answer is $T/2$, because the orbits intersect twice. This is incorrect because while the orbits do intersect geometrically halfway through, the space station and the astronaut won't arrive at that point at the same time.)

Example 7: Wang and Ricardo 8.4

A particle moves in a circle of radius R, under the influence of a central force. If its minimum and maximum speeds are v_1 and v_2, what is the period T?

Solution

At first the problem statement might sound confusing, until you realize that the origin need not be at the center of the circle; it must be off-center. Now, it would be intractable to find the trajectory for a general central force law, but we can infer T by thinking about how quickly area is swept out, as in Kepler's second law. This works because conservation of angular momentum holds for all central force laws, not just the inverse square.

At the furthest and closest points, the distances from the origin must be r_1 and r_2, and by conservation of angular momentum, the speeds v_1 and v_2 are achieved at these points, so

$$r_1 v_1 = r_2 v_2, \quad r_1 + r_2 = 2R, \quad \frac{dA}{dt} = \frac{1}{2} r_1 v_1.$$

Using the first two equations, we can solve for r_1 and plug it into the third, for

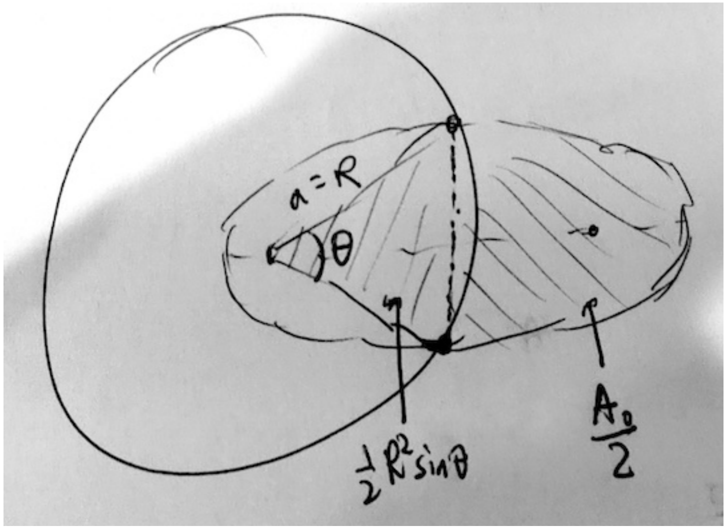
$$r_1 = \frac{2R}{1 + v_1/v_2}, \quad \frac{dA}{dt} = \frac{R}{1/v_1 + 1/v_2}.$$

Since $dA/dt = \pi R^2/T$, we have

$$T = \pi R \left(\frac{1}{v_1} + \frac{1}{v_2} \right).$$

[3] **Problem 8** (PPP 88)**.** A rocket is launched from and returns to a spherical planet of radius R so that its velocity vector on return is parallel to its velocity vector at launch. The angular separation at the center of the planet between the launch and arrival points is θ. How long does the flight take, if the period of a satellite flying around the planet just above its surface is T_0?

Solution. Note that the starting and ending points of the trajectory are the two ends of the minor axis of that trajectory (which is an ellipse by Kepler's first law). Therefore, the distance from focus to an end of the minor axis is R, so the sum of the distances from the two foci to an end of the minor axis is $2R$, so $2a = 2R$, or $a = R$. By Kepler's third law the period depends only on a, so the period of this orbit would also be T_0 if the Earth were a point mass. However, only part of the orbit is actually performed. Hence we need to find the fraction of the ellipse's area that is swept out.



We see that the area swept out is $A = \frac{1}{2}A_0 + \frac{1}{2}a^2 \sin\theta$ where $A_0 = \pi ab = \pi a^2 \sin\frac{\theta}{2}$ is the area of the full ellipse. Thus,

$$T/T_0 = A/A_0 = \boxed{\frac{1}{2} + \frac{\cos\frac{\theta}{2}}{\pi}}.$$

[4] **Problem 9** (Physics Cup 2012). A cannon at the equator fires a cannonball, which hits the North pole. Neglecting air resistance and the Earth's rotation, at what angle to the horizontal should the cannonball be fired to minimize the required speed?

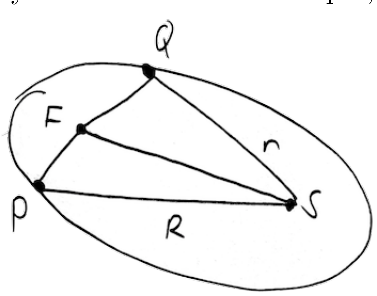
Solution. The answer is 22.5°. See the official solutions here.

[4] **Problem 10** (EFPhO 2015). An asteroid is initially stationary, a distance R from a star of mass M. The asteroid suddenly explodes into many pieces, with speed ranging from zero to v_0. What is the set of all points that can be hit by a piece of the asteroid? (Hint: this problem requires more geometry than the rest. For simplicity, you can begin by treating the problem as two-dimensional, but the solution you find will work just as well for three.)

Solution. Consider a given piece of the asteroid, with speed v_0. Its trajectory is an ellipse of major axis $2a$, where a satisfies

$$\frac{1}{2}mv_0^2 - \frac{GMm}{R} = \frac{GMm}{2a}.$$

Let S be the position of the sun, and let P be the original point of the asteroid. Let F be the other focus of this elliptical trajectory. By the definition of an ellipse, $SP + FP = 2a$, so $FP = 2a - R$.



Let Q be a point that is reached by the piece, and suppose $SQ = r$. By the triangle inequality,

$$PQ \leq PF + FQ = 2a - R + 2a - r = 4a - r - R.$$

Therefore, we see that

$$QP + QS \leq 4a - R.$$

This constraint applies to all points Q that can be hit. Thus, the points that can be hit lie within an ellipse with foci at the sun and the asteroid, with major axis $4a - R$.

Is it possible to hit *every* point in this ellipse? It's intuitive that it's sufficient to show that we can hit every point on the boundary, since that's the hardest thing to do; points inside can be reached by launching with reduced speed. Let Q be a given point on the boundary. Then the inequalities above become equalities as long as $PQ = PF + FQ$, which occurs when F is on PQ. So the question is reduced to whether we can put the other focus of the orbit at any angle we want, relative to P. If you play around with a few drawings of orbits, you can see this is always possible by varying the launch angle, no matter what the launch speed is, so we can get the full ellipse.

In three dimensions, the answer is the set of points enclosed by rotating the ellipse about the axis PS. The resulting shape is called a spheroid.

Idea 6: Reduced Mass

Consider two objects of mass m_1 and m_2 with positions $\mathbf{r}_1$ and $\mathbf{r}_2$ with relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, interacting by a central potential $V(r)$. For the purposes of computing $\mathbf{r}$ alone, we may replace this system with a single mass μ in the same central potential $V(r)$, where μ is the reduced mass, obeying

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

Both systems have the same solutions for $\mathbf{r}(t)$.

Example 8

Consider two planets of mass m. If one planet is somehow fixed in place, the other can perform a circular orbit of radius R with period T. If both planets are allowed to move, they can simultaneously perform circular orbits of radius $R/2$ about their center of mass. What is the period of this motion?

Solution

First let's try an explicit solution. In the first case,

$$\frac{mv^2}{R} = \frac{Gm^2}{R^2}, \quad v = \sqrt{\frac{Gm}{R}}.$$

In the second case, we have

$$\frac{mv^2}{R/2} = \frac{Gm^2}{R^2}, \quad v = \frac{1}{\sqrt{2}}\sqrt{\frac{Gm}{R}}.$$

The velocity in this case is a factor of $1/\sqrt{2}$ smaller, but the arc length of the orbit is a factor of 2 smaller, so the period is $T/\sqrt{2}$.

We can also handle the problem with reduced mass. Consider the relative position $\mathbf{r}_1 - \mathbf{r}_2$ in the second case, which orbits in a circle of radius R. Applying the above idea, we can work in the reduced system. In this system, there is a single mass $\mu = (1/m + 1/m)^{-1} = m/2$ in a circular orbit of radius R, experiencing the same force Gm^2/R^2, so

$$\frac{\mu v^2}{R} = \frac{Gm^2}{R^2}, \quad v = \sqrt{\frac{2Gm}{R}}.$$

The speed is $\sqrt{2}$ bigger than in the first case, but the arc length of the orbit is the same, so the period is $T/\sqrt{2}$.

Reduced mass is a bit unintuitive, since you need to work in two very different pictures. On the other hand, some people like it because it's mathematically concrete, and can reduce problems to one-liners. Whether you use it is up to you.

[2] **Problem 11** (MPPP 27)**.** Two permanent magnets are aligned on a horizontal frictionless table, separated by a distance d. The magnets are held in such a way so that the net force between them is attractive, and there are no torques generated.

If one of the magnets is held and the other is released, the two collide after time t_1. If instead the roles are reversed, the two collide after time t_2. If instead both magnets are released from rest, how long does it take for them to collide? (For a simpler related problem, see $F = ma$ 2019 B4.)

Solution. Let $U(r)$ be the potential energy of the two magnets as a function of r, the distance between the two edges of the magnets (r starts off as d, ends at 0). Let the magnet on the right have mass m_1, and let the one on the left have mass m_2.

Consider the case where m_2 is fixed. Say the time it takes to collide is t_1, and let $v(r)$ be the speed of m_1. By conservation of energy, we have that

$$\frac{1}{2} m_1 v(r)^2 = U(d) - U(r) \equiv V(r),$$

so $v(r) = \sqrt{\frac{2V(r)}{m_1}}$. Then, $v = -dr/dt$, so

$$t_1 = \int_0^d dr/v(r) = \int_0^d \sqrt{\frac{m_1}{2V(r)}}\, dr \equiv \sqrt{m_1}\tau.$$

Similarly, if $t_2 = \sqrt{m_2}\tau$ is the time it takes if m_1 is fixed.

Now consider the case where they are both free, and let the velocity of m_i be v_i. Then,

$$m_1 v_1 = m_2 v^2$$
$$m_1 v_1^2 + m_2 v_2^2 = 2V(r).$$

Solving this, we get

$$v_1(r) = \sqrt{\frac{2m_2 V(r)}{m_1(m_1 + m_2)}}$$
$$v_2(r) = \sqrt{\frac{2m_1 V(r)}{m_2(m_1 + m_2)}}.$$

Now, $-dr/dt = v_1(r) + v_2(r)$, so the time T we are interested in is

$$\begin{aligned}
T &= \int_0^d \frac{dr}{v_1(r) + v_2(r)} \\
&= \int_0^d \frac{dr}{\sqrt{\frac{2V(r)}{m_1+m_2}}\left(\sqrt{\frac{m_2}{m_1}} + \sqrt{\frac{m_1}{m_2}}\right)} \\
&= \int_0^d \sqrt{\frac{m_1 m_2}{(m_1+m_2)\cdot 2V(r)}}\, dr \\
&= \sqrt{\frac{m_1 m_2}{m_1+m_2}}\, \tau \\
&= \boxed{\frac{t_1 t_2}{\sqrt{t_1^2 + t_2^2}}}.
\end{aligned}$$

[3] **Problem 12.** USAPhO 2012, problem A4.

4 Rocket Science

So far you've done some challenging problems, but they haven't exactly been rocket science. These questions literally *are* rocket science. Since almost all of these kinds of questions rely on the same few essential ideas, I've grouped them into a few longer questions. The 4 and 5 point problems below are representative of older IPhO problems in length.

[2] **Problem 13.** A rocket begins at rest in empty space. The engine is turned on and exerts a constant thrust, so $P = Fv$ increases over time. After a long time, the power of the engine can become arbitrarily high, even though it's doing the same thing at all times. This is called the Oberth effect, and has real practical consequences; all else equal, a rocket should be preferentially used when the velocity is high, since it will produce extra power. Where does the extra power come from?

Solution. When fuel in a stationary rocket is burned, it is ejected out the back of the rocket with a sizable kinetic energy. On the other hand, once the rocket starts moving, it can get more power out of the same fuel for two reasons. First, the fuel already has kinetic energy, because it's moving with the rocket. Second, once the fuel is ejected, it has less kinetic energy than in the stationary case. The extra energy in both cases goes into the rocket.

We saw a similar problem in **M3** with a car viewed from a different reference frame, in which case the source of the extra energy was the Earth itself. In general, the "extra" energy comes from whatever the vehicle pushes on to move itself forward.

Note that above, we said that the Oberth effect means the rocket should be fired when its velocity is high. But what is that velocity with respect to? After all, for any rocket, you can find some frame where it's moving fast, and some frame where it isn't moving at all. The answer is that the "correct" frame depends on what you want to do. For example, if you want to escape the solar system, you need to achieve escape velocity in the Sun's frame, because the Sun's gravity dominates. (For some additional details, see here.)

[4] **Problem 14.** A rocket with a full fuel tank has a mass M and is initially stationary. The fuel is ejected at a rate σ, where σ has units of kg/s, at a relative velocity of u.

(a) If the rocket begins in space, show that the velocity of the rocket when its total mass is M' is

$$v = u \log \frac{M}{M'}.$$

This is the Tsiolkovsky rocket equation.

(b) Repeat part (a) for a rocket in a uniform gravitational field g. Do you get the best final velocity if σ is high or low? (Ignore gravity for the rest of this problem.)

(c) In a multi-stage rocket, an empty fuel tank detaches from the rocket once it is used up, after which a second engine starts up. Explain why this can achieve a much higher final velocity than just firing both engines at once. (If you want a quantitative treatment of this, you can see INPhO 2016, problem 3.)

(d) It is desired for a rocket to begin at zero speed and accelerate to speed v, to deliver a given payload. If the exhaust comes out with a relative velocity of u, how should u be chosen to minimize the fuel energy that must be spent to perform this maneuver? (Hint: let the *final* mass of the rocket be fixed, since that's the mass of the payload we want to transport. You will have to solve an equation numerically.)

(e) If u has this value, what fraction η of the spent fuel's energy ends up in the rocket's final kinetic energy?

(f) Now suppose u can be freely varied over time. Qualitatively, how should it be chosen to maximize η, and what is the maximum possible value of η?

Solution. (a) Let $p = mv$ be the momentum of the rocket and all the fuel instantaneously inside it. As some fuel of mass dm is ejected from the rocket, the total momentum is conserved, so

$$dp = (v - u)\, dm.$$

On the other hand, we also have

$$dp = m\, dv + v\, dm.$$

Combining these equations gives

$$m\, dv = -u\, dm$$

so integrating gives

$$\log m = -\frac{v}{u} + C.$$

Fixing C with the initial condition gives the desired result.

(b) The reasoning is similar except that there is now an additional term representing the change in momentum due to the gravitational force. We have $dp + (-dm)(v - u) = -mg\, dt$, so $m\, dv = -u\, dm - mg\, dt$. Therefore,

$$\frac{dm}{m} = -\frac{dv}{u} - \frac{g}{u}\, dt,$$

so $\log m = -v/u - gt/u + \log(M)$. Solving for v gives

$$\boxed{v = u \log \frac{M}{M'} - \frac{g}{\sigma}(M - M')}.$$

It's better if σ is high, since you are constantly losing momentum to gravity.

(c) The idea is that $\frac{M-M_0}{M'-M_0} > \frac{M}{M'}$ where M_0 is the mass of the ejected tank, so the change in speed is higher. Basically, the empty fuel tank is now dead weight, so ejecting it means you don't waste energy speeding it up.

(d) Let the initial and final masses be M and M'. In order for the rocket to reach a velocity of v, $v = u \log \frac{M}{M'}$, or $M = M' e^{v/u}$.

Now, the energy released by burning a small mass dm of fuel is precisely $(dm)u^2/2$. One way to see this is to work in the frame instantaneously moving with the rocket; then the only final energy is in the kinetic energy $(dm)u^2/2$ of the ejected fuel itself, since the rocket picks up negligible speed. This energy must have come from the internal energy of the burning of the fuel, and this quantity is the same in all frames, as we've discussed in **M3**.

Therefore, the total fuel energy burnt is, in any frame,

$$E = \frac{1}{2}(M - M')u^2 = \frac{1}{2}M'(e^{v/u} - 1)u^2$$

This is minimized when $dE/du = 0$ (treating M' as fixed), which gives

$$2u(e^{v/u} - 1) = ve^{v/u}.$$

Letting $x = v/u$, we need to find the solution to

$$x = 2(1 - e^{-x}).$$

This can be done by plugging in $2(1 - e^{-\text{Ans}})$ repeatedly in our calculator to get $x = 1.5936$. Thus the value of u should satisfy

$$u \approx \frac{v}{1.5936}$$

(e) At the end, the rocket will have a kinetic energy of $\frac{1}{2}M'v^2$ and the total fuel burnt will be $\frac{1}{2}M'(e^x - 1)v^2/x^2$, and divide the former by the latter to get

$$\eta = \frac{x^2}{(e^x - 1)} = 0.6476$$

(f) We should always set u equal to the velocity of the rocket at that moment. Then when the fuel comes out, it's at a dead stop, so all of the kinetic energy burned goes into the rocket. Thus the maximum value of η is 100%. This is called a "perfect rocket", though it's not the kind of thing one would want to use in practice. It's not trivial to change u arbitrarily, from an engineering point of view, and a perfect rocket at low speeds would have low power.

[3] **Problem 15.** USAPhO 2015, problem B1. A basic, two-step rocket maneuver.

[5] **Problem 16.** The classic cosmic speeds. For each part, evaluate your answers numerically, using

$$M_{\text{Earth}} = 5.97 \times 10^{24}\,\text{kg}, \quad M_{\text{Sun}} = 1.99 \times 10^{30}\,\text{kg}, \quad R_{\text{Earth}} = 6.37 \times 10^{6}\,\text{m}, \quad d_{\text{Sun}} = 1.50 \times 10^{11}\,\text{m}.$$

Neglect the rotation of the Earth about its own axis for all parts except for part (b).

(a) What is the minimum launch speed required to put a satellite into orbit around the Earth? This is the first cosmic speed. (It's useful to think in terms of speeds because the Tsiolkovsky rocket equation tells us that directly determines the amount of fuel needed. Multistage rocket maneuvers are often described in terms of their "total Δv".)

(b) If you account for the rotation of the Earth, what is the new minimum speed and how should the satellite be launched?

(c) What is the minimum launch speed required for a rocket to escape the gravitational field of the Earth? This is the second cosmic speed.

(d) What is the minimum launch speed required for a rocket to leave the solar system? This is the third cosmic speed. How should the satellite be launched? (Hint: doing this exactly is very hard; instead use the approximation $R_{\text{Earth}} \ll d_{\text{Sun}}$. To check, the answer is 16.7 km/s.)

(e) What is the minimum launch speed required for a rocket to hit the Sun? Assume you cannot make any adjustments to the rocket's path after launch. (To check, the answer is 31.8 km/s.)

(f) If subsequent adjustments are allowed, the minimum launch speed to hit the Sun can be dramatically reduced. Find the minimum launch speed required to hit the Sun if an infinitesimal adjustment later is allowed.

Solution. The answer will be a function of

$$v_0 = \sqrt{\frac{GM_{\text{Earth}}}{R_{\text{Earth}}}} = 7.9\,\text{km/s}, \quad u_0 = \sqrt{\frac{GM_{\text{Sun}}}{d_{\text{Sun}}}} = 29.8\,\text{km/s}.$$

Note that u_0 is the speed the Earth orbits the Sun.

(a) By Newton's second law, $mv^2/R = GMm/R^2$, so the answer is simply $v_0 = 7.9\,\text{km/s}$.

(b) Let v_r be the speed of rotation from the earth. To launch from the poles, we need to launch with speed v_0, but from the equator, we need to launch with only $v_0 - v_r$, giving 7.4 km/s.

(c) The total energy must be 0, so $-GMm/R + \frac{1}{2}mv^2 = 0$, or $v = \sqrt{2}v_0 = 11.2\,\text{km/s}$.

(d) We work in two stages: first the rocket leaves the field of the Earth, then it leaves the field of the Sun. This is valid since $R_{\text{Earth}} \ll d_{\text{Sun}}$. In fact, this is necessary: we cannot do the problem in a single step using energy conservation, because we would necessarily have to work in a frame where either the Earth or Sun has a significant velocity. Then there may be large changes in the kinetic energy of the Earth or Sun, which can be extremely subtle to deal with. (Recall the problem we had with the accelerating car in **M3**!)

Once the rocket has left the field of the Earth, its velocity relative to the Sun must be $\sqrt{2}u_0$. Since the Earth already has velocity u_0, the minimum relative velocity to the Earth is $(\sqrt{2}-1)u_0$. Now work in the frame of the Earth for the first stage. If the launch velocity is v, then energy conservation gives

$$\frac{1}{2}(v^2 - ((\sqrt{2}-1)u_0)^2) = \frac{GM_{\text{Earth}}}{R_{\text{Earth}}} = v_0^2.$$

Solving for v, we get

$$v = \sqrt{2v_0^2 + (3 - 2\sqrt{2})u_0^2} = 16.7\,\text{km/s}$$

which gives the advertised numeric answer.

If you found this part quite tricky, don't worry: there have been whole papers written about it, and many textbooks that got it wrong, including Halliday and Resnick!

(e) In this case, after leaving the Earth we need zero velocity, so velocity u_0 relative to the Earth. By similar reasoning, we get

$$v = \sqrt{2v_0^2 + u_0^2} = 31.8\,\text{km/s}.$$

(f) The best option is actually to do the procedure of part (d), in order to leave the solar system. After the rocket is a very large distance away, it can perform a very small boost to cancel out its angular momentum and fall into the Sun. This gives an answer of 16.7 km/s. (This solution is simply the first two thirds of an Edelbaum maneuver, as described in the remark below.)

Remark

There's a whole science of multi-stage rocket maneuvers. For example, suppose your goal is to quickly escape the solar system. As you found in part (d) of problem 16, the minimum launch speed necessary is the third cosmic speed. However, you can also start by doing the maneuver of part (e). Once the rocket is very close to the Sun, it'll be moving extremely quickly, which means that a second impulse can provide a huge amount of energy. This is called the Oberth maneuver, as it uses the Oberth effect. Doing it this way costs more fuel, in terms of total Δv, but can allow the rocket to leave much faster.

In practice, you can only get within some distance r_{min} of the Sun without the rocket burning up, so there's a limit to how much you can employ the Oberth effect. Thus, in some cases a three-impulse maneuver, called the Edelbaum maneuver, can be even better. In the Edelbaum maneuver, you begin with a forward impulse to get to a higher elliptical orbit, then perform a backward impulse to drop to r_{min}. This gives a higher speed at r_{min}, since the rocket is on an elliptical orbit with higher total energy. Then a final forward impulse can be used to escape the solar system. You can read more about these maneuvers here. However, neither the Oberth or Edelbaum maneuvers have ever been used, because the Δv requirement is too high for them to be feasible. For an authoritative reference on rocket maneuvers, see *An Introduction to the Mathematics and Methods of Astrodynamics* by Battin.

[4] **Problem 17** (MPPP 36). Consider a solar system with two planets, in circular orbits with radii R_1 and $R_2 = xR_1$, where $x > 1$. A space probe is planned to be launched from the first planet, which we will call the Earth, and use a gravitational slingshot from the second planet to exit the solar system. The goal is to do this with the smallest fuel energy expenditure possible.

(a) The space probe is launched so that, after it has exited the gravitational field of the Earth, but before it has moved very far, it has speed v_0 in the Sun's frame. Furthermore, its velocity is parallel to the Earth's velocity in the Sun's frame. Explain why this direction of launch minimizes the energy needed.

(b) Assume the space probe arrives near the second planet, with radial and tangential speeds v_r and v_t with respect to the Sun. Find v_r and v_t.

(c) Suppose the planet have speed v_p. In terms of v_p, v_r, and v_t, what is the largest possible speed v_f of the space probe (relative to the Sun) after the gravitational slingshot ends?

(d) Find the value of x that minimizes the required initial launch speed v_0, for the probe to be able to escape the solar system. (Hint: to save space, consider nondimensionalizing variables. Unless you are very good at algebra, you will have to optimize something numerically.)

(e) Which real solar system planet is closest to this ideal planet?

Solution. (a) We can achieve any velocity relative to the Earth with the same energy expenditure (ignoring the small effect of the Earth's rotation). But what matters for escaping the solar system is the velocity relative to the Sun. This is biggest if the velocity relative to the Earth and the Earth's velocity relative to the Sun are parallel, so that the speeds add.

(b) By angular momentum conservation,

$$v_t = \frac{v_0}{x}.$$

By energy conservation,

$$\frac{1}{2}mv_0^2 - \frac{GMm}{R} = \frac{1}{2}m(v_r^2 + v_t^2) - \frac{GMm}{xR}.$$

This can be solved straightforwardly. Introducing the Earth's speed $v_E = \sqrt{GM/R}$,

$$v_r = \sqrt{v_0^2\left(1 - \frac{1}{x^2}\right) - 2v_E^2\left(1 - \frac{1}{x}\right)}.$$

(c) A gravitational slingshot is simply an elastic collision, so as we saw in **M3**, the best frame to use is the center of mass frame, which in this case is effectively the planet's frame. In this frame the speed of the probe is

$$v_{\text{rel}} = \sqrt{(v_t - v_p)^2 + v_r^2}.$$

As shown in **M3**, the most general thing that can happen is that the velocity of the probe (in this frame) is rotated.

The final speed of the space probe, relative to the Sun, is a vector of length v_{rel} plus the velocity of the planet v_p. So the highest possible speed is achieved when these are parallel,

$$v_f = v_p + \sqrt{(v_t - v_p)^2 + v_r^2}.$$

(d) Escape velocity is achieved when $v_f = \sqrt{2}v_p$. Plugging this in gives

$$(\sqrt{2} - 1)v_p = \sqrt{(v_t - v_p)^2 + v_r^2}.$$

Squaring both sides, we have

$$(2 - 2\sqrt{2})v_p^2 = v_r^2 + v_t^2 - 2v_tv_p.$$

Plugging in the results of part (b),

$$(2 - 2\sqrt{2})v_p^2 = \frac{v_0^2}{x^2} + v_0^2\left(1 - \frac{1}{x^2}\right) - 2v_E^2\left(1 - \frac{1}{x}\right) - \frac{2}{x}v_0v_p.$$

After a little simplification, and using $v_p = v_E/\sqrt{x}$, this becomes

$$\frac{v_E^2}{x}(2 - 2\sqrt{2}) = v_0^2 - 2v_E^2\left(1 - \frac{1}{x}\right) - \frac{2v_0 v_E}{x^{3/2}}.$$

Let's work with the dimensionless variable $u = v_0/v_E$, which obeys

$$u^2 - \frac{2u}{x^{3/2}} + \frac{2\sqrt{2}}{x} - 2 = 0.$$

This is a quadratic in u. Applying the quadratic formula and taking the physical sign gives

$$u = \frac{1}{x^{3/2}} + \sqrt{\frac{1}{x^3} - \frac{2\sqrt{2}}{x} + 2}.$$

This is the function we want to minimize with respect to x. Taking the derivative and setting it to zero is possible, though extremely painful; this yields

$$x = \frac{9 + \sqrt{81 - 24\sqrt{8}}}{8} \approx 1.58.$$

Alternatively, one can simply perform binary search on a calculator, giving the same result.

(e) This is the closest to Mars, which has $x = 1.52$.

Remark

Above we discussed the Oberth and Edelbaum maneuvers, which use two and three impulses, respectively. In general, if you only deal with the gravity of the Sun, optimal maneuvers never require more than three impulses, so they can't get *too* complicated. But in reality, it would be impractical to exit the solar system or reach the Sun without also using gravitational slingshots. The Voyager probes used multiple slingshots off the gas giants to do the former, while the Parker Solar Probe did an incredible seven gravitational slingshots off Venus in a row to do the latter! These kinds of trajectories need to be calculated years in advance.

Still, you might be thinking, is this really the hardest stuff in the world, when it just boils down to Newtonian mechanics? Well, as Lee DuBridge, the president of Caltech once said:

> I [like] to talk about space to nonscientific audiences. In the first place, they can't check up on whether what you are saying is right or not. And in the second place, they can't make head or tail out of what you are telling them anyway—-so they just gasp with surprise and wonderment, and give you a big hand for being smart enough to say such incomprehensible things. And I never let on that all you have to do to work the whole thing out is to set the centrifugal force equal to the gravitational force and solve for the velocity. That's all there is to it!

I'm just being glib here – the moon landing is unquestionably one of the greatest engineering feats in history. The physical laws at play are elementary, but their application is subtle, and the engineering required getting thousands of tricky real-world details right.

Example 9

An object quickly flies past a star of mass M, with nearly constant speed v, so that its distance of closest approach is R. Estimate the angle by which the object is deflected.

Solution

To solve this exactly, we would write down Newton's second law in polar coordinates and solve a differential equation. However, we can get a rough estimate a simpler way. Since the object is flying quickly, its path is approximately a straight line. Most of the transverse impulse it experiences occurs when it is at a distance of order R from the star, and we can approximate this as

$$\Delta p_\perp = \int F_\perp \, dt \sim F_\perp \, \Delta t \sim \frac{GMm}{R^2} \frac{R}{v}.$$

The small angle of deflection is

$$\Delta\theta \approx \frac{\Delta p_\perp}{mv} \sim \frac{GM}{Rv^2}.$$

This is a decent approximation for the true answer, which turns out to be $2GM/Rv^2$.

In Newtonian gravity, we can think of light as consisting of massless particles moving at speed c, so we can find the deflection of light by setting $v = c$. However, in general relativity the bending of light is actually twice as large, $\Delta\theta = 4GM/Rc^2$. The observation of this factor of 2 by Eddington during a solar eclipse was one of the first tests of general relativity, but it's pretty tricky; Einstein himself missed it in his original paper of the subject!

Roughly speaking, the source of the factor of 2 is that general relativity is a theory of spacetime curvature. The "temporal" curvature corresponds, in the Newtonian limit, to motion under a potential. But the star also sources "spatial" curvature. As explained here, it corresponds to an "angular defect", so that moving tangentially by $2\pi r$ doesn't bring you back where you started. We could neglect this effect in problem 5 because we were considering nonrelativistic particles, which move much faster through time than space. But for light, the two effects are comparable and contribute equally to the deflection.

Remark

Another famous prediction of general relativity is the perihelion precession of Mercury. As you will show in **R3**, the orbit of Mercury advances by a tiny angle $\Delta\theta$ on each cycle. However, knowing that general relativity is a relativistic theory of gravity, we can estimate this angle by dimensional analysis. The only dimensionful parameters are the strength of gravity GM, the radius R of Mercury's orbit, and the speed of light c. (Other parameters we might care about can be expressed in terms of these; for instance, the speed of Mercury is $v = \sqrt{GM/R}$.) By similar logic to the above problem, the only possible expression is

$$\Delta\theta \sim \frac{GM}{Rc^2} \sim 10^{-8}$$

which is incredibly tiny. The true answer is larger by a factor of $6\pi/(1-e^2)$, where $e \approx 0.2$ is the eccentricity.

5 Fictitious Forces

Idea 7

Consider an inertial frame and a rotating frame with angular velocity $\boldsymbol{\omega}$. For any vector $\mathbf{V}$, the time derivatives of $\mathbf{V}$ in these two frames are related by

$$\left(\frac{d\mathbf{V}}{dt}\right)_{\text{in}} = \left(\frac{d\mathbf{V}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{V}.$$

For example, when $\mathbf{V}$ is the position $\mathbf{r}$, we have the familiar result

$$\mathbf{v}_{\text{in}} = \mathbf{v}_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{r}.$$

Applying this equation to the velocity $\mathbf{v}$, we find

$$\mathbf{a}_{\text{in}} = \mathbf{a}_{\text{rot}} + 2\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

The two terms on the right correspond to the Coriolis and centrifugal forces,

$$\mathbf{F}_{\text{rot}} = \mathbf{F} - 2m\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

In the case where $\boldsymbol{\omega}$ can change, we also have the azimuthal force $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$. (If you prefer, these forces can also be derived by working in components in polar coordinates, as shown in chapter 11 of Wang and Ricardo, volume 1.)

Idea 8

Sometimes, the best way to deal with fictitious forces is to just avoid them by using an inertial frame instead. This is especially true when the Coriolis force is not small; it's straightforward to treat it approximately if it's small, but otherwise it's quite complicated. If a problem presents a situation in a rotating frame, there's no reason you have to stay in that frame! (One example of this is $F = ma$ 2018, problem B12.)

Example 10

Angular momentum conservation tells us that an ice skater increases their angular velocity as they pull their arms inward. Derive this result by working in the frame that always rotates with the skater, as the skater pulls their arms in radially. Specifically, model the skater as two point masses m a distance r from the axis. Show that balancing the Coriolis and azimuthal forces yields a result equivalent to using angular momentum conservation in an inertial frame.

Solution

Let ω be the (time-dependent) angular velocity of the skater's frame. Balancing the forces on one arm,

$$2m\omega\dot{r} = -m\dot{\omega}r$$

which is equivalent, by the product rule, to the statement that ωr^2 is constant. Then $mr^2\omega$ is constant, which is exactly the angular momentum in an inertial frame.

Example 11

A projectile is dropped from height h at the equator. Let the Earth have angular velocity ω, and let the local gravitational acceleration be g. Counting only the Coriolis force, which direction is it deflected when it hits the ground, and by about how far? Is the correction due to the centrifugal force significant?

Solution

The earth rotates from west to east, so the angular velocity points from the south pole to the north pole. The velocity of the falling ball points radially inward, so the Coriolis force points east. We naturally assume the height h is much less than the radius of the Earth, so the inward gravitational acceleration is constant. The Coriolis acceleration is thus

$$a_c = 2\omega v = 2\omega g t$$

in the eastward direction, and integrating this twice gives a deflection

$$d(t) = \frac{1}{3}\omega g t^3.$$

The projectile hits the ground at $t = \sqrt{2h/g}$, giving a final eastward deflection of

$$d = \frac{\omega}{3}\sqrt{\frac{(2h)^3}{g}}.$$

This is the right answer to first order in ω. For a neat, geometric method that arrives at the same result, see the solutions to NBPhO 2016, problem 9.

Remark

What if you wanted to refine the answer above? It turns out this is quite subtle, because a slew of other effects appear at higher order in ω and h, including the centrifugal force (which affects both the mass and the shape of the Earth), and the variation in g with height. If you want to explore this in detail, see problems 10.12 and 10.13 of Morin.

Example 12

The Eotvos effect is the fact that the apparent weight of an object on Earth depends on its motion. How large is this effect at latitude ϕ, and what directions of motion have an effect?

Solution

The Eotvos effect is due to the Coriolis force. As we mentioned in the previous example, the angular velocity of the Earth points out of the north pole. For concreteness, let's suppose we're in the northern hemisphere, $\phi > 0$. Then a object moving east with speed v will yield an outward Coriolis force $2m\omega v$, making the apparent weight lighter, while a westward velocity will make the apparent weight heavier. (Moving north or south, or up and down, just deflects the object east or west.) Eotvos first measured this effect in the lab in 1915, by rotating a balance. It must be accounted for by surveys of g, which are used to find oil deposits.

Example 13

Explain where the factor of 2 in the Coriolis force comes from, working in an inertial frame.

Solution

For concreteness, consider a rotating cylindrical space station of radius R with angular velocity ω. An astronaut initially stands on its rim, then jumps upward, picking up an inward radial velocity u in the space station's rotating frame. The Coriolis force implies that the astronaut will have tangential acceleration $2\omega u$.

In an inertial frame, this 2 comes from the combination of two effects of equal magnitude. Let $v = \omega R$ be the initial tangential velocity of the astronaut in this frame. As the astronaut moves radially inward, angular momentum conservation implies that their tangential velocity increases, so that after a time dt it is

$$\omega R \, \frac{R}{R - u\, dt} = \omega R + \omega u\, dt.$$

At the same time, the tangential speed of the rotating frame decreases, to

$$\omega(R - u\, dt) = \omega R - \omega u\, dt.$$

The relative tangential acceleration is thus $2\omega u$, giving the desired result.

[2] **Problem 18.** A cylindrical space station of radius R can create artificial gravity by rotating with angular velocity ω about its axis.

(a) For an observer rotating along with the spaceship on the rim, what gravitational acceleration g do they perceive?

(b) The observer throws a ball parallel to the floor. For some launch speed v, the observer will see the ball perform a circular orbit along the spaceship, always parallel to the floor. Find v.

(c) What does the motion of part (b) look like, in a frame that isn't rotating with the ship?

Solution. (a) The centrifugal acceleration is $\omega^2 R$, and this is perceived as gravity, $g = \omega^2 R$.

(b) The Coriolis force points upward, so the acceleration upward is

$$a = 2\omega v - \omega^2 R.$$

For the ball to perform a circular orbit, it needs to have a centripetal acceleration of $a = \omega^2 R$. Combining these and solving gives

$$v = \omega R.$$

(c) This is an example of a case where working in an inertial frame is easiest. In an inertial frame, the ball just hovers in place (since there isn't any gravity), while the space station's floor rotates with speed v right under it.

[2] **Problem 19.** Every satellite in orbit around the Earth is slowly falling due to drag. Consider a satellite steadily falling, with a large tangential velocity and small inward radial velocity.

(a) Show that for a satellite initially in a circular orbit, losing energy U to drag *increases* the kinetic energy of the satellite. By how much is it increased?

(b) The result of part (a) seems almost paradoxical. How can it be explained in an inertial frame, given that the drag force always acts to slow down the satellite?

(c) Now consider a uniformly rotating frame, whose angular velocity is equal to the initial angular velocity of the satellite. In this frame, the drag force always points tangentially backwards, but the satellite ends up going tangentially forward. What force is responsible?

Solution. (a) This follows from the virial theorem, namely that the time average of the kinetic energy is negative of the time average of the total energy. So losing U total energy means gaining U kinetic energy. This recalcitrant behavior, where the mass seems to want to accelerate in the direction opposite the way it's pushed, is called the "donkey effect" in galactic dynamics.

(b) Gravity always points radially, but since the satellite's velocity has an inward radial component, that means gravity has a component along the velocity, and hence increases the speed. If you go through the calculation, which is a slightly more complex version of an example in **M5**, you'll find that the speed-increasing effect of gravity is precisely twice the speed-decreasing effect of the drag force.

(c) The inward component of the velocity gives rise to a Coriolis force pointing tangentially forward. Again, if you go through the calculation, you'll find it's twice as large as the drag force, effectively flipping its direction. The explanation looks totally different in the rotating frame, but the result is the same.

[2] **Problem 20.** A frictionless tube of length R is rotated with angular velocity ω about one of its ends. A package is placed in the tube at a distance r_0 from the axis of rotation, with no initial radial velocity. What the package flies out the other end of the tube, what is its speed?

Solution. In the frame rotating with the tube, there is only a centrifugal force, which produces a radial acceleration $a = \omega^2 r$. The solution to this equation is

$$r(t) = Ae^{\omega t} + Be^{-\omega t}$$

and fitting to the initial conditions $r(0) = r_0$ and $v(0) = 0$ gives

$$r(t) = r_0 \cosh(\omega t), \quad v(t) = \omega r_0 \sinh(\omega t).$$

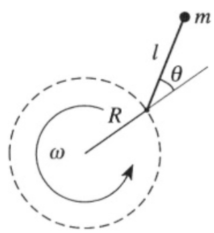
When the package flies out the end, we have $R = r_0 \cosh(\omega t_0)$, which implies

$$v(t_0) = \omega\sqrt{R^2 - r_0^2}.$$

Of course, we have to go back to the lab frame, where there's a tangential speed ωR, giving

$$v = \omega\sqrt{2R^2 - r_0^2}.$$

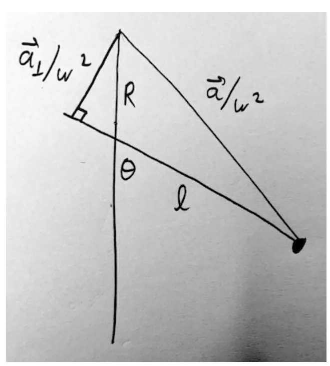
[2] **Problem 21** (Cahn)**.** A pendulum is designed for use on a gravity-free spacecraft. The pendulum consists of a mass at the end of a rod of length ℓ. The pivot at the other end of the rod is forced to move in a circle of radius R with angular frequency ω. Let θ be the angle the rod makes with the radial direction.



Show this system behaves exactly like a pendulum of length ℓ in a uniform gravitational field $g = \omega^2 R$. That is, show that $\theta(t)$ is a solution for one system if and only if it is for the other.

Solution. This system experiences no gravitational force, but instead experiences a Coriolis and centrifugal force. The Coriolis force plays no role, because it is always perpendicular to the velocity of the mass and the angular velocity, which implies it is directed along the rigid rod; it merely changes the tension in the rod.

The centrifugal acceleration $\mathbf{a}$ is directed away from the origin; the relevant part of it is the component $\mathbf{a}_\perp$ perpendicular to the rod. Referring to the below diagram, we see that $a_\perp = \omega^2(R\sin\theta)$.



This is exactly the same $a_\perp$ as for a pendulum in gravity $g = \omega^2 R$, so the systems are equivalent.

[4] **Problem 22.** IPhO 2016, problem 1B. A useful set of Coriolis force exercises.

[3] **Problem 23.** USAPhO 2020, problem A2. A tricky question on the Foucault pendulum. For an algebraic derivation of the final result, see section 9.9 of Taylor; it uses the complex number method introduced for a problem in **M1**. For a beautiful but more abstract geometric derivation, see section 11.5.1 of Griffiths' *Introduction to Quantum Mechanics* (3rd edition).

As a warning, this problem and its solution are a lot rougher than in a typical USAPhO, making the question more confusing than it should be. That's because the 2020 USAPhO was cancelled for the pandemic, and AAPT released a rough draft to give people something to practice on. So if you have unexpected trouble on this problem, or find the solution puzzling, don't worry about it. I only include the 2020 problems in these problem sets because they illustrate some new ideas.

[4] **Problem 24** (Morin 10.26)**.** A coin stands upright on a turntable rotating with angular frequency ω, and rolls without slipping so that its center is motionless in the lab frame. Thus, in the frame of the turntable, the coin rolls without slipping in a large circle with angular frequency ω.

(a) In the lab frame, explain how $\mathbf{F} = d\mathbf{p}/dt$ and $\boldsymbol{\tau} = d\mathbf{L}/dt$ are satisfied. (This is the easy part.)

(b) In the frame of the turntable, verify that $\mathbf{F} = d\mathbf{p}/dt$.

(c) In the frame of the turntable, verify that $\boldsymbol{\tau} = d\mathbf{L}/dt$. (This is the hard part.)

If you slogged through part (c), you'll understand why we rarely want to think about torques for extended bodies in rotating frames.

