PHYS-330: QFT I — Problem Set 1 (due 10/4)

Stanford University, 2022 — Bernhard Mistlberger, Kevin Zhou

1. Using natural units. (8 points)

In this course, we will work in "natural" units, where $\hbar = k_B = c = \mu_0 = \epsilon_0 = 1$. As a result, any physical quantity $\mathcal{A}$ has the same dimensions as $(1\,\text{eV})^n$ for some n, which we write as $[\mathcal{A}] = n$. For example, we have

$$[\text{energy}] = [\text{momentum}] = [\text{mass}] = [\text{temperature}] = 1 \tag{1}$$

and

$$[\text{length}] = [\text{time}] = -1. \tag{2}$$

These results immediately imply, e.g., [frequency] = 1, [speed] = 0, and [volume] = −3.

If the mass of a particle in natural units is $m = 1\,\text{eV}$, that means its mass in SI units is

$$m = \frac{1\,\text{eV}}{c^2} = 1.8 \times 10^{-36}\,\text{kg}. \tag{3}$$

One physical interpretation is that a particle of this mass has rest energy 1 eV.

a) Find the wavelength and period of a photon of energy 1 eV in SI units.

b) Express the temperature $T = 1\,\text{eV}$ in SI units.

If you commit the above results to memory, you should always be able to recover numeric values in SI units. The other skill you need is going from SI units to natural units.

c) Find $[G]$, where G is Newton's constant.

d) Find $[n]$, $[P]$, and $[\rho]$, where n is number density, P is pressure, and ρ is mass density.

e) Find $[\phi]$, where ϕ is a real scalar field.

f) Find $[q]$, $[A]$, $[E]$ and $[B]$, where q is electric charge, A is vector potential, and E and B are electric and magnetic fields.

Once you're comfortable with natural units, they'll be an incredibly convenient tool for making rough estimates. For example, the mass of the proton is $m_p \sim$ GeV, and everything in nuclear physics is roughly governed by this scale. From this, we can immediately conclude that the radius of the proton is roughly $r \sim \text{GeV}^{-1}$, in natural units.

g) Write down rough expressions for the density and electric field within a nucleus, and the temperature above which nuclei melt into quark-gluon plasma, in natural units.

Technically, all of the estimates here will be a bit off, because some of these quantities are actually determined by the pion mass $m_\pi \sim 10^{-1}\,\text{GeV}$, as the pion governs the forces between nucleons. For a more careful treatment, see *Astronomical reach of fundamental physics* by Burrows and Ostriker.

2. The harmonic oscillator in quantum mechanics. (15 points)

This exercise reviews the quantum harmonic oscillator, which has Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}. \tag{4}$$

a) Write the Hamiltonian in terms of the ladder operators a and $a^\dagger$, where

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\, x + i\,\frac{p}{\sqrt{m\omega}}\right). \tag{5}$$

b) The normalized vacuum state $|0\rangle$ is defined to satisfy $a|0\rangle = 0$. The number states are then defined by $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ for any integer n. Show that $[a, a^\dagger] = 1$, and use this fact alone to show that the number states are properly normalized.

c) Calculate the expectation values of x, p, and the number operator $N = a^\dagger a$ in the number state $|n\rangle$.

d) Calculate the standard deviations Δx, Δp and ΔN in the number state $|n\rangle$. For what n is the Heisenberg uncertainty product $\Delta x\,\Delta p$ minimal?

e) Suppose the particle begins in the vacuum state $|0\rangle$, and at time $t = 0$, we apply an impulse α. This can be modeled by a Hamiltonian term $-\alpha x \delta(t)$, and the state immediately after the impulse is

$$|\alpha\rangle = e^{i\alpha x}|0\rangle. \tag{6}$$

Show that $|\alpha\rangle$ is an eigenvector of a, and find the eigenvalue.

f) Find the expectation values of x, p, and N, and their standard deviations, for all $t > 0$. (Hint: after you find the answers for the initial state $|\alpha\rangle$, it is easiest to generalize to arbitrary t using Heisenberg picture.)

Your result in part (e) shows that $|\alpha\rangle$ is a so-called coherent state. You might have heard that they are important because they are the "most classical" states. A more important reason is that they are the states you automatically get when you drive a quantum system. As you can see from your results, in the limit of strong driving, Δx and Δp become negligible compared to x and p, and we recover classical physics. Later we will see how a similar result allows quantum fields to behave like classical fields.

3. The relativistic classical point particle. (12 points)

The spacetime trajectory of a relativistic point particle is $x^\mu(\tau) = (x^0(\tau), \mathbf{x}(\tau))$, where τ is an arbitrary parameter. The corresponding action is proportional to the relativistic "length" of the trajectory, where the relativistic line element is

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2. \tag{7}$$

The action is therefore

$$S = -\alpha \int_{\mathcal{P}} ds = -\alpha \int_{\tau_1}^{\tau_2} d\tau \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \tag{8}$$

where α is a constant, and τ_1 and τ_2 are the initial and final values of the parameter.

a) The easiest way to understand the nonrelativistic limit $|\partial_t x^i| \ll 1$ is to set $\tau = t$. By demanding that the action reduces to that of a free nonrelativistic particle of mass m (plus a constant), determine the value of the constant α.

If we don't just set $\tau = t$, there are four Euler–Lagrange equations and canonical momenta,

$$\frac{dp^\mu}{d\tau} = \frac{\partial L}{\partial x_\mu}, \quad p^\mu = \frac{\partial L}{\partial(dx_\mu/d\tau)}. \tag{9}$$

b) Find the Euler–Lagrange equations for a general parameter τ, then show that they are equivalent to the conservation of the physical four-momentum of the particle.

c) A simple local, Lorentz invariant way to include a force on the particle is to add

$$S_{\text{int}} = -q \int_{\mathcal{P}} A_\mu(x^\mu)\, dx^\mu = -q \int_{\tau_1}^{\tau_2} A_\mu(x^\mu)\, \frac{dx^\mu}{d\tau}\, d\tau \tag{10}$$

to the action, where $A^\mu(x^\mu)$ is a given four-vector field. Calculate p^μ and $\partial L/\partial x^\mu$, continuing to assume general τ.

d) Now set τ to be the proper time s experienced by the particle (so that $ds = d\tau$) and evaluate the Euler–Lagrange equations, simplifying as much as possible.

A warning: if you set τ to proper time *before* doing part (c), and apply the Euler–Lagrange equations anyway, you'll get nonsense. The reason is that the derivation of the Euler–Lagrange equation assumes all the variables $x^\mu(\tau)$ can be varied independently, but when $d\tau = ds$ we automatically have the constraint $\sqrt{\eta_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau)} = 1$. Lagrangians with constraints are subtle and important, but beyond the scope of this course. For much more about them, see *Quantization of Gauge Systems* by Henneaux and Teitelboim.

4. The complex scalar field. (5 points)

The Lagrangian density for a canonically normalized free real scalar field of mass m is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)\,(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2. \tag{11}$$

Now consider a theory of two free real scalar fields ϕ_1 and ϕ_2, both with mass m.

a) Write their Lagrangian density in terms of the complex scalar field $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$ and its complex conjugate Φ^*.

Complex fields are always equivalent to a pair of equal mass real fields, and are useful because such pairs occur frequently in nature, for reasons we'll see later. (At low energies, we actually don't know of any complex scalar fields, but the electron is described by a Dirac field, which is a complex fermion field built from two equal mass real fermion fields.)

Complex fields are convenient once you get to know them, but they come with an annoying problem: it is not obvious how to vary the action with respect to Φ, because any change in Φ also changes Φ^*. It turns out that you will always get the right results (i.e. results that are equivalent to what you'd get working in terms of the two real fields) by treating Φ and Φ^* as if they were *independent* real fields, even though they clearly aren't. (For an explanation why, see page 56 of Sidney Coleman's lecture notes.)

b) Compute the conjugate momenta Π and Π^* of Φ and Φ^*, and the Euler–Lagrange equations for Φ and Φ^*.

c) Show that the action is invariant under the transformation $\Phi \to e^{i\alpha}\Phi$, for any real α. What is the equivalent symmetry in terms of the real scalar fields ϕ_1 and ϕ_2?

PHYS-330: QFT I — Solutions 1

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One physical interpretation is that a particle of this mass has rest energy $1\,\mathrm{eV}$.

a) Find the wavelength and period of a photon of energy $1\,\mathrm{eV}$ in SI units.

Solution: Since $E = \hbar\omega$, the period is

$$T = \frac{2\pi\hbar}{1\,\mathsf{eV}} = 4 \times 10^{-15}\,\mathrm{s}. \tag{S1}$$

Since $\lambda = cT$, the wavelength is

$$\lambda = \frac{2\pi\hbar c}{1\,\mathsf{eV}} = 1.2 \times 10^{-6}\,\mathrm{m}. \tag{S2}$$

b) Express the temperature $T = 1\,\mathrm{eV}$ in SI units.

Solution: Since energy has dimensions of $k_B T$, the temperature is

$$T = \frac{1\mathsf{eV}}{k_B} = 12000\,\mathrm{K}. \tag{S3}$$

This is the temperature where the typical thermal energy of a degree of freedom is $\sim 1\,\mathsf{eV}$, which would be sufficient to break apart molecules and ionize most atoms.

If you commit the above results to memory, you should always be able to recover numeric values in SI units. The other skill you need is going from SI units to natural units.

c) Find $[G]$, where G is Newton's constant.

Solution: You can do this part, and all the parts below, by brute force. But it's faster and more fun to get the answers by thinking about equations you already know.

For example, the gravitational potential energy formula implies $[E] = [GM^2/R]$, so $[G] = [R/M] = -2$. (In natural units, $G \sim 1/M_{\mathrm{pl}}^2$ where M_{pl} is the Planck mass.)

d) Find $[n]$, $[P]$, and $[\rho]$, where n is number density, P is pressure, and ρ is mass density.

Solution: Since n is a dimensionless number per volume, $[n] = [1/\text{volume}] = 3$. Noting that $dU = P\,dV$ for thermodynamic work, $[P] = [\text{energy}/\text{volume}] = 4$. Similarly, $[\rho] = [\text{mass}/\text{volume}] = 4$.

e) Find $[\phi]$, where ϕ is a real scalar field.

Solution: Note that the action S is the integral of a Lagrangian (with units of energy) over time, so

$$[S] = [\text{energy} \cdot \text{time}] = 0. \tag{S4}$$

In field theory, the action is the integral over 4-dimensional spacetime of a Lagrangian density $\mathcal{L}$, which implies $[\mathcal{L}] = 4$. Examining the kinetic term, we have

$$[(\partial_\mu \phi)^2] = 4 \tag{S5}$$

and we have $[\partial_\mu] = 1$, which implies $[\phi] = 1$.

f) Find $[q]$, $[A]$, $[E]$ and $[B]$, where q is electric charge, A is vector potential, and E and B are electric and magnetic fields.

Solution: In natural units, the force between charges is $F = q^2/4\pi r^2$. Since $[F] = [dp/dt] = 2$, we conclude $[q] = 0$, i.e. charge is dimensionless. (In case you were wondering, the specific value of the electron charge is about -0.3. This result is more commonly expressed in terms of the "fine structure constant" $\alpha = e^2/4\pi \approx 1/137$, and later, the fact that $\alpha \ll 1$ will tell us that electromagnetic interactions in QFT can be treated in perturbation theory.)

For the vector potential, one could recall the Lorenz gauge equation of motion $\partial^2 A^\mu = J^\mu$ where J^μ is the current density. We have $[\partial^2] = 2$ and $[J] = [\text{charge/volume}] = 3$, so we conclude $[A] = 1$.

The electric and magnetic fields are spatial or time derivatives of the vector potential, so $[E] = [B] = 2$.

Once you're comfortable with natural units, they'll be an incredibly convenient tool for making rough estimates. For example, the mass of the proton is $m_p \sim \text{GeV}$, and everything in nuclear physics is roughly governed by this scale. From this, we can immediately conclude that the radius of the proton is roughly $r \sim \text{GeV}^{-1}$, in natural units.

g) Write down rough expressions for the density and electric field within a nucleus, and the temperature above which nuclei melt into quark-gluon plasma, in natural units.

Solution: It's just $\rho \sim \text{GeV}^4$, $E \sim \text{GeV}^2$, and $T \sim \text{GeV}$. These are all very rough estimates, dropping dimensionless factors like $\alpha \sim 1/137$ or dependence on the pion mass, but they're all within a few orders of magnitude.

Technically, all of the estimates here will be a bit off, because some of these quantities are actually determined by the pion mass $m_\pi \sim 10^{-1}\,\text{GeV}$, as the pion governs the forces between nucleons. For a more careful treatment, see *Astronomical reach of fundamental physics* by Burrows and Ostriker.

2. The harmonic oscillator in quantum mechanics. (15 points)

This exercise reviews the quantum harmonic oscillator, which has Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}. \tag{4}$$

a) Write the Hamiltonian in terms of the ladder operators a and $a^\dagger$, where

$$a = \frac{1}{\sqrt{2}} \left(\sqrt{m\omega}\, x + i \frac{p}{\sqrt{m\omega}} \right). \tag{5}$$

Solution: Of course, we have

$$a^\dagger = \frac{1}{\sqrt{2}} \left(\sqrt{m\omega}\, x - i \frac{p}{\sqrt{m\omega}} \right). \tag{S6}$$

Following the usual textbook steps, you should find

$$H = \omega \left(a^\dagger a + \frac{1}{2} \right). \tag{S7}$$

Equivalent forms, such as having $a^\dagger a + a a^\dagger$ in parentheses, are also acceptable.

b) The normalized vacuum state $|0\rangle$ is defined to satisfy $a|0\rangle = 0$. The number states are then defined by $a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$ for any integer n. Show that $[a, a^\dagger] = 1$, and use this fact alone to show that the number states are properly normalized.

Solution: Using the fact that $[x, p] = i$, we have

$$[a, a^\dagger] = \frac{1}{2} \left[\sqrt{m\omega}\, x + i \frac{p}{\sqrt{m\omega}}, \sqrt{m\omega}\, x - i \frac{p}{\sqrt{m\omega}} \right] \tag{S8}$$

$$= \frac{1}{2} \left([x, -ip] + [ip, x] \right) = 1 \tag{S9}$$

as desired.

Now, by repeatedly using the definition of the number states, we have

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle. \tag{S10}$$

Note that we have

$$[a, (a^\dagger)^n] = n (a^\dagger)^{n-1}. \tag{S11}$$

The intuition is that to convert $a(a^\dagger)^n$ into $(a^\dagger)^n a$, we need to move an a through an $a^\dagger$ a total of n times. For each move, we use the identity $a a^\dagger = a^\dagger a + [a, a^\dagger] = a^\dagger a + 1$, and therefore pick up a factor of $(a^\dagger)^{n-1}$. The factor of n in the final result is because we need to do n such moves.

To check the normalization, we just compute

$$\langle n|n\rangle = \frac{1}{n!} \langle 0| a^n (a^\dagger)^n |0\rangle \tag{S12}$$

$$= \frac{1}{n!} \langle 0| a^{n-1}\, a (a^\dagger)^n |0\rangle \tag{S13}$$

$$= \frac{1}{n!} \langle 0| a^{n-1} \left((a^\dagger)^n a + n (a^\dagger)^{n-1} \right) |0\rangle \tag{S14}$$

$$= \frac{1}{(n-1)!} \langle 0| a^{n-1} (a^\dagger)^{n-1} |0\rangle \tag{S15}$$

$$= \langle n-1|n-1\rangle. \tag{S16}$$

This shows that the norms of all the number states are equal, so they're all normalized because $|0\rangle$ is.

c) Calculate the expectation values of x, p, and the number operator $N = a^\dagger a$ in the number state $|n\rangle$.

Solution: We can write x and p in terms of ladder operators as

$$x = \frac{1}{\sqrt{2m\omega}} (a^\dagger + a), \qquad p = i \sqrt{\frac{m\omega}{2}} (a^\dagger - a). \tag{S17}$$

Then $\langle n|x|n\rangle$ has two terms, one proportional to $\langle n|n+1\rangle$ and the other proportional to $\langle n|n-1\rangle$, which both vanish. Thus, $\langle n|x|n\rangle = 0$, and by similar reasoning $\langle n|p|n\rangle = 0$. As for the number operator,

$$\langle n|N|n\rangle = \langle n|a^\dagger a|n\rangle = \langle n-1|\sqrt{n}\,\sqrt{n}|n-1\rangle = n. \tag{S18}$$

d) Calculate the standard deviations Δx, Δp and ΔN in the number state $|n\rangle$. For what n is the Heisenberg uncertainty product $\Delta x\, \Delta p$ minimal?

Solution: For a given state, the standard deviation of any operator A is defined by

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \tag{S19}$$

where the expectation values are taken with respect to the state. Now first, we note that

$$\langle n|x^2|n\rangle = \frac{1}{2m\omega}\langle n|(a+a^\dagger)(a+a^\dagger)|n\rangle \tag{S20}$$

$$= \frac{1}{2m\omega}\langle n|a^\dagger a + aa^\dagger|n\rangle \tag{S21}$$

$$= \frac{1}{2m\omega}(n + (n+1)) \tag{S22}$$

$$= \frac{2n+1}{2m\omega}. \tag{S23}$$

From this and the previous part, we conclude

$$\Delta x = \sqrt{\frac{2n+1}{2m\omega}}. \tag{S24}$$

A very similar computation for p gives

$$\Delta p = \sqrt{\frac{m\omega\,(2n+1)}{2}}. \tag{S25}$$

Clearly the Heisenberg uncertainty product is minimal for $n = 0$, in which case $\Delta x\,\Delta p = 1/2$, saturating the uncertainty relation.

Finally, we have

$$\langle n|N^2|n\rangle = n\langle n|N|n\rangle = n^2 \tag{S26}$$

from which we conclude $\Delta N = 0$, as expected.

e) Suppose the particle begins in the vacuum state $|0\rangle$, and at time $t = 0$, we apply an impulse α. This can be modeled by a Hamiltonian term $-\alpha x\delta(t)$, and the state immediately after the impulse is

$$|\alpha\rangle = e^{i\alpha x}|0\rangle. \tag{6}$$

Show that $|\alpha\rangle$ is an eigenvector of a, and find the eigenvalue.

Solution: For simplicity, let's define $\beta = \alpha/\sqrt{2m\omega}$, so that

$$|\alpha\rangle = e^{i\beta(a+a^\dagger)}|0\rangle. \tag{S27}$$

We'll show a direct solution. First, note that $[a, a+a^\dagger] = 1$. This implies that

$$[a, (a+a^\dagger)^n] = n(a+a^\dagger)^{n-1} \tag{S28}$$

where the factor of n comes from commuting the a past each copy of $a + a^\dagger$. Now we have

$$a|\alpha\rangle = a\sum_{n=0}^{\infty}\frac{(i\beta)^n}{n!}(a+a^\dagger)^n|0\rangle \tag{S29}$$

$$= \sum_{n=0}^{\infty}\frac{(i\beta)^n}{n!}\left([a,(a+a^\dagger)^n] + (a+a^\dagger)^n a\right)|0\rangle \tag{S30}$$

$$= \sum_{n=0}^{\infty}\frac{(i\beta)^n}{n!}\left(n(a+a^\dagger)^{n-1}\right)|0\rangle \tag{S31}$$

$$= \sum_{n=0}^{\infty}\frac{(i\beta)^n}{(n-1)!}(a+a^\dagger)^{n-1}|0\rangle \tag{S32}$$

$$= \sum_{n=0}^{\infty}\frac{(i\beta)^{n+1}}{n!}(a+a^\dagger)^n|0\rangle \tag{S33}$$

$$= i\beta|\alpha\rangle \tag{S34}$$

where we Taylor expanded the exponential, used the definition of the commutator, used $a|0\rangle = 0$ and Eq. (S28), and shifted the variable n. Thus, $|\alpha\rangle$ is an eigenvector of a with eigenvalue $i\beta = i\alpha/\sqrt{2m\omega}$.

f) Find the expectation values of x, p, and N, and their standard deviations, for all $t > 0$. (Hint: after you find the answers for the initial state $|\alpha\rangle$, it is easiest to generalize to arbitrary t using Heisenberg picture.)

Solution: First, let's calculate the expectation values in the initial state $|\alpha\rangle$. Since x commutes with $e^{i\alpha x}$, we have

$$\langle\alpha|x|\alpha\rangle = \langle 0|e^{-i\alpha x} x e^{i\alpha x}|0\rangle = \langle 0|x|0\rangle = 0. \tag{S35}$$

Similarly, we have

$$\langle\alpha|x^2|\alpha\rangle = \langle 0|x^2|0\rangle = \frac{1}{2m\omega} \tag{S36}$$

where we used Eq. (S24). This gives a standard deviation

$$\Delta x = \frac{1}{\sqrt{2m\omega}}. \tag{S37}$$

Next, for p it's useful to use the decomposition into ladder operators, as

$$\langle\alpha|a|\alpha\rangle = \langle\alpha|i\beta|\alpha\rangle = i\beta, \qquad \langle\alpha|a^\dagger|\alpha\rangle = (a|\alpha\rangle)^\dagger|\alpha\rangle = -i\beta \tag{S38}$$

where we used the result of part (e). We therefore have

$$\langle\alpha|p|\alpha\rangle = i\sqrt{\frac{m\omega}{2}}\langle\alpha|(a^\dagger - a)|\alpha\rangle = (-2i\beta)\, i\sqrt{\frac{m\omega}{2}} = \alpha \tag{S39}$$

which is unsurprising, as the state $|\alpha\rangle$ as defined by exerting an impulse α on the ground state. Next,

$$\begin{aligned}
\langle\alpha|p^2|\alpha\rangle &= -\frac{m\omega}{2}\langle\alpha|(a^\dagger - a)^2|\alpha\rangle && \text{(S40)}\\
&= -\frac{m\omega}{2}\langle\alpha|(a^\dagger a^\dagger + aa - aa^\dagger - a^\dagger a)|\alpha\rangle && \text{(S41)}\\
&= \frac{m\omega}{2}\langle\alpha|(-a^\dagger a^\dagger - aa + 2a^\dagger a + 1)|\alpha\rangle && \text{(S42)}\\
&= \frac{m\omega}{2}(4\beta^2 + 1) && \text{(S43)}\\
&= \frac{m\omega}{2} + \alpha^2. && \text{(S44)}
\end{aligned}$$

Therefore, the standard deviation is

$$\Delta p = \sqrt{\frac{m\omega}{2}} \tag{S45}$$

which is unchanged from the ground state. Finally, for the number states,

$$\langle\alpha|N|\alpha\rangle = \langle\alpha|a^\dagger a|\alpha\rangle = \beta^2 = \frac{\alpha^2}{2m\omega} \tag{S46}$$

and

$$\begin{aligned}
\langle\alpha|N^2|\alpha\rangle &= \langle\alpha|a^\dagger a a^\dagger a|\alpha\rangle && \text{(S47)}\\
&= \beta^2\langle\alpha|aa^\dagger|\alpha\rangle && \text{(S48)}\\
&= \beta^2\langle\alpha|(1 + a^\dagger a)|\alpha\rangle && \text{(S49)}\\
&= \beta^2 + \beta^4 && \text{(S50)}
\end{aligned}$$

from which we conclude

$$\Delta N = \beta = \frac{\alpha}{\sqrt{2m\omega}}. \tag{S51}$$

You could also derive all of the above results using the Baker–Campbell–Hausdorff theorem, but such machinery isn't necessary. We simply leaned on the result of part (e) to get all of the results, and the solution to part (e) is effectively a proof of that theorem in the simple case we need.

Now let's consider how these expectation values and uncertainties evolve over time. The evolution of an expectation value of any operator $\mathcal{O}$ is

$$\langle \mathcal{O} \rangle(t) = \langle \psi | e^{iHt} \mathcal{O} e^{-iHt} | \psi \rangle. \tag{S52}$$

Since N commutes with H, this implies that N and ΔN just stay constant over time. As for x and p, the easiest way is to compute the "Heisenberg" operators $\mathcal{O}_H(t) = e^{iHt} \mathcal{O} e^{-iHt}$. The basic result is that Heisenberg operators satisfy Hamilton's equations, which here imply

$$\frac{dp_H}{dt} = -m\omega^2 x_H, \qquad \frac{dx_H}{dt} = \frac{p_H}{m}. \tag{S53}$$

The solution of this pair of equations is

$$p_H(t) = p \cos \omega t - m\omega x \sin \omega t, \qquad x_H(t) = x \cos \omega t + \frac{p}{m\omega} \sin \omega t. \tag{S54}$$

This means the expectation values we're after follow immediately from those we already calculated. We know that $\langle \alpha | x | \alpha \rangle = 0$ and $\langle \alpha | p | \alpha \rangle = \alpha$, from which we conclude

$$\langle x \rangle(t) = \frac{\alpha}{m\omega} \sin \omega t, \qquad \langle p \rangle(t) = \alpha \cos \omega t \tag{S55}$$

Similarly, we have

$$\langle x^2 \rangle(t) = \langle \alpha | x^2 \cos^2 \omega t + \frac{(xp + px)}{m\omega} \cos \omega t \, \sin \omega t + \frac{p^2}{m^2 \omega^2} \sin^2 \omega t \, | \alpha \rangle \tag{S56}$$

$$= \cos^2 \omega t \langle \alpha | x^2 | \alpha \rangle + \frac{\sin^2 \omega t}{m^2 \omega^2} \langle \alpha | p^2 | \alpha \rangle \tag{S57}$$

$$= \left(\frac{\alpha \, \sin \omega t}{m\omega} \right)^2 + \frac{1}{2m\omega} \tag{S58}$$

which implies that the position uncertainty does not change over time,

$$\Delta x(t) = \frac{1}{\sqrt{2m\omega}}. \tag{S59}$$

By an extremely similar computation, we find the momentum uncertainty doesn't change over time either,

$$\Delta p(t) = \sqrt{\frac{m\omega}{2}}. \tag{S60}$$

Your result in part (e) shows that $|\alpha\rangle$ is a so-called coherent state. You might have heard that they are important because they are the "most classical" states. A more important reason is that they are the states you automatically get when you drive a quantum system. As you can see from your results, in the limit of strong driving, Δx and Δp become negligible compared to x and p, and we recover classical physics. Later we will see how a similar result allows quantum fields to behave like classical fields.

3. The relativistic classical point particle. (12 points)

The spacetime trajectory of a relativistic point particle is $x^\mu(\tau) = (x^0(\tau), \mathbf{x}(\tau))$, where τ is an arbitrary parameter. The corresponding action is proportional to the relativistic "length" of the trajectory, where the relativistic line element is

$$ds^2 = \eta_{\mu\nu} \, dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2. \tag{7}$$

The action is therefore

$$S = -\alpha \int_{\mathcal{P}} ds = -\alpha \int_{\tau_1}^{\tau_2} d\tau \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \tag{8}$$

where α is a constant, and τ_1 and τ_2 are the initial and final values of the parameter.

a) The easiest way to understand the nonrelativistic limit $|\partial_t x^i| \ll 1$ is to set $\tau = t$. By demanding that the action reduces to that of a free nonrelativistic particle of mass m (plus a constant), determine the value of the constant α.

Solution: Setting $\tau = t$, we have

$$S = -\alpha \int dt \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} = -\alpha \int dt \sqrt{1 - v(t)^2} \tag{S61}$$

where $v(t)$ is the ordinary speed. In the nonrelativistic limit $v \ll 1$,

$$S \approx -\alpha \int dt \left(1 - \frac{v^2}{2} + O(v^4)\right). \tag{S62}$$

The first term is just an irrelevant constant, while the next gives a Lagrangian $\alpha v^2/2$. On the other hand, we know the Lagrangian for a free nonrelativistic particle is $mv^2/2$, so we conclude $\alpha = m$.

If we don't just set $\tau = t$, there are four Euler–Lagrange equations and canonical momenta,

$$\frac{dp^\mu}{d\tau} = \frac{\partial L}{\partial x_\mu}, \quad p^\mu = \frac{\partial L}{\partial (dx_\mu / d\tau)}. \tag{9}$$

b) Find the Euler–Lagrange equations for a general parameter τ, then show that they are equivalent to the conservation of the physical four-momentum of the particle.

Solution: For simplicity, let a dot denote a derivative with respect to τ, so that

$$L = -m\sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} = -m\sqrt{\dot{x}^2}. \tag{S63}$$

The canonical momenta are

$$p^\mu = \frac{\partial L}{\partial \dot{x}_\mu} = -\frac{m}{2\sqrt{\dot{x}^2}}(2\dot{x}^\mu) = -\frac{m\dot{x}^\mu}{\sqrt{\dot{x}^2}}. \tag{S64}$$

The Lagrangian has no direct dependence on x^μ, so the Euler–Lagrange equations just state that p^μ is conserved, $dp^\mu/d\tau = 0$.

This isn't the end of the problem, though; we still need to show how the canonical momentum p^μ is related to *physical* four-momentum. One way to do this is to let τ be the proper time s of the particle. Then $\dot{x}^2 = 1$, and $p^\mu = -m\, dx^\mu/ds$-momentum. Alternatively, we can let τ be the coordinate time t, in which case $p^\mu = -m(dx^\mu/dt)/\sqrt{1-v^2}$. In both cases, it's clear that p^μ is the *negative* of the physical four-momentum. (This negative sign is weird, but harmless, since the physical results are the same. It occurs because we used a mostly negative metric, which will pay off later in the course.)

c) A simple local, Lorentz invariant way to include a force on the particle is to add

$$S_{\text{int}} = -q \int_{\mathcal{P}} A_\mu(x^\mu)\, dx^\mu = -q \int_{\tau_1}^{\tau_2} A_\mu(x^\mu) \frac{dx^\mu}{d\tau}\, d\tau \tag{10}$$

to the action, where $A^\mu(x^\mu)$ is a given four-vector field. Calculate p^μ and $\partial L/\partial x^\mu$, continuing to assume general τ.

Solution: There are a lot of μ's floating around, so to avoid confusion, let's use the fact that we can always rename a dummy index to rewrite the interaction as

$$S_{\text{int}} = -q \int A_\nu(x^\mu)\dot{x}^\nu\, d\tau. \tag{S65}$$

First, the canonical momenta are changed to

$$p^\mu = \frac{\partial L}{\partial \dot{x}_\mu} = -\frac{m\dot{x}^\mu}{\sqrt{\dot{x}^2}} - qA^\mu(x^\mu). \tag{S66}$$

Next, the right-hand side of the Euler–Lagrange equations is now nontrivial,

$$\frac{\partial L}{\partial x_\mu} = -q\,\dot{x}^\nu \partial^\mu A_\nu(x^\mu) \tag{S67}$$

where the partial derivative on the right is with respect to x^μ.

A common issue with this part was mismatched indices (having μ up on one side and down on the other side), which can be avoided by taking care to make the index positions consistent at each step. Another issue was duplicated indices (having more than two copies of μ, which is meaningless), which can be avoided by renaming dummy indices when required.

d) Now set τ to be the proper time s experienced by the particle (so that $ds = d\tau$) and evaluate the Euler–Lagrange equations, simplifying as much as possible.

Solution: From this point on, we'll leave the x^μ argument of the vector potential implicit. Specializing the proper time, the canonical momenta are

$$p^\mu = -m\dot{x}^\mu - qA^\mu. \tag{S68}$$

We therefore have

$$\frac{dp^\mu}{ds} = -m\ddot{x}^\mu - q\frac{dA^\mu}{ds} = -m\ddot{x}^\mu - q\dot{x}^\nu \partial_\nu A^\mu. \tag{S69}$$

Equating this with Eq. (S67) gives

$$m\ddot{x}^\mu = -q\dot{x}^\nu\left(\partial_\nu A^\mu - \partial^\mu A_\nu\right) = q\dot{x}_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu). \tag{S70}$$

This is an acceptable final answer; the right-hand side can also be written as $q\dot{x}_\nu F^{\mu\nu}$ where F is the electromagnetic field strength tensor. If you treat A^μ as a vector potential and plug in the definitions of the electric and magnetic fields, you'll recover the Lorentz force law.

A warning: if you set τ to proper time *before* doing part (c), and apply the Euler–Lagrange equations anyway, you'll get nonsense. The reason is that the derivation of the Euler–Lagrange equation assumes all the variables $x^\mu(\tau)$ can be varied independently, but when $d\tau = ds$ we automatically have the constraint $\sqrt{\eta_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau)} = 1$. Lagrangians with constraints are subtle and important, but beyond the scope of this course. For much more about them, see *Quantization of Gauge Systems* by Henneaux and Teitelboim.

4. The complex scalar field. (5 points)

The Lagrangian density for a canonically normalized free real scalar field of mass m is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2. \tag{11}$$

Now consider a theory of two free real scalar fields ϕ_1 and ϕ_2, both with mass m.

a) Write their Lagrangian density in terms of the complex scalar field $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$ and its complex conjugate Φ^*.

Solution: The mass terms involve $\phi_1^2 + \phi_2^2$, which clearly indicates the answer involves the complex norm $\Phi^*\Phi$. Treating the kinetic term the same way, and fixing the numeric constants, gives

$$\mathcal{L} = (\partial_\mu\Phi^*)(\partial^\mu\Phi) - m^2\Phi^*\Phi. \tag{S71}$$

Complex fields are always equivalent to a pair of equal mass real fields, and are useful because such pairs occur frequently in nature, for reasons we'll see later. (At low energies, we actually don't know of any complex scalar fields, but the electron is described by a Dirac field, which is a complex fermion field built from two equal mass real fermion fields.)

Complex fields are convenient once you get to know them, but they come with an annoying problem: it is not obvious how to vary the action with respect to Φ, because any change in Φ also changes Φ^*. It turns out that you will always get the right results (i.e. results that are equivalent to what you'd get working in terms of the two real fields) by treating Φ and Φ^* as if they were *independent* real fields, even though they clearly aren't. (For an explanation why, see page 56 of Sidney Coleman's lecture notes.)

b) Compute the conjugate momenta Π and Π^* of Φ and Φ^*, and the Euler–Lagrange equations for Φ and Φ^*.

Solution: Note that the kinetic term contains $\dot{\Phi}^*\dot{\Phi}$, where a dot denotes a time derivative. Thus, the canonical momenta are

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \dot{\Phi}^*, \qquad \Pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^*} = \dot{\Phi}. \tag{S72}$$

The Euler–Lagrange equations are

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi)} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0, \qquad \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi^*)} - \frac{\partial \mathcal{L}}{\partial \Phi^*} = 0 \tag{S73}$$

from which we read off

$$(\partial^2 + m^2)\Phi^* = 0, \qquad (\partial^2 + m^2)\Phi = 0 \tag{S74}$$

which is just the Klein–Gordan equation.

c) Show that the action is invariant under the transformation $\Phi \to e^{i\alpha}\Phi$, for any real α. What is the equivalent symmetry in terms of the real scalar fields ϕ_1 and ϕ_2?

Solution: Each term in the Lagrangian has one power of Φ and one power of Φ^*, and $\Phi^* \to e^{-i\alpha}\Phi$, so the phase cancels out. In terms of the real scalar fields, an equivalent symmetry is a rotation,

$$\phi_1 \to \phi_1 \cos\alpha - \phi_2 \sin\alpha, \quad \phi_2 \to \phi_2 \cos\alpha + \phi_1 \sin\alpha. \tag{S75}$$

PHYS-330: QFT I — Problem Set 2 (due 10/11)

Stanford University, 2022 — Bernhard Mistlberger, Kevin Zhou

1. Lorentz transformations. (10 points)

Lorentz transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ are linear transformations that leave inner products invariant, meaning that $x^\mu y_\mu = x'^\mu y'_\mu$ for any four-vectors x and y.

a) Show that this implies

$$\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu. \tag{1}$$

b) All proper, orthochronous Lorentz transformations (i.e. all those which preserve the orientation of space and the direction of time) can be decomposed into infinitesimal Lorentz transformations. These take the form

$$x^\mu \to x'^\mu = x^\mu + \epsilon\, \omega^\mu{}_\nu x^\nu \tag{2}$$

where ϵ is infinitesimal. Show that $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

c) The elements of any infinitesimal Lorentz transformation $\omega^\mu{}_\nu$ can be written as a 4×4 matrix, where μ and ν index the row and column, respectively. For an infinitesimal rotation $\epsilon = d\theta$ about the z-axis, write out this matrix, and denote it by iJ^3 for later. What exponential of J^3 corresponds to a finite rotation by an angle θ?

d) Write down the matrix iK^1 corresponding to an infinitesimal boost by $\epsilon = dv$ about the x-axis. What exponential of K^1 corresponds to a finite boost by a velocity v?

e) Defining J^1, J^2, K^2, and K^3 similarly, the generators obey commutation relations

$$[J^i, J^j] = i\epsilon^{ijk}J^k, \qquad [J^i, K^j] = i\epsilon^{ijk}K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk}J^k. \tag{3}$$

This is the Lie algebra of the Lorentz group. Physically, these results tells us that infinitesimal rotations and boosts are vectors (i.e. angular velocity and velocity are vectors), and that composing boosts can yield a rotation. Prove these results for the cases $(i,j) = (1,1)$ and $(i,j) = (1,2)$. (The proofs for other cases are similar.)

2. Quantization of the complex scalar field. (30 points)

This will be an involved problem, but it will teach you everything there is to know about free field mode expansions. (You don't have to write every detail; when it's clear other computations will go the same way as one you just did, it's fine to just say so and move on.) In problem set 1 we considered a complex scalar field Φ with Lagrangian density

$$\mathcal{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2\Phi^*\Phi, \tag{4}$$

and found the canonical fields Φ and Φ^* and momenta Π and Π^*. Letting $\omega_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + m^2}$, introduce the following mode expansion for the canonical fields and momenta:

$$\Phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{5}$$

$$\Phi^*(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left[b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{6}$$

$$\Pi(\mathbf{x}) = -i\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\left[b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{7}$$

$$\Pi^*(\mathbf{x}) = -i\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right]. \tag{8}$$

We quantize the fields by imposing the canonical commutation relations

$$[\Phi(\mathbf{x}), \Pi(\mathbf{y})] = [\Phi^*(\mathbf{x}), \Pi^*(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \tag{9}$$

with all other commutators between these fields vanishing. (Technically, we should write Hermitian conjugates $\Phi^\dagger$ and $\Pi^\dagger$ here, since Φ and Π are now operators, but we'll continue to use stars to emphasize the links between the classical and quantum theories.)

a) Show that the operators $a(\mathbf{k})$, $a^\dagger(\mathbf{k})$, $b(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ obey

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{p})\right] = \left[b(\mathbf{k}), b^\dagger(\mathbf{p})\right] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{p}), \tag{10}$$

with all other commutators between these operators vanishing. This implies that we have two sets of independent creation and annihilation operators for each $\mathbf{k}$.

