Modern I: Semiclassical Mechanics

The basics of quantum mechanics can be found in chapters 46 and 47 of Halliday and Resnick, and are covered more thoroughly in chapters 3 through 6 in Krane. Chapter 10 of Krane covers quantum statistical mechanics as used in the final section. For a complete, but advanced treatment of the WKB approximation, see chapter 9 of Griffiths' *Introduction to Quantum Mechanics* (3rd edition). For some nice conceptual discussion, see chapters I-37 and I-38 of the Feynman lectures, or if you're ambitious, essentially all of volume III. There is a total of **80** points.

1 The WKB Approximation

A proper introduction to quantum mechanics would take a whole book. Luckily, there is a "semiclassical" regime of quantum mechanics which can be handled with much less machinery. Historically, this regime was discovered first, by scientists like Bohr, and it suffices to explain many quantum effects. To introduce the WKB approximation, we'll start by considering classical standing waves.

Idea 1

The variation of the phase ϕ of a wave is described by its wavenumber and angular frequency,

$$k = \frac{d\phi}{dx}, \quad \omega = \frac{d\phi}{dt}$$

As covered in **W1**, the group velocity is

$$v = \frac{d\omega}{dk}.$$

A standing wave can form if the wave's phase lines back up with itself after one round trip,

$$\oint k\, dx = 2\pi n, \quad n \in \mathbb{Z}.$$

A simple case is a string of length L with fixed ends, where we have

$$2kL = 2\pi n$$

which gives the wavenumbers $k_n = \pi n/L$ and hence the standing wave angular frequencies $\omega_n = \pi v n/L$, as we saw in **W1**.

[1] **Problem 1.** A slightly more subtle case is the case of a string of length L with one fixed and one free end. Show that the standing wave angular frequencies are

$$\omega_n = \frac{\pi v}{L}(n + 1/2).$$

The reason our principle above doesn't give the right answer is that a wave picks up an extra phase shift π when it reflects off a fixed end, so we really should have written

$$\oint k\, dx = 2\pi(n + 1/2)$$

in this case. We didn't run into any problems for two fixed ends, because in that case we get two phase shifts of π, which have no overall effect.

[2] **Problem 2.** Suppose a string of length L is hung from the ceiling. The string has mass density μ, and the bottom of the string is held fixed and pulled down with a force $F \gg gL\mu$. If the string were weightless, then the standing wave angular frequencies would simply be $\pi v n/L$, where $v = \sqrt{F/\mu}$. However, the weight causes the tension and hence the wave speed to vary throughout the rope.

(a) Explain why the wave's angular frequency ω is uniform, i.e. why standing wave solutions are proportional to $\cos(\omega t)$.

(b) Find the angular frequencies of standing waves, including corrections up to first order in $gL\mu/F$.

This is a more quantitative version of a problem we encountered in **W1**.

In quantum mechanics, the state of a particle is described by a wavefunction $\psi(x,t)$ which obeys the Schrodinger equation. When a particle is confined in a finite volume, there are standing wave solutions analogous to those of classical wave mechanics, which have discrete frequencies.

Idea 2: WKB Approximation

The momentum and energy of a quantum particle obey the de Broglie relations

$$p = \hbar k, \quad E = \hbar \omega$$

where E and p are related just as in classical mechanics,

$$E = \frac{p^2}{2m} + V(x).$$

For a particle with reasonably well-defined momentum, the wavefunction is a wavepacket which travels at the group velocity

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp}.$$

This corresponds to the classical velocity of the particle in the classical limit. (For example, you should check that the above equations imply $p = mv_g$, or $p = \gamma m v_g$ when the relativistic momentum and energy are used.)

Just as for a classical standing wave, ω is the same everywhere for quantum standing waves. Since energy is related to frequency, these standing waves are also states of definite energy. In the semiclassical limit, the standing waves must satisfy

$$\oint p\, dx = (2\pi n + \alpha)\hbar = \left(n + \frac{\alpha}{2\pi}\right) h.$$

The extra phase α depends on how the particle gets reflected at the endpoints of its motion.

Remark

The left-hand side of the quantization condition above is precisely the adiabatic invariant from **M4**, which stays the same if we change the system parameters slowly. This ensures the quantization condition is preserved over time, as it must be for self-consistency. If you instead change the system parameters quickly, the integral is not preserved, but that's because the change causes transitions from one energy level to another (i.e. to waves with different n).

[2] **Problem 3.** Consider a one-dimensional box of length L, with hard walls. We can think of these hard walls as a potential $V(x)$ that is zero inside the box and infinite outside the box. For this potential, we simply have $\alpha = 0$.

(a) Find the energy levels of a particle of mass m.

(b) Now suppose the particle is replaced with a photon, with $E = pc$. Find the allowed energies.

The frequencies you found in part (b) correspond to the standing wave frequencies for electromagnetic waves in a box with reflecting (i.e. perfectly conducting) walls.

[2] **Problem 4.** Now consider a particle of mass m in the potential $V(x) = kx^2/2$. In this case, the particle turns around at a point where the potential energy gradually increases from below the particle's energy E to above it. It can be shown that each "soft" boundaries contributes $\pi/2$ to α, so that for this potential we can take $\alpha = \pi$.

Show that the energy levels are

$$E_n = \hbar\omega_0 \left(n + \frac{1}{2}\right), \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

This system is called the quantum harmonic oscillator, and remarkably, this is the exact answer, even though we used an approximation to get it. This result will be used in several problems below.

[3] **Problem 5.** USAPhO 2015, problem A1.

[5] **Problem 6.** IPhO 2006, problem 1. This is a neat problem which illustrates the effect of a gravitational field on quantum particles, as well as the basics of interferometry, a subject developed further in **W2**. Give this a try even if it looks tough; only the ideas introduced above are needed!

Idea 3: Bohr Quantization

In general, $p\,dx$ may be replaced by any generalized momentum/position pair. For example,

$$\oint L\,d\theta = nh.$$

When angular momentum is conserved, the left-hand side is simply $2\pi L$, immediately giving

$$L = n\hbar$$

which is Bohr's quantization condition. In a system of particles rotating together, L stands for the total angular momentum of the system.

Compared to back-and-forth linear motion, covered in idea 2, rotation is different because it's inherently periodic. For rotation, the integer n can be positive or negative, representing a particle going clockwise or counterclockwise. Also, there is no analogue of the α phase factor because the particle just rotates all the way around; it never gets reflected.

Example 1

Find the energy levels and orbit radii of the electron in the hydrogen atom using Bohr quantization.

Solution

We postulate a circular orbit, and quantize the angular momentum. We have

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2}, \quad L = mvr = n\hbar.$$

Solving the second equation for v and plugging into the first gives

$$r = \frac{4\pi\epsilon_0\hbar^2}{me^2} n^2 = a_0 n^2$$

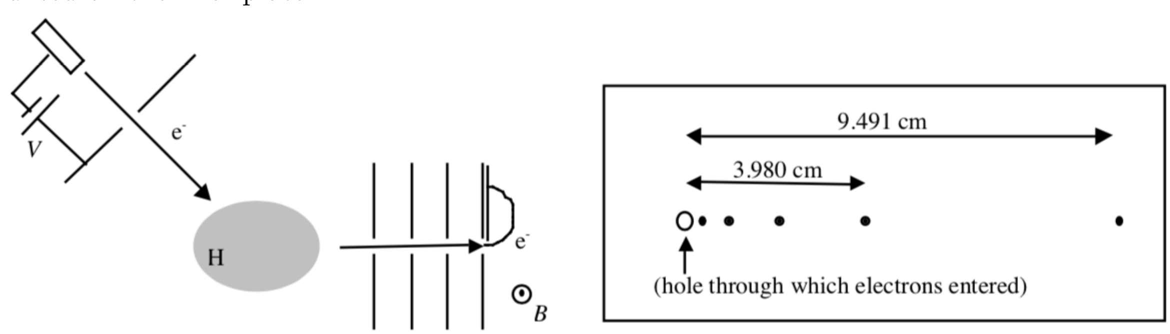
where $a_0 = 5.3 \times 10^{-11}$ m is called the Bohr radius; these are the allowed orbit radii. To get the energies, we use the standard result for circular motion with an inverse square force that the total energy is half the potential energy, so

$$E = -\frac{e^2}{8\pi\epsilon_0 r} = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2}.$$

Evidently, they get more and more closely spaced together as n increases. The big constant in front is called the Rydberg, and is equal to 13.6 eV.

[1] **Problem 7.** Find the energy levels of positronium, a bound state of a positron and electron.

[2] **Problem 8** (USAPhO 2004)**.** Electrons are accelerated from rest through a potential V into a cloud of cold atomic hydrogen. A series of plates with aligned holes select a beam of scattered electrons moving perpendicular to the plates. Immediately beyond the final plate, the electrons enter a uniform magnetic field B perpendicular to the beam; they curve and strike a piece of film mounted on the final plate.



When the film is developed, a series of spots is observed. The distances between the hole and the two most distant spots are measured. You may assume that the film is large enough to have intercepted all of the electrons, i.e. that there are no spots farther from the hole than those shown. The number of spots shown is not necessarily accurate.

Make the approximation that the mass of the hydrogen atom is much larger than the mass of the electron. Assume that each electron scatters off only one atom, which is initially in the ground state (lowest energy state) and has negligible thermal velocity. Determine B, V, and the total number of spots on the film.

[2] **Problem 9.** A rotor consists of two particles of mass m connected by a rigid rod of length L.

(a) Find the energy levels if the particles are not identical.

(b) Find the energy levels if the particles are identical. (Hint: recall that the closed loop integrals in the previous ideas are over paths that take the system back to its original state.)

You might find it disturbing that the result is so different if the particles are or aren't completely identical, but it's a well-verified fact about molecular rotational energy levels. Without the effect of part (b), many predictions of quantum mechanics would come out totally wrong!

[3] **Problem 10.** INPhO 2020, problem 3.

[3] **Problem 11.** INPhO 2016, problem 6. Unfortunately, this question automatically comes with the solutions, but it's still useful to work through.

Remark

In popular science, people sometimes speak of "quantizing" a system as similar to making everything discrete. But as you've seen above, it's more complicated than that. For instance, position never becomes discrete; instead, we integrate over it.

The general rule in quantum mechanics is that confinement to a finite "size" causes the conjugate variable to become discrete. For example, above you looked at several examples of particles bound to potentials. These are confined in space, and hence have discrete orbits in phase space by idea 2, and thus discrete energies. But a free particle not bound to a potential can have any energy, because $E = p^2/2m$ and there is no condition at all on p. On the other hand, angles are always confined to the finite range $[0, 2\pi]$, which is why the angular momentum of any system is quantized.

2 Higher Dimensions

Idea 4

For a system with more than one degree of freedom, the WKB quantization condition holds for each individually,

$$\oint p_i \, dx_i = \left(n_i + \frac{\alpha_i}{2\pi}\right) h.$$

In this case, there can be multiple quantum states with a given energy, in which case we say that energy level is degenerate; the number of states with that energy is called the degeneracy.

[2] **Problem 12.** Consider a particle of mass m in a two-dimensional box of width and length L, with hard walls. This is the two-dimensional analogue of problem 3, and again α_i is zero.

(a) Write down the energy of the state corresponding to n_1 and n_2.

(b) What is the lowest energy level with a degeneracy of greater than 2?

[4] **Problem 13.** Consider a particle of mass m in the potential $V(x,y) = kr^2/2$. This is the two-dimensional analogue of problem 4. It is also the potential experienced by an electron in the obsolete "plum pudding" model of the atom, where they are embedded in a ball of uniform charge density.

(a) By working in Cartesian coordinates, find all of the energy levels, as well as the number of states within each energy level, called the degeneracy.

(b) [A] Now repeat the exercise in polar coordinates. In this case the integrals

$$\oint p_r \, dr, \quad \oint L \, d\theta$$

are quantized. Find the energy levels and their degeneracies. Note that for the radial motion, you will have to use the effective potential, as covered in **M6**. You will run into a difficult integral, so you may use the fact that

$$\int_{C-\sqrt{C^2-D^2}}^{C+\sqrt{C^2-D^2}} dx \, \sqrt{\frac{2C}{x} - \frac{D^2}{x^2} - 1} = (C - |D|)\pi$$

valid for $|D| \leq C$. How does your answer compare to that of part (a)?

Remark

Sommerfeld applied an analysis like that of part (b) of problem 13 to the Bohr model, yielding the semiclassical orbits which are ellipses with the nucleus at the focus. (In fact, if you're so inclined, you can do this too, using the same provided integral.) This accounted for the quantum numbers n and ℓ in hydrogen. The quantum number m comes from additionally quantizing L_z, which implies that the elliptical orbits can only occur in certain planes, an idea known as "space quantization". Sommerfeld even managed to compute the first relativistic corrections to the energy levels, explaining their so-called "fine structure".

With all this included, the Bohr theory provides a complete description of the energy levels of hydrogen, except that (1) the $\ell = 0$ orbitals are missing, since they would have to go straight through the nucleus, (2) space quantization seems artificial and breaks rotational symmetry, and (3) the number of states isn't quite right, a deficiency that would later be fixed by including spin. Many complicated attempts were made to patch these problems, or to extend the theory to multi-electron atoms, but they were forgotten after the modern theory of quantum mechanics (in terms of the Schrodinger equation) appeared.

However, what you've learned above is not completely irrelevant today. The correspondence principle is the idea that quantum results should yield smoothly transition to classical ones in the limit $\hbar \to 0$, which in this context means sending the quantum numbers to infinity. And that's exactly what happens. For high quantum numbers, you can superpose atomic orbitals of nearby energy to create a sharply peaked wavefunction, just like how we could create wavepackets from plane waves in **W1**. These peaks act like localized classical particles, following the Bohr model's orbits. Thus, the Bohr model is still useful for studying Rydberg atoms, which are hydrogen-like atoms excited to very high energy levels.

[3] **Problem 14** (Cahn)**.** A crude model of an electron bound to an atom is a particle of mass m attached to a one-dimensional spring, with spring constant k and hence angular frequency $\omega = \sqrt{k/m}$. Consider two such atoms.

(a) Write down the energy levels of the system, assuming the atoms are completely independent. How many states correspond to each energy?

(b) Let the electrons have positions x_i relative to their respective equilibrium positions. Now suppose the atoms are brought close together, causing the electrons to repel. For simplicity, we represent this in terms of an extra potential energy term $k' x_1 x_2$, where k' is small. Find the new energy levels of the system exactly. (Hint: this can be done with a clever change of variables. However, you have to be careful because changing to new coordinates x'_i also requires changing the momenta; after all, if we didn't, then the quantization condition of idea 2 would change, leading to different energy levels! If K is the kinetic energy, and you are using momentum variables x_i, then the momenta should be defined as $p_i = \partial K / \partial \dot{x}_i$.)

(c) Your answer should not make sense for large k'. Physically, what is going on?

Part (b) gives a simple example of how energy levels "split" in the presence of interactions.

Example 2

A nonrelativistic particle of mass m is in a cubical box with side length L and hard walls. Find the approximate number of quantum states with energy at most E_0, where E_0 is large.

Solution

Using the same reasoning as in previous problems, we apply "hard wall" boundary conditions, requiring the wavefunction to go to zero at the boundary. Thus, the wavefunction is

$$\psi \propto \sin(k_x x) \sin(k_y y) \sin(k_z z)$$

where

$$k_i = \frac{\pi}{L} n_i, \quad n_i \text{ positive integer}$$

and the energy is

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}.$$

The simplest way to proceed is to think in terms of "momentum space", an abstract space whose axes are p_x, p_y, and p_z. The allowed states form a grid in the first octant of momentum space, with a volume of $(\pi\hbar/L)^3$ for each state. The surface $E = E_0$ corresponds to a sphere of radius $\sqrt{2mE_0}$. Therefore the number of states with energy at most E_0 is approximately

$$N = \frac{1}{8} \left(\frac{4}{3} \pi (2mE_0)^{3/2} \right) \left(\frac{\pi\hbar}{L} \right)^{-3}.$$

Now let's solve the problem a slightly different way: suppose the box has *periodic* boundary conditions, so that the right side is identified with the left side, and so on. In this case, the wavefunctions can all be written in the form

$$\psi \propto \exp(i(k_x x + k_y y + k_z z))$$

but now the allowed values of the wavenumbers are

$$k_i = \frac{2\pi}{L} n_i, \quad n_i \text{ integer}.$$

The allowed states form a grid in all of momentum space, not just the first octant, with a volume of $(2\pi\hbar/L)^3$ for each state. That is, while the volume around each state is eight times as large, the states now occupy eight octants instead of one. Then the overall density of states is still the same, and the number of states with energy at most E_0 is approximately

$$N = \left(\frac{4}{3}\pi(2mE_0)^{3/2}\right)\left(\frac{2\pi\hbar}{L}\right)^{-3}$$

which matches the result for hard walls. The point of this computation is to show that when we care about the statistical properties of many states, the boundary conditions won't matter. In practice, you'll see both kinds of boundary conditions quite often.

If you find the differences between the two boundary conditions confusing, you're not alone. In his original derivation of blackbody radiation, Lord Rayleigh used "hard wall" boundary conditions but allowed negative n_i, leading to a factor of 8 error. Jeans corrected it, which is why the result is now called the Rayleigh–Jeans law.

[4] **Problem 15.** Do the following JPhO problem. This pedagogical problem introduces the WKB approximation and phase space, reviewing everything covered above, and applies it to "clusters" of atoms. You can skip sections I and III, which are covered elsewhere on this problem set.

Remark

Because matter has wave properties, particles such as electrons can exhibit interference effects, like those covered in **W2**. For example, you can run a double-slit experiment firing individual electrons at a time and seeing where they hit the screen, which will gradually build up an interference pattern. You'll see an example of this in the next problem.

[4] **Problem 16.** IPhO 1993, problem 3. (Hint: to do the final part of this problem, you should *not* try to compare the total path lengths traversed by the electrons. That would be very hard, and worse, it won't give the right answer, because the potential from the wire also affects the electrons' phases. Instead, you should use the facts about wavefronts mentioned in **W2**. That is, waves always propagate perpendicular to wavefronts, and all points on a wavefront have the same phase.)

3 The Uncertainty Principle

Idea 5: Heisenberg Uncertainty

So far we have treated a quantum particle as having a well-defined position and momentum, but in reality the uncertainties in the position and momentum obey

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}$$

where, as in **P2**, the uncertainties may be interpreted as standard deviations. The "semi-classical limit" used in the rest of this problem set simply corresponds to the case where the required uncertainty is relatively small, which is reached for energy levels $n \gg 1$. Occasionally, Olympiad questions will ask you to use the Heisenberg uncertainty principle to make a very rough estimate. In these cases, the constant factors will not matter.

Idea 6: Energy-Time Uncertainty

There are two commonly used versions of the energy-time uncertainty principle. If the energy of a system is only measured for a finite time Δt, it must have a finite uncertainty ΔE in its energy. In addition, if a system significantly changes its state in time Δt, then its energy must have been uncertain by a finite amount ΔE. In both cases, we have

$$\Delta E\,\Delta t \geq \frac{\hbar}{2}.$$

A third common statement of the energy-time uncertainty principle is "for a short time Δt, a system can violate energy conservation by an amount ΔE". This is wrong, because quantum systems always conserve energy; systems that naively seem to violate energy conservation simply didn't have a well-defined energy in their initial state to begin with. However, thinking this way will usually get you the right answers, essentially because of dimensional analysis.

Example 3

Consider once again a particle of mass m attached to a one-dimensional spring, with natural angular frequency ω. Use the uncertainty principle to estimate the minimum possible energy of the particle, and compare it with the result of problem 4.

Solution

Suppose the uncertainties in position and momentum are Δx and Δp. Then the potential energy is of order $k(\Delta x)^2/2$ and the kinetic energy is of order $(\Delta p)^2/2m$. Dropping constants,

$$E \sim k(\Delta x)^2 + \frac{(\Delta p)^2}{m} \gtrsim k(\Delta x)^2 + \frac{\hbar^2}{(\Delta x)^2 m}$$

where we applied the uncertainty principle. The ground state minimizes the energy, which is achieved when $(\Delta x)^2 \sim \hbar/\sqrt{km}$. In this case, the energy is of order $k\hbar/\sqrt{km} \sim \hbar\sqrt{k/m} \sim \hbar\omega$, which is just what we found earlier. (A similar derivation can be used to derive the energy of the ground state of hydrogen, along with the Bohr radius; try it!)

Remark

We can also "solve" the above problem with the energy-time uncertainty principle incorrectly. The only timescale in the problem is $1/\omega$, so

$$\Delta E \gtrsim \frac{\hbar}{\Delta t} \sim \hbar\omega$$

so $E \gtrsim \hbar\omega$. However, in reality the ground state has *no* energy uncertainty; its energy is simply the ground state energy. Another way of saying this is that a particle can hang out in the ground state forever, so Δt is infinite and hence ΔE is zero. This incorrect derivation gives the right answer just because it's the only possible answer by dimensional analysis.

Thus, a sloppy problem might ask you to do it.

Example 4

Consider a single slit diffraction experiment, where photons of wavelength λ pass through a slit of width a. If the screen is a large distance D away, roughly how wide is the resulting diffraction pattern on the screen?

Solution

The photon has a momentum $p_x = \hbar k = h/\lambda$, and passing through the slit necessarily gives it a transverse momentum uncertainty of

$$\Delta p_y \sim \frac{h}{a}$$

where we dropped order one constants, which means an angle uncertainty of

$$\Delta\theta \sim \frac{\Delta p_y}{p_x} \sim \frac{\lambda}{a}.$$

Therefore, using basic geometry, the size of the pattern on the screen is

$$\Delta y \sim D\Delta\theta \sim \frac{D\lambda}{a}.$$

This is the approximate width of the central maximum for single slit diffraction, as we found in **W2**. The reason the result is the same is that light acts like a wave both classically and quantum mechanically; the quantum version of the derivation is just the same as the classical version, but with "everything multiplied by h". What's new about this derivation is that it also applies for matter particles, which have $\lambda = h/p$.

Example 5

The Higgs boson has a mass of $125\,\text{GeV}$ and a lifetime of about $\tau = 1.6 \times 10^{-22}\,\text{s}$. About what percentage uncertainty must a measurement of a Higgs boson's mass have?