Solution. Let the coin have radius r and mass m, and let its orbit have radius R.

(a) The normal force cancels the gravitational force, while the friction force evidently vanishes. Hence all forces and torques cancel.

(b) The centrifugal force is $m\omega^2 R$ and directed outward, while the Coriolis force due to the orbital motion is $2m\omega v = 2m\omega^2 R$ and directed inward. Hence there is an inward force of $m\omega^2 R$, as required.

(c) A torque is required to turn around the coin's spin angular momentum. This is provided by the Coriolis force associated with the spin motion, which provides zero net force but does provide a net torque. Let the coin rotate with spin angular velocity ω_s and consider the noninertial frame following the center of mass.

Putting the origin at the center of the coin, and assuming the coin spans the xy plane,

$$\mathbf{v}_{\text{spin}} = \omega_s(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}).$$

The spin Coriolis acceleration at that point is

$$\mathbf{a} = 2\boldsymbol{\omega} \times \mathbf{v}_{\text{spin}} = 2\omega\omega_s y\hat{\mathbf{z}}, \quad \boldsymbol{\omega} = \omega\hat{\mathbf{y}}.$$

The differential torque is

$$d\boldsymbol{\tau} = \mathbf{r} \times \mathbf{a}\, dm = (x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) \times (2\omega\omega_s y\hat{\mathbf{z}})dm.$$

The term proportional to xy will integrate to zero, so we might as well ignore it,

$$d\boldsymbol{\tau} = 2\omega\omega_s y^2\hat{\mathbf{x}}\, dm.$$

Upon integrating over the coin, we have

$$\boldsymbol{\tau} = 2\omega\omega_s \int y^2\, dm.$$

This can be simplified using a trick. By rotational symmetry, we have

$$\boldsymbol{\tau} = \omega\omega_s\hat{\mathbf{x}} \int x^2 + y^2\, dm = I_{\text{cm}}\omega\omega_s\hat{\mathbf{x}} = L_{\text{spin}}\omega_s\hat{\mathbf{x}}$$

just as required. This Coriolis torque makes it a pain to think about rotating extended bodies in rotating frames.

Idea 9

An object freely falling in a gravitational field will experience tidal forces, due to the spatial variation of gravitational fields. Specifically, suppose a mass M is placed at $(R, 0)$. Then the gravitational acceleration near the origin is

$$\mathbf{g} \approx \frac{GM}{R^2}\hat{\mathbf{x}} + \mathbf{g}_{\text{tidal}}, \quad \mathbf{g}_{\text{tidal}} = \frac{GM}{R^3}\left(2x\hat{\mathbf{x}} - y\hat{\mathbf{y}}\right).$$

In particular, if a small rigid object with center of mass at the origin is released, it will have acceleration $(GM/R^2)\hat{\mathbf{x}}$. In the frame of the center of mass, the rest of the object will experience the residual gravitational acceleration $\mathbf{g}_{\text{tidal}}$. Intuitively $\mathbf{g}_{\text{tidal}}$ tends to stretch the object in the longitudinal direction and squeeze it in the transverse direction.

Example 14

Estimate the height of the tides on Earth induced by the Moon.

Solution

We will simplify the problem by treating the Earth and Moon as stationary, and the Earth as a ball of fluid. The result above tells us that the tidal acceleration due to the Moon is

$$\mathbf{g} = \frac{GM_M}{D^3}(2x\hat{\mathbf{x}} - y\hat{\mathbf{y}})$$

where D is the distance to the Moon.

Suppose that without the Moon's presence, the Earth is a sphere of radius R_E. By integrating this, the tidal gravitational potential induced on the Earth's surface is

$$\Delta\Phi = -\frac{GM_M R_E^2}{D^3}\frac{3\cos^2\theta - 1}{2}.$$

Changing the gravitational potential on the surface by $\Delta\Phi$ means the equilibrium height of the water shifts by $\Delta h = \Delta\Phi/g$, where g is the gravitational acceleration of the Earth. Thus,

$$\Delta h \sim \frac{GM_M R_E^2/D^3}{GM_E/R_E^2} = \frac{M_M}{M_E}\left(\frac{R_E}{D}\right)^3 R_E.$$

This can also be written in terms of the densities,

$$\Delta h \sim \frac{\rho_M}{\rho_E}\left(\frac{R_M}{D}\right)^3 R_E.$$

The first term is of order one. The second term is the angular radius of the Moon in the sky, which is about a quarter of a degree, giving

$$\Delta h \sim \left(\frac{2\pi}{360 \times 4}\right)^3 R_E \sim 0.5\,\mathrm{m}$$

which is of the right order of magnitude.

Remark

The tidal effects of the Sun and Moon are comparable, but the Moon's are somewhat larger. This can be seen by the final equation above: the Sun and Moon have comparable angular diameter, since eclipses can just barely happen, but the Moon is 2.4 times denser.

When the Moon and Sun's tidal effects reinforce, one has a larger tidal effect, called a spring tide; otherwise, one has a neap tide. Given what's been said so far, you should be able to figure out what phase(s) of the Moon and time(s) of day correspond to each (at least for an idealized "spherical cow" Earth), and also roughly how much stronger tides are at spring tides than at neap tides.

It's worth noting that the *actual* tides are far more complicated than just computing the tidal force, because they depend on the dynamical response of the water, which in turn depends on the detailed lay of the land. For some more details, see here.

One should be careful when applying tidal forces, because many astronomical objects are also spinning or orbiting, which provides an additional centrifugal force. For a good discussion of tides, see section 10.3 of Morin.

[3] **Problem 25** (Morin 10.31)**.** A small spherical rock covered with sand falls in radially toward a planet. Let the planet have radius R and density ρ_p and let the rock have density ρ_r. When the rock gets close enough to the planet, the tidal force will pull the sand off the rock. The cutoff distance is called the Roche limit; it gives the radial distance below which loose objects can't coalesce into larger ones.

(a) Show that the Roche limit is

$$d = R\left(\frac{2\rho_p}{\rho_r}\right)^{1/3}.$$

(b) Now suppose the rock is orbiting the planet and rotating so that the same side always faces the planet. Show that the Roche limit is

$$d = R\left(\frac{3\rho_p}{\rho_r}\right)^{1/3}.$$

Solution. Let the rock have radius r, and consider the point on the rock nearest to the planet.

(a) The gravitational acceleration at this point due to the rock is

$$g_{\text{rock}} = \frac{4\pi}{3}Gr\rho_r.$$

The tidal acceleration (relative to the center of the form) is

$$g_{\text{tidal}} = \frac{8\pi}{3}Gr\rho_p\left(\frac{R}{d}\right)^3$$

where we set $x \to r$ and $R \to d$ in the formula given in the idea above. When these two are equal, the normal force on the point vanishes, so the sand can lift off; solving gives the desired result.

(b) The rock as a whole accelerates due to the planet's gravity with

$$g_{\text{planet}} = \frac{4\pi}{3} G R^3 \rho_p \frac{1}{d^2} = \omega^2 d$$

where ω is the orbital angular velocity of the rock about the planet. Since the planet is tidally locked, it also has a spin angular velocity of ω, so the point has an acceleration of $\omega^2 r$, giving

$$a_{\text{centripetal}} = \omega^2 r = \frac{4\pi}{3} G r \rho_p \left(\frac{R}{d}\right)^3.$$

The sand at this point lifts off when $g_{\text{rock}} - g_{\text{tidal}} = a_{\text{centripetal}}$, as the normal force vanishes in this case. Solving this gives the desired result.

[4] **Problem 26.** IPhO 2009, problem 1. A neat problem about how the Moon has slowed down the Earth's rotation. This is a fascinating subject; in the deep past, not only were days shorter, but the weather on Earth was completely different, owing to the much higher Coriolis forces.

[5] **Problem 27.** IPhO 1992, problem 1. A problem on a strange propulsion mechanism, which is in some sense the reverse of the previous problem. Fill in your answers on the attached answer sheet, along with your solutions.

Remark: Discovering Gravity

In elementary school, we hear that Newton understood gravity in a flash of inspiration, after being hit on the head with an apple. Later, you might learn that it didn't quite work that way: there was an apple tree in Newton's childhood home, but an apple didn't hit him, and Newton didn't publish his ideas on gravity until decades afterward.

However, the story is an oversimplification in a much more significant way. Newton's law of gravity actually contains many independent insights. For example, you need to realize that gravitational forces occur between *pairs* of objects, rather than emanating from an object, or reflecting an object's desire to move towards its "natural" place of being. To explain the orbits, you need to understand that the force is radial, not tangential, and moreover that it is *not* balanced by any other radial force. You need to see that the force acts between *all* pairs of objects, and not just certain pairs of objects with the right qualities, like iron and magnets, that the force is proportional to mass and falls off with distance, and that it occurs "at a distance" with nothing in between.

All of these insights, which we think of as obvious today, were viewed as unintuitive or downright occult by intelligent thinkers of the time. For example, you probably think the astrological idea that Jupiter governs blood and Venus governs phlegm is laughable, as did many 17$^{\text{th}}$ century astronomers, but would the idea that the Moon governs the rise and fall of water on Earth sound any more plausible, if you hadn't been told early on that it's true by people you trust? (If you flip this logic around, you can understand why so many people believe in astrology.) Or, going further back to antiquity, if you claimed then that everything is affected by gravity, how could you explain why flames go up? (Before you embark on an explanation of buoyancy, you would first have to explain how air exerts a massive yet somehow unobservable pressure on everything, why air has mass but doesn't fall, and that buoyant forces for air exist at all. The helium balloon wouldn't be around for thousands of years, and without modern tools, it's surprisingly tricky to make a stable vacuum.)

Between Galileo and Newton, there were many incremental steps towards the development of universal gravitation. For instance, Borelli proposed that Jupiter's moons obeyed Kepler's laws, and Horrocks found that Jupiter and Saturn slightly deviated from Kepler's laws because of their mutual attraction. Newton played an important role by putting everything on a solid foundation, such as by deriving Kepler's first law and the shell theorems. But as you can see from Newton's notebooks, these insights came from years of experience tinkering with concrete calculations. Indeed, think about the massive distance between the law of gravity and Aristotle's idea that things fall because they seek their "natural" place. To get from Aristotle's "rocks want to go home" theory to Newtonian mechanics requires not just genius, but many geniuses. And of course, there were just as many steps needed to get from noticing static electricity existed to writing down Coulomb's law, including a bewildering array of homemade experiments with medieval technology. Nothing is trivial in physics.

Remark: Negative Mass

One intriguing difference between electrostatics and gravity is that charge can be positive or negative, while mass is apparently only positive. Could there be a "negative mass", to complete the analogy, which falls upward in a gravitational field? If general relativity is true, the answer is no: it respects the equivalence principle, which implies that particles all accelerate the same way in a gravitational field. (Every month or so, people will write clickbait articles saying that negative mass has been discovered, but most of those examples are like balloons. They don't go up in a gravitational field because they actually have negative mass, they go up because they have less mass than the air they're displacing.)

However, there's an alternative formulation of negative mass that does respect the equivalence principle: let $\mathbf{F} = m\mathbf{g}$ and $\mathbf{F} = m\mathbf{a}$ with the same negative m in both equations. Then the force on the mass points upward, but it accelerates downward anyway.

This leads to some seriously strange consequences. For example, suppose we had a positive and a negative mass interacting. Then the two would repel each other, which implies that both will start zooming off in the *same* direction! This is completely compatible with momentum and energy conservation, since the negative mass has flipped momentum and negative energy, but it's quite strange. In a universe with an equal amount of both kinds of masses, they'll both eventually end up zooming around at the speed of light, but the energy will be zero. Anyway, at the moment we don't know of any ordinary matter with these weird properties.

Remark: Lagrangian Points

Earnshaw's theorem states that a point charge in an electrostatic field cannot be in stable equilibrium, i.e. it is impossible for the charge to be in equilibrium and feel a restoring force when displaced in an arbitrary direction. The proof is simply Gauss's law: for some candidate equilibrium point P, draw a small Gaussian surface around it. Since there is no charge inside (the test charge itself doesn't count, since we're only caring about the fields of *other* charges, which act on it), the flux must be zero. But a stable equilibrium requires the field to have an inward-pointing component in all directions, which implies nonzero flux.

Earnshaw's theorem also applies to gravity; by the same logic, in space there are no gravitational potential extrema. However, you may have heard that there are stable Lagrangian points in the Earth-Sun system, where satellites can be stored. How is this possible, given that stability usually requires a potential minimum?

The first subtlety is that Lagrangian points rotate with the Earth about the Sun, so we really should be working in a rotating frame. In this frame, there is an additional centrifugal force. However, this doesn't help, because you can check that the divergence of the centrifugal force is *positive* everywhere, i.e. it causes instability rather than stability.

As usual for subtleties involving fictitious forces, the resolution comes from the Coriolis force. Near the stable Lagrangian points, the Coriolis force deflects satellites sideways. The stability of the points is not because there is an ordinary restoring force, but because of this continuous sideways deflection, which keeps particles from moving far away.

How can this can be analyzed mathematically? For a typical restoring force, we would expand the force linearly in $\Delta\mathbf{r}$ and determine stability using the methods of **M4**. Here that doesn't work, because we have a velocity-dependent force, but we can still write the force as linear in the *four* coordinates $(\Delta\mathbf{r}, \mathbf{v})$. This gives a set of four linear differential equations, which can be solved in terms of normal modes as in **M4**. The motion is stable if no modes are exponentially growing. For a more complete analysis, see here.

Mechanics VII: Fluids

The fundamental material is covered in chapters 15 and 16 of Halliday, Resnick, and Krane, and at a somewhat higher level in chapter 9 of Wang and Ricardo, volume 1. For a neat explanation of lift and the Coanda effect, see this video. For interesting discussion, see chapters II-40 and II-41 of the Feynman lectures. For a much more advanced introduction which uses vector calculus heavily, see chapters 2–5 and 12–15 of *Physics of Continuous Matter* by Lautrup. There is a total of **86** points.

1 Fluid Statics

Idea 1

In equilibrium, the pressure in a static fluid varies with height as

$$\frac{dp}{dy} = -\rho g.$$

This always holds in equilibrium. For instance, if we squeeze a sealed container of fluid, increasing the pressure locally, then this pressure increase must propagate throughout the entire fluid to maintain $dp/dy = -\rho g$. This is Pascal's principle.

Idea 2: Archimedes' Principle

An object in a fluid experiences an upward buoyant force due to the different pressures on its top and bottom sides. The force is equal in magnitude to the weight of the fluid that would fill the volume of the immersed portion of the object.

This can be surprisingly tricky, so we'll begin with some conceptual questions.

Example 1

A large rock is tied to a balloon filled with air. Both are placed in a lake. As the balloon sinks, how do the air pressure in the balloon, the average density of the balloon, air, and rock system, and magnitude of the net force on the system vary?

Solution

For simplicity, we ignore the elastic force in the balloon itself. Then for the balloon to be in equilibrium, its pressure must match that of the water pressure, so the air pressure in the balloon increases. As the balloon sinks, the rock stays the same volume but the balloon is squeezed smaller, so the density of the system increases. Finally, since the density of water is very approximately constant, the buoyant force on the system is decreasing since its volume is decreasing, so the net force is increasing; the system accelerates downward faster and faster.

[1] **Problem 1** (HRK). The average human body floats in water. SCUBA divers wear weights and a flotation vest that can fill with a varying amount of air to establish neutral buoyancy. A diver is originally neutrally buoyant at a certain depth. How should the diver manipulate the amount of air in their flotation vest to move to a lower depth, then stay there at neutral buoyancy?

[2] **Problem 2.** A beaker contains liquid water at its freezing point and has a big ice cube floating in it, also at its freezing point. If the ice cube

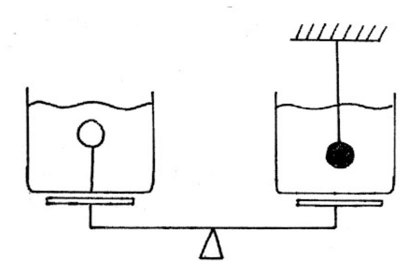
(a) is solid ice,

(b) contains a small metal ball, or

(c) contains a lot of olive oil (which will float on the water in a thin layer),

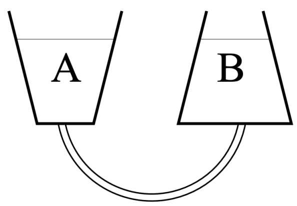
then how does the fluid level change when the cube melts? In all cases, neglect the density of air.

[2] **Problem 3** (Povey, Moscow 1939)**.** Consider a pair of scales with identical vessels in which there are equal quantities of water.



In the left-hand vessel you suspend a very light ping-pong ball on a thin, light wire attached to the base of the vessel. In the right-hand vessel you suspend a ping-pong ball filled with lead, again by a light thin wire. Do the scales stay level, go down on the left, or go down on the right?

[2] **Problem 4** (BAUPC)**.** Two trapezoidal containers, connected by a tube as shown, hold water.

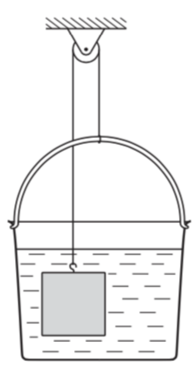


Assume that the containers do not undergo thermal expansion.

(a) If the water in container A is heated, causing it to expand, will water flow through the tube? If so, in which direction?

(b) What if the water in container B is heated instead?

[2] **Problem 5** (MPPP 85)**.** A solid cube of volume V_i and density ρ_i is fastened to one end of a cord, the other end of which is attached to a light bucket containing water, of density $\rho_w = \rho_i/10$.



The system is in equilibrium.

(a) Find the volume V_w of the water in the bucket.

(b) What would happen if more water were poured into the bucket?

(c) What would happen if some or all of the water evaporated?

Example 2

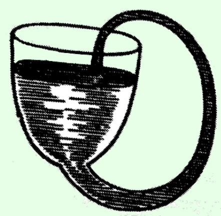
A perfectly spherical, nonrotating planet is covered with water. Geological activity causes a small underwater mountain to form, made of rock that is denser than water. Does the ocean surface above this mountain become higher or lower?

Solution

Systems minimize their energy in equilibrium. This means that in hydrostatic equilibrium, the surface of the water is an equipotential. Since the gravitational field of the mountain increases the gravitational potential near it, the water surface is higher near the mountain.

Example 3

Robert Boyle is best known for Boyle's law, but he also invented a remarkably simple perpetual motion machine, called the *perpetual vase*.



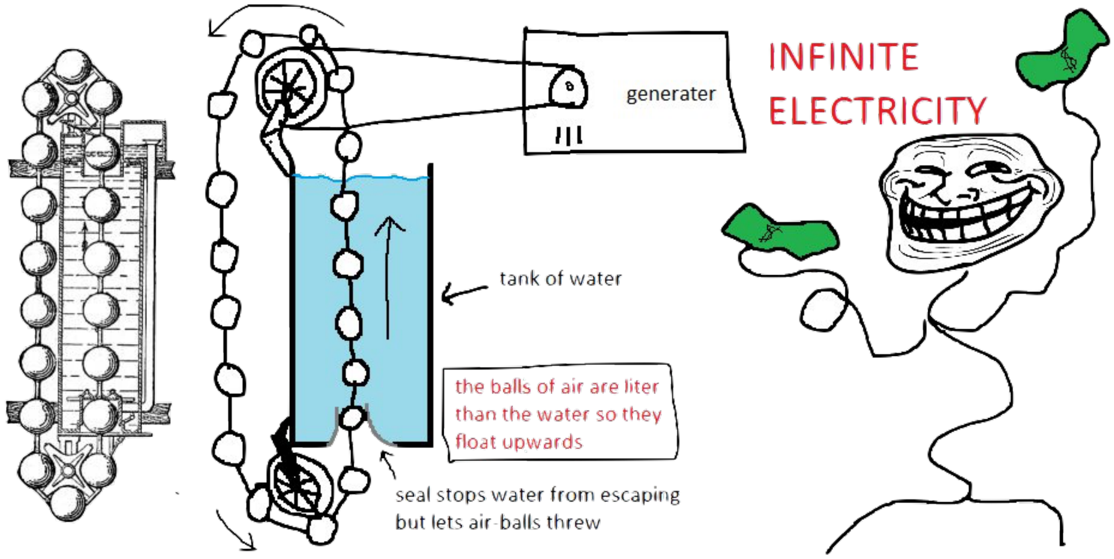
Since the volume of the vase is much greater than the neck, the pressure in the neck cannot possibly hold up all of the water in the vase. Hence the water will flow through the neck and fall back into the vase, causing perpetual motion. Why doesn't this work?

Solution

This is an example of the hydrostatic paradox. Most of the upward force on the water is *not* provided by the pressure in the water in the neck, but from the normal force from the walls; each piece of wall provides enough normal force to hold up all of the water above it. (Of course, ultimately each piece of the glass is held in place by internal forces with other pieces of the glass, which ultimately are balanced by whatever is holding the glass.)

Thus, the water in the neck only supports the water directly above it. That's precisely what is balanced by the heightened pressure in the neck, so the water doesn't start moving. (There have been many more attempts at fluid-based perpetual motion, as you can see here.)

[2] **Problem 6.** Below is another perpetual motion machine, in both original and modern form.



The balls are less dense than water. The balls on the left are pulled downward by gravity, while the balls on the right are pushed upward by the buoyant force.

(a) Why doesn't this work?

(b) Would it work if the balls and chain were replaced with a flexible tube of constant thickness?

[2] **Problem 7** (HRK). A fluid is rotating at constant angular velocity ω about the vertical axis of a cylindrical container. Defining $z = 0$ to be the water level at the cylinder's axis, show that the liquid surface is the paraboloid

$$z = \frac{\omega^2 r^2}{2g}.$$

Since a paraboloid perfectly focuses incoming light which is parallel to its axis, a rotating fluid can be used as a telescope, as was first pointed out by Isaac Newton. Such liquid-mirror telescopes are cheap, but have the disadvantage that they can only point up. Alternatively, one can gradually cool molten glass in a rotating container so that it solidifies into a paraboloidal lens.

[3] **Problem 8.** USAPhO 2013, problem A4. In order to make measurements, print out the problem before starting.

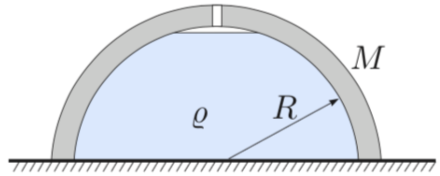
2 Fluid Mechanics

Next we'll consider some situations involving fluids and other objects, where the fluids can be treated at least quasistatically but the objects must be treated dynamically.

Idea 3

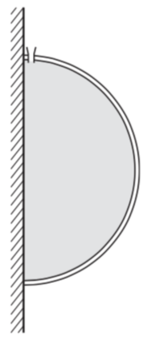
The buoyant force can be regarded as acting at the center of gravity of the fluid displaced by the submerged part of a floating object, called the center of buoyancy. A floating configuration is stable if, when the configuration is slightly rotated, the buoyant force provides a restoring torque about the center of mass.

[2] **Problem 9** (Kalda). A hemispherical container is placed upside-down on a smooth horizontal surface. Water is poured in through a small hole at the bottom of the container. Exactly when the container fills, water starts leaking from between the table and the edge of the container.



Find the mass of the container if the water has density ρ and the hemisphere has radius R.

[2] **Problem 10** (MPPP 89). A thin-walled hemispherical shell of mass m and radius R is pressed against a smooth vertical wall.



It is filled with water through a small aperture at its top, with total mass M. Find the minimum magnitude of the force that has to be applied to the shell to keep the liquid in place.

[3] **Problem 11.** 🕓 USAPhO 2002, problem A4.

[3] **Problem 12.** 🕓 USAPhO 2004, problem A2.

[3] **Problem 13.** A log with a square cross section and very low density will float stably with one of its sides parallel to the water.

(a) If the density of the log is increased, show that when

$$\rho_{\text{log}} = \frac{3-\sqrt{3}}{6}\rho_{\text{water}}$$

the orientation becomes unstable. (Hint: to keep the calculations short, choose a good coordinate system and work to the lowest relevant order everywhere.)

(b) How do you think the orientation of the log varies as ρ_{log} is varied? In particular, what's the orientation when $\rho_{\text{log}}/\rho_{\text{water}} = 1/2$? How about when $\rho_{\text{log}} \approx \rho_{\text{water}}$?

Finding the stable orientation of the log for general values of ρ_{log} is quite complicated, but you can play with a nice simulation here; you can also use this to check your answer.

Remark

Some Olympiad questions involving oscillating fluids, which are more subtle. These questions are often impossible to solve exactly, because one must keep track of the entire motion of the water to know how much kinetic and potential energy are in play. In **M4**, you solved IPhO 1984, problem 2, which only asked for an order of magnitude estimate. Physics Cup 2018, problem 4 considers a V-shaped container, where the calculation can be done exactly.

[4] **Problem 14.** EuPhO 2022, problem 1. A nice fluid oscillations problem which can be solved nearly exactly without too much trouble.

Idea 4: Virtual Mass

When an object moves through water, it effectively has extra inertia because it forces water to move as well. This is the "virtual mass" effect (also called added mass, or hydrodynamic mass) which we first mentioned in **M4**. It can be computed exactly in a few special cases:

$$\Delta m = \begin{cases} \rho V/2 & \text{sphere} \\ \rho V & \text{long cylinder moving perpendicular to axis} \end{cases}$$

where ρ is the water density and V is the volume of the object. You don't have to memorize these results, but the idea of virtual mass does occasionally show up. For instance, IPhO 1995, problem 3 involves oscillations of a cylindrical buoy of mass m which is only partially submerged in water; they ask you to simply assume a virtual mass $m/3$. Furthermore, Physics Cup 2019, problem 1 introduces a slick method, based on vector calculus analogies, which can be used to compute the virtual mass exactly for a few more special shapes.

Example 4

Derive the expression for the virtual mass of a sphere.

Solution

Consider a spherical object of radius a moving uniformly with speed v_0 through water of density ρ. The object forces the water to move: the water ahead of it has to get out of the way, while the water behind it needs to fill the space it leaves behind. By the ideas of **M4**, the total kinetic energy of the water is $(\Delta m)v_0^2/2$, where Δm is the virtual mass.

It turns out the fluid's velocity field $\mathbf{v}(\mathbf{r})$ has to satisfy $\nabla \cdot \mathbf{v} = 0$, reflecting the incompressibility of water, and $\nabla \times \mathbf{v} = 0$, reflecting the absence of vorticity. It also has to go to zero far from the sphere, and have zero relative normal velocity at the sphere itself. These differential equations and boundary conditions yield a unique solution. The methods for finding the solution are standard, and typically taught in an undergraduate electromagnetism course, but since they're outside the Olympiad syllabus, I'll just display the answer. The velocity is

$$\mathbf{v}(\mathbf{r}) = \frac{v_0 a^3}{2r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

in polar coordinates, where we placed the origin at the center of the sphere and aligned the $\hat{\mathbf{z}}$ axis with its direction of motion. If you've done **E1**, you might notice this is just like the electric dipole field; this coincidence isn't *too* surprising because that field satisfies the similar equations $\nabla \cdot \mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$, which are quite restrictive.

Now, to derive the virtual mass, we just have to carry out the kinetic energy integral, which is easiest in spherical coordinates,

$$\begin{aligned} K &= \int \frac{\rho v^2}{2} \, dV \\ &= \frac{\rho v_0^2 a^6}{8} \int_a^\infty \frac{r^2 \, dr}{r^6} \int_0^{2\pi} d\phi \int_0^\pi (\sin\theta \, d\theta)(4\cos^2\theta + \sin^2\theta) \\ &= \frac{\rho v_0^2 a^6}{8} \left(\frac{1}{3a^3} \right) (2\pi)(4). \end{aligned}$$

This yields a virtual mass of $(2\pi/3)\rho a^3 = \rho V/2$, as stated above.

Example 5

What is the initial upward acceleration of a spherical air bubble in water?

Solution

The upward buoyant force on the bubble is $\rho V g$, and the mass of the bubble is negligible, so if we didn't know about virtual mass, we would be tempted to conclude the acceleration is enormous. Instead, the buoyant force is used to move the virtual mass $\rho V/2$ out of the way, so the upward acceleration is $2g$.

Like most things in fluid dynamics, this isn't an exact result. The usual expression for the buoyant force assumes no motion at all, while the virtual mass derivation assumes uniform motion, neither of which are true for an accelerating bubble. For the result above to be accurate, the bubble has to be small, so that the pressure and flow fields have time to reach a quasi-steady state, but not too small, so that we can still ignore viscous forces.

3 Fluid Dynamics

Idea 5: Continuity

In steady flow, the quantity $\rho A v$ is constant along tubes of streamlines.

Idea 6: Bernoulli's Principle

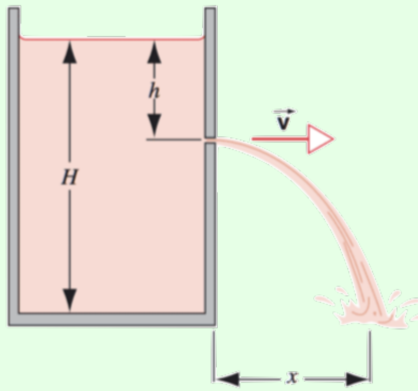
For steady, nonviscous, incompressible flow, the quantity

$$p + \frac{1}{2}\rho v^2 + \rho g y$$

is constant along streamlines. Another version of Bernoulli's principle, valid for compressible flow, is given in **T3**. As explained there, the incompressible result here is applicable for water flow, and for gas flow as long as the velocity is much less than the speed of sound.

Example 6: HRK

A tank is filled with water to a height H. A small hole is punched in one of the walls at a depth h below the water surface as shown.



Find the distance x from the foot of the wall at which the stream strikes the floor.

Solution

The flow isn't perfectly steady, but it's close enough since the hole is small. We thus apply Bernoulli's principle along a streamline, where one point is at the water's top surface, and the other point is just outside the hole. Both points are at atmospheric pressure, because they are directly exposed to the atmosphere. Since the hole is small compared to the tank, the velocity at the first point is small by continuity, so we neglect it, giving

$$\frac{1}{2}\rho v^2 = \rho g h$$

which implies Torricelli's law,

$$v = \sqrt{2gh}.$$

The time t to fall is $t = \sqrt{2(H-h)/g}$, so

$$x = vt = 2\sqrt{h(H-h)}$$

which incidentally is maximized at $h = H/2$.

Incidentally, Bernoulli himself was aware that the answer was different for a large hole, and treated the general case in his 1738 book, *Hydrodynamica*. The method is to apply energy conservation to all of the water at once (i.e. equating the rate of decrease of gravitational potential energy to the rate of increase of total kinetic energy), rather than attempt to apply it along streamlines. You can see this general analysis here.

Example 7

Why should you close your barn door during a storm?

Solution

The wind in a storm can flow into the barn, at which point it stops. By Bernoulli's principle, this increases its pressure by $\rho v^2/2$. On the other hand, the wind that doesn't enter the barn has to flow faster along the top to get around it, which decreases its pressure. These two effects both create a net upward force on the roof, which can tear it off the barn. The latter effect caused my trampoline to achieve liftoff during Hurricane Sandy, destroying a backyard fence.

Incidentally, this example brings up a little puzzle about Bernoulli's principle. We argued that the air slows down when it enters the barn, so the pressure goes up. But in the reference frame moving with the wind, the air speeds up when it enters the barn – so shouldn't its pressure go down? The issue with this reasoning is two-fold. First, in the wind's frame, the barn is moving, so the flow isn't steady and Bernoulli's principle doesn't apply. Second, even if the barn were moving slowly, so that the flow were almost steady, the barn's motion would still be doing work on the air, and this changes Bernoulli's principle because it is ultimately a restatement of energy conservation. So in either case, the reasoning fails. When obstacles are present, Bernoulli's principle should always be invoked in the frame of the obstacles.

Example 8: JEE 2020

When a train enters a narrow tunnel, your ears pop because of the pressure change. Find the pressure change, assuming the air has constant density ρ, the atmospheric pressure is P_0, the train speed is v, and the cross-sectional areas of the train and tunnel are A_t and A_0.

Solution

We work in the reference frame of the train. In this frame, the air in the tunnel begins moving towards the train at speed v. When it gets to the train, it has to speed up to speed v_f because it flows through a smaller area $A_0 - A_t$, and this causes its pressure to decrease by Bernoulli's principle. Specifically, we have

$$A_0 v = (A_0 - A_t) v_f, \quad P_f + \frac{1}{2}\rho v_f^2 = P_0 + \frac{1}{2}\rho v^2$$

which gives a pressure drop of

$$P_f - P_0 = -\frac{1}{2}\rho v^2 \left(\frac{1}{(1 - A_t/A_0)^2} - 1 \right).$$

We neglected the change in density of the air, which is a good approximation when the train is much slower than the speed of sound. We'll treat fluid flow with changing density in **T3**.

Example 9

A whirly tube is a long, narrow, flexible tube that produces musical tones when swung. Model a whirly tube as a cylinder of length L, rotated about one end with angular velocity ω. For simplicity, neglect gravity. What is the speed of the air when it shoots out the other end?

Solution

The air is slowly sucked from all directions around the entry hole, and shot out at the exit hole. Applying Bernoulli's principle between a point near the entry hole, and the exit hole,

$$P_{\text{atm}} \approx P_{\text{atm}} + \frac{1}{2}\rho v_{\text{out}}^2.$$

But that implies $v_{\text{out}} \approx 0$, which doesn't make sense. The problem is that Bernoulli's principle applies to steady flows, and this situation is definitely not steady: by the time the air goes through the tube, the tube has rotated by a significant amount.

Instead, we apply Bernoulli's principle in a reference frame rotating with the tube. The centrifugal force gives an additional term, turning it into

$$P + \frac{1}{2}\rho v^2 - \frac{1}{2}\rho\omega^2 r^2 = \text{const}.$$

Applying Bernoulli's principle between the same two points gives

$$P_{\text{atm}} \approx P_{\text{atm}} + \frac{1}{2}\rho v^2 - \frac{1}{2}\rho\omega^2 L^2$$

from which we conclude $v = \omega L$. Transforming back to the original reference frame, the exit speed of the air is $\sqrt{v^2 + (\omega L)^2} = \sqrt{2}\, v$.

Example 10

A big fan produces a stream of air with speed v. If the atmospheric pressure in the room is P_{atm}, what's the pressure P in the middle of the fan's air stream?

Solution

This question frequently appears in middle school physics lessons. Obviously, if we apply Bernoulli's principle to the air before and after it goes through the fan, we get

$$P + \frac{1}{2}\rho v^2 = P_{\text{atm}}$$

so that the pressure is lower than atmospheric pressure. Easy, right? But it's wrong!

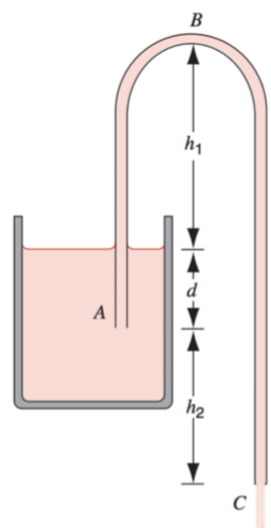
The air in the stream is traveling forward with constant velocity, exposed to the rest of the air in the room, which has atmospheric pressure. If there actually was such a pressure difference, the fan's air stream would be compressed by the air in the room, until it reached atmospheric pressure again. If you look back carefully at the above examples, you'll see this is always the case: air can only be at a different pressure if it's confined away from the atmosphere at large (e.g. in a train tunnel or a whirly tube), or if it's actively being accelerated (e.g. when it flies into or over a barn, in which case the pressure difference is precisely what causes the force). The other case where you can maintain a pressure difference is when the air is moving extremely quickly, which will be discussed in **T3**.

So the correct answer is that $P = P_{\text{atm}}$. But why doesn't Bernoulli's principle work? Because it's a statement of energy conservation, and the fan itself is doing work on the air to get it moving. The correct statement would be

$$P_{\text{atm}} + \frac{1}{2}\rho v^2 = P_{\text{atm}} + w$$

where w is the work done by the fan per unit volume of air.

[2] **Problem 15** (HRK)**.** A siphon is a device for removing liquid from a container that cannot be tipped. An example of a siphon, with constant cross-section, is shown below.



The tube must initially be filled, but once this has been done the liquid will flow until its level drops below the tube opening at A. The liquid has density ρ and negligible viscosity.

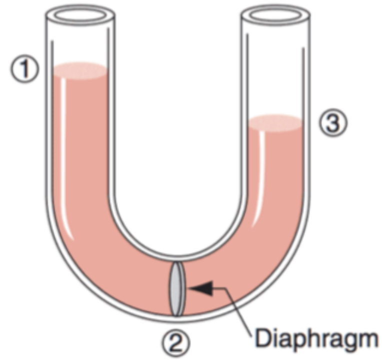
(a) With what speed does the liquid emerge from the tube at C?

(b) What is the pressure of the liquid at the topmost point B?

(c) What is the maximum possible h_1 so that the siphon can operate?

(d) Would the siphon still work if h_2 were slightly negative? How negative can it be, for the siphon to keep on working?

[2] **Problem 16** (HRK)**.** Consider a uniform U-tube with a diaphragm shown below.



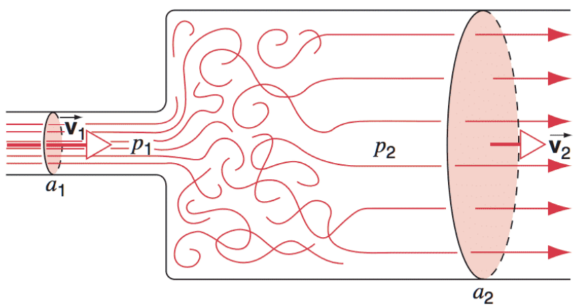
(a) Suppose the diaphragm is opened and the liquid begins to flow from left to right. Show that applying Bernoulli's principle yields a contradiction.

(b) Explain why Bernoulli's principle doesn't apply if the diaphragm has a very wide opening.

(c) Explain why Bernoulli's principle doesn't apply if the diaphragm has a tiny opening.

For a similar idea to this problem, see $F = ma$ 2018 A22.

[2] **Problem 17** (HRK). A stream of fluid of density ρ with speed v_1 passes abruptly from a cylindrical pipe of cross-sectional area a_1 into a wider cylindrical pipe of cross-sectional area a_2 as shown.



The jet will mix with the surrounding fluid, forming a turbulent region where the pressure is approximately p_1. Further to the right, the flow becomes almost uniform again, with average speed v_2 and pressure p_2.

(a) By considering force and momentum, show that

$$p_2 - p_1 = \rho v_2(v_1 - v_2).$$

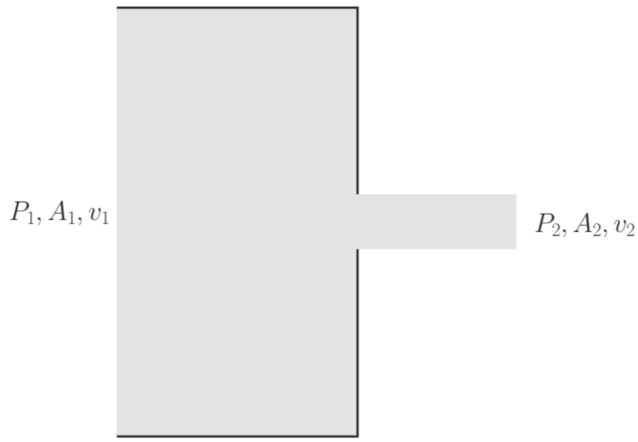
(b) Show from Bernoulli's principle that in a gradually widening pipe we would instead get

$$p_2 - p_1 = \frac{1}{2}\rho(v_1^2 - v_2^2).$$

(c) Find the loss of pressure due to the abrupt enlargement of the pipe. Can you draw an analogy with elastic and inelastic collisions in particle mechanics?

[2] **Problem 18** (PPP 49). After a tap above an empty rectangular basin has been opened, the basin fills with water in a time T_1. After the tap has been closed, opening a plug at the bottom of the basin empties it in time T_2. If both the tap and plug are open, what ratio of T_1/T_2 can cause the basin to overflow?

[4] **Problem 19.** This problem is about the subtle phenomenon of *vena contracta*. An incompressible fluid of density ρ is flowing through a tube of area A_1, which suddenly contracts to area $A_2 \ll A_1$. Naively, the flow looks as shown below.



(a) Argue by energy conservation that

$$v_2^2 \approx 2\frac{P_1 - P_2}{\rho}.$$

(b) Argue that the net force on the fluid shown in the picture is approximately $(P_1 - P_2)A_2$. Then argue by momentum conservation that

$$v_2^2 \approx \frac{P_1 - P_2}{\rho}.$$

(c) The resolution of the paradox is that the "v_2^2" in the first equation is the speed, while the "v_2^2" in the second is really $(v_2)_x^2$. That is, the naive picture above is wrong: the fluid does not exit through the orifice horizontally, but rather contracts as it leaves. Show that if the fluid contracts to a minimum area A_3 with $P_3 \ll P_1$, then $A_3 = A_2/2$, so that momentum conservation is satisfied.

(d) Even assuming ideal fluid flow satisfying Bernoulli's principle, the result above for A_3 is not exact, but is instead off by about 20%. Is the true value of A_3 higher or lower than $A_2/2$?

(e) How could the shape of the orifice be modified so that A_3 is exactly $A_2/2$? How could the orifice be modified to remove the vena contracta effect entirely?

Remark

Vena contracta is too subtle for introductory textbooks, but it makes a big difference in the results. For example, if you estimate how long it takes water in a bucket to empty through a hole using Torricelli's law, then you'll be off by about a factor of 2 if you don't include vena contracta! And Halliday, Resnick, and Krane don't consider the vena contracta in their example titled "thrust on a rocket", getting a thrust which is also off by a factor of 2.

Of course, real rocket scientists are well-aware of these effects. They actually follow the fluid flow in detail, improving on the rough estimates made here, and you can read whole books about rocket nozzle design. We'll revisit this subject in **T3**.

4 Fluid Systems

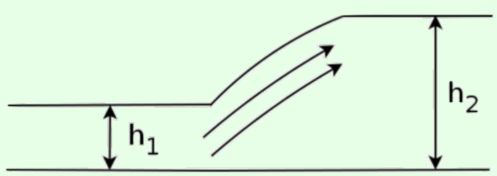
Now we put it all together and consider complex mechanical systems with moving fluids.

Idea 7

If a fluid is moving in a complex way, it's usually difficult to say anything by directly considering the flow. Instead, it's easier to apply conservation laws.