The vacuum state $|0\rangle$ is defined to be the unique state where $a(\mathbf{k})|0\rangle = b(\mathbf{k})|0\rangle = 0$ for all $\mathbf{k}$. The states $a^\dagger(\mathbf{k})|0\rangle$ and $b^\dagger(\mathbf{k})|0\rangle$ each contain one particle, while the state $a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0\rangle$ contains two particles, and so on.

b) In problem set 1, you showed that the complex scalar field Lagrangian had the symmetry $\Phi \to e^{i\alpha}\Phi$, $\Phi^* \to e^{-i\alpha}\Phi^*$. Compute the Noether current J^μ and conserved charge Q associated with the symmetry.

c) Write Q in terms of the creation and annihilation operators. You should find the result is indeterminate up to a constant; resolve this by defining the vacuum to have zero charge, $Q|0\rangle = 0$. What is the charge in the three other states mentioned above?

d) To check that this is the same symmetry operation that we started out with, we can see how it acts on the operators of the theory. Quantum mechanically, symmetries act on operators by conjugation, and we expect to have

$$e^{i\alpha Q}\, \Phi(\mathbf{x})\, e^{-i\alpha Q} = e^{i\alpha}\, \Phi(\mathbf{x}), \qquad e^{i\alpha Q}\, \Phi^*(\mathbf{x})\, e^{-i\alpha Q} = e^{-i\alpha}\, \Phi^*(\mathbf{x}). \tag{11}$$

Show that this implies the commutation relations

$$[Q, \Phi(\mathbf{x})] = \Phi(\mathbf{x}), \qquad [Q, \Phi^*(\mathbf{x})] = -\Phi^*(\mathbf{x}) \tag{12}$$

and show that these relations hold. Thus, Q generates phase rotations of the field.

e) Use Noether's theorem to find the stress-energy tensor $T^{\mu\nu}$ and the associated conserved total four-momentum P^μ.

f) Write the Hamiltonian $H = P^0$ and the spatial momenta $\mathbf{P}$ in terms of the creation and annihilation operators, again defining the vacuum to have zero four-momentum. What is the four-momentum in the three other states mentioned above?

g) By definition, the operator H generates time translations of the field – this is the content of the Schrodinger equation, $i\partial_t|\Psi\rangle = H|\Psi\rangle$, which holds unchanged in quantum field theory. As for the momenta $\mathbf{P}$, we expect

$$e^{i\mathbf{a}\cdot\mathbf{P}}\, \Phi(\mathbf{x})\, e^{-i\mathbf{a}\cdot\mathbf{P}} = \Phi(\mathbf{x} - \mathbf{a}) \tag{13}$$

for any vector $\mathbf{a}$, with a similar result for all the other fields. Show that this implies

$$[P^i, \Phi(\mathbf{x})] = -i\partial^i \Phi(\mathbf{x}) \tag{14}$$

and show that this relation holds. Thus, **P** generates spatial translations.

3. ⋆ Conserved currents of Lorentz transformations. (10 points)

This somewhat tricky problem combines the ideas of the first two. It is completely optional: the problem set will be graded out of 40 points, so that you will receive up to 100% credit if you don't do this problem, and up to 125% credit if you do.

Under a Lorentz transformation, a scalar field profile ϕ gets mapped to ϕ', so that $\phi'(x') = \phi(x)$. This implies that

$$\phi'(x) = \phi(\Lambda^{-1}x). \tag{15}$$

For an infinitesimal Lorentz transformation (2), this corresponds to

$$\phi'(x) = \phi(x) - \epsilon\, \omega^{\mu\nu} x_\nu \partial_\mu \phi(x) \tag{16}$$

to first order in ϵ. Because an infinitesimal Lorentz transformation is parametrized by a rank 2 tensor $\omega^{\mu\nu}$, the corresponding Noether current will be a rank 3 tensor $J^{\mu\nu\rho}$, where the first index is the usual index that comes from Noether's theorem, and the last two describe the Lorentz transformation. For simplicity, you can do the entire problem for a real scalar field. (This corresponds to a complex scalar field with $a(\mathbf{k}) = b(\mathbf{k})$, which in turn implies $\Phi = \Phi^*$ and $\Pi = \Pi^*$.)

a) Show that for a scalar field,

$$J^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} \tag{17}$$

where $T^{\mu\nu}$ is the stress-energy tensor. Let the associated conserved charges be $M^{\mu\nu}$.

b) Since $M^{\mu\nu}$ is antisymmetric, there are six independent conserved charges. The three independent M^{ij} physically correspond to the angular momentum of the field. What is the physical meaning of the other three conserved quantities M^{0i}?

c) Show that after normal ordering, the angular momentum is

$$M^{ij} = i \int \frac{d^3k}{(2\pi)^3}\, a(\mathbf{k})^\dagger \left(k^j \frac{\partial}{\partial k_i} - k^i \frac{\partial}{\partial k_j} \right) a(\mathbf{k}) \tag{18}$$

The form of this answer implies that the particles created and annihilated by scalar fields do not carry any intrinsic angular momentum (i.e. spin).

PHYS-330: QFT I — Solutions 2

Stanford University, 2022 — Bernhard Mistlberger, Kevin Zhou

1. Lorentz transformations. (10 points)

Lorentz transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ are linear transformations that leave inner products invariant, meaning that $x^\mu y_\mu = x'^\mu y'_\mu$ for any four-vectors x and y.

a) Show that this implies

$$\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu. \tag{1}$$

Solution: Since the inner product stays the same, we have

$$\eta_{\mu\nu} x^\mu y^\nu = \eta_{\rho\sigma} x'^\rho y'^\sigma = \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu x^\mu y^\nu. \tag{S1}$$

In other words, we have

$$(\eta_{\mu\nu} - \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu) x^\mu y^\nu = 0. \tag{S2}$$

for any four-vectors x and y, which implies the quantity in parentheses vanishes. (Concretely, we could take $x^\mu = \delta^\mu_0$ and $y^\nu = \delta^\nu_0$ to show the $\mu = \nu = 0$ component vanishes, and so on for all 16 pairs of values.)

b) All proper, orthochronous Lorentz transformations (i.e. all those which preserve the orientation of space and the direction of time) can be decomposed into infinitesimal Lorentz transformations. These take the form

$$x^\mu \to x'^\mu = x^\mu + \epsilon\, \omega^\mu{}_\nu x^\nu \tag{2}$$

where ϵ is infinitesimal. Show that $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

Solution: Plugging $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon\, \omega^\mu{}_\nu$ into Eq. (1) yields

$$\eta_{\mu\nu} = \eta_{\rho\sigma}(\delta^\rho_\mu \delta^\sigma_\nu + \epsilon\, \omega^\rho{}_\mu \delta^\sigma_\nu + \epsilon\, \delta^\rho_\mu \omega^\sigma{}_\nu) \tag{S3}$$

where we dropped a term of order ϵ^2. Contracting the indices on the right-hand side,

$$\eta_{\mu\nu} = \eta_{\mu\nu} + \epsilon\, \omega^\rho{}_\mu \eta_{\rho\nu} + \epsilon\, \eta_{\mu\sigma}\omega^\sigma{}_\nu \tag{S4}$$

from which we conclude $\omega_{\nu\mu} + \omega_{\mu\nu} = 0$, as desired.

c) The elements of any infinitesimal Lorentz transformation $\omega^\mu{}_\nu$ can be written as a 4×4 matrix, where μ and ν index the row and column, respectively. For an infinitesimal rotation $\epsilon = d\theta$ about the z-axis, write out this matrix, and denote it by iJ^3 for later. What exponential of J^3 corresponds to a finite rotation by an angle θ?

Solution: An infinitesimal rotation keeps t and z the same, and maps $x \to x + \epsilon y$ and $y \to y - \epsilon x$, so we can immediately write down

$$iJ^3 = \begin{pmatrix} & & & \\ & & 1 & \\ & -1 & & \\ & & & \end{pmatrix}. \tag{S5}$$

The appropriate matrix exponential is $e^{i\theta J^3}$. The purpose of the factor of i in the definition is to ensure J^3 is a Hermitian matrix, in accordance with how it's treated in quantum mechanics.

d) Write down the matrix iK^1 corresponding to an infinitesimal boost by $\epsilon = dv$ about the x-axis. What exponential of K^1 corresponds to a finite boost by a velocity v?

Solution: An infinitesimal boost keeps y and z the same, and maps $x \to x + \epsilon t$ and $t \to t + \epsilon x$, so

$$iK^1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & \\ & & & \end{pmatrix}. \tag{S6}$$

Note that this is *not* an antisymmetric matrix. The quantity that is antisymmetric is $\omega_{\mu\nu}$, with two indices down, but raising an index to get $\omega^{\mu}{}_{\nu}$ produces some signs. The appropriate exponential for a finite boost is $e^{i\alpha K^1}$, where $\alpha = \tanh^{-1} v$ is the rapidity. (The most common mistake was instead writing e^{ivK^1}.)

There's a minor subtlety: the usual taught form of the Lorentz transformation has $x' = \gamma(x - vt)$, which suggests the matrix should have minus signs. But in that form, we're keeping the situation the same, and x' denotes the coordinate seen by an observer moving to the right. In this problem, we're considering the case where there's just one observer but physical objects are boosted to the right. Nonetheless, if you used the "passive" convention instead, it's no big deal since none of the other results in this problem will be affected.

Note that the factor of i in the definition of K^1 is a common convention, but it implies K^1 is *not* Hermitian. This is also sensible, as K^1 doesn't actually correspond to a physical observable. At a deeper level, it is related to the fact that non-compact non-Abelian Lie groups don't have finite-dimensional unitary representations, which will be discussed in section.

e) Defining J^1, J^2, K^2, and K^3 similarly, the generators obey commutation relations

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \qquad [J^i, K^j] = i\epsilon^{ijk} K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk} J^k. \tag{3}$$

This is the Lie algebra of the Lorentz group. Physically, these results tells us that infinitesimal rotations and boosts are vectors (i.e. angular velocity and velocity are vectors), and that composing boosts can yield a rotation. Prove these results for the cases $(i, j) = (1, 1)$ and $(i, j) = (1, 2)$. (The proofs for other cases are similar.)

Solution: The full set of generators is, up to a factor of i,

$$iJ^1 = \begin{pmatrix} & & & \\ & & & \\ & & & 1 \\ & & -1 & \end{pmatrix} \qquad iJ^2 = \begin{pmatrix} & & & \\ & & & -1 \\ & & & \\ & 1 & & \end{pmatrix} \qquad iJ^3 = \begin{pmatrix} & & & \\ & & 1 & \\ & -1 & & \\ & & & \end{pmatrix} \tag{S7}$$

$$iK^1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & \\ & & & \end{pmatrix} \qquad iK^2 = \begin{pmatrix} & & 1 & \\ & & & \\ 1 & & & \\ & & & \end{pmatrix} \qquad iK^3 = \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{pmatrix} \tag{S8}$$

For the case $(i, j) = (1, 1)$, the first and third relations are clearly satisfied, because the right-hand side is just zero, and $[J^1, K^1] = 0$ because these matrices are nonzero only in independent blocks. For the case $(i, j) = (1, 2)$, we need to check $[J^1, J^2] = iJ^3$, $[J^1, K^2] = iK^3$, and $[K^1, K^2] = -iJ^3$, which is straightforward to do manually.

2. Quantization of the complex scalar field. (30 points)

This will be an involved problem, but it will teach you everything there is to know about free field mode expansions. (You don't have to write every detail; when it's clear other computations will go the same way as one you just did, it's fine to just say so and move on.) In problem set 1 we considered a complex scalar field Φ with Lagrangian density

$$\mathcal{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 \Phi^* \Phi, \tag{4}$$

and found the canonical fields Φ and Φ^* and momenta Π and Π^*. Letting $\omega_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + m^2}$, introduce the following mode expansion for the canonical fields and momenta:

$$\Phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + b^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \tag{5}$$

$$\Phi^*(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \tag{6}$$

$$\Pi(\mathbf{x}) = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left[b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \tag{7}$$

$$\Pi^*(\mathbf{x}) = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left[a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - b^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right]. \tag{8}$$

We quantize the fields by imposing the canonical commutation relations

$$[\Phi(\mathbf{x}), \Pi(\mathbf{y})] = [\Phi^*(\mathbf{x}), \Pi^*(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \tag{9}$$

with all other commutators between these fields vanishing. (Technically, we should write Hermitian conjugates $\Phi^\dagger$ and $\Pi^\dagger$ here, since Φ and Π are now operators, but we'll continue to use stars to emphasize the links between the classical and quantum theories.)

a) Show that the operators $a(\mathbf{k})$, $a^\dagger(\mathbf{k})$, $b(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ obey

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{p})\right] = \left[b(\mathbf{k}), b^\dagger(\mathbf{p})\right] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{p}), \tag{10}$$

with all other commutators between these operators vanishing. This implies that we have two sets of independent creation and annihilation operators for each $\mathbf{k}$.

Solution: In order to keep the expressions as simple as possible, we'll define

$$d\mathbf{x} = d^3x, \qquad \bar{d}\mathbf{p} = \frac{d^3p}{(2\pi)^3}, \qquad \not{\delta}(\mathbf{p}) = (2\pi)^3 \delta^{(3)}(\mathbf{p}). \tag{S9}$$

As a simple of this notation, we have

$$\int d\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}} = \not{\delta}(\mathbf{p}), \qquad \int \bar{d}\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{x}} = \delta(\mathbf{x}). \tag{S10}$$

If you just remember that every momentum delta function comes with a slash, and every momentum integral comes with a bar, all the factors of 2π will take care of themselves.

Now, we note that

$$\int d\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}} \Phi(\mathbf{x}) = \int \frac{d\mathbf{x}\, \bar{d}\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}} \left[a(\mathbf{k}) e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} + b^\dagger(\mathbf{k}) e^{-i(\mathbf{k}'-\mathbf{k})\mathbf{x}} \right] \tag{S11}$$

$$= \int \frac{\bar{d}\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}} \left[a(\mathbf{k}) \not{\delta}(\mathbf{k} + \mathbf{k}') + b^\dagger(\mathbf{k}) \not{\delta}(\mathbf{k}' - \mathbf{k}) \right] \tag{S12}$$

$$= \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} (a(-\mathbf{k}) + b^\dagger(\mathbf{k})). \tag{S13}$$

By a similar argument, we have

$$\int d\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}} (i\Pi(\mathbf{x})) = \sqrt{\frac{\omega_{\mathbf{k}}}{2}} (b(-\mathbf{k}) - a^\dagger(\mathbf{k})). \tag{S14}$$

Combining these equations to solve for the creation and annihilation operators yields

$$a(\mathbf{k}) = \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \left[\omega_{\mathbf{k}}\Phi(\mathbf{x}) + i\Pi^*(\mathbf{x})\right], \tag{S15}$$

$$b(\mathbf{k}) = \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \left[\omega_{\mathbf{k}}\Phi^*(\mathbf{x}) + i\Pi(\mathbf{x})\right], \tag{S16}$$

$$a^\dagger(\mathbf{k}) = \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}}\, e^{i\mathbf{k}\cdot\mathbf{x}} \left[\omega_{\mathbf{k}}\Phi^*(\mathbf{x}) - i\Pi(\mathbf{x})\right], \tag{S17}$$

$$b^\dagger(\mathbf{k}) = \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}}\, e^{i\mathbf{k}\cdot\mathbf{x}} \left[\omega_{\mathbf{k}}\Phi(\mathbf{x}) - i\Pi^*(\mathbf{x})\right]. \tag{S18}$$

Let's start by considering commutators involving $a(\mathbf{k})$. We only have nonzero commutators between Φ and Π, and between Φ^* and Π^*. This automatically implies that $[a(\mathbf{k}), a(\mathbf{p})] = [a(\mathbf{k}), b^\dagger(\mathbf{p})] = 0$, leaving just two to manually check. First, we have

$$\begin{aligned}
[a(\mathbf{k}), b(\mathbf{p})] &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{d\mathbf{y}}{\sqrt{2\omega_{\mathbf{p}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{y}} [\omega_{\mathbf{k}}\Phi(\mathbf{x}) + i\Pi^*(\mathbf{x}), \omega_{\mathbf{p}}\Phi^*(\mathbf{y}) + i\Pi(\mathbf{y})] && \text{(S19)}\\
&= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{d\mathbf{y}}{\sqrt{2\omega_{\mathbf{p}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{y}} (\delta(\mathbf{y}-\mathbf{x})(\omega_{\mathbf{p}} - \omega_{\mathbf{k}})) && \text{(S20)}\\
&= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{p}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{x}} (\omega_{\mathbf{p}} - \omega_{\mathbf{k}}) && \text{(S21)}\\
&= \not{\delta}(\mathbf{k}+\mathbf{p})\, \frac{\omega_{\mathbf{p}} - \omega_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{p}}}} = 0 && \text{(S22)}
\end{aligned}$$

where the final step follows from $\omega_{\mathbf{k}} = \omega_{-\mathbf{k}}$. Next, we have

$$\begin{aligned}
[a(\mathbf{k}), a^\dagger(\mathbf{p})] &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{d\mathbf{y}}{\sqrt{2\omega_{\mathbf{p}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}} [\omega_{\mathbf{k}}\Phi(\mathbf{x}) + i\Pi^*(\mathbf{x}), \omega_{\mathbf{p}}\Phi^*(\mathbf{y}) - i\Pi(\mathbf{y})] && \text{(S23)}\\
&= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{d\mathbf{y}}{\sqrt{2\omega_{\mathbf{p}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}} (\delta(\mathbf{y}-\mathbf{x})(\omega_{\mathbf{p}} + \omega_{\mathbf{k}})) && \text{(S24)}\\
&= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{p}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} (\omega_{\mathbf{p}} + \omega_{\mathbf{k}}) && \text{(S25)}\\
&= \not{\delta}(\mathbf{k}-\mathbf{p})\, \frac{\omega_{\mathbf{p}} + \omega_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{p}}}} = \not{\delta}(\mathbf{k}-\mathbf{p}) && \text{(S26)}
\end{aligned}$$

as desired. All of the other commutators can be derived similarly.

The vacuum state $|0\rangle$ is defined to be the unique state where $a(\mathbf{k})|0\rangle = b(\mathbf{k})|0\rangle = 0$ for all $\mathbf{k}$. The states $a^\dagger(\mathbf{k})|0\rangle$ and $b^\dagger(\mathbf{k})|0\rangle$ each contain one particle, while the state $a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0\rangle$ contains two particles, and so on.

b) In problem set 1, you showed that the complex scalar field Lagrangian had the symmetry $\Phi \to e^{i\alpha}\Phi$, $\Phi^* \to e^{-i\alpha}\Phi^*$. Compute the Noether current J^μ and conserved charge Q associated with the symmetry.

Solution: The infinitesimal transformation is $\delta\Phi = i\Phi$ and $\delta\Phi^* = -i\Phi$, so Noether's theorem gives

$$\begin{aligned}
J^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\,\delta\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\,\delta\Phi^* && \text{(S27)}\\
&= i((\partial^\mu\Phi^*)\Phi - (\partial^\mu\Phi)\Phi^*). && \text{(S28)}
\end{aligned}$$

The charge is just the integral of the charge density,

$$Q = i\int d\mathbf{x}\, ((\partial^0\Phi^*)\Phi - (\partial^0\Phi)\Phi^*). \tag{S29}$$

Of course, if we want to evaluate this in the canonically quantized theory, we should express it in terms of the fields and canonical momenta, giving the final answer

$$Q = i\int d\mathbf{x}\, (\Pi(\mathbf{x})\Phi(\mathbf{x}) - \Pi^*(\mathbf{x})\Phi^*(\mathbf{x})). \tag{S30}$$

c) Write Q in terms of the creation and annihilation operators. You should find the result is indeterminate up to a constant; resolve this by defining the vacuum to have zero charge, $Q|0\rangle = 0$. What is the charge in the three other states mentioned above?

Solution: The fundamental reason the answer will be undetermined up to a constant is that Noether's theorem, which applies to classical fields, tells us (S30), but not the order that $\Pi(\mathbf{x})$ and $\Phi(\mathbf{x})$ are multiplied together. In the classical theory, that simply doesn't matter, but in the quantum theory it does because the fields are noncommuting operators. So without further information, the "right" answer in the quantum theory could contain $\Pi(\mathbf{x})\Phi(\mathbf{x})$, or $\Phi(\mathbf{x})\Pi(\mathbf{x})$, or even a mixture like $(\Pi(\mathbf{x})\Phi(\mathbf{x})+\Phi(\mathbf{x})\Pi(\mathbf{x}))/2$.

This is called an ordering ambiguity, and in general there is no way to resolve it. (Ordering information is intrinsically quantum; it simply doesn't exist in the classical theory!) But in simple cases like this, all of the possible orderings give you the same operator Q, just shifted by a constant. So we can fix the ambiguity by demanding the reasonable assumption $Q|0\rangle = 0$, which is known as normal ordering.

Anyway, let's not worry about this for now, and just directly evaluate the result. First,

$$\int d\mathbf{x}\, i\Pi(\mathbf{x})\Phi(\mathbf{x}) = \int \frac{d\mathbf{x}\bar{d}\mathbf{k}\bar{d}\mathbf{k}'}{2}\sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}}\,(b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})(a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) \tag{S31}$$

$$\begin{aligned} &= \int \frac{\bar{d}\mathbf{k}\bar{d}\mathbf{k}'}{2}\sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}}\big[(b(\mathbf{k})a(\mathbf{k}') - a^\dagger(\mathbf{k})b^\dagger(\mathbf{k}'))\bar{\delta}(\mathbf{k}+\mathbf{k}') \\ &\qquad + (b(\mathbf{k})b^\dagger(\mathbf{k}') - a^\dagger(\mathbf{k})a(\mathbf{k}'))\bar{\delta}(\mathbf{k}-\mathbf{k}')\big] \end{aligned} \tag{S32}$$

$$= \int \frac{\bar{d}\mathbf{k}}{2}\,(b(\mathbf{k})a(-\mathbf{k}) - a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k}) + b(\mathbf{k})b^\dagger(\mathbf{k}) - a^\dagger(\mathbf{k})a(\mathbf{k})). \tag{S33}$$

By a very similar computation, the second term is

$$\int d\mathbf{x}\,(-i\Pi^*(\mathbf{x}))\Phi^*(\mathbf{x}) = \int \frac{\bar{d}\mathbf{k}}{2}\,(-a(\mathbf{k})b(-\mathbf{k}) + b(\mathbf{k})^\dagger a(-\mathbf{k})^\dagger - a(\mathbf{k})a(\mathbf{k})^\dagger + b(\mathbf{k})^\dagger b(\mathbf{k})). \tag{S34}$$

The first term of (S33) cancels with the first term of (S34). To see this very explicitly:

$$\int \bar{d}\mathbf{k}\, b(\mathbf{k})a(-\mathbf{k}) = \int \bar{d}\mathbf{k}'\, a(\mathbf{k}')b(-\mathbf{k}') = \int \bar{d}\mathbf{k}\, a(\mathbf{k})b(-\mathbf{k}') \tag{S35}$$

where we changed the integration variable to $\mathbf{k}' = -\mathbf{k}$, used the commutation relation, and then renamed $\mathbf{k}'$ to $\mathbf{k}$. Similarly, the second term of (S33) cancels with the second term of (S34). This leaves

$$Q = \int \frac{\bar{d}\mathbf{k}}{2}(b(\mathbf{k})b^\dagger(\mathbf{k}) + b(\mathbf{k})^\dagger b(\mathbf{k}) - a^\dagger(\mathbf{k})a(\mathbf{k}) - a(\mathbf{k})a(\mathbf{k})^\dagger). \tag{S36}$$

At this point, we need to impose normal ordering. We will have $Q|0\rangle = 0$ if all the annihilation operators are on the right, so we move them to the right and throw out the constants this produces, giving

$$Q = \int \bar{d}\mathbf{k}\,(b^\dagger(\mathbf{k})b(\mathbf{k}) - a^\dagger(\mathbf{k})a(\mathbf{k})). \tag{S37}$$

Evidently, the conserved charge is the number of type b particles minus the number of type a particles. So the charges in the three states mentioned above are -1, 1, and -2, respectively.

d) To check that this is the same symmetry operation that we started out with, we can see how it acts on the operators of the theory. Quantum mechanically, symmetries act on operators by conjugation, and we expect to have

$$e^{i\alpha Q}\,\Phi(\mathbf{x})\,e^{-i\alpha Q} = e^{i\alpha}\,\Phi(\mathbf{x}), \qquad e^{i\alpha Q}\,\Phi^*(\mathbf{x})\,e^{-i\alpha Q} = e^{-i\alpha}\,\Phi^*(\mathbf{x}). \tag{11}$$

Show that this implies the commutation relations

$$[Q, \Phi(\mathbf{x})] = \Phi(\mathbf{x}), \qquad [Q, \Phi^*(\mathbf{x})] = -\Phi^*(\mathbf{x}) \tag{12}$$

and show that these relations hold. Thus, Q generates phase rotations of the field.

Solution: Taking α infinitesimal, we have

$$(1+i\alpha Q)\Phi(\mathbf{x})(1-i\alpha Q) = (1+i\alpha)\Phi(\mathbf{x}) \tag{S38}$$

and matching up the order α terms gives

$$iQ\Phi - i\Phi Q = i\Phi \tag{S39}$$

which is precisely the desired result. The proof for Φ^* is similar.

To show the first commutation relation, we have

$$\begin{aligned}
[Q,\Phi(\mathbf{x})] &= \int \frac{d\!\!{}^-\mathbf{k}\, d\!\!{}^-\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\, [b^\dagger(\mathbf{k})b(\mathbf{k}) - a^\dagger(\mathbf{k})a(\mathbf{k}), a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}] && \text{(S40)}\\
&= \int \frac{d\!\!{}^-\mathbf{k}\, d\!\!{}^-\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}} \left([b^\dagger(\mathbf{k})b(\mathbf{k}), b^\dagger(\mathbf{k}')]e^{-i\mathbf{k}'\cdot\mathbf{x}} - [a^\dagger(\mathbf{k})a(\mathbf{k}), a(\mathbf{k}')]e^{i\mathbf{k}'\cdot\mathbf{x}}\right) && \text{(S41)}\\
&= \int \frac{d\!\!{}^-\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \left(b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\right) = \Phi(\mathbf{x}) && \text{(S42)}
\end{aligned}$$

as desired. The proof of the other commutation relation is similar, but we don't have to write it because it follows immediately from the result we just proved. Taking the Hermitian conjugate of both sides,

$$\Phi^*(\mathbf{x}) = [Q,\Phi(\mathbf{x})]^\dagger = (Q\Phi(\mathbf{x}) - \Phi(\mathbf{x})Q)^\dagger = \Phi^*(\mathbf{x})Q - Q\Phi^*(\mathbf{x}) = -[Q,\Phi^*(\mathbf{x})] \tag{S43}$$

which is the desired result. (Here we used the fact that Q is Hermitian.)

e) Use Noether's theorem to find the stress-energy tensor $T^{\mu\nu}$ and the associated conserved total four-momentum P^μ.

Solution: Using the same procedure as in class, and suppressing $\mathbf{x}$ arguments, we have

$$\begin{aligned}
T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial^\nu\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\partial^\nu\Phi^* - \eta^{\mu\nu}\mathcal{L} && \text{(S44)}\\
&= \partial^\mu\Phi^*\,\partial^\nu\Phi + \partial^\mu\Phi\,\partial^\nu\Phi^* - \eta^{\mu\nu}\left((\partial_\rho\Phi^*)(\partial^\rho\Phi) - m^2\Phi^*\Phi\right). && \text{(S45)}
\end{aligned}$$

The total energy is

$$\begin{aligned}
P^0 &= \int d\mathbf{x}\, T^{00} && \text{(S46)}\\
&= \int d\mathbf{x}\, \partial^0\Phi\partial^0\Phi^* + \partial^0\Phi^*\partial^0\Phi - (\partial_\rho\Phi^*)(\partial^\rho\Phi) + m^2\Phi^*\Phi && \text{(S47)}\\
&= \int d\mathbf{x}\, \Pi^*\,\Pi + (\nabla\Phi^*)\cdot(\nabla\Phi) + m^2\Phi^*\Phi. && \text{(S48)}
\end{aligned}$$

The total momentum is

$$\begin{aligned}
P^i &= \int d\mathbf{x}\, T^{0i} && \text{(S49)}\\
&= \int d\mathbf{x}\, \partial^0\Phi^*\partial^i\Phi + \partial^0\Phi\partial^i\Phi^* && \text{(S50)}\\
&= \int d\mathbf{x}\, \Pi\,\partial^i\Phi + \Pi^*\,\partial^i\Phi^*. && \text{(S51)}
\end{aligned}$$

f) Write the Hamiltonian $H = P^0$ and the spatial momenta $\mathbf{P}$ in terms of the creation and annihilation operators, again defining the vacuum to have zero four-momentum. What is the four-momentum in the three other states mentioned above?

Solution: Starting with the Hamiltonian, we have three terms:

$$\int d\mathbf{x}\,\Pi^*\Pi = -\int d\mathbf{x} đ\mathbf{k} đ\mathbf{k}'\,\frac{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}{2}\,(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})(b(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} - a^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) \quad (S52)$$

$$\int d\mathbf{x}(\nabla\Phi^*)\cdot(\nabla\Phi) = -\int \frac{d\mathbf{x} đ\mathbf{k} đ\mathbf{k}'}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}(\mathbf{k}b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - \mathbf{k}a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})\cdot(\mathbf{k}'a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} - \mathbf{k}'b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) \quad (S53)$$

$$\int d\mathbf{x}\,m^2\Phi^*\Phi = \int \frac{d\mathbf{x} đ\mathbf{k} đ\mathbf{k}'}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}\,m^2(b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})(a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}). \quad (S54)$$

When we do the $d\mathbf{x}$ integrations, each line gives four terms, along with appropriate delta functions,

$$\int d\mathbf{x}\,\Pi^*\Pi = \int \frac{đ\mathbf{k}}{2\omega_{\mathbf{k}}}\,\omega_{\mathbf{k}}^2(-a(\mathbf{k})b(-\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k}) - b^\dagger(\mathbf{k})a^\dagger(-\mathbf{k})) \quad (S55)$$

$$\int d\mathbf{x}\,(\nabla\Phi^*)\cdot(\nabla\Phi) = \int \frac{đ\mathbf{k}}{2\omega_{\mathbf{k}}}|\mathbf{k}|^2(b(\mathbf{k})a(-\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k}) + b(\mathbf{k})b^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})) \quad (S56)$$

$$\int d\mathbf{x}\,m^2\Phi^*\Phi = \int \frac{đ\mathbf{k}}{2\omega_{\mathbf{k}}}\,m^2(b(\mathbf{k})a(-\mathbf{k}) + b(\mathbf{k})b^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k}) + a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})). \quad (S57)$$

Summing the terms, all the cross terms (those with one a and one b) cancel, and normal ordering leaves

$$H = \int \frac{đ\mathbf{k}}{2\omega_{\mathbf{k}}}(\omega_{\mathbf{k}}^2 + |\mathbf{k}|^2 + m^2)(a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k})) \quad (S58)$$

$$= \int đ\mathbf{k}\,\omega_{\mathbf{k}}(a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k})) \quad (S59)$$

This tells us that each one of the a-type and b-type particles has energy $\omega_{\mathbf{k}}$. Thankfully, the momentum is a little easier to deal with. We have

$$\int d\mathbf{x}\,\Pi\,\partial^i\Phi = -i\int \frac{d\mathbf{x} đ\mathbf{k} đ\mathbf{k}'}{2}\sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}}(b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})\partial^i(a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) \quad (S60)$$

$$= \int \frac{d\mathbf{x} đ\mathbf{k} đ\mathbf{k}'}{2}\sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}}k'^i(b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})(-a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) \quad (S61)$$

$$= \int \frac{đ\mathbf{k}}{2}\,k^i(b(\mathbf{k})a(-\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k}) + a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})) \quad (S62)$$

The easy way to handle the other term in P^i is to notice that when you conjugate this term, you almost get the other term, but the operators are in the reverse order. But of course, we're going to fix the ordering ambiguity with normal ordering later, so this doesn't matter. So we can treat the other term as

$$\left(\int d\mathbf{x}\,\Pi\,\partial^i\Phi\right)^\dagger = \int \frac{đ\mathbf{k}}{2}\,k^i(a^\dagger(-\mathbf{k})b^\dagger(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k}) + b(\mathbf{k})b^\dagger(\mathbf{k}) + b(-\mathbf{k})a(\mathbf{k})) \quad (S63)$$

By substituting $\mathbf{k} = -\mathbf{k}'$ and then renaming $\mathbf{k}'$ back to $\mathbf{k}$, the first and last terms here cancel with the first and last terms of (S62). The second and third terms add, and normal ordering gives

$$P^i = \int đ\mathbf{k}\,k^i(a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k})) \quad (S64)$$

which tells us that a-type and b-type particles both carry momentum $\mathbf{k}$. Therefore, the states mentioned above have four-momentum $(\omega_{\mathbf{k}}, \mathbf{k})$, $(\omega_{\mathbf{k}}, \mathbf{k})$, and $(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}, \mathbf{k}_1 + \mathbf{k}_2)$.

g) By definition, the operator H generates time translations of the field – this is the content of the Schrodinger equation, $i\partial_t|\Psi\rangle = H|\Psi\rangle$, which holds unchanged in quantum field theory. As for the momenta $\mathbf{P}$, we expect

$$e^{i\mathbf{a}\cdot\mathbf{P}}\,\Phi(\mathbf{x})\,e^{-i\mathbf{a}\cdot\mathbf{P}} = \Phi(\mathbf{x} - \mathbf{a}) \quad (13)$$

for any vector $\mathbf{a}$, with a similar result for all the other fields. Show that this implies

$$[P^i, \Phi(\mathbf{x})] = -i\partial^i\Phi(\mathbf{x}) \quad (14)$$

and show that this relation holds. Thus, $\mathbf{P}$ generates spatial translations.

Solution: We again take infinitesimal $\mathbf{a}$ and expand, giving

$$(1 + i\mathbf{a} \cdot \mathbf{P})\Phi(\mathbf{x})(1 - i\mathbf{a} \cdot \mathbf{P}) = \Phi(\mathbf{x}) - \mathbf{a} \cdot \nabla\Phi(\mathbf{x}). \tag{S65}$$

Equating the order $\mathbf{a}$ components and taking care to remember that $\partial^i = -\partial_i$ gives the desired result.

By the way, you might wonder why we're treating the Hamiltonian and momentum slightly differently. The reason is that in this problem set, we're still working in Schrodinger picture, where the operators $\Phi(\mathbf{x})$ depend on space but not time. In Heisenberg picture, the Heisenberg equation of motion for operators gives the analogous result $[H, \Phi(\mathbf{x}, t)] = -i\partial^0\Phi(\mathbf{x}, t)$, so that in general we have $[P^\mu, \Phi(x)] = -i\partial^\mu\Phi(x)$.

Now, evaluating the commutator, we have

$$\begin{aligned}
[P^i, \Phi(\mathbf{x})] &= \int đ\mathbf{k}đ\mathbf{k}' \, \frac{k^i}{\sqrt{2\omega_{\mathbf{k}'}}}[a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k}), a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}] && \text{(S66)} \\
&= \int đ\mathbf{k}đ\mathbf{k}' \, \frac{k^i}{\sqrt{2\omega_{\mathbf{k}'}}}[a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k}), a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}] && \text{(S67)} \\
&= \int đ\mathbf{k} \, \frac{k^i}{\sqrt{2\omega_{\mathbf{k}}}}(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}) && \text{(S68)} \\
&= -i\partial^i \int \frac{đ\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}) = -i\partial^i\Phi(\mathbf{x}). && \text{(S69)}
\end{aligned}$$

3. ⋆ Conserved currents of Lorentz transformations. (10 points)

This somewhat tricky problem combines the ideas of the first two. It is completely optional: the problem set will be graded out of 40 points, so that you will receive up to 100% credit if you don't do this problem, and up to 125% credit if you do.

Under a Lorentz transformation, a scalar field profile ϕ gets mapped to ϕ', so that $\phi'(x') = \phi(x)$. This implies that

$$\phi'(x) = \phi(\Lambda^{-1}x). \tag{15}$$

For an infinitesimal Lorentz transformation (2), this corresponds to

$$\phi'(x) = \phi(x) - \epsilon\, \omega^{\mu\nu} x_\nu \partial_\mu \phi(x) \tag{16}$$

to first order in ϵ. Because an infinitesimal Lorentz transformation is parametrized by a rank 2 tensor $\omega^{\mu\nu}$, the corresponding Noether current will be a rank 3 tensor $J^{\mu\nu\rho}$, where the first index is the usual index that comes from Noether's theorem, and the last two describe the Lorentz transformation. For simplicity, you can do the entire problem for a real scalar field. (This corresponds to a complex scalar field with $a(\mathbf{k}) = b(\mathbf{k})$, which in turn implies $\Phi = \Phi^*$ and $\Pi = \Pi^*$.)

a) Show that for a scalar field,

$$J^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} \tag{17}$$

where $T^{\mu\nu}$ is the stress-energy tensor. Let the associated conserved charges be $M^{\mu\nu}$.

Solution: This is just like the derivation of the stress-energy tensor, except that the constant displacement a^μ is replaced with $\omega^{\mu\nu}x_\nu$. Thus, the current for a given ω is

$$J^\mu = \omega^{\rho\nu} x_\nu T^\mu{}_\rho = \frac{1}{2}\omega_{\rho\nu}(T^{\mu\rho}x^\nu - T^{\mu\nu}x^\rho) \tag{S70}$$

where we used the antisymmetry of ω in the second step. The general rank-three conserved tensor can be found by "stripping off" the $\omega_{\rho\nu}$, giving

$$J^{\mu\nu\rho} = T^{\mu\rho}x^{\nu} - T^{\mu\nu}x^{\rho} \tag{S71}$$

as desired.

b) Since $M^{\mu\nu}$ is antisymmetric, there are six independent conserved charges. The three independent M^{ij} physically correspond to the angular momentum of the field. What is the physical meaning of the other three conserved quantities M^{0i}?

Solution: We have

$$M^{0i} = \int d\mathbf{x}\, J^{00i} = \int d\mathbf{x}\, T^{0i}x^0 - T^{00}x^i. \tag{S72}$$

The first term is just the total momentum P^i times the time t. The second term is an integral of the energy density times x^i, so it is proportional to the location of the "center of energy",

$$X^i_{\mathrm{CE}} = \frac{\int d\mathbf{x}\, T^{00}x^i}{\int d\mathbf{x}\, T^{00}} = \frac{\int d\mathbf{x}\, T^{00}x^i}{E} \tag{S73}$$

where E is the total energy. Plugging in these definitions, we have

$$M^{0i} = P^i t - E X^i_{\mathrm{CE}}. \tag{S74}$$

Since this is conserved, its time derivative vanishes, giving

$$\frac{P^i}{E} = \frac{dX^i_{\mathrm{CE}}}{dt}. \tag{S75}$$

In other words, the conservation of M^{0i} means that the center of energy moves at constant speed. This is a very general principle in relativistic theories.

c) Show that after normal ordering, the angular momentum is

$$M^{ij} = i\int \frac{d^3k}{(2\pi)^3}\, a(\mathbf{k})^\dagger \left(k^j \frac{\partial}{\partial k_i} - k^i \frac{\partial}{\partial k_j}\right) a(\mathbf{k}) \tag{18}$$

The form of this answer implies that the particles created and annihilated by scalar fields do not carry any intrinsic angular momentum (i.e. spin).