Solution

Decay is a significant change in the particle's state, and this change happens over a time τ, which means the energy uncertainty is

$$\Delta E \sim \frac{\hbar}{\tau} = 7 \times 10^{-13}\,\text{J} = 0.004\,\text{GeV}.$$

When we measure the Higgs boson's mass, we really measure the $E = mc^2$ energy released when it decays, so the unavoidable uncertainty of the mass is $\Delta E/E \sim 0.003\%$. (But the actual measured uncertainties are much higher, due to a variety of other effects.)

[1] **Problem 17** (Krane 4.39). An apparatus is used to prepare an atomic beam by heating a collection

of atoms to a temperature T and allowing the beam to emerge through a hole of diameter d in one side of the oven. Show that the uncertainty principle causes the diameter of the beam, after traveling a length L, to be larger than d by an amount of order $L\hbar/d\sqrt{mk_BT}$, where m is the mass of an atom.

[2] **Problem 18** (Insight 8.26)**.** When helium is cooled to extremely cold temperatures, it becomes a superfluid, an exotic type of liquid that can flow with zero dissipation. These strange properties occur because quantum mechanical effects are large, making the quantum uncertainty in the position of each helium atom on the same order as the separation between atoms.

(a) Given that superfluid helium has density ρ and a helium atom has mass m, estimate the temperature T at which helium becomes a superfluid. This is closely related to, but not quite the same thing as Bose–Einstein condensation, a phase transition that bosons undergo at low temperatures.

(b) Numerically evaluate T, given that $\rho \sim 100\,\text{kg/m}^3$ and $m \sim 7 \times 10^{-27}\,\text{kg}$.

[4] **Problem 19.** A neutron is inside a small cubical box of side length d. Ignore gravity.

(a) Estimate the minimum possible pressure on the walls using the uncertainty principle, dropping all numeric factors. In the next two parts, we'll calculate the pressure more carefully.

(b) Calculate the average pressure on the walls by treating the neutron as a classical particle bouncing back and forth, with the same momentum as expected for the ground state in the WKB approximation.

(c) Calculate the average pressure on the walls by finding the energy E of the ground state using the WKB approximation, and the definition of pressure, $P = -\partial E/\partial V$. (This actually gives the exact answer. Of course, by dimensional analysis, taking $P \sim E/V$ would also produce the right answer, up to a constant factor.)

(d) Now suppose that $N \gg 1$ neutrons are inside the box. Neutrons are fermions, as explained in the next section, and hence no two can share the same quantum state. Neglecting any interactions between the neutrons, estimate the minimum possible pressure on the walls, using either the method of part (b) or (c). How does it scale with the number density $n = N/V$?

The large pressure you will find in part (d) is known as degeneracy pressure. It supports compact objects such as white dwarfs and neutron stars, as you'll investigate in **X3**.

[3] **Problem 20.** USAPhO 2018, problem B2.

[3] **Problem 21.** Classically, an electron orbiting a proton with angular frequency ω_o emits radiation with angular frequency $\omega_c = \omega_o$, as covered in **E7**. On the other hand, quantum mechanically the energy levels are discrete, and using the de Broglie relation $\Delta E = \hbar\omega$ indicates the angular frequencies of radiation emitted when the electron drops between energy levels are discrete as well. The classical and quantum models thus seem to be radically different, but in the limit $n \to \infty$ where quantum effects become negligible, the two should match.

(a) Suppose that the electron can orbit the proton in circular orbits with discrete radii r_n. For the n^{th} orbit, compute the angular frequency ω_c of the emitted radiation according to classical mechanics.

(b) Now suppose the electron drops from the n^{th} energy level to the $(n-1)^{\text{th}}$ energy level. Compute the angular frequency ω_q of the emitted radiation according to quantum mechanics, assuming the orbits have radii r_n.

(c) In the limit $n \to \infty$, the results of parts (a) and (b) should coincide. Therefore, by equating these results, infer how r_n depends on n, and thus how L depends on n. If all goes well, you should recover the result of Bohr quantization.

The reasoning here is exactly how Bohr came up with Bohr quantization in the first place. (The de Broglie relation we had to use was motivated earlier through Planck's law, as we showed in **T2**.)

[4] **Problem 22.** IPhO 2005, problem 3. You may skip part 4, since it's quite similar to another problem on this problem set.

4 Bosons and Fermions

So far, we've only solved for the energy levels of individual particles. Now we'll consider what happens when we put many of these particles together. We will assume the particles do not interact, which means their energy levels are just the same as the energy levels for individual particles. If the particles are fermions, they obey the Pauli exclusion principle, which means no two can occupy the same energy level. If they are bosons, there is no such restriction; we'll consider bosons first.

[3] **Problem 23.** This problem is a modification and clarification of USAPhO 2011 A4. (In fact, the answers differ, so don't compare against the USAPhO solution!) Consider a simplified model of the electromagnetic radiation inside a cubical box of side length L at temperature T. In this model, modes of the electric field have spatial dependence

$$E(x, y, z) = E_0 \sin(k_x x) \sin(k_y y) \sin(k_z z)$$

where one corner of the box lies at the origin and the box is aligned with the x, y, and z axes. For simplicity, we will treat the electric field as a *scalar*.

(a) The electric field must be zero everywhere at the sides of the box. What condition does this impose on the k_i?

(b) Each permitted value of the triple (k_x, k_y, k_z) corresponds to a mode, which can be occupied by any number of photons. Each photon has an energy $E = \hbar\omega$, where $\omega = ck$ is the angular frequency of the mode. How many modes have an energy per photon of at most $k_B T$?

(c) As a crude approximation, suppose that in thermal equilibrium, each mode with energy per photon at most $k_B T$ contains exactly one photon, while all other modes contain no photons. Compute the total energy of the photons in the box. (Answer: $(k_B T)^4 L^3 / 8\pi^2 \hbar^3 c^3$.)

Note that the procedure here is different from what we did above. Before, we started with particles and quantized $\oint p\, dx$ to get the allowed quantum states. Here, we're treating a situation with many particles (photons), which are excitations of an underlying field (the electromagnetic field). In this case, we found the (normal) modes of the classical field, then quantized by saying that photons could occupy these modes. This is the methodology of quantum field theory.

[4] **Problem 24.** The final result of the problem above is correct dimensionally, but has incorrect numerical factors because of the crude approximations made. In this problem we'll do a more careful analysis to get the right result. This question is self-contained, but background from **T1** and **T2** will be helpful.

(a) Consider a quantum mode that can support photons of energy E. The mode can be occupied by any whole number of photons. Thus, using the Boltzmann distribution, the probability of having n photons is

$$p_n \propto e^{-nE/k_B T}.$$

Show that the expected number of photons in the mode is

$$\langle n \rangle = \frac{1}{e^{E/k_B T} - 1}.$$

This is the Bose–Einstein distribution.

(b) Sketch $\langle n \rangle$ as a function of E. How does it behave at high and low E, and do those results make physical sense?

(c) Using the Bose–Einstein distribution, show that the total energy is

$$U = \frac{L^3 \hbar}{\pi^2 c^3} \int_0^\infty d\omega \, \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1}$$

where ω is the angular frequency. You'll have to multiply by a factor of two, because there are two independent photon polarizations for each mode we found above. (Note that if we open the box, the photons will fly out, and the frequency distribution of the emitted light will be given by the integrand; this yields Planck's law for blackbody radiation.)

(d) [A] Using an appropriate substitution, show that U is a dimensionful constant times the dimensionless integral

$$\int_0^\infty dx \, \frac{x^3}{e^x - 1}.$$

To evaluate this integral, expand the denominator as a power series, integrate each term individually, and use the fact that the Riemann zeta function obeys

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \quad \zeta(4) = \frac{\pi^4}{90}.$$

When the smoke clears, you should find that

$$U = \frac{\pi^2}{15} \frac{(k_B T)^4 \, L^3}{(\hbar c)^3}.$$

[5] **Problem 25.** APhO 2002, problem 1. This useful problem covers the other common example of a quantized bosonic field. In the above problems, we quantized the electromagnetic field to get photons. Here, we quantize a displacement field to get phonons.

Idea 7

In problems 23 and 24, we handled a system of bosons (specifically photons) by considering the modes the photons could occupy, then calculating how many photons were in each mode. This was the easiest route. If we had instead fixed the number of photons, then counted the ways they could be distributed among the mode, the combinatorics would have been a complete nightmare, because multiple photons can occupy the same mode.

Fermions, which obey the Pauli exclusion principle, are simpler, because no two can be in the same state. For instance, if there are n noninteracting fermions in a system, then the lowest energy state of the whole system consists of having one fermion occupy the lowest energy state, the second occupy the second-lowest energy state, and so on. (Accounting for interactions makes the problem much more complicated, because it means the energy of a state depends on whether other states are occupied. However, you can explain a surprising amount while completely neglecting interactions.)

Example 6: Tremaine–Gunn Bound

Suppose all of the dark matter in the galaxy is composed of a single kind of fermionic particle, of mass m. The escape velocity of the galaxy is of order $v_{\text{esc}} \sim 10^{-3}c$, and the dark matter density near Earth is $\rho \sim 0.3\,\text{GeV}/(c^2\,\text{cm}^3)$. What's the minimum possible value of m?

Solution

The reason there's a minimum possible value of m is that, as m gets smaller, we need more dark matter particles. But the Pauli exclusion principle tells us that if we want to add more particles, they need to have higher and higher energy, and at some point the particles will have so much energy they won't be bound to the galaxy at all.

To get a rough estimate, let's suppose the galaxy has length scale L, so that we need at least $N \sim \rho L^3/m$ dark matter particles. They need to have energy less than $E_0 \sim m_0 v_{\text{esc}}^2$. Plugging this into the final result of example 2 and dropping all numeric factors gives

$$\frac{\rho L^3}{m} \gtrsim \frac{m^3 v_{\text{esc}}^3 L^3}{\hbar^3}$$

which yields the bound

$$m \lesssim \left(\frac{\rho\,\hbar^3}{v_{\text{esc}}^3}\right)^{1/4} \sim 10\,\text{eV}/c^2.$$

A few decades ago, neutrinos were leading dark matter candidates, since they are light fermionic particles that interact very weakly with ordinary matter. But we now know that the neutrino mass is well below this bound, so that nice idea doesn't work. On the other hand, dark matter could still be composed of bosonic particles of much lighter mass.

[2] **Problem 26.** Consider a system with many noninteracting fermions, and many quantum states. Each quantum state can be either empty or occupied by a fermion. We want to find the probability that a given quantum state, of energy E, is occupied.

(a) To put a fermion in this state, we need to remove a fermion from some other state. Suppose the energy released by doing this, suitably averaged, is μ. (This is the chemical potential, and it depends on the temperature, the number of fermions, and the number of states and their energies.) Using the Boltzmann distribution, show that the probability of occupancy is

$$\langle n \rangle = \frac{1}{e^{(E-\mu)/k_B T} + 1}.$$

This is the Fermi–Dirac distribution.

(b) Sketch $\langle n \rangle$ as a function of E for small but nonzero temperature, as well as the limit attained for zero temperature.

[3] **Problem 27.** In this problem we'll consider the energy of the conducting electrons in a solid at low temperatures. Model a solid as a cubical box of side length L with periodic boundary conditions.

(a) Find the number of quantum states with energy at most E_F, making sure to account for the two spin states of the electron.

(b) Suppose there are N electrons in total. They will fill all of the energy levels up to $\mu = E_F$, where E_F is called the Fermi energy. Show that

$$E_F = \frac{h^2}{2m} \left(\frac{3N}{8\pi V} \right)^{2/3}.$$

(c) A sodium crystal has one conduction electron per atom. The density and molar mass are

$$\rho = 0.971 \times 10^3 \, \mathrm{kg/m^3}, \quad M = 0.023 \, \mathrm{kg/mol}.$$

Find N/V and E_F, and use this to evaluate the typical speed v of an electron.

[3] **Problem 28** (MIT)**.** [A] This is an advanced problem that is only placed here because the final result is neat. An integer N can be *partitioned* by writing it as a sum of positive integers, and the partition function $p(N)$ is the number of unique ways this can be done. For example,

$$4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 2 + 2 = 1 + 3 = 4$$

which implies $p(4) = 5$. Counting the number of partitions of an integer is a very hard combinatorics problem, but we can get an estimate for large N using string theory.

(a) Consider an ideal string with hard boundary conditions and fundamental angular frequency ω. Show that the number of distinct quantum states with energy $N\hbar\omega$ is $p(N)$.

(b) Now suppose the string is at temperature T, where T is chosen so that the expected energy is $N\hbar\omega$. In the thermodynamic limit $N \gg 1$, find a relation between N and T. You may use the result $\zeta(2) = \pi^2/6$.

(c) By approximating the entropy as $S \approx k_B \log p(N)$, show that

$$\frac{\hbar\omega}{kT} = \frac{d \log p(N)}{dN}.$$

Combine this with your previous result to find an estimate for $p(N)$.

To check your answer, the celebrated Hardy–Ramanujan formula (which is more accurate than the very rough treatment we give above) is

$$p(N) \sim \frac{1}{4\sqrt{3}\,N} \exp\left(\pi\sqrt{\frac{2N}{3}}\right).$$

While we only considered a simple nonrelativistic string here, calculations of this sort show up in the thermodynamics of string theory. For further discussion, see chapter 22 of Zwiebach.

Example 7: Casimir Forces

Consider an infinite uniform string, on which waves propagate with speed v. The string is held fixed by pins at two points separated by a distance L. When the string is in its ground state, what is the resulting force between the pins?

Solution

Of course, in classical mechanics the ground state would just be $y(x) = 0$, and the force would just be the tension T. But there is an additional quantum mechanical contribution, which arises because each of the standing waves between the pins, with angular frequencies $\omega_n = n\pi v/L$, carries a so-called "zero point" energy $\hbar\omega_n$.

As usual, force can be found by differentiating energy, $F = -dE/dx$. By summing up the zero point energy in all of the standing waves, we naively get

$$E = \sum_n \frac{\hbar\omega_n}{2} = \frac{\pi\hbar v}{2L} \sum_{n=1}^{\infty} n = E_0 \sum_{n=1}^{\infty} n = \infty$$

which is rather unhelpful. This result is off for two reasons. First, even when there aren't any pins, the string still has standing waves, and these waves *also* have a naively infinite energy density. When we move the pins a bit, we change both the zero point energy outside the pins and inside, by infinite amounts, but the *net* change is finite, giving a finite force.

Quantitatively, the energy in between the pins due to the standing waves is

$$E_{\text{plate}} = E_0 \sum_{n=1}^{\infty} n = \infty$$

and the energy we would have had there if the pins didn't exist is the "continuous" sum,

$$E_{\text{vac}} = E_0 \int_0^{\infty} x\, dx = \infty.$$

The difference should be finite, but we can't just subtract infinity with infinity, which brings us to the second problem: none of these quantities are *actually* infinite. Any real string will have a finite maximum oscillation frequency – for instance, the wavelength certainly can't get smaller than the atomic spacing. Alternatively, even if we had an idealized string where E_{vac} was actually infinite, no real pin can perfectly block waves of all frequencies. For sufficiently high frequencies the waves won't be affected by the pins, so that the sum in

E_{plate} eventually behaves like the integral in E_{vac}, leaving a finite difference between the two.

In other words, the difference between E_{plate} and E_{vac} in reality comes from only low n and x, Therefore, let's "regulate" the two expressions above so that they're unchanged in this regime, but match each other at high n and x. The simplest way to do this is to take

$$E_{\text{plate}} = E_0 \sum_{n=1}^{\infty} n e^{-\epsilon n}, \quad E_{\text{vac}} = E_0 \int_0^{\infty} x e^{-\epsilon x} \, dx = \frac{E_0}{\epsilon^2}$$

for small ϵ. To handle the sum, let $\alpha = e^{-\epsilon}$, so that

$$E_{\text{plate}}/E_0 = \alpha + 2\alpha^2 + 3\alpha^3 + \ldots.$$

Now we use the usual trick for arithmetic-geometric series. Note that

$$\alpha E_{\text{plate}}/E_0 = \alpha^2 + 2\alpha^3 + 3\alpha^4 + \ldots.$$

Subtracting, we find

$$(1-\alpha) E_{\text{plate}}/E_0 = \alpha + \alpha^2 + \alpha^3 + \ldots = \frac{\alpha}{1-\alpha}.$$

We thus conclude that

$$E_{\text{plate}} = \frac{e^{-\epsilon}}{(1-e^{-\epsilon})^2} E_0 = E_{\text{vac}} - \frac{1}{12} E_0 + O(\epsilon)$$

where we used a result from **P1**. Finally, when we take ϵ to zero, the difference is simply

$$E = E_{\text{plate}} - E_{\text{vac}} = -\frac{1}{12} E_0.$$

Differentiating gives the force,

$$F = \frac{\pi \hbar v}{24 L^2}$$

which turns out to be attractive. Not only is this finite, it's right! Experiments have measured this "Casimir force" precisely for light between two conductors, where $v = c$, and confirmed the expected results.

You're probably suspicious about this derivation because it depends on the arbitrary choice of an exponential suppression. What if the sums and integrals were regulated at high n and x in a different way? Shouldn't the answer depend on the details of the string and pin? Remarkably, the answer is no: the regulator doesn't matter. If you try others, such as $e^{-\epsilon n^2}$ or $1/n^{\epsilon}$, you'll get the same result; you can find a general proof in chapter 15 of Schwartz's *Quantum Field Theory and the Standard Model*. The reason is that the effect comes from physics at low frequencies, so it doesn't matter how you regulate the high frequencies.

It is for precisely this reason that you will sometimes see the mysterious equation

$$1 + 2 + 3 + \ldots = -\frac{1}{12}.$$

It's not really true. Instead, what it physically means is that the difference between the regulated sum and integral is $-1/12$ for any reasonable regulator.

Modern I: Semiclassical Mechanics

The basics of quantum mechanics can be found in chapters 46 and 47 of Halliday and Resnick, and are covered more thoroughly in chapters 3 through 6 in Krane. Chapter 10 of Krane covers quantum statistical mechanics as used in the final section. For a complete, but advanced treatment of the WKB approximation, see chapter 9 of Griffiths' *Introduction to Quantum Mechanics* (3rd edition). For some nice conceptual discussion, see chapters I-37 and I-38 of the Feynman lectures, or if you're ambitious, essentially all of volume III. There is a total of **80** points.

1 The WKB Approximation

A proper introduction to quantum mechanics would take a whole book. Luckily, there is a "semiclassical" regime of quantum mechanics which can be handled with much less machinery. Historically, this regime was discovered first, by scientists like Bohr, and it suffices to explain many quantum effects. To introduce the WKB approximation, we'll start by considering classical standing waves.

Idea 1

The variation of the phase ϕ of a wave is described by its wavenumber and angular frequency,

$$k = \frac{d\phi}{dx}, \quad \omega = \frac{d\phi}{dt}$$

As covered in **W1**, the group velocity is

$$v = \frac{d\omega}{dk}.$$

A standing wave can form if the wave's phase lines back up with itself after one round trip,

$$\oint k\, dx = 2\pi n, \quad n \in \mathbb{Z}.$$

A simple case is a string of length L with fixed ends, where we have

$$2kL = 2\pi n$$

which gives the wavenumbers $k_n = \pi n/L$ and hence the standing wave angular frequencies $\omega_n = \pi v n/L$, as we saw in **W1**.

[1] **Problem 1.** A slightly more subtle case is the case of a string of length L with one fixed and one free end. Show that the standing wave angular frequencies are

$$\omega_n = \frac{\pi v}{L}(n + 1/2).$$

Solution. Here, we see that the end is free, so it corresponds to an anti-node. Thus, the length of the string is a half integer amount of half wavelengths, so $L = \frac{1}{2}(n + 1/2)\lambda_n$, which means that

$$f_n = v/\lambda_n = \frac{v}{2L}(n + 1/2).$$

Thus, $\omega_n = 2\pi f_n = \frac{\pi v}{L}(n + 1/2)$, as desired.

The reason our principle above doesn't give the right answer is that a wave picks up an extra phase shift π when it reflects off a fixed end, so we really should have written

$$\oint k\,dx = 2\pi(n + 1/2)$$

in this case. We didn't run into any problems for two fixed ends, because in that case we get two phase shifts of π, which have no overall effect.

[2] **Problem 2.** Suppose a string of length L is hung from the ceiling. The string has mass density μ, and the bottom of the string is held fixed and pulled down with a force $F \gg gL\mu$. If the string were weightless, then the standing wave angular frequencies would simply be $\pi v n/L$, where $v = \sqrt{F/\mu}$. However, the weight causes the tension and hence the wave speed to vary throughout the rope.

(a) Explain why the wave's angular frequency ω is uniform, i.e. why standing wave solutions are proportional to $\cos(\omega t)$.

(b) Find the angular frequencies of standing waves, including corrections up to first order in $gL\mu/F$.

This is a more quantitative version of a problem we encountered in **W1**.

Solution. (a) Recall that the solutions of the wave equation were proportional to $e^{i(kx-\omega t)}$ because the wave equation was linear, and had no explicit dependence on x and t. The wave equation describing waves on this string does have explicit dependence on x, because the tension (and hence the wave velocity) varies along the string. But it still doesn't have any explicit dependence on t, so guessing a solution proportional to $e^{-i\omega t}$ (or equivalently $\cos(\omega t)$ in real variables) still works.

(b) Let h be the height from the bottom of the string. By Newton's second law, the tension at height h is $T(h) = F + \mu g h$, so the speed is $v(h) = \sqrt{F/\mu + gh}$. Thus, the quantization condition is

$$\oint k\,dx = 2\int_0^L k\,dh = 2\int_0^L \frac{\omega}{v}\,dh = 2\int_0^L \frac{\omega}{\sqrt{F/\mu + gh}}dh = 2\pi n.$$

Note that

$$\begin{aligned} 2\int_0^L \frac{\omega}{\sqrt{F/\mu + gh}}dh &= \frac{2\omega}{\sqrt{F/\mu}}\int_0^L \frac{dh}{\sqrt{1 + gh\mu/F}} \\ &= 2\omega\sqrt{F/\mu}/g \int_0^{gL\mu/F} \frac{dx}{\sqrt{1+x}} \\ &\approx 2\omega\sqrt{F/\mu}/g \int_0^{gL\mu/F} (1 - x/2)dx \\ &= 2\omega\sqrt{F/\mu}/g \left(gL\mu/F - \frac{1}{4}(gL\mu/F)^2\right) \\ &= 2\omega L\sqrt{\mu/F}\left(1 - \frac{1}{4}(gL\mu/F)\right). \end{aligned}$$

We therefore conclude

$$\omega = \frac{\pi n\sqrt{F/\mu}}{L}\left(1 + \frac{1}{4}(gL\mu/F)\right).$$

In quantum mechanics, the state of a particle is described by a wavefunction $\psi(x, t)$ which obeys the Schrodinger equation. When a particle is confined in a finite volume, there are standing wave solutions analogous to those of classical wave mechanics, which have discrete frequencies.