Example 11

A fluid of density ρ flowing with a fast velocity v_1 and height h_1 can undergo a "hydraulic jump", where the height of the fluid increases to h_2. At the same time, the fluid flow slows down and becomes turbulent.



This phenomenon is very common in everyday life. For example, it happens whenever you turn on the water faucet in a sink; the hydraulic jump occurs on a circle centered on the faucet. Find the final height h_2.

Solution

During this process, the bulk kinetic energy of the water is not conserved, because it is converted to turbulent motion. However, the horizontal momentum of the water is approximately conserved. Consider a stream of water of width w flowing in the x direction, where the hydraulic jump occurs at $x = 0$. By mass conservation,

$$v_1 h_1 = v_2 h_2$$

where v_2 is the final speed. Now we consider a fixed subset of the water encompassing the hydraulic jump. The atmospheric pressure does not yield a net horizontal force on the water, so we focus on the pressure in excess of atmospheric pressure. The total excess pressure force on the left end is

$$F_\ell = \int_0^{h_1} \rho g h w \, dh = \frac{1}{2}\rho g w h_1^2.$$

Therefore, we have total force

$$F = \frac{1}{2}\rho g w (h_1^2 - h_2^2).$$

On the other hand, the mass of water that flows through the hydraulic jump per unit time is $\rho h_1 w v_1$, and its velocity decreases by $v_1 - v_2$, so

$$\frac{dp}{dt} = \rho h_1 w v_1 (v_1 - v_2) = \rho w v_1 v_2 (h_2 - h_1)$$

where we used mass conservation. Equating $F = dp/dt$ and simplifying gives

$$g(h_1 + h_2) = 2v_1 v_2.$$

Applying mass conservation again leads to a quadratic in h_2,

$$h_2^2 + h_1 h_2 - \frac{2v_1^2 h_1}{g} = 0$$

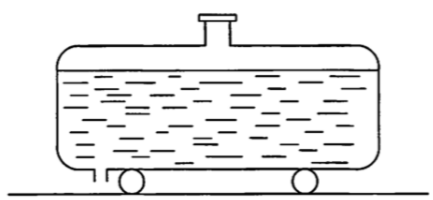
and the physically relevant positive solution is the answer,

$$h_2 = -\frac{h_1}{2} + \sqrt{\frac{h_1^2}{4} + \frac{2h_1 v_1^2}{g}}.$$

For $v_1^2 > gh_1$, we have $h_2 > h_1$ and an ordinary hydraulic jump. For $v_1^2 < gh_1$, you might expect a "reverse" hydraulic jump to occur, but this is impossible by the second law of thermodynamics. In a hydraulic jump, some of the kinetic energy of laminar flow energy is converted to turbulent flow, which is essentially heat; thus the reverse can't happen. So in addition to deriving h_2, we've found the minimum v_1 for a hydraulic jump to be possible!

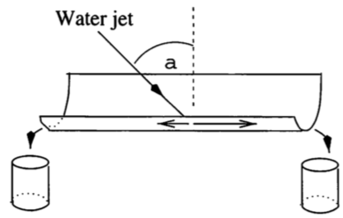
Note that this conservation law approach doesn't tell us about how far a fluid will flow before it undergoes a hydraulic jump. That would require understanding the fluid flow in detail, accounting for turbulence and viscosity, which is generally analytically intractable. For more on this subject, see sections 26.1 and 26.2 of Lautrup.

[3] **Problem 20** (PPP 70). A tanker full of liquid is at rest on a frictionless horizontal road.



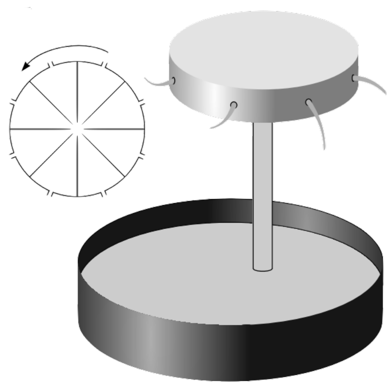
A small vertical outlet pipe at the rear of the tanker is opened. Describe qualitatively how the tanker will move (a) immediately afterward, and (b) after a long time. Assume that the water always falls out of the cart with zero horizontal velocity in the cart's frame.

[3] **Problem 21** (PPP 74). A jet of water strikes a horizontal gutter of semicircular cross-section obliquely, as shown.



The jet lies in the vertical plane that contains the center-line of the gutter. Assume the angle is relatively shallow, so that the water hits the gutter smoothly, and doesn't splatter. Find the ratio of the quantities of water flowing out at the two ends of the gutter as a function of the angle of incidence α of the jet.

[3] **Problem 22** (EFPhO 2005)**.** A water pump consists of a vertical tube of cross-sectional area S_1 topped with a cylindrical rotating tank of radius r. All the vessels are filled with water; there are holes of total cross-sectional area $S_2 \ll S_1$ along the perimeter of the tank, which are open for the operating regime of the pump. The height of the tank from the water surface of the reservoir is h. An electric engine keeps the vessel rotation at angular velocity ω. The water density is ρ, the air pressure p_0, and the saturated vapor pressure p_k. Inside the tank, there are metal blades, which make the water rotate with the tank.



(a) Find the pressure p_2 at the perimeter of the tank when all the holes are closed.

(b) Now suppose the holes are opened. Find the velocity v_2 of the water jets with respect to the ground.

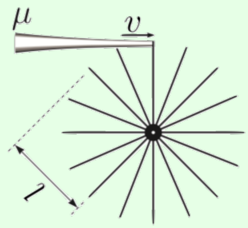
(c) If the tank rotates too fast, the pump efficiency drops due to cavitation; the water starts "boiling" in some parts of the pump. Find the highest cavitation-free angular speed $\omega_{\max}$.

(d) If the power of the electric engine is P, what is the theoretical upper limit of the volume pumped per unit time, assuming S_2 can be freely adjusted?

[3] **Problem 23.** A helicopter with length scale ℓ and density ρ_h can hover using power P, in air of density ρ_a. Find a rough estimate for P in terms of the given parameters. (For a nice followup discussion of lift, see section 3.6 of The Art of Insight.)

Example 12: Kalda 82

A water turbine consists of a large number of paddles that could be considered as light flat boards with length ℓ, that are at one end attached to a rotating axis. The paddles' free ends are positions on the surface of an imaginary cylinder that is coaxial with the turbine's axis. A stream of water with velocity v and flow rate μ (kg/s) is directed on the turbine such that it only hits the edges of the paddles.



Find the maximum possible power that can be extracted.

Solution

Let v_t be the speed of the edge of the turbine. In time dt, the amount of mass of water that collides with the turbine is

$$dm = \frac{\mu}{v}(v - v_t)\, dt.$$

The horizontal force on the paddle is

$$F = \frac{dp}{dt} = \frac{dm}{dt}\Delta v = \frac{\mu}{v}(v - v_t)^2$$

so the power delivered to the turbine is

$$P = F v_t = \frac{\mu v_t}{v}(v - v_t)^2.$$

Maximizing this by setting $dP/dv_t = 0$ gives $v_t = v/3$, so the maximum power is $4\mu v^2/27$. This is 8/27 of the total power in the incoming water.

[3] **Problem 24.** Air of constant density ρ and wind speed v_i is heading directly towards a windmill of area A. When the wind gets to the windmill blades, it is traveling forward with speed v_f. Well after it leaves the vicinity of the blades, it has speed v_o. The design of the windmill, such as the shape and speed at which its blades turn, can be adjusted to set the value of v_f.

(a) Find the power going from the wind to the turbine by using energy conservation, assuming that there are no extraneous energy losses, e.g. to turbulence.

(b) Find the power going from the wind to the turbine by considering the force of the windmill on the air and using momentum conservation, again assuming no extraneous energy losses.

(c) Find an upper bound on the ratio of the wind power that can be harvested by the windmill, to the amount of wind power that would pass through it if it weren't running.

This result is called the Betz limit.

[5] **Problem 25.** GPhO 2017, problem 2. A very tricky composite fluids/mechanics problem.

5 Wet Water

So far we've mostly ignored viscosity and turbulence, an unrealistic limit that some refer to as "dry water". Now we'll consider some problems involving real, wet water.

Idea 8

A velocity gradient is associated with a drag force

$$F = \eta A \frac{dv}{dy}$$

where η is the (dynamic) viscosity. In addition, for fluid flowing next to a wall, the layer of fluid right next to the wall is approximately at rest.

Example 13: HRK

Prairie dogs live in large colonies in complex interconnected burrow systems. They face the problem of maintaining a sufficient air supply to their burrows to avoid suffocation. They avoid this by building conical earth mounds about some of their many burrow openings. How does this air conditioning scheme work?

Solution

Because of viscous effects, the wind speed is small near the ground, and hence grows with height. By Bernoulli's principle, this means the pressure at the top of a mound is slightly lower than the pressure at an opening without a mound. This difference in pressure drives air flow through the burrows.

Example 14

If you've used a standard garden hose, you might have noticed that the water shoots higher if you partially block the outlet with your finger. Why does this happen?

Solution

The water company provides water to your house at a fixed pressure $P_{\text{atm}} + \Delta P$. Thus, naively the water should always shoot equally far, because Bernoulli's principle says the exit speed is $v = \sqrt{2\Delta P/\rho}$, corresponding to a peak height $\Delta P/\rho g$, independent of the area of the hole. (There is a vena contracta effect, as mentioned in problem 19, but this also doesn't depend on the area.)

The resolution is that for a typical long, thin garden hose, viscous losses dominate. As you'll see in problem 26, a higher mass flow rate leads to a higher drop in pressure. When you partially block the outlet, you're simply decreasing the flow rate, so that viscosity has a smaller effect, allowing the water to get closer to the maximum possible height $\Delta P/\rho g$.

In plumbing, the quantity $\Delta P/\rho g$ is called the "pressure head", and effects like viscosity give rise to "head loss". Unfortunately, for most realistic pipes it is intractable to calculate the head loss, because the water flow is turbulent. Instead, the amount of head loss is parametrized by the so-called Darcy friction factor, whose values are tabulated in references.

Example 15

If you stir a cup of coffee, around how long does it take the rotational motion to settle down?

Solution

The rotational motion stops because of viscous drag against the walls. For concreteness, let's suppose the coffee has density ρ, viscosity η, and is in a mug of radius R and height $H \gg R$ (so most of the drag comes from the vertical wall of the mug). The angular momentum is

$$L \sim I\omega \sim \rho R^4 H\omega.$$

The damping torque due to viscous forces is

$$\tau \sim RF \sim \eta A \frac{dv}{dr} R$$

and since the drag is from the vertical wall, $A \sim HR$. Estimating the velocity gradient dv/dr is a little trickier. As mentioned above, the coffee *right* next to the wall has zero velocity, while the coffee slightly inward from the wall has speed $v \sim R\omega$. The velocity transitions between these two values in a thin "boundary layer".

Finding the exact thickness of this boundary layer would require solving complicated differential equations, but it suffices to use dimensional analysis. Note that R and H can't possibly play a role, since the layer is so thin it doesn't "see" the shape of the mug. The fluid properties η and ρ surely matter. Perhaps more subtly, ω matters. If the fluid weren't spinning, but rather were uniformly translating in a plane, then the boundary layer would just grow over time until it was the size of the whole fluid. That's what we saw in problem 26, where the velocity changes gradually along the whole pipe radius R. The boundary layer doesn't grow to the whole mug's size here, because the velocity it's trying to match is constantly changing over the timescale $1/\omega$.

Using dimensional analysis, we thus conclude the boundary layer has thickness

$$\Delta r \sim \sqrt{\frac{\eta}{\rho\omega}}.$$

The damping torque is

$$\tau \sim \eta\,(HR)\frac{R\omega}{\Delta r}R \sim \sqrt{\rho\eta\omega^3}\,HR^3$$

so the timescale for damping is

$$T \sim \frac{L}{\tau} \sim \sqrt{\frac{\rho}{\eta\omega}}\,R.$$

Numerically, if we use the rough estimates

$$\rho \sim 10^3\,\mathrm{kg/m^3}, \quad \omega \sim 10\,\mathrm{s^{-1}}, \quad R \sim 0.1\,\mathrm{m}, \quad \eta \sim 10^{-3}\,\mathrm{Pa\,s}$$

where η is the value for room temperature water, then we get the reasonable results

$$\Delta r \sim 0.3\,\mathrm{mm}, \quad T \sim 30\,\mathrm{s}.$$

[3] **Problem 26.** Water flows through a cylindrical pipe of radius R and length $L \gg R$, across which a pressure difference Δp is applied.

(a) If the flow is slow, viscous effects dominate. By balancing forces on a cylinder of fluid, show that

$$v(r) = \frac{\Delta p}{4\eta L}(R^2 - r^2).$$

Then show that the total mass flux is

$$\frac{dm}{dt} = \frac{\rho \pi R^4 \Delta p}{8\eta L}.$$

This is called Poiseuille's law.

(b) If the flow is very fast, the flow is turbulent. Viscous effects are negligible, and the work done by the pressure difference is dissipated by turbulence into internal energy. Find a rough estimate of the mass flow rate.

[4] **Problem 27.** When a spherical object of radius R moves with velocity v through a fluid of viscosity η and density ρ, it experiences a drag force.

(a) Apply dimensional analysis to constrain the possible forms of the drag force F. You should find there is one dimensionless quantity inversely proportional to η, in accordance with the Buckingham Pi theorem of **P1**. This dimensionless quantity is called the Reynolds number, and it determines what kind of drag dominates.

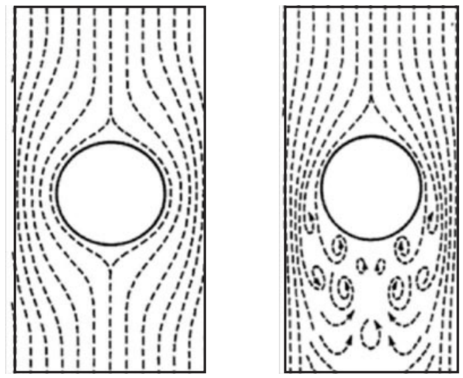
(b) It turns out that $F \propto v$ at low velocities and $F \propto v^2$ at high velocities. Using this information, find the form of the drag force in both cases. (For reference below: the answers are

$$F = 6\pi\eta R v, \quad F = \frac{1}{2} C_d \rho A v^2$$

where C_d is a dimensionless drag coefficient, which is about 1/2 for a sphere. The drag coefficient depends strongly on the shape of the object, being much smaller for streamlined shapes, and weakly on the velocity.)

(c) Hot water has density $\rho = 10^3\,\mathrm{kg/m^3}$ and viscosity $\eta = 0.3 \times 10^{-3}\,\mathrm{Pa\,s}$. (Room temperature water has about 3 times the viscosity.) For an object of radius 1 cm, find the characteristic velocity that divides the two types of drag.

(d) The two cases correspond to flow patterns as shown below.



In the latter case, a region of turbulent flow is created. Using this picture, explain why the drag force is proportional to v^2.

(e) The results above apply to both liquids and gases. In a gas, the relevant quantities are the mass m of the gas molecules, their typical speed u, their number density n, and radius r (which determines how often their collide with each other). Use dimensional analysis to constrain the possible forms of the viscosity η. How do you think η scales with n?

Drag is nicely discussed throughout The Art of Insight; see sections 3.5, 5.3.2, and 8.3.1.2.

Remark

Without knowing the answer to part (b) above, one might expect that the drag force can depend on η, ρ, v, and the shape of the object. In the linear case, the drag force does not depend on ρ. In the quadratic case, the drag force does not depend on η.

These differences can be understood by thinking of where the energy dissipated is going. In the quadratic case, the fluid picks up macroscopic kinetic energy, in the form of a turbulent flow pattern, which is why the drag force does not depend on η. In the linear case, the fluid slows smoothly and hence does not pick up any macroscopic energy; instead the energy is dissipated as heat. Since the macroscopic kinetic energy is not involved, the drag force does not depend on ρ. (Of course, in the quadratic case the turbulent motion eventually stops; at this point it has been converted to heat. The time it takes this to happen is set by η, but it occurs well after the object has passed by and hence does not affect the drag force.)

Example 16

If raindrops fall, why don't clouds fall?

Solution

This isn't a stupid question! It's actually a tough one, which stumped the ancient Greeks and Romans. To give context, we'll cover a bit of atmospheric physics, a topic we will continue in **T1** and **T3**. This is all a bit of a simplification of an interesting story, told in more detail in chapter II-9 of the Feynman lectures.

First, it's useful to review the water cycle. Sunlight directly warms up the ground, and the ground thereby warms the air near the ground. Since warmer air at the same pressure is less dense, it begins to rise by convection. This air also expands roughly adiabatically as it rises, lowering its temperature. Warmer air can also hold more water, so if the original air was moist, water vapor will condense into droplets as the air rises. (This last point is important, because the condensation releases energy, partially counteracting the cooling of the rising air. This keeps it warmer and hence lighter than its surroundings, allowing it to continue to rise.)

Humid air is nearly transparent. However, when water molecules join into small droplets (of order $n \lesssim 100$ molecules), then the amount of electromagnetic radiation scattered by the droplets grows as n^2, as long as the droplet is roughly smaller than the wavelength of the light. (We will justify this in more detail in **E7**.) Therefore, there is an overall enhancement of scattering per water molecule by a factor of n, which is why clouds *aren't* transparent.

Now consider a droplet of radius r. Depending on the droplet size and velocity, the drag force scales as r or r^2, while the gravitational force scales as r^3. Hence the tiny water droplets in clouds are hence carried upward with the ascending moist air, since the drag force dominates. They fall down once they accrete into sufficiently large raindrops, where gravity dominates.

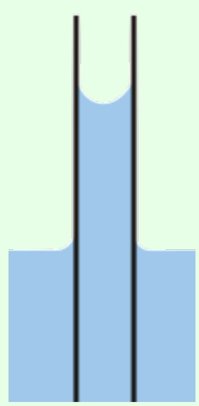
Incidentally, raindrops do not have the teardrop shape shown in typical illustrations. Small raindrops are nearly spherical, because of surface tension. Large raindrops are squashed by air resistance into a "hamburger" shape.

6 Surface Tension

We now return to surface tension, first covered in **M2**, which we'll see yet again in **T3**.

Example 17

A very thin, hollow glass tube of radius r is dipped vertically inside a container of water.



Find the height to which water can climb in the tube.

Solution

In **M2**, we considered problems that could be solved knowing only the "surface tension of water" γ, which is the energy cost per unit area of having a water-air interface. But in this problem there is also a water-glass interface, and the answer to the question depends on precisely how water and glass interact. Specifically, you need to know the surface tension coefficient γ_{wg} which determines the energy cost of having a water-glass interface.

Fortunately, it turns out you don't need to know γ_{wg} if you know the contact angle θ, i.e. the angle between the glass and water surface at the top of the meniscus, which is drawn as acute in the diagram above. We'll just treat θ as a given, but for an explanation of how θ is determined, see **T3** or section 5.5 of Lautrup.

Since the glass tube is very thin, surface tension determines the shape of the water-air surface, so it is spherical since spheres minimize area. By some elementary geometry, one can show that the radius of curvature of this sphere is $R = r/\cos\theta$.

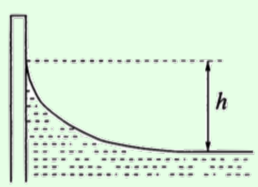
We showed using force balance arguments in **M2** that the pressure inside the curved water surface is lower than atmospheric pressure by $\Delta P = 2\gamma/R$. On the other hand, we also know from Pascal's principle that $\Delta P = \rho g h$. Equating the two gives

$$h = \frac{2\gamma\cos\theta}{\rho g r}.$$

This is Jurin's law. Ideally, water and glass have zero contact angle. This implies that water perfectly wets glass, i.e. that a droplet of water placed on a horizontal glass surface will spread to cover it completely (though this doesn't happen in reality because glass tends to quickly get coated in a layer of impurities). Making this assumption, which we will use for problems below, the answer reduces to $h = 2\gamma/\rho g r$.

Example 18: PPP 130

Water in a glass beaker forms a meniscus, as shown below.



Find the height h to which the meniscus rises above the flat water surface.

Solution

We consider all of the external horizontal forces acting on the water. The surface tension force acting at the top of the meniscus is purely vertical, because water and glass have zero contact angle. The other surface tension force acting on the flat part of the water is γ per length. This balances the excess hydrostatic pressure (i.e. the pressure above atmospheric pressure) at the wall, which is $\rho g h^2/2$ per unit length. Thus,

$$h = \sqrt{\frac{2\gamma}{\rho g}}.$$

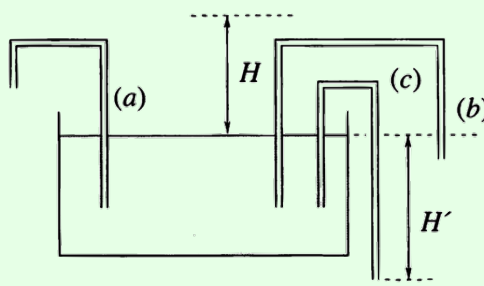
We could have also gotten this with dimensional analysis, up to the prefactor.

Remark

You might be wondering how to compute the shape of the meniscus. There are two methods. First, the pressure right above the water surface is P_{atm}, so the pressure right below the water surface can be determined from the radii of curvature of the surface, using the Young–Laplace equation from **M2**. This pressure can also be computed from the height of the surface using Pascal's principle. Combining these two yields a differential equation for the shape with a rather complicated solution, as explained in sections 5.6 and 5.7 of Lautrup. As you'll see in problem 32, you can also derive this result by considering force balance on the water.

Example 19: PPP 29

Water can rise to a height H in a certain capillary tube. Three "gallows" are made from this tubing by bending it, and placed into a tank of water.



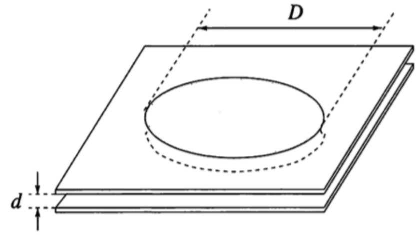
Note that $H' > H$. For which tubes, if any, does water flow out of the other end?

Solution

Clearly no water can fall out of (a), because this would produce a perpetual motion machine. The gallows (b) and (c) are a bit more subtle. Water will *not* fall out of a capillary tube if its end is less than a height H below the free water surface; this follows from the same derivation as Jurin's law, with the surface tension acting to hold the water in the tube. So water only falls out of (c).

[2] **Problem 28** (Russia 2006). A soap bubble of radius R and surface tension γ has a small tube of radius $r \ll R$ passing through its surface. If the air has density ρ, find the rate of decrease of R.

[2] **Problem 29** (PPP 63). Water is stuck between two parallel glass plates. The distance between the plates is d, and the diameter of the trapped water disc is $D \gg d$.



In terms of the surface tension γ of water, what is the force acting between the two plates? This effect can cause wet glass plates to stick together.

[3] **Problem 30** (EFPhO 2009)**.** A soap film of thickness $h = 1\,\mu$m is formed inside a ring of diameter $D = 10\,$cm, and the surface tension of the film is $\gamma = 0.025\,$N/m. If the film is broken at the center, it will begin to fall apart; estimate the time needed for this to happen.

[3] **Problem 31.** USAPhO 2020, problem B1. A nice, slightly mathematically involved surface tension problem with a real-world impact. This setup is discussed in detail in section 5.4 of Lautrup.

[4] **Problem 32.** IPhO 2023, problem 3, parts B and C. A nice problem on the shape of a meniscus, which also explains why pieces of cereal attract each other in a bowl of milk.

Example 20: IPhO 2022 3B

Slightly wet sand is much stronger than either dry sand or very wet sand, which allows the construction of large structures like sand castles. Why is this, and how does the strength depend on the typical size r of the sand grains?

Solution

When a pile of sand is dry, the only force keeping it in place is friction, which is weak. When it's very wet, it's essentially just water, which will simply collapse. But when it's slightly wet, adjacent sand grains have a small layer of water connecting them. Since sand grains are small, this implies a huge total surface area, and thus large surface tension effects.

There are actually two conceptually distinct components to the effect. First, the bit of water connecting two sand grains will provide a surface tension force $F \sim \gamma r$. Second, as you saw in problem 29, the water has a pressure lower by $\Delta P \sim \gamma/r$, leading to an attractive pressure force $(\Delta P)A \sim \gamma r$. In either case, that means the force needed to displace a single grain of sand scales with r. The number of sand grains in a fixed cross-sectional area scales as $1/r^2$, so the weight a sand castle can bear scales as $1/r$. Thus, fine-grained sand is stronger.

This is another example of the subtleties of granular media, first mentioned in **M2**. Neither sand nor water are strong on their own, but they're strong together. Water provides the forces, while the sand provide the structure which lets those forces be effective.

Mechanics VII: Fluids

The fundamental material is covered in chapters 15 and 16 of Halliday, Resnick, and Krane, and at a somewhat higher level in chapter 9 of Wang and Ricardo, volume 1. For a neat explanation of lift and the Coanda effect, see this video. For interesting discussion, see chapters II-40 and II-41 of the Feynman lectures. For a much more advanced introduction which uses vector calculus heavily, see chapters 2–5 and 12–15 of *Physics of Continuous Matter* by Lautrup. There is a total of **86** points.

1 Fluid Statics

Idea 1

In equilibrium, the pressure in a static fluid varies with height as

$$\frac{dp}{dy} = -\rho g.$$

This always holds in equilibrium. For instance, if we squeeze a sealed container of fluid, increasing the pressure locally, then this pressure increase must propagate throughout the entire fluid to maintain $dp/dy = -\rho g$. This is Pascal's principle.

Idea 2: Archimedes' Principle

An object in a fluid experiences an upward buoyant force due to the different pressures on its top and bottom sides. The force is equal in magnitude to the weight of the fluid that would fill the volume of the immersed portion of the object.

This can be surprisingly tricky, so we'll begin with some conceptual questions.

Example 1

A large rock is tied to a balloon filled with air. Both are placed in a lake. As the balloon sinks, how do the air pressure in the balloon, the average density of the balloon, air, and rock system, and magnitude of the net force on the system vary?

Solution

For simplicity, we ignore the elastic force in the balloon itself. Then for the balloon to be in equilibrium, its pressure must match that of the water pressure, so the air pressure in the balloon increases. As the balloon sinks, the rock stays the same volume but the balloon is squeezed smaller, so the density of the system increases. Finally, since the density of water is very approximately constant, the buoyant force on the system is decreasing since its volume is decreasing, so the net force is increasing; the system accelerates downward faster and faster.

[1] **Problem 1** (HRK)**.** The average human body floats in water. SCUBA divers wear weights and a flotation vest that can fill with a varying amount of air to establish neutral buoyancy. A diver is originally neutrally buoyant at a certain depth. How should the diver manipulate the amount of air in their flotation vest to move to a lower depth, then stay there at neutral buoyancy?

Solution. To move to a lower depth, the amount of air should be decreased. To stay at neutral buoyancy, the amount of air should be increased. In fact, the amount of air in the vest should end up *greater* than the original amount, because at a lower depth, the same amount of air would be squeezed to a smaller volume, making the vest less buoyant.

[2] **Problem 2.** A beaker contains liquid water at its freezing point and has a big ice cube floating in it, also at its freezing point. If the ice cube

(a) is solid ice,

(b) contains a small metal ball, or

(c) contains a lot of olive oil (which will float on the water in a thin layer),

then how does the fluid level change when the cube melts? In all cases, neglect the density of air.

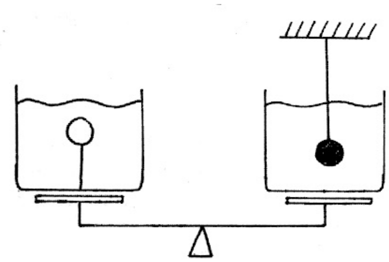
Solution. (a) The water level does not change. The ice cube creates a "hole" in the water that it sits in, and when it melts it exactly fills this hole.

(b) Initially the metal ball displaces its weight in water, which is large, but after the ice cube melts it falls to the bottom and only displaces its volume in water. Hence the water level goes down.

(c) Initially, the oil displaces its weight in water, but after the ice cube melts it sits on top, thereby occupying its whole volume. Since the oil is less dense than water, the fluid level goes up. (Specifically, the water level goes down, and an oil layer sits on top of it, at a height greater than the original water level.) Note that the answer would be different if there was only a little olive oil, because in that case the oil would coalesce into a few drops on the water surface. In that case each drop displaces its weight, so that the water level stays the same.

As a sidenote, if we accounted for the density of air, then the answer to part (a) would actually be that the water level goes up a tiny bit. The reason is that part of the ice cube poking out above the water surface experiences an extra buoyant force from the air itself, which means the ice cube is not as deep in the water as one would expect. So when it melts, it slightly more than fills the hole, causing the water level to go up.

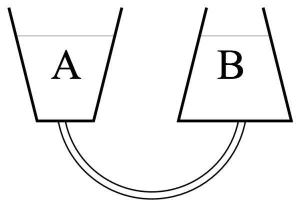
[2] **Problem 3** (Povey, Moscow 1939)**.** Consider a pair of scales with identical vessels in which there are equal quantities of water.



In the left-hand vessel you suspend a very light ping-pong ball on a thin, light wire attached to the base of the vessel. In the right-hand vessel you suspend a ping-pong ball filled with lead, again by a light thin wire. Do the scales stay level, go down on the left, or go down on the right?

Solution. The ball on the right experiences an upward buoyant force, so it exerts a downward force on the water. As for the ball on the left, it has no effect whatsoever on the force on the scale, because this force is simply equal to the weight of all the water. Hence the scales go down on the right.

[2] **Problem 4** (BAUPC)**.** Two trapezoidal containers, connected by a tube as shown, hold water.



Assume that the containers do not undergo thermal expansion.

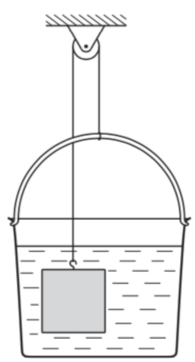
(a) If the water in container A is heated, causing it to expand, will water flow through the tube? If so, in which direction?

(b) What if the water in container B is heated instead?

Solution. (a) The pressure at the bottom of the containers is $P = \rho g h = mgh/V$, where h is the height of the water level above the container, V the volume, and m is the mass of the water in the container. When A is heated and the water level in A rises, V/h (average area) will increase as seen by the shape of the container. Thus $P_A = mgh/V$ will decrease, and water will flow from B to A.

(b) In this case, V/h for B will decrease, since when the water level rises we include a section with a smaller cross-sectional area. Thus P_B will increase, and water will flow from B to A again.

[2] **Problem 5** (MPPP 85)**.** A solid cube of volume V_i and density ρ_i is fastened to one end of a cord, the other end of which is attached to a light bucket containing water, of density $\rho_w = \rho_i/10$.



The system is in equilibrium.

(a) Find the volume V_w of the water in the bucket.

(b) What would happen if more water were poured into the bucket?

(c) What would happen if some or all of the water evaporated?

Solution. (a) There is a buoyant force of $\rho_w V_i$ on the block, which pushes the block up and the bucket down. In equilibrium, the tension in the cord T must balance against the weight of the block and the buoyant force: $T = (\rho_i V_i - \rho_w V_i)g$. Similarly for the water/bucket, $T = (\rho_w V_w + \rho_w V_i)g$. Equating the two gets

$$\rho_w V_w = \rho_i V_i - 2\rho_w V_i$$

$$V_w = 8V_i$$

(b) The effective weight on the left side won't change, but the bucket will be heavier. If the final volume of water V_f is less than $10V_i$, where $V_f = 10V_i$ is when the system will be in equilibrium if the block is out of the water, then the system will be in equilibrium at some point with the block partially submerged. Once $V_f > 10V_i$, the bucket will just keep falling.

(c) The cube will fall until it hits the bottom of the bucket (the amount of evaporation doesn't matter), and then the system will be stuck there since the cube can't pass through the bucket.

Example 2

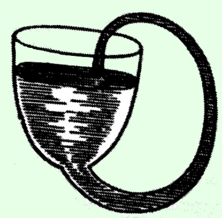
A perfectly spherical, nonrotating planet is covered with water. Geological activity causes a small underwater mountain to form, made of rock that is denser than water. Does the ocean surface above this mountain become higher or lower?

Solution

Systems minimize their energy in equilibrium. This means that in hydrostatic equilibrium, the surface of the water is an equipotential. Since the gravitational field of the mountain increases the gravitational potential near it, the water surface is higher near the mountain.

Example 3

Robert Boyle is best known for Boyle's law, but he also invented a remarkably simple perpetual motion machine, called the *perpetual vase*.



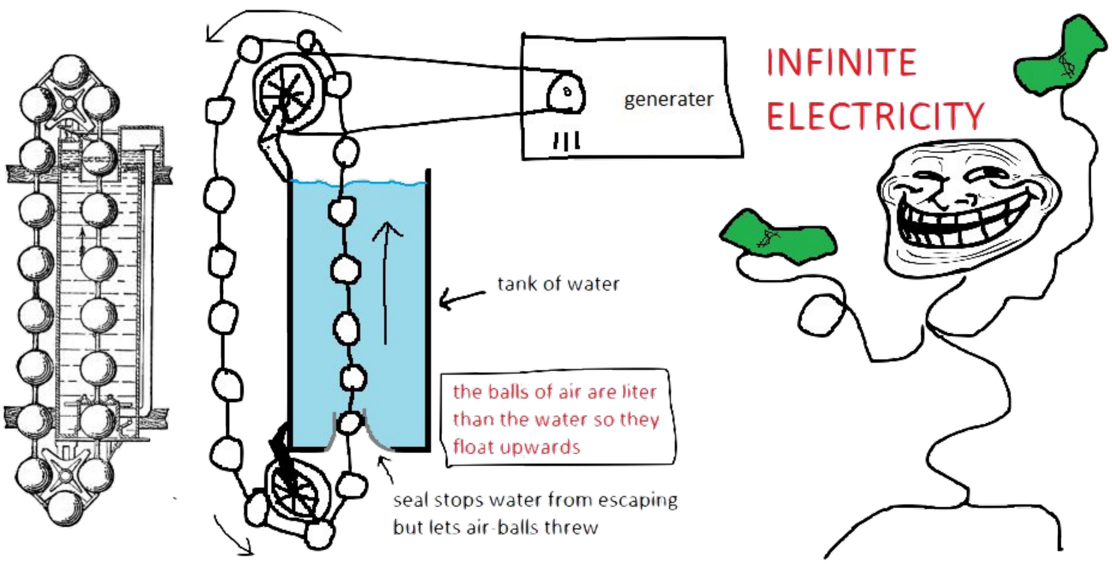
Since the volume of the vase is much greater than the neck, the pressure in the neck cannot possibly hold up all of the water in the vase. Hence the water will flow through the neck and fall back into the vase, causing perpetual motion. Why doesn't this work?

Solution

This is an example of the hydrostatic paradox. Most of the upward force on the water is *not* provided by the pressure in the water in the neck, but from the normal force from the walls; each piece of wall provides enough normal force to hold up all of the water above it. (Of course, ultimately each piece of the glass is held in place by internal forces with other pieces of the glass, which ultimately are balanced by whatever is holding the glass.)

Thus, the water in the neck only supports the water directly above it. That's precisely what is balanced by the heightened pressure in the neck, so the water doesn't start moving. (There have been many more attempts at fluid-based perpetual motion, as you can see here.)

[2] **Problem 6.** Below is another perpetual motion machine, in both original and modern form.



The balls are less dense than water. The balls on the left are pulled downward by gravity, while the balls on the right are pushed upward by the buoyant force.

(a) Why doesn't this work?

(b) Would it work if the balls and chain were replaced with a flexible tube of constant thickness?

Solution. (a) Let the balls have volume V, and the column have height h. The positive work done on a ball by the buoyant force, as it climbs the length of the column, is

$$W_{\text{up}} = F\,\Delta x = (\rho g V)h.$$

On the other hand, it costs work to insert the ball into the column at the bottom,

$$W_{\text{in}} = P\,\Delta V = (\rho g h)V.$$

Thus, the energy you get from letting a ball go all the way up is just the energy you put in by pushing the ball in at the bottom, so there's no free energy.

What this means in practice is that if you actually set up the system, it will start moving a bit until the first ball hits the bottom of the column, and then it won't be able to go in. If you push it in, then the chain will start going around, but only at a constant speed, until friction slows it down.

(b) In this case, the objection raised in part (a) doesn't hold. The tube can just slide in at the bottom, so that $W_{\text{in}} = 0$. However, the machine still doesn't work because now the buoyant force vanishes, so $W_{\text{up}} = 0$ too. The point is that the buoyant force is only $\rho g V$ if the *entirety* of the object with volume V is surrounded by water. Since the tube just goes right through the bottom of the column, there's no water present to push up on the bottom of the tube, and hence no buoyant force.

[2] **Problem 7** (HRK)**.** A fluid is rotating at constant angular velocity ω about the vertical axis of a cylindrical container. Defining $z = 0$ to be the water level at the cylinder's axis, show that the liquid surface is the paraboloid

$$z = \frac{\omega^2 r^2}{2g}.$$

Since a paraboloid perfectly focuses incoming light which is parallel to its axis, a rotating fluid can be used as a telescope, as was first pointed out by Isaac Newton. Such liquid-mirror telescopes are cheap, but have the disadvantage that they can only point up. Alternatively, one can gradually cool molten glass in a rotating container so that it solidifies into a paraboloidal lens.

Solution. Work in the frame rotating with the fluid, where it is static. Balancing the pressure and centrifugal force on a cylindrical shell of thickness dr, at radius r and height h, gives

$$dp(2\pi r h) = \omega^2 r(\rho(2\pi r h\, dr)) \implies \frac{dp}{dr} = \rho\omega^2 r.$$

On the other hand, we also know that in hydrostatic equilibrium, the pressure obeys

$$\frac{dp}{dz} = -\rho g.$$

The pressure has to stay the same along the surface, so

$$\frac{dp}{dr} + \frac{dp}{dz}\frac{dz}{dr} = 0.$$

We thus have $dz/dr = \omega^2 r/g$, and integrating gives the desired result.

[3] **Problem 8.** USAPhO 2013, problem A4. In order to make measurements, print out the problem before starting.

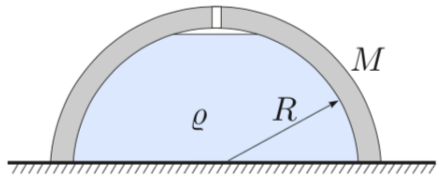
2 Fluid Mechanics

Next we'll consider some situations involving fluids and other objects, where the fluids can be treated at least quasistatically but the objects must be treated dynamically.

Idea 3

The buoyant force can be regarded as acting at the center of gravity of the fluid displaced by the submerged part of a floating object, called the center of buoyancy. A floating configuration is stable if, when the configuration is slightly rotated, the buoyant force provides a restoring torque about the center of mass.

[2] **Problem 9** (Kalda)**.** A hemispherical container is placed upside-down on a smooth horizontal surface. Water is poured in through a small hole at the bottom of the container. Exactly when the container fills, water starts leaking from between the table and the edge of the container.



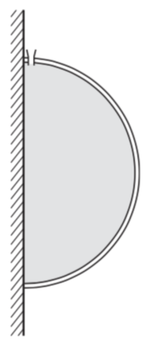
Find the mass of the container if the water has density ρ and the hemisphere has radius R.

Solution. Note that right when the water is full, the normal force between the ground and the container is 0. Thus, the weight of the container and water is balanced by the normal force on the water. However, this is just $\rho g R(\pi R^2)$, so we have

$$\left(M + \frac{2}{3}\pi R^3 \rho\right) g = \rho g \pi R^3,$$

so $M = \boxed{\rho\pi R^3/3}$. Note that the atmosphere has a negligible effect here, because if all atmospheric effects are accounted for, the net effect is just a tiny buoyant force on the container and water.

[2] **Problem 10** (MPPP 89)**.** A thin-walled hemispherical shell of mass m and radius R is pressed against a smooth vertical wall.



It is filled with water through a small aperture at its top, with total mass M. Find the minimum magnitude of the force that has to be applied to the shell to keep the liquid in place.

Solution. We consider the system of the water and shell. The external force F exerted must counteract the vertical force of gravity, and the horizontal force of the hydrostatic pressure from the wall. First, vertical force balance gives

$$F_y = (M + m)g.$$

Evaluating horizontal force balance is slightly trickier. However, note that by symmetry, the average pressure at the part of the wall touching the water is precisely the pressure at the vertical center of the hemisphere, so

$$F_x = \overline{P}A = (\rho g R)(\pi R^2) = \pi R^3 \rho g = \frac{3}{2}Mg.$$

Thus, the total force needed is

$$F = \sqrt{F_x^2 + F_y^2} = g\sqrt{\frac{13}{4}M^2 + 2Mm + m^2}.$$

Note that we ignored the effect of the atmosphere in this question, which would be tiny in any case; one can tell that it should be ignored since the problem statement never specified the density of air.

Technically, we should verify that the torques can be balanced too, by choosing an appropriate point to apply the force F. Take the origin O to be the center of the hemisphere, so that the radial pressure of the curved part produces no torque. The applied force needs to cancel the torques from gravity and the pressure from the wall. It turns out there always exists a point of application for F that does this, but showing it explicitly is messy and unenlightening. In this problem, you're just meant to see intuitively that torque can be balanced.

[3] **Problem 11.** USAPhO 2002, problem A4.

[3] **Problem 12.** USAPhO 2004, problem A2.

[3] **Problem 13.** A log with a square cross section and very low density will float stably with one of its sides parallel to the water.

(a) If the density of the log is increased, show that when

$$\rho_{\text{log}} = \frac{3-\sqrt{3}}{6}\rho_{\text{water}}$$

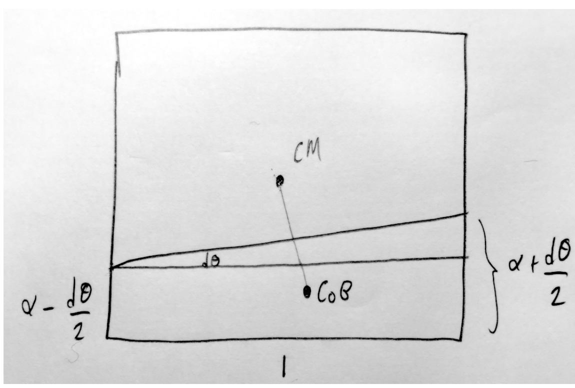
the orientation becomes unstable. (Hint: to keep the calculations short, choose a good coordinate system and work to the lowest relevant order everywhere.)

(b) How do you think the orientation of the log varies as ρ_{log} is varied? In particular, what's the orientation when $\rho_{\text{log}}/\rho_{\text{water}} = 1/2$? How about when $\rho_{\text{log}} \approx \rho_{\text{water}}$?