Solution: We have

$$M^{ij} = \int d\mathbf{x}\, J^{0ij} = \int d\mathbf{x}\, x^i T^{0j} - x^j T^{0i}. \tag{S76}$$

The resulting integrals look very similar to those we already did in 2(f), but the factor of x^i complicates things: it means we can no longer do the $d\mathbf{x}$ integral trivially. The trick is to convert the factor of x^i into a momentum derivative. From a starting point much like (S60), we have

$$\begin{aligned}
\int d\mathbf{x}\, x^i \pi\, \partial^j \phi &= -\frac{i}{2}\int d\mathbf{x} đ\mathbf{k} đ\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} x^i (a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})\partial^j (a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + a^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) && \text{(S77)}\\
&= \frac{1}{2}\int d\mathbf{x} đ\mathbf{k} đ\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} x^i k'^j (a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})(-a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + a^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) && \text{(S78)}\\
&= -\frac{i}{2}\int d\mathbf{x} đ\mathbf{k} đ\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} k'^j (a(\mathbf{k})\partial_{k_i} e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})\partial_{k_i} e^{-i\mathbf{k}\cdot\mathbf{x}}) \\
&\qquad\qquad \times (-a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + a^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) && \text{(S79)}\\
&= -\frac{i}{2}\int đ\mathbf{k} đ\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} k'^j \big((a(\mathbf{k})a(\mathbf{k}') - a^\dagger(\mathbf{k})a^\dagger(\mathbf{k}'))\partial_{k_i} \not{\delta}(\mathbf{k}+\mathbf{k}') \\
&\qquad\qquad + (a(\mathbf{k})a^\dagger(\mathbf{k}') - a^\dagger(\mathbf{k})a(\mathbf{k}'))\partial_{k_i} \not{\delta}(\mathbf{k}-\mathbf{k}')\big). && \text{(S80)}
\end{aligned}$$

To evaluate the delta function, we need to integrate by parts, which causes the ∂_{k_i}'s to hit the rest of the integrand. Now, the ∂_{k_i} can either act on $\sqrt{\omega_{\mathbf{k}}}$ or the ladder operators. If it hits the $\sqrt{\omega_{\mathbf{k}}}$, it produces a factor of k^i which makes the integrand proportional to $k^i k'^j$. However, since M^{ij} involves an antisymmetrization in i and j, these terms don't contribute to the final answer. We only care about the terms where ∂_{k_i} acts on the ladder operators, giving

$$M^{ij} = \frac{i}{2} \int d\!\!\bar{}\mathbf{k}\, k^j (-(\partial_{k_i} a(\mathbf{k}))a(-\mathbf{k}) + (\partial_{k_i} a^\dagger(\mathbf{k}))a^\dagger(-\mathbf{k}) \\ + (\partial_{k_i} a(\mathbf{k}))a^\dagger(\mathbf{k}) - (\partial_{k_i} a^\dagger(\mathbf{k}))a(\mathbf{k})) - (i \leftrightarrow j\,\text{term}). \quad \text{(S81)}$$

Let's first show that the first term vanishes. For convenience we define

$$I^{ij} = \int d\mathbf{k}\, k^j a(-\mathbf{k})\partial_{k_i} a(\mathbf{k}). \quad \text{(S82)}$$

Integrating by parts again, we have

$$\begin{aligned} I^{ij} &= -\int d\mathbf{k}\, \partial_{k_i}(k^j a(-\mathbf{k}))a(\mathbf{k}) && \text{(S83)} \\ &= -\int d\mathbf{k}\, \delta^{ij} a(-\mathbf{k})a(\mathbf{k}) + k^j (\partial_{k_i} a(-\mathbf{k}))a(\mathbf{k}) && \text{(S84)} \\ &= -\int d\mathbf{k}\, \delta^{ij} a(-\mathbf{k})a(\mathbf{k}) + k^j (\partial_{k_i} a(\mathbf{k}))a(-\mathbf{k}) && \text{(S85)} \\ &= -I^{ij} - \int d\mathbf{k}\, \delta^{ij} a(-\mathbf{k})a(\mathbf{k}) && \text{(S86)} \end{aligned}$$

where we did the usual $\mathbf{k} \to -\mathbf{k}$ trick in the third step. This shows that I^{ij} is proportional to δ^{ij}. Since the final answer M^{ij} involves an antisymmetrization in i and j, this shows the first term does not contribute to M^{ij}. Similarly, the second term doesn't contribute either.

You might be worried about what happens to the boundary terms when we integrate by parts. Usually, we justify throwing away boundary terms by saying the integrand goes to zero at infinity. But here, it doesn't: the integrand is always a nonzero operator, for any $\mathbf{k}$. However, as the integrand goes to infinity, we get creation and annihilation operators for extremely high momentum particles. These operators have zero matrix elements if you sandwich them between any two physically accessible states, so they might as well just be zero.

We are now left with the last two terms in (S81). Normal ordering now gives

$$\begin{aligned} M^{ij} &= \frac{i}{2} \int d\!\!\bar{}\mathbf{k}\, k^j (a^\dagger(\mathbf{k})\partial_{k_i} a(\mathbf{k}) - (\partial_{k_i} a^\dagger(\mathbf{k}))a(\mathbf{k})) - (i \leftrightarrow j\,\text{term}) && \text{(S87)} \\ &= \frac{i}{2} \int d\!\!\bar{}\mathbf{k}\, k^j (a^\dagger(\mathbf{k})\partial_{k_i} a(\mathbf{k}) + a^\dagger(\mathbf{k})\partial_{k_i} a(\mathbf{k})) - (i \leftrightarrow j\,\text{term}) && \text{(S88)} \\ &= i \int d\!\!\bar{}\mathbf{k}\, a^\dagger(\mathbf{k})(k^j \partial_{k_i} - k^i \partial_{k_j})a(\mathbf{k}) && \text{(S89)} \end{aligned}$$

where we integrated by parts again. This is the final result.

The only reason this result looks more complicated than our result for the spatial momentum is that we expanded the field in plane wave modes, which are eigenfunctions of translations but not rotations. Alternatively, we could have expanded the field in spherical waves (i.e. equal to a spherical Bessel function in r times a spherical harmonic), in which case the angular momentum expansion looks simple and the momentum expansion looks complicated. Plane waves are useful for describing the initial states in a particle physics experiment, but spherical waves can be useful in other situations, such as the emission of photons from a excited atom.

PHYS-330: QFT I — Problem Set 3 (due 10/18)

Stanford University, 2022 — Bernhard Mistlberger, Kevin Zhou

1. Locality in relativistic quantum field theory. (20 points)

The vacuum two-point correlation function of a real scalar field is

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle. \tag{1}$$

It only depends on the difference of positions $z = x - y$, by translational invariance, and it quantifies correlations between the field values at x and y in the vacuum state.

a) Show that

$$D(z) = \int \frac{d^4p}{(2\pi)^4}\,(2\pi)\delta(p^2 - m^2)\,\theta(p^0)\,e^{-ip\cdot z} \tag{2}$$

where the Heaviside step function θ is 1 if the argument is positive and 0 otherwise.

b) Show that if $z'^\mu = \Lambda^\mu{}_\nu z^\nu$ for a proper orthochronous Lorentz transformation Λ, then $D(z') = D(z)$. (Hint: show that d^4p and $\delta(p^2 - m^2)\,\theta(p^0)$ are each Lorentz invariant.)

c) Show that for spacelike separation, $z^2 < 0$, the correlation function is exponentially decaying but nonzero. You can do this in two ways. First, you can show that

$$D(z) = \frac{m}{4\pi^2\sqrt{-z^2}}\,K_1(\sqrt{-z^2}\,m) \tag{3}$$

where K_1 is a modified Bessel function of the second kind. Alternatively, you may numerically integrate $D(z)$ and graph the result. In both cases you may use any books or computer programs needed, such as Mathematica or Abramowitz and Stegun.

A key requirement for a relativistic theory is that it is local, meaning that effects don't propagate faster than the speed of light. That means any change applied to the field at x should only affect observable results at y if x and y aren't spacelike separated, $(y-x)^2 \geq 0$.

You might thus be concerned about part (c), which implies a field in the vacuum "knows" about the values of the field at spacelike separation. But there's nothing wrong with this. For example, to define a reference frame in special relativity, one synchronizes the clocks by sending light pulses throughout all space. After synchronization, all the clocks "know" about the values on the other clocks, even at spacelike separation. But that doesn't mean that changes propagate faster than light; it's just a consequence of how we set up the system. Similarly, ensuring a field is in the vacuum state requires absorbing all particles throughout all space, and this process sets up correlations between field values.

To test if our theory is local, we must see whether *changes* in the state propagate faster than light; that is the subject of the rest of the question.

d) One way to interact with a quantum field is to measure its value at a point, a process which changes the state. From quantum mechanics, we know that measurements of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ do not affect each other if $[\mathcal{O}_1, \mathcal{O}_2] = 0$. Show that

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = 0 \tag{4}$$

for spacelike separation, $(x-y)^2 < 0$, as expected from locality.

e) Another way to interact with a field is to couple it to a classical source $J(x)$, which for a real scalar field corresponds to taking the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + \phi(x)J(x). \tag{5}$$

This is the scalar analogue of driving the electromagnetic field with a current $J^\mu(x)$. For simplicity, we'll treat $\phi(x)$ as a classical field for now, though similar conclusions will hold when it is a quantum field. Show that the classical equation of motion is

$$(\partial_\mu \partial^\mu + m^2)\,\phi(x) = J(x). \tag{6}$$

f) Show that the equation of motion is solved by

$$\phi(x) = i\int d^4y\, G(x-y)J(y) \tag{7}$$

where G is a Green's function of the Klein–Gordan operator, which means

$$(\partial_\mu \partial^\mu + m^2)\, G(z) = -i\,\delta^{(4)}(z). \tag{8}$$

The factors of $-i$ here are purely conventional, and will simplify results later.

g) By taking Fourier transforms, we may heuristically write the Green's function as

$$G(z) = \int \frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot z}}{p^2 - m^2} \tag{9}$$

which formally obeys Eq. (8). However, this expression is not mathematically well-defined because the integral blows up when $p^2 = m^2$. To get a definite result, we must add "$i\epsilon$" terms to the denominator to keep it from vanishing. There are multiple ways to do this, which physically corresponds to the fact that there are multiple possible Green's functions, depending on the field's boundary conditions. Three key examples are the retarded, advanced, and Feynman Green's functions,

$$G_R(z) = \lim_{\epsilon\to 0^+}\int \frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot z}}{(p^0+i\epsilon)^2 - |\mathbf{p}|^2 - m^2} \tag{10}$$

$$G_A(z) = \lim_{\epsilon\to 0^+}\int \frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot z}}{(p^0-i\epsilon)^2 - |\mathbf{p}|^2 - m^2} \tag{11}$$

$$G_F(z) = \lim_{\epsilon\to 0^+}\int \frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot z}}{p^2 - m^2 + i\epsilon} \tag{12}$$

where the limit notation means ϵ approaches zero from the positive end. By using the residue theorem to perform the integral over p^0, show that

$$G_R(z) = \theta(z^0)(D(z) - D(-z)) \tag{13}$$

$$G_A(z) = \theta(-z^0)(D(-z) - D(z)) \tag{14}$$

$$G_F(z) = \theta(z^0)D(z) + \theta(-z^0)D(-z) \tag{15}$$

h) The retarded Greens function applies when the field is zero before the source acts. Show that $G_R(x-y)$ vanishes for spacelike separation, as expected from locality.

i) The Feynman propagator will play a crucial role when we introduce Feynman diagrams because it is equal to $\langle 0|T\phi(x)\phi(y)|0\rangle$. Does it vanish for spacelike separation?

2. Recovering classical field theory. (10 points)

If an operator $\mathcal{O}$ is time-independent in Schrodinger picture, then in Heisenberg picture,

$$\frac{d\mathcal{O}(t)}{dt} = i\,[H(t), \mathcal{O}(t)]. \tag{16}$$

In quantum field theory, working in the Heisenberg picture allows fields to depend on spacetime, making results look more Lorentz invariant. For example, we saw that in Heisenberg picture, a real scalar field obeys the Klein–Gordan equation $(\partial^2 + m^2)\phi = 0$. Now consider the Lagrangian of Eq. (5), which additionally includes a source term $J(x)$. In this case, the Hamiltonian is explicitly time-dependent,

$$H(t) = H_0 - \int d^3\mathbf{x}\,\phi(\mathbf{x}, t)J(\mathbf{x}, t) \tag{17}$$

where H_0 is the free Hamiltonian.

a) Show that in Heisenberg picture, the field $\phi(x)$ obeys Eq. (6). (Thus, by the logic of problem 1, expectation values of a quantum field $\phi(x)$ respond locally to sources.)

With the source, quantum fields can evolve in time nontrivially. Suppose we start in the vacuum state $|0\rangle$, and at time $t = 0$ apply an impulse to the field via

$$J(x) = \delta(t)\,j(\mathbf{x}). \tag{18}$$

This is the scalar analogue of suddenly turning an electric current on and off in electromagnetism. It is also closely related to the last half of problem 2 of problem set 1.

b) The impulse causes operators to instantaneously shift in value at $t = 0$. Show that

$$\phi(\mathbf{x}, 0^+) = \phi(\mathbf{x}, 0^-), \qquad \pi(\mathbf{x}, 0^+) = \pi(\mathbf{x}, 0^-) + j(\mathbf{x}). \tag{19}$$

Here, 0^+ means a time right after $t = 0$, and 0^- means a time right before, i.e.

$$f(0^+) - f(0^-) = \lim_{\epsilon\to 0^+} \int_{-\epsilon}^{\epsilon} \frac{df(t)}{dt}\,dt \tag{20}$$

for any function of time.

c) Show that this is equivalent to the annihilation operators shifting by

$$a(\mathbf{p}, 0^+) = a(\mathbf{p}, 0^-) + \frac{i}{\sqrt{2\omega_{\mathbf{p}}}}\,\tilde{j}(\mathbf{p}) \tag{21}$$

where $\tilde{j}$ is the Fourier transform of j.

d) Show that the field is in a coherent state after the impulse, in the sense that the state is an eigenvector of $a(\mathbf{p}, 0^+)$. Then evaluate $\langle 0|a^\dagger(\mathbf{p}, 0^+)a(\mathbf{p}, 0^+)|0\rangle$, which gives the number of particles of momentum $\mathbf{p}$ produced, per unit volume of momentum space.

As you saw in problem set 1, coherent states of the quantum harmonic oscillator have the same position and momentum uncertainties as the vacuum state. Similarly, in quantum field theory, coherent states of the field have the same ϕ and π uncertainties as the vacuum, while their expectation values behave classically. When the driving is strong, the uncertainties become negligible compared to the expectation values, and the number of particles becomes large so that we can no longer see their discreteness. We therefore recover a classical field.

3. Practice with time-ordered exponentials. (10 points)

Suppose the Hamiltonian is $H = H_0 + H_{\text{int}}$, where H_0 is a time-independent free Hamiltonian, and H_{int} is an interaction which could be time-dependent. In interaction picture, operators evolve under the free Hamiltonian H_0 alone, and the states are $|\psi(t)\rangle_I$. Let's first review some results derived in lecture.

a) Show that

$$i\,\partial_t|\psi(t)\rangle_I = H_{\text{int},I}(t)|\psi(t)\rangle_I. \tag{22}$$

where $H_{\text{int},I}(t)$ is the interaction Hamiltonian in the interaction picture.

b) Show that if the interaction picture time evolution operator is defined as

$$|\psi(t_f)\rangle_I = U(t_f, t_i)|\psi(t_i)\rangle_I \tag{23}$$

for $t_f > t_i$, then it is given by Dyson's formula,

$$U(t_f, t_i) = T\exp\left(-i\int_{t_i}^{t_f} dt\, H_{\text{int},I}(t)\right) \tag{24}$$

where T denotes time ordering.

Now let's do some concrete calculations with the time evolution operator.

c) Suppose the interaction is only turned on for two moments, i.e. it has the form

$$H_{\text{int},I}(t) = g\,(h_1\,\delta(t-t_1) + h_2\,\delta(t-t_2)). \tag{25}$$

Write out $U(t_f, t_i)$ up to and including terms of order g^2, assuming $t_i < t_1 < t_2 < t_f$.

d) Now consider a general $H_{\text{int},I}(t)$ which is proportional to a coupling g. It is generally true that $U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)$. Explicitly show that this result is true, up to and including terms of order g^2, in the case $t_1 < t_2 < t_3$.

PHYS-330: QFT I — Solutions 3

Stanford University, 2022 — Bernhard Mistlberger, Kevin Zhou

1. Locality in relativistic quantum field theory. (20 points)

The vacuum two-point correlation function of a real scalar field is

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle. \tag{1}$$

It only depends on the difference of positions $z = x - y$, by translational invariance, and it quantifies correlations between the field values at x and y in the vacuum state.

a) Show that

$$D(z) = \int \frac{d^4p}{(2\pi)^4}\,(2\pi)\delta(p^2 - m^2)\,\theta(p^0)\,e^{-ip\cdot z} \tag{2}$$

where the Heaviside step function θ is 1 if the argument is positive and 0 otherwise.

Solution: Using the same $\bar{d}$ and $\not{\delta}$ notation as in the solutions to the second problem set,

$$\begin{aligned}
\langle 0|\phi(x)\phi(y)|0\rangle &= \int \frac{\bar{d}\mathbf{p}\,\bar{d}\mathbf{q}}{\sqrt{2\omega_\mathbf{p}}\sqrt{2\omega_\mathbf{q}}} e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\omega_p x^0} e^{-i\mathbf{q}\cdot\mathbf{y}} e^{i\omega_q y^0} \langle 0|a(\mathbf{q})a^\dagger(\mathbf{p})|0\rangle && \text{(S1)}\\
&= \int \frac{\bar{d}\mathbf{p}\,\bar{d}\mathbf{q}}{\sqrt{2\omega_\mathbf{p}}\sqrt{2\omega_\mathbf{q}}} e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\omega_p x^0} e^{-i\mathbf{q}\cdot\mathbf{y}} e^{i\omega_q y^0} \not{\delta}(\mathbf{p}-\mathbf{q}) && \text{(S2)}\\
&= \int \frac{\bar{d}\mathbf{p}}{2\omega_\mathbf{p}} e^{-i\omega_p z^0} e^{i\mathbf{p}\cdot\mathbf{z}}. && \text{(S3)}
\end{aligned}$$

On the other hand, starting from the desired expression, we have

$$\begin{aligned}
D(z) &= \int \bar{d}\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{z}} \int_0^\infty \frac{dp^0}{2\pi}\,(2\pi)\delta(p^2-m^2) e^{-ip^0 z^0} && \text{(S4)}\\
&= \int \frac{\bar{d}\mathbf{p}}{2\omega_\mathbf{p}}\, e^{-i\omega_p z^0} e^{i\mathbf{p}\cdot\mathbf{z}} && \text{(S5)}
\end{aligned}$$

which matches.

b) Show that if $z'^\mu = \Lambda^\mu{}_\nu z^\nu$ for a proper orthochronous Lorentz transformation Λ, then $D(z') = D(z)$. (Hint: show that d^4p and $\delta(p^2 - m^2)\,\theta(p^0)$ are each Lorentz invariant.)

Solution: For $D(z')$, let's change the integration variable from p to $p'^\mu = \Lambda^\mu{}_\nu p^\nu$. Then

$$\begin{aligned}
D(z') &= \int \frac{d^4p'}{(2\pi)^4}\,(2\pi)\delta(p'^2-m^2)\theta(p'^0) e^{-ip'\cdot z'} && \text{(S6)}\\
&= \int \frac{d^4p'}{(2\pi)^4}\,(2\pi)\delta(p'^2-m^2)\theta(p'^0) e^{-ip\cdot z} && \text{(S7)}
\end{aligned}$$

which is precisely the same thing as $D(z)$ provided that

$$d^4p'\,\delta(p'^2-m^2)\,\theta(p'^0) = d^4p\,\delta(p^2-m^2)\,\theta(p^0). \tag{S8}$$

Lorentz transformations always preserve inner products, $p^2 = p'^2$, so the two delta functions are the same. Furthermore, for timelike momenta (which are the only ones that contribute to the integral, because of the presence of the delta functions), orthochronous Lorentz transformations always preserve the sign of the time component, by definition. Physically this corresponds to the fact that you can't boost a particle from positive energy to negative energy, and you can check it by explicitly computing the effects of Lorentz boosts. (Of course, Lorentz boosts certainly can change the sign of the time component of *spacelike* vectors.)

The tricky part is to show that proper orthochronous Lorentz transformations preserve spacetime volume, $d^4p' = d^4p$. There are a few different ways to show this. The most formal way is to just compute the Jacobian determinant. The Jacobian matrix for the transformations between the variables is just

$$\frac{\partial p'^\mu}{\partial p^\nu} = \Lambda^\mu{}_\nu \tag{S9}$$

and its determinant is

$$J = -\frac{1}{24}\epsilon_{\mu\nu\rho\sigma}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\Lambda^\rho{}_{\rho'}\Lambda^\sigma{}_{\sigma'}\epsilon^{\mu'\nu'\rho'\sigma'} \tag{S10}$$

as you can show by expanding out the definition of the determinant and using the antisymmetry of ϵ. On the other hand, we know the Levi–Civita symbol is an invariant tensor, so we just get

$$J = -\frac{1}{24}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 1. \tag{S11}$$

Another method is to note that in terms of matrix multiplications, $\eta = \Lambda^T\eta\Lambda$, which implies

$$\det\eta = (\det\Lambda^T)(\det\eta)(\det\Lambda) \tag{S12}$$

and therefore $J = \det\Lambda = \pm 1$. Since the proper orthochronous Lorentz transformations are connected to the identity, we must have $J = 1$. Finally, you could just manually show that $J = 1$ when $\Lambda^\mu{}_\nu$ is a rotation or a boost, i.e. by showing that the matrices

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\gamma & \sinh\gamma & & \\ \sinh\gamma & \cosh\gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad \Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta & \sin\theta \\ & & -\sin\theta & \cos\theta \end{pmatrix} \tag{S13}$$

have unit determinant. This is sufficient, since any proper orthochronous Lorentz transformation can be built out of rotations and boosts.

c) Show that for spacelike separation, $z^2 < 0$, the correlation function is exponentially decaying but nonzero. You can do this in two ways. First, you can show that

$$D(z) = \frac{m}{4\pi^2\sqrt{-z^2}}\,K_1(\sqrt{-z^2}\,m) \tag{3}$$

where K_1 is a modified Bessel function of the second kind. Alternatively, you may numerically integrate $D(z)$ and graph the result. In both cases you may use any books or computer programs needed, such as Mathematica or Abramowitz and Stegun.

Solution: By our result from part (b), we can choose a frame where $z^\mu = (0, \mathbf{z})$, and set up spherical coordinates with the axis aligned with $\mathbf{z}$. Defining $r = \sqrt{-z^2} = |\mathbf{z}|$ and $p = |\mathbf{p}|$, we have

$$D(z) = \int \frac{đ\mathbf{p}}{2\omega_{\mathbf{p}}}e^{i\mathbf{p}\cdot\mathbf{z}} \tag{S14}$$

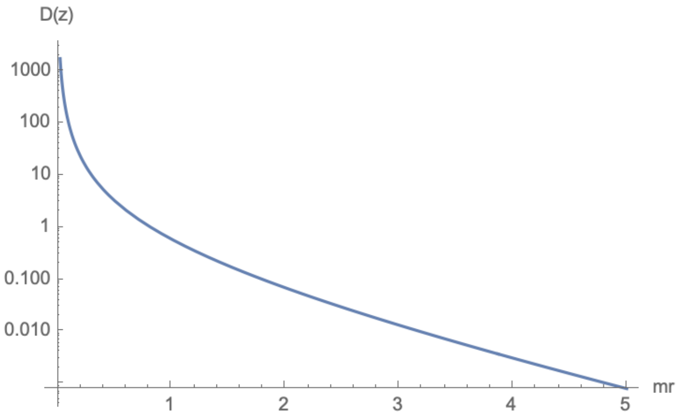
$$= \frac{1}{(2\pi)^3}\int_0^\infty dp\int_{-1}^1 d\,\cos\theta\int_0^{2\pi} d\phi\,\frac{p^2}{2\sqrt{p^2+m^2}}e^{ipr\cos\theta} \tag{S15}$$

$$= \frac{1}{(2\pi)^2}\int_0^\infty dp\,\frac{p}{\sqrt{p^2+m^2}}\frac{\sin(pr)}{r}. \tag{S16}$$

At this point we can straightforwardly evaluate the integral numerically, e.g. in Mathematica.

```
Dz[r_, m_] := NIntegrate[p Sin[p r] / (r Sqrt[p^2 + m^2]),
  {p, 0, Infinity}]
```

```
LogPlot[Dz[r, 1], {r, 0, 5}, AxesLabel → {"mr", "D(z)"}]
```



Alternatively, if we want to stay analytic to the end, a little more work is necessary. First, we replace the integral with an equivalent complex integral over the entire real axis,

$$D(z) = -\frac{i}{8\pi^2 r} \int_{-\infty}^{\infty} dp \, \frac{p \, e^{ipr}}{\sqrt{p^2 + m^2}}. \tag{S17}$$

To evaluate the integral, we deform the contour so that it wraps around the vertical branch cut that starts at $p = im$. Across this branch cut, the square root flips sign, so the contour integral going down the left end of the branch cut is equal to the contour integral going up the right end, giving

$$D(z) = \frac{1}{4\pi^2 r} \int_{m}^{\infty} dx \, \frac{x \, e^{-xr}}{\sqrt{x^2 - m^2}} \tag{S18}$$

where $p = ix$. Finally, looking up this integral in a book or Mathematica yields the final result,

$$D(z) = \frac{m}{4\pi^2 r} K_1(mr). \tag{S19}$$

A key requirement for a relativistic theory is that it is local, meaning that effects don't propagate faster than the speed of light. That means any change applied to the field at x should only affect observable results at y if x and y aren't spacelike separated, $(y-x)^2 \geq 0$.

You might thus be concerned about part (c), which implies a field in the vacuum "knows" about the values of the field at spacelike separation. But there's nothing wrong with this. For example, to define a reference frame in special relativity, one synchronizes the clocks by sending light pulses throughout all space. After synchronization, all the clocks "know" about the values on the other clocks, even at spacelike separation. But that doesn't mean that changes propagate faster than light; it's just a consequence of how we set up the system. Similarly, ensuring a field is in the vacuum state requires absorbing all particles throughout all space, and this process sets up correlations between field values.

To test if our theory is local, we must see whether *changes* in the state propagate faster than light; that is the subject of the rest of the question.

d) One way to interact with a quantum field is to measure its value at a point, a process which changes the state. From quantum mechanics, we know that measurements of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ do not affect each other if $[\mathcal{O}_1, \mathcal{O}_2] = 0$. Show that

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = 0 \tag{4}$$

for spacelike separation, $(x-y)^2 < 0$, as expected from locality.

Solution: We have

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = D(z) - D(-z). \tag{S20}$$

On the other hand, when z is spacelike, it can always be transformed into $-z$ by a Lorentz transformation. (A simple way to see this is to note that both z and $-z$ can be transformed into a standard four-vector like $(0, 0, 0, \sqrt{-z^2})$, by first boosting to get rid of the time component, and rotating to get the spatial component pointing the right way.) Therefore, $D(z) = D(-z)$, so the commutator vanishes.

Note that when z is timelike, you can't Lorentz transform it into $-z$, so the argument that $D(z) = D(-z)$ fails. That makes sense because the commutator *should* be nonzero for timelike separations; if it was zero for all separations, then the theory would be trivial, as no measurements would ever affect anything.

e) Another way to interact with a field is to couple it to a classical source $J(x)$, which for a real scalar field corresponds to taking the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + \phi(x)J(x). \tag{5}$$

This is the scalar analogue of driving the electromagnetic field with a current $J^\mu(x)$. For simplicity, we'll treat $\phi(x)$ as a classical field for now, though similar conclusions will hold when it is a quantum field. Show that the classical equation of motion is

$$(\partial_\mu \partial^\mu + m^2)\,\phi(x) = J(x). \tag{6}$$

Solution: The Euler–Lagrange equations are

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{S21}$$

where we now have

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \partial^\mu \phi, \qquad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi + J \tag{S22}$$

which together give the desired result.

f) Show that the equation of motion is solved by

$$\phi(x) = i \int d^4y\, G(x-y) J(y) \tag{7}$$

where G is a Green's function of the Klein–Gordan operator, which means

$$(\partial_\mu \partial^\mu + m^2)\, G(z) = -i\, \delta^{(4)}(z). \tag{8}$$

The factors of $-i$ here are purely conventional, and will simplify results later.

Solution: We have

$$\begin{aligned} (\partial^2 + m^2)\phi(x) &= i(\partial^2 + m^2) \int d^4y\, G(x-y)J(y) && \text{(S23)} \\ &= i \int d^4y\, (-i\delta^{(4)}(x-y))J(y) && \text{(S24)} \\ &= J(x) && \text{(S25)} \end{aligned}$$

as desired. Note that there is no term from the derivatives acting on the current because the derivatives are with respect to x, while J is a function of y.

g) By taking Fourier transforms, we may heuristically write the Green's function as

$$G(z) = \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot z}}{p^2 - m^2} \tag{9}$$

which formally obeys Eq. (8). However, this expression is not mathematically well-defined because the integral blows up when $p^2 = m^2$. To get a definite result, we must add "$i\epsilon$" terms to the denominator to keep it from vanishing. There are multiple ways to do this, which physically corresponds to the fact that there are multiple possible Green's functions, depending on the field's boundary conditions. Three key examples are the retarded, advanced, and Feynman Green's functions,

$$G_R(z) = \lim_{\epsilon\to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot z}}{(p^0 + i\epsilon)^2 - |\mathbf{p}|^2 - m^2} \tag{10}$$

$$G_A(z) = \lim_{\epsilon\to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot z}}{(p^0 - i\epsilon)^2 - |\mathbf{p}|^2 - m^2} \tag{11}$$

$$G_F(z) = \lim_{\epsilon\to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot z}}{p^2 - m^2 + i\epsilon} \tag{12}$$

where the limit notation means ϵ approaches zero from the positive end. By using the residue theorem to perform the integral over p^0, show that

$$G_R(z) = \theta(z^0)(D(z) - D(-z)) \tag{13}$$

$$G_A(z) = \theta(-z^0)(D(-z) - D(z)) \tag{14}$$

$$G_F(z) = \theta(z^0)D(z) + \theta(-z^0)D(-z) \tag{15}$$

Solution: Applying partial fractions, we have

$$G_R(z) = i \int \frac{d^3p\, dp^0}{(2\pi)^4} \frac{1}{2\omega_p} \left[\frac{1}{(p^0 - \omega_p + i\epsilon)} - \frac{1}{(p^0 + \omega_p + i\epsilon)} \right] e^{-ip\cdot z} \tag{S26}$$

which implies both poles are shifted below the real p^0 axis. Now we perform the integral over p^0. When $z^0 > 0$, the integrand is exponentially damped below the real p^0 axis, so it makes no difference if we replace the contour over the real line with a closed contour which additionally includes a semicircle (of infinite radius) below the real axis, as shown.

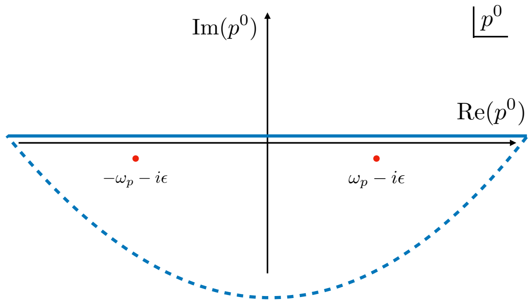


Figure 1: Contour for the retarded propagator for $z^0 > 0$.

This closed contour integral can be calculated using the residue theorem. Each pole contributes $-2\pi i$ times its residue (where the minus sign is because the contour goes around the poles clockwise), giving

$$G_R(z) \;=\; \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{2\omega_p} e^{-i\omega_p z^0 + i\mathbf{p}\cdot\mathbf{z}} - \frac{1}{2\omega_p} e^{i\omega_p z^0 + i\mathbf{p}\cdot\mathbf{z}} \right] \tag{S27}$$

Reindexing $\mathbf{p} \to -\mathbf{p}$ in the second term gives

$$\begin{aligned} G_R(z) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[e^{-i\omega_p z^0 + i\mathbf{p}\cdot\mathbf{z}} - e^{i\omega_p z^0 - i\mathbf{p}\cdot\mathbf{z}} \right] &\text{(S28)} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[e^{-ip\cdot z} - e^{ip\cdot z} \right] = D(z) - D(-z) &\text{(S29)} \end{aligned}$$

where here we are defining $p = (\omega_p, \mathbf{p})$. On the other hand, when $z^0 < 0$, the integral is exponentially damped above the real p^0 axis. We can thus close the contour above the real axis, and it encircles no poles, so the integral vanishes. Thus,

$$G_R(z) = \theta(z^0)(D(z) - D(-z)) \qquad \text{(S30)}$$

as desired. The proof for $G_A(z)$ is extremely similar, but now with both poles above the real axis.

Finally, the Feynman propagator is

$$\begin{aligned} G_F(z) &= i \int \frac{d^4p}{(2\pi)^4} \left[\frac{1}{(p^0 - \omega_p + i\epsilon)(p^0 + \omega_p - i\epsilon)} \right] e^{-ip\cdot z} &\text{(S31)} \\ &= i \int \frac{d^3p\,dp^0}{(2\pi)^4} \frac{1}{2\omega_p} \left[\frac{1}{(p^0 - \omega_p + i\epsilon)} - \frac{1}{(p^0 + \omega_p - i\epsilon)} \right] e^{-ip\cdot z} &\text{(S32)} \end{aligned}$$

which has one pole above the real axis, and one pole below. In figure 2 we show the contours of integration

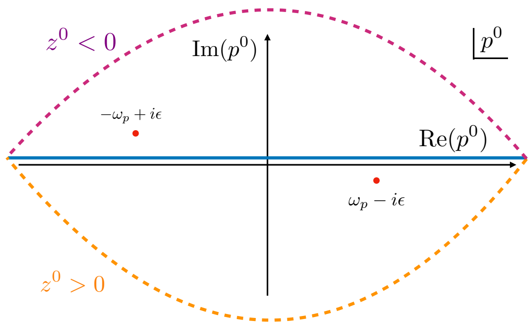


Figure 2: Contour for the Feynman propagator.

for the p^0 integral. If $z^0 > 0$ the dashed, orange line in fig. 2 represents a contour of a semi-circle with infinite radius in the lower half of the complex plane and does not contribute to the integral. The integral along the the real p^0 axis consequently yields minus one times the residue at $p^0 = \omega_p + i\epsilon$ and the other residue does not contribute. Conversely, if $z^0 < 0$, the purple dashed contour in the positive imaginary half-plane contributes zero to the integral and the integral along the real axis yields the residue at $p^0 = -\omega_p + i\epsilon$. We thus conclude

$$\begin{aligned} G_F(z) &= \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_p} \left[\theta(z^0) e^{-ipz} + \theta(-z^0) e^{ipz} \right] &\text{(S33)} \\ &= \theta(z^0) D(z) + \theta(-z^0) D(-z). &\text{(S34)} \end{aligned}$$

h) The retarded Greens function applies when the field is zero before the source acts. Show that $G_R(x-y)$ vanishes for spacelike separation, as expected from locality.

Solution: When z^0 is negative, $G_R(z)$ is just zero by the definition of the step function. When z^0 is positive, $G_R(z)$ is just the same thing as $\langle 0|[\phi(x), \phi(y)]|0\rangle$, which we showed was zero for spacelike separation in part (d).

i) The Feynman propagator will play a crucial role when we introduce Feynman diagrams because it is equal to $\langle 0|T\phi(x)\phi(y)|0\rangle$. Does it vanish for spacelike separation?

Solution: It doesn't vanish for spacelike separation. For example, when z^0 is positive, the Feynman propagator is just $D(z)$, and we already showed in part (c) that this can be nonzero for spacelike separations. For this reason, the Feynman propagator has little use in classical field theory (which is why it doesn't show up in electromagnetism classes, while the retarded and advanced propagators do). But it will be useful when we set up perturbation theory.

Some students tried to solve this problem by setting $z^0 = 0$, but this is a bit tricky because the step functions are discontinuous there. There's no need to do this, because we already showed that $D(z)$ can be nonzero for general spacelike separation, not just when $z^0 = 0$.

2. Recovering classical field theory. (10 points)

If an operator $\mathcal{O}$ is time-independent in Schrodinger picture, then in Heisenberg picture,

$$\frac{d\mathcal{O}(t)}{dt} = i\,[H(t), \mathcal{O}(t)]. \tag{16}$$

In quantum field theory, working in the Heisenberg picture allows fields to depend on spacetime, making results look more Lorentz invariant. For example, we saw that in Heisenberg picture, a real scalar field obeys the Klein–Gordon equation $(\partial^2 + m^2)\phi = 0$. Now consider the Lagrangian of Eq. (5), which additionally includes a source term $J(x)$. In this case, the Hamiltonian is explicitly time-dependent,

$$H(t) = H_0 - \int d^3\mathbf{x}\, \phi(\mathbf{x}, t) J(\mathbf{x}, t) \tag{17}$$

where H_0 is the free Hamiltonian.

a) Show that in Heisenberg picture, the field $\phi(x)$ obeys Eq. (6). (Thus, by the logic of problem 1, expectation values of a quantum field $\phi(x)$ respond locally to sources.)

Solution: Note that the new term in the Hamiltonian commutes with $\phi(\mathbf{y}, t)$ for any $\mathbf{y}$. Therefore, the Heisenberg equation of motion for ϕ is exactly the same as before,

$$\dot{\phi}(x) = \pi(x) \tag{S35}$$

As for the canonical momentum, we have

$$\dot{\pi}(x) = \nabla^2\phi(x) - m^2\phi(x) + i\left[-\int d^3\mathbf{y}\, \phi(\mathbf{y}, t) J(\mathbf{y}, t), \pi(\mathbf{x}, t)\right] \tag{S36}$$

$$= \nabla^2\phi(x) - m^2\phi(x) + (-i)(i)\int d^3\mathbf{y}\, \delta^{(3)}(\mathbf{y} - \mathbf{x}) J(\mathbf{y}, t) \tag{S37}$$

$$= \nabla^2\phi(x) - m^2\phi(x) + J(x). \tag{S38}$$

Combining these two equations gives the result.

With the source, quantum fields can evolve in time nontrivially. Suppose we start in the vacuum state $|0\rangle$, and at time $t = 0$ apply an impulse to the field via

$$J(x) = \delta(t)\, j(\mathbf{x}). \tag{18}$$

This is the scalar analogue of suddenly turning an electric current on and off in electromagnetism. It is also closely related to the last half of problem 2 of problem set 1.

b) The impulse causes operators to instantaneously shift in value at $t = 0$. Show that

$$\phi(\mathbf{x}, 0^+) = \phi(\mathbf{x}, 0^-), \qquad \pi(\mathbf{x}, 0^+) = \pi(\mathbf{x}, 0^-) + j(\mathbf{x}). \tag{19}$$

Here, 0^+ means a time right after $t = 0$, and 0^- means a time right before, i.e.

$$f(0^+) - f(0^-) = \lim_{\epsilon \to 0^+} \int_{-\epsilon}^{\epsilon} \frac{df(t)}{dt}\, dt \tag{20}$$

for any function of time.

Solution: Integrating the Heisenberg equation of motion for an infinitesimal time centered at $t = 0$,

$$\phi(\mathbf{x}, 0^+) - \phi(\mathbf{x}, 0^-) = -i\left[\int d^3\mathbf{y}\, \phi(\mathbf{y}, 0) j(\mathbf{y}), \phi(\mathbf{x}, 0)\right] = 0. \tag{S39}$$

On the other hand, we have

$$\pi(\mathbf{x}, 0^+) - \pi(\mathbf{x}, 0^-) = -i\left[\int d^3\mathbf{y}\, \phi(\mathbf{y}, 0) j(\mathbf{y}), \pi(\mathbf{x}, 0)\right] \tag{S40}$$

$$= -i \int d^3\mathbf{y}\, i\delta(\mathbf{y} - \mathbf{x}) j(\mathbf{y}) \tag{S41}$$

$$= j(\mathbf{x}). \tag{S42}$$

c) Show that this is equivalent to the annihilation operators shifting by

$$a(\mathbf{p}, 0^+) = a(\mathbf{p}, 0^-) + \frac{i}{\sqrt{2\omega_{\mathbf{p}}}} \tilde{j}(\mathbf{p}) \tag{21}$$

where $\tilde{j}$ is the Fourier transform of j.