Idea 2: WKB Approximation

The momentum and energy of a quantum particle obey the de Broglie relations

$$p = \hbar k, \quad E = \hbar\omega$$

where E and p are related just as in classical mechanics,

$$E = \frac{p^2}{2m} + V(x).$$

For a particle with reasonably well-defined momentum, the wavefunction is a wavepacket which travels at the group velocity

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp}.$$

This corresponds to the classical velocity of the particle in the classical limit. (For example, you should check that the above equations imply $p = mv_g$, or $p = \gamma m v_g$ when the relativistic momentum and energy are used.)

Just as for a classical standing wave, ω is the same everywhere for quantum standing waves. Since energy is related to frequency, these standing waves are also states of definite energy. In the semiclassical limit, the standing waves must satisfy

$$\oint p\, dx = (2\pi n + \alpha)\hbar = \left(n + \frac{\alpha}{2\pi}\right) h.$$

The extra phase α depends on how the particle gets reflected at the endpoints of its motion.

Remark

The left-hand side of the quantization condition above is precisely the adiabatic invariant from **M4**, which stays the same if we change the system parameters slowly. This ensures the quantization condition is preserved over time, as it must be for self-consistency. If you instead change the system parameters quickly, the integral is not preserved, but that's because the change causes transitions from one energy level to another (i.e. to waves with different n).

[2] **Problem 3.** Consider a one-dimensional box of length L, with hard walls. We can think of these hard walls as a potential $V(x)$ that is zero inside the box and infinite outside the box. For this potential, we simply have $\alpha = 0$.

(a) Find the energy levels of a particle of mass m.

(b) Now suppose the particle is replaced with a photon, with $E = pc$. Find the allowed energies.

The frequencies you found in part (b) correspond to the standing wave frequencies for electromagnetic waves in a box with reflecting (i.e. perfectly conducting) walls.

Solution. (a) The quantization condition is that

$$2pL = 2\pi n\hbar \implies p = \pi n\hbar/L.$$

Thus, $E = p^2/2m = \frac{\pi^2 n^2 \hbar^2}{2mL^2}$. Note that this solution only makes sense for $n \geq 1$.

(b) We see that p is the same, and $E = pc = \pi n\hbar c/L$.

[2] **Problem 4.** Now consider a particle of mass m in the potential $V(x) = kx^2/2$. In this case, the particle turns around at a point where the potential energy gradually increases from below the particle's energy E to above it. It can be shown that each "soft" boundaries contributes $\pi/2$ to α, so that for this potential we can take $\alpha = \pi$.

Show that the energy levels are

$$E_n = \hbar\omega_0 \left(n + \frac{1}{2}\right), \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

This system is called the quantum harmonic oscillator, and remarkably, this is the exact answer, even though we used an approximation to get it. This result will be used in several problems below.

Solution. Here we have the quantization condition

$$2\int_{-L}^{L} m\omega_0 \sqrt{L^2 - x^2}\, dx = 2\pi\hbar(n + 1/2).$$

Note that the integral is just the area of a half-ellipse, so

$$\pi L^2 m\omega_0 = 2\pi\hbar(n + 1/2),$$

so

$$E = \frac{1}{2}m\omega_0^2 L^2 = \hbar\sqrt{k/m}(n + 1/2) = \hbar\omega_0(n + 1/2)$$

as desired. Note that this solution makes sense for $n \geq 0$.

[3] **Problem 5.** USAPhO 2015, problem A1.

[5] **Problem 6.** IPhO 2006, problem 1. This is a neat problem which illustrates the effect of a gravitational field on quantum particles, as well as the basics of interferometry, a subject developed further in **W2**. Give this a try even if it looks tough; only the ideas introduced above are needed!

Idea 3: Bohr Quantization

In general, $p\,dx$ may be replaced by any generalized momentum/position pair. For example,

$$\oint L\, d\theta = nh.$$

When angular momentum is conserved, the left-hand side is simply $2\pi L$, immediately giving

$$L = n\hbar$$

which is Bohr's quantization condition. In a system of particles rotating together, L stands for the total angular momentum of the system.

Compared to back-and-forth linear motion, covered in idea 2, rotation is different because it's inherently periodic. For rotation, the integer n can be positive or negative, representing a particle going clockwise or counterclockwise. Also, there is no analogue of the α phase factor because the particle just rotates all the way around; it never gets reflected.

Example 1

Find the energy levels and orbit radii of the electron in the hydrogen atom using Bohr quantization.

Solution

We postulate a circular orbit, and quantize the angular momentum. We have

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2}, \quad L = mvr = n\hbar.$$

Solving the second equation for v and plugging into the first gives

$$r = \frac{4\pi\epsilon_0\hbar^2}{me^2} n^2 = a_0 n^2$$

where $a_0 = 5.3 \times 10^{-11}$ m is called the Bohr radius; these are the allowed orbit radii. To get the energies, we use the standard result for circular motion with an inverse square force that the total energy is half the potential energy, so

$$E = -\frac{e^2}{8\pi\epsilon_0 r} = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2}.$$

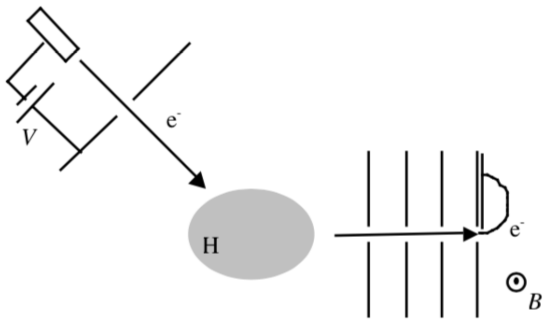
Evidently, they get more and more closely spaced together as n increases. The big constant in front is called the Rydberg, and is equal to 13.6 eV.

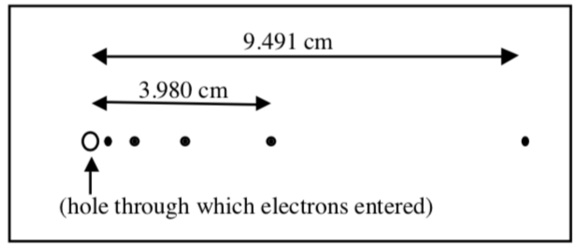
[1] **Problem 7.** Find the energy levels of positronium, a bound state of a positron and electron.

Solution. You can do this through an explicit analysis very similar to the example. On the other hand, we can also use the idea of reduced mass introduced in **M6**. The reduced mass of positronium is $m/2$, so replacing m with $m/2$ in the example's answer gives

$$E = -\frac{me^4}{4(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2}.$$

[2] **Problem 8** (USAPhO 2004)**.** Electrons are accelerated from rest through a potential V into a cloud of cold atomic hydrogen. A series of plates with aligned holes select a beam of scattered electrons moving perpendicular to the plates. Immediately beyond the final plate, the electrons enter a uniform magnetic field B perpendicular to the beam; they curve and strike a piece of film mounted on the final plate.





When the film is developed, a series of spots is observed. The distances between the hole and the two most distant spots are measured. You may assume that the film is large enough to have intercepted all of the electrons, i.e. that there are no spots farther from the hole than those shown. The number of spots shown is not necessarily accurate.

Make the approximation that the mass of the hydrogen atom is much larger than the mass of the electron. Assume that each electron scatters off only one atom, which is initially in the ground state (lowest energy state) and has negligible thermal velocity. Determine B, V, and the total number of spots on the film.

Solution. This is USAPhO 2004, problem B1, and you can check its official solutions.

[2] **Problem 9.** A rotor consists of two particles of mass m connected by a rigid rod of length L.

(a) Find the energy levels if the particles are not identical.

(b) Find the energy levels if the particles are identical. (Hint: recall that the closed loop integrals in the previous ideas are over paths that take the system back to its original state.)

You might find it disturbing that the result is so different if the particles are or aren't completely identical, but it's a well-verified fact about molecular rotational energy levels. Without the effect of part (b), many predictions of quantum mechanics would come out totally wrong!

Solution. (a) We see that the angular momentum L_z is constant, so the quantization condition says $L_z \cdot 2\pi = nh$, or $L_z = n\hbar$. Then, $E = \frac{1}{2I}L_z^2 = \frac{n^2\hbar^2}{mL^2}$.

(b) The point here is that after just a π rotation, the system is back in its original state, because the particles are identical. Thus, $L_z \cdot \pi = nh$, or $L_z = 2n\hbar$, or $E = \frac{4n^2\hbar^2}{mL^2}$.

[3] **Problem 10.** INPhO 2020, problem 3.

Solution. See the official solutions here.

[3] **Problem 11.** INPhO 2016, problem 6. Unfortunately, this question automatically comes with the solutions, but it's still useful to work through.

Remark

In popular science, people sometimes speak of "quantizing" a system as similar to making everything discrete. But as you've seen above, it's more complicated than that. For instance, position never becomes discrete; instead, we integrate over it.

The general rule in quantum mechanics is that confinement to a finite "size" causes the conjugate variable to become discrete. For example, above you looked at several examples of particles bound to potentials. These are confined in space, and hence have discrete orbits in phase space by idea 2, and thus discrete energies. But a free particle not bound to a potential can have any energy, because $E = p^2/2m$ and there is no condition at all on p. On the other hand, angles are always confined to the finite range $[0, 2\pi]$, which is why the angular momentum of any system is quantized.

2 Higher Dimensions

Idea 4

For a system with more than one degree of freedom, the WKB quantization condition holds for each individually,

$$\oint p_i \, dx_i = \left(n_i + \frac{\alpha_i}{2\pi}\right) h.$$

In this case, there can be multiple quantum states with a given energy, in which case we say that energy level is degenerate; the number of states with that energy is called the degeneracy.

[2] **Problem 12.** Consider a particle of mass m in a two-dimensional box of width and length L, with hard walls. This is the two-dimensional analogue of problem 3, and again α_i is zero.

(a) Write down the energy of the state corresponding to n_1 and n_2.

(b) What is the lowest energy level with a degeneracy of greater than 2?

Solution. (a) In each dimension, we have $p_x = \pi n_x \hbar/L$ and $p_y = \pi n_y \hbar/L$ as before. So the energy is

$$E = \frac{\pi^2 \hbar^2}{2mL^2}(n_x^2 + n_y^2)$$

where again, $n_x, n_y \geq 1$.

(b) The first number that can be written as the sum of two squares in two nontrivial ways is

$$50 = 7^2 + 1^2 = 5^2 + 5^2$$

so the first degenerate energy level is $\frac{25\pi^2\hbar^2}{mL^2}$.

[4] **Problem 13.** Consider a particle of mass m in the potential $V(x, y) = kr^2/2$. This is the two-dimensional analogue of problem 4. It is also the potential experienced by an electron in the obsolete "plum pudding" model of the atom, where they are embedded in a ball of uniform charge density.

(a) By working in Cartesian coordinates, find all of the energy levels, as well as the number of states within each energy level, called the degeneracy.

(b) [A] Now repeat the exercise in polar coordinates. In this case the integrals

$$\oint p_r \, dr, \quad \oint L \, d\theta$$

are quantized. Find the energy levels and their degeneracies. Note that for the radial motion, you will have to use the effective potential, as covered in **M6**. You will run into a difficult integral, so you may use the fact that

$$\int_{C-\sqrt{C^2-D^2}}^{C+\sqrt{C^2-D^2}} dx \sqrt{\frac{2C}{x} - \frac{D^2}{x^2} - 1} = (C - |D|)\pi$$

valid for $|D| \leq C$. How does your answer compare to that of part (a)?

Solution. (a) Let $\omega = \sqrt{k/m}$. As always, each direction contributes energy $\hbar\omega(n + 1/2)$, so the total energy is $\hbar\omega(n_x + n_y + 1)$. The number of states with energy $\hbar\omega n$ is the number of solutions to $n_x + n_y = n - 1$, which is simply n.

(b) By conservation of energy and angular momentum, we have

$$E = \frac{1}{2}kr^2 + \frac{1}{2}m(\dot{r}^2 + v_\theta^2) = \frac{1}{2}kr^2 + \frac{L^2}{2mr^2} + \frac{p_r^2}{2m}.$$

Quantizing the angular motion gives

$$\oint L\,d\theta = 2\pi L = n_\theta h$$

which means $L = n_\theta \hbar$ as usual. Let $a \leq r \leq b$ where $\dot{r} = 0$ at $r = a$ or $r = b$. Then quantizing the radial motion (accounting for two "soft" boundaries) gives

$$(n_r + 1/2)h = 2\int_a^b dr \sqrt{2mE - \frac{n_\theta^2\hbar^2}{r^2} - \omega^2 m^2 r^2}.$$

To clean this up a bit, let's set $\hbar = k = m = 1$ for now and put them back by dimensional analysis at the end. Then the equation reduces to

$$2\pi(n_r + 1/2) = 2\int_a^b dr \sqrt{2E - \frac{n_\theta^2}{r^2} - r^2}.$$

By substituting $u = 1/r^2$ we find

$$2\pi(n_r + 1/2) = 2\int_{a'}^{b'} \frac{du}{2\sqrt{u}} \sqrt{2E - \frac{n_\theta^2}{u} - u} = \int_{a'}^{b'} du \sqrt{\frac{2E}{u} - \frac{n_\theta^2}{u^2} - 1}$$

where a' and b' are the zeroes of the quantity inside the square root,

$$a' = E - \sqrt{E^2 - n_\theta^2}, \quad b' = E + \sqrt{E^2 - n_\theta^2}.$$

This incidentally shows that we need $|n_\theta| \leq E$ for the result to make sense. Using the provided integral, we find

$$2\pi(n_r + 1/2) = (E - |n_\theta|)\pi.$$

The constraint $|n_\theta| \leq E$ then translates to $n_r \geq 0$, which makes sense. Solving for E gives the final result,

$$E = 2n_r + n_\theta + 1.$$

By dimensional analysis, the right-hand side needs a factor of $\hbar\omega$ to become an energy, so

$$E = \hbar\omega(2n_r + |n_\theta| + 1).$$

The result is identical to that of part (a). The lowest energy level is $E = \hbar\omega$, corresponding to $(n_r, n_\theta) = (0, 0)$. The next is $E = 2\hbar\omega$, corresponding to $(n_r, n_\theta) = (0, \pm 1)$. The next is $E = 3\hbar\omega$, corresponding to $(n_r, n_\theta) = (1, 0)$ or $(0, \pm 2)$, and so on.

Remark

Sommerfeld applied an analysis like that of part (b) of problem 13 to the Bohr model, yielding the semiclassical orbits which are ellipses with the nucleus at the focus. (In fact, if you're so inclined, you can do this too, using the same provided integral.) This accounted for the quantum numbers n and ℓ in hydrogen. The quantum number m comes from additionally quantizing L_z, which implies that the elliptical orbits can only occur in certain planes, an idea known as "space quantization". Sommerfeld even managed to compute the first relativistic corrections to the energy levels, explaining their so-called "fine structure".

With all this included, the Bohr theory provides a complete description of the energy levels of hydrogen, except that (1) the $\ell = 0$ orbitals are missing, since they would have to go straight through the nucleus, (2) space quantization seems artificial and breaks rotational symmetry, and (3) the number of states isn't quite right, a deficiency that would later be fixed by including spin. Many complicated attempts were made to patch these problems, or to extend the theory to multi-electron atoms, but they were forgotten after the modern theory of quantum mechanics (in terms of the Schrodinger equation) appeared.

However, what you've learned above is not completely irrelevant today. The correspondence principle is the idea that quantum results should yield smoothly transition to classical ones in the limit $\hbar \to 0$, which in this context means sending the quantum numbers to infinity. And that's exactly what happens. For high quantum numbers, you can superpose atomic orbitals of nearby energy to create a sharply peaked wavefunction, just like how we could create wavepackets from plane waves in **W1**. These peaks act like localized classical particles, following the Bohr model's orbits. Thus, the Bohr model is still useful for studying Rydberg atoms, which are hydrogen-like atoms excited to very high energy levels.

[3] **Problem 14** (Cahn)**.** A crude model of an electron bound to an atom is a particle of mass m attached to a one-dimensional spring, with spring constant k and hence angular frequency $\omega = \sqrt{k/m}$. Consider two such atoms.

(a) Write down the energy levels of the system, assuming the atoms are completely independent. How many states correspond to each energy?

(b) Let the electrons have positions x_i relative to their respective equilibrium positions. Now suppose the atoms are brought close together, causing the electrons to repel. For simplicity, we represent this in terms of an extra potential energy term $k' x_1 x_2$, where k' is small. Find the new energy levels of the system exactly. (Hint: this can be done with a clever change of variables. However, you have to be careful because changing to new coordinates x_i' also requires changing the momenta; after all, if we didn't, then the quantization condition of idea 2 would change, leading to different energy levels! If K is the kinetic energy, and you are using momentum variables x_i, then the momenta should be defined as $p_i = \partial K / \partial \dot{x}_i$.)

(c) Your answer should not make sense for large k'. Physically, what is going on?

Part (b) gives a simple example of how energy levels "split" in the presence of interactions.

Solution. (a) This is just two copies of an ordinary harmonic oscillator, so $E_{n,m} = \hbar\omega(n+m+1)$ for $n, m \geq 0$. The lowest energy has one corresponding state ($n = m = 0$), the next one has two ($(n, m) = (1, 0)$ or $(0, 1)$), the next has three, and so on.

(b) The energy of the system has the form

$$E = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + k'x_1x_2.$$

This can be brought into a simplified form by expanding in normal modes, just as you saw in **M4**. Specifically, we define

$$y_1 = \frac{x_1 + x_2}{\sqrt{2}}, \quad y_2 = \frac{x_1 - x_2}{\sqrt{2}}.$$

The corresponding momenta, defined as suggested in the hint, are

$$p_1' = \frac{p_1 + p_2}{\sqrt{2}}, \quad p_2' = \frac{p_1 - p_2}{\sqrt{2}}.$$

In terms of these new variables, we have

$$E = \frac{{p_1'}^2}{2m} + \frac{{p_2'}^2}{2m} + \frac{1}{2}(k + k')y_1^2 + \frac{1}{2}(k - k')y_2^2.$$

But this is just the form of two independent harmonic oscillators, with resonant angular frequencies $\sqrt{(k \pm k')/m}$. So the energy levels are

$$E_{n,m} = \frac{\hbar}{\sqrt{m}} \left((n + 1/2)\sqrt{k + k'} + (m + 1/2)\sqrt{k - k'} \right).$$

This can be written a bit more simply by Taylor expanding, which gives

$$E_{n,m} \approx \hbar\omega \left(n\left(1 + \frac{k'}{2k}\right) + m\left(1 - \frac{k'}{2k}\right) + 1 \right) = \hbar\omega \left((n + m + 1) + (n - m)\frac{k'}{2k} \right).$$

In other words, an energy level that contains N states splits into N separate, closely spaced energy levels. This behavior is ubiquitous in quantum mechanics. This trick of turning everything into a bunch of independent quantum harmonic oscillators by using normal modes is also very important; it'll basically be the bedrock of many graduate physics courses.

(c) For $k' > k$, the energy becomes an imaginary number. When k' is this big, the repulsion between the electrons is so strong that they both just shoot off to infinity in opposite directions. That is, the energy is not bounded below; the energy can be lowered to negative infinity by increasing the separation. That means the electrons are not bound at all, so there aren't discrete energy levels.

Example 2

A nonrelativistic particle of mass m is in a cubical box with side length L and hard walls. Find the approximate number of quantum states with energy at most E_0, where E_0 is large.

Solution

Using the same reasoning as in previous problems, we apply "hard wall" boundary conditions, requiring the wavefunction to go to zero at the boundary. Thus, the wavefunction is

$$\psi \propto \sin(k_x x)\sin(k_y y)\sin(k_z z)$$

where

$$k_i = \frac{\pi}{L} n_i, \quad n_i \text{ positive integer}$$

and the energy is

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}.$$

The simplest way to proceed is to think in terms of "momentum space", an abstract space whose axes are p_x, p_y, and p_z. The allowed states form a grid in the first octant of momentum space, with a volume of $(\pi\hbar/L)^3$ for each state. The surface $E = E_0$ corresponds to a sphere of radius $\sqrt{2mE_0}$. Therefore the number of states with energy at most E_0 is approximately

$$N = \frac{1}{8}\left(\frac{4}{3}\pi(2mE_0)^{3/2}\right)\left(\frac{\pi\hbar}{L}\right)^{-3}.$$

Now let's solve the problem a slightly different way: suppose the box has *periodic* boundary conditions, so that the right side is identified with the left side, and so on. In this case, the wavefunctions can all be written in the form

$$\psi \propto \exp(i(k_x x + k_y y + k_z z))$$

but now the allowed values of the wavenumbers are

$$k_i = \frac{2\pi}{L} n_i, \quad n_i \text{ integer}.$$

The allowed states form a grid in all of momentum space, not just the first octant, with a volume of $(2\pi\hbar/L)^3$ for each state. That is, while the volume around each state is eight times as large, the states now occupy eight octants instead of one. Then the overall density of states is still the same, and the number of states with energy at most E_0 is approximately

$$N = \left(\frac{4}{3}\pi(2mE_0)^{3/2}\right)\left(\frac{2\pi\hbar}{L}\right)^{-3}$$

which matches the result for hard walls. The point of this computation is to show that when we care about the statistical properties of many states, the boundary conditions won't matter. In practice, you'll see both kinds of boundary conditions quite often.

If you find the differences between the two boundary conditions confusing, you're not alone. In his original derivation of blackbody radiation, Lord Rayleigh used "hard wall" boundary conditions but allowed negative n_i, leading to a factor of 8 error. Jeans corrected it, which is why the result is now called the Rayleigh–Jeans law.

[4] **Problem 15.** Do the following JPhO problem. This pedagogical problem introduces the WKB approximation and phase space, reviewing everything covered above, and applies it to "clusters" of atoms. You can skip sections I and III, which are covered elsewhere on this problem set.

Solution. See the official solutions here.