Finding the stable orientation of the log for general values of ρ_{log} is quite complicated, but you can play with a nice simulation here; you can also use this to check your answer.

Solution. For simplicity, we'll set the side length of the log to 1.

(a) The task reduces to finding how the center of mass and center of buoyancy move after an infinitesimal rotation $d\theta$. For simplicity, we align the coordinate system with the log and place the origin at the center of mass.



The fraction of the log submerged is $\alpha = \rho_{\text{log}}/\rho_{\text{water}}$. To compute the coordinates of the center of buoyancy we split it into two pieces as shown above. Then

$$x_B = \frac{1}{\alpha}\left(\frac{1}{6}\cdot\frac{d\theta}{2} + 0\cdot\left(\alpha - \frac{d\theta}{2}\right)\right) = \frac{d\theta}{12\alpha}$$

and

$$y_B = \frac{1}{2} - \frac{\alpha}{2} + O(d\theta).$$

where y is positive downward. For neutral stability, (x_B, y_B) must lie on a vertical line from the center of mass, which implies $x_B/y_B = d\theta$, so

$$\frac{1}{12\alpha}\left(\frac{1}{2} - \frac{\alpha}{2}\right)^{-1} = 1.$$

This is a quadratic equation with solution $\alpha = (3 - \sqrt{3})/6$.

(b) When $\rho_{\text{log}}/\rho_{\text{water}} = 1/2$, it's fairly intuitive that the log sits at a 45° angle, with a corner facing directly down. And when $\rho_{\text{log}} \approx \rho_{\text{water}}$, the result is the same as in the low density case: the log sits with a side parallel to the water surface. However, it's much less intuitive for other values of the density. As you can see in the linked simulation, the equilibrium orientation can actually be at any angle, depending on the density.

This is a subtle and unintuitive result. In fact, an entire paper has been written on this problem, which you can see if you want more details!

Remark

Some Olympiad questions involving oscillating fluids, which are more subtle. These questions are often impossible to solve exactly, because one must keep track of the entire motion of the water to know how much kinetic and potential energy are in play. In **M4**, you solved IPhO 1984, problem 2, which only asked for an order of magnitude estimate. Physics Cup 2018, problem 4 considers a V-shaped container, where the calculation can be done exactly.

[4] **Problem 14.** ⏱ EuPhO 2022, problem 1. A nice fluid oscillations problem which can be solved nearly exactly without too much trouble.

Solution. See the official solutions here. (It's interesting that here, the water's potential and kinetic energy get multiplied by the exact same factor, resulting in an oscillation period that doesn't depend on the cylinder's exact dimensions. For a generic container geometry, the two won't precisely match, but they will change the oscillation frequency by a comparable amount in opposite directions.)

Idea 4: Virtual Mass

When an object moves through water, it effectively has extra inertia because it forces water to move as well. This is the "virtual mass" effect (also called added mass, or hydrodynamic mass) which we first mentioned in **M4**. It can be computed exactly in a few special cases:

$$\Delta m = \begin{cases} \rho V/2 & \text{sphere} \\ \rho V & \text{long cylinder moving perpendicular to axis} \end{cases}$$

where ρ is the water density and V is the volume of the object. You don't have to memorize these results, but the idea of virtual mass does occasionally show up. For instance, IPhO 1995, problem 3 involves oscillations of a cylindrical buoy of mass m which is only partially submerged in water; they ask you to simply assume a virtual mass $m/3$. Furthermore, Physics Cup 2019, problem 1 introduces a slick method, based on vector calculus analogies, which can be used to compute the virtual mass exactly for a few more special shapes.

Example 4

Derive the expression for the virtual mass of a sphere.

Solution

Consider a spherical object of radius a moving uniformly with speed v_0 through water of density ρ. The object forces the water to move: the water ahead of it has to get out of the way, while the water behind it needs to fill the space it leaves behind. By the ideas of **M4**, the total kinetic energy of the water is $(\Delta m)v_0^2/2$, where Δm is the virtual mass.

It turns out the fluid's velocity field $\mathbf{v}(\mathbf{r})$ has to satisfy $\nabla\cdot\mathbf{v} = 0$, reflecting the incompressibility of water, and $\nabla \times \mathbf{v} = 0$, reflecting the absence of vorticity. It also has to go to zero far from the sphere, and have zero relative normal velocity at the sphere itself. These differential equations and boundary conditions yield a unique solution. The methods for finding the solution are standard, and typically taught in an undergraduate electromagnetism course, but since they're outside the Olympiad syllabus, I'll just display the answer. The velocity is

$$\mathbf{v}(\mathbf{r}) = \frac{v_0 a^3}{2r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

in polar coordinates, where we placed the origin at the center of the sphere and aligned the $\hat{\mathbf{z}}$ axis with its direction of motion. If you've done **E1**, you might notice this is just like the electric dipole field; this coincidence isn't *too* surprising because that field satisfies the similar equations $\nabla \cdot \mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$, which are quite restrictive.

Now, to derive the virtual mass, we just have to carry out the kinetic energy integral, which is easiest in spherical coordinates,

$$\begin{aligned} K &= \int \frac{\rho v^2}{2}\,dV \\ &= \frac{\rho v_0^2 a^6}{8}\int_a^\infty \frac{r^2\,dr}{r^6}\int_0^{2\pi} d\phi \int_0^\pi (\sin\theta\,d\theta)(4\cos^2\theta + \sin^2\theta) \\ &= \frac{\rho v_0^2 a^6}{8}\left(\frac{1}{3a^3}\right)(2\pi)(4). \end{aligned}$$

This yields a virtual mass of $(2\pi/3)\rho a^3 = \rho V/2$, as stated above.

Example 5

What is the initial upward acceleration of a spherical air bubble in water?

Solution

The upward buoyant force on the bubble is $\rho V g$, and the mass of the bubble is negligible, so if we didn't know about virtual mass, we would be tempted to conclude the acceleration is enormous. Instead, the buoyant force is used to move the virtual mass $\rho V/2$ out of the way, so the upward acceleration is $2g$.

Like most things in fluid dynamics, this isn't an exact result. The usual expression for the buoyant force assumes no motion at all, while the virtual mass derivation assumes uniform motion, neither of which are true for an accelerating bubble. For the result above to be accurate, the bubble has to be small, so that the pressure and flow fields have time to reach a quasi-steady state, but not too small, so that we can still ignore viscous forces.

3 Fluid Dynamics

Idea 5: Continuity

In steady flow, the quantity $\rho A v$ is constant along tubes of streamlines.

Idea 6: Bernoulli's Principle

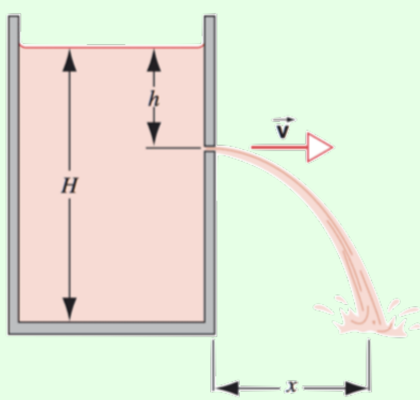
For steady, nonviscous, incompressible flow, the quantity

$$p + \frac{1}{2}\rho v^2 + \rho g y$$

is constant along streamlines. Another version of Bernoulli's principle, valid for compressible flow, is given in **T3**. As explained there, the incompressible result here is applicable for water flow, and for gas flow as long as the velocity is much less than the speed of sound.

Example 6: HRK

A tank is filled with water to a height H. A small hole is punched in one of the walls at a depth h below the water surface as shown.



Find the distance x from the foot of the wall at which the stream strikes the floor.

Solution

The flow isn't perfectly steady, but it's close enough since the hole is small. We thus apply Bernoulli's principle along a streamline, where one point is at the water's top surface, and the other point is just outside the hole. Both points are at atmospheric pressure, because they are directly exposed to the atmosphere. Since the hole is small compared to the tank, the velocity at the first point is small by continuity, so we neglect it, giving

$$\frac{1}{2}\rho v^2 = \rho g h$$

which implies Torricelli's law,

$$v = \sqrt{2gh}.$$

The time t to fall is $t = \sqrt{2(H-h)/g}$, so

$$x = vt = 2\sqrt{h(H-h)}$$

which incidentally is maximized at $h = H/2$.

Incidentally, Bernoulli himself was aware that the answer was different for a large hole, and treated the general case in his 1738 book, *Hydrodynamica*. The method is to apply energy conservation to all of the water at once (i.e. equating the rate of decrease of gravitational potential energy to the rate of increase of total kinetic energy), rather than attempt to apply it along streamlines. You can see this general analysis here.

Example 7

Why should you close your barn door during a storm?

Solution

The wind in a storm can flow into the barn, at which point it stops. By Bernoulli's principle, this increases its pressure by $\rho v^2/2$. On the other hand, the wind that doesn't enter the barn has to flow faster along the top to get around it, which decreases its pressure. These two effects both create a net upward force on the roof, which can tear it off the barn. The latter effect caused my trampoline to achieve liftoff during Hurricane Sandy, destroying a backyard fence.

Incidentally, this example brings up a little puzzle about Bernoulli's principle. We argued that the air slows down when it enters the barn, so the pressure goes up. But in the reference frame moving with the wind, the air speeds up when it enters the barn – so shouldn't its pressure go down? The issue with this reasoning is two-fold. First, in the wind's frame, the barn is moving, so the flow isn't steady and Bernoulli's principle doesn't apply. Second, even if the barn were moving slowly, so that the flow were almost steady, the barn's motion would still be doing work on the air, and this changes Bernoulli's principle because it is ultimately a restatement of energy conservation. So in either case, the reasoning fails. When obstacles are present, Bernoulli's principle should always be invoked in the frame of the obstacles.

Example 8: JEE 2020

When a train enters a narrow tunnel, your ears pop because of the pressure change. Find the pressure change, assuming the air has constant density ρ, the atmospheric pressure is P_0, the train speed is v, and the cross-sectional areas of the train and tunnel are A_t and A_0.

Solution

We work in the reference frame of the train. In this frame, the air in the tunnel begins moving towards the train at speed v. When it gets to the train, it has to speed up to speed v_f because it flows through a smaller area $A_0 - A_t$, and this causes its pressure to decrease by Bernoulli's principle. Specifically, we have

$$A_0 v = (A_0 - A_t) v_f, \quad P_f + \frac{1}{2}\rho v_f^2 = P_0 + \frac{1}{2}\rho v^2$$

which gives a pressure drop of

$$P_f - P_0 = -\frac{1}{2}\rho v^2 \left(\frac{1}{(1 - A_t/A_0)^2} - 1 \right).$$

We neglected the change in density of the air, which is a good approximation when the train is much slower than the speed of sound. We'll treat fluid flow with changing density in **T3**.

Example 9

A whirly tube is a long, narrow, flexible tube that produces musical tones when swung. Model a whirly tube as a cylinder of length L, rotated about one end with angular velocity ω. For simplicity, neglect gravity. What is the speed of the air when it shoots out the other end?

Solution

The air is slowly sucked from all directions around the entry hole, and shot out at the exit hole. Applying Bernoulli's principle between a point near the entry hole, and the exit hole,

$$P_{\text{atm}} \approx P_{\text{atm}} + \frac{1}{2}\rho v_{\text{out}}^2.$$

But that implies $v_{\text{out}} \approx 0$, which doesn't make sense. The problem is that Bernoulli's principle applies to steady flows, and this situation is definitely not steady: by the time the air goes through the tube, the tube has rotated by a significant amount.

Instead, we apply Bernoulli's principle in a reference frame rotating with the tube. The centrifugal force gives an additional term, turning it into

$$P + \frac{1}{2}\rho v^2 - \frac{1}{2}\rho\omega^2 r^2 = \text{const}.$$

Applying Bernoulli's principle between the same two points gives

$$P_{\text{atm}} \approx P_{\text{atm}} + \frac{1}{2}\rho v^2 - \frac{1}{2}\rho\omega^2 L^2$$

from which we conclude $v = \omega L$. Transforming back to the original reference frame, the exit speed of the air is $\sqrt{v^2 + (\omega L)^2} = \sqrt{2}\, v$.

Example 10

A big fan produces a stream of air with speed v. If the atmospheric pressure in the room is P_{atm}, what's the pressure P in the middle of the fan's air stream?

Solution

This question frequently appears in middle school physics lessons. Obviously, if we apply Bernoulli's principle to the air before and after it goes through the fan, we get

$$P + \frac{1}{2}\rho v^2 = P_{\text{atm}}$$

so that the pressure is lower than atmospheric pressure. Easy, right? But it's wrong!

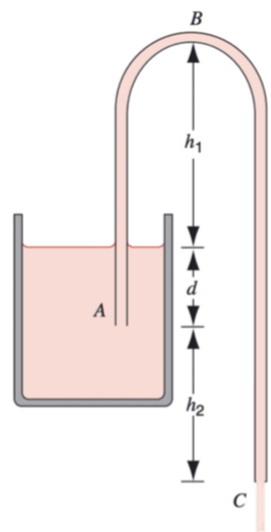
The air in the stream is traveling forward with constant velocity, exposed to the rest of the air in the room, which has atmospheric pressure. If there actually was such a pressure difference, the fan's air stream would be compressed by the air in the room, until it reached atmospheric pressure again. If you look back carefully at the above examples, you'll see this is always the case: air can only be at a different pressure if it's confined away from the atmosphere at large (e.g. in a train tunnel or a whirly tube), or if it's actively being accelerated (e.g. when it flies into or over a barn, in which case the pressure difference is precisely what causes the force). The other case where you can maintain a pressure difference is when the air is moving extremely quickly, which will be discussed in **T3**.

So the correct answer is that $P = P_{\text{atm}}$. But why doesn't Bernoulli's principle work? Because it's a statement of energy conservation, and the fan itself is doing work on the air to get it moving. The correct statement would be

$$P_{\text{atm}} + \frac{1}{2}\rho v^2 = P_{\text{atm}} + w$$

where w is the work done by the fan per unit volume of air.

[2] **Problem 15** (HRK). A siphon is a device for removing liquid from a container that cannot be tipped. An example of a siphon, with constant cross-section, is shown below.



The tube must initially be filled, but once this has been done the liquid will flow until its level drops below the tube opening at A. The liquid has density ρ and negligible viscosity.

(a) With what speed does the liquid emerge from the tube at C?

(b) What is the pressure of the liquid at the topmost point B?

(c) What is the maximum possible h_1 so that the siphon can operate?

(d) Would the siphon still work if h_2 were slightly negative? How negative can it be, for the siphon to keep on working?

Solution. (a) Applying Bernoulli's principle between the surface of the water and point C gives

$$\frac{1}{2}\rho v^2 = \rho g(h_2 + d)$$

which implies

$$v = \sqrt{2g(h_2 + d)}.$$

(b) By continuity the speed v in the tube is constant. Applying Bernoulli's principle between points B and C gives

$$p_{\text{atm}} = p_B + \rho g(h_1 + h_2 + d)$$

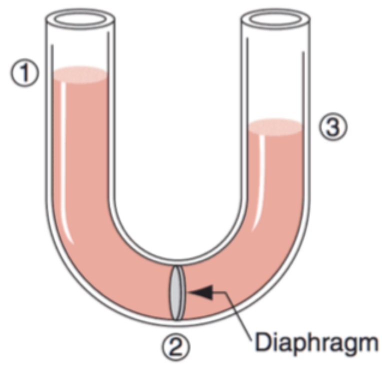
which gives

$$p_B = p_{\text{atm}} - \rho g(h_1 + h_2 + d).$$

(c) For the siphon to just barely work, the flow speed v should be tiny, so $h_2 + d \approx 0$. The highest value of h_1 is when the pressure is zero at point B, since pressure can't be negative, so $h_1 \leq p_{\text{atm}}/\rho g$. (If we start with this maximum possible value of h_1 but then increase $h_2 + d$ above zero, then the siphon will stop working, because the water flow will break up along the exit tube.)

(d) Yes. It is still energetically favorable for water to flow through the siphon as long as point C is below the surface of the water. As mentioned above, the siphon works as long as $h_2 + d > 0$.

[2] **Problem 16** (HRK). Consider a uniform U-tube with a diaphragm shown below.



(a) Suppose the diaphragm is opened and the liquid begins to flow from left to right. Show that applying Bernoulli's principle yields a contradiction.

(b) Explain why Bernoulli's principle doesn't apply if the diaphragm has a very wide opening.

(c) Explain why Bernoulli's principle doesn't apply if the diaphragm has a tiny opening.

For a similar idea to this problem, see $F = ma$ 2018 A22.

Solution. (a) Since the pressures and velocities at points 1 and 3 are the same, Bernoulli's principle would imply the heights are also the same, which is false.

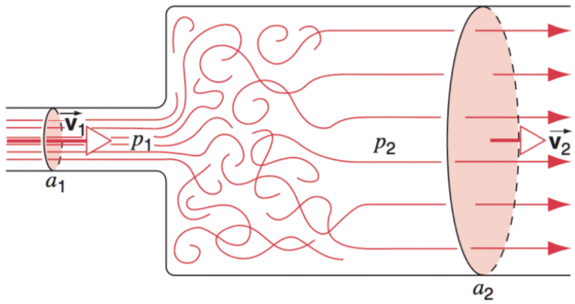
(b) Bernoulli's principle is just energy conservation, applied to water moving along a streamline. In this case, the liquid just oscillates back and forth, with point 1 and point 3 alternating periodically in height. Bernoulli's principle can't be applied between points 1 and 3 because water never moves all the way from point 1 to point 3; it just wiggles back and forth.

(c) In this case, viscous effects are not negligible. Energy is dissipated to heat, so Bernoulli's principle doesn't apply. This should be intuitively clear, but more formally, the result of problem 26 shows the pressure loss $\Delta p \propto v/R^2$, where v is the fluid velocity through the diaphragm, and here R is small.

But what if we used a very nonviscous fluid? In that case, you would still lose energy, but to turbulence; the flow pattern after the diaphragm's opening would look like the setup of problem 17. Energy in turbulent eddies eventually dissipates to heat, so we again lose energy and Bernoulli's principle doesn't apply.

But what if the diaphragm is also shaped like a smooth curve, to prevent turbulence? In that case, you don't lose energy, but the flow isn't steady. The fluid continually accelerates as it goes through the diaphragm; in the long run the heights of points 1 and 3 alternate, as in part (b). Fluid does go all the way from point 1 to point 3, but Bernoulli's principle can't be applied because the flow isn't steady.

[2] **Problem 17** (HRK). A stream of fluid of density ρ with speed v_1 passes abruptly from a cylindrical pipe of cross-sectional area a_1 into a wider cylindrical pipe of cross-sectional area a_2 as shown.



The jet will mix with the surrounding fluid, forming a turbulent region where the pressure is approximately p_1. Further to the right, the flow becomes almost uniform again, with average speed v_2 and pressure p_2.

(a) By considering force and momentum, show that

$$p_2 - p_1 = \rho v_2(v_1 - v_2).$$

(b) Show from Bernoulli's principle that in a gradually widening pipe we would instead get

$$p_2 - p_1 = \frac{1}{2}\rho(v_1^2 - v_2^2).$$

(c) Find the loss of pressure due to the abrupt enlargement of the pipe. Can you draw an analogy with elastic and inelastic collisions in particle mechanics?

Solution. (a) Let's consider the fluid in the region bounded by the two shaded circles. After a small time dt, this fluid moves to the right, and some of the fluid originally traveling at v_1 ends up traveling at v_2. The rate of change in momentum is

$$\frac{\Delta p}{\Delta t} = (v_2 - v_1)\frac{\Delta m}{\Delta t} = \rho v_2 a_2 (v_2 - v_1).$$

This must be equal to the net force on the fluid, which has three contributions:

- A leftward force $p_2 a_2$ from the fluid on its right side.
- A rightward force $p_1 a_1$ from the fluid on its left sisde.
- A rightward force $p_1(a_2 - a_1)$ from the vertical part of the wall.

This is a net rightward force of $(p_1 - p_2)a_2$. Equating these expressions and dividing by a_2 gives the desired result.

(b) This is simply a direct application of Bernoulli's principle.

(c) The extra loss of pressure is the difference,

$$\Delta p = \frac{1}{2}\rho(v_1 - v_2)^2.$$

As in an inelastic collision, the loss of energy (reflected in the loss of pressure, which is essentially like elastic potential energy) goes as the square of the relative speed.

[2] **Problem 18** (PPP 49)**.** After a tap above an empty rectangular basin has been opened, the basin fills with water in a time T_1. After the tap has been closed, opening a plug at the bottom of the basin empties it in time T_2. If both the tap and plug are open, what ratio of T_1/T_2 can cause the basin to overflow?

Solution. Let water come out of the tap at a volumetric flow rate of $\dot{V}$, so that the rectangular basin with cross-sectional area A and height h_0 will can be filled in time $T_1 = Ah_0/\dot{V}$.

When the plug at the bottom with cross sectional area a has been opened, water will flow out at a velocity of $v = \sqrt{2gh}$ giving a volumetric flow rate of $-a\sqrt{2gh}$, where h is the water depth. Then

$$\frac{d}{dt}(Ah) = A\frac{dh}{dt} = -a\sqrt{2gh},$$

$$-\int_{h_0}^{0} \frac{dh}{\sqrt{2gh}} = \int_0^{T_2} \frac{a}{A} dt,$$

$$\sqrt{\frac{2h_0}{g}} = \frac{a}{A} T_2.$$

In order to overflow the water basin, the tap needs to add water faster than the plug drains water when water is draining at the fastest (when the basin is almost full). The overflow condition is then

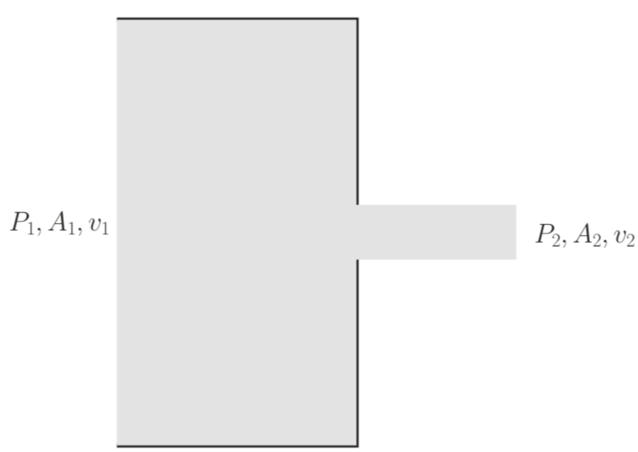
$$\dot{V} = \frac{Ah_0}{T_1} > a\sqrt{2gh_0}.$$

Plugging in our result for T_2 gives

$$\frac{T_1}{T_2} < \frac{1}{2}.$$

This differs from the naive answer $T_1/T_2 = 1$ because the rate of emptying depends on the current water height. This also implies that all those elementary school questions about filling and emptying a basin simultaneously are wrong. For example, you might have once been asked, "if a basin can be filled in 2 minutes and drains in 3 minutes, how long does it take to fill if the drain is open?" The naive answer is 6 minutes, but the real-life answer is that it never fills up all the way.

[4] **Problem 19.** This problem is about the subtle phenomenon of *vena contracta*. An incompressible fluid of density ρ is flowing through a tube of area A_1, which suddenly contracts to area $A_2 \ll A_1$. Naively, the flow looks as shown below.



(a) Argue by energy conservation that

$$v_2^2 \approx 2\frac{P_1 - P_2}{\rho}.$$

(b) Argue that the net force on the fluid shown in the picture is approximately $(P_1 - P_2)A_2$. Then argue by momentum conservation that

$$v_2^2 \approx \frac{P_1 - P_2}{\rho}.$$

(c) The resolution of the paradox is that the "v_2^2" in the first equation is the speed, while the "v_2^2" in the second is really $(v_2)_x^2$. That is, the naive picture above is wrong: the fluid does not exit through the orifice horizontally, but rather contracts as it leaves. Show that if the fluid contracts to a minimum area A_3 with $P_3 \ll P_1$, then $A_3 = A_2/2$, so that momentum conservation is satisfied.

(d) Even assuming ideal fluid flow satisfying Bernoulli's principle, the result above for A_3 is not exact, but is instead off by about 20%. Is the true value of A_3 higher or lower than $A_2/2$?

(e) How could the shape of the orifice be modified so that A_3 is exactly $A_2/2$? How could the orifice be modified to remove the vena contracta effect entirely?

Solution. (a) Applying Bernoulli's principle,

$$\frac{1}{2}\rho(v_2^2 - v_1^2) = P_1 - P_2.$$

Since $A_1 v_1 = A_2 v_2$ and $A_2 \ll A_1$, we have $v_2^2 - v_1^2 \approx v_2^2$, so the desired equation follows.

(b) Taking a tube bounded by A_1 and A_2 and apply $F = dp/dt$ to the fluid within it. The pressure at the walls is approximately P_1 everywhere, so the force cancels out except at the area A_2. The net force is

$$F \approx (P_1 - P_2)A_2.$$

On the other hand, the rate of change of momentum is $(dm/dt)v_2 = \rho A_2 v_2^2$, where we again use $v_1 \ll v_2$, giving the result.

(c) Neglecting P_3, Bernoulli's principle gives

$$\frac{1}{2}\rho v_3^2 \approx P_1.$$

To use momentum conservation, apply $F = dp/dt$ to a tube bounded by A_1 and A_3, giving

$$F \approx P_1 A_2, \quad \frac{dp}{dt} = \rho A_3 v_3^2$$

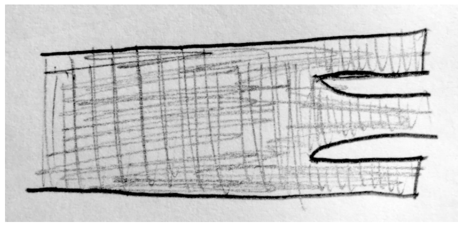
which gives us

$$\frac{A_3}{A_2}\rho v_3^2 \approx P_1.$$

Combining these equations gives $A_3/A_2 \approx 1/2$ as desired.

(d) In reality, the pressure on the right wall is not precisely P_1, but instead slightly lower near the hole because the fluid has velocity there. Hence the net force is actually larger than expected, so $A_3 > A_2/2$.

(e) We can force the net force to be almost exactly $(P_1 - P_2)A_2$ with a "Borda mouthpiece."



On the other hand, the vena contracta effect can be removed by just making a curved opening that follows the path the water would have taken without it, ending at the area A_3. The water comes out of such a hole exactly straight.

Remark

Vena contracta is too subtle for introductory textbooks, but it makes a big difference in the results. For example, if you estimate how long it takes water in a bucket to empty through a hole using Torricelli's law, then you'll be off by about a factor of 2 if you don't include vena contracta! And Halliday, Resnick, and Krane don't consider the vena contracta in their example titled "thrust on a rocket", getting a thrust which is also off by a factor of 2.

Of course, real rocket scientists are well-aware of these effects. They actually follow the fluid flow in detail, improving on the rough estimates made here, and you can read whole books about rocket nozzle design. We'll revisit this subject in **T3**.

4 Fluid Systems

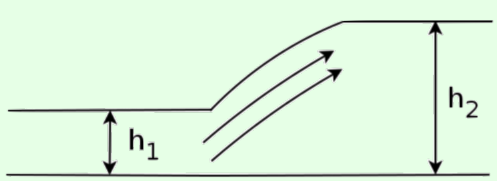
Now we put it all together and consider complex mechanical systems with moving fluids.

Idea 7

If a fluid is moving in a complex way, it's usually difficult to say anything by directly considering the flow. Instead, it's easier to apply conservation laws.

Example 11

A fluid of density ρ flowing with a fast velocity v_1 and height h_1 can undergo a "hydraulic jump", where the height of the fluid increases to h_2. At the same time, the fluid flow slows down and becomes turbulent.



This phenomenon is very common in everyday life. For example, it happens whenever you turn on the water faucet in a sink; the hydraulic jump occurs on a circle centered on the faucet. Find the final height h_2.

Solution

During this process, the bulk kinetic energy of the water is not conserved, because it is converted to turbulent motion. However, the horizontal momentum of the water is approximately conserved. Consider a stream of water of width w flowing in the x direction, where the hydraulic jump occurs at $x = 0$. By mass conservation,

$$v_1 h_1 = v_2 h_2$$

where v_2 is the final speed. Now we consider a fixed subset of the water encompassing the hydraulic jump. The atmospheric pressure does not yield a net horizontal force on the water, so we focus on the pressure in excess of atmospheric pressure. The total excess pressure force on the left end is

$$F_\ell = \int_0^{h_1} \rho g h w \, dh = \frac{1}{2} \rho g w h_1^2.$$

Therefore, we have total force

$$F = \frac{1}{2} \rho g w (h_1^2 - h_2^2).$$

On the other hand, the mass of water that flows through the hydraulic jump per unit time is $\rho h_1 w v_1$, and its velocity decreases by $v_1 - v_2$, so

$$\frac{dp}{dt} = \rho h_1 w v_1 (v_1 - v_2) = \rho w v_1 v_2 (h_2 - h_1)$$

where we used mass conservation. Equating $F = dp/dt$ and simplifying gives

$$g(h_1 + h_2) = 2 v_1 v_2.$$

Applying mass conservation again leads to a quadratic in h_2,

$$h_2^2 + h_1 h_2 - \frac{2 v_1^2 h_1}{g} = 0$$

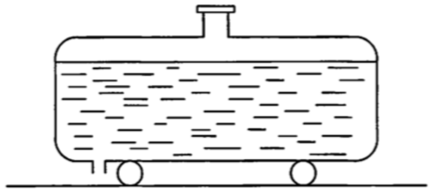
and the physically relevant positive solution is the answer,

$$h_2 = -\frac{h_1}{2} + \sqrt{\frac{h_1^2}{4} + \frac{2 h_1 v_1^2}{g}}.$$

For $v_1^2 > g h_1$, we have $h_2 > h_1$ and an ordinary hydraulic jump. For $v_1^2 < g h_1$, you might expect a "reverse" hydraulic jump to occur, but this is impossible by the second law of thermodynamics. In a hydraulic jump, some of the kinetic energy of laminar flow energy is converted to turbulent flow, which is essentially heat; thus the reverse can't happen. So in addition to deriving h_2, we've found the minimum v_1 for a hydraulic jump to be possible!

Note that this conservation law approach doesn't tell us about how far a fluid will flow before it undergoes a hydraulic jump. That would require understanding the fluid flow in detail, accounting for turbulence and viscosity, which is generally analytically intractable. For more on this subject, see sections 26.1 and 26.2 of Lautrup.

[3] **Problem 20** (PPP 70)**.** A tanker full of liquid is at rest on a frictionless horizontal road.



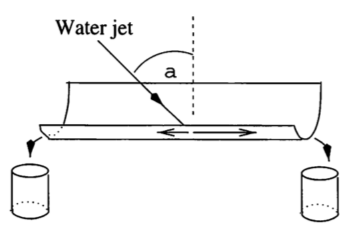
A small vertical outlet pipe at the rear of the tanker is opened. Describe qualitatively how the tanker will move (a) immediately afterward, and (b) after a long time. Assume that the water always falls out of the cart with zero horizontal velocity in the cart's frame.

Solution. (a) The total horizontal momentum of the tanker and liquid is conserved, and is initially zero, so the center of mass of the tanker and liquid cannot move horizontally. When water starts to flow out, it comes out on the tanker's left side. Thus, the tanker has to initially move to the right.

(b) However, it's impossible for the tanker to always move to the right. If that were the case, then after a long time, when the water has all left, both the tanker and all of the water will be moving to the right, violating momentum conservation. Thus, at some point the tanker has to turn around, and its final velocity is to the left. (However, if the draining process was so violent that the water started sloshing around, then the tanker would jerk back and forth, which would screw up the above argument. We're implicitly assuming that the draining process is slow, so that the water still inside the tanker moves with it.)

By the way, it's worth thinking about the forces that make the tanker move. These forces must be due to water pressure. When the water starts flowing towards the drain, it has a higher velocity near the drain, and thus a lower pressure by Bernoulli's principle. Thus, the total pressure force on the tanker's left wall is lower than that on its right wall, which is why the tanker starts moving to the right. On the other hand, when there's only a thin layer of water left, the part of the water to the right of the drain will have a big leftward horizontal velocity. That leftward momentum is transferred to the tanker near the drain, where the water is forced to turn around and fall down vertically; that's why the tanker starts moving to the left near the end. If you're interested in seeing more, there's a complete analysis here which even includes explicit expressions for the tanker's position over time.

[3] **Problem 21** (PPP 74)**.** A jet of water strikes a horizontal gutter of semicircular cross-section obliquely, as shown.



The jet lies in the vertical plane that contains the center-line of the gutter. Assume the angle is relatively shallow, so that the water hits the gutter smoothly, and doesn't splatter. Find the ratio of the quantities of water flowing out at the two ends of the gutter as a function of the angle of incidence α of the jet.

Solution. Let the original water jet have area A_0 and speed v. Let v_1 be the speed of the stream to the right, and let A_1 be its area. Similarly define v_2 and A_2. First, we claim that

$$v = v_1 = v_2.$$

This follows directly from Bernoulli's principle. The incoming jet has atmospheric pressure, because it's exposed to the air, and so do the two streams. Since they have the same pressures, the have the same speeds. (Of course, this wouldn't be true if energy was dissipated. For instance, if the water jet were fast and directed straight down, water would splatter everywhere.)

Next, conservation of mass gives

$$A_0 = A_1 + A_2.$$

Conservation of horizontal momentum gives

$$\rho A v^2 \sin\alpha = \rho A_1 v^2 - \rho A_2 v^2$$

which implies

$$A \sin\alpha = A_1 - A_2.$$

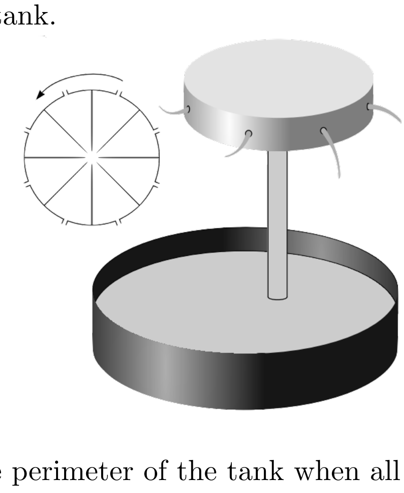
Combining this with mass conservation gives

$$A_1 = \frac{1+\sin\alpha}{2} A, \quad A_2 = \frac{1-\sin\alpha}{2} A.$$

Since the speeds are the same, the ratio of flow rates is just the ratio of areas,

$$\frac{A_1}{A_2} = \frac{1+\sin\alpha}{1-\sin\alpha}.$$

[3] **Problem 22** (EFPhO 2005)**.** A water pump consists of a vertical tube of cross-sectional area S_1 topped with a cylindrical rotating tank of radius r. All the vessels are filled with water; there are holes of total cross-sectional area $S_2 \ll S_1$ along the perimeter of the tank, which are open for the operating regime of the pump. The height of the tank from the water surface of the reservoir is h. An electric engine keeps the vessel rotation at angular velocity ω. The water density is ρ, the air pressure p_0, and the saturated vapor pressure p_k. Inside the tank, there are metal blades, which make the water rotate with the tank.



(a) Find the pressure p_2 at the perimeter of the tank when all the holes are closed.

(b) Now suppose the holes are opened. Find the velocity v_2 of the water jets with respect to the ground.

(c) If the tank rotates too fast, the pump efficiency drops due to cavitation; the water starts "boiling" in some parts of the pump. Find the highest cavitation-free angular speed ω_{max}.

(d) If the power of the electric engine is P, what is the theoretical upper limit of the volume pumped per unit time, assuming S_2 can be freely adjusted?

Solution. See the solution to problem 4 here.

[3] **Problem 23.** A helicopter with length scale ℓ and density ρ_h can hover using power P, in air of density ρ_a. Find a rough estimate for P in terms of the given parameters. (For a nice followup discussion of lift, see section 3.6 of The Art of Insight.)

Solution. Helicopters push themselves upward by pushing air downward. We need

$$\frac{dp}{dt} \sim \rho_h \ell^3 g$$

to support the aircraft, while considering the rate of air pushed downward gives

$$\frac{dp}{dt} \sim \frac{dm}{dt} v \sim \rho_a \ell^2 v^2$$

where v is the velocity of the air. By comparing both sides,

$$v \sim \sqrt{\frac{\rho_h \ell g}{\rho_a}}.$$

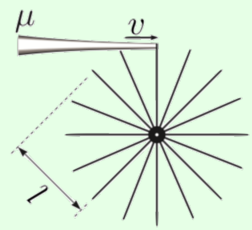
The power needed goes into putting kinetic energy into the air,

$$P \sim \frac{dm}{dt} v^2 \sim \rho_a \ell^2 v^3 \sim \sqrt{\frac{\ell^7 g^3 \rho_h^3}{\rho_a}}.$$

The only thing that might be surprising is the dependence on ρ_a, where more energy is required if the air is thinner. (This is why helicopters have trouble rescuing people from Mount Everest.) The reason is that thinner air needs to be pushed down faster to get the same lift, but this requires more power because energy is quadratic in speed. Also, note that this problem couldn't have been solved by dimensional analysis alone, since two densities were present.

Example 12: Kalda 82

A water turbine consists of a large number of paddles that could be considered as light flat boards with length ℓ, that are at one end attached to a rotating axis. The paddles' free ends are positions on the surface of an imaginary cylinder that is coaxial with the turbine's axis. A stream of water with velocity v and flow rate μ (kg/s) is directed on the turbine such that it only hits the edges of the paddles.



Find the maximum possible power that can be extracted.

Solution

Let v_t be the speed of the edge of the turbine. In time dt, the amount of mass of water that collides with the turbine is
$$dm = \frac{\mu}{v}(v - v_t)\, dt.$$
The horizontal force on the paddle is
$$F = \frac{dp}{dt} = \frac{dm}{dt}\Delta v = \frac{\mu}{v}(v - v_t)^2$$
so the power delivered to the turbine is
$$P = Fv_t = \frac{\mu v_t}{v}(v - v_t)^2.$$
Maximizing this by setting $dP/dv_t = 0$ gives $v_t = v/3$, so the maximum power is $4\mu v^2/27$. This is $8/27$ of the total power in the incoming water.

[3] **Problem 24.** Air of constant density ρ and wind speed v_i is heading directly towards a windmill of area A. When the wind gets to the windmill blades, it is traveling forward with speed v_f. Well after it leaves the vicinity of the blades, it has speed v_o. The design of the windmill, such as the shape and speed at which its blades turn, can be adjusted to set the value of v_f.

(a) Find the power going from the wind to the turbine by using energy conservation, assuming that there are no extraneous energy losses, e.g. to turbulence.

(b) Find the power going from the wind to the turbine by considering the force of the windmill on the air and using momentum conservation, again assuming no extraneous energy losses.

(c) Find an upper bound on the ratio of the wind power that can be harvested by the windmill, to the amount of wind power that would pass through it if it weren't running.

This result is called the Betz limit.

Solution. (a) Since there's nowhere else for the energy to go, the power must be the rate of change of the wind's energy. The mass flow rate is $\mu = \rho A v_f$, so
$$P = \frac{1}{2}\mu(v_i^2 - v_o^2) = \frac{1}{2}\rho A v_f (v_i - v_o)(v_i + v_o).$$

(b) The (negative) power on the wind is Fv_f where F is the rate of change of momentum of the wind. Ideally, all of this power goes to the windmill, so

$$P = Fv_f = \mu(v_i - v_o)v_f = \rho A v_f^2 (v_i - v_o).$$

(c) By comparing these equations, we find $v_f = (v_i + v_o)/2$, which allows us to eliminate v_f. Plugging this back in gives

$$P = \frac{1}{4}\rho A (v_i + v_o)^2 (v_i - v_o)$$

which can then be maximized with respect to v_o. Setting the derivative to zero gives $v_o = v_i/3$ and thus $P = (8/27)\rho A v_i^3$. If the windmill were not running, the rate at which wind energy flows through it is $(\rho A v_i)v_i^2/2$, which means the maximum fraction harvested is 16/27.

[5] **Problem 25.** GPhO 2017, problem 2. A very tricky composite fluids/mechanics problem.

Solution. See the official solutions here.

5 Wet Water

So far we've mostly ignored viscosity and turbulence, an unrealistic limit that some refer to as "dry water". Now we'll consider some problems involving real, wet water.

Idea 8

A velocity gradient is associated with a drag force

$$F = \eta A \frac{dv}{dy}$$

where η is the (dynamic) viscosity. In addition, for fluid flowing next to a wall, the layer of fluid right next to the wall is approximately at rest.

Example 13: HRK

Prairie dogs live in large colonies in complex interconnected burrow systems. They face the problem of maintaining a sufficient air supply to their burrows to avoid suffocation. They avoid this by building conical earth mounds about some of their many burrow openings. How does this air conditioning scheme work?

Solution

Because of viscous effects, the wind speed is small near the ground, and hence grows with height. By Bernoulli's principle, this means the pressure at the top of a mound is slightly lower than the pressure at an opening without a mound. This difference in pressure drives air flow through the burrows.

Example 14

If you've used a standard garden hose, you might have noticed that the water shoots higher if you partially block the outlet with your finger. Why does this happen?

Solution

The water company provides water to your house at a fixed pressure $P_{\text{atm}} + \Delta P$. Thus, naively the water should always shoot equally far, because Bernoulli's principle says the exit speed is $v = \sqrt{2\Delta P/\rho}$, corresponding to a peak height $\Delta P/\rho g$, independent of the area of the hole. (There is a vena contracta effect, as mentioned in problem 19, but this also doesn't depend on the area.)

The resolution is that for a typical long, thin garden hose, viscous losses dominate. As you'll see in problem 26, a higher mass flow rate leads to a higher drop in pressure. When you partially block the outlet, you're simply decreasing the flow rate, so that viscosity has a smaller effect, allowing the water to get closer to the maximum possible height $\Delta P/\rho g$.

In plumbing, the quantity $\Delta P/\rho g$ is called the "pressure head", and effects like viscosity give rise to "head loss". Unfortunately, for most realistic pipes it is intractable to calculate the head loss, because the water flow is turbulent. Instead, the amount of head loss is parametrized by the so-called Darcy friction factor, whose values are tabulated in references.

Example 15

If you stir a cup of coffee, around how long does it take the rotational motion to settle down?