Solution: First, note that

$$a^\dagger(\mathbf{p}, 0^+) = a^\dagger(\mathbf{p}, 0^-) - \frac{i}{\sqrt{2\omega_{\mathbf{p}}}} \tilde{j}^*(\mathbf{p}) \tag{S43}$$

Furthermore, because $j(\mathbf{x})$ is real, we have

$$\tilde{j}^*(\mathbf{p}) = \left(\int d\mathbf{x}\, j(\mathbf{x}) e^{-i\mathbf{p}\cdot\mathbf{x}}\right)^* = \int d\mathbf{x}\, j(\mathbf{x}) e^{i\mathbf{p}\cdot\mathbf{x}} = \tilde{j}(-\mathbf{p}). \tag{S44}$$

The change in the field operator is

$$\phi(\mathbf{x}, 0^+) - \phi(\mathbf{x}, 0^-) = \int \frac{đ\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \left(\frac{i\,\tilde{j}(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}} e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{i\,\tilde{j}^*(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}} e^{-i\mathbf{k}\cdot\mathbf{x}}\right). \tag{S45}$$

Substituting $\mathbf{k}' = -\mathbf{k}$ in the second term and renaming $\mathbf{k}'$ to $\mathbf{k}$ gives

$$\phi(\mathbf{x}, 0^+) - \phi(\mathbf{x}, 0^-) = i \int \frac{đ\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \left(\frac{\tilde{j}(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}} e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{\tilde{j}^*(-\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}} e^{i\mathbf{k}\cdot\mathbf{x}}\right) = 0. \tag{S46}$$

On the other hand, we have

$$\pi(\mathbf{x}, 0^+) - \pi(\mathbf{x}, 0^-) = -i \int đ\mathbf{k} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left(\frac{i\,\tilde{j}(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}} e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{-i\,\tilde{j}^*(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}} e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \tag{S47}$$

$$= \frac{1}{2} \int đ\mathbf{k}\, \left(\tilde{j}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + \tilde{j}^*(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \tag{S48}$$

$$= \frac{1}{2} \int đ\mathbf{k}\, (\tilde{j}(\mathbf{k}) + \tilde{j}^*(-\mathbf{k})) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{S49}$$

$$= j(\mathbf{x}) \tag{S50}$$

as desired.

d) Show that the field is in a coherent state after the impulse, in the sense that the state is an eigenvector of $a(\mathbf{p}, 0^+)$. Then evaluate $\langle 0|a^\dagger(\mathbf{p}, 0^+)a(\mathbf{p}, 0^+)|0\rangle$, which gives the number of particles of momentum $\mathbf{p}$ produced, per unit volume of momentum space.

Solution: Because we are in the vacuum state before the impulse, $a(\mathbf{p}, 0^-)|0\rangle = 0$, we have

$$a(\mathbf{p}, 0^+)|0\rangle = \frac{i}{\sqrt{2\omega_\mathbf{p}}}\,\tilde{j}(\mathbf{p})|0\rangle \tag{S51}$$

which implies the field is in a coherent state after the impulse. Then we have

$$\langle 0|a^\dagger(\mathbf{p}, 0^+)a(\mathbf{p}, 0^+)|0\rangle = \langle 0|\,\frac{(-i)\,\tilde{j}^*(\mathbf{p})}{\sqrt{2\omega_\mathbf{p}}}\,\frac{i\,\tilde{j}(\mathbf{p})}{\sqrt{2\omega_\mathbf{p}}}|0\rangle = \frac{|\tilde{j}(\mathbf{p})|^2}{2\omega_\mathbf{p}}. \tag{S52}$$

Incidentally, the fact that the field is in a coherent state implies that the number of particles in a given range of momenta is Poisson distributed.

As you saw in problem set 1, coherent states of the quantum harmonic oscillator have the same position and momentum uncertainties as the vacuum state. Similarly, in quantum field theory, coherent states of the field have the same ϕ and π uncertainties as the vacuum, while their expectation values behave classically. When the driving is strong, the uncertainties become negligible compared to the expectation values, and the number of particles becomes large so that we can no longer see their discreteness. We therefore recover a classical field.

3. Practice with time-ordered exponentials. (10 points)

Suppose the Hamiltonian is $H = H_0 + H_{\text{int}}$, where H_0 is a time-independent free Hamiltonian, and H_{int} is an interaction which could be time-dependent. In interaction picture, operators evolve under the free Hamiltonian H_0 alone, and the states are $|\psi(t)\rangle_I$. Let's first review some results derived in lecture.

a) Show that

$$i\,\partial_t|\psi(t)\rangle_I = H_{\text{int},I}(t)|\psi(t)\rangle_I. \tag{22}$$

where $H_{\text{int},I}(t)$ is the interaction Hamiltonian in the interaction picture.

Solution: By definition, matrix elements of operators are the same in all pictures, so for any states $|\psi\rangle$ and $|\psi'\rangle$, and any operator $\mathcal{O}$, we have

$${}_I\langle\psi'(t)|\mathcal{O}_I(t)|\psi(t)\rangle_I = {}_S\langle\psi'(t)|\mathcal{O}_S|\psi(t)\rangle_S. \tag{S53}$$

On the other hand, we know that

$$\mathcal{O}_I(t) = e^{iH_0t}\mathcal{O}_S e^{-iH_0t} \tag{S54}$$

so comparing the expressions gives

$$|\psi(t)\rangle_I = e^{iH_0t}|\psi(t)\rangle_S. \tag{S55}$$

Applying $i\,\partial_t$ to both sides gives

$$\begin{aligned}
i\,\partial_t|\psi(t)\rangle_I &= -H_0e^{iH_0t}|\psi(t)\rangle_S + ie^{iH_0t}\partial_t|\psi(t)\rangle_S && \text{(S56)}\\
&= -H_0|\psi(t)\rangle_I + e^{iH_0t}H_S|\psi(t)\rangle_S && \text{(S57)}\\
&= -e^{iH_0t}H_0e^{-iH_0t}|\psi(t)\rangle_I + e^{iH_0t}H_Se^{-iH_0t}|\psi(t)\rangle_I && \text{(S58)}\\
&= H_{\text{int},I}(t)|\psi(t)\rangle_I && \text{(S59)}
\end{aligned}$$

as desired.

b) Show that if the interaction picture time evolution operator is defined as

$$|\psi(t_f)\rangle_I = U(t_f, t_i)|\psi(t_i)\rangle_I \tag{23}$$

for $t_f > t_i$, then it is given by Dyson's formula,

$$U(t_f, t_i) = T \exp\left(-i \int_{t_i}^{t_f} dt\, H_{\text{int},I}(t)\right) \tag{24}$$

where T denotes time ordering.

Solution: Abbreviating $H_{\text{int},I}(t)$ as $H(t)$, note that the time-ordered exponential is

$$U(t_f, t_i) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_i}^{t_f} dt_n \int_{t_i}^{t_f} dt_{n-1} \cdots \int_{t_i}^{t_f} dt_1\, T(H(t_n)H(t_{n-1}) \cdots H(t_1)). \tag{S60}$$

We can incorporate the effect of the time ordering by changing the integration bounds,

$$U(t_f, t_i) = \sum_{n=0}^{\infty} (-i)^n \int_{t_i}^{t_f} dt_n \int_{t_i}^{t_n} dt_{n-1} \cdots \int_{t_i}^{t_2} dt_1\, H(t_n)H(t_{n-1}) \cdots H(t_1) \tag{S61}$$

where the factor of $1/n!$ cancelled because the new integration region has $1/n!$ times the volume of the original one. Now, part (a) tells us that we want to prove

$$i\partial_{t_f} U(t_f, t_i) = H(t_f)U(t_f, t_i). \tag{S62}$$

When we differentiating Eq. (S61) with respect to t_f, we only pick up one term from the upper bound of the outer integral, giving

$$i\partial_{t_f} U(t_f, t_i) = H(t_f) \sum_{n=1}^{\infty} (-i)^{n-1} \int_{t_i}^{t_f} dt_{n-1} \cdots \int_{t_i}^{t_2} dt_1\, H(t_{n-1}) \cdots H(t_1). \tag{S63}$$

Shifting the sum in n downward by one recovers $U(t_f, t_i)$, as desired.

Now let's do some concrete calculations with the time evolution operator.

c) Suppose the interaction is only turned on for two moments, i.e. it has the form

$$H_{\text{int},I}(t) = g\,(h_1\,\delta(t - t_1) + h_2\,\delta(t - t_2)). \tag{25}$$

Write out $U(t_f, t_i)$ up to and including terms of order g^2, assuming $t_i < t_1 < t_2 < t_f$.

Solution: We have

$$\int_{t_i}^{t_f} dt\, H_{\text{int},I}(t) = g(h_1 + h_2) \tag{S64}$$

so the exponential without the time-ordering would be

$$1 - ig(h_1 + h_2) - \frac{g^2}{2}(h_1 + h_2)^2 + O(g^3). \tag{S65}$$

The time-ordering puts contributions that occur later on the left, which means all the h_2 terms are to the left of the h_1 terms, giving

$$1 - ig(h_1 + h_2) - \frac{g^2}{2}(h_1^2 + 2h_2h_1 + h_2^2) + O(g^3). \tag{S66}$$

d) Now consider a general $H_{\text{int},I}(t)$ which is proportional to a coupling g. It is generally true that $U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)$. Explicitly show that this result is true, up to and including terms of order g^2, in the case $t_1 < t_2 < t_3$.

Solution: The desired right-hand side is

$$U(t_3, t_1) = 1 - ig \int_{t_1}^{t_3} dt\, H(t) - g^2 \int_{t_1}^{t_3} dt \int_{t_1}^{t} dt'\, H(t)H(t') + O(g^3). \tag{S67}$$

On the left-hand side, expanding the product of the two terms gives

$$\begin{aligned} U(t_3, t_2)U(t_2, t_1) &= 1 - ig \int_{t_1}^{t_2} dt\, H(t) - ig \int_{t_2}^{t_3} dt\, H(t) \\ - g^2 \int_{t_1}^{t_2} dt \int_{t_2}^{t_3} dt'\, H(t)H(t') &- g^2 \int_{t_1}^{t_2} dt \int_{t_1}^{t} dt'\, H(t)H(t') - g^2 \int_{t_2}^{t_3} dt \int_{t_2}^{t} dt'\, H(t)H(t') + O(g^3). \end{aligned} \tag{S68}$$

Clearly, the constant and $O(g)$ terms agree, so the tricky part is showing the equality of the $O(g^2)$ terms,

$$\begin{aligned} \int_{t_1}^{t_3} dt \int_{t_1}^{t} dt'\, H(t)H(t') = &\int_{t_1}^{t_2} dt \int_{t_2}^{t_3} dt'\, H(t)H(t') \\ &+ \int_{t_1}^{t_2} dt \int_{t_1}^{t} dt'\, H(t)H(t') + \int_{t_2}^{t_3} dt \int_{t_2}^{t} dt'\, H(t)H(t') \end{aligned} \tag{S69}$$

The left-hand side integrates over all $t_1 < t' < t < t_3$, and t_2 is between t_1 and t_3. All terms on the right-hand side enforce $t' < t$. The second term includes the region where both t and t' are less than t_2, the third term includes the regions where both are greater than t_2, and the first term corresponds to when t_2 is between t and t'.

PHYS-330: QFT I — Problem Set 4 (due 10/25)

Stanford University, 2022 — Bernhard Mistlberger, Kevin Zhou

1. Using Wick's theorem. (7 points)

For both parts of this problem, you can use Wick's theorem for two real scalar fields.

a) Show that $T(\phi(x_1)\phi(x_2))$ and $:\phi(x_1)\phi(x_2):$ both remain the same when x_1 and x_2 are exchanged. Using these results, explain why $D_F(x_1 - x_2) = D_F(x_2 - x_1)$.

b) Prove Wick's theorem for a time ordered product of three scalar fields,

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) = :\phi(x_1)\phi(x_2)\phi(x_3): + \phi(x_1)D_F(x_2 - x_3) + \phi(x_2)D_F(x_3 - x_1) + \phi(x_3)D_F(x_1 - x_2). \quad (1)$$

2. Diagrams in ϕ^3 theory. (18 points)

Consider a real scalar field ϕ of mass m with a cubic self-interaction, so that

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2, \qquad \mathcal{L}_{\text{int}} = \frac{\lambda}{3!}\,\phi^3. \quad (2)$$

In this problem, you will draw some diagrams in this interacting theory and compute some symmetry factors. As in lecture, $|0\rangle$ is the free vacuum, the interaction Hamiltonian density is $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$, and H_I is the interaction Hamiltonian in interaction picture.

a) Using Wick's theorem, evaluate the vacuum correlation function

$$\langle 0|\, T\exp\left(-i\int dt\, H_I(t)\right)|0\rangle \quad (3)$$

up to and including terms of order λ^2. Your result should be an explicit expression in terms of integrals of the position-space Feynman propagator. (You will run into terms proportional to $D_F(0)$, which is infinite. Don't worry about this for now; we'll come back to this issue when we discuss renormalization.)

b) Draw all distinct diagrams that contribute to the vacuum correlation function at order λ^3 and λ^4. For this part, you do not need to evaluate the diagrams or compute any symmetry factors, but you should neatly display all of your diagrams in one box.

c) Now consider the two-point correlation function

$$\langle\Omega|T\left(\phi_H(x)\phi_H(y)\right)|\Omega\rangle = \frac{\langle 0|T\left(\phi_I(x)\phi_I(y)\exp\left(-i\int dt\, H_I\right)\right)|0\rangle}{\langle 0|T\exp\left(-i\int dt\, H_I\right)|0\rangle} \quad (4)$$

where ϕ_H and ϕ_I are Heisenberg and interaction picture fields. As discussed in class, only connected diagrams contribute to the left-hand side, i.e. diagrams where all fields are connected to at least one of the external points. Find all diagrams that contribute up to and including order λ^4, along with their symmetry factors, which you can compute with any method. Again, give your answer by drawing everything neatly inside one box. (Hint: there are 13 fully connected diagrams, i.e. diagrams where all fields are connected to *both* of the external points.)

3. Recovering nonrelativistic quantum mechanics. (15 points)

A free complex scalar field Φ of mass m satisfies the Klein–Gordan equation, so its plane wave modes $e^{-ip\cdot x}$ satisfy $\omega^2 = |\mathbf{k}|^2 + m^2$. This implies we have plane waves with both positive and negative ω, which is universal in relativistic theories. Upon quantization, excitations of these modes correspond to matter and antimatter, with opposite charges.

The nonrelativistic limit is $|\mathbf{k}| \ll m$, but since antimatter is inherently a feature of relativistic theories, we should also throw away the negative ω modes. To do this, take

$$\Phi(x) = e^{-imt}\,\chi(x)/\sqrt{2m} \tag{5}$$

and assume $|\dot{\chi}| \ll m\chi$, so that only nonrelativistic positive ω modes are excited. The $\sqrt{2m}$ factor transfers us back to nonrelativistic normalization.

a) Simplify the complex scalar field action to

$$S = \int d^4x \left(i\chi^*\dot{\chi} - \frac{1}{2m}\nabla\chi^* \cdot \nabla\chi \right) \tag{6}$$

by dropping a term that is small for $|\dot{\chi}| \ll m\chi$.

b) Find the Euler–Lagrange equations for χ and χ^*, and the conserved current $J^\mu = (\rho, \mathbf{J})$ corresponding to the symmetry $\chi \to e^{-i\alpha}\chi$ and $\chi^* \to e^{i\alpha}\chi^*$. (Be careful to account for minus signs from the relativistic metric.)

For the relativistic complex scalar field, we had to canonically quantize Φ, Π, Φ^*, and Π^* as two pairs of phase space variables. But for this nonrelativistic field, the canonical momentum for χ is just $i\chi^*$, so we only have one pair. This is because we threw away the negative ω solutions, and it implies the canonical commutation relations are

$$[\chi(\mathbf{x}), \chi^*(\mathbf{y})] = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{7}$$

with all other commutators vanishing. These can be satisfied with the mode expansion

$$\chi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, a(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}}, \qquad \chi^*(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, a^\dagger(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{8}$$

c) Show that $[a(\mathbf{p}), a^\dagger(\mathbf{q})] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})$. (Of course, the other commutators vanish.) Then show that the single-particle states $|\mathbf{x}\rangle = \chi^*(\mathbf{x})|0\rangle$ and $|\mathbf{p}\rangle = a^\dagger(\mathbf{p})|0\rangle$ behave like nonrelativistic position and momentum states, obeying

$$\langle \mathbf{y}|\mathbf{x}\rangle = \delta^{(3)}(\mathbf{y} - \mathbf{x}), \quad \langle \mathbf{q}|\mathbf{p}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{q} - \mathbf{p}), \quad \langle \mathbf{p}|\mathbf{x}\rangle = e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{9}$$

The factors of 2π are just due to our Fourier transform convention.

d) A general single-particle state takes the form

$$|\psi(t)\rangle = \int d^3\mathbf{x}\, \psi(\mathbf{x}, t)|\mathbf{x}\rangle \tag{10}$$

where $\psi(\mathbf{x}, t)$ is the particle's position-space wavefunction at time t. As always in quantum theories, the state evolves over time as $i\,\partial_t|\psi\rangle = H|\psi\rangle$, where H is the Hamiltonian. Write this equation as a partial differential equation involving $\psi(\mathbf{x}, t)$.

e) Take the term in the action you dropped in part (a) and simplify it by applying the Euler–Lagrange equations you found in part (b). Find how the Hamiltonian in part (d) changes when this term is added. What is its physical interpretation?

4. ⋆ The force between sources. (5 points)

In this optional problem, we'll compute a directly measurable quantity with quantum field theory, for the first time in this course. So far we have only learned to compute correlation functions, which often do not have a direct physical interpretation. For example, the vacuum correlation function Eq. (3) can be used to find the energy of the interacting vacuum relative to the free vacuum, but this isn't directly measurable. (You can't turn off the interaction in real life.)

Instead, let's again consider a free real scalar field with source $J(x)$,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + \phi(x)J(x). \tag{11}$$

If we change the vacuum by including a static source $J(x)$, the resulting shift in vacuum energy can be directly interpreted as the energy of the source, which is measurable.

a) Using Wick's theorem, show that

$$\langle 0|\, T \exp\left(i \int d^4x\, \phi(x)J(x)\right)|0\rangle = \exp\left(-\frac{1}{2}\int d^4x\, d^4y\, J(x)D_F(x-y)J(y)\right). \tag{12}$$

b) Now consider the case where two point sources are turned on for a long time T,

$$J(x) = g\, f(t)\,(\delta^{(3)}(\mathbf{x}) + \delta^{(3)}(\mathbf{x}-\mathbf{R})), \qquad f(t) = \begin{cases} 1 & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}. \tag{13}$$

If you plug this into Eq. (12), you'll get infinite terms involving $D_F(0)$, reflecting the fact that point sources have infinite energy. Discarding these terms so we can focus on the sources' interaction energy, evaluate Eq. (12) assuming $T \gg R \gg 1/m$.

c) Relate your answer to the interaction energy $V(R)$ between the sources, and find it. Is the force between them attractive or repulsive?

1. Using Wick's theorem. (7 points)

For both parts of this problem, you can use Wick's theorem for two real scalar fields.

a) Show that $T(\phi(x_1)\phi(x_2))$ and $:\phi(x_1)\phi(x_2):$ both remain the same when x_1 and x_2 are exchanged. Using these results, explain why $D_F(x_1-x_2) = D_F(x_2-x_1)$.

Solution: For the time ordered product this is true by definition; it puts the later operator on the left no matter what their original order was. As for the normal ordered product, we can split each field into parts with only creation and annihilation operators,

$$\phi(x_i) = \phi_i^+ + \phi_i^-, \qquad 0 = \phi_i^-|0\rangle = \langle 0|\phi_i^+. \tag{S1}$$

The normal ordered product is defined by

$$:\phi_1\phi_2: = \phi_1^+\phi_2^+ + \phi_1^+\phi_2^- + \phi_2^+\phi_1^- + \phi_1^-\phi_2^-. \tag{S2}$$

Since $[\phi_i^+, \phi_j^+] = [\phi_i^-, \phi_j^-] = 0$, this expression is identical if we exchange x_1 and x_2. Finally, Wick's theorem tells us that the Feynman propagator is the difference of a time ordered product and a normal ordered product, so it is also symmetric in its arguments.

b) Prove Wick's theorem for a time ordered product of three scalar fields,

$$\begin{aligned} T(\phi(x_1)\phi(x_2)\phi(x_3)) = \; &:\phi(x_1)\phi(x_2)\phi(x_3): + \phi(x_1)D_F(x_2-x_3) \\ &+ \phi(x_2)D_F(x_3-x_1) + \phi(x_3)D_F(x_1-x_2). \end{aligned} \tag{1}$$

Solution: We simply work through the cases for the time ordering of x_1, x_2, and x_3. By definition,

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) = \phi_1 T\{\phi_2\phi_3\}\Big|_{t_1>t_2,t_3} + \phi_2 T\{\phi_1\phi_3\}\Big|_{t_2>t_1,t_3} + \phi_3 T\{\phi_1\phi_2\}\Big|_{t_3>t_2,t_1}. \tag{S3}$$

Now let's apply Wick's theorem to the last term,

$$\phi_3 T\{\phi_1\phi_2\}\Big|_{t_3>t_2,t_1} = (\phi_3^+ + \phi_3^-)(\phi_1^+\phi_2^+ + \phi_1^+\phi_2^- + \phi_2^+\phi_1^- + \phi_1^-\phi_2^- + D_F(x_2-x_3))\Big|_{t_3>t_2,t_1}. \tag{S4}$$

A term is normal ordered if all the ϕ_i^+'s are to the left of all the ϕ_i^-'s, so

$$\begin{aligned} :\phi_1\phi_2\phi_3: = \; &(\phi_3^+ + \phi_3^-)(\phi_1^+\phi_2^+ + \phi_1^+\phi_2^- + \phi_2^+\phi_1^- + \phi_1^-\phi_2^-)\Big|_{t_3>t_2,t_1} \\ &+ [\phi_3^-, \phi_2^+]\phi_1 + [\phi_3^-, \phi_1^+]\phi_2\Big|_{t_3>t_2,t_1}. \end{aligned} \tag{S5}$$

We have therefore shown that

$$\begin{aligned} \phi_3 T\{\phi_1\phi_2\}\Big|_{t_3>t_2,t_1} = \; &\big(:\phi_1\phi_2\phi_3: + \phi_1[\phi_3^-, \phi_2^+] + \phi_2[\phi_3^-, \phi_1^+] + \phi_3 D_F(x_1-x_2)\big) \\ &\times \theta(x_3^0 - x_2^0)\theta(x_3^0 - x_1^0). \end{aligned} \tag{S6}$$

When we add the other two terms, the first terms of each one combine to give $:\phi_1\phi_2\phi_3:$ with no step functions. In addition, pairs of commutator terms combine into Feynman propagators by

$$D_F(x,y) = \theta(x^0-y^0)[\phi^-(x), \phi^+(y)] + \theta(y^0-x^0)[\phi^-(y), \phi^+(x)]. \tag{S7}$$

To see how this works, let's consider the terms that eventually combine to yield $\phi(x_3)D_F(x_1-x_2)$,

$$\begin{aligned} \phi_3\Big(&D_F(x_1-x_2)\theta(t_3-t_2)\theta(t_3-t_1) \\ &+ [\phi_1^-, \phi_2^+]\theta(t_1-t_3)\theta(t_1-t_2) + [\phi_2^-, \phi_1^+]\theta(t_2-t_1)\theta(t_2-t_3)\Big). \end{aligned} \tag{S8}$$

The first term here is the desired term when t_3 is the latest time. The second and third terms combine to give the Feynman propagator for any ordering of t_1 and t_2, under the condition that t_3 is *not* the latest. So the sum is $\phi(x_3)D_F(x_1 - x_2)$, and the logic for the other terms is similar.

2. Diagrams in ϕ^3 theory. (18 points)

Consider a real scalar field ϕ of mass m with a cubic self-interaction, so that

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2, \qquad \mathcal{L}_{\text{int}} = \frac{\lambda}{3!}\phi^3. \tag{2}$$

In this problem, you will draw some diagrams in this interacting theory and compute some symmetry factors. As in lecture, $|0\rangle$ is the free vacuum, the interaction Hamiltonian density is $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$, and H_I is the interaction Hamiltonian in interaction picture.

a) Using Wick's theorem, evaluate the vacuum correlation function

$$\langle 0| T \exp\left(-i\int dt\, H_I(t)\right)|0\rangle \tag{3}$$

up to and including terms of order λ^2. Your result should be an explicit expression in terms of integrals of the position-space Feynman propagator. (You will run into terms proportional to $D_F(0)$, which is infinite. Don't worry about this for now; we'll come back to this issue when we discuss renormalization.)

Solution: Let the desired correlation function be $c_0 + \lambda c_1 + \lambda^2 c_2 + \mathcal{O}(\lambda^3)$. Then

$$c_0 = \langle 0|0\rangle = 1 \tag{S9}$$

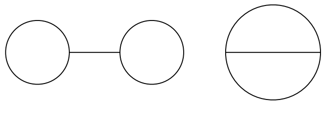
and

$$c_1 = \frac{i}{3!}\langle 0|T\left(\int d^4x\, \phi(x)\phi(x)\phi(x)\right)|0\rangle = 0 \tag{S10}$$

because we have an odd number of copies of ϕ. Finally,

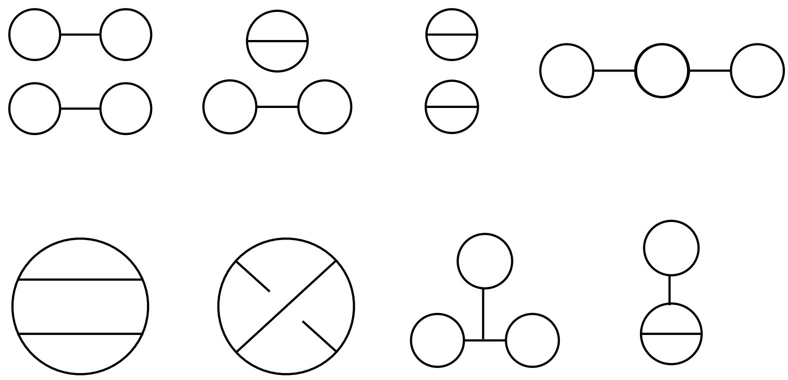
$$\begin{aligned} c_2 &= \frac{1}{2!}\left(\frac{i}{3!}\right)^2 \langle 0|T\left(\int d^4x\, d^4y\, \phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\right)|0\rangle & \text{(S11)} \\ &= -\frac{1}{72}\int d^4x\, d^4y\, \left[3\cdot 3\cdot D_F(x-x)D_F(x-y)D_F(y-y) + 3!\cdot D_F(x-y)^3\right] & \text{(S12)} \\ &= -\int d^4x\, d^4y\, \left(\frac{1}{8}D_F(0)^2 D_F(x-y) + \frac{1}{12}D_F(x-y)^3\right). & \text{(S13)} \end{aligned}$$

The terms in c_2 correspond to the diagrams shown below.



b) Draw all distinct diagrams that contribute to the vacuum correlation function at order λ^3 and λ^4. For this part, you do not need to evaluate the diagrams or compute any symmetry factors, but you should neatly display all of your diagrams in one box.

Solution: There are no diagrams at any odd order. The 8 diagrams at order λ^4 are shown below.

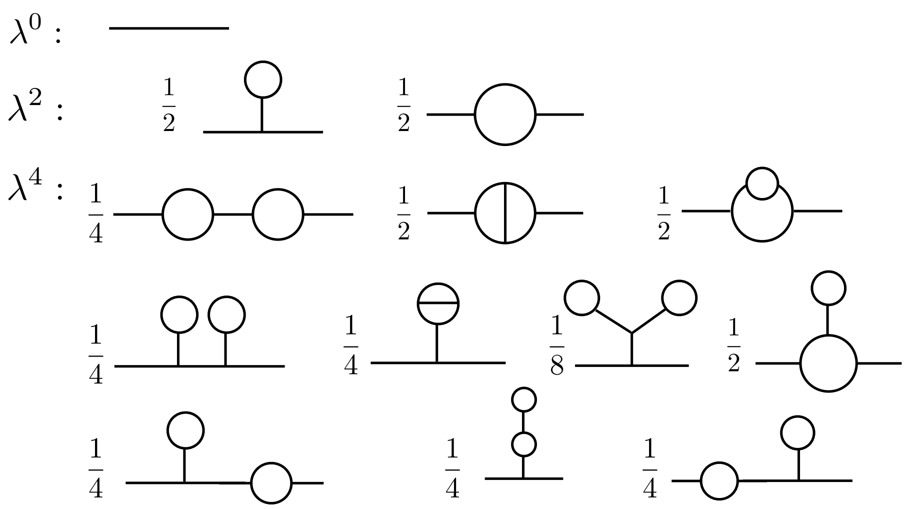


c) Now consider the two-point correlation function

$$\langle\Omega|T\left(\phi_H(x)\phi_H(y)\right)|\Omega\rangle = \frac{\langle 0|T\left(\phi_I(x)\phi_I(y)\exp\left(-i\int dt\, H_I\right)\right)|0\rangle}{\langle 0|T\exp\left(-i\int dt\, H_I\right)|0\rangle} \tag{4}$$

where ϕ_H and ϕ_I are Heisenberg and interaction picture fields. As discussed in class, only connected diagrams contribute to the left-hand side, i.e. diagrams where all fields are connected to at least one of the external points. Find all diagrams that contribute up to and including order λ^4, along with their symmetry factors, which you can compute with any method. Again, give your answer by drawing everything neatly inside one box. (Hint: there are 13 fully connected diagrams, i.e. diagrams where all fields are connected to *both* of the external points.)

Solution: Again, odd orders in λ always yield correlation functions on the right-hand side with an odd number of fields, which vanish since $\langle 0|\phi|0\rangle = 0$. The fully connected diagrams that contribute at orders λ^0, λ^2, and λ^4 are shown below, along with their symmetry factors.



These symmetry factors can all be found with the rules discussed in class:

- $1/2$ for every vertex connected to itself.
- $1/k!$ for k propagators connecting the same two vertices.

- 1/2 for two interchangeable vertices.

Of course, you can also compute them by hand if you want to make sure. **(to do: also include the four non-fully connected diagrams)**

3. Recovering nonrelativistic quantum mechanics. (15 points)

A free complex scalar field Φ of mass m satisfies the Klein–Gordan equation, so its plane wave modes $e^{-ip\cdot x}$ satisfy $\omega^2 = |\mathbf{k}|^2 + m^2$. This implies we have plane waves with both positive and negative ω, which is universal in relativistic theories. Upon quantization, excitations of these modes correspond to matter and antimatter, with opposite charges.

The nonrelativistic limit is $|\mathbf{k}| \ll m$, but since antimatter is inherently a feature of relativistic theories, we should also throw away the negative ω modes. To do this, take

$$\Phi(x) = e^{-imt}\,\chi(x)/\sqrt{2m} \tag{5}$$

and assume $|\dot{\chi}| \ll m\chi$, so that only nonrelativistic positive ω modes are excited. The $\sqrt{2m}$ factor transfers us back to nonrelativistic normalization.

a) Simplify the complex scalar field action to

$$S = \int d^4x\, \left(i\chi^*\dot{\chi} - \frac{1}{2m}\nabla\chi^*\cdot\nabla\chi\right) \tag{6}$$

by dropping a term that is small for $|\dot{\chi}| \ll m\chi$.

Solution: The Lagrangian is

$$\begin{aligned}
\mathcal{L} &= (\partial_\mu\Phi)(\partial^\mu\Phi)^* - m^2\Phi^*\Phi && \text{(S14)}\\
&= \frac{1}{2m}(\partial_\mu(e^{-imt}\chi))(\partial^\mu(e^{imt}\chi^*)) - \frac{m}{2}\chi^*\chi && \text{(S15)}\\
&= -\frac{1}{2m}\nabla\chi\cdot\nabla\chi^* + \frac{1}{2m}(\partial_t(e^{-imt}\chi))(\partial_t(e^{imt}\chi^*)) - \frac{m}{2}\chi^*\chi && \text{(S16)}
\end{aligned}$$

where we split the summation over μ into a sum over $\mu = i$ and the $\mu = 0$ term. The middle term is

$$\frac{1}{2m}(\dot{\chi} - im\chi)(\dot{\chi}^* + im\chi^*) = \frac{\dot{\chi}^*\dot{\chi}}{2m} + \frac{i}{2}(\dot{\chi}\chi^* - \dot{\chi}^*\chi) + \frac{m}{2}\chi^*\chi. \tag{S17}$$

Of these terms, the first is negligible in the nonrelativistic limit, and the last term cancels with the last term of Eq. (S16). To get the second term in the desired form, we integrate the action by parts,

$$\frac{i}{2}\int d^4x\,(\dot{\chi}\chi^* - \dot{\chi}^*\chi) = i\int d^4x\,\chi^*\dot{\chi}. \tag{S18}$$

b) Find the Euler–Lagrange equations for χ and χ^*, and the conserved current $J^\mu = (\rho, \mathbf{J})$ corresponding to the symmetry $\chi \to e^{-i\alpha}\chi$ and $\chi^* \to e^{i\alpha}\chi^*$. (Be careful to account for minus signs from the relativistic metric.)

Solution: As usual, we treat χ and χ^* as independent variables. For both parts of this question, we need to compute the derivatives of $\mathcal{L}$ with respect to $\partial_\mu\chi$ and $\partial_\mu\chi^*$. These are

$$\frac{\partial\mathcal{L}}{\partial\dot{\chi}} = i\chi^*, \qquad \frac{\partial\mathcal{L}}{\partial(\nabla\chi)} = -\frac{1}{2m}\nabla\chi^* \tag{S19}$$

and

$$\frac{\partial\mathcal{L}}{\partial\dot{\chi}^*} = 0, \qquad \frac{\partial\mathcal{L}}{\partial(\nabla\chi^*)} = -\frac{1}{2m}\nabla\chi. \tag{S20}$$

The Euler–Lagrange equation for χ is

$$\frac{\partial \mathcal{L}}{\partial \chi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi)} = \partial_t \frac{\partial \mathcal{L}}{\partial \dot{\chi}} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial(\nabla \chi)}. \tag{S21}$$

The left-hand side is zero, and plugging in the results for the right-hand side gives

$$i\dot{\chi}^* = \frac{1}{2m}\nabla^2 \chi^*. \tag{S22}$$

Similarly, the Euler–Lagrange equation for χ^* is

$$\frac{\partial \mathcal{L}}{\partial \chi^*} = \partial_t \frac{\partial \mathcal{L}}{\partial \dot{\chi}^*} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial(\nabla \chi^*)}. \tag{S23}$$

Now the left-hand side is nonzero, but the first term on the right-hand side is zero, giving

$$i\dot{\chi} = -\frac{1}{2m}\nabla^2 \chi. \tag{S24}$$

As expected, the two Euler–Lagrange equations are conjugates of each other, and the equation of motion for χ is just the Schrodinger equation.

For infinitesimal α, the change in the fields is $\delta\chi = -i\alpha\chi$ and $\delta\chi^* = i\alpha\chi^*$, so Noether's theorem gives

$$J^\mu = -i\chi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi)} + i\chi^* \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi^*)}. \tag{S25}$$

The density of the conserved charge is

$$\rho = -i\chi \frac{\partial \mathcal{L}}{\partial \dot{\chi}} + i\chi^* \frac{\partial \mathcal{L}}{\partial \dot{\chi}^*} = \chi^* \chi. \tag{S26}$$

The current density is

$$\mathbf{J} = -i\chi \frac{\partial \mathcal{L}}{\partial(\nabla \chi)} + i\chi^* \frac{\partial \mathcal{L}}{\partial(\nabla \chi^*)} = \frac{i}{2m}(\chi \nabla \chi^* - \chi^* \nabla \chi). \tag{S27}$$

These are the same form as the probability density and probability current in nonrelativistic quantum mechanics. (But here they don't mean the same thing: we're still talking about classical field theory, not quantum mechanics.)

For the relativistic complex scalar field, we had to canonically quantize Φ, Π, Φ^*, and Π^* as two pairs of phase space variables. But for this nonrelativistic field, the canonical momentum for χ is just $i\chi^*$, so we only have one pair. This is because we threw away the negative ω solutions, and it implies the canonical commutation relations are

$$[\chi(\mathbf{x}), \chi^*(\mathbf{y})] = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{7}$$

with all other commutators vanishing. These can be satisfied with the mode expansion

$$\chi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, a(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}}, \qquad \chi^*(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, a^\dagger(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{8}$$

c) Show that $[a(\mathbf{p}), a^\dagger(\mathbf{q})] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$. (Of course, the other commutators vanish.) Then show that the single-particle states $|\mathbf{x}\rangle = \chi^*(\mathbf{x})|0\rangle$ and $|\mathbf{p}\rangle = a^\dagger(\mathbf{p})|0\rangle$ behave like nonrelativistic position and momentum states, obeying

$$\langle \mathbf{y}|\mathbf{x}\rangle = \delta^{(3)}(\mathbf{y} - \mathbf{x}), \quad \langle \mathbf{q}|\mathbf{p}\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{q} - \mathbf{p}), \quad \langle \mathbf{p}|\mathbf{x}\rangle = e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{9}$$

The factors of 2π are just due to our Fourier transform convention.

Solution: We use the same $đ$ and $\delta\!\!\!/$ notation as in the solutions to the second problem set. Inverting the Fourier transform gives

$$a(\mathbf{p}) = \int d\mathbf{x}\, \chi(\mathbf{x}) e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{S28}$$

which implies

$$[a(\mathbf{p}), a^\dagger(\mathbf{q})] = \int d\mathbf{x}\, d\mathbf{y}\, [\chi(\mathbf{x}), \chi^*(\mathbf{y})]\, e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{y}} \tag{S29}$$

$$= \int d\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{x}} \tag{S30}$$

$$= \delta\!\!\!/(\mathbf{p} - \mathbf{q}). \tag{S31}$$

Now, we have

$$\langle \mathbf{y}|\mathbf{x}\rangle = \langle 0|\chi^*(\mathbf{y})\chi(\mathbf{x})|0\rangle = \delta(\mathbf{y} - \mathbf{x}) \tag{S32}$$

where we applied the commutation relation and used $\chi^*(\mathbf{y})|0\rangle = 0$. Next,

$$\langle \mathbf{q}|\mathbf{p}\rangle = \langle 0|a(\mathbf{q})a^\dagger(\mathbf{p})|0\rangle = \delta\!\!\!/(\mathbf{q} - \mathbf{p}) \tag{S33}$$

where we applied the commutation relation and used $a(\mathbf{q})|0\rangle = 0$. Finally,

$$\langle \mathbf{p}|\mathbf{x}\rangle = \langle 0|a(\mathbf{p}) \int đ\mathbf{q}\, a^\dagger(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}}|0\rangle = e^{i\mathbf{p}\cdot\mathbf{x}} \tag{S34}$$

d) A general single-particle state takes the form

$$|\psi(t)\rangle = \int d^3\mathbf{x}\, \psi(\mathbf{x}, t)|\mathbf{x}\rangle \tag{10}$$

where $\psi(\mathbf{x}, t)$ is the particle's position-space wavefunction at time t. As always in quantum theories, the state evolves over time as $i\,\partial_t|\psi\rangle = H|\psi\rangle$, where H is the Hamiltonian. Write this equation as a partial differential equation involving $\psi(\mathbf{x}, t)$.