Remark

Because matter has wave properties, particles such as electrons can exhibit interference effects, like those covered in **W2**. For example, you can run a double-slit experiment firing individual electrons at a time and seeing where they hit the screen, which will gradually build up an interference pattern. You'll see an example of this in the next problem.

[4] **Problem 16.** IPhO 1993, problem 3. (Hint: to do the final part of this problem, you should *not* try to compare the total path lengths traversed by the electrons. That would be very hard, and worse, it won't give the right answer, because the potential from the wire also affects the electrons' phases. Instead, you should use the facts about wavefronts mentioned in **W2**. That is, waves always propagate perpendicular to wavefronts, and all points on a wavefront have the same phase.)

3 The Uncertainty Principle

Idea 5: Heisenberg Uncertainty

So far we have treated a quantum particle as having a well-defined position and momentum, but in reality the uncertainties in the position and momentum obey

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

where, as in **P2**, the uncertainties may be interpreted as standard deviations. The "semi-classical limit" used in the rest of this problem set simply corresponds to the case where the required uncertainty is relatively small, which is reached for energy levels $n \gg 1$. Occasionally, Olympiad questions will ask you to use the Heisenberg uncertainty principle to make a very rough estimate. In these cases, the constant factors will not matter.

Idea 6: Energy-Time Uncertainty

There are two commonly used versions of the energy-time uncertainty principle. If the energy of a system is only measured for a finite time Δt, it must have a finite uncertainty ΔE in its energy. In addition, if a system significantly changes its state in time Δt, then its energy must have been uncertain by a finite amount ΔE. In both cases, we have

$$\Delta E \, \Delta t \geq \frac{\hbar}{2}.$$

A third common statement of the energy-time uncertainty principle is "for a short time Δt, a system can violate energy conservation by an amount ΔE". This is wrong, because quantum systems always conserve energy; systems that naively seem to violate energy conservation simply didn't have a well-defined energy in their initial state to begin with. However, thinking this way will usually get you the right answers, essentially because of dimensional analysis.

Example 3

Consider once again a particle of mass m attached to a one-dimensional spring, with natural angular frequency ω. Use the uncertainty principle to estimate the minimum possible energy of the particle, and compare it with the result of problem 4.

Solution

Suppose the uncertainties in position and momentum are Δx and Δp. Then the potential energy is of order $k(\Delta x)^2/2$ and the kinetic energy is of order $(\Delta p)^2/2m$. Dropping constants,

$$E \sim k(\Delta x)^2 + \frac{(\Delta p)^2}{m} \gtrsim k(\Delta x)^2 + \frac{\hbar^2}{(\Delta x)^2 m}$$

where we applied the uncertainty principle. The ground state minimizes the energy, which is achieved when $(\Delta x)^2 \sim \hbar/\sqrt{km}$. In this case, the energy is of order $k\hbar/\sqrt{km} \sim \hbar\sqrt{k/m} \sim \hbar\omega$, which is just what we found earlier. (A similar derivation can be used to derive the energy of the ground state of hydrogen, along with the Bohr radius; try it!)

Remark

We can also "solve" the above problem with the energy-time uncertainty principle incorrectly. The only timescale in the problem is $1/\omega$, so

$$\Delta E \gtrsim \frac{\hbar}{\Delta t} \sim \hbar\omega$$

so $E \gtrsim \hbar\omega$. However, in reality the ground state has *no* energy uncertainty; its energy is simply the ground state energy. Another way of saying this is that a particle can hang out in the ground state forever, so Δt is infinite and hence ΔE is zero. This incorrect derivation gives the right answer just because it's the only possible answer by dimensional analysis. Thus, a sloppy problem might ask you to do it.

Example 4

Consider a single slit diffraction experiment, where photons of wavelength λ pass through a slit of width a. If the screen is a large distance D away, roughly how wide is the resulting diffraction pattern on the screen?

Solution

The photon has a momentum $p_x = \hbar k = h/\lambda$, and passing through the slit necessarily gives it a transverse momentum uncertainty of

$$\Delta p_y \sim \frac{h}{a}$$

where we dropped order one constants, which means an angle uncertainty of

$$\Delta\theta \sim \frac{\Delta p_y}{p_x} \sim \frac{\lambda}{a}.$$

Therefore, using basic geometry, the size of the pattern on the screen is

$$\Delta y \sim D\Delta\theta \sim \frac{D\lambda}{a}.$$

This is the approximate width of the central maximum for single slit diffraction, as we found in **W2**. The reason the result is the same is that light acts like a wave both classically and quantum mechanically; the quantum version of the derivation is just the same as the classical version, but with "everything multiplied by h". What's new about this derivation is that it also applies for matter particles, which have $\lambda = h/p$.

Example 5

The Higgs boson has a mass of 125 GeV and a lifetime of about $\tau = 1.6 \times 10^{-22}\,\text{s}$. About what percentage uncertainty must a measurement of a Higgs boson's mass have?

Solution

Decay is a significant change in the particle's state, and this change happens over a time τ, which means the energy uncertainty is

$$\Delta E \sim \frac{\hbar}{\tau} = 7 \times 10^{-13}\,\text{J} = 0.004\,\text{GeV}.$$

When we measure the Higgs boson's mass, we really measure the $E = mc^2$ energy released when it decays, so the unavoidable uncertainty of the mass is $\Delta E/E \sim 0.003\%$. (But the actual measured uncertainties are much higher, due to a variety of other effects.)

[1] **Problem 17** (Krane 4.39)**.** An apparatus is used to prepare an atomic beam by heating a collection of atoms to a temperature T and allowing the beam to emerge through a hole of diameter d in one side of the oven. Show that the uncertainty principle causes the diameter of the beam, after traveling a length L, to be larger than d by an amount of order $L\hbar/d\sqrt{mk_BT}$, where m is the mass of an atom.

Solution. The energy of the particles is on order k_BT, so $v \sim \sqrt{k_BT/m}$. Thus, the time taken to travel the length L is $t \sim L/v \sim L\sqrt{m/k_BT}$. Now, the uncertainty in the vertical direction is d, so the range of vertical momenta is $\sim \hbar/d$, so the range of vertical speeds is $\sim \hbar/md$. Thus, in the time t, we get a spread of order

$$\frac{t\hbar}{md} = \frac{L\hbar}{d\sqrt{m\,k_BT}},$$

as desired.

[2] **Problem 18** (Insight 8.26)**.** When helium is cooled to extremely cold temperatures, it becomes a superfluid, an exotic type of liquid that can flow with zero dissipation. These strange properties occur because quantum mechanical effects are large, making the quantum uncertainty in the position of each helium atom on the same order as the separation between atoms.

(a) Given that superfluid helium has density ρ and a helium atom has mass m, estimate the temperature T at which helium becomes a superfluid. This is closely related to, but not quite the same thing as Bose–Einstein condensation, a phase transition that bosons undergo at low temperatures.

(b) Numerically evaluate T, given that $\rho \sim 100\,\text{kg/m}^3$ and $m \sim 7 \times 10^{-27}\,\text{kg}$.

Solution. (a) The energy is of order $k_B T$, so the momentum is of order $p \sim \sqrt{2mE} \sim \sqrt{mk_BT}$. This leads to a spread in position by the uncertainty principle of

$$\Delta x \sim \frac{\hbar}{p} \sim \frac{\hbar}{\sqrt{m\,k_B T}}.$$

The volume per helium atom is m/ρ, giving a typical separation of $(m/\rho)^{1/3}$. Setting this equal to Δx and solving for T gives

$$T \sim \frac{\rho^{2/3}\hbar^2}{k_B m^{5/3}}.$$

(b) Plugging in the numbers gives $T \sim 0.7\,\text{K}$. The actual answer is $2.172\,\text{K}$, so this isn't bad for such a rough estimate!

[4] **Problem 19.** A neutron is inside a small cubical box of side length d. Ignore gravity.

(a) Estimate the minimum possible pressure on the walls using the uncertainty principle, dropping all numeric factors. In the next two parts, we'll calculate the pressure more carefully.

(b) Calculate the average pressure on the walls by treating the neutron as a classical particle bouncing back and forth, with the same momentum as expected for the ground state in the WKB approximation.

(c) Calculate the average pressure on the walls by finding the energy E of the ground state using the WKB approximation, and the definition of pressure, $P = -\partial E/\partial V$. (This actually gives the exact answer. Of course, by dimensional analysis, taking $P \sim E/V$ would also produce the right answer, up to a constant factor.)

(d) Now suppose that $N \gg 1$ neutrons are inside the box. Neutrons are fermions, as explained in the next section, and hence no two can share the same quantum state. Neglecting any interactions between the neutrons, estimate the minimum possible pressure on the walls, using either the method of part (b) or (c). How does it scale with the number density $n = N/V$?

The large pressure you will find in part (d) is known as degeneracy pressure. It supports compact objects such as white dwarfs and neutron stars, as you'll investigate in **X3**.

Solution. (a) The uncertainty of position in each dimension is around $d/2$, and for momentum, p_i (factors of 2 can be ignored). Consider the pressure on the faces in the yz-plane. The time between every collision is $t \sim d/v_x = md/p_x$, and the impulse is $\sim p_x$. Thus the force is $\sim p_x^2/md$, giving a pressure of around p_x^2/md^3. Thus Heisenberg's uncertainty principle gives

$$P_{\text{min}} \sim \frac{\hbar^2}{md^5}$$

(b) Earlier, we found that $p_i = \frac{\pi}{d} n_i \hbar$. The time between collisions is $t = 2md/p_i$ and the impulse is $2p_i$, giving a pressure of p_i^2/md^3. Note that this is a directional pressure, i.e. if the p_i were different, the pressures on each wall would be different.

The pressure on the wall perpendicular to the i direction is

$$P_i = \frac{\pi^2 \hbar^2}{md^5} n_i^2.$$

For the ground state, the n_i are all equal to one, so we have a uniform pressure,

$$P = \frac{\pi^2 \hbar^2}{md^5}$$

which is the same order of magnitude as in part (a).

(c) Earlier, we found that for a two-dimensional box,

$$E = \frac{\pi^2 \hbar^2}{2md^2}(n_x^2 + n_y^2).$$

This generalizes straightforwardly to a three-dimensional box. In the ground state, $n_x = n_y = n_z = 1$, giving

$$E = \frac{\pi^2 \hbar^2}{2md^2}(1+1+1) = \frac{3\pi^2}{2} \frac{\hbar^2}{mV^{2/3}}.$$

Carrying out the derivative,

$$P = \frac{3\pi^2}{2} \frac{2}{3} \frac{\hbar^2}{mV^{5/3}} = \frac{\pi^2 \hbar^2}{md^5}.$$

As expected, this coincides with the answer to (b), since both ultimately originate from the same approximation.

(d) In this case, it's clearest to use the method of part (c), even though it's rougher. We don't want to keep track of the different pressures on each wall from part (b), because we know that once we average over many particles, it's going to average out to a uniform pressure anyway.

If we keep track of the n_i dependence, we have

$$E_{\mathbf{n}} = \frac{\pi^2}{2} \frac{\hbar^2}{mV^{2/3}} n^2, \quad n^2 = n_x^2 + n_y^2 + n_z^2$$

where we're treating the n_i like the components of a vector. This contributes a pressure

$$P_{\mathbf{n}} = \frac{\pi^2}{3} \frac{\hbar^2}{mV^{5/3}} n^2.$$

Just as in example 2, an eighth of a sphere of $\mathbf{n}$ values is filled, where for N particles in total, the radius n_{max} of the sphere obeys

$$N = \frac{1}{8}\left(\frac{4}{3}\pi n_{\text{max}}^3\right)$$

which tells us that

$$n_{\text{max}} = \sqrt[3]{\frac{6N}{\pi}}.$$

The total pressure can be found by summing over all the lattice points within this eighth of a sphere. Since N is large, this sum can be approximated as an integral,

$$P = \int_0^{n_{\max}} \frac{4\pi n^2\, dn}{8} P_{\mathbf{n}} = \frac{\pi^3 \hbar^2}{6mV^{5/3}} \int_0^{n_{\max}} n^4\, dn = \frac{\pi^3\hbar^2}{30md^5}\left(\frac{6N}{\pi}\right)^{5/3} \propto \frac{\hbar^2 n^{5/3}}{m}.$$

Thus, the degeneracy pressure scales as $n^{5/3}$. (If you solved this problem instead using part (b), or using periodic boundary conditions from example 2, you should get the exact same answer.)

[3] **Problem 20.** USAPhO 2018, problem B2.

[3] **Problem 21.** Classically, an electron orbiting a proton with angular frequency ω_o emits radiation with angular frequency $\omega_c = \omega_o$, as covered in **E7**. On the other hand, quantum mechanically the energy levels are discrete, and using the de Broglie relation $\Delta E = \hbar\omega$ indicates the angular frequencies of radiation emitted when the electron drops between energy levels are discrete as well. The classical and quantum models thus seem to be radically different, but in the limit $n \to \infty$ where quantum effects become negligible, the two should match.

(a) Suppose that the electron can orbit the proton in circular orbits with discrete radii r_n. For the n^{th} orbit, compute the angular frequency ω_c of the emitted radiation according to classical mechanics.

(b) Now suppose the electron drops from the n^{th} energy level to the $(n-1)^{\text{th}}$ energy level. Compute the angular frequency ω_q of the emitted radiation according to quantum mechanics, assuming the orbits have radii r_n.

(c) In the limit $n \to \infty$, the results of parts (a) and (b) should coincide. Therefore, by equating these results, infer how r_n depends on n, and thus how L depends on n. If all goes well, you should recover the result of Bohr quantization.

The reasoning here is exactly how Bohr came up with Bohr quantization in the first place. (The de Broglie relation we had to use was motivated earlier through Planck's law, as we showed in **T2**.)

Solution. (a) Classically, we need to balance the centripetal force with the Coulomb force,

$$m\omega_0^2 r = \frac{e^2}{4\pi\epsilon_0 r^2}.$$

Since $\omega_c = \omega_0$, we get

$$\omega_c = \sqrt{\frac{e^2}{4\pi\epsilon_0 r_n^3 m}}.$$

(b) The de Broglie relation tells us that $\hbar\omega_q = E_n - E_{n-1}$, so using standard results for circular orbits in an inverse square potential,

$$\omega_q = \frac{e^2}{4\pi\hbar\epsilon_0}\frac{1}{2}\left(\frac{1}{r_{n-1}} - \frac{1}{r_n}\right).$$

(c) Thinking of n as a large number, we can approximate

$$\left(\frac{1}{r_{n-1}} - \frac{1}{r_n}\right) = \frac{r_n - r_{n-1}}{r_{n-1} r_n} \approx \frac{r_n - r_{n-1}}{r_n^2} \approx \frac{1}{r_n^2} \frac{dr_n}{dn}.$$

Plugging this into the equation $\omega_c = \omega_q$ and simplifying, we get

$$2\hbar \sqrt{\frac{4\pi\epsilon_0}{e^2 m}} = \frac{1}{\sqrt{r_n}} \frac{dr_n}{dn}.$$

Separating and integrating, we have

$$r_n = \frac{4\pi\epsilon_0 \hbar^2}{e^2 m} n^2.$$

There could be a constant of integration, but for large n, it's negligible, and we can conclude that for large n, $r_n \propto n^2$. On the other hand, we have

$$L_n = m v_n r_n = m \omega_c r_n^2 \propto r_n^{1/2}$$

which means that for high n, $L_n \propto n$.

This is as far as we can go, "rigorously". The amazing thing is that this derivation is based on $\omega_c = \omega_q$, which only holds at large n, along with approximations that only work at large n, and also involves an unknown constant of integration. But if we just set the constant of integration to zero, and assume the derivation works for all n, then you can check that we recover $L = n\hbar$, which happens to be exactly true in the real world!

[4] **Problem 22.** IPhO 2005, problem 3. You may skip part 4, since it's quite similar to another problem on this problem set.

4 Bosons and Fermions

So far, we've only solved for the energy levels of individual particles. Now we'll consider what happens when we put many of these particles together. We will assume the particles do not interact, which means their energy levels are just the same as the energy levels for individual particles. If the particles are fermions, they obey the Pauli exclusion principle, which means no two can occupy the same energy level. If they are bosons, there is no such restriction; we'll consider bosons first.

[3] **Problem 23.** This problem is a modification and clarification of USAPhO 2011 A4. (In fact, the answers differ, so don't compare against the USAPhO solution!) Consider a simplified model of the electromagnetic radiation inside a cubical box of side length L at temperature T. In this model, modes of the electric field have spatial dependence

$$E(x, y, z) = E_0 \sin(k_x x) \sin(k_y y) \sin(k_z z)$$

where one corner of the box lies at the origin and the box is aligned with the x, y, and z axes. For simplicity, we will treat the electric field as a *scalar*.

(a) The electric field must be zero everywhere at the sides of the box. What condition does this impose on the k_i?

(b) Each permitted value of the triple (k_x, k_y, k_z) corresponds to a mode, which can be occupied by any number of photons. Each photon has an energy $E = \hbar\omega$, where $\omega = ck$ is the angular frequency of the mode. How many modes have an energy per photon of at most $k_B T$?

(c) As a crude approximation, suppose that in thermal equilibrium, each mode with energy per photon at most $k_B T$ contains exactly one photon, while all other modes contain no photons. Compute the total energy of the photons in the box. (Answer: $(k_B T)^4 L^3 / 8\pi^2 \hbar^3 c^3$.)

Note that the procedure here is different from what we did above. Before, we started with particles and quantized $\oint p\, dx$ to get the allowed quantum states. Here, we're treating a situation with many particles (photons), which are excitations of an underlying field (the electromagnetic field). In this case, we found the (normal) modes of the classical field, then quantized by saying that photons could occupy these modes. This is the methodology of quantum field theory.

Solution. (a) We see that $k_x L = n_x \pi$, $k_y L = n_y \pi$, and $k_z L = n_z \pi$.

(b) Define the vector $\mathbf{n} = (n_x, n_y, n_z)$. For each mode, the energy per photon is

$$E = \hbar\omega = \hbar c k = \frac{\pi\hbar c}{L}\sqrt{n_x^2 + n_y^2 + n_z^2}.$$

Moreover, the values of the n_i are positive integers. Therefore, the quantum states we are looking for form an eighth of a sphere in phase space, bounded by states with

$$n_{\text{max}} = \frac{E_{\text{max}} L}{\pi\hbar c}.$$

The number of modes (i.e. the number of values of $\mathbf{n}$) with an energy per photon of at most E_{max} is thus

$$N(E_{\text{max}}) = \frac{1}{8}\left(\frac{4}{3}\pi n_{\text{max}}^3\right) = \frac{1}{6\pi^2}\left(\frac{EL}{\hbar c}\right)^3.$$

In this case, the requested answer is

$$N(k_B T) = \frac{1}{6\pi^2}\left(\frac{k_B T L}{\hbar c}\right)^3.$$

Note that this is different from the official USAPhO solution, because they also allowed negative values for the n_i. This is incorrect, because flipping the sign of one of the n_i gives you exactly the same mode, up to an irrelevant -1 factor.

(c) We sum over the modes. For each occupied mode, we assume the energy stored is E, so

$$U = \int E\, dN = \int E \frac{dN}{dE}\, dE = \frac{1}{6\pi^2}\left(\frac{L}{\hbar c}\right)^3 \int_0^{k_B T} E(3E^2)\, dE = \frac{1}{8\pi^2}\frac{(k_B T)^4 L^3}{\hbar^3 c^3}.$$

[4] **Problem 24.** The final result of the problem above is correct dimensionally, but has incorrect numerical factors because of the crude approximations made. In this problem we'll do a more careful analysis to get the right result. This question is self-contained, but background from **T1** and **T2** will be helpful.

(a) Consider a quantum mode that can support photons of energy E. The mode can be occupied by any whole number of photons. Thus, using the Boltzmann distribution, the probability of having n photons is

$$p_n \propto e^{-nE/k_BT}.$$

Show that the expected number of photons in the mode is

$$\langle n \rangle = \frac{1}{e^{E/k_BT} - 1}.$$

This is the Bose–Einstein distribution.

(b) Sketch $\langle n \rangle$ as a function of E. How does it behave at high and low E, and do those results make physical sense?

(c) Using the Bose–Einstein distribution, show that the total energy is

$$U = \frac{L^3 \hbar}{\pi^2 c^3} \int_0^\infty d\omega \, \frac{\omega^3}{e^{\hbar\omega/k_BT} - 1}$$

where ω is the angular frequency. You'll have to multiply by a factor of two, because there are two independent photon polarizations for each mode we found above. (Note that if we open the box, the photons will fly out, and the frequency distribution of the emitted light will be given by the integrand; this yields Planck's law for blackbody radiation.)

(d) **[A]** Using an appropriate substitution, show that U is a dimensionful constant times the dimensionless integral

$$\int_0^\infty dx \, \frac{x^3}{e^x - 1}.$$

To evaluate this integral, expand the denominator as a power series, integrate each term individually, and use the fact that the Riemann zeta function obeys

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \quad \zeta(4) = \frac{\pi^4}{90}.$$

When the smoke clears, you should find that

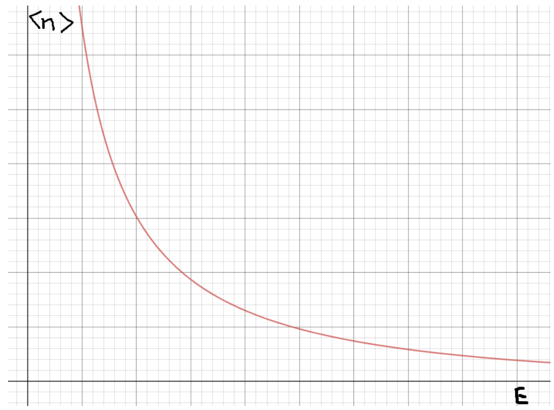
$$U = \frac{\pi^2}{15} \frac{(k_BT)^4 \, L^3}{(\hbar c)^3}.$$

Solution. (a) We see that

$$\langle n \rangle = \frac{\sum_n n e^{-nE/k_BT}}{\sum_n e^{-nE/k_BT}} = \frac{\left(\frac{1}{e^{E/k_BT}-1}\right)^2}{\frac{1}{e^{E/k_BT}-1}} = \frac{1}{e^{E/k_BT} - 1},$$

as desired.

(b) $\langle n \rangle$ as a function of E looks like this:



For modes with low E, we have $\langle n \rangle \approx k_B T/E \ll 1$. This means the energy stored in this mode is approximately $\langle n \rangle E \approx k_B T$. That makes physical sense: in this limit where there are lots of photons, their discreteness doesn't matter, so the mode can be treated classically, getting energy $k_B T$ by the equipartition theorem.