Solution

The rotational motion stops because of viscous drag against the walls. For concreteness, let's suppose the coffee has density ρ, viscosity η, and is in a mug of radius R and height $H \gg R$ (so most of the drag comes from the vertical wall of the mug). The angular momentum is

$$L \sim I\omega \sim \rho R^4 H\omega.$$

The damping torque due to viscous forces is

$$\tau \sim RF \sim \eta A \frac{dv}{dr} R$$

and since the drag is from the vertical wall, $A \sim HR$. Estimating the velocity gradient dv/dr is a little trickier. As mentioned above, the coffee *right* next to the wall has zero velocity, while the coffee slightly inward from the wall has speed $v \sim R\omega$. The velocity transitions between these two values in a thin "boundary layer".

Finding the exact thickness of this boundary layer would require solving complicated differential equations, but it suffices to use dimensional analysis. Note that R and H can't possibly play a role, since the layer is so thin it doesn't "see" the shape of the mug. The fluid properties η and ρ surely matter. Perhaps more subtly, ω matters. If the fluid weren't spinning, but rather were uniformly translating in a plane, then the boundary layer would just grow over time until it was the size of the whole fluid. That's what we saw in problem 26, where the velocity changes gradually along the whole pipe radius R. The boundary layer doesn't grow to the whole mug's size here, because the velocity it's trying to match is constantly changing over the timescale $1/\omega$.

Using dimensional analysis, we thus conclude the boundary layer has thickness

$$\Delta r \sim \sqrt{\frac{\eta}{\rho\omega}}.$$

The damping torque is

$$\tau \sim \eta\,(HR)\frac{R\omega}{\Delta r}R \sim \sqrt{\rho\eta\omega^3}\,HR^3$$

so the timescale for damping is

$$T \sim \frac{L}{\tau} \sim \sqrt{\frac{\rho}{\eta\omega}}\,R.$$

Numerically, if we use the rough estimates

$$\rho \sim 10^3\,\mathrm{kg/m^3}, \quad \omega \sim 10\,\mathrm{s^{-1}}, \quad R \sim 0.1\,\mathrm{m}, \quad \eta \sim 10^{-3}\,\mathrm{Pa\,s}$$

where η is the value for room temperature water, then we get the reasonable results

$$\Delta r \sim 0.3\,\mathrm{mm}, \quad T \sim 30\,\mathrm{s}.$$

[3] **Problem 26.** Water flows through a cylindrical pipe of radius R and length $L \gg R$, across which a pressure difference Δp is applied.

(a) If the flow is slow, viscous effects dominate. By balancing forces on a cylinder of fluid, show that

$$v(r) = \frac{\Delta p}{4\eta L}(R^2 - r^2).$$

Then show that the total mass flux is

$$\frac{dm}{dt} = \frac{\rho\pi R^4 \Delta p}{8\eta L}.$$

This is called Poiseuille's law.

(b) If the flow is very fast, the flow is turbulent. Viscous effects are negligible, and the work done by the pressure difference is dissipated by turbulence into internal energy. Find a rough estimate of the mass flow rate.

Solution. (a) We see $-\eta(2\pi r L)dv/dr = \pi r^2 \Delta p$, so

$$dv/dr = -\frac{\Delta p}{2\eta L}r.$$

Integrating and using the fact that $v(R) = 0$ yields the desired result. Now, the mass flux is

$$\begin{aligned} dm/dt &= \int_0^R (2\pi r dr)\rho v(r) \\ &= 2\pi\rho \frac{\Delta p}{4\eta L} \int_0^R (R^2 - r^2) r\, dr \\ &= \frac{\rho\pi R^4 \Delta p}{8\eta L} \end{aligned}$$

as desired.

(b) We perform dimensional analysis, leaving μ out because viscosity is negligible. The parameters of the problem are Δp, L, R, and ρ, which is one more than the number of independent dimensions. However, we can note that the flow rate ought to stay the same if we connect two identical pipes in series, with the same pressure drop Δp across each one. This implies that the mass flow rate only depends on the ratio $\Delta p/L$, i.e. the pressure gradient. Carrying out dimensional analysis as usual gives

$$\frac{dm}{dt} \propto \sqrt{\frac{\rho R^5 \Delta p}{L}}.$$

[4] **Problem 27.** When a spherical object of radius R moves with velocity v through a fluid of viscosity η and density ρ, it experiences a drag force.

(a) Apply dimensional analysis to constrain the possible forms of the drag force F. You should find there is one dimensionless quantity inversely proportional to η, in accordance with the Buckingham Pi theorem of **P1**. This dimensionless quantity is called the Reynolds number, and it determines what kind of drag dominates.

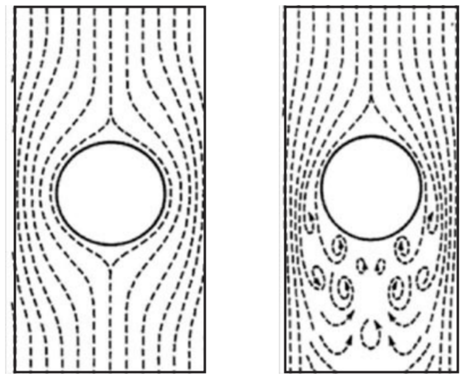
(b) It turns out that $F \propto v$ at low velocities and $F \propto v^2$ at high velocities. Using this information, find the form of the drag force in both cases. (For reference below: the answers are

$$F = 6\pi\eta R v, \quad F = \frac{1}{2} C_d \rho A v^2$$

where C_d is a dimensionless drag coefficient, which is about $1/2$ for a sphere. The drag coefficient depends strongly on the shape of the object, being much smaller for streamlined shapes, and weakly on the velocity.)

(c) Hot water has density $\rho = 10^3\,\text{kg/m}^3$ and viscosity $\eta = 0.3 \times 10^{-3}\,\text{Pa s}$. (Room temperature water has about 3 times the viscosity.) For an object of radius $1\,\text{cm}$, find the characteristic velocity that divides the two types of drag.

(d) The two cases correspond to flow patterns as shown below.



In the latter case, a region of turbulent flow is created. Using this picture, explain why the drag force is proportional to v^2.

(e) The results above apply to both liquids and gases. In a gas, the relevant quantities are the mass m of the gas molecules, their typical speed u, their number density n, and radius r (which determines how often their collide with each other). Use dimensional analysis to constrain the possible forms of the viscosity η. How do you think η scales with n?

Drag is nicely discussed throughout The Art of Insight; see sections 3.5, 5.3.2, and 8.3.1.2.

Solution. (a) By running a standard dimensional analysis, we find the most general expression with the right dimensions of force is

$$F = \eta R v f\left(\frac{\rho R v}{\eta}\right).$$

In accordance with the Buckingham Pi theorem of **P1**, we can't pin down the answer exactly; we can only determine it up to an unknown function of $\mathrm{Re} = \rho R v/\eta$, the unique dimensionless quantity in the problem. This quantity is called the Reynolds number; when it is low, viscosity dominates.

(b) At low velocities, viscosity dominates, so we are in the low Reynolds number regime. The fact that $F \propto v$ in this regime means that the function f must approach a constant,

$$\lim_{x \to 0} f(x) = c_1.$$

This means we now know how the force depends on all variables,

$$F \propto \eta R v.$$

At high velocities, we have a high Reynolds number. To get a force proportional $F \propto v^2$, we must have

$$\lim_{x \to \infty} f(x) = c_2 x.$$

Therefore we have

$$F \propto \rho R^2 v^2.$$

This is an illustration of how dimensional analysis plus a few limiting cases lets us solve a tricky problem. For intermediate velocities, of course, we would need to know the form of $f(x)$, which is quite complicated and in practice is found from simulations or experiments.

(c) One way of doing this is by noting that the characteristic velocity is when the forces are of the same order,

$$6\pi\eta Rv = \frac{1}{2}C_d\rho Av^2 \approx \frac{1}{4}\pi\rho R^2v^2$$

which gives

$$v = \frac{24\eta}{\rho R} = 7.2\times 10^{-4}\,\mathrm{m/s}$$

for hot water.

Since the Reynolds number Re is the only dimensionless quantity in the problem, the crossover must correspond to some value for Re. Our rough estimate above corresponds to taking $\mathrm{Re} = 24$. (In reality, the crossover happens at $\mathrm{Re} \sim 10^3$, but unfortunately there's no easy way to deduce this from first principles; it was measured, not calculated.)

(d) In the ball's frame, the average vertical velocity of the water decreases significantly behind the ball (let's say to 0) due to the turbulent flow. Thus, by momentum conservation, the drag force on the ball is $F = dp/dt \sim v(dm/dt) \sim v(\rho Av) \propto v^2$.

(e) A dimensionless quantity that we can find is nr^3. Since $[\eta] = \mathrm{kg/m\ s}$, with m and u being the only variables with mass and time respectively, we have $\eta \propto mu$. To make the length dimensions work, we can divide by r^2, and hence

$$\eta \sim \frac{mu}{r^2}g(nr^3)$$

where g is another general function.

We can't say too much about g in general, but we can understand the limit of a very sparse gas. At a molecular level, viscosity in such a gas is due to individual gas molecules colliding with the object and carrying away some of the momentum. Doubling n doubles the rate of collisions, and hence doubles the viscosity. So for a very sparse gas, we should have $\eta \propto n$, which means g is linear, so $\eta \propto munr$. You'll derive this properly in **T1**, along with the result for a non-sparse gas.

Remark

Without knowing the answer to part (b) above, one might expect that the drag force can depend on η, ρ, v, and the shape of the object. In the linear case, the drag force does not depend on ρ. In the quadratic case, the drag force does not depend on η.

These differences can be understood by thinking of where the energy dissipated is going. In the quadratic case, the fluid picks up macroscopic kinetic energy, in the form of a turbulent flow pattern, which is why the drag force does not depend on η. In the linear case, the fluid slows smoothly and hence does not pick up any macroscopic energy; instead the energy is dissipated as heat. Since the macroscopic kinetic energy is not involved, the drag force does not depend on ρ. (Of course, in the quadratic case the turbulent motion eventually stops; at this point it has been converted to heat. The time it takes this to happen is set by η, but it occurs well after the object has passed by and hence does not affect the drag force.)

Example 16

If raindrops fall, why don't clouds fall?

Solution

This isn't a stupid question! It's actually a tough one, which stumped the ancient Greeks and Romans. To give context, we'll cover a bit of atmospheric physics, a topic we will continue in **T1** and **T3**. This is all a bit of a simplification of an interesting story, told in more detail in chapter II-9 of the Feynman lectures.

First, it's useful to review the water cycle. Sunlight directly warms up the ground, and the ground thereby warms the air near the ground. Since warmer air at the same pressure is less dense, it begins to rise by convection. This air also expands roughly adiabatically as it rises, lowering its temperature. Warmer air can also hold more water, so if the original air was moist, water vapor will condense into droplets as the air rises. (This last point is important, because the condensation releases energy, partially counteracting the cooling of the rising air. This keeps it warmer and hence lighter than its surroundings, allowing it to continue to rise.)

Humid air is nearly transparent. However, when water molecules join into small droplets (of order $n \lesssim 100$ molecules), then the amount of electromagnetic radiation scattered by the droplets grows as n^2, as long as the droplet is roughly smaller than the wavelength of the light. (We will justify this in more detail in **E7**.) Therefore, there is an overall enhancement of scattering per water molecule by a factor of n, which is why clouds *aren't* transparent.

Now consider a droplet of radius r. Depending on the droplet size and velocity, the drag force scales as r or r^2, while the gravitational force scales as r^3. Hence the tiny water droplets in clouds are hence carried upward with the ascending moist air, since the drag force dominates. They fall down once they accrete into sufficiently large raindrops, where gravity dominates.

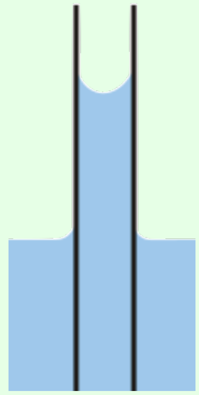
Incidentally, raindrops do not have the teardrop shape shown in typical illustrations. Small raindrops are nearly spherical, because of surface tension. Large raindrops are squashed by air resistance into a "hamburger" shape.

6 Surface Tension

We now return to surface tension, first covered in **M2**, which we'll see yet again in **T3**.

Example 17

A very thin, hollow glass tube of radius r is dipped vertically inside a container of water.



Find the height to which water can climb in the tube.

Solution

In **M2**, we considered problems that could be solved knowing only the "surface tension of water" γ, which is the energy cost per unit area of having a water-air interface. But in this problem there is also a water-glass interface, and the answer to the question depends on precisely how water and glass interact. Specifically, you need to know the surface tension coefficient γ_{wg} which determines the energy cost of having a water-glass interface.

Fortunately, it turns out you don't need to know γ_{wg} if you know the contact angle θ, i.e. the angle between the glass and water surface at the top of the meniscus, which is drawn as acute in the diagram above. We'll just treat θ as a given, but for an explanation of how θ is determined, see **T3** or section 5.5 of Lautrup.

Since the glass tube is very thin, surface tension determines the shape of the water-air surface, so it is spherical since spheres minimize area. By some elementary geometry, one can show that the radius of curvature of this sphere is $R = r/\cos\theta$.

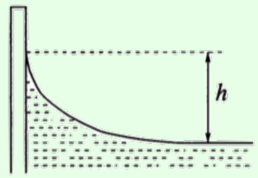
We showed using force balance arguments in **M2** that the pressure inside the curved water surface is lower than atmospheric pressure by $\Delta P = 2\gamma/R$. On the other hand, we also know from Pascal's principle that $\Delta P = \rho g h$. Equating the two gives

$$h = \frac{2\gamma\cos\theta}{\rho g r}.$$

This is Jurin's law. Ideally, water and glass have zero contact angle. This implies that water perfectly wets glass, i.e. that a droplet of water placed on a horizontal glass surface will spread to cover it completely (though this doesn't happen in reality because glass tends to quickly get coated in a layer of impurities). Making this assumption, which we will use for problems below, the answer reduces to $h = 2\gamma/\rho g r$.

Example 18: PPP 130

Water in a glass beaker forms a meniscus, as shown below.



Find the height h to which the meniscus rises above the flat water surface.

Solution

We consider all of the external horizontal forces acting on the water. The surface tension force acting at the top of the meniscus is purely vertical, because water and glass have zero contact angle. The other surface tension force acting on the flat part of the water is γ per length. This balances the excess hydrostatic pressure (i.e. the pressure above atmospheric pressure) at the wall, which is $\rho g h^2/2$ per unit length. Thus,

$$h = \sqrt{\frac{2\gamma}{\rho g}}.$$

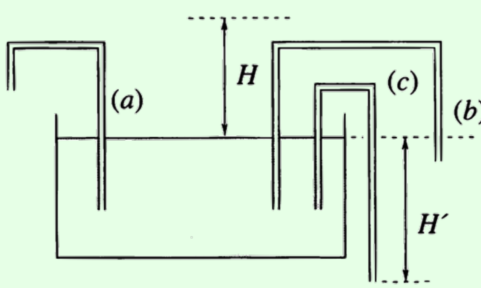
We could have also gotten this with dimensional analysis, up to the prefactor.

Remark

You might be wondering how to compute the shape of the meniscus. There are two methods. First, the pressure right above the water surface is P_{atm}, so the pressure right below the water surface can be determined from the radii of curvature of the surface, using the Young–Laplace equation from **M2**. This pressure can also be computed from the height of the surface using Pascal's principle. Combining these two yields a differential equation for the shape with a rather complicated solution, as explained in sections 5.6 and 5.7 of Lautrup. As you'll see in problem 32, you can also derive this result by considering force balance on the water.

Example 19: PPP 29

Water can rise to a height H in a certain capillary tube. Three "gallows" are made from this tubing by bending it, and placed into a tank of water.



Note that $H' > H$. For which tubes, if any, does water flow out of the other end?

Solution

Clearly no water can fall out of (a), because this would produce a perpetual motion machine. The gallows (b) and (c) are a bit more subtle. Water will *not* fall out of a capillary tube if its end is less than a height H below the free water surface; this follows from the same derivation as Jurin's law, with the surface tension acting to hold the water in the tube. So water only falls out of (c).

[2] **Problem 28** (Russia 2006)**.** A soap bubble of radius R and surface tension γ has a small tube of radius $r \ll R$ passing through its surface. If the air has density ρ, find the rate of decrease of R.

Solution. By Bernoulli's principle, the flow rate is

$$v = \sqrt{\frac{2\Delta p}{\rho}} = \sqrt{\frac{8\sigma}{\rho R}}$$

where we used a result from **M2**. By mass conservation,

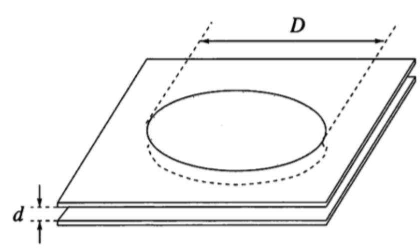
$$\left|\frac{dR}{dt}\right|(4\pi R^2) = (\pi r^2)v$$

from which we conclude

$$\frac{dR}{dt} = -\frac{r^2}{R^2}\sqrt{\frac{\gamma}{2\rho R}}.$$

Technically, the true answer is a bit different because the air inside the bubble is under a higher pressure, and so slightly denser. But $\Delta p \ll p_{\text{atm}}$ for any bubble you can reasonably make, so this isn't a significant source of error.

[2] **Problem 29** (PPP 63)**.** Water is stuck between two parallel glass plates. The distance between the plates is d, and the diameter of the trapped water disc is $D \gg d$.



In terms of the surface tension γ of water, what is the force acting between the two plates? This effect can cause wet glass plates to stick together.

Solution. If you imagine slicing the puddle of water along a diameter, then its boundaries with the air are arcs of circles, since this minimizes the surface area. Since water perfectly wets glass, these circles are tangent to the two glass plates, which mean they have radius of curvature $d/2$. In addition, the surface of the water has radius of curvature $D/2$ in the orthogonal direction. Thus, by the Young–Laplace equation,

$$\Delta P = \gamma\left(\frac{2}{D} - \frac{2}{d}\right) \approx -\frac{2\gamma}{d}.$$

This lowered pressure inside the water puddle causes a "suction" force between the two plates, of magnitude

$$F = |\Delta P| A = \pi (D/2)^2 \frac{2\gamma}{d} = \frac{\gamma \pi D^2}{2d}.$$

[3] **Problem 30** (EFPhO 2009)**.** A soap film of thickness $h = 1\,\mu$m is formed inside a ring of diameter $D = 10\,$cm, and the surface tension of the film is $\gamma = 0.025\,$N/m. If the film is broken at the center, it will begin to fall apart; estimate the time needed for this to happen.

Solution. Like the helicopter question, this can't be solved with pure dimensional analysis, because there are four quantities (h, D, γ, and the density ρ). Instead, we need to think about the dynamics. The edge of the break will expand outward, pulled by surface tension. This competes with the inertia of the film itself, and the inertia per area only depends on the combination ρh. Thus, we can perform dimensional analysis on the combinations ρh, D, and γ, giving

$$t \sim \sqrt{\frac{\rho h}{\gamma}} D \sim 0.02\,\text{s}.$$

This is good enough for an estimate, but for completeness, we present a more precise solution below.

Assume the film is broken at the center, so the edge of the break will be an expanding circle of radius r. The surface tension will provide a force of $4\pi r \gamma$ outwards, pulling on the mass that was originally inside the circle of $m = \rho \pi r^2 h$. Thus, by considering forces along the radial direction (i.e. treating r as a generalized coordinate in the spirit of **M4**), we have $F_r = dp_r/dt$, or

$$4\pi r \gamma = \frac{dm}{dt} v + m \frac{dv}{dt} = 2\pi r v^2 \rho h + \pi r^2 \rho h \frac{dv}{dt}.$$

Cleaning this up a bit, we have

$$v_0^2 = v^2 + \frac{r}{2}\frac{dv}{dt}, \quad v_0 = \sqrt{\frac{2\gamma}{\rho h}}.$$

This equation tells us that the speed of the break quickly approaches v_0 when r is small. Our result for v_0 is called the Taylor–Culick formula; you can see the constant speed in action in slow-motion videos. Thus, the total time taken is

$$t \approx \frac{D/2}{v_0} = \sqrt{\frac{\rho h}{8\gamma}} D \sim 0.01\,\text{s}$$

where we used $\rho \approx 10^3\,\text{kg/m}^3$, since soap films are mostly water.

If you want to be even more precise, we can also solve the differential equation exactly. We can get rid of the t-dependence entirely by writing $dv/dt = (dv/dr)(dr/dt) = v\,dv/dr$, giving

$$v_0^2 = v^2 + \frac{rv}{2}\frac{dv}{dr}.$$

This is now a separable, first-order differential equation which can be integrated to find $v(r)$,

$$\int \frac{dr}{r} = \int \frac{dv}{2} \frac{v}{v_0^2 - v^2}.$$

The broken part starts with $v = 0$ and small radius r_0. Then, integrating and simplifying gives

$$v(r) = v_0\sqrt{1 - (r_0/r)^4}$$

which indicates that once r becomes larger than the tiny value r_0, the velocity rapidly approaches v_0, as stated above. You can go a step further, integrating to find $r(t)$, but the result is a hypergeometric function, which isn't very enlightening.

[3] **Problem 31.** USAPhO 2020, problem B1. A nice, slightly mathematically involved surface tension problem with a real-world impact. This setup is discussed in detail in section 5.4 of Lautrup.

[4] **Problem 32.** IPhO 2023, problem 3, parts B and C. A nice problem on the shape of a meniscus, which also explains why pieces of cereal attract each other in a bowl of milk.

Example 20: IPhO 2022 3B

Slightly wet sand is much stronger than either dry sand or very wet sand, which allows the construction of large structures like sand castles. Why is this, and how does the strength depend on the typical size r of the sand grains?

Solution

When a pile of sand is dry, the only force keeping it in place is friction, which is weak. When it's very wet, it's essentially just water, which will simply collapse. But when it's slightly wet, adjacent sand grains have a small layer of water connecting them. Since sand grains are small, this implies a huge total surface area, and thus large surface tension effects.

There are actually two conceptually distinct components to the effect. First, the bit of water connecting two sand grains will provide a surface tension force $F \sim \gamma r$. Second, as you saw in problem 29, the water has a pressure lower by $\Delta P \sim \gamma/r$, leading to an attractive pressure force $(\Delta P)A \sim \gamma r$. In either case, that means the force needed to displace a single grain of sand scales with r. The number of sand grains in a fixed cross-sectional area scales as $1/r^2$, so the weight a sand castle can bear scales as $1/r$. Thus, fine-grained sand is stronger.

This is another example of the subtleties of granular media, first mentioned in **M2**. Neither sand nor water are strong on their own, but they're strong together. Water provides the forces, while the sand provide the structure which lets those forces be effective.

Mechanics VIII: Synthesis

Three-dimensional rotation is covered in chapter 7 of Kleppner and chapter 9 of Morin. For further discussion and examples, see chapter I-20 of the Feynman lectures, and this awesome video. There is a total of **101** points.

1 3D Rotation

In **M6**, we considered mostly two-dimensional rotation. Now we will tackle the full three-dimensional case, starting with the general description of rigid body motion.

Idea 1: Chasles' Theorem

The instantaneous velocity of a three-dimensional rigid body can always be decomposed in one of two ways. First, for any given point, it can be written in terms of a translational velocity plus a pure rotation about an axis going through that point. In practice, this point is almost always chosen to be the center of mass, giving the decomposition

$$\mathbf{v} = \mathbf{v}_{\text{CM}} + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\text{CM}}).$$

Alternatively, there always exists an axis so that the motion can be written as rotation about that axis, plus a translational velocity *parallel* to the axis, giving

$$\mathbf{v} = \mathbf{v}_0 + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0)$$

where $\mathbf{v}_0$ and $\boldsymbol{\omega}$ are parallel. (This is known as a "screw" motion.) These two decompositions are analogous to the two we saw in **M5**, though in the three-dimensional case the first tends to be much more useful.

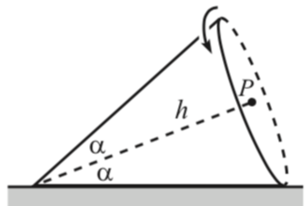
Example 1

You have a small globe, which is mounted so that it can spin on the polar axis and can be spun about a horizontal axis (so that the south pole can be on top). Give the globe a quick spin about the polar axis, and then, before it stops, give it another quick spin about the horizontal axis. Are there any points on the globe that are at rest?

Solution

The first spin gives the angular velocity a vertical component $\boldsymbol{\omega}_1$. The second spin gives the angular velocity an additional horizontal component $\boldsymbol{\omega}_2$. The globe now rotates about its center of mass with angular velocity $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. Precisely two points on the globe are stationary, namely the points that are parallel and antiparallel to this vector.

[2] **Problem 1** (Morin 9.3)**.** A cone rolls without slipping on a table; this means that all the points of the cone that touch the table are instantaneously stationary. The half-angle of the vertex is α, and the axis has length h.



Let the speed of the center of the base, point P in the figure, be v.

(a) Compute the angular velocity $\boldsymbol{\omega}$ by thinking of the motion as pure rotation about some axis.

(b) Compute the angular velocity $\boldsymbol{\omega}$ by thinking of the motion as translation of P, plus rotation about an axis passing through P.

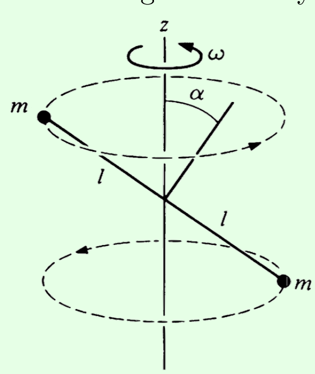
(c) The apex of the cone is fixed, and the cone continues to rotate. As this motion goes on, the angular velocity vector rotates uniformly, keeping a constant magnitude. Find the angular velocity $\boldsymbol{\Omega}$ of the angular velocity vector.

In the precession problems below, it's important to keep track of the difference between $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$.

Most of our statements about rotational dynamics from **M6** remain true. The main new aspect is that angular momentum is not necessarily parallel to angular velocity.

Example 2: KK Example 7.4

Consider a rigid body consisting of two particles of mass m connected by a massless rod of length 2ℓ, rotating about the z-axis with angular velocity ω as shown.



Find the angular momentum of the system.

Solution

We simply add $\mathbf{r} \times \mathbf{p}$ for both masses. Let the rod lie in the xz plane at this moment. Then for the top left mass,

$$\mathbf{r} = -\ell \cos\alpha\,\hat{\mathbf{x}} + \ell \sin\alpha\,\hat{\mathbf{z}}.$$

The momentum is

$$\mathbf{p} = m\mathbf{v} = m\boldsymbol{\omega} \times \mathbf{r} = -m\omega\ell\cos\alpha\,\hat{\mathbf{y}}.$$

Then the angular momentum is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\omega\ell^2 \cos\alpha\,(\sin\alpha\,\hat{\mathbf{x}} + \cos\alpha\,\hat{\mathbf{z}})\,.$$

The other mass has the opposite $\mathbf{r}$ and $\mathbf{p}$ and hence the same $\mathbf{L}$, so the total angular momentum is

$$\mathbf{L} = 2m\omega\ell^2 \cos\alpha\,(\sin\alpha\,\hat{\mathbf{x}} + \cos\alpha\,\hat{\mathbf{z}})\,.$$

It is directed perpendicular to the rod, and in particular, it isn't parallel to the angular velocity!

Here is another way to derive the same result. We can decompose the angular velocity vector into a component along the rod, and a component perpendicular to the rod. The former contributes no angular momentum, because rotating about the rod's axis doesn't move the masses. The latter contributes all the angular momentum. So the angular momentum is

$$L = I_\perp \omega_\perp = (2m\ell^2)(\omega\cos\alpha)$$

directed perpendicular to the rod, which is what we just saw explicitly.

We can summarize the lessons drawn from this example as follows.

Idea 2

For a three-dimensional object, $\mathbf{L}$ is not necessarily parallel to $\boldsymbol{\omega}$. In general, for pure rotation about an axis passing through the origin, we have $\mathbf{L} = I\boldsymbol{\omega}$ where I is a 3×3 matrix called the "moment of inertia tensor about the origin". In components, this means that

$$L_i = \sum_j I_{ij}\omega_j.$$

While this is simple and general, the I_{ij} are a pain to calculate. You can learn more in the reading, but to my knowledge, no Olympiad problem has ever required computing a general moment of inertia tensor.

For the purposes of Olympiad problems, there is a better way to think about the angular momentum. We use the second decomposition of idea 1, and think of the motion as translation plus rotation about the center of mass. If the object has an axis of symmetry, which it will in almost all Olympiad problems, then the angular velocity can then be decomposed into a component parallel to the axis, and perpendicular to the axis,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\parallel + \boldsymbol{\omega}_\perp.$$

The key is that, in such situations, the spin angular momentum has two pieces, which are each parallel to the corresponding piece of the angular velocity,

$$\mathbf{L}_\parallel = I_\parallel \boldsymbol{\omega}_\parallel, \quad \mathbf{L}_\perp = I_\perp \boldsymbol{\omega}_\perp$$

where $I_\parallel$ and $I_\perp$ are ordinary moments of inertia about the center of mass. For example, for a flat uniform disc, $I_\parallel = MR^2/2$ and $I_\perp = MR^2/4$.

The total angular momentum about the origin is then

$$\mathbf{L} = \mathbf{r}_{\text{CM}} \times M\mathbf{v}_{\text{CM}} + I_\parallel \boldsymbol{\omega}_\parallel + I_\perp \boldsymbol{\omega}_\perp$$

where the first term is from the motion of the center of mass, and the next two are from rotation about the center of mass. Note that this is exactly the same as what we saw in **M5**, except that the "spin" angular momentum is broken into two parts.

Idea 3

Sometimes it can be hard to visualize $\boldsymbol{\omega}$, so here are two tricks. First, if any two points on the object are stationary, then $\boldsymbol{\omega}$ must be parallel to the axis connecting the two points. Second, if the rotation is complicated, one can use rotating frames to simplify the analysis. If a body has angular velocity $\boldsymbol{\omega}_1$ in one frame, and that frame has an angular velocity $\boldsymbol{\omega}_2$ with respect to a second frame, then the body has angular velocity $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ with respect to the second frame.

Idea 4

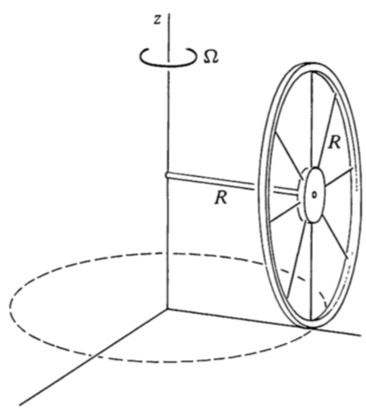
The rate of change of angular momentum is

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}$$

where the torque $\boldsymbol{\tau}$ is defined as in **M5**. The kinetic energy is

$$K = \frac{1}{2}Mv_{\text{CM}}^2 + \frac{1}{2}I_{\parallel}\omega_{\parallel}^2 + \frac{1}{2}I_{\perp}\omega_{\perp}^2.$$

[3] **Problem 2** (KK 7.1). A thin hoop of mass M and radius R rolls without slipping about the z-axis. It is supported by an axle of length R through its center, as shown.



The axle circles around the z-axis with angular speed Ω, so that the bottom point of the wheel traces out a circle of radius R. Let O be the pivot point of the rod, i.e. the point where the rod meets the z-axis.

(a) Find the instantaneous angular velocity $\boldsymbol{\omega}$ of the hoop.

(b) As the motion continues, the angular velocity vector rotates in a circle. Find the angular velocity of the angular velocity vector of the hoop.

(c) Find the instantaneous angular momentum $\mathbf{L}$ of the hoop, about the point O.

(d) Find the instantaneous torque on the hoop about the contact point with the ground, and verify that $\boldsymbol{\tau} = d\mathbf{L}/dt$.

This is a foundational question, worth thoroughly understanding before trying the rest of the section.

Finally, we need just a few more facts to get going with dynamics problems.

Idea 5

In any dynamics problem, there are many choices you can make in the setup. For example, if you're using an inertial frame, you need to choose where the origin is; usually it's best to place it along the axis of symmetry if possible. You are also free to use a noninertial frame with acceleration $\mathbf{a}$. The only difference is that there will be a fictitious force $-M\mathbf{a}$ acting at the center of mass. For that reason, it's usually best to have the accelerating frame follow the center of mass, keeping it at its origin, so no new torques are introduced.

However, you should avoid *rotating* reference frames for dynamics problems. Not only will there be position-dependent Coriolis forces, but they'll add up and contribute a Coriolis *torque*, which is a pain to calculate, as you saw in **M6**. In general, rotating frames are only good for getting a handle on the kinematics, as mentioned in idea 3.

Example 3: KK Example 7.5

Calculate the magnitude of the torque on the rod in example 2.

Solution

We recall that the angular momentum was

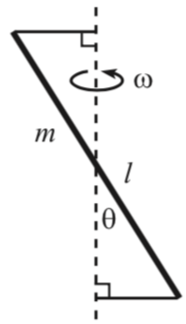
$$\mathbf{L} = 2m\omega\ell^2 \cos\alpha\,(\sin\alpha\,\hat{\mathbf{x}} + \cos\alpha\,\hat{\mathbf{z}}).$$

The rod as a whole rotates with angular velocity $\omega\hat{\mathbf{z}}$. In particular, the angular momentum vector rotates with this angular velocity as well; its horizontal component moves in a circle with angular velocity ω. Then

$$|\boldsymbol{\tau}| = \left|\frac{d\mathbf{L}}{dt}\right| = \omega L_x = 2m\omega^2\ell^2\,\cos\alpha\sin\alpha = m\omega^2\ell^2\sin(2\alpha).$$

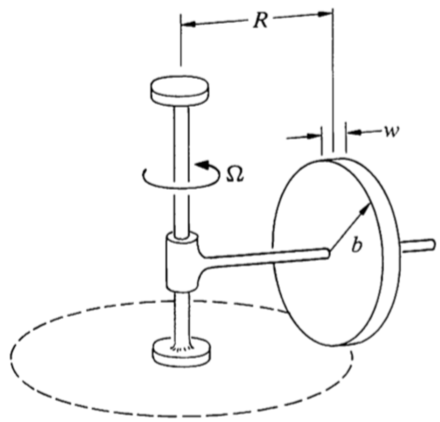
It might be surprising that there needs to be a torque given that $\boldsymbol{\omega}$ is constant, but that's just because $\mathbf{L}$ and $\boldsymbol{\omega}$ aren't necessarily parallel. Conversely, there can be situations where there is no torque, yet $\boldsymbol{\omega}$ changes over time.

[2] **Problem 3** (Morin 9.10)**.** A stick of mass m and length ℓ spins with angular frequency ω around an axis in zero gravity, as shown.



The stick makes an angle θ with the axis and is kept in its motion by two strings that are perpendicular to the axis. Find the tension in the strings.

[2] **Problem 4** (KK 7.4). In an old-fashioned rolling mill, grain is ground by a disk-shaped millstone which rolls in a circle on a flat surface driven by a vertical shaft. Because of the stone's angular momentum, the contact force with the surface can be greater than the weight of the wheel.



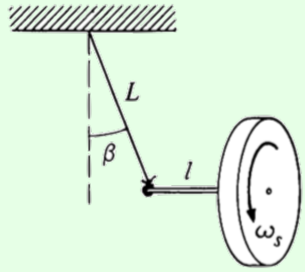
Assume the millstone is a uniform disk of mass M, radius b, and width w, and it rolls without slipping in a circle of radius R with angular velocity Ω. Find the contact force. Assume the millstone is closely fitted to the axle so that it cannot tip, and $w \ll R$. Neglect friction.

[2] **Problem 5** (Morin 9.29). A uniform ball rolls without slipping on a table. It rolls onto a piece of paper, which you then slide around in an arbitrary (horizontal) manner. You may even give the paper abrupt, jerky motions, so that the ball slips with respect to it. After you allow the ball to come off the paper, it will eventually resume rolling without slipping on the table. Show that the final velocity equals the initial velocity. (Hint: this remarkably simple result is because of a conservation law. We saw a lower-dimensional version of this problem in **M5**.)

When a system such as a gyroscope is given a high angular momentum, it can exhibit precession. In this case, the angular velocity $\boldsymbol{\omega}$ precesses (i.e. rotates) with a small angular velocity $\boldsymbol{\Omega}$. Precession is a famously counterintuitive phenomenon, but it can be handled using just the same principles we've laid out above.

Example 4: KK 7.3

A gyroscope wheel is at one end of an axle of length ℓ. The other end of the axle is suspended from a string of length L.



The wheel is set into motion so that it executes slow, uniform precession in the horizontal plane. The wheel has mass M and moment of inertia I_0 about its center of mass, and turns with angular speed ω_s. Neglect the mass of the shaft and string. Find the angle β the string makes with the vertical, assuming β is very small.

Solution

Let T be the tension in the rope. The entire system precesses with associated angular velocity $\Omega\hat{\mathbf{z}}$. (That is, this is the angular velocity of the angular velocity and angular momentum vectors.) Since the center of mass does not accelerate vertically, and the center of mass moves in a horizontal circle,

$$T\cos\beta = Mg, \quad T\sin\beta = M\Omega^2(\ell + L\sin\beta).$$

We'll work to lowest possible order in β everywhere, which means approximating $\cos\beta \approx 1$ and ignoring the $L\sin\beta$ term, giving

$$T = Mg, \quad T\beta = M\Omega^2\ell.$$

Combining these equations gives the precession angular frequency

$$\Omega = \sqrt{\frac{g\beta}{\ell}}.$$

This is as far as we can go with forces alone.

Now we use $\boldsymbol{\tau} = d\mathbf{L}/dt$. First we need to determine the angular velocity of the wheel. Note that if we went in the rotating frame with angular velocity $\Omega\hat{\mathbf{z}}$, the wheel would just spin with angular speed ω_s in place. So the angular velocity vector in the original frame is

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} + \omega_s\hat{\mathbf{x}}$$

where $\hat{\mathbf{x}}$ is directed along the rod. The center of mass of the wheel moves with speed $(\ell + L\sin\beta)\Omega$ in a horizontal circle.

Since the precession is assumed to be slow, Ω is much smaller than ω_s. Thus, we can simply ignore contributions to the angular momentum proportional to Ω. That is, we can take

$$\boldsymbol{\omega} \approx \omega_s\hat{\mathbf{x}}$$

for the purposes of computing angular momentum, giving

$$\mathbf{L} \approx I_0 \omega_s \hat{\mathbf{x}}.$$

This rotates in a horizontal circle with angular speed Ω, so

$$|\boldsymbol{\tau}| \approx I_0 \Omega \omega_s.$$

The torques in the relevant direction come from gravity and the vertical component of the tension force, $|\boldsymbol{\tau}| = Mg\ell$. Equating these, we have

$$Mg\ell = I_0 \Omega \omega_s.$$

Plugging the result for Ω in above and solving for β gives

$$\beta = \frac{m^2 g \ell^3}{\omega^2 I_0^2}.$$

Note that assuming Ω was small kept the equations simple. This is what Kleppner calls the gyroscope approximation. It can be applied in some, but not all, of the below problems.

Remark

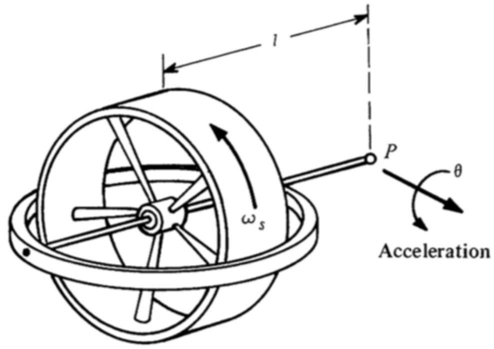
In most gyroscope problems, such as the one above, we assume the motion has reached a steady state, but you might wonder just how it gets started in the first place. For example, suppose we had the same setup as the previous problem, with the wheel spinning and the axle horizontal. For simplicity, let's get rid of the string and suppose the end of the axle is held at a fixed support. Now suppose the axle and wheel are released with no translational motion.

The following chain of events ensues:

- Of course, the axle starts to tip downward because of the weight of the wheel. (Rotational mechanics is counterintuitive, but not *that* counterintuitive!)
- This produces a downward component of angular momentum, which is balanced by the axle/wheel system twisting about its center of mass, rotating slightly about the z-axis.
- This twist tries to push the end of the axle out of the page, causing the support to exert a force on the axle pointing into the page. That force propagates down the axle as an internal shear stress, eventually causing the center of mass of the wheel to start moving into the page, starting the precession.
- In reality, the process overshoots and overcorrects, leading to oscillations called nutation on top of the precession. (For more details, see Note 2 of chapter 7 of Kleppner.)
- For a typical pivot, energy can be dissipated at the pivot point, but the angular momentum of the system stays roughly the same. Assuming this is the case, the oscillations will eventually damp away, leaving the uniform precession.

Notice that in this example, the initial angular momentum is perfectly horizontal. The final angular momentum includes an upward component due to the uniform precession, which implies that the axle must tilt slightly downward, by an angle of order $(\omega/\Omega)^2$. Therefore, if you want to set up uniform precession with the axle perfectly horizontal, as in the above example, you should point the axle slightly upward when releasing it from rest.

[2] **Problem 6** (KK 8.5). An "integrating gyro" can be used to measure the speed of a vehicle. Consider a gyroscope spinning at high speed ω_s. The gyroscope is attached to a vehicle by a universal pivot. If the vehicle accelerates in the direction perpendicular to the spin axis at rate a, then the gyroscope will precess about the acceleration axis, as shown.



The total angle of precession is θ. Show that if the vehicle starts from rest, its final speed is

$$v = \frac{I_s \omega_s}{M\ell}\theta$$

where $I_s\omega_s$ is the gyroscope's spin angular momentum, M is the total mass, and ℓ is the distance from the pivot to the center of mass.

[3] **Problem 7** (KK 7.5). When an automobile rounds a curve at high speed, the weight distribution on the wheels is changed. For sufficiently high speeds, the loading on the inside wheels goes to zero, at which point the car starts to roll over. This tendency can be avoided by mounting a large spinning flywheel on the car.

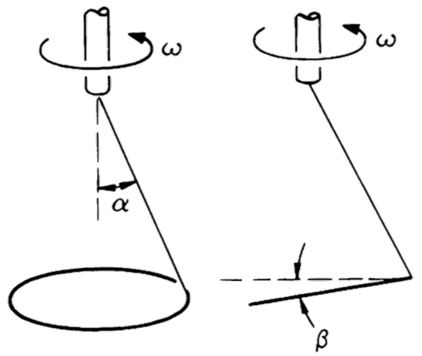
(a) In what direction should the flywheel be mounted, and what should be the sense of rotation, to help equalize the loading? (Check your method works for the car turning in either direction.)