Solution: First, we need to find the Hamiltonian. Applying the Legendre transform (or equivalently considering the element T^{00} of the stress-energy tensor),

$$\mathcal{H} = \dot{\chi}\frac{\partial \mathcal{L}}{\partial \dot{\chi}} - \mathcal{L} = \frac{1}{2m}\nabla\chi^* \cdot \nabla\chi. \tag{S35}$$

Therefore, in terms of creation and annihilation operators, the Hamiltonian is

$$H = \frac{1}{2m}\int d\mathbf{x}\, \nabla\chi^* \cdot \nabla\chi \tag{S36}$$

$$= \frac{1}{2m}\int d\mathbf{x} \int đ\mathbf{p}\, đ\mathbf{q}\, a^\dagger(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}\, (i\mathbf{p}\cdot(-i\mathbf{q}))\, a(\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{x}} \tag{S37}$$

$$= \frac{1}{2m}\int d\mathbf{x} \int đ\mathbf{p}\, đ\mathbf{q}\, (\mathbf{p}\cdot\mathbf{q})\, e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\, a^\dagger(\mathbf{p}) a(\mathbf{q}) \tag{S38}$$

$$= \frac{1}{2m}\int đ\mathbf{p}\, |\mathbf{p}|^2\, a^\dagger(\mathbf{p}) a(\mathbf{p}). \tag{S39}$$

The Hamiltonian acts on position eigenstates as

$$H|\mathbf{x}\rangle = \frac{1}{2m}\int đ\mathbf{p}\, |\mathbf{p}|^2\, a^\dagger(\mathbf{p}) a(\mathbf{p}) \int đ\mathbf{q}\, a^\dagger(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}}|0\rangle \tag{S40}$$

$$= \frac{1}{2m}\int đ\mathbf{p}\, |\mathbf{p}|^2\, a^\dagger(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}|0\rangle \tag{S41}$$

$$= -\frac{\nabla^2}{2m}\int đ\mathbf{p}\, a^\dagger(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}|0\rangle \tag{S42}$$

$$= -\frac{\nabla^2}{2m}|\mathbf{x}\rangle. \tag{S43}$$

Finally, the time evolution equation is

$$\int d\mathbf{x}\, i\dot{\psi}\, |\mathbf{x}\rangle = \int d\mathbf{x}\, \psi\, H|\mathbf{x}\rangle = -\frac{1}{2m}\int d\mathbf{x}\, \psi\, \nabla^2|\mathbf{x}\rangle = -\frac{1}{2m}\int d\mathbf{x}\, (\nabla^2\psi)|\mathbf{x}\rangle \tag{S44}$$

where we integrated by parts in the last step. Matching the coefficients of $|\mathbf{x}\rangle$ gives

$$i\dot{\psi} = -\frac{\nabla^2\psi}{2m} \tag{S45}$$

which is of course the nonrelativistic single-particle Schrodinger equation. Thus, our nonrelativistic quantum scalar field theory contains single-particle nonrelativistic quantum mechanics. In addition, it naturally contains states with arbitrarily many particles.

Notice that the single-particle quantum wavefunction ψ obeys the same equation of motion as the classical field χ, and both have a conserved current of the same form. This formal correspondence is a general structural feature of quantum field theory, and one of the original clues used to construct it. It is also the reason the pioneers of quantum field theory called it "second quantization," under the mistaken impression they were quantizing ψ a second time rather than χ for a first time.

e) Take the term in the action you dropped in part (a) and simplify it by applying the Euler–Lagrange equations you found in part (b). Find how the Hamiltonian in part (d) changes when this term is added. What is its physical interpretation?

Solution: We had dropped the term $\dot{\chi}^*\dot{\chi}/2m$, and plugging in the Euler–Lagrange equations gives

$$\mathcal{L} \supset \frac{1}{8m^3}\nabla^2\chi^*\, \nabla^2\chi. \tag{S46}$$

This contributes an additional term to the Hamiltonian,

$$H \supset -\frac{1}{8m^3}\int d\mathbf{x}\, \nabla^2\chi^*\, \nabla^2\chi \tag{S47}$$

and following the same steps as in part (d) yields

$$H \supset -\frac{1}{8m^3}\int đ\mathbf{p}\, |\mathbf{p}|^4\, a^\dagger(\mathbf{p})a(\mathbf{p}). \tag{S48}$$

This is simply the leading relativistic correction to the kinetic energy, which you often see, e.g. when computing the fine structure of the hydrogen atom. Applying a similar (but more technically complex) procedure to the Dirac equation, known as the Foldy—Wouthuysen transformation, one can derive the fine structure corrections for the hydrogen atom. You can continue turning the crank to get relativistic corrections at all orders.

4. ⋆ The force between sources. (5 points)

In this optional problem, we'll compute a directly measurable quantity with quantum field theory, for the first time in this course. So far we have only learned to compute correlation functions, which often do not have a direct physical interpretation. For example, the vacuum correlation function Eq. (3) can be used to find the energy of the interacting vacuum relative to the free vacuum, but this isn't directly measurable. (You can't turn off the interaction in real life.)

Instead, let's again consider a free real scalar field with source $J(x)$,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + \phi(x)J(x). \tag{11}$$

If we change the vacuum by including a static source $J(x)$, the resulting shift in vacuum energy can be directly interpreted as the energy of the source, which is measurable.

a) Using Wick's theorem, show that

$$\langle 0|\, T \exp\left(i \int d^4x\, \phi(x) J(x)\right) |0\rangle = \exp\left(-\frac{1}{2} \int d^4x\, d^4y\, J(x) D_F(x-y) J(y)\right). \tag{12}$$

Solution: Again, odd-order terms vanish, so the left-hand side is

$$\sum_n \frac{i^{2n}}{(2n)!} \int d^4x_1 \cdots d^4x_{2n}\, J(x_1) \cdots J(x_{2n}) \langle 0|\, T\phi(x_1) \cdots \phi(x_{2n})|0\rangle. \tag{S49}$$

There are $(2n-1)(2n-3)\cdots(3)(1)$ possible Wick contractions, all of which yield the same thing, giving

$$\sum_n \frac{i^{2n}}{(2n)(2n-2)\cdots(2)} \left(\int d^4x\, d^4y\, J(x) D_F(x-y) J(y)\right)^n. \tag{S50}$$

Simplifying the numeric factors gives

$$\sum_n \frac{1}{n!} \left(-\frac{1}{2} \int d^4x\, d^4y\, J(x) D_F(x-y) J(y)\right)^n \tag{S51}$$

which is indeed the Taylor series of the desired right-hand side.

b) Now consider the case where two point sources are turned on for a long time T,

$$J(x) = g\, f(t)\, (\delta^{(3)}(\mathbf{x}) + \delta^{(3)}(\mathbf{x} - \mathbf{R})), \qquad f(t) = \begin{cases} 1 & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}. \tag{13}$$

If you plug this into Eq. (12), you'll get infinite terms involving $D_F(0)$, reflecting the fact that point sources have infinite energy. Discarding these terms so we can focus on the sources' interaction energy, evaluate Eq. (12) assuming $T \gg R \gg 1/m$.

Solution: It's helpful to go to momentum space, where we get

$$\exp\left(-\frac{i}{2} \int d^4x\, d^4y\, J(x) J(y) \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}\right). \tag{S52}$$

Taking only the two interaction terms and imposing the position-space delta functions gives

$$\exp\left(-ig^2 \int dx^0\, dy^0\, f(x^0) f(y^0) \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip^0(x^0-y^0)} e^{-i\mathbf{p}\cdot\mathbf{R}}}{p^2 - m^2 + i\epsilon}\right). \tag{S53}$$

At this point, we can perform the temporal integrals; they are a representation of the delta function,

$$\int_{-T/2}^{T/2} dx^0 \int_{-T/2}^{T/2} dy^0\, e^{-ip^0(x^0-y^0)} = 2\pi T\, \delta(p^0) \tag{S54}$$

for large T, where the normalization can be found by integrating both sides against p^0. We thus get

$$\exp\left(ig^2 T \int \frac{d^3p}{(2\pi)^3} \frac{e^{-i\mathbf{p}\cdot\mathbf{R}}}{|\mathbf{p}|^2 + m^2}\right) \tag{S55}$$

where we dropped the $i\epsilon$ since the denominator won't vanish. This is a standard integral you can look up or do with spherical coordinates and contour integration (we'll see it again later in the course), so

$$\exp\left(ig^2 T\, \frac{e^{-mr}}{4\pi r}\right). \tag{S56}$$

c) Relate your answer to the interaction energy $V(R)$ between the sources, and find it. Is the force between them attractive or repulsive?

Solution: If the interaction energy is V, then the correlation function should be e^{-iVT}, reflecting the change in energy of the vacuum. Comparing with above expression gives

$$V(r) = -\frac{g^2 e^{-mr}}{4\pi r}. \tag{S57}$$

The force falls off exponentially on the scale $1/m$, and it is attractive. (Incidentally, since we just have a free field theory and classical sources, we could have done this all much more easily in classical field theory. But the approach we've just taken is much more powerful and general.)

PHYS-330: QFT I — Problem Set 5 (due 11/1)

Stanford University, 2022 — Bernhard Mistlberger, Kevin Zhou

1. Mandelstam variables. (4 points)

Consider any $2 \to 2$ scattering process, where the two incoming particles have momenta p_1 and p_2, and the two outgoing particles have momenta p_3 and p_4. By momentum conservation, $p_1 + p_2 = p_3 + p_4$, and $p_i^2 = m_i^2$ where m_i is the mass of particle i. In this situation, it is often useful to work in terms of the Mandelstam variables

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2. \tag{1}$$

a) Show that $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$.

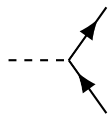
b) In the center of mass frame, the total energy is E_{cm} and the angle of $\mathbf{p}_1$ to $\mathbf{p}_3$ is θ. Write s, t, and u in terms of E_{cm} and θ, assuming all four particles are massless.

2. Scalar Yukawa amplitudes. (7 points)

Let ϕ be a real scalar field of mass M, and ψ be a complex scalar field of mass m, with

$$\mathcal{L}_{\mathrm{int}} = -g\,\psi^* \psi \phi. \tag{2}$$

The field ψ can annihilate a particle, or create a particle with opposite $U(1)$ charge; these particles are conventionally called the ψ and ψ^*, respectively. Similarly, the field ϕ can create or annihilate a ϕ particle. (That is, particles are named after the field that annihilates them.) As discussed in section, the Feynman rules are

 $= \dfrac{i}{p^2 - M^2 + i\epsilon}$  $= \dfrac{i}{p^2 - m^2 + i\epsilon}$  $= -ig$ (3)

where a dashed line stands for ϕ and a solid line stands for ψ.

a) Assuming $M > 2m$, calculate the decay rate of a ϕ particle to leading order in g. (Start from equation (4.86) of Peskin and Schroeder, and do the phase space integrals.)

b) Find the amplitude for $\psi(p_1)\psi^*(p_2) \to \psi(p_3)\psi^*(p_4)$ scattering to leading order in g, in terms of Mandelstam variables.

c) Suppose the energies we can reach in an experiment are higher than m, but much lower than M. In this case, we might not know that ϕ particles exist, since we can't produce them, so we would have to describe the scattering process in part (b) using an "effective" field theory in terms of ψ alone. Show that at leading order,

$$\mathcal{L}_{\mathrm{int}} = -\lambda\,\psi^* \psi \psi^* \psi \tag{4}$$

will yield the same answer for part (b) for some value of λ, and find that value in terms of g and M. Assume all elements of the momenta p_i are much less than M.

3. Solving a trivial theory. (4 points)

Consider a free real scalar field of mass m, but treat the mass term as the perturbation,

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi), \qquad \mathcal{L}_{\mathrm{int}} = -\frac{1}{2} m^2 \phi^2. \tag{5}$$

a) Write down the Feynman rules for this theory.

b) Evaluate the momentum-space propagator exactly, summing all of the (infinitely many) connected Feynman diagrams.

4. Pion scattering in the linear sigma model. (25 points)

Consider a theory with N free real scalar fields $\Phi_i(x)$ of equal mass m,

$$\mathcal{L}_0 = \sum_{i=1}^{N} \frac{1}{2}\partial_\mu \Phi_i \partial^\mu \Phi_i - \frac{m^2}{2}\Phi_i\Phi_i. \tag{6}$$

This Lagrangian is symmetric under rotations of the scalar fields among themselves. We can show this more clearly by defining an N-element vector $\mathbf{\Phi}$ with elements of Φ_i, so

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \mathbf{\Phi})^2 - \frac{m^2}{2}\mathbf{\Phi}\cdot\mathbf{\Phi}. \tag{7}$$

This is just shorthand for Eq. (6). Note that the index on Φ_i is not a spatial index (such as on ∂_i). It is a “flavor” index, meaning it just identifies which field we're talking about.

a) We quantize the theory by imposing the equal time commutation relations

$$[\Pi_i(\mathbf{x},0), \Phi_j(\mathbf{y},0)] = i\delta_{ij}\,\delta^{(3)}(\mathbf{x}-\mathbf{y}), \qquad \Pi_i(x) = \partial_t \Phi_i(x), \tag{8}$$

with all other commutators vanishing. Show that the propagator is

$$\langle 0|T\{\Phi_i(x)\Phi_j(y)\}|0\rangle = \overbracket{\Phi_i(x)\Phi_j}(y) = \delta_{ij}\,D_F(x-y). \tag{9}$$

where δ_{ij} is 1 if $i=j$, and 0 otherwise. The momentum space Feynman rule is

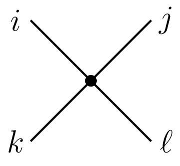
$$i \;\rule[0.5ex]{3em}{0.4pt}\; j \;=\; \frac{i\,\delta_{ij}}{p^2 - m^2 + i\epsilon} \tag{10}$$

where the i and j are flavor indices.

b) The linear sigma model additionally contains the interaction

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4}\left(\mathbf{\Phi}\cdot\mathbf{\Phi}\right)^2. \tag{11}$$

which is also symmetric under rotations of $\mathbf{\Phi}$. Show that this interaction corresponds to the momentum space Feynman rule



$$= -2i\lambda\left(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right). \tag{12}$$

(Hint: consider the cases where all four fields have the same flavor, and when two pairs of fields have the same flavor.)

c) Find the *total* cross section in the centre of mass frame for the processes

$$\Phi_1\Phi_2 \to \Phi_1\Phi_2, \qquad \Phi_1\Phi_1 \to \Phi_2\Phi_2, \qquad \Phi_1\Phi_1 \to \Phi_1\Phi_1 \tag{13}$$

to leading order in λ, in terms of E_{cm}. (Hint: start from equation (4.85) of Peskin and Schroeder, and be careful with factors of 2.)

In the linear sigma model, the potential energy of a uniform classical field is

$$V(\Phi) = \frac{m^2}{2}\mathbf{\Phi}\cdot\mathbf{\Phi} + \frac{\lambda}{4}\left(\mathbf{\Phi}\cdot\mathbf{\Phi}\right)^2. \tag{14}$$

When we quantize the harmonic oscillator, the usual definition of the creation and annihilation operators in terms of x and p only makes sense if the potential's minimum is at $x = 0$. If the minimum is somewhere else, then many things go wrong. For instance, $a|0\rangle$ won't be zero, and more generally $a^\dagger$ and a won't have simple commutation relations with H, so won't properly raise and lower the energy. Similarly, it only makes sense to quantize fields in the usual way about minima of the potential $V(\Phi)$.

If $m^2 > 0$ and $\lambda > 0$, there is a unique minimum of the potential at $\mathbf{\Phi} = \mathbf{0}$, so our treatment above makes sense. Now suppose that $m^2 < 0$ and $\lambda > 0$.

d) Defining $\mu^2 = -m^2$, show that the minima of the potential are at

$$|\mathbf{\Phi}| = v = \sqrt{\frac{\mu^2}{\lambda}}. \tag{15}$$

The quantum theory thus has multiple vacuum states, but all have nonvanishing $\langle\mathbf{\Phi}\rangle$.

By symmetry under rotations of $\mathbf{\Phi}$, we can suppose without loss of generality that we are in a vacuum state where $\langle\Phi_N\rangle = v$, with all others vanishing. Define the new fields

$$\sigma(x) = \Phi_N(x) - v, \qquad \pi_i(x) = \Phi_i(x), \qquad i \in \{1, 2, \ldots, N-1\} \tag{16}$$

which all have vanishing expectation values in this vacuum.

e) Show that in terms of these fields, the Lagrangian is

$$\begin{aligned}\mathcal{L} = {}& \frac{1}{2}(\partial_\mu \boldsymbol{\pi})^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}m_\sigma^2\sigma^2 \\ & - \frac{\lambda_{4\pi}}{4}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2 - \lambda_{3\sigma}\sigma^3 - \frac{\lambda_{4\sigma}}{4}\sigma^4 - \lambda_{\pi\sigma1}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma - \lambda_{\pi\sigma2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma^2 + C. \end{aligned} \tag{17}$$

and give expressions for m_σ, $\lambda_{4\pi}$, $\lambda_{3\sigma}$, $\lambda_{4\sigma}$, $\lambda_{\pi\sigma1}$, $\lambda_{\pi\sigma2}$ and C in terms of μ and λ. We see that σ is massive, but we have $N-1$ massless pion fields π_i.

f) Write the momentum-space Feynman rules in terms of μ and λ. Draw the pions with a solid line and flavor index i, and the σ particle with a double solid line. (Hint: if you've seen the pattern from the other Feynman rules in the problem set, it should be possible to write down the answer with some thought, but no explicit calculation. There is no Feynman rule corresponding to the constant C, which has no effect here.)

g) Compute the decay rate of a σ particle to leading order in λ, and give the corresponding lifetime in seconds if $\lambda = 0.1$, $\mu = 10$ GeV and $N = 3$.

h) Find the amplitude for $\pi_i(p_1)\pi_j(p_2) \to \pi_k(p_3)\pi_\ell(p_4)$ scattering to leading order in λ, in terms of Mandelstam variables. (Hint: there are four Feynman diagrams.)

i) Show that the amplitude in part (h) vanishes when all spatial momenta $\mathbf{p}_i$ go to zero.

The theory above is an example of "spontaneous" symmetry breaking. The original theory has an $SO(N)$ rotational symmetry among the fields $\mathbf{\Phi}$. After fixing a vacuum state, we only have an $SO(N-1)$ rotational symmetry among the fields $\boldsymbol{\pi}$. The symmetries that rotate Φ_N into the π_i are still there, but operating with them moves us between different vacuum states; they tell us that the vacua all have the same energy. Thus, there is no energy cost for shifting a pion field π_i by a small constant, corresponding to the fact that there is no mass term for the pion fields. This is an example of a Goldstone's theorem, which states that each spontaneously broken continuous symmetry corresponds to a massless boson, called a Goldstone boson.

Pions are the lightest mesons, and mediate interactions between protons and neutrons. The linear sigma model was a phenomenological model which, among other things, explained why the pion was so light. The complete picture we have today is that pions are the Goldstone bosons corresponding to the spontaneous breaking of chiral symmetry, a symmetry of quantum chromodynamics which appears for massless up and down quarks.

In reality, the up and down quarks have small masses. This additional, "explicit" breaking of chiral symmetry explains why real pions have nonzero mass. In addition, the quarks differ in electric charge, which explains why the pions have different masses.

j) We can see a similar phenomenon in the linear sigma model. Show that when we add a small term $a\,\Phi_N$ to the Lagrangian, the pion fields get a mass term m_π^2 proportional to a, and find this term. (Hint: to keep things from getting messy, work to lowest order in a as much as possible.)

5. ⋆ A contrived calculation. (5 points)

This problem is optional. In scalar ϕ^3 theory there is a diagram that contributes to the vacuum correlation function at order λ^8, shaped like a cube. Find its symmetry factor. (This is somewhat involved, and the heuristic rules given in lecture will not be enough.)

PHYS-330: QFT I Solutions 5

Stanford University, 2022 — Bernhard Mistlberger, Kevin Zhou

1. Mandelstam variables. (4 points)

Consider any $2 \to 2$ scattering process, where the two incoming particles have momenta p_1 and p_2, and the two outgoing particles have momenta p_3 and p_4. By momentum conservation, $p_1 + p_2 = p_3 + p_4$, and $p_i^2 = m_i^2$ where m_i is the mass of particle i. In this situation, it is often useful to work in terms of the Mandelstam variables

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2. \tag{1}$$

a) Show that $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$.

Solution: Just directly using the definitions,

$$\begin{aligned} s + t + u &= p_1^2 + 2p_1 \cdot p_2 + p_2^2 + p_1^2 - 2p_1 \cdot p_3 + p_3^2 + p_1^4 - 2p_1 \cdot p_4 + p_4^2 &\text{(S1)} \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1 \cdot (p_1 + p_2 - p_3 - p_4) &\text{(S2)} \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2. &\text{(S3)} \end{aligned}$$

b) In the center of mass frame, the total energy is E_{cm} and the angle of $\mathbf{p}_1$ to $\mathbf{p}_3$ is θ. Write s, t, and u in terms of E_{cm} and θ, assuming all four particles are massless.

Solution: This is easiest if we use concrete coordinate expressions. Aligning the z-axis with $\mathbf{p}_1$,

$$p_1^\mu = \frac{E_{\text{cm}}}{2}(1, 0, 0, 1), \qquad p_2^\mu = \frac{E_{\text{cm}}}{2}(1, 0, 0, -1) \tag{S4}$$

where the spatial momenta are opposite because we're working in the center of mass frame, and equal in magnitude to the energy because the particles are massless. After the scattering, we have

$$p_3^\mu = \frac{E_{\text{cm}}}{2}(1, 0, \sin\theta, \cos\theta), \qquad p_4^\mu = \frac{E_{\text{cm}}}{2}(1, 0, -\sin\theta, -\cos\theta). \tag{S5}$$

At this point we can just compute the Mandelstam variables explicitly, giving

$$s = E_{\text{cm}}^2, \qquad t = -E_{\text{cm}}^2 \frac{1 - \cos\theta}{2} = -E_{\text{cm}}^2 \sin^2\frac{\theta}{2}, \qquad u = -E_{\text{cm}}^2 \frac{1 + \cos\theta}{2} = -E_{\text{cm}}^2 \cos^2\frac{\theta}{2}. \tag{S6}$$

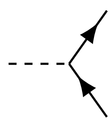
Note that they sum to zero, as expected from part (a).

2. Scalar Yukawa amplitudes. (7 points)

Let ϕ be a real scalar field of mass M, and ψ be a complex scalar field of mass m, with

$$\mathcal{L}_{\text{int}} = -g\, \psi^* \psi \phi. \tag{2}$$

The field ψ can annihilate a particle, or create a particle with opposite $U(1)$ charge; these particles are conventionally called the ψ and ψ^*, respectively. Similarly, the field ϕ can create or annihilate a ϕ particle. (That is, particles are named after the field that annihilates them.) As discussed in section, the Feynman rules are

 $= \dfrac{i}{p^2 - M^2 + i\epsilon}$  $= \dfrac{i}{p^2 - m^2 + i\epsilon}$  $= -ig$ (3)

where a dashed line stands for ϕ and a solid line stands for ψ.

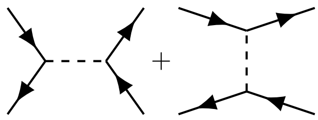
a) Assuming $M > 2m$, calculate the decay rate of a ϕ particle to leading order in g. (Start from equation (4.86) of Peskin and Schroeder, and do the phase space integrals.)

Solution: The Feynman diagram is just one copy of the interaction vertex, so $i\mathcal{M} = -ig$. Let the initial momentum be $k^\mu = (M, \mathbf{0})$ and the final momenta be p_1^μ and p_2^μ. Then

$$\begin{aligned}
\Gamma &= \frac{1}{2M} \int \frac{d^3\mathbf{p}_1}{(2\pi)^3\, 2E_1} \frac{d^3\mathbf{p}_2}{(2\pi)^3\, 2E_2} \left|-ig\right|^2 (2\pi)^4 \delta^{(4)}(k - p_1 - p_2) && \text{(S7)} \\
&= \frac{g^2}{32\pi^2 M} \int \frac{d^3\mathbf{p}_1}{E_1} \frac{d^3\mathbf{p}_2}{E_2}\, \delta^{(4)}(k - p_1 - p_2) && \text{(S8)} \\
&= \frac{g^2}{32\pi^2 M} \int \frac{d^3\mathbf{p}_1}{E_1^2}\, \delta(M - 2E_1) && \text{(S9)} \\
&= \frac{g^2}{8\pi M} \int_0^\infty d|\mathbf{p}_1|\, \frac{|\mathbf{p}_1|^2}{E_1^2}\, \delta(M - 2E_1) && \text{(S10)} \\
&= \frac{g^2}{8\pi M} \int_m^\infty dE_1\, \frac{|\mathbf{p}_1|}{E_1}\, \delta(M - 2E_1) && \text{(S11)} \\
&= \frac{g^2}{16\pi M} \sqrt{1 - (2m/M)^2}. && \text{(S12)}
\end{aligned}$$

b) Find the amplitude for $\psi(p_1)\psi^*(p_2) \to \psi(p_3)\psi^*(p_4)$ scattering to leading order in g, in terms of Mandelstam variables.

Solution: There are two Feynman diagrams, giving amplitude

$i\mathcal{M} =$  (S13)

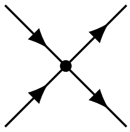
$$\begin{aligned}
&= (-ig)^2 \left(\frac{i}{(p_1 + p_2)^2 - M^2 + i\epsilon} + \frac{i}{(p_1 - p_3)^2 - M^2 + i\epsilon} \right) && \text{(S14)} \\
&= -g^2 \left(\frac{i}{s - M^2 + i\epsilon} + \frac{i}{t - M^2 + i\epsilon} \right). && \text{(S15)}
\end{aligned}$$

c) Suppose the energies we can reach in an experiment are higher than m, but much lower than M. In this case, we might not know that ϕ particles exist, since we can't produce them, so we would have to describe the scattering process in part (b) using an "effective" field theory in terms of ψ alone. Show that at leading order,

$$\mathcal{L}_{\text{int}} = -\lambda\, \psi^*\psi\psi^*\psi \tag{4}$$

will yield the same answer for part (b) for some value of λ, and find that value in terms of g and M. Assume all elements of the momenta p_i are much less than M.

Solution: The Feynman rule for the interaction in the effective theory is just

 $= -4i\lambda.$ (S16)

The factor of 4 comes about because there are 2 ways to choose which copy of ψ contracts with what, and similarly 2 ways for ψ^*.

This is also the amplitude for the scattering process in the effective theory. On the other hand, we have $s, t \ll M$, so the answer to part (b) becomes $2ig^2/M^2$. Equating these results gives $\lambda = -g^2/2M^2$. (The fact that λ is suppressed by powers of M is a generic feature of effective field theory.)

3. Solving a trivial theory. (4 points)

Consider a free real scalar field of mass m, but treat the mass term as the perturbation,

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi), \qquad \mathcal{L}_{\text{int}} = -\frac{1}{2}m^2\phi^2. \tag{5}$$

a) Write down the Feynman rules for this theory.

Solution: The Feynman rules are that the propagator is

 $$= \frac{i}{p^2 + i\epsilon} \tag{S17}$$

and the interaction vertex is

 $$= -im^2. \tag{S18}$$

b) Evaluate the momentum-space propagator exactly, summing all of the (infinitely many) connected Feynman diagrams.

Solution: The diagrams are

 $+$ $+$ $+ \ldots$ (S19)

which correspond to

$$\frac{i}{p^2+i\epsilon} + \frac{i}{p^2+i\epsilon}(-im^2)\frac{i}{p^2+i\epsilon} + \frac{i}{p^2+i\epsilon}(-im^2)\frac{i}{p^2+i\epsilon}(-im^2)\frac{i}{p^2+i\epsilon} + \ldots. \tag{S20}$$

This is a geometric series where the ratio between terms is $m^2/(p^2 + i\epsilon)$, so the sum is

$$\frac{i}{p^2+i\epsilon}\frac{1}{1 - \frac{m^2}{p^2+i\epsilon}} = \frac{i}{p^2 - m^2 + i\epsilon}. \tag{S21}$$

Of course, this is just the familiar propagator for a free massive scalar field.

4. Pion scattering in the linear sigma model. (25 points)

Consider a theory with N free real scalar fields $\Phi_i(x)$ of equal mass m,

$$\mathcal{L}_0 = \sum_{i=1}^{N} \frac{1}{2}\partial_\mu\Phi_i\partial^\mu\Phi_i - \frac{m^2}{2}\Phi_i\Phi_i. \tag{6}$$

This Lagrangian is symmetric under rotations of the scalar fields among themselves. We can show this more clearly by defining an N-element vector $\mathbf{\Phi}$ with elements of Φ_i, so

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\mathbf{\Phi})^2 - \frac{m^2}{2}\mathbf{\Phi}\cdot\mathbf{\Phi}. \tag{7}$$

This is just shorthand for Eq. (6). Note that the index on Φ_i is not a spatial index (such as on ∂_i). It is a "flavor" index, meaning it just identifies which field we're talking about.

a) We quantize the theory by imposing the equal time commutation relations

$$[\Pi_i(\mathbf{x},0), \Phi_j(\mathbf{y},0)] = i\delta_{ij}\,\delta^{(3)}(\mathbf{x}-\mathbf{y}), \qquad \Pi_i(x) = \partial_t\Phi_i(x), \tag{8}$$

with all other commutators vanishing. Show that the propagator is

$$\langle 0|T\{\Phi_i(x)\Phi_j(y)\}|0\rangle = \Phi_i(x)\Phi_j(y) = \delta_{ij}\,D_F(x-y). \tag{9}$$

where δ_{ij} is 1 if $i = j$, and 0 otherwise. The momentum space Feynman rule is

$$i \;\text{———}\; j \;=\; \frac{i\,\delta_{ij}}{p^2 - m^2 + i\epsilon} \tag{10}$$

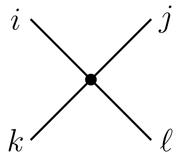
where the i and j are flavor indices.

Solution: The commutation relations imply that the mode expansion of the field Φ_i involves a set of creation and annihilation operators $a_i^\dagger(\mathbf{k})$ and $a_i(\mathbf{k})$, whose commutators vanish for $i \neq j$. Therefore, when $i \neq j$, the contraction of two fields vanishes because Φ_i^+ and Φ_j^- automatically commute. When $i = j$, the derivation of the propagator is just the same as for a single real scalar field.

b) The linear sigma model additionally contains the interaction

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4}\left(\mathbf{\Phi}\cdot\mathbf{\Phi}\right)^2. \tag{11}$$

which is also symmetric under rotations of $\mathbf{\Phi}$. Show that this interaction corresponds to the momentum space Feynman rule



$$= -2i\lambda\left(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right). \tag{12}$$

(Hint: consider the cases where all four fields have the same flavor, and when two pairs of fields have the same flavor.)

Solution: Expanding out the interaction explicitly gives

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4}\left(\sum_i \Phi_i\Phi_i\right)^2 = -\frac{\lambda}{4}\sum_{i,j}\Phi_i\Phi_i\Phi_j\Phi_j = -\frac{\lambda}{4}\sum_i \Phi_i^4 - \frac{\lambda}{2}\sum_{i<j}\Phi_i^2\Phi_j^2. \tag{S22}$$

The first term yields a vertex for four fields of the same flavor. Whenever we introduce a copy of this vertex, we get a factor of $4! = 24$ because of the ways to choose how to contract the copies of Φ_i with other things, so the overall vertex factor is $-6i\lambda$.

The second term yields a vertex for four fields where two pairs have the same flavor. These come with an extra factor of $2\cdot 2 = 4$ because of the two copies of Φ_i and two copies of Φ_j, and thus the overall vertex factor is $-2i\lambda$.

Now let's check if this is compatible with the provided Feynman rule. When $i = j = k = \ell$, all the Kronecker deltas are equal to 1, so we get $-6i\lambda$. When $i = j$ and $k = \ell$, but $i \neq k$, then only the first Kronecker delta is equal to 1, and the others are equal to 0, so we get $-2i\lambda$.

c) Find the *total* cross section in the centre of mass frame for the processes

$$\Phi_1\Phi_2 \to \Phi_1\Phi_2, \qquad \Phi_1\Phi_1 \to \Phi_2\Phi_2, \qquad \Phi_1\Phi_1 \to \Phi_1\Phi_1 \tag{13}$$

to leading order in λ, in terms of E_{cm}. (Hint: start from equation (4.85) of Peskin and Schroeder, and be careful with factors of 2.)

Solution: The Feynman diagrams for all three of these processes are just one copy of the interaction vertex, and the amplitudes are just $-2i\lambda$, $-2i\lambda$, and $-6i\lambda$ respectively. Because all four masses are identical, we may use equation (4.85), which states

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{cm}}^2}. \tag{S23}$$

The scattering is isotropic, so the angular integrals are trivial.

For the first process, the angular integral just gives a factor of 4π, so

$$\sigma(\Phi_1\Phi_2 \to \Phi_1\Phi_2) = \frac{4\pi\lambda^2}{16\pi^2 E_{\rm cm}^2} = \frac{\lambda^2}{4\pi E_{\rm cm}^2}. \tag{S24}$$

For the second process, the angular integral only gives a factor of 2π because the particles in the final state are identical. (The state where the first Φ_2 exits left and the second exits right is exactly the same as the state where the first Φ_2 exits right and the second exits left. So integrating over all possible directions for the momentum of the first Φ_2 would be double counting.) Thus,

$$\sigma(\Phi_1\Phi_1 \to \Phi_2\Phi_2) = \frac{\lambda^2}{8\pi E_{\rm cm}^2}. \tag{S25}$$

Finally, the last cross section is 9 times bigger, giving

$$\sigma(\Phi_1\Phi_1 \to \Phi_1\Phi_1) = \frac{9\lambda^2}{8\pi E_{\rm cm}^2}. \tag{S26}$$

In the linear sigma model, the potential energy of a uniform classical field is

$$V(\Phi) = \frac{m^2}{2}\mathbf{\Phi}\cdot\mathbf{\Phi} + \frac{\lambda}{4}\left(\mathbf{\Phi}\cdot\mathbf{\Phi}\right)^2. \tag{14}$$

When we quantize the harmonic oscillator, the usual definition of the creation and annihilation operators in terms of x and p only makes sense if the potential's minimum is at $x = 0$. If the minimum is somewhere else, then many things go wrong. For instance, $a|0\rangle$ won't be zero, and more generally $a^\dagger$ and a won't have simple commutation relations with H, so won't properly raise and lower the energy. Similarly, it only makes sense to quantize fields in the usual way about minima of the potential $V(\Phi)$.

If $m^2 > 0$ and $\lambda > 0$, there is a unique minimum of the potential at $\mathbf{\Phi} = \mathbf{0}$, so our treatment above makes sense. Now suppose that $m^2 < 0$ and $\lambda > 0$.

d) Defining $\mu^2 = -m^2$, show that the minima of the potential are at

$$|\mathbf{\Phi}| = v = \sqrt{\frac{\mu^2}{\lambda}}. \tag{15}$$

The quantum theory thus has multiple vacuum states, but all have nonvanishing $\langle\mathbf{\Phi}\rangle$.

Solution: Setting $dV/d|\mathbf{\Phi}|$ to zero gives $m^2|\mathbf{\Phi}| + \lambda|\mathbf{\Phi}|^3 = 0$, which yields the desired result.

By symmetry under rotations of $\mathbf{\Phi}$, we can suppose without loss of generality that we are in a vacuum state where $\langle\Phi_N\rangle = v$, with all others vanishing. Define the new fields

$$\sigma(x) = \Phi_N(x) - v, \qquad \pi_i(x) = \Phi_i(x), \qquad i \in \{1, 2, \ldots, N-1\} \tag{16}$$

which all have vanishing expectation values in this vacuum.

e) Show that in terms of these fields, the Lagrangian is

$$\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial_\mu \boldsymbol{\pi})^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}m_\sigma^2\sigma^2 \\ - \frac{\lambda_{4\pi}}{4}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2 - \lambda_{3\sigma}\sigma^3 - \frac{\lambda_{4\sigma}}{4}\sigma^4 - \lambda_{\pi\sigma 1}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma - \lambda_{\pi\sigma 2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma^2 + C.\end{aligned} \tag{17}$$

and give expressions for m_σ, $\lambda_{4\pi}$, $\lambda_{3\sigma}$, $\lambda_{4\sigma}$, $\lambda_{\pi\sigma 1}$, $\lambda_{\pi\sigma 2}$ and C in terms of μ and λ. We see that σ is massive, but we have $N-1$ massless pion fields π_i.

Solution: This is straightforward algebra, and the answers are

$$m_\sigma = \sqrt{2}\,\mu, \quad \lambda_{4\pi} = \lambda, \quad \lambda_{3\sigma} = \mu\sqrt{\lambda}, \quad \lambda_{4\sigma} = \lambda \tag{S27}$$

and

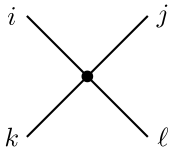
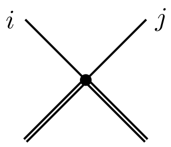
$$\lambda_{\pi\sigma 1} = \mu\sqrt{\lambda}, \quad \lambda_{\pi\sigma 2} = \frac{\lambda}{2}, \quad C = -\frac{\mu^4}{4\lambda}. \tag{S28}$$

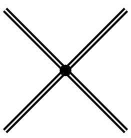
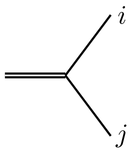
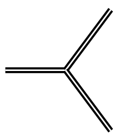
f) Write the momentum-space Feynman rules in terms of μ and λ. Draw the pions with a solid line and flavor index i, and the σ particle with a double solid line. (Hint: if you've seen the pattern from the other Feynman rules in the problem set, it should be possible to write down the answer with some thought, but no explicit calculation. There is no Feynman rule corresponding to the constant C, which has no effect here.)

Solution: The propagators are

i j $= \dfrac{i\,\delta_{ij}}{p^2 + i\epsilon}$ $= \dfrac{i}{p^2 - m_\sigma^2 + i\epsilon}$ (S29)

and the interaction vertices are

 $= -2i\lambda\,(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$  $= -2i\lambda\,\delta_{ij}$ (S30)

 $= -6i\lambda$  $= -2i\mu\sqrt{\lambda}\,\delta_{ij}$  $= -6i\mu\sqrt{\lambda}$ (S31)

g) Compute the decay rate of a σ particle to leading order in λ, and give the corresponding lifetime in seconds if $\lambda = 0.1$, $\mu = 10$ GeV and $N = 3$.