For modes with high E, we have $\langle n \rangle \approx e^{-E/k_B T} \ll 1$. This also makes sense, because in this limit there isn't enough thermal energy to significantly excite the modes.

(c) The logic is exactly the same as part (c) of the previous problem, but instead of saying that some modes have 1 photon and others have 0 photons, we just assign the proper $\langle n \rangle$ photons to every mode. Then

$$U = \int \langle n \rangle E \, dN = \frac{1}{2\pi^2} \left(\frac{L}{\hbar c} \right)^3 \int_0^\infty \langle n \rangle E^3 \, dE.$$

As stated, we multiply by a factor of 2 to account for the two polarizations per mode. We also change variables from E to ω using $E = \hbar\omega$. This gives

$$U = \frac{L^3 \hbar}{\pi^2 c^3} \int_0^\infty d\omega \, \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1}$$

just as desired.

(d) Now we make the substitution $x = \hbar\omega/k_B T$, where $dx = \hbar d\omega/k_B T$.

$$U = \frac{L^3 \hbar}{\pi^2 c^3} \int_0^\infty \left(\frac{k_B T}{\hbar} dx \right) \frac{(k_B T x/\hbar)^3}{e^x - 1} = \frac{(k_B T)^4 L^3}{\pi^2 c^3 \hbar^3} \int_0^\infty dx \frac{x^3}{e^x - 1}.$$

We can rearrange the integral into

$$I = \int_0^\infty dx \frac{x^3}{e^x - 1} = \int_0^\infty e^{-x} x^3 \frac{1}{1 - e^{-x}} dx.$$

Recognizing $1/(1 - e^{-x})$ as a geometric series $\sum_{n=0}^\infty (e^{-x})^n$, we can represent the integral as

$$I = \int_0^\infty \sum_{n=0}^\infty x^3 e^{-(n+1)x} dx.$$

We can reindex this since the only instance of n is $n + 1$, so we can start the summation at $n = 1$. Since we can integrate this term by term, we can change the order of the integral and summation, then integrate by parts to get

$$I = \sum_{n=1}^{\infty} \int_0^{\infty} x^3 e^{-nx} dx = \sum_{n=0}^{\infty} \int_0^{\infty} (3x^2 dx) \left(\frac{e^{-nx}}{n} \right) = \sum_{n=0}^{\infty} \int_0^{\infty} (6x dx) \left(\frac{e^{-nx}}{n^2} \right)$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} (6 dx) \left(\frac{e^{-nx}}{n^3} \right) = \sum_{n=0}^{\infty} \frac{6}{n^4} = 6\zeta(4).$$

Putting this into our original expression gets

$$U = \frac{\pi^2 k_B^4}{15 c^3 \hbar^3} L^3 T^4.$$

[5] **Problem 25.** APhO 2002, problem 1. This useful problem covers the other common example of a quantized bosonic field. In the above problems, we quantized the electromagnetic field to get photons. Here, we quantize a displacement field to get phonons.

Idea 7

In problems 23 and 24, we handled a system of bosons (specifically photons) by considering the modes the photons could occupy, then calculating how many photons were in each mode. This was the easiest route. If we had instead fixed the number of photons, then counted the ways they could be distributed among the mode, the combinatorics would have been a complete nightmare, because multiple photons can occupy the same mode.

Fermions, which obey the Pauli exclusion principle, are simpler, because no two can be in the same state. For instance, if there are n noninteracting fermions in a system, then the lowest energy state of the whole system consists of having one fermion occupy the lowest energy state, the second occupy the second-lowest energy state, and so on. (Accounting for interactions makes the problem much more complicated, because it means the energy of a state depends on whether other states are occupied. However, you can explain a surprising amount while completely neglecting interactions.)

Example 6: Tremaine–Gunn Bound

Suppose all of the dark matter in the galaxy is composed of a single kind of fermionic particle, of mass m. The escape velocity of the galaxy is of order $v_{\text{esc}} \sim 10^{-3}c$, and the dark matter density near Earth is $\rho \sim 0.3\,\text{GeV}/(c^2\,\text{cm}^3)$. What's the minimum possible value of m?

Solution

The reason there's a minimum possible value of m is that, as m gets smaller, we need more dark matter particles. But the Pauli exclusion principle tells us that if we want to add more particles, they need to have higher and higher energy, and at some point the particles will have so much energy they won't be bound to the galaxy at all.

To get a rough estimate, let's suppose the galaxy has length scale L, so that we need at least $N \sim \rho L^3/m$ dark matter particles. They need to have energy less than $E_0 \sim m_0 v_{\text{esc}}^2$.

Plugging this into the final result of example 2 and dropping all numeric factors gives

$$\frac{\rho L^3}{m} \gtrsim \frac{m^3 v_{\text{esc}}^3 L^3}{\hbar^3}$$

which yields the bound

$$m \lesssim \left(\frac{\rho\,\hbar^3}{v_{\text{esc}}^3}\right)^{1/4} \sim 10\,\text{eV}/c^2.$$

A few decades ago, neutrinos were leading dark matter candidates, since they are light fermionic particles that interact very weakly with ordinary matter. But we now know that the neutrino mass is well below this bound, so that nice idea doesn't work. On the other hand, dark matter could still be composed of bosonic particles of much lighter mass.

[2] **Problem 26.** Consider a system with many noninteracting fermions, and many quantum states. Each quantum state can be either empty or occupied by a fermion. We want to find the probability that a given quantum state, of energy E, is occupied.

(a) To put a fermion in this state, we need to remove a fermion from some other state. Suppose the energy released by doing this, suitably averaged, is μ. (This is the chemical potential, and it depends on the temperature, the number of fermions, and the number of states and their energies.) Using the Boltzmann distribution, show that the probability of occupancy is

$$\langle n \rangle = \frac{1}{e^{(E-\mu)/k_B T} + 1}.$$

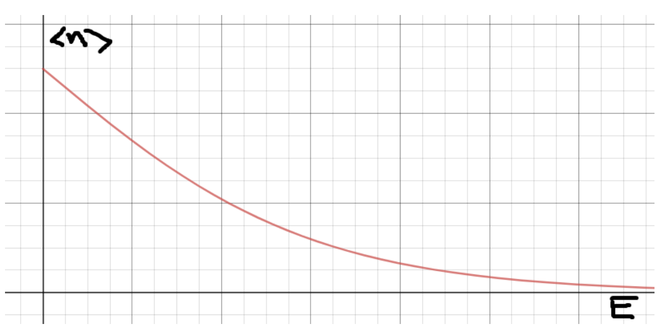
This is the Fermi–Dirac distribution.

(b) Sketch $\langle n \rangle$ as a function of E for small but nonzero temperature, as well as the limit attained for zero temperature.

Solution. (a) The two possibilities are being occupied and not occupied, and the latter comes with a Boltzmann factor of $e^{-(E-\mu)/k_B T}$. Thus,

$$\langle n \rangle = \frac{e^{-(E-\mu)/k_B T}}{1 + e^{-(E-\mu)/k_B T}} = \frac{1}{1 + e^{(E-\mu)/k_B T}}.$$

(b) The graph looks like this for $\mu = 0$:



For a nonzero value of μ, the graph should look like this:

As $T \to 0$, the form of $\langle n \rangle$ will start to look like a step function, $\theta(\mu - E)$. This simply means that the fermions fill up the lowest energy states first, to minimize their total energy. The chemical potential is set by how many fermions there are in total.

[3] **Problem 27.** In this problem we'll consider the energy of the conducting electrons in a solid at low temperatures. Model a solid as a cubical box of side length L with periodic boundary conditions.

(a) Find the number of quantum states with energy at most E_F, making sure to account for the two spin states of the electron.

(b) Suppose there are N electrons in total. They will fill all of the energy levels up to $\mu = E_F$, where E_F is called the Fermi energy. Show that

$$E_F = \frac{h^2}{2m} \left(\frac{3N}{8\pi V} \right)^{2/3}.$$

(c) A sodium crystal has one conduction electron per atom. The density and molar mass are

$$\rho = 0.971 \times 10^3 \, \mathrm{kg/m^3}, \quad M = 0.023 \, \mathrm{kg/mol}.$$

Find N/V and E_F, and use this to evaluate the typical speed v of an electron.

Solution. (a) In a cubical box with period boundary conditions, the wavenumbers satisfy $k_i = \frac{2\pi}{L} n_i$ where n_i can also be a negative integer. As seen in the example, the number of states is $\frac{4}{3}\pi(\sqrt{2mE_F})^3(2\pi\hbar/L)^{-3}$, but we multiply that by 2 due to the two spin states of the electron.

$$N = \frac{8}{3}\pi(2mE_F)^{3/2} \left(\frac{2\pi\hbar}{L} \right)^{-3}.$$

(b) In a cube, $V = L^3$. Rearranging the above equation gets

$$E_F = \frac{h^2}{2m} \left(\frac{3N}{8\pi V} \right)^{2/3}.$$

(c) If N_A is Avogadro's number, then the mass of each atom is $m = M/N_A$, which gives an atom density of ρ/m. Since there's one conduction atom per atom, we have $N/V = \rho/m = N_A\rho/M$, which gives us $N/V = 2.54 \times 10^{28} \, \mathrm{m^{-3}}$. Putting this into our previous formula gives $E_F = 5.05 \times 10^{-19}$ J. The typical speed v_F satisfies

$$E_F \sim \frac{1}{2} m v_F^2$$

which gives $v_F \sim 10^6 \, \mathrm{m/s}$, or on the order of 1% of the speed of light! This verifies a statement made in **E4** about electrical conduction in solids.

[3] **Problem 28** (MIT). [A] This is an advanced problem that is only placed here because the final result is neat. An integer N can be *partitioned* by writing it as a sum of positive integers, and the partition function $p(N)$ is the number of unique ways this can be done. For example,

$$4 = 1+1+1+1 = 1+1+2 = 2+2 = 1+3 = 4$$

which implies $p(4) = 5$. Counting the number of partitions of an integer is a very hard combinatorics problem, but we can get an estimate for large N using string theory.

(a) Consider an ideal string with hard boundary conditions and fundamental angular frequency ω. Show that the number of distinct quantum states with energy $N\hbar\omega$ is $p(N)$.

(b) Now suppose the string is at temperature T, where T is chosen so that the expected energy is $N\hbar\omega$. In the thermodynamic limit $N \gg 1$, find a relation between N and T. You may use the result $\zeta(2) = \pi^2/6$.

(c) By approximating the entropy as $S \approx k_B \log p(N)$, show that

$$\frac{\hbar\omega}{kT} = \frac{d \log p(N)}{dN}.$$

Combine this with your previous result to find an estimate for $p(N)$.

To check your answer, the celebrated Hardy–Ramanujan formula (which is more accurate than the very rough treatment we give above) is

$$p(N) \sim \frac{1}{4\sqrt{3}\,N} \exp\left(\pi\sqrt{\frac{2N}{3}}\right).$$

While we only considered a simple nonrelativistic string here, calculations of this sort show up in the thermodynamics of string theory. For further discussion, see chapter 22 of Zwiebach.

Solution. (a) The modes of the string have angular frequencies $m\omega$, which means that upon quantization, each quantum in mode m will have energy $m\hbar\omega$. Thus, putting in quanta to reach total energy $N\hbar\omega$ is the same as building a partition of N out of integers m. For example, $4 = 1+1+1+1$ corresponds to putting 4 quanta in the fundamental mode, while $4 = 4$ corresponds to putting 1 quantum in the fourth harmonic.

(b) By borrowing some results from previous problems, we have

$$N = \sum_m \frac{m}{e^{m\hbar\omega/kT} - 1} \approx \int_0^\infty \frac{m\,dm}{e^{m\hbar\omega/kT} - 1}.$$

We have already done a similar integral in problem 24, and using the same trick of expanding the denominator in a Taylor series and integrating term by term gives

$$N \approx \frac{\pi^2}{6}\left(\frac{kT}{\hbar\omega}\right)^2.$$

(c) Using the definition of temperature,

$$T = \frac{dE}{dS} = \frac{\hbar\omega}{k} \frac{dN}{d \log p(N)}.$$

Rearranging slightly gives the desired result. Eliminating T using the above result,

$$\frac{d \log p}{dN} \approx \frac{\pi}{\sqrt{6N}}.$$

Separating and integrating,

$$\log p(N) \sim \int_0^N \frac{\pi}{\sqrt{6N}}\, dN = \pi\sqrt{\frac{2N}{3}}$$

which agrees with the Hardy–Ramanujan formula. (Of course, this only gets the leading term right, because we made approximations. The most severe approximation we did was taking $S \approx k_B \log p(N)$. In reality, the energy at a given temperature is not fixed, but rather has fluctuations, so we should have instead summed $p(n)$ for a range of n near N.)

Example 7: Casimir Forces

Consider an infinite uniform string, on which waves propagate with speed v. The string is held fixed by pins at two points separated by a distance L. When the string is in its ground state, what is the resulting force between the pins?

Solution

Of course, in classical mechanics the ground state would just be $y(x) = 0$, and the force would just be the tension T. But there is an additional quantum mechanical contribution, which arises because each of the standing waves between the pins, with angular frequencies $\omega_n = n\pi v/L$, carries a so-called "zero point" energy $\hbar\omega_n$.

As usual, force can be found by differentiating energy, $F = -dE/dx$. By summing up the zero point energy in all of the standing waves, we naively get

$$E = \sum_n \frac{\hbar\omega_n}{2} = \frac{\pi\hbar v}{2L} \sum_{n=1}^{\infty} n = E_0 \sum_{n=1}^{\infty} n = \infty$$

which is rather unhelpful. This result is off for two reasons. First, even when there aren't any pins, the string still has standing waves, and these waves *also* have a naively infinite energy density. When we move the pins a bit, we change both the zero point energy outside the pins and inside, by infinite amounts, but the *net* change is finite, giving a finite force.

Quantitatively, the energy in between the pins due to the standing waves is

$$E_{\text{plate}} = E_0 \sum_{n=1}^{\infty} n = \infty$$

and the energy we would have had there if the pins didn't exist is the "continuous" sum,

$$E_{\text{vac}} = E_0 \int_0^\infty x\, dx = \infty.$$

The difference should be finite, but we can't just subtract infinity with infinity, which brings us to the second problem: none of these quantities are *actually* infinite. Any real string will have a finite maximum oscillation frequency – for instance, the wavelength certainly can't get smaller than the atomic spacing. Alternatively, even if we had an idealized string where E_{vac} was actually infinite, no real pin can perfectly block waves of all frequencies. For sufficiently high frequencies the waves won't be affected by the pins, so that the sum in E_{plate} eventually behaves like the integral in E_{vac}, leaving a finite difference between the two.

In other words, the difference between E_{plate} and E_{vac} in reality comes from only low n and x, Therefore, let's "regulate" the two expressions above so that they're unchanged in this regime, but match each other at high n and x. The simplest way to do this is to take

$$E_{\text{plate}} = E_0 \sum_{n=1}^\infty n e^{-\epsilon n}, \quad E_{\text{vac}} = E_0 \int_0^\infty x e^{-\epsilon x}\, dx = \frac{E_0}{\epsilon^2}$$

for small ϵ. To handle the sum, let $\alpha = e^{-\epsilon}$, so that

$$E_{\text{plate}}/E_0 = \alpha + 2\alpha^2 + 3\alpha^3 + \ldots.$$

Now we use the usual trick for arithmetic-geometric series. Note that

$$\alpha E_{\text{plate}}/E_0 = \alpha^2 + 2\alpha^3 + 3\alpha^4 + \ldots.$$

Subtracting, we find

$$(1-\alpha)E_{\text{plate}}/E_0 = \alpha + \alpha^2 + \alpha^3 + \ldots = \frac{\alpha}{1-\alpha}.$$

We thus conclude that

$$E_{\text{plate}} = \frac{e^{-\epsilon}}{(1-e^{-\epsilon})^2}\, E_0 = E_{\text{vac}} - \frac{1}{12}E_0 + O(\epsilon)$$

where we used a result from **P1**. Finally, when we take ϵ to zero, the difference is simply

$$E = E_{\text{plate}} - E_{\text{vac}} = -\frac{1}{12}E_0.$$

Differentiating gives the force,

$$F = \frac{\pi \hbar v}{24 L^2}$$

which turns out to be attractive. Not only is this finite, it's right! Experiments have measured this "Casimir force" precisely for light between two conductors, where $v = c$, and confirmed the expected results.

You're probably suspicious about this derivation because it depends on the arbitrary choice of an exponential suppression. What if the sums and integrals were regulated at high n and x in a different way? Shouldn't the answer depend on the details of the string and pin? Remarkably, the answer is no: the regulator doesn't matter. If you try others, such as $e^{-\epsilon n^2}$ or $1/n^{\epsilon}$, you'll get the same result; you can find a general proof in chapter 15 of Schwartz's *Quantum Field Theory and the Standard Model*. The reason is that the effect comes from physics at low frequencies, so it doesn't matter how you regulate the high frequencies.

It is for precisely this reason that you will sometimes see the mysterious equation

$$1 + 2 + 3 + \ldots = -\frac{1}{12}.$$

It's not really true. Instead, what it physically means is that the difference between the regulated sum and integral is $-1/12$ for any reasonable regulator.

Modern II: Atoms, Particles, and Nuclei

Chapters 48, 50, 51, and 52 of Halliday and Resnick are a useful introduction. For further reading, see chapters 12 through 14 of Krane for nuclear and particle physics, and section 5.2 of Griffiths' *Introduction to Quantum Mechanics* (3rd edition) for atomic physics. If you'd like to learn a lot more about these subjects, see the MIT OCW 22.01 course on nuclear engineering, chapters 1 and 2 of Griffiths' *Introduction to Elementary Particles*, or David Tong's Lectures on Particle Physics. For some neat reading about symmetries in particle physics, see chapter I-52 of the Feynman lectures. For all problems, you can consult the periodic table. There is a total of **87** points.

1 Nuclear Decay

Idea 1

Atomic nuclei are written as ${}^A_N\mathrm{X}$ where X is the name of the element, A is the mass number (number of neutrons plus protons), and N is the atomic number (number of protons). Since N can be inferred from X, we often don't write it.

Idea 2

The most common nuclear decay channels are alpha decay,

$${}^A_Z\mathrm{X} \to {}^{A-4}_{Z-2}\mathrm{X}' + {}^4_2\mathrm{He}$$

and beta decay,

$${}^A_Z\mathrm{X} \to {}^{A}_{Z+1}\mathrm{X}' + e^- + \overline{\nu}_e.$$

Here, $\overline{\nu}_e$ is a light neutral particle called an anti-electron neutrino. A variant of beta decay, called β^+ decay or positron emission, is

$${}^A_Z\mathrm{X} \to {}^{A}_{Z-1}\mathrm{X}' + e^+ + \nu_e$$

where ν_e is called an electron neutrino, and e^+ is a positron. If electrons are present, the nuclei may also capture them, leading to the process

$${}^A_Z\mathrm{X} + e^- \to {}^{A}_{Z-1}\mathrm{X}' + \nu_e.$$

Finally, nuclei can decay from excited states by emitting photons, in gamma decay.

There are many more processes, such as inverse beta decay or double beta decay. However, the general principles underlying which decays are allowed are simple: baryon number, electric charge, and electron number are all conserved. In the restricted setting of nuclear processes,

$$\begin{aligned}
\text{baryon number} &= \text{number of protons and neutrons} \\
\text{electric charge} &= \text{number of proton and positrons} - \text{number of electrons} \\
\text{electron number} &= \text{number of electrons and electron neutrinos} \\
&\quad - \text{number of positrons and anti-electron neutrinos.}
\end{aligned}$$

Idea 3

The amount of energy released in a nuclear decay can be inferred from the drop in mass energy, $\Delta E = (\Delta m)c^2$. A nuclear decay can only spontaneously occur if it lowers the energy of the *entire* nucleus. To emphasize this point, note that at the level of individual nucleons, $\beta^\pm$ decay involve the processes

$$n \to p + e^- + \overline{\nu}_e, \quad p \to n + e^+ + \nu$$

respectively. Either of these processes could be energetically favorable inside a nucleus, depending on its composition. But an isolated proton will never decay, because protons are heavier than neutrons.

[1] **Problem 1** (Krane 12.38)**.** Complete the following decays:

(a) $^{27}\text{Si} \to {}^{27}\text{Al} +$

(b) $^{74}\text{As} \to {}^{74}\text{Se} +$

(c) $^{228}\text{U} \to \alpha +$

(d) $^{93}\text{Mo} + e^- \to$

(e) $^{131}\text{I} \to {}^{131}\text{Xe} +$

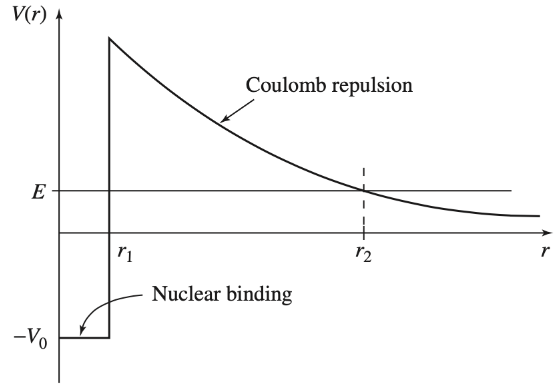
Example 1: PPP 189

^{7}Be is a radioactive element with a half-life of 53.37 days. When isotope 7 of beryllium is heated to a few thousand degrees, its half-life changes. This is strange, because nuclear processes typically involve much higher energy scales. What is the explanation for this?

Solution

Temperatures at this scale are not enough to affect nuclear physics, but are enough to affect atomic physics. The electrons gain enough energy to leave the atoms, then hit other nuclei, causing electron capture, changing the isotope of beryllium and hence its half-life.

[4] **Problem 2.** In Gamow's theory of alpha decay, alpha particles can escape from nuclei by quantum tunneling. The alpha particle is bound to the nucleus by a nuclear force, which we model as a finite square well, $V(r) = -V_0$ for $r < r_1$, and repelled by the Coulomb force, $V(r) = k(Ze)(2e)/r = \alpha/r$. The combination of the two creates a potential barrier the alpha particle must tunnel though. Let the alpha particle have mass m and energy E.



(a) Using classical mechanics, calculate the time between collisions with the wall. This is also correct in quantum mechanics; one can take the wavefunction to be a wavepacket, which really does collide with the walls with the same frequency.