(b) Show that for a disk-shaped flywheel of mass m and radius R, the requirement for equal loading is that the angular velocity ω of the flywheel is related to the velocity of the car v by

$$\omega = 2v\frac{ML}{mR^2}$$

where M is the total mass of the car and flywheel, and L is the height of their center of mass.

[3] **Problem 8** (KK 7.7). A thin hoop of mass M and radius R is suspended from a string through a point on the rim of the hoop. If the support is turned with high angular velocity ω, the hoop will spin as shown, with its plane nearly horizontal and its center nearly on the axis of the support.

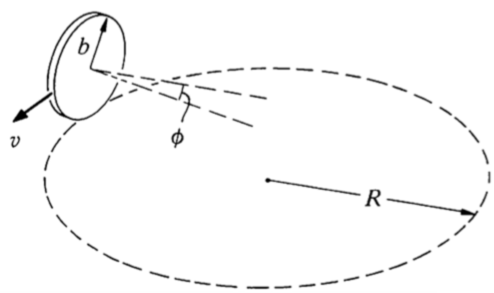


The string makes an angle α with the vertical.

(a) Find, approximately, the small angle β between the plane of the hoop and the horizontal.

(b) Find, approximately, the radius of the small circle traced out by the center of mass.

[4] **Problem 9** (KK 7.6, Morin 9.23)**.** With the right initial conditions, a coin on a table can roll in a circle.

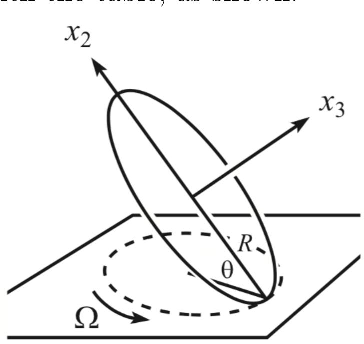


As shown, the coin leans inward, with its axis tilted to the horizontal by an angle ϕ. The radius of the coin is b, the radius of the circle it follows on the table is R, and its velocity is v.

(a) Assuming the coin rolls without slipping and $b \ll R$, show $\tan\phi = 3v^2/2gR$.

(b) No longer assuming $b \ll R$, show that the described motion is only possible if $R > (5/6)b\sin\phi$.

[4] **Problem 10** (Morin 9.24)**.** If you spin a coin around a vertical diameter on a table, it will slowly lose energy and begin a wobbling motion. The angle between the coin and the table will gradually decrease, and eventually it will come to rest. Assume this process is slow, and consider the motion when the coin makes an angle θ with the table, as shown.



You may assume that the center of mass is essentially motionless. Let R be the radius of the coin, and let Ω be the angular frequency at which the contact point on the table traces out its circle. Assume the coin rolls without slipping.

(a) Show that the angular velocity of the coin is $\boldsymbol{\omega} = \Omega \sin\theta \, \hat{\mathbf{x}}_2$, where $\hat{\mathbf{x}}_2$ always points upward along the coin, directly away from the contact point.

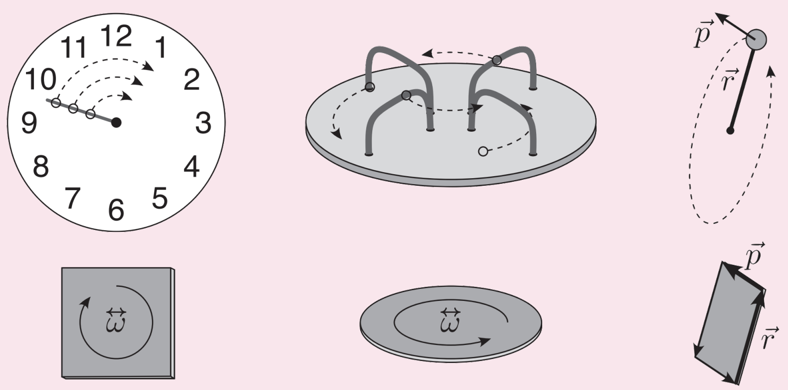
(b) Show that $\Omega = 2\sqrt{g/R \sin\theta}$.

(c) Show that the face on the coin appears to rotate, when viewed from above, with angular frequency $(1 - \cos\theta)\Omega$.

Remark: Bivectors

Vector quantities defined by the cross product have some unusual properties. For example, under a spatial inversion, which flips the signs of $\mathbf{r}$ and $\mathbf{p}$, the sign of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ doesn't get flipped, so $\mathbf{L}$ transforms differently from other vectors. The same applies to the velocity $\boldsymbol{\omega}$ and magnetic field $\mathbf{B}$. All three of these quantities are "pseudovectors", not true vectors.

The underlying reason is that all of these quantities are fundamentally a different kind of mathematical object. They are really rank 2 differential forms, also called bivectors in three dimensions. While a vector is specified by an arrow with magnitude and direction, a bivector is specified by a planar tile with area and orientation. The following figure, taken from this paper, shows how it can be constructed visually from the cross product.



In three dimensions, we can always convert between bivectors and pseudovectors using the right-hand rule, so any calculation can be done with either form. Bivectors have the advantage of visually representing rotational quantities: the angular velocity bivector lies along an object's plane of rotation, while the magnetic field bivector lies along the plane in which it makes charged particles circularly orbit. However, it is easier to add vectors, both visually and mathematically, which also makes it easier to think about decomposing vectors into components. This advantage is so important in practice that I don't recommend using bivectors at all for three-dimensional problems.

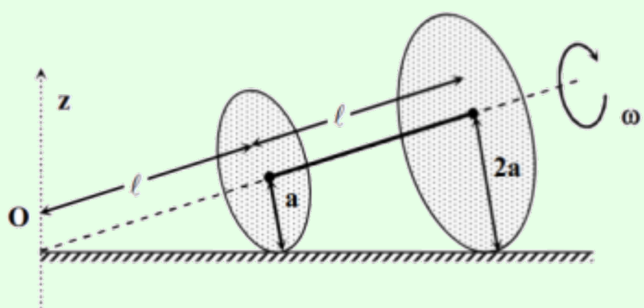
On the other hand, when you work in higher-dimensional spaces, the differential form perspective becomes indispensible. In general, in d dimensions the angular velocity has $\binom{d}{2}$ components, corresponding to the rotation rate in each independent plane.

- Of course, when $d = 1$ there is no such thing as rotation at all, while when $d = 2$ the angular velocity has one component, so we treat it as a scalar.
- When $d = 3$ the angular velocity has three components, so we treat it as a vector.
- When $d = 4$ the angular velocity has four components, so we can't even pretend it's a vector; we have to use the differential form description.

By the way, if you want to look into this material more, be sure to steer clear of "geometric algebra", which dominates the Google search results. Geometric algebra is a strange internet cult which recruits unsuspecting young people by telling them about bivectors, which are indeed cool. Once they have your attention, they'll claim that "mainstream" physics has hit a dead end because it refuses to go beyond vector notation, and then you'll spend years relearning all of physics in their wacky alternative notation. The truth is that physicists don't teach geometric algebra because it's not that useful when $d = 3$, while in higher dimensions we use tensor calculus and differential forms, which are much more powerful than either vectors or geometric algebra. More generally, if a physics movement has tons of internet presence but no actual textbooks or novel results, it's not worth paying attention to.

Example 5: IIT JEE 2016

Two thin circular discs, with radii a and $2a$, are connected by a rod of length $\ell = \sqrt{24}\,a$ through their centers. This rigid object rolls without slipping on a flat table.



The center of mass of the object rotates about the z-axis with an angular speed of Ω. The angular speed of the object about the axis of the rod is ϖ. How are Ω and ϖ related?

Solution

This is the most famous problem ever set on the IIT JEE (condensed for clarity), celebrated by generations of students for its difficulty. But it's also an example of how *not* to write a 3D rotation problem. Under the standard definition of angular velocity, none of the options provided in the question were correct, while the intended answer requires a nonstandard, arbitrary definition. You can find a detailed explanation of this here, by one of the former top scorers on the JEE, and I'll give a condensed explanation below.

First, let's figure out what's going on. The kinematics of this problem isn't any different from problem 1. Defining the x-axis to be horizontal in the figure above, the instantaneous angular velocity is $\boldsymbol{\omega} = \omega\,\hat{\mathbf{x}}$, while the "angular velocity of the angular velocity", describing the precession, is $\boldsymbol{\Omega} = (\omega/\sqrt{24})\,\hat{\mathbf{z}}$. The hard part is figuring out what the question writers meant by "the angular speed ϖ about the axis of the rod".

If we're only talking about the object's *instantaneous* motion, then the only possible answer is $\varpi = \boldsymbol{\omega} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector pointing along the rod. In that case we have $\Omega/\varpi = 5/24$, which wasn't an answer choice in the exam. On the other hand, if we are comparing the object's orientation at different times, then there isn't a unique answer. At a finite time later, the object will be in a different place, and computing a relative angle requires defining a convention for comparing orientations.

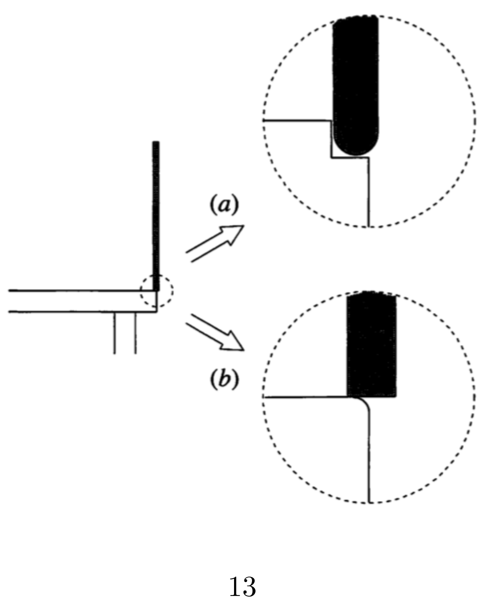
Here's what the problem authors meant. We work in the frame rotating with angular velocity $\boldsymbol{\Omega}$. In this frame, the system is spinning in place, with angular velocity $\boldsymbol{\omega} + \boldsymbol{\Omega}$ parallel to $\hat{\mathbf{n}}$. The definition of ϖ is $|\boldsymbol{\omega} + \boldsymbol{\Omega}|$, which gives $\Omega/\varpi = 1/5$, the intended answer.

Another way of saying this is that when we compare the orientation of the system at one moment to its orientation at another moment, we bring them to the same position by *rotating about the z-axis*, at which point they differ by a rotation about $\hat{\mathbf{n}}$. But this procedure is totally arbitrary, and not specified by the problem. To pose the problem properly, the writers could have either defined ϖ explicitly in the rotating frame mentioned above, or replaced it with a quantity with equivalent but unambiguous physical meaning, such as the interval between times a given point on the rim of a disc touches the ground. Fortunately, you'll almost never see problems this ambiguous on Olympiads.

2 Composite Rotation

These are rotational dynamics problems like the ones you saw in **M5**, but more complex.

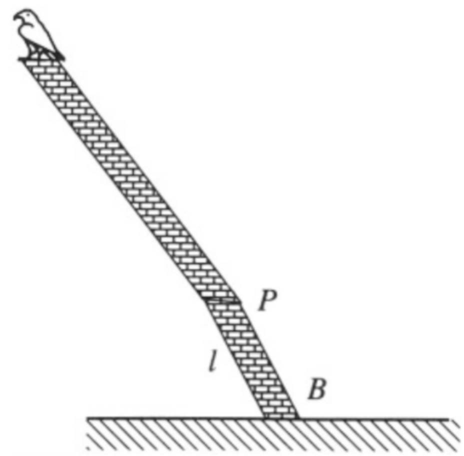
[3] **Problem 11** (PPP 60). A uniform thin rod is placed with one end on the edge of a table in a nearly vertical position and then released from rest. Find the angle it makes with the vertical at the moment it loses contact with the table. Investigate the following two extreme cases.



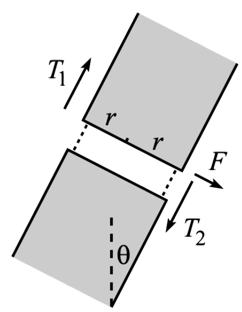
(a) The edge of the table is smooth (friction is negligible) but has a small, singe-step groove.

(b) The edge of the table is rough (friction is large) and very sharp, which means the radius of curvature of the edge is much smaller than the flat end-face of the rod. Half of the end-face protrudes beyond the table edge, so that when it is released the rod pivots about the edge.

[3] **Problem 12** (Cahn)**.** A tall, thin brick chimney of height L is slightly perturbed from its vertical equilibrium position so that it topples over, rotating rigidly about its base B until it breaks at a point P.



(a) For concreteness, we will model the internal forces in the chimney as shown below. Assume throughout that r is very small.



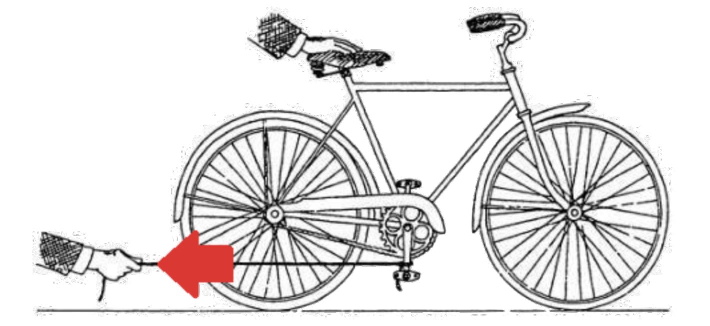
We assume that each piece of the chimney experiences a shear force F and longitudinal tension/compression forces T_1 and T_2 from its neighbors. Find the point on the chimney with the greatest $|T_1|$ or $|T_2|$, assuming the chimney is very thin.

(b) Find the point on the chimney experiencing the greatest shear force F.

(c) At what point is the chimney most likely to break? Do you think the limiting factor is the chimney's maximal compressive strength, tensile strength, or shear strength?

[3] **Problem 13.** IPhO 2014, problem 1A.

[3] **Problem 14** (PPP 14)**.** A bicycle is supported so that it can move forward or backwards but cannot fall sideways; its pedals are in their highest and lowest positions.



A student crouches beside the bicycle and pulls a string attached to the lower pedal, providing a backward horizontal force.

(a) Which way does the bicycle move?

(b) Does the chain-wheel rotate in the same or opposite sense as the rear wheel?

(c) Which way does the lower pedal move relative to the ground?

In particular, be sure to account for the gearing of the bike! To check your answer, watch this video.

[4] **Problem 15.** APhO 2005, problem 1B. A problem on parametric resonance, an idea we first encountered in **M4**. The problem was so subtle that the APhO problem writers themselves could not agree on what the correct answer was!

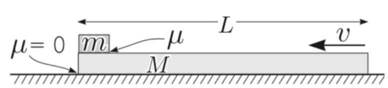
[4] **Problem 16.** INPhO 2020, problem 5. A tough angular collision problem.

[5] **Problem 17.** EuPhO 2019, problem 2. A tough problem about the motion of an rigid body in a magnetic field.

3 Frictional Losses

These miscellaneous problems are grouped under the theme of friction or energy dissipation.

[2] **Problem 18** (Kalda). A plank of length L and mass M lies on a frictionless horizontal surface; on one end sits a small block of mass m.



The coefficient of friction between the block and plank is μ. The plank is sharply hit and given horizontal velocity v. What is the minimum v required for the block to slide across the plank and fall off the other end?

[3] **Problem 19** (BAUPC). A uniform sheet of metal of length ℓ lies on a roof inclined at angle θ, with coefficient of kinetic friction $\mu > \tan\theta$. During the daytime, thermal expansion causes the sheet to uniformly expand by an amount $\Delta\ell \ll \ell$. At night, the sheet contracts back to its original length. What is the displacement of the sheet after one day and night?

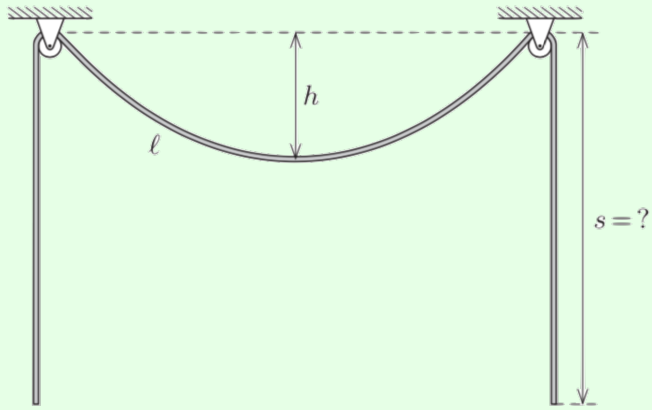
[3] **Problem 20.** APhO 2010, problem 1A. A question about a different kind of inelastic collision.

[5] **Problem 21.** IdPhO 2020, problem 2. A nice problem on anisotropic friction.

4 Ropes, Wires, and Chains

Example 6: MPPP 78

A uniform flexible rope passes over two small frictionless pulleys mounted at the same height.



The length of rope between the pulleys is ℓ, and its sag is h. In equilibrium, what is the length s of the rope segments that hang down on either side?

Solution

The problem can be attacked by differential equations, but there is an elegant solution using only algebra. We let our unknowns be s, the tension $\mathbf{T}_1 = (T_{1,x}, T_{1,y})$ in the rope at the pulley, and the tension T_2 at the lowest point.

Considering the entire sagging portion as the system, vertical force balance gives

$$2T_{1,y} = \lambda \ell g, \quad T_{1,y} = \lambda \ell g/2.$$

Now consider half of the sagging portion as the system. Horizontal force balance gives

$$T_2 = T_{1,x}.$$

Finally, consider one of the hanging portions as the system. Then

$$T_1 = \lambda g s.$$

We hence have three equations, but four unknowns.

For the final equation, we need to consider how the tension changes throughout the rope. This would usually be done by a differential equation, but there is a clever approach using conservation of energy. Suppose we cut the rope somewhere, pull out a segment dx, and reattach the two ends. This requires work $T\,dx$, where T is the magnitude of the local tension. Now suppose we cut the rope somewhere else, separate the ends by dx, and paste our segment inside. This requires work $-T'\,dx$. After this process, the rope is exactly in the same state it was before, so the total work done must be zero.

This would seem to prove that $T = T'$, which is clearly wrong. The extra contribution is that if the two locations have a difference in height Δy, then it takes work $\lambda g(\Delta y)\,dx$ to move the segment from the first to the second. So in equilibrium, for any two points of the rope,

$$\Delta T = \lambda g\,\Delta y.$$

Therefore, we have

$$T_1 - T_2 = \lambda g h.$$

Now we're ready to solve. We have

$$T_1^2 - T_2^2 = (\lambda \ell g/2)^2$$

from our first three equations, and dividing by this new relation gives

$$T_1 + T_2 = \lambda g\,\frac{\ell^2}{4h}.$$

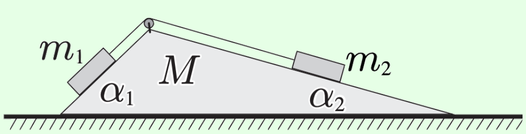
This allows us to solve for T_1, which gives

$$s = \frac{T_1}{\lambda g} = \frac{h}{2} + \frac{\ell^2}{8h}.$$

This is a useful result in real engineering projects: it means that the tension in a cable can be estimated by seeing how much it sags.

Example 7: Kalda 27

A wedge with mass M and acute angles α_1 and α_2 lies on a horizontal surface. A string has been drawn across a pulley situated at the top of the wedge, and its ends are tied to blocks with masses m_1 and m_2.



There is no friction anywhere. What is the acceleration of the wedge?

Solution

This is a classic example of a problem best solved with the Lagrangian-like techniques of **M4**. By working in generalized coordinates, we won't have to solve any systems of equations.

Let s be the distance the rope moves through the pulley, so that both blocks have speed $\dot{s}$ in the noninertial frame of the wedge. The "generalized force" is

$$F_{\text{eff}} = -\frac{dV}{ds} = (m_1 \sin\alpha_1 - m_2 \sin\alpha_2) g.$$

Now, the kinetic energy in the lab frame will be of the form

$$K = \frac{1}{2} M_{\text{eff}} \dot{s}^2$$

which means that, by the Euler–Lagrange equations,

$$\ddot{s} = \frac{F_{\text{eff}}}{M_{\text{eff}}}.$$

Our task is now to calculate M_{eff}. Since the center of mass of the system can't move horizontally, the wedge has speed

$$v_w = \frac{m_1 \cos\alpha_1 + m_2 \cos\alpha_2}{M + m_1 + m_2} \dot{s}.$$

Now, it's a bit annoying to directly compute the kinetic energy K in the lab frame, but it's easy to compute the kinetic energy in the frame of the wedge: it's simply $(m_1 + m_2)\dot{s}^2/2$. But the two are also related simply,

$$K + \frac{1}{2}(M + m_1 + m_2) v_w^2 = \frac{1}{2}(m_1 + m_2)\dot{s}^2.$$

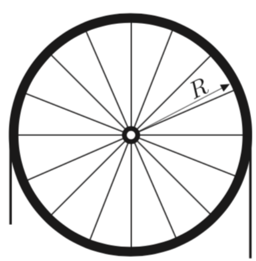
Using this to solve for K, we conclude

$$M_{\text{eff}} = m_1 + m_2 - \frac{(m_1 \cos\alpha_1 + m_2 \cos\alpha_2)^2}{M + m_1 + m_2}.$$

Finally, the desired answer is

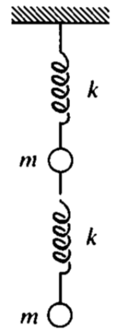
$$a_w = \frac{m_1 \cos\alpha_1 + m_2 \cos\alpha_2}{M + m_1 + m_2} \ddot{s}.$$

[3] **Problem 22** (Kalda). A rope of mass per unit length ρ and length L is thrown over a pulley so that the length of one hanging end is ℓ. The rope and pulley have enough friction so that they do not slip against each other.



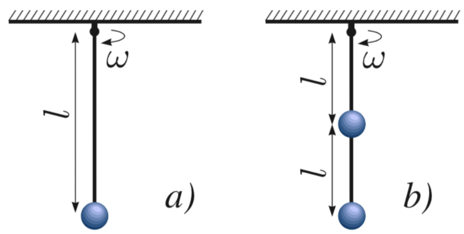
The pulley is a hoop of mass m and radius R attached to a horizontal axle by light spokes. Find the force on the axle immediately after the motion begins.

[3] **Problem 23** (French 5.10). Two equal masses are connected as shown with two identical massless springs of spring constant k.



Considering only motion in the vertical direction, show that the ratio of the frequencies of the two normal modes is $(\sqrt{5}+1)/(\sqrt{5}-1)$.

[3] **Problem 24** (Kalda). A massless rod of length ℓ is attached to the ceiling by a hinge which allows the rod to rotate in a vertical plane.

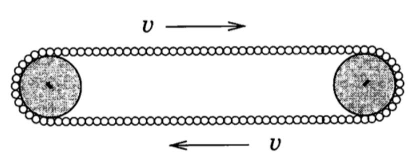


The rod is initially vertical and the hinge is spun with a fixed angular velocity ω.

(a) If a mass m if attached to the bottom of the rod, find the maximum ω for which the configuration is stable.

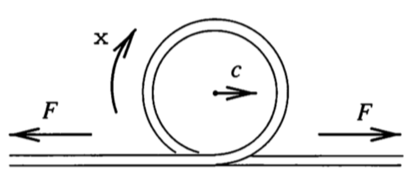
(b) [A] Now suppose another mass m and rod of length ℓ is attached to the first mass by an identical hinge that turns in the same direction, as shown above. Find the maximum ω for which the configuration is stable. (Hint: the configuration is unstable if *any* infinitesimal change in the angles of the rods can lower the energy.)

[3] **Problem 25** (PPP 104). A flexible chain of uniform density is wrapped tightly around two cylinders as shown.



The cylinders are made to rotate and cause the chain to move with speed v. The chain suddenly slips off the cylinders and falls vertically. How does the shape of the chain vary during the fall?

[4] **Problem 26** (PPP 106). A long, heavy flexible rope with mass ρ per unit length is stretched by a constant force F. A sudden movement causes a circular loop to form at one end of the rope.



The center of the loop moves with speed c as shown.

(a) Calculate the speed c, assuming gravity is negligible.

(b) Find the energy E carried by a loop rotating with angular frequency ω.

(c) Show that the momentum p carried by the loop obeys $E = pc$. This is true for waves in general, as we'll see in **W1**.

(d) Find the angular momentum carried by the loop.

5 [A] Advanced Mathematical Techniques

The following problems were cut from earlier problem sets because they required more advanced math; however, they illustrate some very neat and important ideas.

[3] **Problem 27.** In **P1**, you found a general expression for the period of a pendulum oscillating with amplitude θ_0 in terms of an integral, then approximated the integral to find

$$\omega = \omega_0 \left(1 - \frac{\theta_0^2}{16} + O(\theta_0^4)\right)$$

where $\omega_0 = \sqrt{g/L}$. In this problem, we will show a different way to get the same answer, by solving the equation of motion approximately. We write the solution $\theta(t)$ as a series in θ_0. The overall solution is of order θ_0, and the corrections only depend on θ_0^2, so we can write

$$\theta(t) = \theta_0 f_0(t) + \theta_0^3 f_1(t) + \theta_0^5 f_2(t) + \ldots$$

where all the functions $f_i(t)$ are of order 1. Then we plug this expansion into Newton's second law, $\ddot{\theta} + \omega_0^2 \sin\theta = 0$, and expand it out order by order in θ_0.

(a) A naive first guess is to set $f_0(t)$ so that it cancels precisely the order θ_0 terms in this equation, then set $f_1(t)$ to cancel the order θ_0^3 terms, and so on. Using this guess, show that

$$\ddot{f}_0 + \omega_0^2 f_0 = 0, \quad \ddot{f}_1 + \omega_0^2 f_1 = \frac{\omega_0^2 f_0^3}{6}$$

where the first equation has solution $f_0(t) = \cos(\omega_0 t)$.

Unfortunately, this decomposition is not very useful. The problem is that two things are going on at once: the oscillations are not quite sinusoidal, and they have an angular frequency lower than ω_0. The expansion we've done would be useful if we only had the first effect, because then $f_1(t)$ would just capture the small, non-sinusoidal corrections to $f_0(t)$. But our method can't account for the frequency shift; by construction, $f_0(t)$ always oscillates at angular frequency ω_0. Over time, the real oscillation $\theta(t)$ gets out of phase with $f_0(t)$. This manifests itself as a "secular growth" in $f_1(t)$, i.e. it increases in magnitude every cycle until it has a huge value, of order $1/\theta_0^2$, and our perturbative expansion breaks down.

(b) Write the right-hand side of the differential equation for $f_1(t)$ as a sum of sinusoids, and show that it contains a term proportional to $\cos(\omega_0 t)$. This resonantly drives $f_1(t)$, causing the secular growth.

(c) We can salvage our perturbative expansion using the method of "renormalized" frequencies. We impose by fiat that $f_0(t)$ oscillates at the true angular frequency, letting

$$\ddot{f}_0 + \omega^2 f_0 = 0, \quad \omega = \omega_0(1 - c\theta_0^2 + O(\theta_0^4)).$$

Because of this choice, the differential equation for $f_1(t)$, which contains all terms at order θ_0^3, will be altered. The correct choice of ω is precisely the one for which this eliminates the secular growth of $f_1(t)$. Using this idea, show that $c = 1/16$.

If you keep going, you'll find the next term $f_2(t)$ still has secular growth. We can remove it by having both $f_0(t)$ and $f_1(t)$ oscillate at angular frequency $\omega_0(1 - \theta_0^2/16 + d\theta_0^4)$, where d is chosen to cancel the secular growth of $f_2(t)$. In this way, the frequency can be found to any order in θ_0^2. (This technique is called the method of strained coordinates. It's an example of multiple-scale analysis.)

[3] **Problem 28.** You might be wondering how we can solve the weakening spring problem from **M4** without anything fancy like the adiabatic theorem. There is a general technique to solve linear differential equations whose coefficients are slowly varying. First, write the equation of motion as

$$\ddot{x} + \omega^2(t)x = 0.$$

Then expand $x(t)$ as

$$x(t) = A(t)e^{i\phi(t)}, \quad \dot{\phi}(t) = \omega(t).$$

The point of writing $x(t)$ this way is that pulling out the factor of $e^{i\phi(t)}$ will automatically account for the rapid oscillations. The factor $A(t)$ only varies slowly, so it's easier to handle by itself.

(a) Evaluate $\ddot{x}(t)$ and plug it into the equation of motion.

(b) Using the fact that $A(t)$ and $\omega(t)$ vary slowly, throw out small terms in your equation from part (a), until you get a differential equation you can easily integrate. This is a simple example of the WKB approximation.

(c) Show that this gives the expected final result for a weakening spring.

[4] **Problem 29** (BAUPC 1996)**.** A mass M is located at the vertex of an angle $\theta \ll 1$ formed by two massless sticks of length ℓ. The structure is held so that the left stick is initially vertical, then released. The right stick hits the ground at time $t = 0$. The structure then rocks back and forth, coming to a stop at time $t = T$.

(a) Prove the identity

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$$

using the result $\sum_{n \geq 1} 1/n^2 = \pi^2/6$.

(b) Calculate T to leading order in θ. Performing the expansion is rather tricky and requires the previous identity.

[3] **Problem 30.** In this problem, we'll go through Laplace's slick derivation of Kepler's first law. Throughout, we assume the orbit takes place in the xy plane, with the Sun at the origin.

(a) Show that

$$\ddot{x} = -\frac{\gamma x}{r^3}, \quad \ddot{y} = -\frac{\gamma y}{r^3}$$

where γ is a constant that depends on the parameters.

(b) Show that

$$\frac{d}{dt}(r^3 \ddot{x}) = -\gamma \dot{x}, \quad \frac{d}{dt}(r^3 \ddot{y}) = -\gamma \dot{y}.$$

(c) Show that

$$\frac{d}{dt}(r^3 \ddot{r}) = -\gamma \dot{r}.$$

(Hint: this can get messy. As a first step, try showing the left-hand side is equal to $(r^2/2)\, d^3(r^2)/dt^3$. You will have to switch variables to x and y and then switch back; for these purposes it's useful to use the results of part (a), and the definition $r^2 = x^2 + y^2$.)

(d) Define $\psi(t) = r(t)^3$. In parts (b) and (c), we have shown that the differential equation

$$\frac{d}{dt}\left(\psi(t)\,\frac{du}{dt}\right) = -\gamma u$$

has three solutions, namely $\dot{x}$, $\dot{y}$, and $\dot{r}$. Any second-order linear differential equations only has two independent solutions. If $\dot{x}$ and $\dot{y}$ are not independent, the orbit is simply a line, which is trivial. Assuming that doesn't happen, they are independent, so $\dot{r}$ must be a linear combination of them,

$$\dot{r} = A\dot{x} + B\dot{y}.$$

Use this result to argue that the orbit is a conic section.

6 Mechanics and Geometry

For dessert, we'll consider a few cute problems that relate statics to geometry.

Example 8

Given a triangle ABC, the Fermat point is the point X that minimizes $AX + BX + CX$. Design a machine that finds the Fermat point.

Solution

We take a horizontal plane and drill holes at points A, B, and C. A mass M on a rope is fed through each hole, and the three ends of the rope are tied together at point X. The gravitational potential energy is proportional to $AX + BX + CX$, so in equilibrium X lies on the Fermat point. Moreover, since the tensions in each rope are all equal to Mg, force balance requires $\angle AXB = \angle BXC = \angle CXA = 120°$.

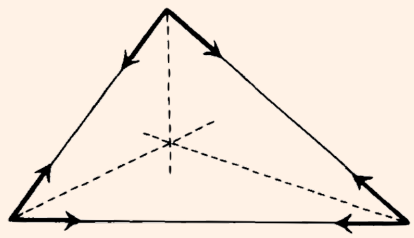
[1] **Problem 31.** Using similar reasoning, design a machine that finds the point X that minimizes $(AX)^2 + (BX)^2 + (CX)^2$. What geometrical property can you conclude about this point?

Example 9

Show that the incenter of a triangle (i.e. the meeting point of the angle bisectors) exists.

Solution

Apply six forces at the vertices of a triangle as shown.



These forces clearly balance, and also produce no net torque on the triangle. Now combine the forces applied at each vertex, yielding three forces that point along the angle bisectors. By the principles of **M2**, the torques of these forces can only balance if their lines of action meet at a point. Therefore the angle bisectors are concurrent, so the incenter exists.

Example 10

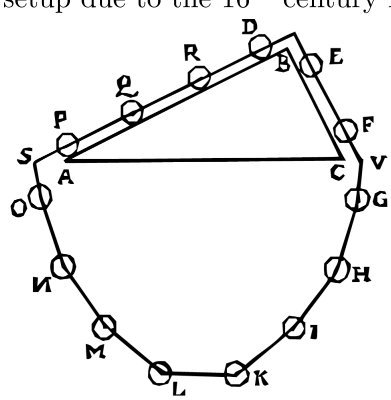
Let AB be a diameter of a circle, and let a mass be free to slide on the circle. The mass is connected to two identical straight springs of zero rest length, which are in turn connected to points A and B. At what points C can the mass be in static equilibrium?

Solution

The potential energy of the system is proportional to $(\mathrm{AC})^2 + (\mathrm{BC})^2$. Since ABC is a right triangle, this is just equal to $(\mathrm{AB})^2$ by the Pythagorean theorem. Since the potential energy doesn't depend on where the mass is, it can be at static equilibrium at any point on the circle. Alternatively, you can show that the mass is in static equilibrium by force balance, and use the reasoning in reverse to derive the Pythagorean theorem.

[1] **Problem 32.** Consider a right triangle ABC filled with a fluid of uniform pressure. Using torque balance, establish the Pythagorean theorem.

[1] **Problem 33.** Shown below is a setup due to the 16$^{\text{th}}$ century mathematician Stevin.



One might argue that because there are more masses on AB than on BC, this is a perpetual motion machine that turns counterclockwise. By using the fact that perpetual motion machines don't actually exist, prove the law of sines.

[2] **Problem 34.** Consider the n-sided polygon P of least possible area that circumscribes a closed convex curve K. Prove that every tangency point of K with a side of P is the midpoint of that side. (Hint: begin by supposing that the area outside P is filled with a gas of uniform pressure, with a vacuum inside P.)

[2] **Problem 35.** In this problem we'll derive Kepler's first law yet again, using no calculus, but a bit of Euclidean geometry. As usual, we suppose a planet of mass m orbits a fixed star of much greater mass M. Placing the star at the origin, let ϕ be the angle between $\mathbf{r}$ and $\mathbf{v}$ for the planet.

(a) Write down the quantities E and L in terms of G, M, m, v, r, and ϕ, and show that

$$\left(r^2 + \frac{GMm}{E} r\right) \sin^2 \phi = \frac{L^2}{2mE}.$$

(b) Now consider an ellipse with semimajor axis a and eccentricity e, meaning that the distance between the foci is $2ae$, with one of the foci F at the origin. Consider a point P on the ellipse, so that the angle between the tangent to the ellipse at P and FP is ϕ. If $r = |FP|$, show that

$$(r^2 - 2ar) \sin^2 \phi = -a^2(1 - e^2).$$

You will have to use the geometrical property that a light ray sent from one focus will reflect at the ellipse to hit the other focus.

(c) By comparing your results for (a) and (b), conclude that the orbit is an ellipse with

$$a = -\frac{GMm}{2E}, \quad e = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}.$$

Mechanics VIII: Synthesis

Three-dimensional rotation is covered in chapter 7 of Kleppner and chapter 9 of Morin. For further discussion and examples, see chapter I-20 of the Feynman lectures, and this awesome video. There is a total of **101** points.

1 3D Rotation

In **M6**, we considered mostly two-dimensional rotation. Now we will tackle the full three-dimensional case, starting with the general description of rigid body motion.

Idea 1: Chasles' Theorem

The instantaneous velocity of a three-dimensional rigid body can always be decomposed in one of two ways. First, for any given point, it can be written in terms of a translational velocity plus a pure rotation about an axis going through that point. In practice, this point is almost always chosen to be the center of mass, giving the decomposition

$$\mathbf{v} = \mathbf{v}_{\text{CM}} + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\text{CM}}).$$

Alternatively, there always exists an axis so that the motion can be written as rotation about that axis, plus a translational velocity *parallel* to the axis, giving

$$\mathbf{v} = \mathbf{v}_0 + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0)$$

where $\mathbf{v}_0$ and $\boldsymbol{\omega}$ are parallel. (This is known as a "screw" motion.) These two decompositions are analogous to the two we saw in **M5**, though in the three-dimensional case the first tends to be much more useful.

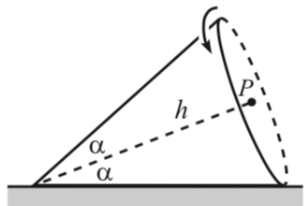
Example 1

You have a small globe, which is mounted so that it can spin on the polar axis and can be spun about a horizontal axis (so that the south pole can be on top). Give the globe a quick spin about the polar axis, and then, before it stops, give it another quick spin about the horizontal axis. Are there any points on the globe that are at rest?

Solution

The first spin gives the angular velocity a vertical component $\boldsymbol{\omega}_1$. The second spin gives the angular velocity an additional horizontal component $\boldsymbol{\omega}_2$. The globe now rotates about its center of mass with angular velocity $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. Precisely two points on the globe are stationary, namely the points that are parallel and antiparallel to this vector.

[2] **Problem 1** (Morin 9.3)**.** A cone rolls without slipping on a table; this means that all the points of the cone that touch the table are instantaneously stationary. The half-angle of the vertex is α, and the axis has length h.



Let the speed of the center of the base, point P in the figure, be v.

(a) Compute the angular velocity $\boldsymbol{\omega}$ by thinking of the motion as pure rotation about some axis.

(b) Compute the angular velocity $\boldsymbol{\omega}$ by thinking of the motion as translation of P, plus rotation about an axis passing through P.

(c) The apex of the cone is fixed, and the cone continues to rotate. As this motion goes on, the angular velocity vector rotates uniformly, keeping a constant magnitude. Find the angular velocity $\boldsymbol{\Omega}$ of the angular velocity vector.

In the precession problems below, it's important to keep track of the difference between $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$.

Solution. (a) The points of the cone that touch the table form a line. This whole line is instantaneously stationary, so the motion is pure rotation about this axis. Let d be the distance from P to the ground. Then the speed of point P is $v = \omega d$, which gives

$$\omega = \frac{v}{d} = \frac{v}{h \sin \alpha}.$$

(b) The point P has speed v. The point directly below P is at rest, and its velocity can also be written as $v - \omega d$. Then $v = \omega d$, giving the same answer as below.

(c) The center of the base moves in a circle of radius $h \cos \alpha$ with speed v, and hence completes one cycle in time $2\pi h \cos \alpha / v$. Hence the angular velocity of the angular velocity is

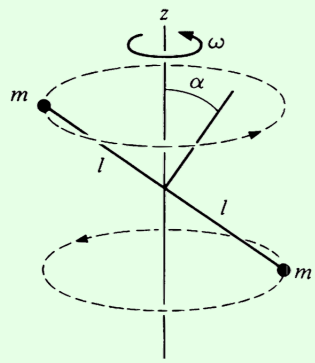
$$\boldsymbol{\Omega} = -\frac{v}{h \cos \alpha} \hat{\mathbf{z}}$$

where we used the right hand rule to find the sign. Note how this is distinct from the angular velocity: not only do they have totally different magnitudes, they point in totally different directions! The angular velocity vector only describes what a body is doing *right now*. It doesn't include $\boldsymbol{\Omega}$, which is about what the body will do in the future.

Most of our statements about rotational dynamics from **M6** remain true. The main new aspect is that angular momentum is not necessarily parallel to angular velocity.

Example 2: KK Example 7.4

Consider a rigid body consisting of two particles of mass m connected by a massless rod of length 2ℓ, rotating about the z-axis with angular velocity ω as shown.



Find the angular momentum of the system.

Solution

We simply add $\mathbf{r} \times \mathbf{p}$ for both masses. Let the rod lie in the xz plane at this moment. Then for the top left mass,

$$\mathbf{r} = -\ell \cos\alpha\,\hat{\mathbf{x}} + \ell \sin\alpha\,\hat{\mathbf{z}}.$$

The momentum is

$$\mathbf{p} = m\mathbf{v} = m\boldsymbol{\omega} \times \mathbf{r} = -m\omega\ell\cos\alpha\,\hat{\mathbf{y}}.$$

Then the angular momentum is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\omega\ell^2 \cos\alpha\,(\sin\alpha\,\hat{\mathbf{x}} + \cos\alpha\,\hat{\mathbf{z}})\,.$$

The other mass has the opposite $\mathbf{r}$ and $\mathbf{p}$ and hence the same $\mathbf{L}$, so the total angular momentum is

$$\mathbf{L} = 2m\omega\ell^2 \cos\alpha\,(\sin\alpha\,\hat{\mathbf{x}} + \cos\alpha\,\hat{\mathbf{z}})\,.$$

It is directed perpendicular to the rod, and in particular, it isn't parallel to the angular velocity!

Here is another way to derive the same result. We can decompose the angular velocity vector into a component along the rod, and a component perpendicular to the rod. The former contributes no angular momentum, because rotating about the rod's axis doesn't move the masses. The latter contributes all the angular momentum. So the angular momentum is

$$L = I_\perp \omega_\perp = (2m\ell^2)(\omega\cos\alpha)$$

directed perpendicular to the rod, which is what we just saw explicitly.

We can summarize the lessons drawn from this example as follows.

Idea 2

For a three-dimensional object, $\mathbf{L}$ is not necessarily parallel to $\boldsymbol{\omega}$. In general, for pure rotation about an axis passing through the origin, we have $\mathbf{L} = I\boldsymbol{\omega}$ where I is a 3×3 matrix called the "moment of inertia tensor about the origin". In components, this means that

$$L_i = \sum_j I_{ij} \omega_j.$$

While this is simple and general, the I_{ij} are a pain to calculate. You can learn more in the reading, but to my knowledge, no Olympiad problem has ever required computing a general moment of inertia tensor.