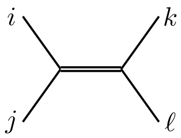
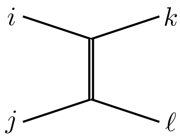
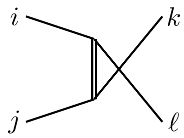
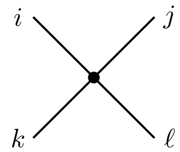
Solution: The amplitude to decay to a pair of some species of pion is $i\mathcal{M} = -2i\mu\sqrt{\lambda}$, so recycling the result to problem 2(a) gives the decay rate

$$\Gamma = \frac{N-1}{2}\frac{(2\sqrt{\lambda}\mu)^2}{8\pi m_\sigma} = \frac{(N-1)\lambda\mu}{4\sqrt{2}\,\pi}. \tag{S32}$$

where the factor of $1/2$ accounts for the two identical particles in the final state, and $N-1$ is the number of species of pion. Inverting this and converting back from natural units gives a lifetime 6×10^{-24} s.

h) Find the amplitude for $\pi_i(p_1)\pi_j(p_2) \to \pi_k(p_3)\pi_\ell(p_4)$ scattering to leading order in λ, in terms of Mandelstam variables. (Hint: there are four Feynman diagrams.)

Solution: The amplitude is

$i\mathcal{M} =$  $+$  $+$  $+$  (S33)

$$= (-2i\sqrt{\lambda}\mu)^2\left(\frac{i\,\delta_{ij}\delta_{k\ell}}{s - m_\sigma^2 + i\epsilon} + \frac{i\,\delta_{ik}\delta_{j\ell}}{t - m_\sigma^2 + i\epsilon} + \frac{i\,\delta_{i\ell}\delta_{jk}}{u - m_\sigma^2 + i\epsilon}\right) - 2i\lambda\,(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \tag{S34}$$

$$= (-2i\lambda)\left(\frac{2\mu^2\,\delta_{ij}\delta_{k\ell}}{s - 2\mu^2 + i\epsilon} + \frac{2\mu^2\,\delta_{ik}\delta_{j\ell}}{t - 2\mu^2 + i\epsilon} + \frac{2\mu^2\,\delta_{i\ell}\delta_{jk}}{u - 2\mu^2 + i\epsilon} + \delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right). \tag{S35}$$

(S36)

i) Show that the amplitude in part (h) vanishes when all spatial momenta $\mathbf{p}_i$ go to zero.

Solution: In this limit, we have $s = t = u = 0$, so the denominators all become $-2\mu^2$. Then each of the first three diagrams cancels with one term from the last diagram.

This is another manifestation of Goldstone's theorem, discussed below. The general statement is that in the effective theory of pions alone (i.e. without σ), sometimes called the nonlinear sigma model, the Lagrangian only depends on the spatial derivatives $\partial_\mu \boldsymbol{\pi}$ of the pion field. Since acting with a derivative on a field produces a factor of the particle momentum p^μ, the Feynman rules for the interactions always come with factors of momenta, so scattering amplitudes like the one we computed must vanish at zero momentum.

The theory above is an example of "spontaneous" symmetry breaking. The original theory has an $SO(N)$ rotational symmetry among the fields $\mathbf{\Phi}$. After fixing a vacuum state, we only have an $SO(N-1)$ rotational symmetry among the fields $\boldsymbol{\pi}$. The symmetries that rotate Φ_N into the π_i are still there, but operating with them moves us between different vacuum states; they tell us that the vacua all have the same energy. Thus, there is no energy cost for shifting a pion field π_i by a small constant, corresponding to the fact that there is no mass term for the pion fields. This is an example of a Goldstone's theorem, which states that each spontaneously broken continuous symmetry corresponds to a massless boson, called a Goldstone boson.

Pions are the lightest mesons, and mediate interactions between protons and neutrons. The linear sigma model was a phenomenological model which, among other things, explained why the pion was so light. The complete picture we have today is that pions are the Goldstone bosons corresponding to the spontaneous breaking of chiral symmetry, a symmetry of quantum chromodynamics which appears for massless up and down quarks.

In reality, the up and down quarks have small masses. This additional, "explicit" breaking of chiral symmetry explains why real pions have nonzero mass. In addition, the quarks differ in electric charge, which explains why the pions have different masses.

j) We can see a similar phenomenon in the linear sigma model. Show that when we add a small term $a\,\Phi_N$ to the Lagrangian, the pion fields get a mass term m_π^2 proportional to a, and find this term. (Hint: to keep things from getting messy, work to lowest order in a as much as possible.)

Solution: The potential now includes $-a\,\Phi_N$, which tilts it in the Φ_N direction. There is now a unique vacuum state where only $\langle \Phi_N \rangle = v$ is nonzero. Setting the derivative of the potential to zero gives

$$-\mu^2 v + \lambda v^3 - a = 0 \tag{S37}$$

which rearranges to

$$\lambda v^2 = \mu^2 + \frac{a}{v}. \tag{S38}$$

To solve this at leading order in a, we replace the v on the right-hand side with the unperturbed expectation value $v_0 = \sqrt{\mu^2/\lambda}$. Solving for v then yields

$$v = \sqrt{\frac{\mu^2}{\lambda} + \frac{a}{\mu\sqrt{\lambda}}} = v_0 + \frac{a}{2\mu^2} + O(a^2). \tag{S39}$$

To find the mass term for the pion, it's easiest to start with Eq. (17) and do a second shift, defining

$$\sigma' = \sigma - \frac{a}{2\mu^2}. \tag{S40}$$

The only term that will produce a pion mass term linear in a is $-\lambda_{\pi\sigma 1}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma$, giving

$$\mathcal{L} \supset -\mu\sqrt{\lambda}\,(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\frac{a}{2\mu^2} = -\frac{a\sqrt{\lambda}}{2\mu}\,\boldsymbol{\pi}\cdot\boldsymbol{\pi}. \tag{S41}$$

In other words, the pion mass is $\sqrt{a\sqrt{\lambda}/\mu}$.

5. ⋆ A contrived calculation. (5 points)

This problem is optional. In scalar ϕ^3 theory there is a diagram that contributes to the vacuum correlation function at order λ^8, shaped like a cube. Find its symmetry factor. (This is somewhat involved, and the heuristic rules given in lecture will not be enough.)

Solution: The most reliable way to do this is to simply count all the Wick contractions that yield a cube, building up the diagram one vertex at a time.

- Consider what the three copies of ϕ in the first vertex can contract with. The first has $7 \cdot 3$ choices, the second has $6 \cdot 3$ choices, and the third has $5 \cdot 3$ choices.
- Now consider the opposite corner of the cube. There are 4 choices for which remaining uncontracted vertex is the opposite corner. Its three copies of ϕ have $3 \cdot 3$, $2 \cdot 3$, and 3 choices for contractions.
- Finally, we attach the two halves of the cube together. There are 6 ways of choosing which vertices attach to which, and 2^6 choices for how ϕ fields are contracted to make the joining edges.

Therefore, the symmetry factor is

$$\frac{7\cdot 3\cdot 6\cdot 3\cdot 5\cdot 3\cdot 4\cdot 3\cdot 3\cdot 2\cdot 3\cdot 3\cdot 6\cdot 2^6}{8!\cdot(3!)^8} = \frac{1}{48}. \tag{S42}$$

This makes sense, as the symmetry group of the cube has 48 elements.

1. The Lorentz algebra. (10 points)

You previously showed the rotation and boost generators obey commutation relations

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \qquad [J^i, K^j] = i\epsilon^{ijk} K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk} J^k. \tag{1}$$

These are Euclidean indices, which is why pairs of upstairs indices can be contracted.

a) Defining $J^i_\pm = (J^i \pm iK^i)/2$, show that

$$[J^i_+, J^j_-] = 0, \qquad [J^i_\pm, J^j_\pm] = i\epsilon^{ijk} J^k_\pm, \tag{2}$$

so that the Lorentz algebra is just two copies of the algebra of the rotation group.

The commutation relations can be expressed more concisely if we collect all six generators into the antisymmetric Lorentz tensor $J^{\mu\nu}$, where

$$J^{ij} = \epsilon^{ijk} J^k, \qquad J^{0i} = K^i \tag{3}$$

The commutation relations (1) can then all be written together as

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\,(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}). \tag{4}$$

By definition, a representation of the Lorentz algebra is a choice of $J^{\mu\nu}$ that satisfies these commutation relations. The simplest example is the trivial representation $J^{\mu\nu} = 0$, which describes the action of Lorentz transformations on scalars.

b) In the vector representation, each generator $J^{\mu\nu}$ is a 4×4 matrix, with elements

$$(J^{\mu\nu})_{\alpha\beta} = i\,(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta). \tag{5}$$

This describes the action of Lorentz transformations on four-vectors V^β, as you've already seen. Starting from (5), verify the generators satisfy the Lorentz algebra (4). (Hint: upper and lower indices should be contracted in pairs to perform matrix multiplication. For example, $(J^{01} J^{23})_{\alpha\beta} = (J^{01})_\alpha{}^\gamma (J^{23})_{\gamma\beta}$. If you get confused, start by plugging in specific values for the indices and doing the sums explicitly.)

c) The gamma matrices are 4×4 matrices defined to satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}. \tag{6}$$

Show that

$$[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] = 2\,(\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - \eta^{\mu\rho}\gamma^\nu\gamma^\sigma + \eta^{\nu\sigma}\gamma^\rho\gamma^\mu - \eta^{\mu\sigma}\gamma^\rho\gamma^\nu). \tag{7}$$

d) In the Dirac spinor representation, the generators are the 4×4 matrices

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{8}$$

This describes the action of Lorentz transformations on Dirac spinors. Show that these generators satisfy the Lorentz algebra (4).

2. Properties of gamma matrices. (15 points)

In this problem, you should only use the defining property (6) of the gamma matrices. Below, $\mathbb{1}_4$ denotes a 4×4 identity matrix.

a) Show that contractions of gamma matrices satisfy

$$\gamma^\mu \gamma_\mu = 4\, \mathbb{1}_4 \tag{9}$$
$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\, \gamma^\nu \tag{10}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\, \eta^{\nu\rho}\, \mathbb{1}_4 \tag{11}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\, \gamma^\sigma \gamma^\rho \gamma^\nu \tag{12}$$

b) Show that the traces of products of gamma matrices obey

$$\operatorname{tr} \gamma^\mu = 0 \tag{13}$$
$$\operatorname{tr} \gamma^\mu \gamma^\nu = 4\, \eta^{\mu\nu} \tag{14}$$
$$\operatorname{tr} \gamma^\mu \gamma^\nu \gamma^\rho = 0 \tag{15}$$
$$\operatorname{tr} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = 4\, (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}). \tag{16}$$

c) It will be useful to introduce the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Show that

$$\{\gamma^5, \gamma^\mu\} = 0 \tag{17}$$
$$(\gamma^5)^2 = \mathbb{1}_4 \tag{18}$$
$$\operatorname{tr} \gamma^5 = 0 \tag{19}$$
$$\operatorname{tr} \gamma^\mu \gamma^\nu \gamma^5 = 0 \tag{20}$$
$$\operatorname{tr} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5 = -4i\, \epsilon^{\mu\nu\rho\sigma}. \tag{21}$$

3. Invariance of the Dirac Lagrangian. (15 points)

The Dirac Lagrangian is

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m\mathbb{1}_4)\Psi \tag{22}$$

where $\bar{\Psi} = \Psi^\dagger \gamma^0$ and $\partial\!\!\!/ = \gamma^\mu \partial_\mu$. The four-component spinor Ψ is acted on by the gamma matrices. In general, a Lorentz transformation Λ will change a spinor according to

$$\Psi(x) \to \Psi'(x') = U(\Lambda)\Psi(x) \tag{23}$$

where $U(\Lambda)$ is some 4×4 matrix, not necessarily unitary.

a) Show that the Dirac Lagrangian is invariant under Lorentz transformations if

$$U^{-1}(\Lambda) = \gamma^0 U^\dagger(\Lambda)\gamma^0, \qquad U^{-1}(\Lambda)\gamma^\mu U(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu. \tag{24}$$

b) Show that for an infinitesimal Lorentz transformation $\Lambda^{\mu\nu} = \eta^{\mu\nu} + \omega^{\mu\nu}$ the above relations are satisfied for

$$U(\Lambda) = \mathbb{1}_4 - \frac{i}{2}\omega^{\mu\nu} S_{\mu\nu}, \tag{25}$$

where $S^{\mu\nu}$ is the Lorentz generator defined by (8). (Hint: use $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$, which holds in every representation of the gamma matrices.)

c) A finite Lorentz transformation is given by exponentiating a generator,

$$U(\Lambda) = \exp\left(-\frac{i}{2}\omega^{\mu\nu} S_{\mu\nu}\right). \tag{26}$$

Explicitly write down the 4×4 matrix $U(\Lambda)$ for a rotation about the x-axis by an angle θ, and a boost along the z axis with rapidity ϕ. Use the Dirac representation of the gamma matrices, as this will yield simple results in the nonrelativistic limit.

d) Show that the Dirac Lagrangian is invariant under $\Psi \to e^{-i\alpha}\Psi$, and find the associated conserved current J_V^μ. Then show explicitly that $\partial_\mu J_V^\mu = 0$ using the Dirac equation.

e) Show that when $m = 0$, the Dirac Lagrangian is also invariant under $\Psi \to e^{-i\alpha\gamma^5}\Psi$, and find the associated conserved current J_A^μ. What is $\partial_\mu J_A^\mu$ when m is nonzero?

4. ⋆ Spinors in three dimensions. (5 points)

In this course, we focus on spinors in four dimensions for good reason. In this optional problem, you'll see how the same mathematical structures appear in three dimensions.

a) Consider spinors in three spacetime dimensions. What are the smallest nonzero matrices that can satisfy (6)? Write down three such matrices γ^0, γ^1, and γ^2 explicitly.

b) We define the spinor Lorentz generators by (8) in any dimension. Since there are now only two spatial dimensions, there is only one rotation generator S^{12}. What phase does a spinor pick up after a 2π rotation?

c) How does the tensor product of two spinor representations decompose into irreducible representations of the Lorentz group?

In two spatial dimensions there are exotic particles called anyons, which can pick up an arbitrary phase after a 2π rotation. However, they can't be described by the conventional quantum fields covered in this course.

PHYS-330: QFT I — Solutions 6

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1. The Lorentz algebra. (10 points)

You previously showed the rotation and boost generators obey commutation relations

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \qquad [J^i, K^j] = i\epsilon^{ijk} K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk} J^k. \tag{1}$$

These are Euclidean indices, which is why pairs of upstairs indices can be contracted.

a) Defining $J^i_\pm = (J^i \pm iK^i)/2$, show that

$$[J^i_+, J^j_-] = 0, \qquad [J^i_\pm, J^j_\pm] = i\epsilon^{ijk} J^k_\pm, \tag{2}$$

so that the Lorentz algebra is just two copies of the algebra of the rotation group.

Solution: In general, for any numbers a and b,

$$\frac{1}{4}\left[J^i + iaK^i, J^j + ibK^j\right] = \frac{1}{4}\left([J^i, J^j] + ia[K^i, J^j] + ib[J^i, K^j] - ab[K^i, K^j]\right) \tag{S1}$$

$$= \frac{i}{4}\epsilon^{ijk}\left((1+ab)J^k + i(a+b)K^k\right). \tag{S2}$$

For $a = -b = 1$ we find

$$[J^i_+, J^j_-] = 0 \tag{S3}$$

as desired. For $a = b = \pm 1$ we find

$$[J^i_\pm, J^j_\pm] = i\epsilon^{ijk} J^k_\pm \tag{S4}$$

also as desired.

The commutation relations can be expressed more concisely if we collect all six generators into the antisymmetric Lorentz tensor $J^{\mu\nu}$, where

$$J^{ij} = \epsilon^{ijk} J^k, \qquad J^{0i} = K^i \tag{3}$$

The commutation relations (1) can then all be written together as

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\,(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}). \tag{4}$$

By definition, a representation of the Lorentz algebra is a choice of $J^{\mu\nu}$ that satisfies these commutation relations. The simplest example is the trivial representation $J^{\mu\nu} = 0$, which describes the action of Lorentz transformations on scalars.

b) In the vector representation, each generator $J^{\mu\nu}$ is a 4×4 matrix, with elements

$$(J^{\mu\nu})_{\alpha\beta} = i\,(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta). \tag{5}$$

This describes the action of Lorentz transformations on four-vectors V^β, as you've already seen. Starting from (5), verify the generators satisfy the Lorentz algebra (4). (Hint: upper and lower indices should be contracted in pairs to perform matrix multiplication. For example, $(J^{01}J^{23})_{\alpha\beta} = (J^{01})_\alpha{}^\gamma (J^{23})_{\gamma\beta}$. If you get confused, start by plugging in specific values for the indices and doing the sums explicitly.)

Solution: We just directly evaluate the commutator,

$$[J^{\mu\nu}, J^{\rho\sigma}]_{\alpha\delta} = \eta^{\beta\gamma}\Big[-(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta)(\delta^\rho_\gamma\delta^\sigma_\delta - \delta^\sigma_\gamma\delta^\rho_\delta) + \delta^\rho_\alpha\delta^\sigma_\beta - \delta^\sigma_\alpha\delta^\rho_\beta)(\delta^\mu_\gamma\delta^\nu_\delta - \delta^\nu_\gamma\delta^\mu_\delta)\Big]. \tag{S5}$$

There are 8 terms in total, each of which has the form $\eta^{\beta\gamma}$ times a product of four delta functions. For each one, we can apply the identity $\delta^\mu_\alpha \delta^\nu_\beta \eta^{\alpha\beta} = \eta^{\mu\nu}$ to reduce the term to a product of two delta functions times a metric tensor. Collecting the terms in four pairs yields the desired result.

This is tedious to write out. An easier way is to note that (S5) is antisymmetric in μ and ν, and in ρ and σ, which means the result also has to be antisymmetric in these pairs of indices. In this way, you can reduce the amount of algebra needed by a factor of 4, though you have to be a bit careful with signs.

c) The gamma matrices are 4×4 matrices defined to satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}. \tag{6}$$

Show that

$$[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] = 2\,(\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - \eta^{\mu\rho}\gamma^\nu\gamma^\sigma + \eta^{\nu\sigma}\gamma^\rho\gamma^\mu - \eta^{\mu\sigma}\gamma^\rho\gamma^\nu). \tag{7}$$

Solution: We just directly use the definition of the commutator, and then use (6) repeatedly to get the two terms in the same form, picking up factors of the metric with each anticommutation:

$$\begin{aligned}
[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] &= \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu && \text{(S6)}\\
&= \gamma^\mu\{\gamma^\nu, \gamma^\rho\}\gamma^\sigma - \gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu && \text{(S7)}\\
&= 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - \gamma^\mu\gamma^\rho\{\gamma^\nu, \gamma^\sigma\} + \gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu && \text{(S8)}\\
&= 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - 2\eta^{\nu\sigma}\gamma^\mu\gamma^\rho + \{\gamma^\mu, \gamma^\rho\}\gamma^\sigma\gamma^\nu - \gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu && \text{(S9)}\\
&= 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - 2\eta^{\nu\sigma}\gamma^\mu\gamma^\rho + 2\eta^{\mu\rho}\gamma^\sigma\gamma^\nu - \gamma^\rho\{\gamma^\mu, \gamma^\sigma\}\gamma^\nu && \text{(S10)}\\
&= 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - 2\eta^{\nu\sigma}\gamma^\mu\gamma^\rho + 2\eta^{\mu\rho}\gamma^\sigma\gamma^\nu - 2\eta^{\mu\sigma}\gamma^\rho\,\gamma^\nu && \text{(S11)}\\
&= 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma + 2\eta^{\nu\sigma}\gamma^\rho\gamma^\mu - 2\eta^{\mu\rho}\gamma^\nu\gamma^\sigma - 2\eta^{\mu\sigma}\gamma^\rho\,\gamma^\nu. && \text{(S12)}
\end{aligned}$$

Note that in the last step, we used the anticommutation relation on each of the middle two terms; the factors of $\eta^{\nu\sigma}\eta^{\mu\rho}$ cancel between the two terms.

d) In the Dirac spinor representation, the generators are the 4×4 matrices

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{8}$$

This describes the action of Lorentz transformations on Dirac spinors. Show that these generators satisfy the Lorentz algebra (4).

Solution: Plugging in the definitions, we have

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] &= -\frac{1}{16}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu, \gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho] && \text{(S13)}\\
&= -\frac{1}{16}\left([\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] - [\gamma^\nu\gamma^\mu, \gamma^\rho\gamma^\sigma] - [\gamma^\mu\gamma^\nu, \gamma^\sigma\gamma^\rho] + [\gamma^\nu\gamma^\mu, \gamma^\sigma\gamma^\rho]\right) && \text{(S14)}\\
&= -\frac{1}{4}[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] && \text{(S15)}\\
&= -\frac{1}{2}\left(\eta^{\nu\rho}\gamma^\mu\gamma^\sigma + \eta^{\nu\sigma}\gamma^\rho\gamma^\mu - \eta^{\mu\rho}\gamma^\nu\gamma^\sigma - \eta^{\mu\sigma}\gamma^\rho\gamma^\nu\right). && \text{(S16)}
\end{aligned}$$

In the third step, we used the anticommutation relations on the last three terms, and in the fourth step we used the result of part (c). To finish the problem, we note that

$$\gamma^\mu\gamma^\nu = \frac{1}{2}[\gamma^\mu, \gamma^\nu] + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = -2iS^{\mu\nu} + \eta^{\mu\nu}. \tag{S17}$$

Applying this to all four terms, the factors of the metric all cancel out, leaving the desired result

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\eta^{\nu\rho}S^{\mu\sigma} + i\eta^{\nu\sigma}S^{\rho\mu} - i\eta^{\mu\rho}S^\nu\sigma - i\eta^{\mu\sigma}S^{\rho\nu}. \tag{S18}$$

2. Properties of gamma matrices. (15 points)

In this problem, you should only use the defining property (6) of the gamma matrices. Below, $\mathbb{1}_4$ denotes a 4×4 identity matrix.

a) Show that contractions of gamma matrices satisfy

$$\gamma^\mu \gamma_\mu = 4\, \mathbb{1}_4 \tag{9}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\, \gamma^\nu \tag{10}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\, \eta^{\nu\rho}\, \mathbb{1}_4 \tag{11}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\, \gamma^\sigma \gamma^\rho \gamma^\nu \tag{12}$$

Solution: Since γ^μ and γ_μ commute, we have

$$\gamma^\mu \gamma_\mu = \frac{1}{2}(\gamma^\mu \gamma_\mu + \gamma_\mu \gamma^\mu) = \frac{1}{2}\{\gamma^\mu, \gamma_\mu\} = \eta^\mu_\mu = 4 \tag{S19}$$

where the right-hand side has an implicit 4×4 spinor identity matrix. Now, the proofs of the other three statements are very similar, but we use the defining anticommutation relation (6) to get the γ^μ and γ_μ next to each other, so we can use (9). The results are:

$$\begin{aligned}
\gamma^\mu \gamma^\nu \gamma_\mu &= \gamma^\mu \{\gamma^\nu, \gamma_\mu\} - \gamma^\mu \gamma_\mu \gamma^\nu && \text{(S20)}\\
&= 2\gamma^\mu \eta^\nu_\mu - 4\gamma^\nu && \text{(S21)}\\
&= -2\gamma^\nu. && \text{(S22)}\\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= \gamma^\mu \gamma^\nu \{\gamma^\rho, \gamma_\mu\} - \gamma^\mu \gamma^\nu \gamma_\mu \gamma^\rho && \text{(S23)}\\
&= 2\eta^\rho_\mu \gamma^\mu \gamma^\nu - \gamma^\mu \{\gamma^\nu, \gamma_\mu\} \gamma^\rho + \gamma^\mu \gamma_\mu, \gamma^\nu \gamma^\rho && \text{(S24)}\\
&= 2\gamma^\rho \gamma^\nu - 2\gamma^\nu \gamma^\rho + 4\gamma^\nu \gamma^\rho && \text{(S25)}\\
&= 4\eta^{\rho\nu}. && \text{(S26)}\\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= 2\eta^\sigma_\mu \gamma^\mu \gamma^\nu \gamma^\rho - \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu \gamma^\sigma && \text{(S27)}\\
&= 2\gamma^\sigma \gamma^\nu \gamma^\rho - 2\eta^\rho_\mu \gamma^\mu \gamma^\nu \gamma^\sigma + \gamma^\mu \gamma^\nu \gamma_\mu \gamma^\rho \gamma^\sigma && \text{(S28)}\\
&= 2\gamma^\sigma \gamma^\nu \gamma^\rho - 2\gamma^\rho \gamma^\nu \gamma^\sigma + 2\eta^\nu_\mu \gamma^\mu \gamma^\rho \gamma^\sigma - \gamma^\mu \gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma && \text{(S29)}\\
&= 2\gamma^\sigma \gamma^\nu \gamma^\rho - 2\gamma^\rho \gamma^\nu \gamma^\sigma - 2\gamma^\nu \gamma^\rho \gamma^\sigma && \text{(S30)}\\
&= 2\gamma^\sigma \{\gamma^\nu, \gamma^\rho\} - 2\gamma^\sigma \gamma^\rho \gamma^\nu - 2\gamma^\rho \gamma^\nu \gamma^\sigma - 2\{\gamma^\nu, \gamma^\rho\}\gamma^\sigma && \text{(S31)}\\
&= 2\gamma^\sigma \{\gamma^\nu, \gamma^\rho\} - 2\gamma^\sigma \gamma^\rho \gamma^\nu - 2\gamma^\rho \gamma^\nu \gamma^\sigma - 2\{\gamma^\nu, \gamma^\rho\}\gamma^\sigma + 2\gamma^\rho \gamma^\nu \gamma^\sigma && \text{(S32)}\\
&= -2\gamma^\sigma \gamma^\rho \gamma^\nu && \text{(S33)}
\end{aligned}$$

b) Show that the traces of products of gamma matrices obey

$$\operatorname{tr} \gamma^\mu = 0 \tag{13}$$

$$\operatorname{tr} \gamma^\mu \gamma^\nu = 4\, \eta^{\mu\nu} \tag{14}$$

$$\operatorname{tr} \gamma^\mu \gamma^\nu \gamma^\rho = 0 \tag{15}$$

$$\operatorname{tr} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = 4\,(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}). \tag{16}$$

Solution: We use the cyclic property of the trace,

$$\operatorname{tr} ABC \ldots XYZ = \operatorname{tr} BC \ldots XYZA \tag{S34}$$

as well as the results we proved in part (a). For example, we have

$$\operatorname{tr} \gamma^\mu = \operatorname{tr}\left[\frac{1}{4}\gamma^\mu \gamma^\nu \gamma_\nu\right] = \operatorname{tr}\left[\frac{1}{4}\gamma_\nu \gamma^\mu \gamma^\nu\right] = -\frac{1}{2}\operatorname{tr} \gamma^\mu \tag{S35}$$

where we used (9), the cyclic property of the trace, and (10). The only way this equation can be satisfied is if $\operatorname{tr}\gamma^\mu = 0$, as desired. A similar idea works for the trace of three gamma matrices,

$$\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho = \frac{1}{4}\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\sigma = \frac{1}{4}\operatorname{tr}\gamma_\sigma\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = -\frac{1}{2}\operatorname{tr}\gamma^\rho\gamma^\nu\gamma^\mu = -\frac{1}{2}\operatorname{tr}\gamma^\mu\gamma^\rho\gamma^\nu. \tag{S36}$$

We can bring the right-hand side to the same form as the left-hand side using the anticommutation relation,

$$-\frac{1}{2}\operatorname{tr}\gamma^\mu\gamma^\rho\gamma^\nu = -\frac{1}{2}\operatorname{tr}\gamma^\mu\{\gamma^\rho,\gamma^\nu\} + \frac{1}{2}\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho = \frac{1}{2}\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho \tag{S37}$$

where the second step uses (13). We thus conclude $\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho = 0$. One can generalize this argument to show that the trace of an odd number of gamma matrices always vanishes. (Of course, there are multiple ways to show this; you could also have done it by inserting factors of $(\gamma^5)^2 = 1$, as in Peskin.)

Now let's consider the equations with an even number of gamma matrices. For two gamma matrices, we can use the cyclic property of the trace to produce an anticommutator, giving

$$\operatorname{tr}\gamma^\mu\gamma^\nu = \frac{1}{2}\operatorname{tr}\left[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right] = \operatorname{tr}[\eta^{\mu\nu}] = 4\eta^{\mu\nu}. \tag{S38}$$

For four gamma matrices, we can use a similar idea, repeatedly using the anticommutation relations to bring γ^μ to the right and then moving it back to the left using the cyclic property of the trace,

$$\begin{aligned}
\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma &= \operatorname{tr}\{\gamma^\mu,\gamma^\nu\}\gamma^\rho\gamma^\sigma - \operatorname{tr}\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma && \text{(S39)}\\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} - \operatorname{tr}\gamma^\nu\{\gamma^\mu,\gamma^\rho\}\gamma^\sigma - \operatorname{tr}\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma && \text{(S40)}\\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 8\eta^{\mu\rho}\eta^{\nu\sigma} + \operatorname{tr}\gamma^\nu\gamma^\rho\{\gamma^\mu,\gamma^\sigma\} - \operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma && \text{(S41)}\\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 8\eta^{\mu\rho}\eta^{\nu\sigma} + 8\eta^{\mu\sigma}\eta^{\rho\nu} - \operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma. && \text{(S42)}
\end{aligned}$$

Adding $\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ to both sides and dividing by two gives the result.

While these relations might seem random, there's a very simple way to summarize what they mean. Consider an arbitrarily long product of arbitrary gamma matrices,

$$\operatorname{tr}(\gamma^1\gamma^3\gamma^0\gamma^0\gamma^2\gamma^3\gamma^2\gamma^1\cdots). \tag{S43}$$

The defining anticommutation relations tells us that we pick up a minus sign when we swap the order of any two adjacent, distinct gamma matrices. So such a trace can always be rewritten in the form

$$\pm\operatorname{tr}((\gamma^0)^{n_0}(\gamma^1)^{n_1}(\gamma^2)^{n_2}(\gamma^3)^{n_3}) \tag{S44}$$

where n_0 through n_3 are nonnegative integers. The anticommutation relations also tell us that $\gamma^0\gamma^0 = 1$ and $\gamma^1\gamma^1 = \gamma^2\gamma^2 = \gamma^3\gamma^3 = -1$, which reduces the trace above (up to a sign) to

$$\pm\operatorname{tr}((\gamma^0)^{m_0}(\gamma^1)^{m_1}(\gamma^2)^{m_2}(\gamma^3)^{m_3}) \tag{S45}$$

where the m_i are 0 or 1, depending on the parity of the n_i. Looking back at what we've just proven, (13) says the trace of any single gamma matrix is zero, (14) says the trace of the product of any two distinct gamma matrices is zero, and (15) and (16) say the analogous results for three and four gamma matrices. Therefore, the trace (S45) is automatically zero unless *all* of the m_i are equal to zero. The punchline is that the only way to get a nonzero trace is to multiply an even number of copies of each type of gamma matrix.

c) It will be useful to introduce the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Show that

$$\{\gamma^5,\gamma^\mu\} = 0 \tag{17}$$

$$(\gamma^5)^2 = \mathbb{1}_4 \tag{18}$$

$$\operatorname{tr}\gamma^5 = 0 \tag{19}$$

$$\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^5 = 0 \tag{20}$$

$$\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5 = -4i\,\epsilon^{\mu\nu\rho\sigma}. \tag{21}$$

Solution: We can check the first statement for each value of μ. For example, for $\mu = 0$ we have

$$\{\gamma^5, \gamma^0\} = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 \tag{S46}$$

The second term can be brought to the first term by performing three anticommutations, each of which flips the sign. Thus, the sum of the two terms is zero. The logic for $\mu = 1, 2, 3$ is similar.

To prove the second statement, we note that

$$\begin{aligned}
(\gamma^5)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 & \text{(S47)}\\
&= \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 & \text{(S48)}\\
&= -\gamma^2\gamma^3\gamma^2\gamma^3 & \text{(S49)}\\
&= -\gamma^3\gamma^3 & \text{(S50)}\\
&= 1 & \text{(S51)}
\end{aligned}$$

where we just used the anticommutation relations, which imply $\gamma^0\gamma^0 = 1$ and $\gamma^1\gamma^1 = \gamma^2\gamma^2 = \gamma^3\gamma^3 = -1$.

To prove the third statement, we can simply use (16),

$$\operatorname{tr}\gamma^5 = i\operatorname{tr}\gamma^0\gamma^1\gamma^2\gamma^3 = 4i(\eta^{01}\eta^{23} - \eta^{02}\eta^{13} + \eta^{03}\eta^{12}) = 0. \tag{S52}$$

For the fourth statement, we can do some casework on the values of μ and ν. If $\mu = \nu$, then the γ^μ and γ^ν multiply to give ± 1, leaving us with $\operatorname{tr}\gamma^5 = 0$. On the other hand, suppose $\mu \neq \nu$, then we can bring the factors of γ^μ and γ^ν within γ^5 to the left, by performing anticommutations. Multiplying them with $\gamma^\mu\gamma^\nu$ yields ± 1 and leaves the trace of a product of two different gamma matrices, which is zero.

Finally, for the last statement, let's first show that the left-hand side is totally antisymmetric in its four indices. For the μ and ν indices, note that

$$\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5 = 2\eta^{\mu\nu}\operatorname{tr}\gamma^\rho\gamma^\sigma\gamma^5 - \operatorname{tr}\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^5 = -\operatorname{tr}\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^5. \tag{S53}$$

By similar reasoning, we get a sign flip when we exchange the ν and ρ indices, and the ρ and σ indices, which suffices to show that the left-hand side is antisymmetric in all four indices. Thus, it must be equal to $A\epsilon^{\mu\nu\rho\sigma}$ for some constant A. To find the constant, we just consider a special case,

$$\operatorname{tr}\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 = A\epsilon^{0123} = A. \tag{S54}$$

The left-hand side is just

$$-i\operatorname{tr}(\gamma^5)^2 = -4i \tag{S55}$$

which yields the desired result.

3. Invariance of the Dirac Lagrangian. (15 points)

The Dirac Lagrangian is

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m\mathbb{1}_4)\Psi \tag{22}$$

where $\bar{\Psi} = \Psi^\dagger\gamma^0$ and $\partial\!\!\!/ = \gamma^\mu\partial_\mu$. The four-component spinor Ψ is acted on by the gamma matrices. In general, a Lorentz transformation Λ will change a spinor according to

$$\Psi(x) \to \Psi'(x') = U(\Lambda)\Psi(x) \tag{23}$$

where $U(\Lambda)$ is some 4×4 matrix, not necessarily unitary.

a) Show that the Dirac Lagrangian is invariant under Lorentz transformations if

$$U^{-1}(\Lambda) = \gamma^0 U^\dagger(\Lambda)\gamma^0, \qquad U^{-1}(\Lambda)\gamma^\mu U(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu. \tag{24}$$

Solution: From basic special relativity, we know that under a Lorentz transformation,

$$x^\mu \to x'^{\,\mu} = \Lambda^\mu{}_\nu x^\nu, \qquad \frac{\partial}{\partial x_\mu} \to \frac{\partial}{\partial x'_\mu} = \Lambda^\mu{}_\nu \frac{\partial}{\partial x^\nu}. \tag{S56}$$

Now let's show that each term in the Dirac Lagrangian is invariant. The mass term is invariant because

$$\bar\Psi\Psi = \Psi^\dagger\gamma^0\Psi \to \Psi^\dagger U^\dagger\gamma^0 U\Psi = \Psi^\dagger\gamma^0 U^{-1}U\Psi = \bar\Psi\Psi. \tag{S57}$$

To show the kinetic term is invariant, note that

$$\begin{aligned}
\bar\Psi\partial\!\!\!/\Psi &\to \Psi^\dagger U^\dagger\gamma^0\gamma_\mu\Lambda^\mu{}_\nu\partial^\nu U\Psi && \text{(S58)}\\
&= \Psi^\dagger\gamma^0 U^{-1}\gamma_\mu\Lambda^\mu{}_\nu U\partial^\nu\Psi && \text{(S59)}\\
&= \bar\Psi\Lambda_{\mu\rho}\gamma^\rho\Lambda^\mu{}_\nu\partial^\nu\Psi && \text{(S60)}\\
&= \bar\Psi\eta_{\nu\rho}\gamma^\rho\partial^\nu\Psi && \text{(S61)}\\
&= \bar\Psi\partial\!\!\!/\Psi. && \text{(S62)}
\end{aligned}$$

b) Show that for an infinitesimal Lorentz transformation $\Lambda^{\mu\nu} = \eta^{\mu\nu} + \omega^{\mu\nu}$ the above relations are satisfied for

$$U(\Lambda) = \mathbb{1}_4 - \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}, \tag{25}$$

where $S^{\mu\nu}$ is the Lorentz generator defined by (8). (Hint: use $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$, which holds in every representation of the gamma matrices.)

Solution: Expanding out the definitions, we have

$$U = \mathbb{1} + \frac{1}{8}[\gamma^\mu,\gamma^\nu]\omega_{\mu\nu} \tag{S63}$$

which implies

$$U^{-1} = \mathbb{1} - \frac{1}{8}[\gamma^\mu,\gamma^\nu]\omega_{\mu\nu} \tag{S64}$$

since we're considering infinitesimal translations. To prove the first result note that

$$\begin{aligned}
\gamma^0 U^\dagger\gamma^0 &= \mathbb{1} + \frac{1}{8}\gamma^0[(\gamma^\nu)^\dagger,(\gamma^\mu)^\dagger]\gamma^0\,\omega_{\mu\nu} && \text{(S65)}\\
&= \mathbb{1} + \frac{1}{8}\gamma^0[\gamma^0\gamma^\nu\gamma^0,\gamma^0\gamma^\mu\gamma^0]\gamma^0\,\omega_{\mu\nu} && \text{(S66)}\\
&= \mathbb{1} - \frac{1}{8}[\gamma^\mu,\gamma^\nu]\omega_{\mu\nu} && \text{(S67)}\\
&= U^{-1}. && \text{(S68)}
\end{aligned}$$

In the first step, we used the fact that $[A,B]^\dagger = [B^\dagger,A^\dagger]$, in the second step we used $\gamma_\mu^\dagger = \gamma^0\gamma_\mu\gamma^0$, and in the third step we used $\gamma^0\gamma^0 = 1$.

For the second relation, note that

$$\gamma_\mu[\gamma_\rho,\gamma_\sigma] = 4(\eta_{\mu\rho}\gamma_\sigma - \eta_{\mu\sigma}\gamma_\rho) + [\gamma_\rho,\gamma_\sigma]\gamma_\mu. \tag{S69}$$

Next,

$$\gamma^\mu U = \gamma^\mu(\mathbb{1} + \frac{1}{8}[\gamma_\rho,\gamma_\sigma]\omega^{\rho\sigma}) = U\gamma^\mu + \frac{1}{2}(\omega^{\mu\sigma}\gamma_\sigma - \omega^{\sigma\mu}\gamma_\sigma) = U\gamma^\mu + \omega^{\mu\sigma}\gamma_\sigma. \tag{S70}$$

Finally, we find

$$U^{-1}\gamma_\mu U = U^{-1}(U\gamma_\mu + \omega_{\mu\sigma}\gamma^\sigma) = (\eta_{\mu\nu} + \omega_{\mu\nu})\gamma^\nu = \Lambda_{\mu\nu}\gamma^\nu \tag{S71}$$

as desired. Note that in the penultimate step, we dropped a term of order ω^2 since the Lorentz transformation is infinitesimal.

c) A finite Lorentz transformation is given by exponentiating a generator,

$$U(\Lambda) = \exp\left(-\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}\right). \tag{26}$$

Explicitly write down the 4×4 matrix $U(\Lambda)$ for a rotation about the x-axis by an angle θ, and a boost along the z axis with rapidity ϕ. Use the Dirac representation of the gamma matrices, as this will yield simple results in the nonrelativistic limit.