(b) In quantum mechanics, each collision has an associated amplitude to escape by quantum tunneling. To compute this, recall from **X1** that the WKB approximation states that the wavefunction picks up a phase $e^{i\theta}$, where

$$\theta = \frac{1}{\hbar} \int p\, dx.$$

Calculate θ by integrating from r_1 to r_2, assuming that $r_1 \ll r_2$ for simplicity. You should find that θ is a complex number, indicating the wavefunction exponentially decays in the barrier. (Hint: you will find a tricky integral, for which you should use a trigonometric substitution.)

(c) The probability of escape scales as the amplitude squared. Write down an approximate expression for the timescale τ for decay to occur.

This model is very rough, so the numeric and slowly varying prefactors should not be expected to be accurate. But the exponential dependence of the timescale on the energy, which you should have found is due to the tunneling probability scaling as $e^{-\sqrt{E_g/E}}$ for some constant E_g, is by far the most important piece, and it fits experimental results.

(d) In nuclear fusion reactions in the Sun, the process above occurs in reverse: an incoming alpha particle (i.e. helium nucleus) needs to tunnel through the Coulomb barrier to fuse with another nucleus. The initial energy is Boltzmann distributed as $e^{-E/k_B T}$, so the fusion rate is

$$\Gamma \sim \int dE\, e^{-\sqrt{E_g/E}} e^{-E/k_B T}.$$

The integrand is the product of a rapidly rising exponential and a rapidly falling exponential. Estimate the exponential part of the dependence of Γ on T.

[3] **Problem 3.** Consider the process by which an electron absorbs a single photon, $e^- + \gamma \to e^-$.

(a) Show that this process is forbidden by energy-momentum conservation. By time reversal, emission of a single photon should be forbidden as well. This is quite puzzling, since we already know of many processes where something like absorption or emission seems to happen.

(b) Can an electron in an isolated atom absorb a single photon? If so, why doesn't the reasoning in part (a) work? If not, how can atoms absorb photons at all, as described in **X1**?

(c) Can isolated nuclei emit single photons? If so, why doesn't the reasoning in part (a) work? If not, how can gamma decay occur?

(d) Can isolated electrons absorb or emit *classical* electromagnetic radiation? If so, why doesn't the reasoning in part (a) work? If not, how can Thomson scattering (covered in **E7**) happen?

Idea 4

Radioactive decay is a memoryless process: in an infinitesimal time interval dt, any nucleus has a probability $\lambda\, dt$ of decaying, regardless of its previous history. As a result, the number of radioactive nuclei falls exponentially as

$$N(t) = N_0 e^{-\lambda t}.$$

The activity $A(t)$ is the rate of decay events, and also falls exponentially,

$$A(t) = A_0 e^{-\lambda t}.$$

The mean lifetime of the nuclei is $\tau = 1/\lambda$.

[3] **Problem 4.** This problem tests your understanding of memoryless processes. Below are several plausible ways to measure τ.

(a) We start a stopwatch at noon and stop it when the next decay happens, giving t_1.

(b) We have an intern watch the sample continuously, then at noon, ask them how long it was since the last decay, giving t_2.

(c) We have an intern watch the sample continuously, then at noon, ask them how long it was since the last decay. We then set our stopwatch so that $t = 0$ when that decay happened, and stop the stopwatch when the next decay happens, giving t_3.

(d) We continuously watch the sample, start a stopwatch when the first decay happens, then stop it when the next decay happens, giving t_4.

We repeat procedure i many times, so the average of t_i is τ_i. Find the τ_i in terms of τ.

[2] **Problem 5** (Krane 12.37)**.** A radioactive sample contains N_0 atoms at time $t = 0$. It is observed that N_1 radioactive atoms remain at time t_1 and then decay by time t_2, N_2 remain at t_2 and then decay by time t_3, and so on. Show that if many observations are made, then τ can be measured as

$$\tau = \frac{1}{N_0} \sum_i N_i t_i.$$

Example 2

Radium can be found in trace quantities throughout the Earth, and has a half-life of 1620 years. Suppose that there is currently 1 kg of radium on the Earth. Then extrapolating backwards, there was $2^{4.5\times10^9/1620}$ kg of radium on the Earth when it was formed, which is greater than the mass of the observable universe! What's wrong with this calculation?

Solution

Nuclear decays don't happen in isolation; there are entire networks of nuclear decay chains. Radium decays quickly, but it is also constantly produced by the decay of other isotopes, which have much longer half-lives.

[3] **Problem 6.** USAPhO 2009, problem A2.

[3] **Problem 7.** IPhO 2000, problem 1c. Don't worry about the official answer sheet; treat this like a regular USAPhO problem.

[3] **Problem 8** (PPP 190)**.** Part of the series of isotopes produced by the decay of thorium-232, along with the corresponding half-lives, is given below:

$$^{232}_{90}\mathrm{Th} \xrightarrow{1.4\times10^{10}\,\mathrm{y}} {}^{228}_{88}\mathrm{Ra} \xrightarrow{5.7\,\mathrm{y}} {}^{228}_{89}\mathrm{Ac} \xrightarrow{6.1\,\mathrm{h}} {}^{228}_{90}\mathrm{Th} \xrightarrow{1.9\,\mathrm{y}} {}^{224}_{88}\mathrm{Ra} \xrightarrow{3.6\,\mathrm{d}} {}^{220}_{86}\mathrm{Rn} \xrightarrow{56\,\mathrm{s}} \ldots.$$

Thorium-232 and thorium-228 in equilibrium are extracted from an ore and purified by a chemical process. Sketch the form of the variation in the number of atoms of radon-220 you would expect to be present in this material over a (logarithmic) range from 10^{-3} to 10^3 years.

2 Nuclear Processes

Example 3: PTD 45

Heavy nuclei can decay if struck by a neutron, releasing lighter nuclei and several more neutrons in the process. If each decay event causes, on average, more than one other decay event, then a runaway chain reaction occurs, causing a nuclear explosion. This happens in samples of mass greater than a given "critical mass". If the sample can be compressed, roughly how does the critical mass depend on density?

Solution

Let the sample have radius r, and let the cross-section of collision between neutrons and heavy nuclei be σ. Then for small r, the probability that a produced neutron will collide with another nucleus before exiting the sample is

$$p \sim n\sigma r$$

where n is the number density of nuclei. Critical mass is achieved when this reaches some fixed threshold value, which means $r_{\text{crit}} \propto 1/n \propto 1/\rho$. The critical mass is thus

$$m_{\text{crit}} \propto \rho r_{\text{crit}}^3 \propto 1/\rho^2.$$

Early nuclear weapons worked on the so-called implosion method, where a conventional explosive was used to compress a sphere of radioactive material.

[1] **Problem 9.** Nuclear reactions can occur when nuclei are collided. Find the missing particle in these reactions.

(a) $^{4}\text{He} + {}^{14}\text{N} \to {}^{17}\text{O} +$

(b) $^{9}\text{Be} + {}^{4}\text{He} \to {}^{12}\text{C} +$

(c) $^{27}\text{Al} + {}^{4}\text{He} \to n +$

(d) $^{12}\text{C} + \quad \to {}^{13}\text{N} + n$

In practice, many nuclear and particle physics problems boil down to "optimal collision" problems as you saw in **R2**, so we'll avoid repeating them.

[3] **Problem 10.** EFPhO 2012, problem 6.

The following problems concern nuclear fusion processes in stars, an important topic.

[3] **Problem 11.** USAPhO 2010, problem A4. This covers the proton-proton chain in our Sun.

[2] **Problem 12.** In larger stars, energy is also produced by the CNO cycle. We start with a population of ^{12}C, in an environment containing many protons. You are given that ^{13}N and ^{15}O quickly undergo β^+ decay, and that when ^{15}N is bombarded by a proton, the reaction

$$^{15}\text{N} + {}^{1}\text{H} \to {}^{12}\text{C} + {}^{4}\text{He}$$

occurs. Write out the steps of the CNO cycle and find the net reaction.

Idea 5

A very basic model for the fission of large nuclei is the liquid drop model. We suppose the protons and neutrons are packed with uniform density; thus, the volume is proportional to A, the surface area to $A^{2/3}$, and the radius to $A^{1/3}$. The binding energy of the nucleus has several contributions:

- Each nucleon is bound to the others by the strong nuclear force. This force is short-ranged, so the binding energy for each nucleon is only due to its neighbors, not on how large the nucleus as a whole is, so it is proportional to A.
- There is a negative contribution scaling as $-A^{2/3}$ because nucleons at the surface don't have neighbors on one side.
- There is another negative contribution scaling as $-Z^2/A^{1/3}$ due to the Coulomb repulsion between protons. This scales quadratically with Z because the electromagnetic force is long-ranged, so every proton interacts with every other one.
- Depending on the sophistication of the model, there can be other terms added, whose origin can only be understood through quantum mechanics.

[3] **Problem 13.** INPhO 2014, problem 7. This is an instructive general application of the liquid drop model. The official solutions are already on the page, so you can check your work as you go.

[4] **Problem 14.** IPhO 1997, problem 2. This applies the liquid drop model to nuclear stability.

You can find a few more exercises on the liquid drop model in part A of IPhO 2023, problem 2, though I think the above problems are enough to get the general idea.

3 Basic Particle Physics

It's important to get a feeling for the basics of the Standard Model. To do this, read through chapter 14 of Krane or chapter 1 of Griffiths.

[3] **Problem 15.** After reading the chapter, do the following as well as you can without references. (You can peek if you need to, but try to do that as little as possible.)

(a) Write down the fundamental particles of the Standard Model, along with their electric charges.

(b) Order the particles from lightest to heaviest.

(c) Which particles make up most of what you see in the everyday world?

(d) Which particles participate in the strong interaction?

(e) Which particles participate in the weak interaction?

[3] **Problem 16** (Griffiths 1.19)**.** Your roommate is a chemistry major. She knows all about protons, neutrons, and electrons, and she sees them in action every day in the laboratory. But she is skeptical when you tell her about positrons, muons, neutrinos, pions, quarks, and intermediate vector bosons. Explain to her why each of these play no direct role in chemistry.

Olympiad questions about particle colliders boil down to questions from **E4**, **E7**, and **R2**, so they should be fairly straightforward if you know the principles.

[3] **Problem 17.** USAPhO 2024, problem B1. Analyzing the collision rate in a muon collider using relativistic kinematics and dynamics.

[5] **Problem 18.** IPhO 2016, problem 3. This problem is about the physics of the LHC. Record your answers on the official answer sheet.

[5] **Problem 19.** IPhO 2018, problem 2. This problem covers LHC data analysis in more depth.

Remark

Now that you know the basics, can you tell the difference between the titles of real high energy physics papers, and randomly generated ones? Test your knowledge here!

4 Atomic Physics

There's not too much about atomic physics that can come up, because most quantitative results beyond the Bohr model need the full machinery of quantum mechanics. However, if you're given the atomic energy levels in advance, there's a bit of physics you can do with the resulting transitions.

Idea 6

Electrons in isolated atoms can spontaneously fall from energy level E_1 to E_0, releasing a photon of angular frequency $\omega = (E_1 - E_0)/\hbar$. Thus, since energy levels are discrete, light from such atoms will have a sharply peaked spectrum (i.e. frequency dependence). Since every atom has its own characteristic discrete energy levels, careful investigation of the spectrum can identify them.

Remark

If you like Olympiad number theory, you might want to chew on the following puzzle: in the hydrogen atom, it's possible that a transition from energy level $n \to m$ emits a photon of the same energy as some other transition $n' \to m'$. How can you find all of the (n, m, n', m') for which this is true? The solution is given here.

[3] **Problem 20.** In this problem, we discuss how atomic physicists observe atomic energy levels.

(a) The discrete wavelenths of light observed in the spectra are called "spectral lines". Why are they called lines?

Ideally, each spectral line has zero width. However, in practice, isolated atoms emit radiation in a range of wavelengths centered about each spectral line. For concreteness, we'll consider the sodium doublet, a spectral line in sodium vapor which corresponds to yellow light with wavelength $\lambda = 589\,\text{nm}$. (Why specifically sodium vapor?)

(b) One contribution to spectral line width is the energy-time uncertainty principle: if an excited state survives for time Δt, then the resulting emitted energy must have a spread $\Delta E \Delta t \gtrsim \hbar$. In the case of the sodium doublet, the lifetime is $16\,\text{ns}$. Estimate the spread in wavelengths $\Delta\lambda$ due to this "lifetime broadening".

(c) Another contribution to spectral line width is Doppler broadening: when a gas of atoms is at a nonzero temperature, the atomic motion causes the wavelengths to be changed by the Doppler effect. Estimate the resulting spread in wavelengths $\Delta\lambda$ at $T = 1000\,\text{K}$. (You can consult the tables in appendix D of Krane.)

(d) The spectrum of the Sun has a rather different form. Instead of having radiation at only a few wavelengths, it has radiation at almost all wavelengths, except for a few wavelengths where the amount of radiation *decreases*. Why?

Idea 7

Conversely, when an atom is placed in an electromagnetic field of angular frequency ω, it may absorb a photon to go from energy level E_0 to E_1. The presence of such a field also increases the rate of decay from E_1 down to E_0 via stimulated emission, as we saw in **T1**.

Finally, an electron can be ejected from an atom entirely by absorbing a photon in the photoelectric effect; if the initial energy was $-E$, then the final kinetic energy of the electron is $\hbar\omega - E$.

[3] **Problem 21.** USAPhO 1997, problem A4.

[3] **Problem 22.** USAPhO 1998, problem A3.

[3] **Problem 23.** USAPhO 1998, problem B2.

[3] **Problem 24.** INPhO 2012, problem 5.

[5] **Problem 25.** IPhO 2009, problem 2. This relatively straightforward problem covers the neat application of Doppler laser cooling, a technique for creating ultracold gases that won the 1997 Nobel prize. (For a very similar problem, see APhO 2006, problem 1.)

[5] **Problem 26.** IdPhO 2020, problem 3. Another relatively straightforward problem focusing on chirped pulse amplification, which won the 2018 Nobel prize.

Remark

In a conventional refrigerator, cooling the inside requires the heating of a hot reservoir, which is usually a metal coil located at the back of the fridge. But in Doppler laser cooling, a sample of atoms is cooled without a hot reservoir heating up! This is actually allowed by the second law of thermodynamics because the entropy of the photons goes up. They begin by coming in by a definite direction (the laser beam) and come out in a random direction, so the entropy associated with their orientation increases.

To reach even lower temperatures, one uses the technique of evaporative cooling. The atoms are held in place by a trap, which you can think of as a static, attractive potential $U(r) \propto r^2$. If the trap has finite height, then only the most energetic atoms can escape. The remaining atoms have less energy on average, and hence are colder, just like how evaporating sweat cools people down. This doesn't violate the second law of thermodynamics because the atoms that escape the trap end up in some random place in the lab, so the entropy associated with their position increases.

[5] **Problem 27.** IZhO 2019, problem 3. A problem on the dynamics on a laser, which is arguably the most important invention for atomic physics in history.

Modern II: Atoms, Particles, and Nuclei

Chapters 48, 50, 51, and 52 of Halliday and Resnick are a useful introduction. For further reading, see chapters 12 through 14 of Krane for nuclear and particle physics, and section 5.2 of Griffiths' *Introduction to Quantum Mechanics* (3rd edition) for atomic physics. If you'd like to learn a lot more about these subjects, see the MIT OCW 22.01 course on nuclear engineering, chapters 1 and 2 of Griffiths' *Introduction to Elementary Particles*, or David Tong's Lectures on Particle Physics. For some neat reading about symmetries in particle physics, see chapter I-52 of the Feynman lectures. For all problems, you can consult the periodic table. There is a total of **87** points.

1 Nuclear Decay

Idea 1

Atomic nuclei are written as ${}^{A}_{N}\mathrm{X}$ where X is the name of the element, A is the mass number (number of neutrons plus protons), and N is the atomic number (number of protons). Since N can be inferred from X, we often don't write it.

Idea 2

The most common nuclear decay channels are alpha decay,

$$^{A}_{Z}\mathrm{X} \to {}^{A-4}_{Z-2}\mathrm{X}' + {}^{4}_{2}\mathrm{He}$$

and beta decay,

$$^{A}_{Z}\mathrm{X} \to {}^{A}_{Z+1}\mathrm{X}' + e^- + \overline{\nu}_e.$$

Here, $\overline{\nu}_e$ is a light neutral particle called an anti-electron neutrino. A variant of beta decay, called β^+ decay or positron emission, is

$$^{A}_{Z}\mathrm{X} \to {}^{A}_{Z-1}\mathrm{X}' + e^+ + \nu_e$$

where ν_e is called an electron neutrino, and e^+ is a positron. If electrons are present, the nuclei may also capture them, leading to the process

$$^{A}_{Z}\mathrm{X} + e^- \to {}^{A}_{Z-1}\mathrm{X}' + \nu_e.$$

Finally, nuclei can decay from excited states by emitting photons, in gamma decay.

There are many more processes, such as inverse beta decay or double beta decay. However, the general principles underlying which decays are allowed are simple: baryon number, electric charge, and electron number are all conserved. In the restricted setting of nuclear processes,

$$\begin{aligned} \text{baryon number} &= \text{number of protons and neutrons} \\ \text{electric charge} &= \text{number of proton and positrons} - \text{number of electrons} \\ \text{electron number} &= \text{number of electrons and electron neutrinos} \\ &\quad - \text{number of positrons and anti-electron neutrinos.} \end{aligned}$$

Idea 3

The amount of energy released in a nuclear decay can be inferred from the drop in mass energy, $\Delta E = (\Delta m)c^2$. A nuclear decay can only spontaneously occur if it lowers the energy of the *entire* nucleus. To emphasize this point, note that at the level of individual nucleons, $\beta^{\pm}$ decay involve the processes

$$n \to p + e^- + \overline{\nu}_e, \quad p \to n + e^+ + \nu$$

respectively. Either of these processes could be energetically favorable inside a nucleus, depending on its composition. But an isolated proton will never decay, because protons are heavier than neutrons.

[1] **Problem 1** (Krane 12.38)**.** Complete the following decays:

(a) $^{27}\text{Si} \to {}^{27}\text{Al} +$

(b) $^{74}\text{As} \to {}^{74}\text{Se} +$

(c) $^{228}\text{U} \to \alpha +$

(d) $^{93}\text{Mo} + e^- \to$

(e) $^{131}\text{I} \to {}^{131}\text{Xe} +$

Solution. (a) $^{27}\text{Si} \to {}^{27}\text{Al} + e^+ + \nu_e$.

(b) $^{74}\text{As} \to {}^{74}\text{Se} + e^- + \overline{\nu}_e$.

(c) $^{228}\text{U} \to \alpha + {}^{224}\text{Th}$.

(d) $^{93}\text{Mo} + e^- \to {}^{93}\text{Nb} + \nu_e$.

(e) $^{131}\text{I} \to {}^{131}\text{Xe} + e^- + \overline{\nu}_e$.

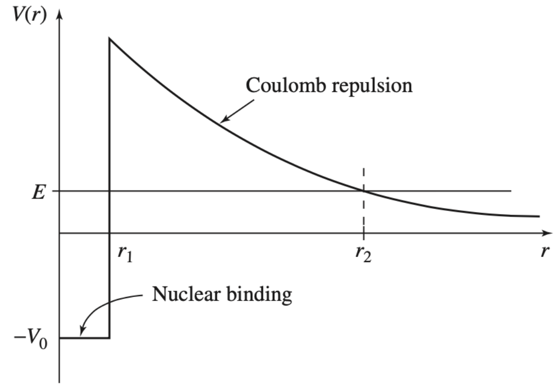
Example 1: PPP 189

^{7}Be is a radioactive element with a half-life of 53.37 days. When isotope 7 of beryllium is heated to a few thousand degrees, its half-life changes. This is strange, because nuclear processes typically involve much higher energy scales. What is the explanation for this?

Solution

Temperatures at this scale are not enough to affect nuclear physics, but are enough to affect atomic physics. The electrons gain enough energy to leave the atoms, then hit other nuclei, causing electron capture, changing the isotope of beryllium and hence its half-life.

[4] **Problem 2.** In Gamow's theory of alpha decay, alpha particles can escape from nuclei by quantum tunneling. The alpha particle is bound to the nucleus by a nuclear force, which we model as a finite square well, $V(r) = -V_0$ for $r < r_1$, and repelled by the Coulomb force, $V(r) = k(Ze)(2e)/r = \alpha/r$. The combination of the two creates a potential barrier the alpha particle must tunnel though. Let the alpha particle have mass m and energy E.



(a) Using classical mechanics, calculate the time between collisions with the wall. This is also correct in quantum mechanics; one can take the wavefunction to be a wavepacket, which really does collide with the walls with the same frequency.

(b) In quantum mechanics, each collision has an associated amplitude to escape by quantum tunneling. To compute this, recall from **X1** that the WKB approximation states that the wavefunction picks up a phase $e^{i\theta}$, where

$$\theta = \frac{1}{\hbar} \int p \, dx.$$

Calculate θ by integrating from r_1 to r_2, assuming that $r_1 \ll r_2$ for simplicity. You should find that θ is a complex number, indicating the wavefunction exponentially decays in the barrier. (Hint: you will find a tricky integral, for which you should use a trigonometric substitution.)

(c) The probability of escape scales as the amplitude squared. Write down an approximate expression for the timescale τ for decay to occur.

This model is very rough, so the numeric and slowly varying prefactors should not be expected to be accurate. But the exponential dependence of the timescale on the energy, which you should have found is due to the tunneling probability scaling as $e^{-\sqrt{E_g/E}}$ for some constant E_g, is by far the most important piece, and it fits experimental results.

(d) In nuclear fusion reactions in the Sun, the process above occurs in reverse: an incoming alpha particle (i.e. helium nucleus) needs to tunnel through the Coulomb barrier to fuse with another nucleus. The initial energy is Boltzmann distributed as $e^{-E/k_B T}$, so the fusion rate is

$$\Gamma \sim \int dE \, e^{-\sqrt{E_g/E}} e^{-E/k_B T}.$$

The integrand is the product of a rapidly rising exponential and a rapidly falling exponential. Estimate the exponential part of the dependence of Γ on T.