For the purposes of Olympiad problems, there is a better way to think about the angular momentum. We use the second decomposition of idea 1, and think of the motion as translation plus rotation about the center of mass. If the object has an axis of symmetry, which it will in almost all Olympiad problems, then the angular velocity can then be decomposed into a component parallel to the axis, and perpendicular to the axis,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\parallel + \boldsymbol{\omega}_\perp.$$

The key is that, in such situations, the spin angular momentum has two pieces, which are each parallel to the corresponding piece of the angular velocity,

$$\mathbf{L}_\parallel = I_\parallel \boldsymbol{\omega}_\parallel, \quad \mathbf{L}_\perp = I_\perp \boldsymbol{\omega}_\perp$$

where $I_\parallel$ and $I_\perp$ are ordinary moments of inertia about the center of mass. For example, for a flat uniform disc, $I_\parallel = MR^2/2$ and $I_\perp = MR^2/4$.

The total angular momentum about the origin is then

$$\mathbf{L} = \mathbf{r}_{\text{CM}} \times M\mathbf{v}_{\text{CM}} + I_\parallel \boldsymbol{\omega}_\parallel + I_\perp \boldsymbol{\omega}_\perp$$

where the first term is from the motion of the center of mass, and the next two are from rotation about the center of mass. Note that this is exactly the same as what we saw in **M5**, except that the "spin" angular momentum is broken into two parts.

Idea 3

Sometimes it can be hard to visualize $\boldsymbol{\omega}$, so here are two tricks. First, if any two points on the object are stationary, then $\boldsymbol{\omega}$ must be parallel to the axis connecting the two points. Second, if the rotation is complicated, one can use rotating frames to simplify the analysis. If a body has angular velocity $\boldsymbol{\omega}_1$ in one frame, and that frame has an angular velocity $\boldsymbol{\omega}_2$ with respect to a second frame, then the body has angular velocity $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ with respect to the second frame.

Idea 4

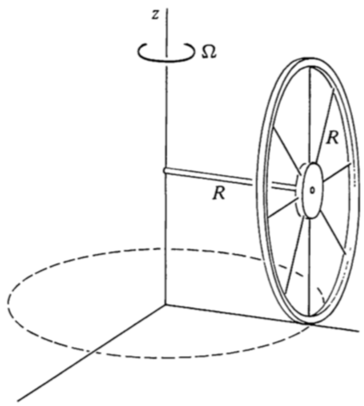
The rate of change of angular momentum is

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}$$

where the torque $\boldsymbol{\tau}$ is defined as in **M5**. The kinetic energy is

$$K = \frac{1}{2}Mv_{\text{CM}}^2 + \frac{1}{2}I_{\parallel}\omega_{\parallel}^2 + \frac{1}{2}I_{\perp}\omega_{\perp}^2.$$

[3] **Problem 2** (KK 7.1)**.** A thin hoop of mass M and radius R rolls without slipping about the z-axis. It is supported by an axle of length R through its center, as shown.



The axle circles around the z-axis with angular speed Ω, so that the bottom point of the wheel traces out a circle of radius R. Let O be the pivot point of the rod, i.e. the point where the rod meets the z-axis.

(a) Find the instantaneous angular velocity $\boldsymbol{\omega}$ of the hoop.

(b) As the motion continues, the angular velocity vector rotates in a circle. Find the angular velocity of the angular velocity vector of the hoop.

(c) Find the instantaneous angular momentum $\mathbf{L}$ of the hoop, about the point O.

(d) Find the instantaneous torque on the hoop about the contact point with the ground, and verify that $\boldsymbol{\tau} = d\mathbf{L}/dt$.

This is a foundational question, worth thoroughly understanding before trying the rest of the section.

Solution. (a) Since both O and the bottom point of the hoop are stationary, the angular velocity must be parallel to the line joining them,

$$\boldsymbol{\omega} \propto \hat{\mathbf{z}} - \hat{\mathbf{y}}.$$

In fact, the motion can be thought of as pure rotation about the line joining them. Now consider the motion of the rod. The y-component of the angular velocity doesn't affect the rod, while the z-component makes it rotate about the z-axis. We are already given that the rod rotates with angular speed Ω about the z-axis, so we must have $\omega_z = \Omega$, and hence

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} - \Omega\hat{\mathbf{y}}.$$

Alternatively, this part can be done using an intermediate rotating frame. We first go to the frame rotating with angular velocity $\Omega\hat{\mathbf{z}}$. In this frame the rod is frozen in place, while the wheel turns in place, with angular velocity $-\Omega\hat{\mathbf{y}}$. So the angular velocity in the original frame is the sum, giving the same answer.

(b) Since the wheel is rigidly attached to the rod, this is simply $\Omega\hat{\mathbf{z}}$.

(c) We apply the result

$$\mathbf{L} = \mathbf{r}_{\text{CM}} \times M\mathbf{v}_{\text{CM}} + I_{\parallel}\boldsymbol{\omega}_{\parallel} + I_{\perp}\boldsymbol{\omega}_{\perp}.$$

We have $I_{\parallel} = MR^2$ and $I_{\perp} = MR^2/2$, so

$$\mathbf{L} = MR^2\Omega\hat{\mathbf{z}} - MR^2\Omega\hat{\mathbf{y}} + \frac{MR^2}{2}\Omega\hat{\mathbf{z}} = MR^2\Omega\left(\frac{3}{2}\hat{\mathbf{z}} - \hat{\mathbf{y}}\right).$$

(d) The reason we take the force about the contact point is so that we won't have to deal with the friction and normal forces. The axle pulls with force $M\Omega^2 R$ for the centripetal force, giving a torque of $M\Omega^2 R^2\hat{\mathbf{x}}$ about the contact point.

To find $d\mathbf{L}/dt$, note that the vertical component of $\mathbf{L}$ won't change; only the $\hat{\mathbf{y}}$ component will change at rate Ω. Thus at that instance, the magnitude of $d\mathbf{L}/dt$ is $MR^2\Omega^2$. By thinking about how $\mathbf{L}$ rotates about the the z-axis, we see that the direction of $d\mathbf{L}/dt$ is $\hat{\mathbf{x}}$. Thus we see that $d\mathbf{L}/dt = M\Omega^2R^2\hat{\mathbf{x}} = \boldsymbol{\tau}$.

Finally, we need just a few more facts to get going with dynamics problems.

Idea 5

In any dynamics problem, there are many choices you can make in the setup. For example, if you're using an inertial frame, you need to choose where the origin is; usually it's best to place it along the axis of symmetry if possible. You are also free to use a noninertial frame with acceleration $\mathbf{a}$. The only difference is that there will be a fictitious force $-M\mathbf{a}$ acting at the center of mass. For that reason, it's usually best to have the accelerating frame follow the center of mass, keeping it at its origin, so no new torques are introduced.

However, you should avoid *rotating* reference frames for dynamics problems. Not only will there be position-dependent Coriolis forces, but they'll add up and contribute a Coriolis *torque*, which is a pain to calculate, as you saw in **M6**. In general, rotating frames are only good for getting a handle on the kinematics, as mentioned in idea 3.

Example 3: KK Example 7.5

Calculate the magnitude of the torque on the rod in example 2.

Solution

We recall that the angular momentum was

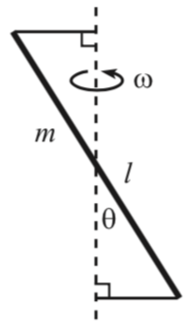
$$\mathbf{L} = 2m\omega\ell^2\cos\alpha\,(\sin\alpha\,\hat{\mathbf{x}} + \cos\alpha\,\hat{\mathbf{z}}).$$

The rod as a whole rotates with angular velocity $\omega\hat{\mathbf{z}}$. In particular, the angular momentum vector rotates with this angular velocity as well; its horizontal component moves in a circle with angular velocity ω. Then

$$|\boldsymbol{\tau}| = \left|\frac{d\mathbf{L}}{dt}\right| = \omega L_x = 2m\omega^2\ell^2\,\cos\alpha\sin\alpha = m\omega^2\ell^2\sin(2\alpha).$$

It might be surprising that there needs to be a torque given that $\boldsymbol{\omega}$ is constant, but that's just because $\mathbf{L}$ and $\boldsymbol{\omega}$ aren't necessarily parallel. Conversely, there can be situations where there is no torque, yet $\boldsymbol{\omega}$ changes over time.

[2] **Problem 3** (Morin 9.10). A stick of mass m and length ℓ spins with angular frequency ω around an axis in zero gravity, as shown.



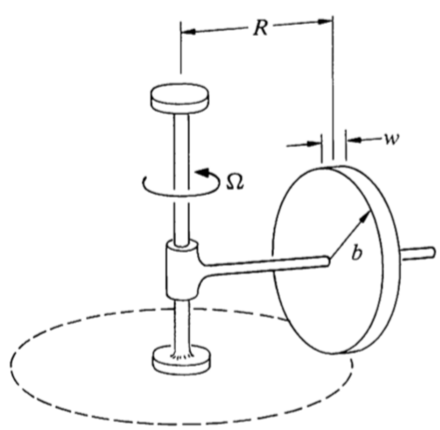
The stick makes an angle θ with the axis and is kept in its motion by two strings that are perpendicular to the axis. Find the tension in the strings.

Solution. The component of angular velocity parallel to the stick contributes no angular momentum, and the component perpendicular to the stick is $\omega \sin\theta$, so it contributes $L = \frac{1}{12} m\ell^2 \omega \sin\theta$ (this can be formalized with the inertia tensor). The direction is perpendicular to the stick. We see then that it traces out a cone, so to find $\dot{\mathbf{L}}$, we note that only the horizontal component of $\mathbf{L}$ is changing, so its rate of change is $\omega L \cos\theta$. The torque is $2T(\ell/2)\cos\theta$, so

$$T\ell\cos\theta = \omega \frac{1}{12} m\ell^2 \omega \sin\theta \cos\theta,$$

or $T = \boxed{\frac{1}{12} m\ell\omega^2 \sin\theta}$.

[2] **Problem 4** (KK 7.4). In an old-fashioned rolling mill, grain is ground by a disk-shaped millstone which rolls in a circle on a flat surface driven by a vertical shaft. Because of the stone's angular momentum, the contact force with the surface can be greater than the weight of the wheel.



Assume the millstone is a uniform disk of mass M, radius b, and width w, and it rolls without slipping in a circle of radius R with angular velocity Ω. Find the contact force. Assume the millstone is closely fitted to the axle so that it cannot tip, and $w \ll R$. Neglect friction.

Solution. Take torques about the point where the vertical and horizontal rods meet. The only torque is from gravity and the normal force,

$$\tau = (N - Mg)R.$$

The angular momentum due to the motion of the center of mass is vertical and constant, so it doesn't matter. As in problem 2, the angular velocity connects this point and the contact point, so

$$\boldsymbol{\omega} = \frac{R}{b}\Omega\hat{\mathbf{x}} + \Omega\hat{\mathbf{z}}.$$

The vertical component of the angular velocity yields another constant vertical component of the angular momentum, so it also doesn't matter. The only part of the angular momentum that changes is the part due to the horizontal component of the angular velocity of the axle. The whole system precesses with angular velocity $\Omega\hat{\mathbf{z}}$, so

$$\tau = \frac{R\Omega}{b}\Omega\left(\frac{1}{2}Mb^2\right).$$

Setting this equal to our other expression for torque and solving gives

$$N = Mg\left(1 + \frac{b\Omega^2}{2g}\right).$$

[2] **Problem 5** (Morin 9.29)**.** A uniform ball rolls without slipping on a table. It rolls onto a piece of paper, which you then slide around in an arbitrary (horizontal) manner. You may even give the paper abrupt, jerky motions, so that the ball slips with respect to it. After you allow the ball to come off the paper, it will eventually resume rolling without slipping on the table. Show that the final velocity equals the initial velocity. (Hint: this remarkably simple result is because of a conservation law. We saw a lower-dimensional version of this problem in **M5**.)

Solution. Let our system be the ball plus the disk, and let the origin be the center of the disk.

We first claim that the component of angular momentum in the plane of the disk is conserved. To show this, it suffices to show that all torques are perpendicular to the plane of the disk. First, the torque at the center of the disk that keeps it spinning is clearly in the perpendicular direction, since the angular momentum of the disk is always in the perpendicular direction, and $\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$. Now, let us find the torque on the ball.

Let $\mathbf{R}$ be the vector from the origin to the center of the ball, and $\mathbf{r}$ the vector from the contact point of the ball to the center of the ball. Now, the normal force and gravity are equal and opposite, but gravity acts at the center, and the normal force acts at the contact point. Thus, the total torque due to these two is

$$\mathbf{R} \times \mathbf{N} + (\mathbf{R} + \mathbf{r}) \times m\mathbf{g} = 0$$

since $\mathbf{g} \parallel \mathbf{r}$, and as noted before, $m\mathbf{g} = -\mathbf{N}$. Finally, there is the friction force, which acts at the contact point. The torque due to it is $\mathbf{R} \times \mathbf{f}$, which points in the perpendicular direction, so the only torques are in the perpendicular direction.

Now, we just compute the parallel component of the angular momentum before and after leaving. This is simply

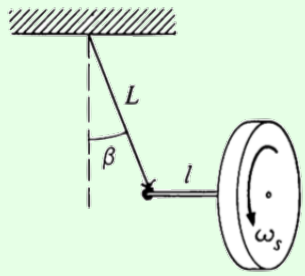
$$((\mathbf{R} + \mathbf{r}) \times \mathbf{v} + I\boldsymbol{\omega})_{\parallel} = (I + mr^2)\boldsymbol{\omega},$$

so $\boldsymbol{\omega}$ doesn't change, so the ball continues on its original course.

When a system such as a gyroscope is given a high angular momentum, it can exhibit precession. In this case, the angular velocity $\boldsymbol{\omega}$ precesses (i.e. rotates) with a small angular velocity $\boldsymbol{\Omega}$. Precession is a famously counterintuitive phenomenon, but it can be handled using just the same principles we've laid out above.

Example 4: KK 7.3

A gyroscope wheel is at one end of an axle of length ℓ. The other end of the axle is suspended from a string of length L.



The wheel is set into motion so that it executes slow, uniform precession in the horizontal plane. The wheel has mass M and moment of inertia I_0 about its center of mass, and turns with angular speed ω_s. Neglect the mass of the shaft and string. Find the angle β the string makes with the vertical, assuming β is very small.

Solution

Let T be the tension in the rope. The entire system precesses with associated angular velocity $\Omega\hat{\mathbf{z}}$. (That is, this is the angular velocity of the angular velocity and angular momentum vectors.) Since the center of mass does not accelerate vertically, and the center of mass moves in a horizontal circle,

$$T\cos\beta = Mg, \quad T\sin\beta = M\Omega^2(\ell + L\sin\beta).$$

We'll work to lowest possible order in β everywhere, which means approximating $\cos\beta \approx 1$ and ignoring the $L\sin\beta$ term, giving

$$T = Mg, \quad T\beta = M\Omega^2\ell.$$

Combining these equations gives the precession angular frequency

$$\Omega = \sqrt{\frac{g\beta}{\ell}}.$$

This is as far as we can go with forces alone.

Now we use $\boldsymbol{\tau} = d\mathbf{L}/dt$. First we need to determine the angular velocity of the wheel. Note that if we went in the rotating frame with angular velocity $\Omega\hat{\mathbf{z}}$, the wheel would just spin with angular speed ω_s in place. So the angular velocity vector in the original frame is

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} + \omega_s\hat{\mathbf{x}}$$

where $\hat{\mathbf{x}}$ is directed along the rod. The center of mass of the wheel moves with speed $(\ell + L \sin\beta)\Omega$ in a horizontal circle.

Since the precession is assumed to be slow, Ω is much smaller than ω_s. Thus, we can simply ignore contributions to the angular momentum proportional to Ω. That is, we can take

$$\boldsymbol{\omega} \approx \omega_s \hat{\mathbf{x}}$$

for the purposes of computing angular momentum, giving

$$\mathbf{L} \approx I_0 \omega_s \hat{\mathbf{x}}.$$

This rotates in a horizontal circle with angular speed Ω, so

$$|\boldsymbol{\tau}| \approx I_0 \Omega \omega_s.$$

The torques in the relevant direction come from gravity and the vertical component of the tension force, $|\boldsymbol{\tau}| = Mg\ell$. Equating these, we have

$$Mg\ell = I_0 \Omega \omega_s.$$

Plugging the result for Ω in above and solving for β gives

$$\beta = \frac{m^2 g \ell^3}{\omega^2 I_0^2}.$$

Note that assuming Ω was small kept the equations simple. This is what Kleppner calls the gyroscope approximation. It can be applied in some, but not all, of the below problems.

Remark

In most gyroscope problems, such as the one above, we assume the motion has reached a steady state, but you might wonder just how it gets started in the first place. For example, suppose we had the same setup as the previous problem, with the wheel spinning and the axle horizontal. For simplicity, let's get rid of the string and suppose the end of the axle is held at a fixed support. Now suppose the axle and wheel are released with no translational motion.

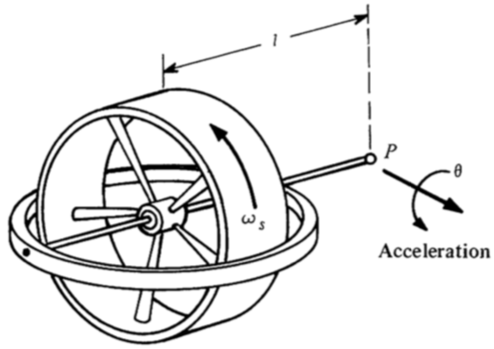
The following chain of events ensues:

- Of course, the axle starts to tip downward because of the weight of the wheel. (Rotational mechanics is counterintuitive, but not *that* counterintuitive!)
- This produces a downward component of angular momentum, which is balanced by the axle/wheel system twisting about its center of mass, rotating slightly about the z-axis.
- This twist tries to push the end of the axle out of the page, causing the support to exert a force on the axle pointing into the page. That force propagates down the axle as an internal shear stress, eventually causing the center of mass of the wheel to start moving into the page, starting the precession.

- In reality, the process overshoots and overcorrects, leading to oscillations called nutation on top of the precession. (For more details, see Note 2 of chapter 7 of Kleppner.)
- For a typical pivot, energy can be dissipated at the pivot point, but the angular momentum of the system stays roughly the same. Assuming this is the case, the oscillations will eventually damp away, leaving the uniform precession.

Notice that in this example, the initial angular momentum is perfectly horizontal. The final angular momentum includes an upward component due to the uniform precession, which implies that the axle must tilt slightly downward, by an angle of order $(\omega/\Omega)^2$. Therefore, if you want to set up uniform precession with the axle perfectly horizontal, as in the above example, you should point the axle slightly upward when releasing it from rest.

[2] **Problem 6** (KK 8.5). An "integrating gyro" can be used to measure the speed of a vehicle. Consider a gyroscope spinning at high speed ω_s. The gyroscope is attached to a vehicle by a universal pivot. If the vehicle accelerates in the direction perpendicular to the spin axis at rate a, then the gyroscope will precess about the acceleration axis, as shown.



The total angle of precession is θ. Show that if the vehicle starts from rest, its final speed is

$$v = \frac{I_s \omega_s}{M\ell}\theta$$

where $I_s\omega_s$ is the gyroscope's spin angular momentum, M is the total mass, and ℓ is the distance from the pivot to the center of mass.

Solution. Work in the accelerating reference frame where the pivot is at rest. Using the gyroscope approximation, we see that $\dot{L} = I_s\omega_s\omega$ where $\omega = \dot{\theta}$, and $\tau = Ma\ell$. Thus,

$$a = \frac{I_s\omega_s\dot{\theta}}{M\ell}.$$

Integrating yields the desired result.

[3] **Problem 7** (KK 7.5). When an automobile rounds a curve at high speed, the weight distribution on the wheels is changed. For sufficiently high speeds, the loading on the inside wheels goes to zero, at which point the car starts to roll over. This tendency can be avoided by mounting a large spinning flywheel on the car.

(a) In what direction should the flywheel be mounted, and what should be the sense of rotation, to help equalize the loading? (Check your method works for the car turning in either direction.)

(b) Show that for a disk-shaped flywheel of mass m and radius R, the requirement for equal loading is that the angular velocity ω of the flywheel is related to the velocity of the car v by

$$\omega = 2v\frac{ML}{mR^2}$$

where M is the total mass of the car and flywheel, and L is the height of their center of mass.

Solution. (a) If the car is turning with a radius of curvature of r at velocity v, then the frictional force must provide the centripetal force of $f = Mv^2/r$. This will exert a torque of fL on the car about the center of mass, where L is the height of the center of mass. The torque points forward for turning left, and backwards for turning right.

Normally, a difference in the normal forces between the wheels will provide the opposing torque to keep the car from rolling over. To keep an equal loading, the flywheel must provide the opposing torque. Another way to think about it is to have the torque from friction to cause precession of the flywheel instead of turning the car (the equal and opposite "reaction torque" will keep the car stable).

The key is that as the car is turning, the flywheel will also turn with the car at angular velocity v/r, thus the direction of its spin angular momentum $\mathbf{L}_s$ will change. From a top view, that means the forwards torque for turning left must turn $\mathbf{L}_s$ counterclockwise, and the backwards torque for turning right must turn $\mathbf{L}_s$ clockwise. This works when $\mathbf{L}_s$ is pointing to the right with respect to the car's motion (the flywheel spins in the opposite direction than that of the wheels).

(b) In order for the torque from friction to turn the flywheel, $\tau = fL = L_s(v/r)$. For a disk-shaped flywheel with angular momentum $L_s = \frac{1}{2}mR^2\omega$, putting in $f = Mv^2/r$ yields

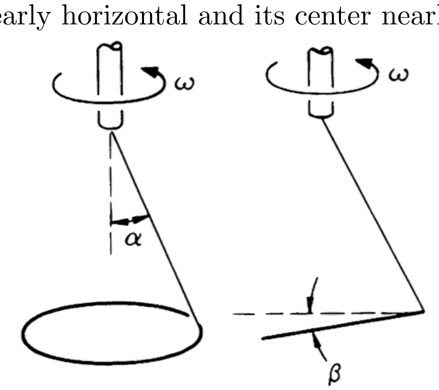
$$\frac{Mv^2L}{r} = \frac{1}{2}mR^2\omega v/r$$

which gives

$$\omega = 2v\frac{ML}{mR^2}$$

as desired.

[3] **Problem 8** (KK 7.7)**.** A thin hoop of mass M and radius R is suspended from a string through a point on the rim of the hoop. If the support is turned with high angular velocity ω, the hoop will spin as shown, with its plane nearly horizontal and its center nearly on the axis of the support.



The string makes an angle α with the vertical.

(a) Find, approximately, the small angle β between the plane of the hoop and the horizontal.

(b) Find, approximately, the radius of the small circle traced out by the center of mass.

Solution. (a) Let the tension in the string be T. To balance gravity, $T\cos\theta = Mg$. The angle between the string and the plane of the hoop is $\pi - (\alpha + \pi/2 - \beta) = \pi/2 - \alpha + \beta$, so the torque on the hoop about the center of mass is $\tau = TR\sin(\pi/2 - \alpha + \beta) = TR\cos(\alpha - \beta)$. This torque rotates the horizontal component of the angular momentum about the center of mass, $MR^2\omega\sin\beta$, with angular velocity ω. Thus

$$TR\cos(\alpha - \beta) \approx TR(\cos\alpha - \beta\sin\alpha) \approx MR^2\omega^2\beta$$

$$R\omega^2\beta = g(1 - \beta\tan\alpha)$$

$$\beta = \frac{g}{R\omega^2 - g\tan\alpha}.$$

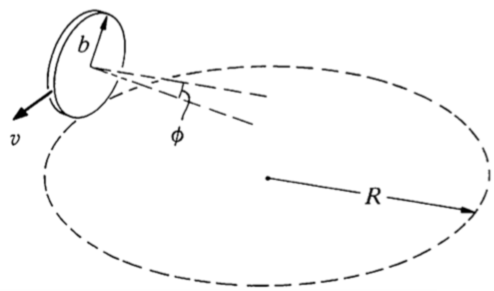
(b) The horizontal component of the tension should provide a centripetal force,

$$T\sin\alpha = M\omega^2 r$$

giving an answer of

$$r = \frac{g\tan\alpha}{\omega^2}.$$

[4] **Problem 9** (KK 7.6, Morin 9.23). With the right initial conditions, a coin on a table can roll in a circle.

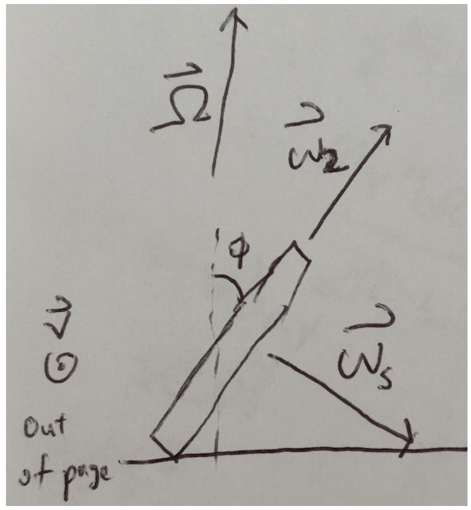


As shown, the coin leans inward, with its axis tilted to the horizontal by an angle ϕ. The radius of the coin is b, the radius of the circle it follows on the table is R, and its velocity is v.

(a) Assuming the coin rolls without slipping and $b \ll R$, show $\tan\phi = 3v^2/2gR$.

(b) No longer assuming $b \ll R$, show that the described motion is only possible if $R > (5/6)b\sin\phi$.

Solution. (a) We work in the nonrotating, but noninertial frame whose origin follows the center of mass. In this frame, the only part of the angular momentum that changes is the horizontal component of the spin angular momentum. The coin spins by "rolling" and "turning", along $\boldsymbol{\omega}_s$ and $\boldsymbol{\omega}_2$ respectively:



Since the motion of the coin is the combination of "rolling" and going around in a circle, the total angular velocity should be $\boldsymbol{\omega}_s + \boldsymbol{\Omega}$ where Ω describes the circular motion/turning of the coin and points vertically upwards. The components of $\boldsymbol{\Omega}$ are $\boldsymbol{\Omega}\cos\phi$ and $-\boldsymbol{\Omega}\sin\phi$ along $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_s$ respectively. The moments of inertia about the coin in the $\boldsymbol{\omega}_s$ and $\boldsymbol{\omega}_2$ directions are $\frac{1}{2}Mb^2$ and $\frac{1}{4}Mb^2$ respectively. Note that we can't simply ignore $\boldsymbol{\Omega}$ because it is vertical; this is because the angular momentum from $\boldsymbol{\Omega}$ does not point in the same direction as $\boldsymbol{\Omega}$ (they point in the same direction only along the principal axes).

With $\mathbf{L} = I\omega$ along those principal axes, the horizontal components of the angular momenta is $L_x = \frac{1}{2}Mb^2(\omega_s - \Omega\sin\phi)\cos\phi + \frac{1}{4}Mb^2\Omega\cos\phi\sin\phi$. The no slip condition is that $\omega_s b = \Omega R$. With $b \gg R$, we can approximate $L_x \approx \frac{1}{2}Mb^2\omega_s\cos\phi$. The torque, $\tau = \Omega L_x$, about the center of mass is $(N\sin\phi - f\cos\phi)b$ where $f = Mv^2/(R - b\sin\phi) \approx Mv^2/R$ is the frictional force, and $N = Mg$ is the normal force. The velocity of the CM is $v = \Omega(R - b\sin\phi) \approx \Omega R$. Then

$$\tau = \Omega L_x = \frac{1}{2}Mb^2(\Omega R/b)\Omega\cos\phi = \frac{1}{2}M\Omega^2 bR\cos\phi = \frac{Mv^2 b}{2R}\cos\phi$$

but we also know that

$$\tau = Mgb\sin\phi - \frac{Mv^2 b}{R}\cos\phi$$

from which we conclude

$$\tan\phi = \frac{3v^2}{2gR}.$$

(b) Now we will do the calculations above without $b \ll R$.

$$\tau = \Omega\left(\frac{1}{2}Mb^2(\omega_s - \Omega\sin\phi)\cos\phi + \frac{1}{4}Mb^2\Omega\cos\phi\sin\phi\right)$$

$$= Mgb\sin\phi - \frac{Mv^2 b}{R - b\sin\phi}\cos\phi.$$

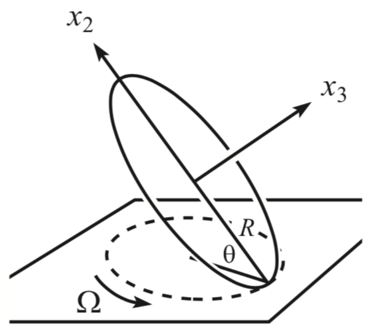
Doing the replacements of $\omega_s = \Omega R/b$ and $v = \Omega(R - b\sin\phi)$, dividing the above equation by $M\Omega^2 b$ yields

$$\frac{1}{2}R\cos\phi - \frac{1}{2}b\sin\phi\cos\phi + \frac{1}{4}b\sin\phi\cos\phi = \frac{g}{\Omega^2}b\sin\phi - (R - b\sin\phi)\cos\phi$$

$$\frac{3}{2}R - \frac{5}{4}b\sin\phi = \frac{g}{\Omega^2}b\tan\phi.$$

Since $\tan\phi > 0$ in order for the motion to make sense, $\frac{3}{2}R - \frac{5}{4}b\sin\phi > 0$, thus we get $R > \frac{5}{6}b\sin\phi$.

[4] **Problem 10** (Morin 9.24)**.** If you spin a coin around a vertical diameter on a table, it will slowly lose energy and begin a wobbling motion. The angle between the coin and the table will gradually decrease, and eventually it will come to rest. Assume this process is slow, and consider the motion when the coin makes an angle θ with the table, as shown.



You may assume that the center of mass is essentially motionless. Let R be the radius of the coin, and let Ω be the angular frequency at which the contact point on the table traces out its circle. Assume the coin rolls without slipping.

(a) Show that the angular velocity of the coin is $\boldsymbol{\omega} = \Omega\sin\theta\,\hat{\mathbf{x}}_2$, where $\hat{\mathbf{x}}_2$ always points upward along the coin, directly away from the contact point.

(b) Show that $\Omega = 2\sqrt{g/R\sin\theta}$.

(c) Show that the face on the coin appears to rotate, when viewed from above, with angular frequency $(1-\cos\theta)\Omega$.

Solution. (a) Since the center of mass is essentially motionless and thee coin is rolling without slipping, the center of mass and the contact point are both stationary. Thus the angular velocity must pass through those lines, and is pointing along $\hat{\mathbf{x}}_2$. Let $\hat{\mathbf{k}}$ be a vertical unit vector. The angular velocity can be seen as the sum of the rotation about the center of mass and pointing along $\hat{\mathbf{k}}$ (turning of the coin's orientation) with angular velocity $\boldsymbol{\omega}_k = \Omega\hat{\mathbf{k}}$, and rotation about $-\hat{v}x_3$ with angular velocity $\boldsymbol{\omega}_3$ to roll without slipping. Thus $\boldsymbol{\omega} = \boldsymbol{\omega}_k + \boldsymbol{\omega}_3$

Since $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_3$ are perpendicular, $\omega = \omega_k\sin\theta$, thus

$$\boldsymbol{\omega} = \Omega\sin\theta\hat{\mathbf{x}}_2$$

(b) The torque about the contact point from gravity is $MgR\cos\theta$, and points horizontally to change the horizontal component of the angular momentum $L_x = I\omega\cos\theta$ at a rate of Ω. The moment of inertia about $\hat{\mathbf{x}}_2$ is $\frac{1}{4}MR^2$, which gives

$$MgR\cos\theta = \frac{1}{4}MR^2\Omega^2\sin\theta\cos\theta.$$

$$\Omega = 2\sqrt{\frac{g}{R\sin\theta}}$$

The same result can be found be taking torques about the center of mass (no friction).

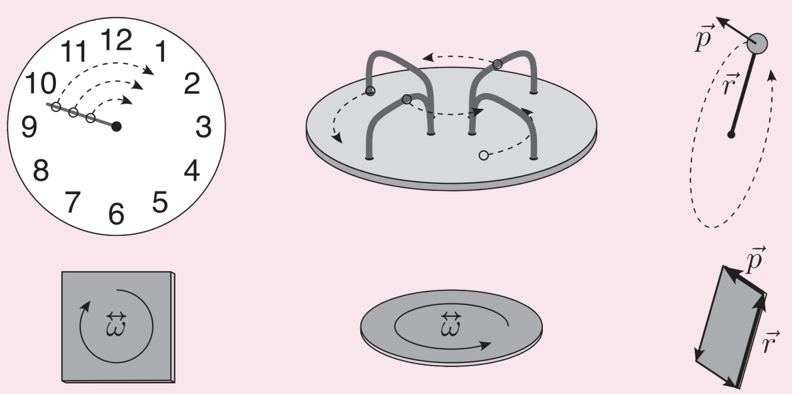
(c) From part (a), we found that $\boldsymbol{\omega} = \boldsymbol{\omega}_k + \boldsymbol{\omega}_3$ and $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_3$ are perpendicular which gets $\omega_3 = -\Omega\cos\theta$. Consider a point on the coin from the top view. $\boldsymbol{\omega}_k$ makes it rotate counterclockwise with angular velocity Ω, and $\boldsymbol{\omega}_3$ rotates it clockwise with angular velocity $\Omega\cos\theta$. Thus the face of the coin appears to be rotating with angular velocity $\Omega(1-\cos\theta)$.

Another way to do this is to consider the difference between the radius of the coin and the radius of the traced out circle. In a full rotation of the contact point in time $T = 2\pi/\Omega$, a distance of $2\pi R\cos\theta$ was covered by the coin. Since the coin didn't slip, that same distance was covered along the coin's edge, so the initial contact point will be a distance of $2\pi R(1-\cos\theta)$ ahead of the new contact point. Thus the angle change is $2\pi(1-\cos\theta)$ in time $T = 2\pi/\Omega$, giving an apparent angular velocity of $(1-\cos\theta)\Omega$.

Remark: Bivectors

Vector quantities defined by the cross product have some unusual properties. For example, under a spatial inversion, which flips the signs of $\mathbf{r}$ and $\mathbf{p}$, the sign of $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ doesn't get flipped, so $\mathbf{L}$ transforms differently from other vectors. The same applies to the velocity $\boldsymbol{\omega}$ and magnetic field $\mathbf{B}$. All three of these quantities are "pseudovectors", not true vectors.

The underlying reason is that all of these quantities are fundamentally a different kind of mathematical object. They are really rank 2 differential forms, also called bivectors in three dimensions. While a vector is specified by an arrow with magnitude and direction, a bivector is specified by a planar tile with area and orientation. The following figure, taken from this paper, shows how it can be constructed visually from the cross product.



In three dimensions, we can always convert between bivectors and pseudovectors using the right-hand rule, so any calculation can be done with either form. Bivectors have the advantage of visually representing rotational quantities: the angular velocity bivector lies along an object's plane of rotation, while the magnetic field bivector lies along the plane in which it makes charged particles circularly orbit. However, it is easier to add vectors, both visually and mathematically, which also makes it easier to think about decomposing vectors into components. This advantage is so important in practice that I don't recommend using bivectors at all for three-dimensional problems.

On the other hand, when you work in higher-dimensional spaces, the differential form perspective becomes indispensible. In general, in d dimensions the angular velocity has $\binom{d}{2}$

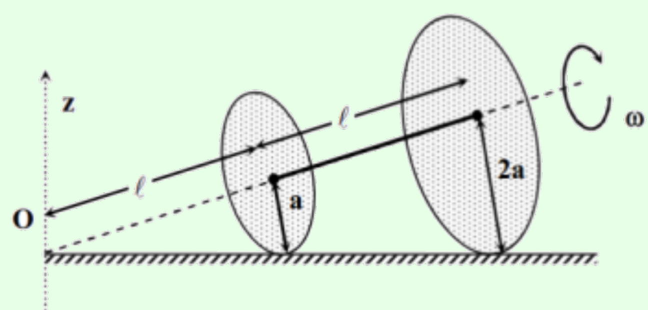
components, corresponding to the rotation rate in each independent plane.

- Of course, when $d = 1$ there is no such thing as rotation at all, while when $d = 2$ the angular velocity has one component, so we treat it as a scalar.
- When $d = 3$ the angular velocity has three components, so we treat it as a vector.
- When $d = 4$ the angular velocity has four components, so we can't even pretend it's a vector; we have to use the differential form description.

By the way, if you want to look into this material more, be sure to steer clear of "geometric algebra", which dominates the Google search results. Geometric algebra is a strange internet cult which recruits unsuspecting young people by telling them about bivectors, which are indeed cool. Once they have your attention, they'll claim that "mainstream" physics has hit a dead end because it refuses to go beyond vector notation, and then you'll spend years relearning all of physics in their wacky alternative notation. The truth is that physicists don't teach geometric algebra because it's not that useful when $d = 3$, while in higher dimensions we use tensor calculus and differential forms, which are much more powerful than either vectors or geometric algebra. More generally, if a physics movement has tons of internet presence but no actual textbooks or novel results, it's not worth paying attention to.

Example 5: IIT JEE 2016

Two thin circular discs, with radii a and $2a$, are connected by a rod of length $\ell = \sqrt{24}\,a$ through their centers. This rigid object rolls without slipping on a flat table.



The center of mass of the object rotates about the z-axis with an angular speed of Ω. The angular speed of the object about the axis of the rod is $\overline{\omega}$. How are Ω and $\overline{\omega}$ related?

Solution

This is the most famous problem ever set on the IIT JEE (condensed for clarity), celebrated by generations of students for its difficulty. But it's also an example of how *not* to write a 3D rotation problem. Under the standard definition of angular velocity, none of the options provided in the question were correct, while the intended answer requires a nonstandard, arbitrary definition. You can find a detailed explanation of this here, by one of the former top scorers on the JEE, and I'll give a condensed explanation below.

First, let's figure out what's going on. The kinematics of this problem isn't any different from problem 1. Defining the x-axis to be horizontal in the figure above, the instantaneous angular velocity is $\boldsymbol{\omega} = \omega\,\hat{\mathbf{x}}$, while the "angular velocity of the angular velocity", describing the precession, is $\boldsymbol{\Omega} = (\omega/\sqrt{24})\,\hat{\mathbf{z}}$. The hard part is figuring out what the question writers meant by "the angular speed ϖ about the axis of the rod".

If we're only talking about the object's *instantaneous* motion, then the only possible answer is $\varpi = \boldsymbol{\omega} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector pointing along the rod. In that case we have $\Omega/\varpi = 5/24$, which wasn't an answer choice in the exam. On the other hand, if we are comparing the object's orientation at different times, then there isn't a unique answer. At a finite time later, the object will be in a different place, and computing a relative angle requires defining a convention for comparing orientations.

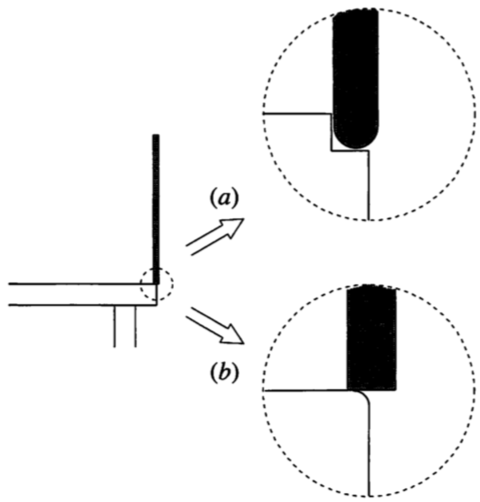
Here's what the problem authors meant. We work in the frame rotating with angular velocity $\boldsymbol{\Omega}$. In this frame, the system is spinning in place, with angular velocity $\boldsymbol{\omega} + \boldsymbol{\Omega}$ parallel to $\hat{\mathbf{n}}$. The definition of ϖ is $|\boldsymbol{\omega} + \boldsymbol{\Omega}|$, which gives $\Omega/\varpi = 1/5$, the intended answer.

Another way of saying this is that when we compare the orientation of the system at one moment to its orientation at another moment, we bring them to the same position by *rotating about the z-axis*, at which point they differ by a rotation about $\hat{\mathbf{n}}$. But this procedure is totally arbitrary, and not specified by the problem. To pose the problem properly, the writers could have either defined ϖ explicitly in the rotating frame mentioned above, or replaced it with a quantity with equivalent but unambiguous physical meaning, such as the interval between times a given point on the rim of a disc touches the ground. Fortunately, you'll almost never see problems this ambiguous on Olympiads.

2 Composite Rotation

These are rotational dynamics problems like the ones you saw in **M5**, but more complex.

[3] **Problem 11** (PPP 60). A uniform thin rod is placed with one end on the edge of a table in a nearly vertical position and then released from rest. Find the angle it makes with the vertical at the moment it loses contact with the table. Investigate the following two extreme cases.



(a) The edge of the table is smooth (friction is negligible) but has a small, singe-step groove.

(b) The edge of the table is rough (friction is large) and very sharp, which means the radius of curvature of the edge is much smaller than the flat end-face of the rod. Half of the end-face protrudes beyond the table edge, so that when it is released the rod pivots about the edge.

Solution. By energy conservation, we have that

$$\frac{1}{2}\frac{1}{3}m\ell^2\omega^2 = mg\frac{\ell}{2}(1-\cos\theta) \implies \omega^2 = \frac{3g}{\ell}(1-\cos\theta).$$

Therefore, the centripetal acceleration of the CM is $a_c = \omega^2\ell/2 = \frac{3}{2}g(1-\cos\theta)$. Differentiating the equation for ω^2, we learn that

$$2\omega\dot{\omega} = \frac{3g}{\ell}(\sin\theta)\dot{\theta},$$

so $\dot{\omega} = \frac{3g}{2\ell}\sin\theta$. Thus, the tangential acceleration of the CM is $a_t = \frac{3}{4}g\sin\theta$. We can solve (b) now.

(b) We see that in this case, the normal force points along the rod. Therefore,

$$N - Mg\cos\theta = -Ma_c,$$

so $N = Mg\left(\frac{5}{2}\cos\theta - \frac{3}{2}\right)$. This becomes 0 at $\theta = \boxed{\cos^{-1}(3/5)}$.