Solution: For rotations around the x-axis by an angle θ we have $\omega^{23} = -\omega^{32} = \theta$ and all other entries zero. In the Dirac representation, we find

$$[\gamma^2, \gamma^3] = \left[\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\right] = -\begin{pmatrix} [\sigma_2, \sigma_3] & 0 \\ 0 & [\sigma_2, \sigma_3] \end{pmatrix} = 2i\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \tag{S72}$$

Consequently,

$$U(\Lambda(\theta)) = \exp\left\{\frac{i\theta}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}\right\} = \cos\frac{\theta}{2}\mathbb{1} + i\sin\frac{\theta}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \tag{S73}$$

This makes sense, because it just says that both of the two-component spinors that make up a Dirac spinor transform like spin $1/2$ particles in ordinary quantum mechanics.

For boosts along the z direction we have $\omega^{03} = -\omega^{30} = \phi$ and all other entries zero, and

$$[\gamma^0, \gamma^3] = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}\right] = 2\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \tag{S74}$$

Consequently,

$$U(\Lambda(\phi)) = \exp\left\{\frac{\phi}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}\right\} = \cosh\frac{\phi}{2}\mathbb{1} + \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}\sinh\frac{\phi}{2}. \tag{S75}$$

In other words, boosts mix the two two-component spinors, while keeping the spin state the same.

d) Show that the Dirac Lagrangian is invariant under $\Psi \to e^{-i\alpha}\Psi$, and find the associated conserved current J_V^μ. Then show explicitly that $\partial_\mu J_V^\mu = 0$ using the Dirac equation.

Solution: The Lagrangian is invariant because we also have $\bar{\Psi} \to e^{i\alpha}\bar{\Psi}$, so the phases cancel out. Now, to avoid confusion when applying Noether's theorem, let's explicitly write out the spinor indices. The changes in the spinors are

$$(\delta\Psi)_a = -i\Psi_a, \qquad (\delta\bar{\Psi})_a = (i\bar{\Psi})_a \tag{S76}$$

and Noether's theorem states

$$J_V^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)_a}(\delta\Psi)_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Psi})_a}(\delta\bar{\Psi})_a \tag{S77}$$

where there is an implicit sum over the spinor index a. (It ranges from 0 to 3, covering the four elements of the spinor, but it is not a Lorentz index.) The second term is just zero, so we get

$$J_V^\mu = (i\bar{\Psi}\gamma^\mu)_a(-i\Psi)_a. \tag{S78}$$

We can rewrite this without explicit spinor indices as

$$J_V^\mu = \bar{\Psi}\gamma^\mu\Psi \tag{S79}$$

where there are now implicit spinor matrix multiplications. To check this is conserved, note that

$$\partial_\mu J_V^\mu = (\partial_\mu\bar{\Psi})\gamma^\mu\Psi + \bar{\Psi}\gamma^\mu\partial_\mu\Psi = im\bar{\Psi}\Psi - im\bar{\Psi}\Psi = 0. \tag{S80}$$

e) Show that when $m = 0$, the Dirac Lagrangian is also invariant under $\Psi \to e^{-i\alpha\gamma^5}\Psi$, and find the associated conserved current J_A^μ. What is $\partial_\mu J_A^\mu$ when m is nonzero?

Solution: Using the fact that γ^5 anticommutes with all gamma matrices, we have

$$\bar{\Psi} = \Psi^\dagger \gamma^0 \to \Psi^\dagger e^{i\alpha\gamma^5} \gamma^0 = \Psi^\dagger \gamma^0 e^{-i\alpha\gamma^5} = \bar{\Psi} e^{-i\alpha\gamma^5}. \tag{S81}$$

Thus, the Lagrangian becomes

$$\Psi^\dagger e^{-i\alpha\gamma^5} \not{\partial} e^{-i\alpha\gamma^5} \Psi. \tag{S82}$$

Moving the $e^{-i\alpha\gamma^5}$ past the $\not{\partial}$ flips the sign of the exponential, so we end up with a factor of $e^{i\alpha\gamma^5} e^{-i\alpha\gamma^5} = 1$. Now, the changes in the spinors per α, for infinitesimal α, are

$$\delta\Psi = -i\gamma^5\Psi, \qquad \delta\bar{\Psi} = -i\bar{\Psi}\gamma^5 \tag{S83}$$

which means the Noether current is

$$J_A^\mu = \bar{\Psi}\gamma^\mu\gamma^5\Psi. \tag{S84}$$

For nonzero m, the divergence of the current is

$$\begin{aligned}
\partial_\mu J_A^\mu &= (\partial_\mu \bar{\Psi})\gamma^\mu\gamma^5\Psi + \bar{\Psi}\gamma^\mu\gamma^5\partial_\mu\Psi && \text{(S85)}\\
&= (\partial_\mu \bar{\Psi})\gamma^\mu\gamma^5\Psi - \bar{\Psi}\gamma^5\gamma^\mu\partial_\mu\Psi && \text{(S86)}\\
&= im\bar{\Psi}\gamma^5\Psi + im\bar{\Psi}\gamma^5\Psi && \text{(S87)}\\
&= 2im\bar{\Psi}\gamma^5\Psi. && \text{(S88)}
\end{aligned}$$

4. ⋆ Spinors in three dimensions. (5 points)

In this course, we focus on spinors in four dimensions for good reason. In this optional problem, you'll see how the same mathematical structures appear in three dimensions.

a) Consider spinors in three spacetime dimensions. What are the smallest nonzero matrices that can satisfy (6)? Write down three such matrices γ^0, γ^1, and γ^2 explicitly.

Solution: The minimum size is 2×2, and one example set is

$$\gamma^0 = \sigma^2, \qquad \gamma^1 = i\sigma^1, \qquad \gamma^2 = i\sigma^3 \tag{S89}$$

where the σ^i are the Pauli matrices.

b) We define the spinor Lorentz generators by (8) in any dimension. Since there are now only two spatial dimensions, there is only one rotation generator S^{12}. What phase does a spinor pick up after a 2π rotation?

Solution: We have

$$S^{12} = \frac{i}{4}[i\sigma^1, i\sigma^3] = -\frac{\sigma^2}{2}. \tag{S90}$$

Then a 2π rotation yields

$$\exp\left(-\pi\sigma^2\right) = -I \tag{S91}$$

and therefore a phase of π, just like in 4 dimensions.

c) How does the tensor product of two spinor representations decompose into irreducible representations of the Lorentz group?

Solution: By the exact same logic as in 4 dimensions, we can extract a Lorentz scalar by $\bar{\psi}\psi$, and a Lorentz vector (which only has three components) by $\bar{\psi}\gamma^\mu\psi$. Since spinors have 2 components, the tensor product has 4 components, so this decomposition is complete: there isn't anything else. For instance, you might consider putting $[\gamma^\mu, \gamma^\nu]$ in the middle, but in three dimensions the commutator of two gamma matrices is just another gamma matrix, so you don't get anything new. Similarly, you can't use γ^5 to get anything new, as its analogue here, $i\gamma^0\gamma^1\gamma^2$, is just proportional to the identity.

In two spatial dimensions there are exotic particles called anyons, which can pick up an arbitrary phase after a 2π rotation. However, they can't be described by the conventional quantum fields covered in this course.

PHYS-330: QFT I — Problem Set 7 (due 11/15)

Stanford University, 2022 — Bernhard Mistlberger, Kevin Zhou

1. Plane wave solutions of the Dirac equation. (10 points)

In the Weyl representation the gamma matrices are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \sigma^\mu = (\mathbb{1}_2, \sigma^1, \sigma^2, \sigma^3), \qquad \bar{\sigma}^\mu = (\mathbb{1}_2, -\sigma^1, -\sigma^2, -\sigma^3). \tag{1}$$

Here, $\mathbb{1}_n$ denotes an $n \times n$ identity matrix.

a) Show that $(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2 \mathbb{1}_2$.

b) Show that the Dirac equation has positive-frequency plane wave solutions

$$\psi(x) = u_s(p) e^{-ip\cdot x}, \qquad u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} \tag{2}$$

where ξ_s is a two-component spinor, i.e. a complex vector with two elements. (Formally, given a matrix A with a complete basis of eigenvectors, with eigenvalues λ_i, we can define a matrix square root $\sqrt{A}$ to have the same eigenvectors, with eigenvalues $\sqrt{\lambda_i}$. But the only thing you need to know to do this problem is $\sqrt{A}\,\sqrt{A} = A$.)

c) Show that if we pick an orthonormal basis of two-component spinors ξ_s with $s \in \{1, 2\}$, satisfying $(\xi_r)^\dagger \xi_s = \delta_{rs}$, then the Dirac spinors satisfy the orthogonality relations

$$u_r(p)^\dagger u_s(p) = 2p_0\,\delta_{rs}, \qquad \bar{u}_r(p) u_s(p) = 2m\,\delta_{rs}. \tag{3}$$

d) Similarly, the Dirac equation has negative-frequency plane wave solutions

$$\psi(x) = v_s(p) e^{ip\cdot x}, \qquad v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix}. \tag{4}$$

Show that these solve the Dirac equation, and satisfy the orthogonality relations

$$v_r(p)^\dagger v_s(p) = 2p_0\,\delta_{rs}, \qquad \bar{v}_r(p) v_s(p) = -2m\,\delta_{rs}. \tag{5}$$

e) Show the completeness relations for the Dirac spinors,

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = \not{p} + m\mathbb{1}_4, \qquad \sum_{s=1}^{2} v_s(p)\bar{v}_s(p) = \not{p} - m\mathbb{1}_4. \tag{6}$$

2. Useful spinor identities. (10 points)

a) Prove the Gordon identity,

$$\bar{u}_r(p')\gamma^\mu u_s(p) = \bar{u}_r(p') \left(\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'_\nu - p_\nu)}{2m} \right) u_s(p) \tag{7}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. (Hint: this can be done using only the Dirac equation and the defining property of the gamma matrices.)

Combinations of gamma matrices can be used to produce a basis Γ^I for the space of 4×4 matrices, where I ranges from 1 to 16. Concretely, we have

$$
\begin{array}{llll}
 & & & \Gamma^{11} = (i/2)[\gamma^0, \gamma^1] \\
 & \Gamma^3 = \gamma^0 & \Gamma^7 = i\gamma^5\gamma^0 & \Gamma^{12} = (i/2)[\gamma^0, \gamma^2] \\
\Gamma^1 = \mathbb{1}_4 & \Gamma^4 = \gamma^1 & \Gamma^8 = i\gamma^5\gamma^1 & \Gamma^{13} = (i/2)[\gamma^0, \gamma^3] \\
\Gamma^2 = \gamma^5 & \Gamma^5 = \gamma^2 & \Gamma^9 = i\gamma^5\gamma^2 & \Gamma^{14} = (i/2)[\gamma^1, \gamma^2] \\
 & \Gamma^6 = \gamma^3 & \Gamma^{10} = i\gamma^5\gamma^3 & \Gamma^{15} = (i/2)[\gamma^1, \gamma^3] \\
 & & & \Gamma^{16} = (i/2)[\gamma^2, \gamma^3]
\end{array}
$$

We define the matrices $\tilde{\Gamma}^I$ similarly, but with lowered Lorentz indices; for example, $\tilde{\Gamma}^4 = \gamma_1 = -\gamma^1 = -\Gamma^4$. (This ensures the equations below won't have annoying extra signs due to the signs in the metric.) An inner product for 4×4 matrices can be defined by

$$\operatorname{tr}(AB) = \sum_{ab} A_{ab} B_{ba}. \tag{8}$$

Under this inner product, the matrices given above are orthogonal, in the sense that

$$\operatorname{tr}(\tilde{\Gamma}^I \Gamma^J) = 4\delta^{IJ} \tag{9}$$

which implies they are linearly independent, and thus indeed form a basis. (You don't have to check (9), but it follows directly from the results you proved in problem set 6.)

b) Show the completeness relation for the Γ matrices,

$$\delta_{ac}\delta_{db} = \frac{1}{4} \sum_I \tilde{\Gamma}^I_{dc} \Gamma^I_{ab}. \tag{10}$$

(Hint: it suffices to show that both sides give the same result when multiplied by a general matrix M_{cd}.)

c) Using (10), show the Fierz identity

$$\tilde{\Gamma}^I_{ab} \Gamma^J_{cd} = \frac{1}{16} \sum_{KL} \operatorname{tr}\left[\tilde{\Gamma}^I \tilde{\Gamma}^K \Gamma^J \Gamma^L\right] \tilde{\Gamma}^L_{ad} \Gamma^K_{cb}. \tag{11}$$

Contracting both sides of the Fierz identity with four spinors $\bar{u}_{1a} u_{2b} \bar{u}_{3c} u_{4d}$ yields

$$(\bar{u}_1 \tilde{\Gamma}^I u_2)(\bar{u}_3 \Gamma^J u_4) = \frac{1}{16} \sum_{KL} \operatorname{tr}\left[\tilde{\Gamma}^I \tilde{\Gamma}^K \Gamma^J \Gamma^L\right] (\bar{u}_1 \tilde{\Gamma}^L u_4)(\bar{u}_3 \Gamma^K u_2). \tag{12}$$

In other words, the Fierz identity relates the product of a contraction of $\bar{u}_1$ with u_2 and a contraction of $\bar{u}_3$ and u_4 (with arbitrary gamma matrices in the middle) to a combination of contractions of $\bar{u}_1$ with u_4 and $\bar{u}_3$ with u_2. This "Fierz transformation" rearranges how the spinors are contracted with each other, which can be useful in calculations.

d) Find the Fierz transformations for $(\bar{u}_1 u_2)(\bar{u}_3 u_4)$ and $(\bar{u}_1 \gamma^\mu u_2)(\bar{u}_3 \gamma_\mu u_4)$ explicitly. (Your final result should only contain the spinors and gamma matrices, not the Γ^I.)

3. Electromagnetism in relativistic notation. (20 points)

In electromagnetism, the Lagrangian is a function of the four-potential $A^\mu = (\phi, \mathbf{A})$,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu \,, \qquad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{13}$$

and $J^\mu = (\rho, \mathbf{J})$ is a classical current density.

a) Show that if the current is conserved, $\partial_\mu J^\mu = 0$, then the action remains the same under the gauge symmetry $A_\mu \to A_\mu + \partial_\mu \alpha$, for any smooth function α.

b) Show that the Euler–Lagrange equation for A^μ is

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{14}$$

c) Defining the electric and magnetic fields by $E^i = F^{i0}$ and $\epsilon^{ijk} B^k = -F^{ij}$, show that (14) is equivalent to two of Maxwell's equations.

d) The other two of Maxwell's equations are

$$\epsilon^{\mu\nu\rho\sigma} \partial_\mu F_{\nu\rho} = 0 \tag{15}$$

which follows directly from the definition of $F_{\mu\nu}$. Interestingly, all four of Maxwell's equations can be written as a single *spinor* equation. Show that the equation

$$\gamma^\nu \gamma^\rho \gamma^\sigma \partial_\nu F_{\rho\sigma} = 2 \gamma^\nu J_\nu \tag{16}$$

contains both (14) and (15). (This is just a mathematical trick with no physical meaning, but it's a nice application of the properties of gamma matrices.)

e) In the Lagrangian (13), we could have also included a term of the form $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$, which is Lorentz invariant and gauge invariant. What is it in terms of $\mathbf{E}$ and $\mathbf{B}$? Show that this term is a total derivative, and thus does not contribute to the action.

For the rest of this problem suppose there is no current, $J^\mu = 0$.

f) Construct the stress-energy tensor by directly applying Noether's theorem.

g) The stress-energy tensor you found in part (e) is conserved, but neither symmetric nor gauge invariant. However, we can define an "improved" stress-energy tensor,

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho (F^{\mu\rho} A^\nu). \tag{17}$$

Assuming the equations of motion hold, show that $\hat{T}^{\mu\nu}$ is symmetric, gauge invariant, conserved ($\partial_\mu \hat{T}^{\mu\nu} = 0$), and traceless ($\eta_{\mu\nu} \hat{T}^{\mu\nu} = 0$). Furthermore, show that it contains the familiar electromagnetic energy and momentum densities,

$$\hat{T}^{00} = \frac{1}{2} (|\mathbf{E}|^2 + |\mathbf{B}|^2) \,, \qquad \hat{T}^{0i} = (\mathbf{E} \times \mathbf{B})^i. \tag{18}$$

In general, the improved stress-energy tensor considered here is called the Belinfante tensor; it is this stress-energy tensor that sources gravity in general relativity. We didn't see it earlier because the improvement terms are related to the spin angular momentum of the field, which is zero for scalar fields.

4. ⋆ Chern–Simons theory. (5 points)

This optional problem presents a way to produce massive gauge fields.

a) The Proca Lagrangian for a massive vector field is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu. \tag{19}$$

Find the equation of motion for A^μ and show that for $m \neq 0$, it implies

$$(\partial^2 + m^2) A^\mu = 0, \qquad \partial_\mu A^\mu = 0. \tag{20}$$

However, this action does not have a gauge symmetry.

b) On the other hand, in two spatial dimensions it is possible to have massive gauge fields. We consider a Lagrangian with a "Chern–Simons" term,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho \tag{21}$$

where all indices take the values 0, 1, and 2, and $\epsilon^{012} = 1$. (This new term is a relative of the term we considered in 3(e), but it is not a total derivative.) Show that the action is gauge invariant, and find the Euler–Lagrange equation for A^μ.

c) The Chern–Simons term is sometimes called a "topological" mass term, because it doesn't depend on the metric. Show that the equation of motion implies

$$(\partial^2 + m^2) F^{\mu\nu} = 0 \tag{22}$$

for some m you should find. Thus the field is massive, yet still has a gauge symmetry.

Since there is no Chern–Simons term in three spatial dimensions, it has little role in particle physics, but it is important in the description of topological effects in two-dimensional condensed matter systems.

1. Plane wave solutions of the Dirac equation. (10 points)

In the Weyl representation the gamma matrices are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \qquad \sigma^\mu = (\mathbb{1}_2, \sigma^1, \sigma^2, \sigma^3), \qquad \bar\sigma^\mu = (\mathbb{1}_2, -\sigma^1, -\sigma^2, -\sigma^3). \tag{1}$$

Here, $\mathbb{1}_n$ denotes an $n \times n$ identity matrix.

a) Show that $(p\cdot\sigma)(p\cdot\bar\sigma) = p^2\,\mathbb{1}_2$.

Solution: Inserting the definitions and suppressing the factors of the identity, the expression is

$$(p^0 - \mathbf{p}\cdot\boldsymbol{\sigma})(p^0 + \mathbf{p}\cdot\boldsymbol{\sigma}) = (p^0)^2 - \frac{1}{2}p^i p^j\{\sigma^i, \sigma^j\} = (p^0)^2 - |\mathbf{p}|^2 = p^2 \tag{S1}$$

as desired, where we used $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$.

b) Show that the Dirac equation has positive-frequency plane wave solutions

$$\psi(x) = u_s(p)e^{-ip\cdot x}, \qquad u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix} \tag{2}$$

where ξ_s is a two-component spinor, i.e. a complex vector with two elements. (Formally, given a matrix A with a complete basis of eigenvectors, with eigenvalues λ_i, we can define a matrix square root $\sqrt{A}$ to have the same eigenvectors, with eigenvalues $\sqrt{\lambda_i}$. But the only thing you need to know to do this problem is $\sqrt{A}\,\sqrt{A} = A$.)

Solution: By the definition of the matrix square root and the result of part (a), we have

$$\sigma\cdot p = \sqrt{\sigma\cdot p}\cdot\sqrt{\sigma\cdot p}, \quad \bar\sigma\cdot p = \sqrt{\bar\sigma\cdot p}\cdot\sqrt{\bar\sigma\cdot p}, \quad m\mathbb{1}_2 = \sqrt{\sigma\cdot p}\cdot\sqrt{\bar\sigma\cdot p} = \sqrt{\bar\sigma\cdot p}\cdot\sqrt{\sigma\cdot p} \tag{S2}$$

where we used $p^2 = m^2$. To show the Dirac equation is satisfied, note that the derivative ∂_μ just pulls down a factor of $-ip^\mu$. Thus, it suffices to show that

$$(p_\mu\gamma^\mu - m\mathbb{1}_4)u_s(p) = 0. \tag{S3}$$

To show this, we just plug in the definitions, giving

$$(p_\mu\gamma^\mu - m\mathbb{1}_4)u_s(p) = \begin{pmatrix} -m\mathbb{1}_2 & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma} & -m\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix} \tag{S4}$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}(-m\mathbb{1}_2 + \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma})\xi_s \\ \sqrt{p\cdot\bar\sigma}(\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} - m\mathbb{1}_2)\xi_s \end{pmatrix} \tag{S5}$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}(-m\mathbb{1}_2 + \sqrt{p^2}\mathbb{1}_2)\xi_s \\ \sqrt{p\cdot\bar\sigma}(\sqrt{p^2}\mathbb{1}_2 - m\mathbb{1}_2)\xi_s \end{pmatrix} \tag{S6}$$

$$= 0 \tag{S7}$$

where we used $p^2 = m^2$ in the last step.

c) Show that if we pick an orthonormal basis of two-component spinors ξ_s with $s \in \{1, 2\}$, satisfying $(\xi_r)^\dagger\xi_s = \delta_{rs}$, then the Dirac spinors satisfy the orthogonality relations

$$u_r(p)^\dagger u_s(p) = 2p_0\,\delta_{rs}, \qquad \bar u_r(p)u_s(p) = 2m\,\delta_{rs}. \tag{3}$$

Solution: To take the Hermitian conjugate, note that $(\sigma^\mu)^\dagger = \sigma^\mu$. Then we simply have

$$u_s^\dagger(p)u_r(p) = \begin{pmatrix} \xi_s^\dagger\sqrt{\sigma\cdot p} & \xi_s^\dagger\sqrt{\bar\sigma\cdot p} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\sigma\cdot p}\,\xi_r \\ \sqrt{\bar\sigma\cdot p}\,\xi_r \end{pmatrix} \tag{S8}$$

$$= \xi_s^\dagger\left(p\cdot\sigma + p\cdot\bar\sigma\right)\xi_r = 2p^0\xi_s^\dagger\xi_r = 2p^0\delta_{rs}. \tag{S9}$$

The proof of the second orthogonality relation is similar:

$$\bar u_s(p)u_r(p) = u_s^\dagger(p)\gamma^0u_r(p) = \begin{pmatrix} \xi_s^\dagger\sqrt{\bar\sigma\cdot p} & \xi_s^\dagger\sqrt{\sigma\cdot p} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\sigma\cdot p}\,\xi_r \\ \sqrt{\bar\sigma\cdot p}\,\xi_r \end{pmatrix} \tag{S10}$$

$$= 2\xi_s^\dagger\left(\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma}\right)\xi_r = 2m\xi_s^\dagger\xi_r = 2m\delta_{rs}. \tag{S11}$$

d) Similarly, the Dirac equation has negative-frequency plane wave solutions

$$\psi(x) = v_s(p)e^{ip\cdot x}, \qquad v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ -\sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix}. \tag{4}$$

Show that these solve the Dirac equation, and satisfy the orthogonality relations

$$v_r(p)^\dagger v_s(p) = 2p_0\,\delta_{rs}, \qquad \bar v_r(p)v_s(p) = -2m\,\delta_{rs}. \tag{5}$$

Solution: The proof that these plane waves solve the Dirac equation is very similar to that of part (b), except that now the derivative ∂_μ pulls down a factor of ip^μ, so we want to show

$$(p_\mu\gamma^\mu + m\mathbb{1}_4)v_s(p) = 0. \tag{S12}$$

This holds because

$$(p_\mu\gamma^\mu + m\mathbb{1}_4)v_s(p) = \begin{pmatrix} m\mathbb{1}_2 & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma} & m\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ -\sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix} \tag{S13}$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}(m\mathbb{1}_2 - \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma})\xi_s \\ \sqrt{p\cdot\bar\sigma}(\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} - m\mathbb{1}_2)\xi_s \end{pmatrix} \tag{S14}$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}(m\mathbb{1}_2 - \sqrt{p^2}\mathbb{1}_2)\xi_s \\ \sqrt{p\cdot\bar\sigma}(\sqrt{p^2}\mathbb{1}_2 - m\mathbb{1}_2)\xi_s \end{pmatrix} \tag{S15}$$

$$= 0. \tag{S16}$$

The proofs of the orthogonality relations are very similar to part (c),

$$v_s^\dagger(p)v_r(p) = \begin{pmatrix} \xi_s^\dagger\sqrt{\sigma\cdot p} & -\xi_s^\dagger\sqrt{\bar\sigma\cdot p} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\sigma\cdot p}\,\xi_r \\ -\sqrt{\bar\sigma\cdot p}\,\xi_r \end{pmatrix} \tag{S17}$$

$$= \xi_s^\dagger\left(p\cdot\sigma + p\cdot\bar\sigma\right)\xi_r = 2p^0\xi_s^\dagger\xi_r = 2p^0\delta_{rs} \tag{S18}$$

and

$$\bar v_s(p)v_r(p) = v_s(p)\gamma^0v_r(p) = -\begin{pmatrix} \xi_s^\dagger\sqrt{\bar\sigma\cdot p} & \xi_s^\dagger\sqrt{\sigma\cdot p} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\sigma\cdot p}\,\xi_r \\ \sqrt{\bar\sigma\cdot p}\,\xi_r \end{pmatrix} \tag{S19}$$

$$= -2\xi_s^\dagger\left(\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma}\right)\xi_r = -2m\xi_s^\dagger\xi_r = -2m\delta_{rs}. \tag{S20}$$

e) Show the completeness relations for the Dirac spinors,

$$\sum_{s=1}^{2} u_s(p)\bar u_s(p) = \not{p} + m\mathbb{1}_4, \qquad \sum_{s=1}^{2} v_s(p)\bar v_s(p) = \not{p} - m\mathbb{1}_4. \tag{6}$$

Solution: This follows straightforwardly from plugging in the definitions,

$$\sum_{s=1}^{2} u_s(p)\bar u_s(p) = \sum_{s=1}^{2}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix}\begin{pmatrix} \xi_s^\dagger\sqrt{p\cdot\bar\sigma} & \xi_s^\dagger\sqrt{p\cdot\sigma} \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix} = p_\mu\gamma^\mu + m\mathbb{1}_4. \tag{S21}$$

$$\sum_{s=1}^{2} v_s(p)\bar{v}_s(p) = \sum_{s=1}^{2} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} \begin{pmatrix} -\xi_s^\dagger\sqrt{p\cdot\bar{\sigma}} & \xi_s^\dagger\sqrt{p\cdot\sigma} \end{pmatrix} = \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} = p_\mu\gamma^\mu - m\mathbb{1}_4. \tag{S22}$$

In both cases we used the completeness relation for the two-component spinors, $\sum_{s=1}^{2}\xi_s\xi_s^\dagger = \mathbb{1}_2$.

2. Useful spinor identities. (10 points)

a) Prove the Gordon identity,

$$\bar{u}_r(p')\gamma^\mu u_s(p) = \bar{u}_r(p')\left(\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'_\nu - p_\nu)}{2m}\right)u_s(p) \tag{7}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. (Hint: this can be done using only the Dirac equation and the defining property of the gamma matrices.)

Solution: It's easiest to start from the second term on the right-hand side. Note that

$$\begin{aligned} \bar{u}_r(p')[\gamma^\mu,\gamma^\nu](p'-p)_\nu u_s(p) &= \bar{u}_r(p')(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p'-p)_\nu u_s(p) && \text{(S23)} \\ &= \bar{u}_r(p')\left[2(p'+p)^\mu - 2(\not{p}'\gamma^\mu + \gamma^\mu\not{p})\right]u_s(p) && \text{(S24)} \\ &= \bar{u}_r(p')(-4m\gamma^\mu + 2(p'^\mu + p^\mu))u_s(p) && \text{(S25)} \end{aligned}$$

where we used the Dirac equation (S3) in the last step. Plugging this into the right hand side of (7) gives the desired result.

Combinations of gamma matrices can be used to produce a basis Γ^I for the space of 4×4 matrices, where I ranges from 1 to 16. Concretely, we have

$$\begin{array}{llll} & & & \Gamma^{11} = (i/2)[\gamma^0,\gamma^1] \\ & \Gamma^3 = \gamma^0 & \Gamma^7 = i\gamma^5\gamma^0 & \Gamma^{12} = (i/2)[\gamma^0,\gamma^2] \\ \Gamma^1 = \mathbb{1}_4 & \Gamma^4 = \gamma^1 & \Gamma^8 = i\gamma^5\gamma^1 & \Gamma^{13} = (i/2)[\gamma^0,\gamma^3] \\ \Gamma^2 = \gamma^5 & \Gamma^5 = \gamma^2 & \Gamma^9 = i\gamma^5\gamma^2 & \Gamma^{14} = (i/2)[\gamma^1,\gamma^2] \\ & \Gamma^6 = \gamma^3 & \Gamma^{10} = i\gamma^5\gamma^3 & \Gamma^{15} = (i/2)[\gamma^1,\gamma^3] \\ & & & \Gamma^{16} = (i/2)[\gamma^2,\gamma^3] \end{array}$$

We define the matrices $\tilde{\Gamma}^I$ similarly, but with lowered Lorentz indices; for example, $\tilde{\Gamma}^4 = \gamma_1 = -\gamma^1 = -\Gamma^4$. (This ensures the equations below won't have annoying extra signs due to the signs in the metric.) An inner product for 4×4 matrices can be defined by

$$\mathrm{tr}(AB) = \sum_{ab} A_{ab}B_{ba}. \tag{8}$$

Under this inner product, the matrices given above are orthogonal, in the sense that

$$\mathrm{tr}(\tilde{\Gamma}^I\Gamma^J) = 4\delta^{IJ} \tag{9}$$

which implies they are linearly independent, and thus indeed form a basis. (You don't have to check (9), but it follows directly from the results you proved in problem set 6.)

b) Show the completeness relation for the Γ matrices,

$$\delta_{ac}\delta_{db} = \frac{1}{4}\sum_I \tilde{\Gamma}^I_{dc}\Gamma^I_{ab}. \tag{10}$$

(Hint: it suffices to show that both sides give the same result when multiplied by a general matrix M_{cd}.)

Solution: We consider a general matrix M, where

$$M_{cd} = \sum_I m^I \Gamma^I_{cd}. \tag{S26}$$

Multiplying both sides with M_{cd}, the left-hand side is just M_{ab}, while the right-hand side is

$$\frac{1}{4}\sum_I \tilde{\Gamma}^I_{dc}\Gamma^I_{ab}\sum_J m^J\Gamma^J_{cd} = \sum_{IJ}\Gamma^I_{ab}m^J\,\mathrm{tr}(\tilde{\Gamma}^I\Gamma^J) = \sum_I \Gamma^I_{ab}m^I = M_{ab}. \tag{S27}$$

Since the two sides match for any matrix M, the completeness relation holds.

c) Using (10), show the Fierz identity

$$\tilde{\Gamma}^I_{ab}\Gamma^J_{cd} = \frac{1}{16}\sum_{KL}\mathrm{tr}\left[\tilde{\Gamma}^I\tilde{\Gamma}^K\Gamma^J\Gamma^L\right]\tilde{\Gamma}^L_{ad}\Gamma^K_{cb}. \tag{11}$$

Solution: We note that

$$\begin{aligned}
\tilde{\Gamma}^I_{ab}\Gamma^J_{cd} &= \tilde{\Gamma}^I_{ef}\Gamma^J_{gh}\delta_{ae}\delta_{fb}\delta_{cg}\delta_{hd} && \text{(S28)}\\
&= \frac{1}{16}\tilde{\Gamma}^I_{ef}\Gamma^J_{gh}\sum_{KL}\tilde{\Gamma}^L_{ad}\Gamma^L_{he}\tilde{\Gamma}^K_{fg}\Gamma^K_{cb} && \text{(S29)}\\
&= \frac{1}{16}\sum_{KL}\mathrm{tr}\left[\tilde{\Gamma}^I\tilde{\Gamma}^K\Gamma^J\Gamma^L\right]\tilde{\Gamma}^L_{ad}\Gamma^K_{cb}. && \text{(S30)}
\end{aligned}$$

where we applied the completeness relation twice, and then used the definition of the trace.

Contracting both sides of the Fierz identity with four spinors $\bar{u}_{1a}u_{2b}\bar{u}_{3c}u_{4d}$ yields

$$(\bar{u}_1\tilde{\Gamma}^I u_2)(\bar{u}_3\Gamma^J u_4) = \frac{1}{16}\sum_{KL}\mathrm{tr}\left[\tilde{\Gamma}^I\tilde{\Gamma}^K\Gamma^J\Gamma^L\right](\bar{u}_1\tilde{\Gamma}^L u_4)(\bar{u}_3\Gamma^K u_2). \tag{12}$$

In other words, the Fierz identity relates the product of a contraction of $\bar{u}_1$ with u_2 and a contraction of $\bar{u}_3$ and u_4 (with arbitrary gamma matrices in the middle) to a combination of contractions of $\bar{u}_1$ with u_4 and $\bar{u}_3$ with u_2. This "Fierz transformation" rearranges how the spinors are contracted with each other, which can be useful in calculations.

d) Find the Fierz transformations for $(\bar{u}_1u_2)(\bar{u}_3u_4)$ and $(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4)$ explicitly. (Your final result should only contain the spinors and gamma matrices, not the Γ^I.)

Solution: For the first case, we can apply the Fierz identity with $\tilde{\Gamma}^I = \Gamma^J = \mathbb{1}_4$. Then the right-hand side becomes

$$\begin{aligned}
\frac{1}{16}\sum_{KL}\mathrm{tr}\left[\tilde{\Gamma}^I\tilde{\Gamma}^K\Gamma^J\Gamma^L\right]\tilde{\Gamma}^L_{ad}\Gamma^K_{cb} &= \frac{1}{16}\sum_{KL}\mathrm{tr}\left[\tilde{\Gamma}^K\Gamma^L\right]\tilde{\Gamma}^L_{ad}\Gamma^K_{cb} && \text{(S31)}\\
&= \frac{1}{4}\sum_{KL}\delta^{KL}\tilde{\Gamma}^L_{ad}\Gamma^K_{cb} && \text{(S32)}\\
&= \frac{1}{4}\sum_{K}\tilde{\Gamma}^K_{ad}\Gamma^K_{cb}. && \text{(S33)}
\end{aligned}$$

Contracting both sides with the spinors, we conclude

$$\begin{aligned}
(\bar{u}_1u_2)(\bar{u}_3u_4) = \frac{1}{4}\Big[&(\bar{u}_1u_4)(\bar{u}_3u_2) + (\bar{u}_1\gamma^5u_4)(\bar{u}_3\gamma^5u_2) + (\bar{u}_1\gamma^\mu u_4)(\bar{u}_3\gamma_\mu u_2) && \text{(S34)}\\
&- (\bar{u}_1\gamma^5\gamma^\mu u_4)(\bar{u}_3\gamma^5\gamma_\mu u_2) - \frac{1}{8}(\bar{u}_1[\gamma^\mu,\gamma^\nu]u_4)(\bar{u}_3[\gamma_\mu,\gamma_\nu]u_2)\Big]. && \text{(S35)}
\end{aligned}$$

Note that there is an extra factor of $1/2$ in the final term to avoid double counting. This result is admittedly rather complex, but it's indeed rearranged as desired.

For the second case, we take $\tilde{\Gamma}^I = \gamma_\mu$ and $\Gamma^J = \gamma^\mu$. Computing the right-hand side requires some casework, and use of the trace identities derived in problem set 6. The results are

$$\tilde{\Gamma}^K = \mathbb{1}_4: \qquad \sum_L \operatorname{tr}\left[\gamma_\mu\gamma^\mu\Gamma^L\right]\tilde{\Gamma}^L_{ad}\delta_{cb} = 16\delta_{cb}\delta_{ad}, \tag{S36}$$

$$\tilde{\Gamma}^K = \gamma_\nu: \qquad \sum_L \operatorname{tr}\left[\gamma_\mu\gamma_\nu\gamma^\mu\Gamma^L\right]\tilde{\Gamma}^L_{ad}\gamma^\nu_{cb} = -8\gamma_{\nu ad}\gamma^\nu_{cb}, \tag{S37}$$

$$\tilde{\Gamma}^K = \gamma^5: \qquad \sum_L \operatorname{tr}\left[\gamma_\mu\gamma^5\gamma^\mu\Gamma^L\right]\tilde{\Gamma}^L_{ad}\gamma_{5cb} = -16\gamma_{5ad}\gamma_{5cb}, \tag{S38}$$

$$\tilde{\Gamma}^K = i\gamma^5\gamma_\nu: \qquad -\sum_L \operatorname{tr}\left[\gamma_\mu\gamma^5\gamma_\nu\gamma^\mu\Gamma^L\right]\tilde{\Gamma}^L_{ad}(\gamma_5\gamma^\nu)_{cb} = 8(i\gamma_5\gamma_\nu)_{ad}(i\gamma_5\gamma^\nu)_{cb}, \tag{S39}$$

$$\tilde{\Gamma}^K = \frac{i}{2}\sum_L[\gamma_\nu,\gamma_\rho]: \qquad \frac{i}{2}\operatorname{tr}\left[\gamma_\mu[\gamma_\nu,\gamma_\rho]\gamma^\mu\Gamma^L\right]\tilde{\Gamma}^L_{ad}\left(\frac{i}{2}[\gamma_\nu,\gamma_\rho]\right)_{cb} = 0. \tag{S40}$$

Putting this all together yields

$$(\bar{u}_1\gamma_\mu u_2)(\bar{u}_3\gamma^\mu u_4) = (\bar{u}_1 u_4)(\bar{u}_3 u_2) - \frac{1}{2}(\bar{u}_1\gamma_\mu u_4)(\bar{u}_3\gamma^\mu u_2) \tag{S41}$$

$$- (\bar{u}_1\gamma^5 u_4)(\bar{u}_3\gamma^5 u_2) - \frac{1}{2}(\bar{u}_1\gamma^5\gamma_\mu u_4)(\bar{u}_3\gamma^5\gamma^\mu u_2). \tag{S42}$$

3. Electromagnetism in relativistic notation. (20 points)

In electromagnetism, the Lagrangian is a function of the four-potential $A^\mu = (\phi, \mathbf{A})$,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\,, \qquad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{13}$$

and $J^\mu = (\rho, \mathbf{J})$ is a classical current density.

a) Show that if the current is conserved, $\partial_\mu J^\mu = 0$, then the action remains the same under the gauge symmetry $A_\mu \to A_\mu + \partial_\mu\alpha$, for any smooth function α.