Solution. (a) We have $v = \sqrt{2E/m}$, so

$$t = \frac{2r_1}{v} = r_1 \sqrt{\frac{2m}{E}}.$$

(b) Within the barrier, we have

$$p = \sqrt{2m(E-V)} = i\sqrt{2m(V-E)}.$$

The second turning point r_2 satisfies $E = \alpha/r_2$. Thus, the WKB phase is

$$\theta = \frac{i}{\hbar}\int_{r_1}^{r_2} \sqrt{2m\left(\frac{\alpha}{r} - E\right)}\,dr = \frac{i}{\hbar}\sqrt{2mE}\int_{r_1}^{r_2}\sqrt{\frac{r_2}{r} - 1}\,dr.$$

Since $r_1 \ll r_2$, we can simply set $r_1 = 0$ in the integral and let $u = r/r_2$, leaving

$$\theta = \frac{i}{\hbar}\sqrt{2mE}\,r_2 \int_0^1 \sqrt{1/u - 1}\,du.$$

This final integral can be performed by letting $u = \sin^2 v$, giving

$$\theta = \frac{i\alpha}{\hbar}\sqrt{\frac{2m}{E}}\int_0^{\pi/2}\sqrt{\frac{1}{\sin^2 v} - 1}\,(2\sin v\cos v)\,dv = \frac{i\alpha}{\hbar}\sqrt{\frac{2m}{E}}\int_0^{\pi/2} 2\cos^2 v\,dv.$$

Since cosine squared averages to 1/2, this integral is $\pi/2$, so

$$\theta = \frac{i\pi\alpha}{\hbar}\sqrt{\frac{m}{2E}}.$$

(c) The timescale is approximately the time between collisions, divided by the probability of escape per collision,

$$\tau \sim te^{2i\theta} \sim r_1\sqrt{\frac{2m}{E}}\exp\left(-\frac{\pi\alpha}{\hbar}\sqrt{\frac{2m}{E}}\right).$$

(d) The integrand is the exponential of a quantity that quickly rises and then falls, which means almost all of the integral's value comes from the region where $-\sqrt{E_g/E} - E/k_BT$ is minimized. Carrying out the derivative, this corresponds to $E \sim E_g^{1/3}(k_BT)^{2/3}$. Plugging this back in, we find the integrand is of order $e^{-(E_g/k_BT)^{1/3}}$ near these energies, so

$$\Gamma \propto e^{-(E_g/k_BT)^{1/3}}.$$

This general idea for treating sharply peaked integrals is called Laplace's method. With a little more work, we can find the prefactor too. However, the exponential is the qualitatively most important part because it has a very sharp dependence on T.

[3] **Problem 3.** Consider the process by which an electron absorbs a single photon, $e^- + \gamma \to e^-$.

(a) Show that this process is forbidden by energy-momentum conservation. By time reversal, emission of a single photon should be forbidden as well. This is quite puzzling, since we already know of many processes where something like absorption or emission seems to happen.

(b) Can an electron in an isolated atom absorb a single photon? If so, why doesn't the reasoning in part (a) work? If not, how can atoms absorb photons at all, as described in **X1**?

(c) Can isolated nuclei emit single photons? If so, why doesn't the reasoning in part (a) work? If not, how can gamma decay occur?

(d) Can isolated electrons absorb or emit *classical* electromagnetic radiation? If so, why doesn't the reasoning in part (a) work? If not, how can Thomson scattering (covered in **E7**) happen?

Solution. (a) Let $c = 1$ and consider the reference frame where the electron was initially at rest with mass m. After the collision with the photon with energy and momentum equal to E_γ, the electron will have energy $m + E_\gamma$ and momentum E_γ. However, since $E^2 = p^2 + m^2$, we get $m^2 + 2E_\gamma m + E_\gamma^2 = E_\gamma^2 + m^2$, reducing to $2E_\gamma m = 0$, which is a contradiction (neither the mass of an electron nor the energy of the photon is 0).

(b) Yes, an electron in an atom can absorb a photon. The issue in part (a) is that to absorb a photon, the rest mass of the system absorbing must increase (which doesn't happen for a lone electron). When an electron is orbiting an atom, it has potential energy associated with its interaction with the nucleus, and when it absorbs a photon, the electron jumps to a higher energy state, which increases the rest mass-energy of the atom.

(c) Yes, when the nuclei breaks apart, the potential energy from nuclear interactions, which will reduce the total rest mass of the nucleus, which allows for the release of a photon while conserving momentum and energy.

(d) No, this process is impossible, because the same relativistic kinematics arguments hold whether the radiation is classical or not. But it isn't in contradiction with Thomson scattering, which is the classical analogue of $e^- + \gamma \to e^- + \gamma$. (Note that whenever we talked about the *absorption* of electromagnetic radiation, it was in the context of electrons inside matter, where the matter can absorb the excess momentum.)

Idea 4

Radioactive decay is a memoryless process: in an infinitesimal time interval dt, any nucleus has a probability $\lambda\, dt$ of decaying, regardless of its previous history. As a result, the number of radioactive nuclei falls exponentially as

$$N(t) = N_0 e^{-\lambda t}.$$

The activity $A(t)$ is the rate of decay events, and also falls exponentially,

$$A(t) = A_0 e^{-\lambda t}.$$

The mean lifetime of the nuclei is $\tau = 1/\lambda$.

[3] **Problem 4.** This problem tests your understanding of memoryless processes. Below are several plausible ways to measure τ.

(a) We start a stopwatch at noon and stop it when the next decay happens, giving t_1.

(b) We have an intern watch the sample continuously, then at noon, ask them how long it was since the last decay, giving t_2.

(c) We have an intern watch the sample continuously, then at noon, ask them how long it was since the last decay. We then set our stopwatch so that $t = 0$ when that decay happened, and stop the stopwatch when the next decay happens, giving t_3.

(d) We continuously watch the sample, start a stopwatch when the first decay happens, then stop it when the next decay happens, giving t_4.

We repeat procedure i many times, so the average of t_i is τ_i. Find the τ_i in terms of τ.

Solution. (a) The probability of decay in a time interval dt is dt/τ, so the probability of not decaying is $(1 - dt/\tau)$. After $N = t/dt$ such time intervals, the probability that a decay still hasn't occurred is $(1 - dt/\tau)^N$. As shown in **P1**, this becomes $e^{-t/\tau}$ in the limit $dt \to 0$. Thus, the probability of a decay occurring after time t in an interval dt is

$$P(t)\,dt = e^{-t/\tau}\frac{dt}{\tau}.$$

The mean lifetime is the average value of this time,

$$\tau_1 = \frac{1}{\tau}\int_0^\infty e^{-t/\tau}t\,dt = \int_0^\infty e^{-t/\tau}dt = \tau.$$

(b) "Waiting" forward or backwards in time are symmetric, so $\tau_2 = \tau_1 = \tau$.

(c) By definition, $t_3 = t_1 + t_2$, and taking expectation values gives $\langle t_3\rangle = \langle t_1 + t_2\rangle = \langle t_1\rangle + \langle t_2\rangle$. Thus, $\tau_3 = \tau_1 + \tau_2 = 2\tau$.

(d) We know that the mean time between decays is τ, so $\tau_4 = \tau$.

Of course, the tricky part of the problem is the following: why is $\tau_3 \neq \tau_4$, even though they seem to be measuring the exact same thing, namely the time between two decays? The difference is in the way we select the decay we look at. For τ_4, we look at a random one of the decay (i.e. if there are a thousand decay events, each one has an equal chance of being the one we look at). But for τ_3, we look at the decay happening during a random *time*, which means that longer time intervals have a larger chance of being randomly picked. This means that $\tau_3 > \tau_4$.

To show this explicitly, note that the probability distribution of decay times is $e^{-t/\tau}/\tau$, as derived in part (a). The probability distribution of decay times *weighted by decay length*, as used in part (c), is $te^{-t/\tau}/\tau^2$. So the expected decay time measured in part (c) is

$$\tau_3 = \int_0^\infty t\left(te^{-t/\tau}/\tau^2\right)\,dt = \frac{1}{\tau^2}\int_0^\infty t^2e^{-t/\tau}\,dt = 2\tau$$

just as argued more intuitively above.

This is quite a tricky factor of 2. Drude got it wrong when formulating the Drude model, which is the simplest classical model of electrical conduction in a metal. It turns out that the Drude model is totally wrong, due to quantum mechanics, but this mistake, plus two other more conceptual issues, made it look like it agreed with experiment.

Another example of a memoryless process is the collisions of a given gas molecule in an ideal gas, according to kinetic theory. For example, all of the subparts above could have been rephrased in terms of observing the distance a gas molecule moves between collisions, with the same conclusions.

[2] **Problem 5** (Krane 12.37)**.** A radioactive sample contains N_0 atoms at time $t = 0$. It is observed that N_1 radioactive atoms remain at time t_1 and then decay by time t_2, N_2 remain at t_2 and then decay by time t_3, and so on. Show that if many observations are made, then τ can be measured as

$$\tau = \frac{1}{N_0}\sum_i N_i t_i.$$

Solution. $N(t)$ should follow $N(t) = N_0 e^{-t/\tau}$, so $dN(t)/dt = -N(t)/\tau$. Thus the number of atoms that decay between time t_i and t_{i+1}, N_i, will be about $N_i = (t_{i+1} - t_i) dN(t_i)/dt = (t_{i+1} - t_i) N(t)/\tau$ as the number of measurements are large. With smaller time intervals, this can be seen as $N_i = N(t) dt/\tau$. Thus looking at the expression $\frac{1}{N_0} \sum_i N_i t_i$ gives

$$\frac{1}{N_0} \sum_i N_i t_i \approx \frac{1}{N_0} \int_0^\infty \left(N(t) \frac{dt}{\tau} \right) t = \frac{1}{\tau} \int_0^\infty e^{-t/\tau} t dt.$$

This integral can be evaluated with parts (differentiating t and integrating $e^{-t/\tau} dt$),

$$\frac{1}{\tau} \int_0^\infty e^{-t/\tau} t dt = \int_0^\infty e^{-t/\tau} dt = \tau,$$

which shows that, as desired,

$$\tau = \frac{1}{N_0} \sum_i N_i t_i.$$

Example 2

Radium can be found in trace quantities throughout the Earth, and has a half-life of 1620 years. Suppose that there is currently 1 kg of radium on the Earth. Then extrapolating backwards, there was $2^{4.5 \times 10^9/1620}$ kg of radium on the Earth when it was formed, which is greater than the mass of the observable universe! What's wrong with this calculation?

Solution

Nuclear decays don't happen in isolation; there are entire networks of nuclear decay chains. Radium decays quickly, but it is also constantly produced by the decay of other isotopes, which have much longer half-lives.

[3] **Problem 6.** USAPhO 2009, problem A2.

[3] **Problem 7.** IPhO 2000, problem 1c. Don't worry about the official answer sheet; treat this like a regular USAPhO problem.

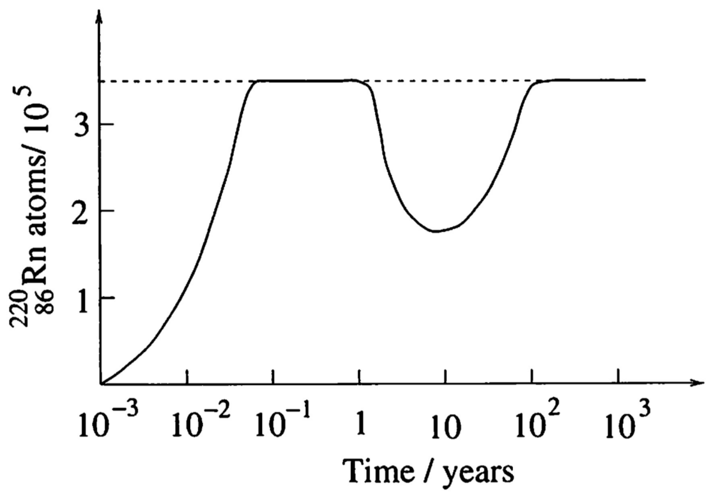
Solution. See the official solutions here and here.

[3] **Problem 8** (PPP 190)**.** Part of the series of isotopes produced by the decay of thorium-232, along with the corresponding half-lives, is given below:

$$^{232}_{90}\mathrm{Th} \xrightarrow{1.4\times10^{10}\,\mathrm{y}} {}^{228}_{88}\mathrm{Ra} \xrightarrow{5.7\,\mathrm{y}} {}^{228}_{89}\mathrm{Ac} \xrightarrow{6.1\,\mathrm{h}} {}^{228}_{90}\mathrm{Th} \xrightarrow{1.9\,\mathrm{y}} {}^{224}_{88}\mathrm{Ra} \xrightarrow{3.6\,\mathrm{d}} {}^{220}_{86}\mathrm{Rn} \xrightarrow{56\,\mathrm{s}} \ldots.$$

Thorium-232 and thorium-228 in equilibrium are extracted from an ore and purified by a chemical process. Sketch the form of the variation in the number of atoms of radon-220 you would expect to be present in this material over a (logarithmic) range from 10^{-3} to 10^3 years.

Solution. The graph should look like this:



It rises at first due to the Radon-224 from the Thorium-228 in the initial sample, which will then decay away before Radon-228 from Thorium-232 plays a significant role. After some time, the effectively "infinite" bank of Thorium-232 (since it's half life is much longer than 10^3 years) will "fill up" all the parts of the chain when the Radon-228 starts contributing to the Thorium-228 stock, and the equilibrium amount of Radon-220 will be reached and kept until after around 10^{10} years.

2 Nuclear Processes

Example 3: PTD 45

Heavy nuclei can decay if struck by a neutron, releasing lighter nuclei and several more neutrons in the process. If each decay event causes, on average, more than one other decay event, then a runaway chain reaction occurs, causing a nuclear explosion. This happens in samples of mass greater than a given "critical mass". If the sample can be compressed, roughly how does the critical mass depend on density?

Solution

Let the sample have radius r, and let the cross-section of collision between neutrons and heavy nuclei be σ. Then for small r, the probability that a produced neutron will collide with another nucleus before exiting the sample is

$$p \sim n\sigma r$$

where n is the number density of nuclei. Critical mass is achieved when this reaches some fixed threshold value, which means $r_{\text{crit}} \propto 1/n \propto 1/\rho$. The critical mass is thus

$$m_{\text{crit}} \propto \rho r_{\text{crit}}^3 \propto 1/\rho^2.$$

Early nuclear weapons worked on the so-called implosion method, where a conventional explosive was used to compress a sphere of radioactive material.

[1] **Problem 9.** Nuclear reactions can occur when nuclei are collided. Find the missing particle in these reactions.

(a) $^{4}\text{He} + {}^{14}\text{N} \to {}^{17}\text{O} +$

(b) $^{9}\text{Be} + {}^{4}\text{He} \to {}^{12}\text{C} +$

(c) $^{27}\text{Al} + {}^{4}\text{He} \to n +$

(d) $^{12}\text{C} + \quad \to {}^{13}\text{N} + n$

Solution. (a) $^{4}\text{He} + {}^{14}\text{N} \to {}^{17}\text{O} \ + p$

(b) $^{9}\text{Be} + {}^{4}\text{He} \to {}^{12}\text{C} \ + n$

(c) $^{27}\text{Al} + {}^{4}\text{He} \to n \ + {}^{30}\text{P}$

(d) $^{12}\text{C} \ + \ {}^{2}\text{H} \to {}^{13}\text{N} + n$

In practice, many nuclear and particle physics problems boil down to "optimal collision" problems as you saw in **R2**, so we'll avoid repeating them.

[3] **Problem 10.** EFPhO 2012, problem 6.

Solution. See the official solutions here.

The following problems concern nuclear fusion processes in stars, an important topic.

[3] **Problem 11.** USAPhO 2010, problem A4. This covers the proton-proton chain in our Sun.

[2] **Problem 12.** In larger stars, energy is also produced by the CNO cycle. We start with a population of ^{12}C, in an environment containing many protons. You are given that ^{13}N and ^{15}O quickly undergo β^+ decay, and that when ^{15}N is bombarded by a proton, the reaction

$$^{15}\text{N} + {}^{1}\text{H} \to {}^{12}\text{C} + {}^{4}\text{He}$$

occurs. Write out the steps of the CNO cycle and find the net reaction.

Solution. Since we start with a lot of ^{12}C and protons, what's going to happen is that they collide. Letting ^{13}N and ^{15}O undergo immediate β^+ decay and continuing the proton bombardment will give the following steps:

$$^{12}\text{C} + {}^{1}\text{H} \to {}^{13}\text{N}.$$
$$^{13}\text{N} \to {}^{13}\text{C} + e^+ + \nu_e.$$
$$^{13}\text{C} + {}^{1}\text{H} \to {}^{14}\text{N}.$$
$$^{14}\text{N} + {}^{1}\text{H} \to {}^{15}\text{O}.$$
$$^{15}\text{O} \to {}^{15}\text{N} + e^+ + \nu_e.$$
$$^{15}\text{N} + {}^{1}\text{H} \to {}^{12}\text{C} + {}^{4}\text{He}.$$

Now eliminating all the "cycled" atoms will get a net reaction of

$$4\ {}^{1}\text{H} \to {}^{4}\text{He} + 2e^+ + 2\nu_e.$$

Idea 5

A very basic model for the fission of large nuclei is the liquid drop model. We suppose the protons and neutrons are packed with uniform density; thus, the volume is proportional to A, the surface area to $A^{2/3}$, and the radius to $A^{1/3}$. The binding energy of the nucleus has several contributions:

- Each nucleon is bound to the others by the strong nuclear force. This force is short-ranged, so the binding energy for each nucleon is only due to its neighbors, not on how large the nucleus as a whole is, so it is proportional to A.
- There is a negative contribution scaling as $-A^{2/3}$ because nucleons at the surface don't have neighbors on one side.
- There is another negative contribution scaling as $-Z^2/A^{1/3}$ due to the Coulomb repulsion between protons. This scales quadratically with Z because the electromagnetic force is long-ranged, so every proton interacts with every other one.
- Depending on the sophistication of the model, there can be other terms added, whose origin can only be understood through quantum mechanics.

[3] **Problem 13.** INPhO 2014, problem 7. This is an instructive general application of the liquid drop model. The official solutions are already on the page, so you can check your work as you go.

[4] **Problem 14.** IPhO 1997, problem 2. This applies the liquid drop model to nuclear stability.

You can find a few more exercises on the liquid drop model in part A of IPhO 2023, problem 2, though I think the above problems are enough to get the general idea.

3 Basic Particle Physics

It's important to get a feeling for the basics of the Standard Model. To do this, read through chapter 14 of Krane or chapter 1 of Griffiths.

[3] **Problem 15.** After reading the chapter, do the following as well as you can without references. (You can peek if you need to, but try to do that as little as possible.)

(a) Write down the fundamental particles of the Standard Model, along with their electric charges.

(b) Order the particles from lightest to heaviest.

(c) Which particles make up most of what you see in the everyday world?

(d) Which particles participate in the strong interaction?

(e) Which particles participate in the weak interaction?

Solution. (a) Quarks: up, charm, top have charges of $+2/3$, and down, strange, bottom have charges of $-1/3$.

Leptons: electron, muon, tau have charges of -1, and their neutrinos have no charge.

The gluon, photon, Z boson, and Higgs boson have no charge, and the $W^{\pm}$ boson has a charge of ± 1.

(b) Photon/gluon (massless), electron neutrino, muon neutrino, tau neutrino, electron, up quark, down quark, strange quark, muon, charm quark, tau, bottom quark, W boson, Z boson, Higgs boson, top quark.

(c) The up and down quarks in nucleons, electrons, and photons.

(d) Gluons mediate the strong interaction, which affects quarks and gluons.

(e) W/Z bosons mediate the weak interaction, which affects quarks and leptons (the electron, muon, tau, and corresponding neutrinos).

[3] **Problem 16** (Griffiths 1.19)**.** Your roommate is a chemistry major. She knows all about protons, neutrons, and electrons, and she sees them in action every day in the laboratory. But she is skeptical when you tell her about positrons, muons, neutrinos, pions, quarks, and intermediate vector bosons. Explain to her why each of these play no direct role in chemistry.

Solution. Positrons will annihilate after contacting electrons, muons are unstable and decay in a few microseconds, neutrinos are too small and have no charge so they're too elusive to be detected, pions are even more unstable than muons, quarks are locked inside protons and neutrons (they won't exist on their own since they'll be unstable that way), and the bosons have a half life of around 3×10^{-25} s.

Olympiad questions about particle colliders boil down to questions from **E4**, **E7**, and **R2**, so they should be fairly straightforward if you know the principles.

[3] **Problem 17.** USAPhO 2024, problem B1. Analyzing the collision rate in a muon collider using relativistic kinematics and dynamics.

[5] **Problem 18.** IPhO 2016, problem 3. This problem is about the physics of the LHC. Record your answers on the official answer sheet.

[5] **Problem 19.** IPhO 2018, problem 2. This problem covers LHC data analysis in more depth.

Remark

Now that you know the basics, can you tell the difference between the titles of real high energy physics papers, and randomly generated ones? Test your knowledge here!

4 Atomic Physics

There's not too much about atomic physics that can come up, because most quantitative results beyond the Bohr model need the full machinery of quantum mechanics. However, if you're given the atomic energy levels in advance, there's a bit of physics you can do with the resulting transitions.

Idea 6

Electrons in isolated atoms can spontaneously fall from energy level E_1 to E_0, releasing a photon of angular frequency $\omega = (E_1 - E_0)/\hbar$. Thus, since energy levels are discrete, light from such atoms will have a sharply peaked spectrum (i.e. frequency dependence). Since every atom has its own characteristic discrete energy levels, careful investigation of the spectrum can identify them.

Remark

If you like Olympiad number theory, you might want to chew on the following puzzle: in the hydrogen atom, it's possible that a transition from energy level $n \to m$ emits a photon of the same energy as some other transition $n' \to m'$. How can you find all of the (n, m, n', m') for which this is true? The solution is given here.

[3] **Problem 20.** In this problem, we discuss how atomic physicists observe atomic energy levels.

(a) The discrete wavelenths of light observed in the spectra are called "spectral lines". Why are they called lines?

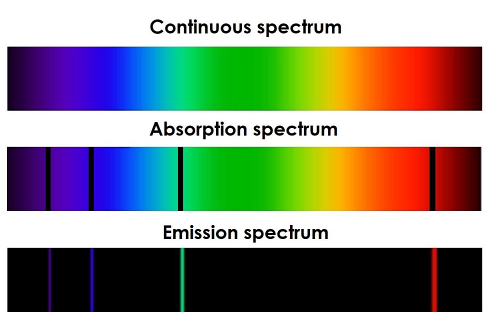
Ideally, each spectral line has zero width. However, in practice, isolated atoms emit radiation in a range of wavelengths centered about each spectral line. For concreteness, we'll consider the sodium doublet, a spectral line in sodium vapor which corresponds to yellow light with wavelength $\lambda = 589\,\text{nm}$. (Why specifically sodium vapor?)