(a) This is identical to the falling ladder problem from **M5**, and hence has the same answer. But we can also solve the problem directly here. We have

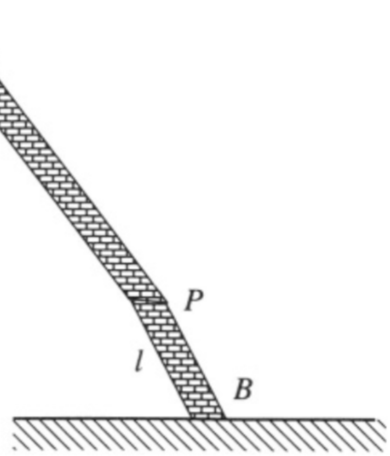
$$N_x = M(a_t\cos\theta - a_c\sin\theta) = \frac{3}{3}Mg\sin\theta(3\cos\theta-2)$$

and

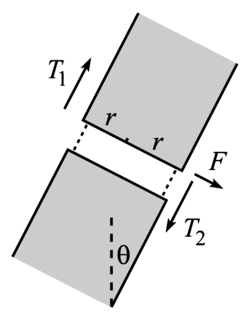
$$N_y = Mg - M(a_c\cos\theta + a_t\sin\theta) = \frac{1}{4}Mg(3\cos\theta-1)^2.$$

The first one to go to 0 is N_x, and this happens at $\theta = \boxed{\cos^{-1}(2/3)}$.

[3] **Problem 12** (Cahn). A tall, thin brick chimney of height L is slightly perturbed from its vertical equilibrium position so that it topples over, rotating rigidly about its base B until it breaks at a point P.



(a) For concreteness, we will model the internal forces in the chimney as shown below. Assume throughout that r is very small.



We assume that each piece of the chimney experiences a shear force F and longitudinal tension/compression forces T_1 and T_2 from its neighbors. Find the point on the chimney with the greatest $|T_1|$ or $|T_2|$, assuming the chimney is very thin.

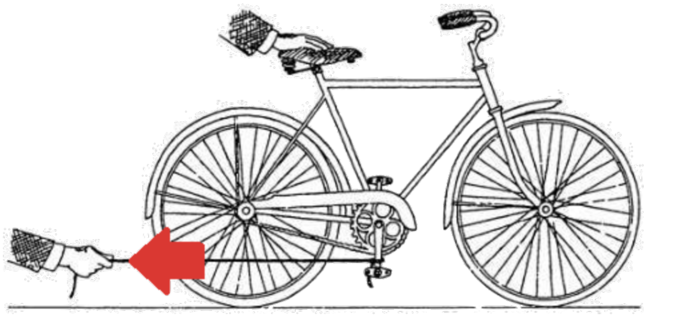
(b) Find the point on the chimney experiencing the greatest shear force F.

(c) At what point is the chimney most likely to break? Do you think the limiting factor is the chimney's maximal compressive strength, tensile strength, or shear strength?

Solution. See the solution here.

[3] **Problem 13.** IPhO 2014, problem 1A.

[3] **Problem 14** (PPP 14). A bicycle is supported so that it can move forward or backwards but cannot fall sideways; its pedals are in their highest and lowest positions.



A student crouches beside the bicycle and pulls a string attached to the lower pedal, providing a backward horizontal force.

(a) Which way does the bicycle move?

(b) Does the chain-wheel rotate in the same or opposite sense as the rear wheel?

(c) Which way does the lower pedal move relative to the ground?

In particular, be sure to account for the gearing of the bike! To check your answer, watch this video.

Solution. (a) This actually depends on the gearing of the bicycle. If the string pulls with force F_0, and the pedals have distance r_p from the pedal axle, it will exert a torque Fr_p about the pedal axle. That torque is then used for the gear with radius r_g about the pedal axle, and the chain wheel will feel a force of $F_0 r_p / r_g$. This force is then transferred to the back wheel. If the wheel gear has a radius of R_g, then the torque on the wheel from the string will be $F_0 r_p R_g / r_g$.

Now the force of friction exerts a torque of fR, where R is the radius of the back wheel. The conditions for moving forward are $f > F_0$ (net force points forward), and $F_0 r_p R_g / r_g > fR$ (the net torque about the center of the back wheel will make it turn forward). Thus the forward criterion is

$$f > F_0 > f \frac{R r_g}{r_p R_g}$$

$$\frac{R}{R_g} < \frac{r_p}{r_g}.$$

The opposite is true for backwards movement:

$$\frac{R}{R_g} > \frac{r_p}{r_g}.$$

Typically, the gearings on bikes will be set so that the bike will move backwards.

(b) The chain-wheel should rotate in the same sense as the rear wheel. The chain rotates the axle, and the wheel rotates with the axle.

(c) The lower pedal will move backwards. This is because positive work must be done on the bike to let it move, so the force and displacement should go in the same direction.

[4] **Problem 15.** APhO 2005, problem 1B. A problem on parametric resonance, an idea we first encountered in **M4**. The problem was so subtle that the APhO problem writers themselves could not agree on what the correct answer was!

Solution. See solution 1B of the official solutions.

[4] **Problem 16.** INPhO 2020, problem 5. A tough angular collision problem.

Solution. See the official solutions here.

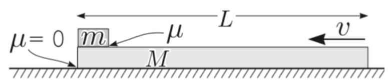
[5] **Problem 17.** EuPhO 2019, problem 2. A tough problem about the motion of an rigid body in a magnetic field.

Solution. See the official solutions here.

3 Frictional Losses

These miscellaneous problems are grouped under the theme of friction or energy dissipation.

[2] **Problem 18** (Kalda)**.** A plank of length L and mass M lies on a frictionless horizontal surface; on one end sits a small block of mass m.



The coefficient of friction between the block and plank is μ. The plank is sharply hit and given horizontal velocity v. What is the minimum v required for the block to slide across the plank and fall off the other end?

Solution. If the block barely is able to slide off, then right before it does, it has relative velocity of 0 with the plank. By momentum conservation the velocities are $\frac{mv}{m+M}$, so the energy loss is

$$\Delta E = \frac{1}{2}mv^2 - \frac{1}{2}(m+M)\left(\frac{mv}{m+M}\right)^2 = \frac{1}{2}\frac{mM}{m+M}v^2.$$

But this is also μmgL, so $v = \sqrt{2\mu gL(1+m/M)}$.

[3] **Problem 19** (BAUPC)**.** A uniform sheet of metal of length ℓ lies on a roof inclined at angle θ, with coefficient of kinetic friction $\mu > \tan\theta$. During the daytime, thermal expansion causes the sheet to uniformly expand by an amount $\Delta\ell \ll \ell$. At night, the sheet contracts back to its original length. What is the displacement of the sheet after one day and night?

Solution. When the sheet expands/contracts, it should do so about a certain point that doesn't move by continuity (the opposite ends move in opposite directions). Additionally, the forces from the expansion/contraction should balance so the point remains stationary.

If the center of expansion is a distance x up from the bottom of the sheet, then the compressional force balance for a sheet with linear mass density ρ will be

$$\mu x\rho g\cos\theta - \rho gx\sin\theta = \mu(\ell - x)\rho g\cos\theta + \rho g(\ell - x)\sin\theta$$

$$\mu - \tan\theta = \mu\frac{\ell}{x} - \mu + \frac{\ell}{x}\tan\theta - \tan\theta$$

$$x = \frac{\mu + \tan\theta}{2\mu}\,\ell.$$

For contraction, the tension at the stationary point a distance y up from the bottom of the sheet has a force balance equation of

$$\mu y\rho g\cos\theta + \rho g y\sin\theta = \mu(\ell - y)\rho g\cos\theta - \rho g(\ell - y)\sin\theta$$

$$y = \ell - x = \frac{\mu - \tan\theta}{2\mu}\,\ell$$

When the sheet expands by an amount $\Delta\ell$, the distance each point moves is proportional to the distance away from the stationary point since the expansion is uniform. The stationary point for contraction is a distance of $x - y = \ell\tan\theta/\mu$ away from the stationary point for contraction, and will move a distance of $\Delta\ell(x-y)/\ell$ down (away from the expansionary point), and stay stationary for contraction. The net displacement for all the points is this distance, which is

$$\frac{\tan\theta}{\mu}\,\Delta\ell.$$

[3] **Problem 20.** APhO 2010, problem 1A. A question about a different kind of inelastic collision.

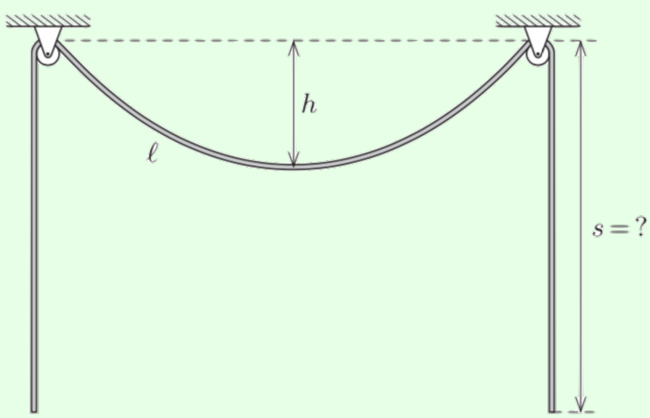
[5] **Problem 21.** IdPhO 2020, problem 2. A nice problem on anisotropic friction.

Solution. See the official solutions here.

4 Ropes, Wires, and Chains

Example 6: MPPP 78

A uniform flexible rope passes over two small frictionless pulleys mounted at the same height.



The length of rope between the pulleys is ℓ, and its sag is h. In equilibrium, what is the length s of the rope segments that hang down on either side?

Solution

The problem can be attacked by differential equations, but there is an elegant solution using only algebra. We let our unknowns be s, the tension $\mathbf{T}_1 = (T_{1,x}, T_{1,y})$ in the rope at the pulley, and the tension T_2 at the lowest point.

Considering the entire sagging portion as the system, vertical force balance gives

$$2T_{1,y} = \lambda \ell g, \quad T_{1,y} = \lambda \ell g/2.$$

Now consider half of the sagging portion as the system. Horizontal force balance gives

$$T_2 = T_{1,x}.$$

Finally, consider one of the hanging portions as the system. Then

$$T_1 = \lambda g s.$$

We hence have three equations, but four unknowns.

For the final equation, we need to consider how the tension changes throughout the rope. This would usually be done by a differential equation, but there is a clever approach using conservation of energy. Suppose we cut the rope somewhere, pull out a segment dx, and reattach the two ends. This requires work $T\,dx$, where T is the magnitude of the local tension. Now suppose we cut the rope somewhere else, separate the ends by dx, and paste our segment inside. This requires work $-T'\,dx$. After this process, the rope is exactly in the same state it was before, so the total work done must be zero.

This would seem to prove that $T = T'$, which is clearly wrong. The extra contribution is that if the two locations have a difference in height Δy, then it takes work $\lambda g(\Delta y)\,dx$ to move the segment from the first to the second. So in equilibrium, for any two points of the rope,

$$\Delta T = \lambda g\,\Delta y.$$

Therefore, we have

$$T_1 - T_2 = \lambda g h.$$

Now we're ready to solve. We have

$$T_1^2 - T_2^2 = (\lambda \ell g/2)^2$$

from our first three equations, and dividing by this new relation gives

$$T_1 + T_2 = \lambda g\,\frac{\ell^2}{4h}.$$

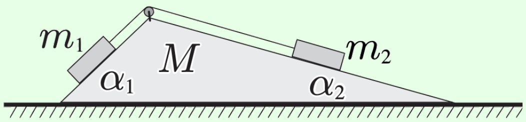
This allows us to solve for T_1, which gives

$$s = \frac{T_1}{\lambda g} = \frac{h}{2} + \frac{\ell^2}{8h}.$$

This is a useful result in real engineering projects: it means that the tension in a cable can be estimated by seeing how much it sags.

Example 7: Kalda 27

A wedge with mass M and acute angles α_1 and α_2 lies on a horizontal surface. A string has been drawn across a pulley situated at the top of the wedge, and its ends are tied to blocks with masses m_1 and m_2.



There is no friction anywhere. What is the acceleration of the wedge?

Solution

This is a classic example of a problem best solved with the Lagrangian-like techniques of **M4**. By working in generalized coordinates, we won't have to solve any systems of equations.

Let s be the distance the rope moves through the pulley, so that both blocks have speed $\dot{s}$ in the noninertial frame of the wedge. The "generalized force" is

$$F_{\text{eff}} = -\frac{dV}{ds} = (m_1 \sin\alpha_1 - m_2 \sin\alpha_2)g.$$

Now, the kinetic energy in the lab frame will be of the form

$$K = \frac{1}{2} M_{\text{eff}} \dot{s}^2$$

which means that, by the Euler–Lagrange equations,

$$\ddot{s} = \frac{F_{\text{eff}}}{M_{\text{eff}}}.$$

Our task is now to calculate M_{eff}. Since the center of mass of the system can't move horizontally, the wedge has speed

$$v_w = \frac{m_1 \cos\alpha_1 + m_2 \cos\alpha_2}{M + m_1 + m_2} \dot{s}.$$

Now, it's a bit annoying to directly compute the kinetic energy K in the lab frame, but it's easy to compute the kinetic energy in the frame of the wedge: it's simply $(m_1 + m_2)\dot{s}^2/2$. But the two are also related simply,

$$K + \frac{1}{2}(M + m_1 + m_2)v_w^2 = \frac{1}{2}(m_1 + m_2)\dot{s}^2.$$

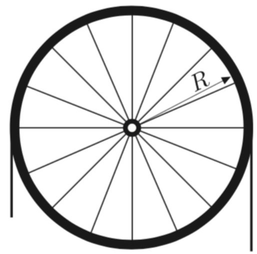
Using this to solve for K, we conclude

$$M_{\text{eff}} = m_1 + m_2 - \frac{(m_1 \cos\alpha_1 + m_2 \cos\alpha_2)^2}{M + m_1 + m_2}.$$

Finally, the desired answer is

$$a_w = \frac{m_1 \cos\alpha_1 + m_2 \cos\alpha_2}{M + m_1 + m_2} \ddot{s}.$$

[3] **Problem 22** (Kalda). A rope of mass per unit length ρ and length L is thrown over a pulley so that the length of one hanging end is ℓ. The rope and pulley have enough friction so that they do not slip against each other.



The pulley is a hoop of mass m and radius R attached to a horizontal axle by light spokes. Find the force on the axle immediately after the motion begins.

Solution. Let the distance the rope moves along the pulley be represented by the coordinate q. The kinetic energy of the rope is $\frac{1}{2}\rho L\dot{q}^2$ since every section of the rope moves with velocity $\dot{q}$. Without slipping, the kinetic energy of the pulley is $\frac{1}{2}m\dot{q}^2$. Another consequence of no slipping is that energy is conserved, so $dK/dt = -dU/dt$.

When the rope moves along a small distance of dq, the change in potential energy can be calculated by considering a segment dq moving from one end to another, having a difference in vertical height of $L - \pi R - 2\ell$. Thus $dU = -\rho g dq(L - \pi R - 2\ell)$.

$$\frac{dK}{dt} = (\rho L + m)\dot{q}\ddot{q} = \rho g\dot{q}(L - \pi R - 2\ell)$$

$$\ddot{q} = g\frac{\rho(L - \pi R - 2\ell)}{\rho L + m}.$$

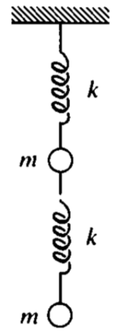
The vertical normal force can be determined by $\sum m_i(a_y)_i$ of the rope. The acceleration of the parts moving up and down will cancel out, so the acceleration of the center of mass can be found by considering the "extra" segment of length $L - \pi R - 2\ell$. The net vertical force on the system is then $\rho(L - \pi R - 2\ell)\ddot{q}$, so the force on the axle N satisfies $(m + \rho L)g - N = \rho(L - \pi R - 2\ell)\ddot{q}$.

$$N_y = g\frac{(\rho L + m)^2 - \rho^2(L - \pi R - 2\ell)^2}{\rho L + m}.$$

Since rope is transferred to the right, there is also a horizontal component of force on the axle. When the rope moves along a distance dq, $\sum m_i dx_i$ is essentially $\rho dq(2R)$ since it can be seen as a segment of length dq moving to the other side. Thus $F_x = \rho\ddot{q}(2R)$.

$$N_x = 2\rho R g\frac{\rho(L - \pi R - 2\ell)}{\rho L + m}.$$

[3] **Problem 23** (French 5.10). Two equal masses are connected as shown with two identical massless springs of spring constant k.



Considering only motion in the vertical direction, show that the ratio of the frequencies of the two normal modes is $(\sqrt{5}+1)/(\sqrt{5}-1)$.

Solution. Let y_1 denote the displacement of the upper mass and y_2 for the lower mass. The equations of motion are

$$m\ddot{y}_1 = -ky_1 - k(y_1 - y_2) = -2ky_1 + ky_2, \quad m\ddot{y}_2 = -ky_2 + ky_1$$

For normal modes, the particles will oscillate at the same frequency. Guessing a form $y_1 = Ae^{i(\omega t+\phi_1)} = \tilde{A}e^{i\omega t}$ and $y_2 = \tilde{B}e^{i\omega t}$ and defining $\alpha = \omega/\sqrt{k/m}$, the equations are

$$-\omega^2\tilde{A} = -2\omega_0^2\tilde{A} + \omega_0^2\tilde{B}, \quad -\omega^2\tilde{B} = -\omega_0^2\tilde{B} + \omega_0^2\tilde{A}.$$

Dividing these equations gives

$$\frac{\tilde{A}}{\tilde{B}} = \frac{1}{2-\alpha^2} = \frac{1-\alpha^2}{1}.$$

Solving for α, we find

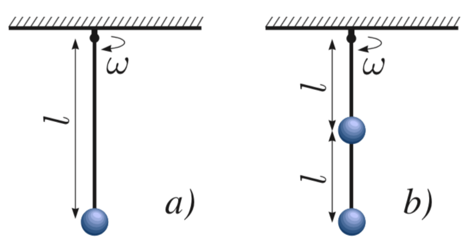
$$\alpha^4 - 3\alpha^2 + 1 = 0, \quad \alpha^2 = \frac{3 \pm \sqrt{5}}{2}$$

Then the ratio of the two normal mode frequencies is

$$\frac{\alpha_1}{\alpha_2} = \sqrt{\frac{1+5+2\sqrt{5}}{1+5-2\sqrt{5}}} = \frac{\sqrt{5}+1}{\sqrt{5}-1}$$

as desired.

[3] **Problem 24** (Kalda)**.** A massless rod of length ℓ is attached to the ceiling by a hinge which allows the rod to rotate in a vertical plane.



The rod is initially vertical and the hinge is spun with a fixed angular velocity ω.

(a) If a mass m if attached to the bottom of the rod, find the maximum ω for which the configuration is stable.

(b) [A] Now suppose another mass m and rod of length ℓ is attached to the first mass by an identical hinge that turns in the same direction, as shown above. Find the maximum ω for which the configuration is stable. (Hint: the configuration is unstable if *any* infinitesimal change in the angles of the rods can lower the energy.)

Solution. (a) In the rotating reference frame rotating with angular velocity ω, the effective potential from the centrifugal force $m\omega^2 r$ is $-\int m\omega^2 r dr = -\frac{1}{2}m\omega^2 r^2$, where r is the distance from the vertical axis through the hinge. In this setup, $r = \ell \sin\theta$ where θ is the angle between the rod and the vertical. The potential energy from gravity is $mg\ell(1-\cos\theta)$. For small angles, the potential energy is

$$U \approx mg\ell\left(1 - \left(1 - \frac{1}{2}\theta^2\right)\right) - \frac{1}{2}m\omega^2\ell^2\theta^2 = \frac{1}{2}\theta^2(mg\ell - m\omega^2\ell^2)$$

The system is stable when $U''(\theta) > 0$, so the maximum value of ω for stability is

$$\omega_{\max} = \sqrt{g/\ell}.$$

(b) Let the angles between the vertical and the upper, lower rods be θ_1, $\theta_2 \ll 1$ respectively. The gravitational potential energy of the lower mass is $mg\ell(1-\cos\theta_1) + mg\ell(1-\cos\theta_2)$, and the Taylor expansion gives $U_2 = \frac{1}{2}mg\ell(\theta_1^2 + \theta_2^2)$. Using the rotating reference frame again, the potential energy from the centrifugal force is $\frac{1}{2}m\omega^2 r^2$, where $r = \ell(\sin\theta_1 + \sin\theta_2) \approx \ell(\theta_1 + \theta_2)$ since the hinges go in the same direction. The total potential of the system (same potential for the first mass) is then

$$\begin{aligned} U(\theta_1, \theta_2) &= mg\ell\theta_1^2 + \frac{1}{2}mg\ell\theta_2^2 - \frac{1}{2}m\omega^2\ell^2\theta_1^2 - \frac{1}{2}m\omega^2\ell^2(\theta_1+\theta_2)^2 \\ &= m\ell^2\left((\omega_0^2 - \omega^2)\theta_1^2 + \frac{1}{2}(\omega_0^2 - \omega^2)\theta_2^2 - \omega^2\theta_1\theta_2\right). \end{aligned}$$

Where $\omega_0^2 = g/\ell$. To be stable, we want the potential energy to be at a local minimum near that point. For this multivariable function, we will do the second derivative test (local minimum if $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} > 0$, where $f_x = \partial f/\partial x$). That means the condition is

$$2(\omega_0^2 - \omega^2)^2 - \omega^4 > 0 \qquad \omega < \omega_0.$$

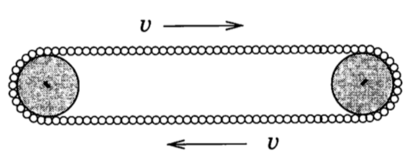
Solving for the roots in the first equation yields

$$\omega^2 = \omega_0^2(2 \pm \sqrt{2}).$$

Since $\omega < \omega_0$, we take the smaller root to find the upper bound for a stable configuration:

$$\omega_{\max} = \sqrt{\frac{g}{l}(2 - \sqrt{2})}$$

[3] **Problem 25** (PPP 104). A flexible chain of uniform density is wrapped tightly around two cylinders as shown.

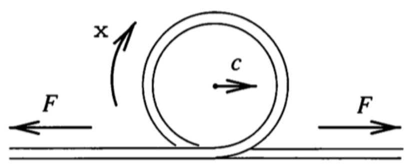


The cylinders are made to rotate and cause the chain to move with speed v. The chain suddenly slips off the cylinders and falls vertically. How does the shape of the chain vary during the fall?

Solution. Firstly, notice that the problem is equivalent to the following question: Given a chain in the plane moving such that each point has velocity directed along the chain and speed v, what happens to the chain. Note that there is no gravity since all the interactions are happening in the xy plane, so we can transform into the CM frame.

Interestingly enough, the answer is that the chain maintains its shape. Consider a little piece of chain dx, and suppose the radius of curvature there is r. We see that if the tension there is T, then $Td\theta$ is the centripetal force, so $Td\theta = (\lambda dx)v^2/r$, so $T = \lambda v^2$ where λ is the linear mass density. In particular, this is independent of r. Therefore, all the pieces are in equilibrium, and so stay there, as they have no reason to move.

[4] **Problem 26** (PPP 106)**.** A long, heavy flexible rope with mass ρ per unit length is stretched by a constant force F. A sudden movement causes a circular loop to form at one end of the rope.



The center of the loop moves with speed c as shown.

(a) Calculate the speed c, assuming gravity is negligible.

(b) Find the energy E carried by a loop rotating with angular frequency ω.

(c) Show that the momentum p carried by the loop obeys $E = pc$. This is true for waves in general, as we'll see in **W1**.

(d) Find the angular momentum carried by the loop.

Solution. (a) By balancing forces on a small piece of the rope,

$$\rho(c^2/R)(Rd\theta) = Fd\theta$$

which gives $F = \rho c^2$, so $c = \sqrt{F/\rho}$.

(b) The mass of the loop is $m = 2\pi R\rho$. Splitting the energy into center of mass energy and rotational energy, we have

$$E = \frac{1}{2}mc^2 + \frac{1}{2}(mR^2)\omega^2 = mc^2 = 2\pi RF = \frac{2\pi Fc}{\omega},$$

since $c = \omega R$.

(c) Since the loop as a whole moves with speed c and has mass m, we have $p = mc$. Since $E = mc^2$, we have $E = pc$ as desired.

(d) The orbital angular momentum is

$$L_o = mcR = \frac{mc^2}{\omega}$$

while the spin angular momentum is

$$L_s = I\omega = mR^2\omega = \frac{mc^2}{\omega}.$$

Thus the total angular momentum is $2mc^2/\omega$.

5 [A] Advanced Mathematical Techniques

The following problems were cut from earlier problem sets because they required more advanced math; however, they illustrate some very neat and important ideas.

[3] **Problem 27.** In **P1**, you found a general expression for the period of a pendulum oscillating with amplitude θ_0 in terms of an integral, then approximated the integral to find

$$\omega = \omega_0 \left(1 - \frac{\theta_0^2}{16} + O(\theta_0^4)\right)$$

where $\omega_0 = \sqrt{g/L}$. In this problem, we will show a different way to get the same answer, by solving the equation of motion approximately. We write the solution $\theta(t)$ as a series in θ_0. The overall solution is of order θ_0, and the corrections only depend on θ_0^2, so we can write

$$\theta(t) = \theta_0 f_0(t) + \theta_0^3 f_1(t) + \theta_0^5 f_2(t) + \ldots$$

where all the functions $f_i(t)$ are of order 1. Then we plug this expansion into Newton's second law, $\ddot{\theta} + \omega_0^2 \sin\theta = 0$, and expand it out order by order in θ_0.

(a) A naive first guess is to set $f_0(t)$ so that it cancels precisely the order θ_0 terms in this equation, then set $f_1(t)$ to cancel the order θ_0^3 terms, and so on. Using this guess, show that

$$\ddot{f}_0 + \omega_0^2 f_0 = 0, \quad \ddot{f}_1 + \omega_0^2 f_1 = \frac{\omega_0^2 f_0^3}{6}$$

where the first equation has solution $f_0(t) = \cos(\omega_0 t)$.

Unfortunately, this decomposition is not very useful. The problem is that two things are going on at once: the oscillations are not quite sinusoidal, and they have an angular frequency lower than ω_0. The expansion we've done would be useful if we only had the first effect, because then $f_1(t)$ would just capture the small, non-sinusoidal corrections to $f_0(t)$. But our method can't account for the frequency shift; by construction, $f_0(t)$ always oscillates at angular frequency ω_0. Over time, the real oscillation $\theta(t)$ gets out of phase with $f_0(t)$. This manifests itself as a "secular growth" in $f_1(t)$, i.e. it increases in magnitude every cycle until it has a huge value, of order $1/\theta_0^2$, and our perturbative expansion breaks down.

(b) Write the right-hand side of the differential equation for $f_1(t)$ as a sum of sinusoids, and show that it contains a term proportional to $\cos(\omega_0 t)$. This resonantly drives $f_1(t)$, causing the secular growth.

(c) We can salvage our perturbative expansion using the method of "renormalized" frequencies. We impose by fiat that $f_0(t)$ oscillates at the true angular frequency, letting

$$\ddot{f}_0 + \omega^2 f_0 = 0, \quad \omega = \omega_0(1 - c\theta_0^2 + O(\theta_0^4)).$$

Because of this choice, the differential equation for $f_1(t)$, which contains all terms at order θ_0^3, will be altered. The correct choice of ω is precisely the one for which this eliminates the secular growth of $f_1(t)$. Using this idea, show that $c = 1/16$.

If you keep going, you'll find the next term $f_2(t)$ still has secular growth. We can remove it by having both $f_0(t)$ and $f_1(t)$ oscillate at angular frequency $\omega_0(1-\theta_0^2/16+d\theta_0^4)$, where d is chosen to cancel the secular growth of $f_2(t)$. In this way, the frequency can be found to any order in θ_0^2. (This technique is called the method of strained coordinates. It's an example of multiple-scale analysis.)

Solution. (a) Plugging everything in and using $\sin\theta = \theta - \theta^3/6 + O(\theta^5)$, we have

$$\theta_0 \ddot{f}_0 + \theta_0^3 \ddot{f}_1 + \omega_0^2 \left(\theta_0 f_0 + \theta_0^3 f_1^3 - \frac{1}{6}\theta_0^3 f_0^3 + O(\theta_0^5) \right) = 0.$$

Collecting the order θ_0 and θ_0^3 terms gives the desired result.

(b) The easiest way to do this is to use the definition of $\cos(\omega_0 t)$ in terms of complex exponentials,

$$\cos^3(\omega_0 t) = \left(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \right)^3 = \frac{e^{3i\omega_0 t} + 3e^{i\omega_0 t} + 3e^{-i\omega_0 t} + e^{-3i\omega_0 t}}{8} = \frac{1}{4}\cos(3\omega_0 t) + \frac{3}{4}\cos(\omega_0 t).$$

Another way is to remember the cosine triple angle identity, but who knows that?

(c) Adjusting ω_0 to the renormalized angular frequency for f_0 moves terms between the two differential equations, so that now we have

$$\ddot{f}_0 + \omega^2 f_0 = 0, \quad \ddot{f}_1 + \omega_0^2 f_1 = \omega_0^2 \left(\frac{f_0^3}{6} - 2c f_0 + O(\theta_0^2) \right).$$

The part of the right-hand side that oscillates at angular frequency ω_0 is

$$\omega_0^2 \left(\frac{1}{6}\frac{3}{4}\cos(\omega_0 t) - 2c\cos(\omega_0 t) \right)$$

from which we conclude $c = 1/16$.

[3] **Problem 28.** You might be wondering how we can solve the weakening spring problem from **M4** without anything fancy like the adiabatic theorem. There is a general technique to solve linear differential equations whose coefficients are slowly varying. First, write the equation of motion as

$$\ddot{x} + \omega^2(t) x = 0.$$

Then expand $x(t)$ as

$$x(t) = A(t) e^{i\phi(t)}, \quad \dot{\phi}(t) = \omega(t).$$

The point of writing $x(t)$ this way is that pulling out the factor of $e^{i\phi(t)}$ will automatically account for the rapid oscillations. The factor $A(t)$ only varies slowly, so it's easier to handle by itself.

(a) Evaluate $\ddot{x}(t)$ and plug it into the equation of motion.

(b) Using the fact that $A(t)$ and $\omega(t)$ vary slowly, throw out small terms in your equation from part (a), until you get a differential equation you can easily integrate. This is a simple example of the WKB approximation.

(c) Show that this gives the expected final result for a weakening spring.

Solution. (a) Just carrying out the time derivatives using the product rule gives

$$\ddot{x} = \ddot{A}e^{i\phi} + 2i\omega\dot{A}e^{i\phi} + i\dot{\omega}Ae^{i\phi} - \omega^2 Ae^{i\phi}.$$

Plugging this back into the equation of motion, the last term cancels, and we can cancel an overall factor of $e^{i\phi}$ to get

$$\ddot{A} + 2i\omega\dot{A} + i\dot{\omega}A = 0.$$

(b) Let's think carefully about how big each of these terms is. If the total time it takes for the spring to weaken is T, where $\omega T \ll 1$, then each time derivative on A or ω multiplies the magnitude of the term by roughly $1/T$. So the first term is of order A/T^2, while the other two are of order $\omega A/T \gg A/T^2$. Therefore, we can throw out the first term and to get

$$\frac{2\dot{A}}{A} = -\frac{\dot{\omega}}{\omega}$$

which is equivalent to

$$\frac{d\log(A^2)}{dt} = \frac{d\log(1/\omega)}{dt}.$$

(c) The above result tells us that $A^2\omega$ is constant, so $A \propto k^{-1/4}$ as found in **M4**.

[4] **Problem 29** (BAUPC 1996)**.** A mass M is located at the vertex of an angle $\theta \ll 1$ formed by two massless sticks of length ℓ. The structure is held so that the left stick is initially vertical, then released. The right stick hits the ground at time $t = 0$. The structure then rocks back and forth, coming to a stop at time $t = T$.

(a) Prove the identity

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$$

using the result $\sum_{n\geq 1} 1/n^2 = \pi^2/6$.

(b) Calculate T to leading order in θ. Performing the expansion is rather tricky and requires the previous identity.

Solution. See the official solutions here.

[3] **Problem 30.** In this problem, we'll go through Laplace's slick derivation of Kepler's first law. Throughout, we assume the orbit takes place in the xy plane, with the Sun at the origin.

(a) Show that

$$\ddot{x} = -\frac{\gamma x}{r^3}, \quad \ddot{y} = -\frac{\gamma y}{r^3}$$

where γ is a constant that depends on the parameters.

(b) Show that

$$\frac{d}{dt}(r^3\ddot{x}) = -\gamma\dot{x}, \quad \frac{d}{dt}(r^3\ddot{y}) = -\gamma\dot{y}.$$

(c) Show that

$$\frac{d}{dt}(r^3\ddot{r}) = -\gamma\dot{r}.$$

(Hint: this can get messy. As a first step, try showing the left-hand side is equal to $(r^2/2)\,d^3(r^2)/dt^3$. You will have to switch variables to x and y and then switch back; for these purposes it's useful to use the results of part (a), and the definition $r^2 = x^2 + y^2$.)

(d) Define $\psi(t) = r(t)^3$. In parts (b) and (c), we have shown that the differential equation

$$\frac{d}{dt}\left(\psi(t)\,\frac{du}{dt}\right) = -\gamma u$$

has three solutions, namely $\dot{x}$, $\dot{y}$, and $\dot{r}$. Any second-order linear differential equations only has two independent solutions. If $\dot{x}$ and $\dot{y}$ are not independent, the orbit is simply a line, which is trivial. Assuming that doesn't happen, they are independent, so $\dot{r}$ must be a linear combination of them,

$$\dot{r} = A\dot{x} + B\dot{y}.$$

Use this result to argue that the orbit is a conic section.

Solution. (a) This just follows from $F = ma$. In terms of the usual parameters, $\gamma = GM$.

(b) This immediately follows from clearing denominators in the results of part (a) and differentiating both sides.

(c) Following the hint, we have

$$\frac{d}{dt}(r^3\ddot{r}) = r^3\,\dddot{r} + 3r^2\dot{r}\ddot{r} = \frac{1}{2}r^2\frac{d^3}{dt^3}(r^2) = r^2\frac{d^2}{dt^2}(r\dot{r}).$$

At this point, we switch back to x and y. By differentiating $r^2 = x^2 + y^2$,

$$r\dot{r} = x\dot{x} + y\dot{y}.$$

Plugging this in gives

$$\frac{d}{dt}(r^3\ddot{r}) = r^2\frac{d^2}{dt^2}(x\dot{x} + y\dot{y}) = r^2\frac{d}{dt}(x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2).$$

We see that we'll have a lot of factors involving $\ddot{x}$ and $\ddot{y}$, but we know how to handle these using part (a). Using part (a) several times, we have

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = -\frac{\gamma}{r^3}(x\dot{x} + y\dot{y}) = -\frac{\gamma}{r^3}(r\dot{r}) = -\frac{\gamma\dot{r}}{r^2}$$

and

$$x\ddot{x} + y\ddot{y} = -\frac{\gamma}{r^3}(x^2 + y^2) = -\frac{\gamma}{r}.$$

Plugging these results in, we find

$$\frac{d}{dt}(r^3\ddot{r}) = r^2\left(-\frac{d}{dt}\left(\frac{-\gamma}{r}\right) - \frac{2\gamma\dot{r}}{r^2}\right) = -\gamma\dot{r}$$

just as desired.

(d) Integrating both sides,

$$r = Ax + By + C.$$

But then squaring both sides shows that the equation of the orbit is just a quadratic in x and y, which is precisely the form of a conic section in Cartesian coordinates. You can also show that the focus is at the origin, though this requires a bit more knowledge about conics.

This question was inspired by this paper, which has a few more derivations of Kepler's first law.

6 Mechanics and Geometry

For dessert, we'll consider a few cute problems that relate statics to geometry.

Example 8

Given a triangle ABC, the Fermat point is the point X that minimizes $AX + BX + CX$. Design a machine that finds the Fermat point.

Solution

We take a horizontal plane and drill holes at points A, B, and C. A mass M on a rope is fed through each hole, and the three ends of the rope are tied together at point X. The gravitational potential energy is proportional to $AX + BX + CX$, so in equilibrium X lies on the Fermat point. Moreover, since the tensions in each rope are all equal to Mg, force balance requires $\angle AXB = \angle BXC = \angle CXA = 120°$.

[1] **Problem 31.** Using similar reasoning, design a machine that finds the point X that minimizes $(AX)^2 + (BX)^2 + (CX)^2$. What geometrical property can you conclude about this point?

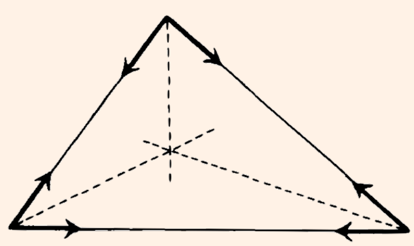
Solution. Attach springs at each of A, B, C with spring constant k, then X minimizes the PE, so it is in equilibrium. Therefore, $k(\mathbf{X} - \mathbf{A}) + k(\mathbf{X} - \mathbf{B}) + k(\mathbf{X} - \mathbf{C}) = 0$, so X is the centroid of ABC. Alternatively, we could have set the gradient of $AX^2 + BX^2 + CX^2$ to be 0.

Example 9

Show that the incenter of a triangle (i.e. the meeting point of the angle bisectors) exists.

Solution

Apply six forces at the vertices of a triangle as shown.



These forces clearly balance, and also produce no net torque on the triangle. Now combine the forces applied at each vertex, yielding three forces that point along the angle bisectors.

By the principles of **M2**, the torques of these forces can only balance if their lines of action meet at a point. Therefore the angle bisectors are concurrent, so the incenter exists.

Example 10

Let AB be a diameter of a circle, and let a mass be free to slide on the circle. The mass is connected to two identical straight springs of zero rest length, which are in turn connected to points A and B. At what points C can the mass be in static equilibrium?

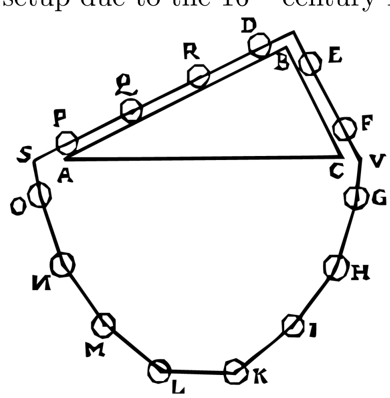
Solution

The potential energy of the system is proportional to $(\text{AC})^2 + (\text{BC})^2$. Since ABC is a right triangle, this is just equal to $(\text{AB})^2$ by the Pythagorean theorem. Since the potential energy doesn't depend on where the mass is, it can be at static equilibrium at any point on the circle. Alternatively, you can show that the mass is in static equilibrium by force balance, and use the reasoning in reverse to derive the Pythagorean theorem.

[1] **Problem 32.** Consider a right triangle ABC filled with a fluid of uniform pressure. Using torque balance, establish the Pythagorean theorem.

Solution. Suppose $\angle C = 90$, and suppose the pressure is p. Taking torques about C, we see that $pa \cdot (a/2) + pb \cdot (b/2) - pc \cdot (c/2) = 0$, or $a^2 + b^2 = c^2$.

[1] **Problem 33.** Shown below is a setup due to the 16$^{\text{th}}$ century mathematician Stevin.



One might argue that because there are more masses on AB than on BC, this is a perpetual motion machine that turns counterclockwise. By using the fact that perpetual motion machines don't actually exist, prove the law of sines.

Solution. For each mass on AB, the component of gravity along AB is proportional to $\sin \angle BAC$. Furthermore, the number of masses is proportional to $\overline{AB}$. This must be balanced by the masses along BC, giving

$$AB \sin \angle BAC = BC \angle BCA$$

which after minor rearrangement is the law of sines.

[2] **Problem 34.** Consider the n-sided polygon P of least possible area that circumscribes a closed convex curve K. Prove that every tangency point of K with a side of P is the midpoint of that side. (Hint: begin by supposing that the area outside P is filled with a gas of uniform pressure, with a vacuum inside P.)

Solution. The minimum energy is achieved when the gas takes up the largest possible area, i.e. when the polygon P has minimum area. Let's model the polygon as being formed by n infinite rods, which don't push on each other. Now, in equilibrium, the torque on each rod must be zero, but the only forces on the rod are the uniform pressure along the part of the rod making up the corresponding polygon side, and the normal force at the contact point. Taking torques about the contact point shows that it must be the midpoint.

[2] **Problem 35.** In this problem we'll derive Kepler's first law yet again, using no calculus, but a bit of Euclidean geometry. As usual, we suppose a planet of mass m orbits a fixed star of much greater mass M. Placing the star at the origin, let ϕ be the angle between $\mathbf{r}$ and $\mathbf{v}$ for the planet.

(a) Write down the quantities E and L in terms of G, M, m, v, r, and ϕ, and show that

$$\left(r^2 + \frac{GMm}{E} r\right) \sin^2 \phi = \frac{L^2}{2mE}.$$

(b) Now consider an ellipse with semimajor axis a and eccentricity e, meaning that the distance between the foci is $2ae$, with one of the foci F at the origin. Consider a point P on the ellipse, so that the angle between the tangent to the ellipse at P and FP is ϕ. If $r = |FP|$, show that

$$(r^2 - 2ar) \sin^2 \phi = -a^2(1 - e^2).$$

You will have to use the geometrical property that a light ray sent from one focus will reflect at the ellipse to hit the other focus.

(c) By comparing your results for (a) and (b), conclude that the orbit is an ellipse with

$$a = -\frac{GMm}{2E}, \quad e = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}.$$

Solution. (a) By definition, we have

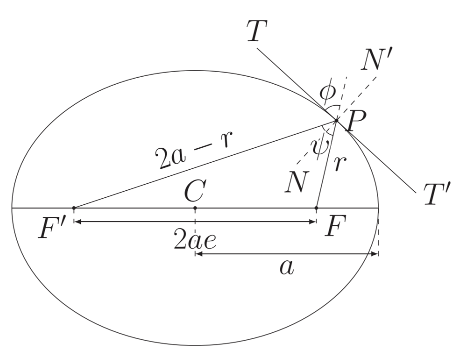
$$L = mrv \sin \phi, \quad E = \frac{1}{2}mv^2 - \frac{GMm}{r}.$$

Solving the second equation for v gives

$$v = \sqrt{\frac{2E}{m} + \frac{2GM}{r}}.$$

Plugging this into the first equation, squaring, and rearranging gives the desired result.

(b) Refer to the below diagram, where TT' is the tangent to the ellipse and NN' is the normal.



By the law of cosines,

$$(2ae)^2 = (2a - r)^2 + r^2 - 2r(2a - r)\cos\psi.$$

By the geometrical properties of the ellipse, NN' is the angle bisector of $\angle F'PF$, so

$$\cos\psi = \cos(\pi - 2\phi) = -\cos(2\phi) = 2\sin^2\phi - 1.$$

Plugging this into the law of cosines and rearranging gives the desired result.

(c) This follows immediately, from inspection. Note that this all breaks down for $E \geq 0$, since in that case the trajectory isn't an ellipse, but similar derivations can be performed for the parabola and hyperbola.