(b) One contribution to spectral line width is the energy-time uncertainty principle: if an excited state survives for time Δt, then the resulting emitted energy must have a spread $\Delta E \Delta t \gtrsim \hbar$. In the case of the sodium doublet, the lifetime is $16\,\text{ns}$. Estimate the spread in wavelengths $\Delta\lambda$ due to this "lifetime broadening".

(c) Another contribution to spectral line width is Doppler broadening: when a gas of atoms is at a nonzero temperature, the atomic motion causes the wavelengths to be changed by the Doppler effect. Estimate the resulting spread in wavelengths $\Delta\lambda$ at $T = 1000\,\text{K}$. (You can consult the tables in appendix D of Krane.)

(d) The spectrum of the Sun has a rather different form. Instead of having radiation at only a few wavelengths, it has radiation at almost all wavelengths, except for a few wavelengths where the amount of radiation *decreases*. Why?

Solution. (a) The spectra of light can be observed through the dispersion of light, where we a band of colors in increasing (or decreasing) wavelength. When discrete wavelengths are emitted or absorbed, we notice one particular wavelength emitted or absorbed, which will make a colored or dark line in the spectrum. It looks like this:



(b) Since $\Delta E = h\Delta f \sim \hbar/\Delta t$ and $f = c/\lambda$, giving $\Delta f = c\Delta\lambda/\lambda^2$, we get

$$\Delta\lambda = \lambda^2 \frac{\hbar}{hc\Delta t} = \frac{\lambda^2}{2\pi c\Delta t} \approx 10^{-5} \text{ nm}.$$

(c) Velocities from thermal motion is much less than c, so we can use $\Delta f = fv/c$. We can estimate $\frac{1}{2}mv^2 = \frac{3}{2}k_BT$, where the mass of a sodium atom is $m = 3.8 \times 10^{-26}$ kg. This gives

$$\Delta\lambda = \lambda^2 \frac{\Delta f}{c} = \lambda v/c = \frac{\lambda}{c}\sqrt{\frac{3k_BT}{m}} \approx 2 \times 10^{-3} \text{ nm}.$$

(d) The electrons in the Sun are stripped off the nuclei because it's hot, so they emit a continuous spectrum of blackbody radiation. But the atoms in the cooler atmosphere of the Sun will absorb certain wavelengths, which will create an absorption spectrum.

Idea 7

Conversely, when an atom is placed in an electromagnetic field of angular frequency ω, it may absorb a photon to go from energy level E_0 to E_1. The presence of such a field also increases the rate of decay from E_1 down to E_0 via stimulated emission, as we saw in **T1**.

Finally, an electron can be ejected from an atom entirely by absorbing a photon in the photoelectric effect; if the initial energy was $-E$, then the final kinetic energy of the electron is $\hbar\omega - E$.

[3] **Problem 21.** USAPhO 1997, problem A4.

[3] **Problem 22.** USAPhO 1998, problem A3.

[3] **Problem 23.** USAPhO 1998, problem B2.

[3] **Problem 24.** INPhO 2012, problem 5.

Solution. See the official solutions here.

[5] **Problem 25.** IPhO 2009, problem 2. This relatively straightforward problem covers the neat application of Doppler laser cooling, a technique for creating ultracold gases that won the 1997 Nobel prize. (For a very similar problem, see APhO 2006, problem 1.)

[5] **Problem 26.** IdPhO 2020, problem 3. Another relatively straightforward problem focusing on chirped pulse amplification, which won the 2018 Nobel prize.

Solution. See the official solutions here.

Remark

In a conventional refrigerator, cooling the inside requires the heating of a hot reservoir, which is usually a metal coil located at the back of the fridge. But in Doppler laser cooling, a sample of atoms is cooled without a hot reservoir heating up! This is actually allowed by the second law of thermodynamics because the entropy of the photons goes up. They begin by coming in by a definite direction (the laser beam) and come out in a random direction, so the entropy associated with their orientation increases.

To reach even lower temperatures, one uses the technique of evaporative cooling. The atoms are held in place by a trap, which you can think of as a static, attractive potential $U(r) \propto r^2$. If the trap has finite height, then only the most energetic atoms can escape. The remaining atoms have less energy on average, and hence are colder, just like how evaporating sweat cools people down. This doesn't violate the second law of thermodynamics because the atoms that escape the trap end up in some random place in the lab, so the entropy associated with their position increases.

[5] **Problem 27.** IZhO 2019, problem 3. A problem on the dynamics on a laser, which is arguably the most important invention for atomic physics in history.

Solution. See the official solutions here.

Modern III: Matter, Astro, and Cosmo

Chapters 35 and 36 of Blundell cover astrophysics, and chapter 15 of Krane covers cosmology. For solid state physics, see chapter 49 of Halliday and Resnick, chapter III-14 of the Feynman lectures, or section 5.3 of Griffiths' *Introduction to Quantum Mechanics* (3rd edition). For more on magnetism, see chapters II-34 through II-37 of the Feynman lectures. For a detailed introduction to the physics of stars and compact objects, see chapters 10 and 16 of *An Introduction to Modern Astrophysics* by Carroll and Ostlie. (Carroll and Ostlie is also a great introduction to astrophysics in general, accessible with just Olympiad physics knowledge.) There is a total of **82** points.

1 Condensed Matter

Condensed matter is an enormous field, touching everything from solid state physics to biophysics and atomic physics, and contains more than half of *all* physicists. However, you hear less about it in the news, and in Olympiad problems, because it requires a substantial amount of background to explain. The following problems cover some classic ideas in condensed matter, using a mix of modern physics and waves.

[5] **Problem 1.** APhO 2016, problem 3. A good question on the quantum mechanics of superconductivity.

[5] **Problem 2.** USAPhO+ 2021, problem 3. A nice question which derives the integer quantum Hall effect using classical electromagnetism and Bohr quantization.

[5] **Problem 3.** APhO 2015, problem 1. A question on the fractional quantum Hall effect, which is even subtler than the integer quantum Hall effect. This question is not very clearly written, and requires a good amount of educated guessing; however, I include it to give you practice with this type of Olympiad question, and some exposure to a very important topic of current research.

[5] **Problem 4.** IZhO 2021, problem 2. A problem on the thermodynamics of plasmas, reviewing some material in **T2**.

[5] **Problem 5.** APhO 2021, problem 2. A very technical problem that fully explains how a famous test of quantum mechanics was conducted. Requires material from **E8**, **W1**, **W3**, and **X1**.

2 Stars

This section contains problems involving stars and star formation. You might think this is a lot about stars, but all of the problems below use a mix of mechanics, electromagnetism, thermodynamics, relativity, and modern physics; they are excellent review for the entire course. For a beautiful graphical overview of these objects, see this paper.

Example 1: PTD 44

The density of stars in the central region of the galaxy is about $n = 10^6\,\mathrm{pc}^{-3}$, and their speeds are about $v = 200\,\mathrm{km s^{-1}}$. Could an advanced civilization develop in this region?

Solution

Impacts between solar systems will occur frequently. For concreteness, suppose catastrophic effects will happen to an Earth-like planet if a different star passes within the equivalent of Jupiter's orbit, which has radius r. The typical time between such events is

$$t \sim \frac{1}{n(\pi r^2)v} \sim 2.4 \times 10^6 \text{ years.}$$

Some argue that this is too short a time for advanced civilization to develop, so the center of the galaxy is outside of the so-called galactic habitable zone.

Example 2: CPhO 2013.3

For stars not too much heavier than the Sun, the luminosity scale with mass as $L \propto M^{3.5}$. If all of these stars release the same fraction α of their rest mass energy by nuclear burning, then how does the lifetime of the star scale with M?

Solution

The amount of energy available is αM. The lifetime is thus

$$\tau = \frac{E}{L} \propto M^{-2.5}$$

so heavier stars live shorter lives. Incidentally, the luminosity scales as $L \propto R^2 T^4$ by the Stefan–Boltzmann law. By considering the details of the interior of the star, we can find how all of these quantities scale with mass, a principle known as stellar homology.

[3] **Problem 6.** USAPhO 2018, problem B3.

[5] **Problem 7.** IPhO 2012, problem 3. This elegant and tricky problem covers the early stages of star formation, and serves as a review of **T1**.

[5] **Problem 8.** GPhO 2017, problem 1. This problem covers the physics of fusion in main sequence stars, relying on **X1** and **X2**.

[5] **Problem 9.** IPhO 2007, problem "pink". This problem covers binary stars, with a strong emphasis on data analysis methods, as covered in **P2**.

[5] **Problem 10.** APhO 2015, problem 2. A nice but somewhat hard to read problem on the aurora and the solar wind.

3 Compact Objects

Compact objects such as white dwarfs and neutron stars must be handled with quantum statistical mechanics, as introduced in **X1**.

[4] **Problem 11.** Do the following JPhO problem. This pedagogical problem reviews the physics of white dwarf stars, deriving the Chandrasekhar limit, using the techniques of **X1**.

[3] **Problem 12.** USAPhO 2024, problem A2. Rough estimates of the dynamics of stars and white dwarfs.

Remark

The estimates performed in the previous problems are quite rough, basically treating the white dwarf as being homogeneous, with uniform density and pressure. In reality, we have $\nabla p = -\rho \mathbf{g}$ just like in any situation in hydrostatic equilibrium, where the degeneracy pressure p is determined by the local density n. It's like the gaseous atmospheres you dealt with in **T1**, but with a different equation of state.

If you additionally allow the white dwarf to have a net charge, then there is an additional contribution from the electrostatic force, and the resulting equations are called the Thomas–Fermi equations of structure. They can also be used to model many-electron atoms, when you can neglect the discreteness of the electrons.

[5] **Problem 13.** GPhO 2016, problem 3. This problem covers the physics of intense magnetic fields in neutron stars, using the results of **E8** and **X1**.

[3] **Problem 14.** The Bekenstein–Hawking formula states that a black hole has an entropy of

$$S = \frac{A}{4}$$

where A is the area of its event horizon. The radius of an uncharged, nonrotating black hole is

$$R = 2M.$$

In these equations, $\hbar$, c, and G have all been set to one; you do not need to restore these factors.

(a) Compute the temperature and heat capacity of such a black hole.

(b) Two uncharged, nonrotating black holes begin very far apart from each other, then merge into a single black hole, emitting gravitational waves in the process; assume there is no initial angular momentum, so the final black hole is nonrotating as well. Find the maximum possible efficiency of this process, defined as the fraction of the initial energy that is converted into gravitational waves, for any set of initial black hole masses.

It may also be useful to review IPhO 2007, problem "blue", which we originally covered in **P1**.

[4] **Problem 15.** US TST 2022, problem 3. Rough estimates of gravitational wave emission.

[3] **Problem 16.** NBPhO 2017, problem 4. Rough estimates of gravitational wave detection.

Remark

The discovery of gravitational waves by LIGO has been one of the most important results this decade, so it's naturally a popular question topic. Once you finish the above problems, you can check out a few others with a different take on the same idea. GPhO 2016, problem 2 (solutions here) does a rougher treatment of gravitational wave emission, while IPhO 2018, problem 1 gives a more accurate treatment using more of the language of general relativity.

For another way to estimate gravitational wave emission, see section 9.3 of The Art of Insight, and for some followup questions, see this paper.

Remark

In 1931, after building a sensitive short-wave radio receiver, Karl Jansky heard an unusual noise on his receiver from a direction that moved across the sky about once a day. He therefore initially thought it was from the Sun. However, over time he found that the direction of the noise moved across the sky only once every 23 hours and 56 minutes.

This was a huge difference! To understand why, note that the length of a day is the time it takes for the same side of the Earth to face the Sun again; it depends on both the Earth's spin and its orbital motion about the Sun. The period of the Earth's spin alone, the so-called sidereal period, is only 23 hours and 56 minutes. Thus, a signal with this period indicates an origin from outside the solar system. Jansky later found that the source was the center of the galaxy; today we know it is due to the supermassive black hole there, Sagittarius A*. The early days of radio astronomy were full of dramatic discoveries like this. To hear about the discovery of pulsars, see this talk.

4 Cosmology

Cosmology is a rather technical topic because a proper treatment requires general relativity, but one can derive special cases of some of the results using just Newtonian gravity.

[2] **Problem 17.** AuPhO 2014, problem 14. A quick problem on the basics of dark matter and galaxy measurements.

[5] **Problem 18.** GPhO 2022, problem 1. More about dark matter.

[5] **Problem 19.** APhO 2016, problem 2. This straightforward problem introduces the basic equations of cosmology, such as the Friedmann equation.

Modern III: Matter, Astro, and Cosmo

Chapters 35 and 36 of Blundell cover astrophysics, and chapter 15 of Krane covers cosmology. For solid state physics, see chapter 49 of Halliday and Resnick, chapter III-14 of the Feynman lectures, or section 5.3 of Griffiths' *Introduction to Quantum Mechanics* (3rd edition). For more on magnetism, see chapters II-34 through II-37 of the Feynman lectures. For a detailed introduction to the physics of stars and compact objects, see chapters 10 and 16 of *An Introduction to Modern Astrophysics* by Carroll and Ostlie. (Carroll and Ostlie is also a great introduction to astrophysics in general, accessible with just Olympiad physics knowledge.) There is a total of **82** points.

1 Condensed Matter

Condensed matter is an enormous field, touching everything from solid state physics to biophysics and atomic physics, and contains more than half of *all* physicists. However, you hear less about it in the news, and in Olympiad problems, because it requires a substantial amount of background to explain. The following problems cover some classic ideas in condensed matter, using a mix of modern physics and waves.

[5] **Problem 1.** APhO 2016, problem 3. A good question on the quantum mechanics of superconductivity.

[5] **Problem 2.** USAPhO+ 2021, problem 3. A nice question which derives the integer quantum Hall effect using classical electromagnetism and Bohr quantization.

[5] **Problem 3.** APhO 2015, problem 1. A question on the fractional quantum Hall effect, which is even subtler than the integer quantum Hall effect. This question is not very clearly written, and requires a good amount of educated guessing; however, I include it to give you practice with this type of Olympiad question, and some exposure to a very important topic of current research.

[5] **Problem 4.** IZhO 2021, problem 2. A problem on the thermodynamics of plasmas, reviewing some material in **T2**.

Solution. See the official solutions here.

[5] **Problem 5.** APhO 2021, problem 2. A very technical problem that fully explains how a famous test of quantum mechanics was conducted. Requires material from **E8**, **W1**, **W3**, and **X1**.

2 Stars

This section contains problems involving stars and star formation. You might think this is a lot about stars, but all of the problems below use a mix of mechanics, electromagnetism, thermodynamics, relativity, and modern physics; they are excellent review for the entire course. For a beautiful graphical overview of these objects, see this paper.

Example 1: PTD 44

The density of stars in the central region of the galaxy is about $n = 10^6\,\mathrm{pc}^{-3}$, and their speeds are about $v = 200\,\mathrm{km s}^{-1}$. Could an advanced civilization develop in this region?

Solution

Impacts between solar systems will occur frequently. For concreteness, suppose catastrophic effects will happen to an Earth-like planet if a different star passes within the equivalent of Jupiter's orbit, which has radius r. The typical time between such events is

$$t \sim \frac{1}{n(\pi r^2)v} \sim 2.4 \times 10^6 \text{ years.}$$

Some argue that this is too short a time for advanced civilization to develop, so the center of the galaxy is outside of the so-called galactic habitable zone.

Example 2: CPhO 2013.3

For stars not too much heavier than the Sun, the luminosity scale with mass as $L \propto M^{3.5}$. If all of these stars release the same fraction α of their rest mass energy by nuclear burning, then how does the lifetime of the star scale with M?

Solution

The amount of energy available is αM. The lifetime is thus

$$\tau = \frac{E}{L} \propto M^{-2.5}$$

so heavier stars live shorter lives. Incidentally, the luminosity scales as $L \propto R^2 T^4$ by the Stefan–Boltzmann law. By considering the details of the interior of the star, we can find how all of these quantities scale with mass, a principle known as stellar homology.

[3] **Problem 6.** USAPhO 2018, problem B3.

[5] **Problem 7.** IPhO 2012, problem 3. This elegant and tricky problem covers the early stages of star formation, and serves as a review of **T1**.

[5] **Problem 8.** GPhO 2017, problem 1. This problem covers the physics of fusion in main sequence stars, relying on **X1** and **X2**.

Solution. See the official solutions here.

[5] **Problem 9.** IPhO 2007, problem "pink". This problem covers binary stars, with a strong emphasis on data analysis methods, as covered in **P2**.

[5] **Problem 10.** APhO 2015, problem 2. A nice but somewhat hard to read problem on the aurora and the solar wind.

Solution. See the official solutions here.

3 Compact Objects

Compact objects such as white dwarfs and neutron stars must be handled with quantum statistical mechanics, as introduced in **X1**.

[4] **Problem 11.** Do the following JPhO problem. This pedagogical problem reviews the physics of white dwarf stars, deriving the Chandrasekhar limit, using the techniques of **X1**.

Solution. See the official solutions here.

[3] **Problem 12.** USAPhO 2024, problem A2. Rough estimates of the dynamics of stars and white dwarfs.

Remark

The estimates performed in the previous problems are quite rough, basically treating the white dwarf as being homogeneous, with uniform density and pressure. In reality, we have $\nabla p = -\rho \mathbf{g}$ just like in any situation in hydrostatic equilibrium, where the degeneracy pressure p is determined by the local density n. It's like the gaseous atmospheres you dealt with in **T1**, but with a different equation of state.

If you additionally allow the white dwarf to have a net charge, then there is an additional contribution from the electrostatic force, and the resulting equations are called the Thomas–Fermi equations of structure. They can also be used to model many-electron atoms, when you can neglect the discreteness of the electrons.

[5] **Problem 13.** GPhO 2016, problem 3. This problem covers the physics of intense magnetic fields in neutron stars, using the results of **E8** and **X1**.

Solution. See the official solutions here.

[3] **Problem 14.** The Bekenstein–Hawking formula states that a black hole has an entropy of

$$S = \frac{A}{4}$$

where A is the area of its event horizon. The radius of an uncharged, nonrotating black hole is

$$R = 2M.$$

In these equations, $\hbar$, c, and G have all been set to one; you do not need to restore these factors.

(a) Compute the temperature and heat capacity of such a black hole.

(b) Two uncharged, nonrotating black holes begin very far apart from each other, then merge into a single black hole, emitting gravitational waves in the process; assume there is no initial angular momentum, so the final black hole is nonrotating as well. Find the maximum possible efficiency of this process, defined as the fraction of the initial energy that is converted into gravitational waves, for any set of initial black hole masses.

It may also be useful to review IPhO 2007, problem "blue", which we originally covered in **P1**.

Solution. (a) The temperature can be found with $T = dE/dS$, and the energy is simply $E = M$ since we're setting $c = 1$. We have $S = A/4 = \pi R^2 = 4\pi M^2$, giving $M = \sqrt{S/4\pi}$. Thus

$$T = \frac{1}{2\sqrt{4\pi S}} = \frac{1}{8\pi M}.$$

The heat capacity can be found with $C = dE/dT$. Using our answer for T, we have $E = 1/(8\pi T)$, giving a heat capacity of

$$C = -\frac{1}{8\pi T^2} = -8\pi M^2.$$

(b) By the second law of thermodynamics, the total change in entropy ΔS must be greater than or equal to zero. At the maximum efficiency, $\Delta S = 0$ so the total entropy of the two black holes must add up to the entropy of the final black hole:

$$S_1 + S_2 = S_f \implies M_1^2 + M_2^2 = M_f^2.$$

All the gravitational energy is included in the mass of the black holes, so the energy that went into the gravitational waves will be $E_w = M_1 + M_2 - M_f$ and $M_1 + M_2$ is the initial energy, which gives an efficiency of

$$\eta = 1 - \frac{\sqrt{M_1^2 + M_2^2}}{M_1 + M_2}.$$

This is maximized when $M_1 = M_2$, giving

$$\eta = 1 - \frac{\sqrt{2}}{2} = 0.29.$$

[4] **Problem 15.** US TST 2022, problem 3. Rough estimates of gravitational wave emission.

Solution. See the official solutions here.

[3] **Problem 16.** NBPhO 2017, problem 4. Rough estimates of gravitational wave detection.

Solution. See the official solutions here.

Remark

The discovery of gravitational waves by LIGO has been one of the most important results this decade, so it's naturally a popular question topic. Once you finish the above problems, you can check out a few others with a different take on the same idea. GPhO 2016, problem 2 (solutions here) does a rougher treatment of gravitational wave emission, while IPhO 2018, problem 1 gives a more accurate treatment using more of the language of general relativity. For another way to estimate gravitational wave emission, see section 9.3 of The Art of Insight, and for some followup questions, see this paper.

Remark

In 1931, after building a sensitive short-wave radio receiver, Karl Jansky heard an unusual noise on his receiver from a direction that moved across the sky about once a day. He therefore initially thought it was from the Sun. However, over time he found that the direction of the noise moved across the sky only once every 23 hours and 56 minutes.

This was a huge difference! To understand why, note that the length of a day is the time it takes for the same side of the Earth to face the Sun again; it depends on both the Earth's spin and its orbital motion about the Sun. The period of the Earth's spin alone, the so-called sidereal period, is only 23 hours and 56 minutes. Thus, a signal with this period indicates an origin from outside the solar system. Jansky later found that the source was the center of the galaxy; today we know it is due to the supermassive black hole there, Sagittarius A*. The early days of radio astronomy were full of dramatic discoveries like this. To hear about the discovery of pulsars, see this talk.

4 Cosmology

Cosmology is a rather technical topic because a proper treatment requires general relativity, but one can derive special cases of some of the results using just Newtonian gravity.

[2] **Problem 17.** AuPhO 2014, problem 14. A quick problem on the basics of dark matter and galaxy measurements.

Solution. See the official solutions here.

[5] **Problem 18.** GPhO 2022, problem 1. More about dark matter.

Solution. See the official solutions here.

[5] **Problem 19.** APhO 2016, problem 2. This straightforward problem introduces the basic equations of cosmology, such as the Friedmann equation.