Solution: The kinetic term is gauge invariant because $F_{\mu\nu}$ is,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu + \partial_\nu\partial_\mu\alpha - \partial_\nu A_\mu - \partial_\nu\partial_\mu\alpha = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{S43}$$

The interaction term is gauge invariant when the current is conserved because

$$J^\mu A_\mu \to J^\mu A_\mu + J^\mu\partial_\mu\alpha = J^\mu A_\mu + \partial_\mu(J^\mu\alpha) \tag{S44}$$

where the change is a total derivative term, which does not affect the action.

b) Show that the Euler–Lagrange equation for A^μ is

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{14}$$

Solution: The Euler–Lagrange equation is

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial\mathcal{L}}{\partial A_\nu}. \tag{S45}$$

The right-hand side is clearly $-J^\nu$. As for the left-hand side, we use the product rule, giving

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}\partial_\mu\left(F^{\rho\sigma}\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)}\right) \tag{S46}$$

$$= -\frac{1}{2}\partial_\mu\left(F^{\rho\sigma}(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho)\right) \tag{S47}$$

$$= -\frac{1}{2}\partial_\mu\left(F^{\mu\nu} - F^{\nu\mu}\right) \tag{S48}$$

$$= -\partial_\mu F^{\mu\nu}. \tag{S49}$$

This gives the desired result.

c) Defining the electric and magnetic fields by $E^i = F^{i0}$ and $\epsilon^{ijk}B^k = -F^{ij}$, show that (14) is equivalent to two of Maxwell's equations.

Solution: First set $\nu = 0$. Since $F^{00} = 0$, we have $\partial_i F^{i0} = J^0$, which in vector calculus notation is Gauss's law, $\nabla \cdot \mathbf{E} = \rho$. Next set $\nu = j$. Here we have

$$J^j = \partial_0 F^{0j} + \partial_i F^{ij} = -\dot{E}^j - \partial_i \epsilon^{ijk} B^k \tag{S50}$$

which in vector calculus notation is

$$\mathbf{J} = -\dot{\mathbf{E}} + \nabla \times \mathbf{B} \tag{S51}$$

which is Ampere's law.

d) The other two of Maxwell's equations are

$$\epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\nu\rho} = 0 \tag{15}$$

which follows directly from the definition of $F_{\mu\nu}$. Interestingly, all four of Maxwell's equations can be written as a single *spinor* equation. Show that the equation

$$\gamma^\nu \gamma^\rho \gamma^\sigma \partial_\nu F_{\rho\sigma} = 2\gamma^\nu J_\nu \tag{16}$$

contains both (14) and (15). (This is just a mathematical trick with no physical meaning, but it's a nice application of the properties of gamma matrices.)

Solution: To recover (14), we want to get the current by itself on the right-hand side. Thus, multiply both sides by γ^μ and take the trace, giving

$$\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)\partial_\nu F_{\rho\sigma} = 2\operatorname{tr}(\gamma^\mu\gamma^\nu)J_\nu = 8J^\mu. \tag{S52}$$

Simplifying the left-hand side, we have

$$\begin{aligned}\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)\partial_\nu F_{\rho\sigma} &= 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})\partial_\nu F_{\rho\sigma} && \text{(S53)}\\ &= 4(-\partial_\nu F^{\mu\nu} + \partial_\nu F^{\nu\mu}) && \text{(S54)}\\ &= 8\,\partial_\nu F^{\nu\mu} && \text{(S55)}\end{aligned}$$

where we used the fact that F is antisymmetric, so $\eta^{\rho\sigma}F_{\rho\sigma} = 0$. Equating the two sides and renaming the indices recovers (14).

To recover (15), we want to get a factor of $\epsilon^{\mu\nu\rho\sigma}$ on the left-hand side. Therefore, multiply both sides by $\gamma^5\gamma^\mu$ and take the trace. By the results we've derived previously, the right-hand side vanishes and the trace on the left-hand side is proportional to $\epsilon^{\mu\nu\rho\sigma}$, as desired.

e) In the Lagrangian (13), we could have also included a term of the form $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$, which is Lorentz invariant and gauge invariant. What is it in terms of $\mathbf{E}$ and $\mathbf{B}$? Show that this term is a total derivative, and thus does not contribute to the action.

Solution: To write this term in terms of $\mathbf{E}$ and $\mathbf{B}$, we just expand out the contractions. There are 24 terms, corresponding to the cases where μ, ν, ρ, and σ are all distinct. However, the antisymmetry of F means that groups of eight of them all give the same thing, e.g.

$$\begin{aligned}\epsilon^{0123}F_{01}F_{23} &= \epsilon^{1023}F_{10}F_{23} = \epsilon^{0132}F_{01}F_{32} = \epsilon^{1032}F_{10}F_{32}\\ &= \epsilon^{2301}F_{23}F_{01} = \epsilon^{2310}F_{23}F_{10} = \epsilon^{3201}F_{32}F_{01} = \epsilon^{3210}F_{32}F_{10}.\end{aligned} \tag{S56}$$

There are thus only three distinct terms to write down. Adjusting the index position for convenience,

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} &= 8(\epsilon_{0123}F^{01}F^{23} + \epsilon_{0213}F^{02}F^{13} + \epsilon_{0312}F^{03}F^{12}) && \text{(S57)}\\ &= 8(-F^{01}F^{23} + F^{02}F^{13} - F^{03}F^{12}) && \text{(S58)}\\ &= 8(F^{10}F^{23} + F^{20}F^{31} + F^{30}F^{12}) && \text{(S59)}\\ &= 8(F^{10}F^{23} + F^{20}F^{31} + F^{30}F^{12}) && \text{(S60)}\\ &= -8\,\mathbf{E}\cdot\mathbf{B}. && \text{(S61)}\end{aligned}$$

Now, to show the term is a total derivative, we note that

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} &= \epsilon^{\mu\nu\rho\sigma}\left[\partial_\mu A_\nu\partial_\rho A_\sigma - \partial_\nu A_\mu\partial_\rho A_\sigma - \partial_\mu A_\nu\partial_\sigma A_\rho + \partial_\nu A_\mu\partial_\sigma A_\rho\right] && \text{(S62)}\\
&= 4\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu\partial_\rho A_\sigma && \text{(S63)}\\
&= 4\epsilon^{\mu\nu\rho\sigma}\partial_\mu(A_\nu\partial_\rho A_\sigma) - 4\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\mu(\partial_\rho A_\sigma) && \text{(S64)}\\
&= \partial_\mu\left(4\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma\right) && \text{(S65)}
\end{aligned}$$

which is a total derivative as desired. Note that the second and third steps in this derivation just follow from the antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$.

For the rest of this problem suppose there is no current, $J^\mu = 0$.

f) Construct the stress-energy tensor by directly applying Noether's theorem.

Solution: Applying Noether's theorem as usual, we have

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho - \eta^{\mu\nu}\mathcal{L} = -F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{S66}$$

g) The stress-energy tensor you found in part (e) is conserved, but neither symmetric nor gauge invariant. However, we can define an "improved" stress-energy tensor,

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho(F^{\mu\rho}A^\nu). \tag{17}$$

Assuming the equations of motion hold, show that $\hat{T}^{\mu\nu}$ is symmetric, gauge invariant, conserved ($\partial_\mu\hat{T}^{\mu\nu} = 0$), and traceless ($\eta_{\mu\nu}\hat{T}^{\mu\nu} = 0$). Furthermore, show that it contains the familiar electromagnetic energy and momentum densities,

$$\hat{T}^{00} = \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2), \qquad \hat{T}^{0i} = (\mathbf{E}\times\mathbf{B})^i. \tag{18}$$

Solution: Adding on the new term gives

$$\hat{T}^{\mu\nu} = (\partial_\rho F^{\mu\rho})A^\nu + F^{\mu\rho}F_\rho{}^\nu + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{S67}$$

When the equations of motion hold, $\partial_\rho F^{\mu\rho} = 0$, so we may simply drop the first term, giving

$$\hat{T}^{\mu\nu} = -\eta_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{S68}$$

where we cleaned up the indices a bit. In this form, the stress-energy tensor is clearly symmetric, and it is gauge invariant because it is written in terms of the gauge invariant F alone. To show conservation, it is easiest to note that

$$\partial_\mu\hat{T}^{\mu\nu} = \partial_\mu T^{\mu\nu} + \partial_\mu\partial_\rho(F^{\mu\rho}A^\nu) = 0 \tag{S69}$$

where the first term vanishes by Noether's theorem, and the second term vanishes by the antisymmetry of F. Finally, to show tracelessness, we note that

$$\eta_{\mu\nu}\hat{T}^{\mu\nu} = -\eta_{\mu\nu}\eta_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} + \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = -F_{\nu\sigma}F^{\nu\sigma} + F_{\rho\sigma}F^{\rho\sigma} = 0. \tag{S70}$$

Now let's consider the components. For the energy density, we have

$$\begin{aligned}
\hat{T}^{00} &= -\eta_{\rho\sigma}F^{0\rho}F^{0\sigma} + \frac{1}{4}\eta^{00}F_{\rho\sigma}F^{\rho\sigma} && \text{(S71)}\\
&= F^{01}F^{01} + F^{02}F^{02} + F^{03}F^{03} + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} && \text{(S72)}\\
&= |\mathbf{E}|^2 + \frac{1}{2}(F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} + F_{12}F^{12} + F_{23}F^{23} + F_{31}F^{31}) && \text{(S73)}\\
&= |\mathbf{E}|^2 + \frac{1}{2}(-|\mathbf{E}|^2 + |\mathbf{B}|^2) && \text{(S74)}\\
&= \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2) && \text{(S75)}
\end{aligned}$$

as desired. For the momentum density, we have

$$\hat{T}^{0i} = -\eta_{\rho\sigma} F^{0\rho} F^{i\sigma} = F^{0j} F^{ij} = E^j \epsilon^{ijk} B^k = (\mathbf{E} \times \mathbf{B})^i. \tag{S76}$$

In general, the improved stress-energy tensor considered here is called the Belinfante tensor; it is this stress-energy tensor that sources gravity in general relativity. We didn't see it earlier because the improvement terms are related to the spin angular momentum of the field, which is zero for scalar fields.

4. ⋆ Chern–Simons theory. (5 points)

This optional problem presents a way to produce massive gauge fields.

a) The Proca Lagrangian for a massive vector field is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu. \tag{19}$$

Find the equation of motion for A^μ and show that for $m \neq 0$, it implies

$$(\partial^2 + m^2) A^\mu = 0, \qquad \partial_\mu A^\mu = 0. \tag{20}$$

However, this action does not have a gauge symmetry.

Solution: The derivation of the equation of motion is similar to problem 3(b), but now a new term appears on the right-hand side, giving

$$-\partial_\mu F^{\mu\nu} = m^2 A^\nu. \tag{S77}$$

This is the equation of motion. Applying ∂_ν to both sides yields

$$m^2 \partial_\nu A^\nu = -\partial_\nu \partial_\mu F^{\mu\nu} = 0 \tag{S78}$$

by the antisymmetry of F, which shows $\partial_\mu A^\mu = 0$ when m is nonzero. Plugging this result back into the original equation of motion gives

$$-\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = -\partial^2 A^\nu = m^2 A^\nu \tag{S79}$$

as desired.

b) On the other hand, in two spatial dimensions it is possible to have massive gauge fields. We consider a Lagrangian with a "Chern–Simons" term,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho \tag{21}$$

where all indices take the values 0, 1, and 2, and $\epsilon^{012} = 1$. (This new term is a relative of the term we considered in 3(e), but it is not a total derivative.) Show that the action is gauge invariant, and find the Euler–Lagrange equation for A^μ.

Solution: The proof that the action is gauge invariant is similar to 3(e). Note that the change in the Lagrangian under a gauge transformation is

$$\delta\mathcal{L} = \frac{\alpha}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} \partial_\rho \alpha = \frac{\alpha}{2} \epsilon^{\mu\nu\rho} (\partial_\mu A_\nu) \partial_\rho \alpha = \partial_\rho \left(\frac{\alpha}{2} \epsilon^{\mu\nu\rho} \alpha \partial_\mu A_\nu \right) \tag{S80}$$

which is a total derivative, as desired. As for the Euler–Lagrange equation, starting from our solution to 3(b), we pick up an extra term on both the left-hand and right-hand sides, giving

$$-\partial_\mu F^{\mu\nu} + \frac{\alpha}{2} \epsilon^{\mu\nu\rho} \partial_\mu A_\rho = \frac{\alpha}{4} \epsilon^{\mu\rho\nu} F_{\mu\rho}. \tag{S81}$$

We can clean this up using the antisymmetry of ϵ to give

$$\partial_\mu F^{\mu\nu} = \frac{\alpha}{2}\epsilon^{\mu\nu\rho}F_{\mu\rho}. \tag{S82}$$

c) The Chern–Simons term is sometimes called a "topological" mass term, because it doesn't depend on the metric. Show that the equation of motion implies

$$(\partial^2 + m^2)F^{\mu\nu} = 0 \tag{22}$$

for some m you should find. Thus the field is massive, yet still has a gauge symmetry.

Solution: This is a bit fiddly. The idea is that we want to use the equation of motion twice. But in its current form, we can't really do much to it, because of the factor of ϵ on the right-hand side which already contracts with most of the indices. However, we can clear factors of ϵ out of the way by contracting with additional copies of ϵ, and using

$$\epsilon^{\mu\nu\rho}\epsilon_{\mu\alpha\beta} = \delta^\nu_\alpha\delta^\rho_\beta - \delta^\nu_\beta\delta^\rho_\alpha. \tag{S83}$$

To get started, we contract both sides of (S82) with $\epsilon_{\nu\alpha\beta}$, giving

$$\epsilon_{\nu\alpha\beta}\partial_\mu F^{\mu\nu} = \alpha F_{\beta\alpha}. \tag{S84}$$

Now we can apply ∂^β to both sides and use the equation of motion again, giving

$$\epsilon_{\nu\alpha\beta}\partial^\beta\partial_\mu F^{\mu\nu} = \frac{\alpha^2}{2}\epsilon_{\mu\alpha\nu}F^{\mu\nu}. \tag{S85}$$

To get rid of the ϵ's again, we contract both sides with $\epsilon^{\alpha\delta\sigma}$, giving

$$\partial^\delta\partial_\mu F^{\mu\sigma} - \partial^\sigma\partial_\mu F^{\mu\delta} = \alpha^2 F^{\sigma\delta}. \tag{S86}$$

Finally, to get the factor of ∂^2 we want, we can apply ∂_δ to both sides, giving

$$\partial^2\partial_\mu F^{\mu\sigma} = \alpha^2\partial_\delta F^{\sigma\delta} \tag{S87}$$

where a term vanished due to the antisymmetry of F. Cleaning this up, we have

$$0 = (\partial^2 + \alpha^2)(\partial_\mu F^{\mu\nu}) = (\partial^2 + \alpha^2)\left(\frac{\alpha}{2}\epsilon^{\mu\nu\rho}F_{\mu\rho}\right) \tag{S88}$$

where we used the equation of motion a final time. Now we just clear away the factor of ϵ by contracting with $\epsilon_{\nu\alpha\beta}$, which finally gives the result with $m = \alpha$.

Since there is no Chern–Simons term in three spatial dimensions, it has little role in particle physics, but it is important in the description of topological effects in two-dimensional condensed matter systems.

PHYS-330: QFT I Problem Set 8 (due 11/29)

Stanford University, 2022 Bernhard Mistlberger, Kevin Zhou

1. Magnetic moments in quantum electrodynamics. (15 points)

The interaction Hamiltonian in quantum electrodynamics for a Dirac field of charge q is

$$H_I = q \int d^3x \, \bar{\Psi} \gamma^\mu \Psi A_\mu. \tag{1}$$

To understand the physical meaning of this expression, we can evaluate its matrix elements in states $|p, s\rangle = \sqrt{2E_p}\, a_s^\dagger(p)|0\rangle$ with a single fermion. We assume the electromagnetic field is in a quantum state with negligible field uncertainty, so that its field operator A_μ can be replaced with a time-independent classical expectation value $A_\mu^{\text{cl}}(\mathbf{x})$.

a) Show that the matrix elements of the Schrodinger picture Hamiltonian are

$$\langle p, s|H_I|p', s'\rangle = q \int d^3x \, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(p) \gamma^\mu u_{s'}(p') A_\mu^{\text{cl}}(\mathbf{x}) \tag{2}$$

when $|p', s'\rangle \neq |p, s\rangle$.

In nonrelativistic quantum mechanics, the single-particle states are $|\mathbf{p}, s\rangle$. If the particle has charge q' and g-factor g, then its magnetic moment is $\mu = (gq'/2m)\,\mathbf{S}$ where $\mathbf{S}$ is the spin. In terms of the particle's position $\mathbf{x}$, the Hamiltonian contains the terms

$$H_I^{\text{nr}} = q' A^0(\mathbf{x}) - \boldsymbol{\mu} \cdot \mathbf{B}(\mathbf{x}). \tag{3}$$

To find the values of q' and g, we equate matrix elements in the nonrelativistic limit,

$$\langle p, s|H_I|p', s'\rangle = 2m \langle \mathbf{p}, s|H_I^{\text{nr}}|\mathbf{p}', s'\rangle \text{ when } |\mathbf{p}|, |\mathbf{p}'| \ll m \tag{4}$$

where the factor of $2m$ converts between relativistic and nonrelativistic normalization. We could evaluate both sides for general $A_\mu^{\text{cl}}(\mathbf{x})$, but it is easier to consider two special cases. In both cases it will be helpful to use the Gordon identity from set 7.

b) Evaluate both sides of (4) in a static electric field, corresponding to general $A^0(\mathbf{x})$ and $\mathbf{A} = 0$. Show that they agree when $q' = q$, as one would expect.

c) Show that if the fermion was a classical spinning ball with uniform mass and charge density, then its g-factor would be 1. This strongly disagrees with the measured value.

d) Evaluate both sides of (4) in a static magnetic field, corresponding to $A^0 = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}(\mathbf{x})$, and infer the value of g. (Hint: to relate spin in the Dirac theory to spin in the nonrelativistic theory, recall that in the nonrelativistic theory, spinors ξ_s have two components and the spin operator is $\mathbf{S} = \boldsymbol{\sigma}/2$. You already showed in problem set 7 how the Dirac spinors $u_s(p)$ are built from ξ_s. You will have to integrate by parts, so assume $\mathbf{A}$ and $\mathbf{B}$ vanish at infinity.)

The agreement of this value you found in part (d) with the experimentally measured value for the electron was one of the early triumphs of the Dirac equation.

2. Decays of the Higgs boson. (10 points)

The Standard Model contains three charged leptons, the electron e, muon μ, and tau τ, which are described by Dirac fields and differ only by their mass. It also contains a spinless particle called the Higgs boson, described by a real scalar field h. The free Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{1}{2}m_h^2\, h^2 + \sum_i \bar{\Psi}_i(i\partial\!\!\!/ - m_i)\Psi_i \tag{5}$$

where $i \in \{e, \mu, \tau\}$, and the numeric values of the masses are

$$m_h = 125\,\mathrm{GeV}, \qquad m_e = 511\,\mathrm{keV}, \qquad m_\mu = 105.7\,\mathrm{MeV}, \qquad m_\tau = 1777\,\mathrm{MeV}. \tag{6}$$

The Higgs field couples to the charged leptons by a Yukawa coupling proportional to mass,

$$\mathcal{L}_{\text{int}} = -\sum_i \frac{m_i}{v}\, h\bar{\Psi}_i\Psi_i \tag{7}$$

Here, v is a constant associated with the breaking of electroweak symmetry, but for this problem you will only need its value, $v = 246\,\mathrm{GeV}$. For this problem, all of your final answers should be numeric, and given to at least two significant figures.

a) Compute the partial decay rate for a Higgs boson to an electron-positron pair $\Gamma_{H\to e^+e^-}$ at leading non-vanishing order in perturbation theory, giving your answer in eV. (Hint: sum over final spin states, and reuse results from problem set 5.)

b) Find the ratios $\Gamma_{H\to\mu^+\mu^-}/\Gamma_{H\to e^+e^-}$ and $\Gamma_{H\to\tau^+\tau^-}/\Gamma_{H\to e^+e^-}$.

c) Find the probability a Higgs boson decays to $\tau^+\tau^-$, also known as the branching ratio

$$\mathrm{BR}_{H\to\tau^+\tau^-} = \frac{\Gamma_{H\to\tau^+\tau^-}}{\sum_X \Gamma_{H\to X}}. \tag{8}$$

The denominator is the total decay rate of the Higgs boson, equal to 4.1 MeV. (To check your answer, you can consult the so-called Yellow Report.)

The decay of Higgs bosons to taus was confirmed experimentally only recently [1,2], and first evidence for the Higgs coupling to muons has been detected [3,4]. Establishing the coupling of the Higgs to electrons remains a monumental experimental challenge.

3. Electron-positron annihilation to muons. (15 points)

One of the key successes of quantum electrodynamics is its description of particle creation and annihilation processes at relativistic energies. In this exercise we will consider the process $e^+e^- \to \mu^+\mu^-$, where the electron and muon are Dirac fields of charge e, and mass m_e and m_μ respectively.

a) Let the initial momenta be $p_{e^+}^\mu$ and $p_{e^-}^\mu$ and the final momenta be $p_{\mu^+}^\mu$ and $p_{\mu^-}^\mu$. Find the scattering matrix element $\mathcal{M}$ for this process, to leading nonvanishing order in e.

b) Let $|\bar{\mathcal{M}}|^2$ be the square of the matrix element, summed over final spin states and averaged over initial spin states. Compute $|\bar{\mathcal{M}}|^2$ in terms of e, m_e, m_μ, and the Mandelstam variables s and t, where

$$s = (p_{e^+} + p_{e^-})^2, \quad t = (p_{e^-} - p_{\mu^-})^2, \quad u = (p_{e^-} - p_{\mu^+})^2. \tag{9}$$

For the rest of this problem, we specialize to the center of mass frame.

c) Rewrite $|\bar{\mathcal{M}}|^2$ in terms of e, m_e, m_μ, s, and the angle θ between $\mathbf{p}_{e^-}$ and $\mathbf{p}_{\mu^-}$.

d) Starting from equation (4.84) of Peskin and Schroeder, compute the differential cross section $d\sigma/d\Omega$ and the total cross section.

You can see section 5.1 of Peskin and Schroeder to get started or check your answer. But note that the book neglects the mass of the electron, while here we account for it.

4. ⋆ Nonminimal couplings. (5 points)

In problem 1, you found the charge and g-factor for a Dirac fermion minimally coupled to the electromagnetic field, i.e. via the simplest possible interaction (1). However, if the fermion is composite, or interacts with heavier particles, we might need additional terms to describe the coupling. For all parts of this problem, you should adopt the formalism of problem 1 and work in the nonrelativistic limit. Detailed calculations are not needed; qualitative final answers (with justification) are sufficient.

a) The g-factors of the proton and neutron are *not* given by the result you found in problem 1. When the Dirac equation was invented, physicists explained this by adding an additional term to $\mathcal{H}_I$, proportional to $i\bar{\Psi}[\gamma^\mu, \gamma^\nu]\Psi\, F_{\mu\nu}$. Show that the physical effect of such a term is indeed to shift the magnetic dipole moment.

b) What is the physical effect of a term proportional to $\bar{\Psi}[\gamma^\mu, \gamma^\nu]\gamma^5\Psi\, F_{\mu\nu}$?

The above two are the only terms with dimension 5, and are therefore the simplest non-minimal couplings one could consider. Next, let's consider some dimension 6 terms.

c) What is the physical effect of a term proportional to $\bar{\Psi}\gamma^\mu\Psi\, \partial^\nu F_{\mu\nu}$?

d) What is the physical effect of a term proportional to $\bar{\Psi}\gamma^\mu\gamma^5\Psi\, \partial^\nu F_{\mu\nu}$?

1. Magnetic moments in quantum electrodynamics. (15 points)

The interaction Hamiltonian in quantum electrodynamics for a Dirac field of charge q is

$$H_I = q \int d^3x\, \bar{\Psi}\gamma^\mu \Psi A_\mu. \tag{1}$$

To understand the physical meaning of this expression, we can evaluate its matrix elements in states $|p,s\rangle = \sqrt{2E_p}\, a_s^\dagger(p)|0\rangle$ with a single fermion. We assume the electromagnetic field is in a quantum state with negligible field uncertainty, so that its field operator A_μ can be replaced with a time-independent classical expectation value $A_\mu^{\text{cl}}(\mathbf{x})$.

a) Show that the matrix elements of the Schrodinger picture Hamiltonian are

$$\langle p,s|H_I|p',s'\rangle = q\int d^3x\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)\gamma^\mu u_{s'}(p')A_\mu^{\text{cl}}(\mathbf{x}) \tag{2}$$

when $|p',s'\rangle \neq |p,s\rangle$.

Solution: Plugging in the Schrodinger picture mode expansion of the Dirac field, and using the abbreviated notation introduced in the solutions to problem set 2,

$$\langle p,s|H_I|p',s'\rangle = q\sum_{r,r'}\int d\mathbf{x}\frac{đ\mathbf{q}}{\sqrt{2E_q}}\frac{đ\mathbf{q}'}{\sqrt{2E_{q'}}}\sqrt{2E_p 2E_{p'}}$$
$$\times\langle 0|a_s(\mathbf{p})\left(a_r^\dagger(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}}\bar{u}_r(q) + b_r(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}\bar{v}_r(q)\right)\not{A}\left(a_{r'}(\mathbf{q}')e^{i\mathbf{q}'\cdot\mathbf{x}}u_{r'}(q) + b_{r'}^\dagger(\mathbf{q}')e^{-i\mathbf{q}'\cdot\mathbf{x}}v_{r'}(q')\right)a_{s'}^\dagger(\mathbf{p}')|0\rangle. \tag{S1}$$

Now let's think about the structure of this matrix element. We can only get nonzero contributions from the terms $a_r^\dagger(\mathbf{q})a_{r'}(\mathbf{q}')$ and $b_r(\mathbf{q})b_{r'}^\dagger(\mathbf{q}')$. On the other hand, the latter term yields

$$\langle 0|a_s(\mathbf{p})b_r(\mathbf{q})b_{r'}^\dagger(\mathbf{q}')a_{s'}^\dagger(\mathbf{p}')|0\rangle = \delta_{ss'}\delta_{rr'}\not{\delta}(\mathbf{p}-\mathbf{p}')\not{\delta}(\mathbf{q}-\mathbf{q}') \tag{S2}$$

and therefore cannot contribute matrix elements with $|p',s'\rangle \neq |p,s\rangle$. Discarding this term, we're left with

$$\langle p,s|H_I|p',s'\rangle = q\int d\mathbf{x}đ\mathbf{q}đ\mathbf{q}'\frac{\sqrt{2E_p2E_{p'}}}{\sqrt{2E_q2E_{q'}}}e^{-i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{x}}\sum_{r,r'}\bar{u}_r(q)\not{A}u_{r'}(q')\langle 0|a_s(\mathbf{p})a_r^\dagger(\mathbf{q})a_{r'}(\mathbf{q}')a_{s'}^\dagger(\mathbf{p}')|0\rangle \tag{S3}$$

$$= q\int d\mathbf{x}đ\mathbf{q}'\frac{\sqrt{2E_{p'}}}{\sqrt{2E_{q'}}}e^{-i(\mathbf{p}-\mathbf{q}')\cdot\mathbf{x}}\sum_{r'}\bar{u}_s(p)\not{A}u_{r'}(q')\langle 0|a_{r'}(\mathbf{q}')a_{s'}^\dagger(\mathbf{p}')|0\rangle \tag{S4}$$

$$= q\int d\mathbf{x}\; e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)\not{A}u_{s'}(p'). \tag{S5}$$

This is the desired result. Using the Gordon identity we find

$$\langle p,s|H_I|p',s'\rangle = \frac{q}{2m}\int d\mathbf{x}\; e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)\left(p'_\mu + p_\mu + i\sigma_{\mu\nu}(p'^\nu - p^\nu)\right)u_{s'}(p')A^\mu(x) \tag{S6}$$

In nonrelativistic quantum mechanics, the single-particle states are $|\mathbf{p},s\rangle$. If the particle has charge q' and g-factor g, then its magnetic moment is $\mu = (gq'/2m)\,\mathbf{S}$ where $\mathbf{S}$ is the spin. In terms of the particle's position $\mathbf{x}$, the Hamiltonian contains the terms

$$H_I^{\text{nr}} = q'A^0(\mathbf{x}) - \boldsymbol{\mu}\cdot\mathbf{B}(\mathbf{x}). \tag{3}$$

To find the values of q' and g, we equate matrix elements in the nonrelativistic limit,

$$\langle p,s|H_I|p',s'\rangle = 2m\langle \mathbf{p},s|H_I^{\rm nr}|\mathbf{p}',s'\rangle \text{ when } |\mathbf{p}|,|\mathbf{p}'| \ll m \tag{4}$$

where the factor of $2m$ converts between relativistic and nonrelativistic normalization. We could evaluate both sides for general $A_\mu^{\rm cl}(\mathbf{x})$, but it is easier to consider two special cases. In both cases it will be helpful to use the Gordon identity from set 7.

b) Evaluate both sides of (4) in a static electric field, corresponding to general $A^0(\mathbf{x})$ and $\mathbf{A}=0$. Show that they agree when $q'=q$, as one would expect.

Solution: If we set $\mathbf{A}=0$, then the matrix element above reduces to

$$\langle p,s|H_I|p',s'\rangle = \frac{q}{2m}\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)\left(p_0'+p_0+i\sigma_{0i}(p'^i-p^i)\right)u_{s'}(p')A^0(x) \tag{S7}$$

since σ_{00} vanishes. In the nonrelativistic limit, $p_0,p_0'\approx m$ are much greater than p^i,p'^i, so we may simply drop the latter terms, giving

$$\langle p,s|H_I|p',s'\rangle = q\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)u_{s'}(p')A^0(x) \tag{S8}$$

$$= 2m\,q\delta_{ss'}\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}A^0(x). \tag{S9}$$

On the other hand, in nonrelativistic quantum mechanics we have

$$2m\langle\mathbf{p},s|q'A^0(\hat{\mathbf{x}})|\mathbf{p}',s'\rangle = 2m\,q'\delta_{ss'}\langle\mathbf{p}|A^0(\hat{\mathbf{x}})|\mathbf{p}'\rangle \tag{S10}$$

$$= 2m\,q'\delta_{ss'}\int d\mathbf{y}d\mathbf{z}\,\langle\mathbf{p}|\mathbf{y}\rangle\langle\mathbf{y}|A^0(\hat{\mathbf{x}})|\mathbf{z}\rangle\langle\mathbf{z}|\mathbf{p}'\rangle \tag{S11}$$

$$= 2m\,q'\delta_{ss'}\int d\mathbf{y}d\mathbf{z}\,e^{-i\mathbf{p}\cdot\mathbf{y}}e^{i\mathbf{p}'\cdot\mathbf{z}}\delta(\mathbf{y}-\mathbf{z})A^0(\mathbf{z}) \tag{S12}$$

$$= 2m\,q'\delta_{ss'}\int d\mathbf{y}\,e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{y}}A^0(\mathbf{y}) \tag{S13}$$

$$= 2m\,q'\delta_{ss'}\int d\mathbf{x}\,e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}A^0(\mathbf{x}). \tag{S14}$$

The two matrix elements match if $q=q'$, as desired.

c) Show that if the fermion was a classical spinning ball with uniform mass and charge density, then its g-factor would be 1. This strongly disagrees with the measured value.

Solution: For simplicity, let's consider a thin ring of radius r, angular velocity ω, mass m, and charge q. Then the angular momentum is

$$S = mvr = m\omega r^2 \tag{S15}$$

and the magnetic moment is

$$\mu = IA = \frac{q}{2\pi/\omega}(\pi r^2) = \frac{q\omega r^2}{2} \tag{S16}$$

Therefore, the ring has $\mu/S = q/2m$, corresponding to a g-factor of 1. Since a ball is just a superposition of such rings, this shows that $g=1$ for a ball, or more generally any spinning object with axial symmetry.

d) Evaluate both sides of (4) in a static magnetic field, corresponding to $A^0=0$ and $\nabla\times\mathbf{A}=\mathbf{B}(\mathbf{x})$, and infer the value of g. (Hint: to relate spin in the Dirac theory to spin in the nonrelativistic theory, recall that in the nonrelativistic theory, spinors ξ_s have two components and the spin operator is $\mathbf{S}=\boldsymbol{\sigma}/2$. You already showed in problem set 7 how the Dirac spinors $u_s(p)$ are built from ξ_s. You will have to integrate by parts, so assume $\mathbf{A}$ and $\mathbf{B}$ vanish at infinity.)

Solution: In this case, since A^0 vanishes, we have

$$\langle p, s|H_I|p', s'\rangle = \frac{q}{2m} \int d\mathbf{x}\; e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(p)\,(p'_i + p_i + i\sigma_{i\nu}(p'^\nu - p^\nu))\,u_{s'}(p')A^i(x). \tag{S17}$$

This might look more complicated than what we had before using the Gordon identity, but this form is useful because it lets us isolate the physically distinct effects of different terms, and figure out which to keep in the nonrelativistic limit. (Though of course you could also do the problem without using the Gordon identity too.) Now, we need to choose which term to keep within the parentheses.

- The p_i and p'_i terms are order mv.
- The term $\sigma_{i0}(p'^0 - p^0)$ is not order m, since the m's cancel. Instead, it's order mv^2.
- The term $\sigma_{ij}(p'^j - p^j)$ is order mv.

The second contribution is therefore negligible in the nonrelativistic limit. The first contribution is not negligible, but it also has nothing to do with spin: it will just yield the $\mathbf{p}\cdot\mathbf{A}$ term in the nonrelativistic Hamiltonian that physically corresponds to the magnetic force on a charged particle. (Considering it won't tell us anything new, because we already know that $q' = q$ from part (b).) We therefore focus on the third term, giving

$$\langle p, s|H_I|p', s'\rangle \supset \frac{q}{2m} \int d\mathbf{x}\; e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(p)\left(i\sigma_{ij}(p'^j - p^j)\right)u_{s'}(p')A^i(x) \tag{S18}$$

$$= \frac{q}{2m} \int d\mathbf{x}\; e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} (p^j - p'^j)(\bar{u}_s(p)\frac{[\gamma^i, \gamma^j]}{2} u_{s'}(p'))A^i(x). \tag{S19}$$

To make further progress, we use the results we derived in the Weyl representation in problem set 7. First, because we're only considering the leading term in the nonrelativistic limit, it suffices to expand the spinor solutions to lowest order in m,

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} \approx \sqrt{m}\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} \tag{S20}$$

We can also explicitly evaluate the commutators. For example, we have

$$[\gamma^1, \gamma^2] = -2i \begin{pmatrix} \sigma^3 & \\ & \sigma^3 \end{pmatrix} \tag{S21}$$

from which we conclude in general that

$$[\gamma^i, \gamma^j] = -2i\epsilon^{ijk} \begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}. \tag{S22}$$

Plugging these results in and simplifying, we find

$$\langle p, s|H_I|p', s'\rangle \supset q \int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} (-i)(p^j - p'^j)\,(\epsilon^{ijk}\xi_s^\dagger\sigma^k\xi_{s'})A^i(x). \tag{S23}$$

To handle the $p^j - p'^j$, we write it as a derivative and integrate by parts, dropping a boundary term,

$$\langle p, s|H_I|p', s'\rangle \supset q \int d\mathbf{x}\, \frac{\partial}{\partial x_j}\left(e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\right)(\epsilon^{ijk}\xi_s^\dagger\sigma^k\xi_{s'})A^i(x) \tag{S24}$$

$$= q\,\xi_s^\dagger\sigma^k\xi_{s'} \int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\epsilon^{ijk}\partial^j A^i(x) \tag{S25}$$

$$= -q\,\xi_s^\dagger\sigma^k\xi_{s'} \int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} B^k. \tag{S26}$$

We can now compare this to the nonrelativistic result,

$$2m\langle \mathbf{p}, s|(-\boldsymbol{\mu}\cdot\mathbf{B})|\mathbf{p}', s'\rangle = -gq'\,\xi_s^\dagger S^k\xi_{s'} \int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} B^k \tag{S27}$$

by the same logic as in part (b). Since $q' = q$ and $\mathbf{S} = \boldsymbol{\sigma}/2$, we conclude that $g = 2$.

The agreement of this value you found in part (d) with the experimentally measured value for the electron was one of the early triumphs of the Dirac equation.

2. Decays of the Higgs boson. (10 points)

The Standard Model contains three charged leptons, the electron e, muon μ, and tau τ, which are described by Dirac fields and differ only by their mass. It also contains a spinless particle called the Higgs boson, described by a real scalar field h. The free Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{1}{2}m_h^2\, h^2 + \sum_i \bar{\Psi}_i(i\partial\!\!\!/ - m_i)\Psi_i \tag{5}$$

where $i \in \{e, \mu, \tau\}$, and the numeric values of the masses are

$$m_h = 125\,\mathrm{GeV}, \qquad m_e = 511\,\mathrm{keV}, \qquad m_\mu = 105.7\,\mathrm{MeV}, \qquad m_\tau = 1777\,\mathrm{MeV}. \tag{6}$$

The Higgs field couples to the charged leptons by a Yukawa coupling proportional to mass,

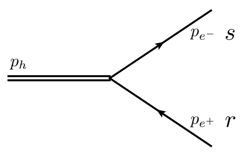
$$\mathcal{L}_{\mathrm{int}} = -\sum_i \frac{m_i}{v}\, h\bar{\Psi}_i\Psi_i \tag{7}$$

Here, v is a constant associated with the breaking of electroweak symmetry, but for this problem you will only need its value, $v = 246\,\mathrm{GeV}$. For this problem, all of your final answers should be numeric, and given to at least two significant figures.

a) Compute the partial decay rate for a Higgs boson to an electron-positron pair $\Gamma_{H\to e^+e^-}$ at leading non-vanishing order in perturbation theory, giving your answer in eV. (Hint: sum over final spin states, and reuse results from problem set 5.)

Solution: There is one Feynman diagram, giving matrix element

$$\mathcal{M}_{h\to e^+e^-} =$$



$$= \bar{u}_s(p_{e^-})\left(-i\frac{m_e}{v}\right)v_r(p_{e^+}). \tag{S28}$$

Squaring the matrix element and summing over final spin states gives

$$|\bar{\mathcal{M}}_{h\to e^+e^-}|^2 = \frac{m_e^2}{v^2}\sum_{r,s}\bar{u}_s(p_{e^-})v_r(p_{e^+})\bar{v}_r(p_{e^+})u_s(p_{e^-}) \tag{S29}$$

$$= \frac{m_e^2}{v^2}\,\mathrm{tr}\left[(p\!\!\!/_{e^-} + m_e)(p\!\!\!/_{e^+} - m_e)\right] \tag{S30}$$

$$= \frac{m_e^2}{v^2}\left(\mathrm{tr}\left[p\!\!\!/_{e^-}p\!\!\!/_{e^+}\right] - 4m_e^2\right) \tag{S31}$$

$$= 2\frac{m_e^2}{v^2}\left(2p_{e^-}\cdot p_{e^+} - 2m_e^2\right) \tag{S32}$$

$$= 2\frac{m_e^2}{v^2}\left(m_h^2 - 4m_e^2\right). \tag{S33}$$

In the last line we used the fact that

$$m_h^2 = p_h^2 = (p_{e^-} + p_{e^+})^2 = 2m_e^2 + 2p_{e^+}\cdot p_{e^-}. \tag{S34}$$

The decay width of a Higgs to leptons l^+l^- is defined by

$$\Gamma_{h\to l^+l^-} = \frac{1}{2m_h}\int \frac{d^4p_{l^-}}{(2\pi)^4}(2\pi)\delta_+(p_{l^-}^2 - m_l^2)\frac{d^4p_{l^+}}{(2\pi)^4}(2\pi)\delta_+(p_{l^+}^2 - m_l^2) \tag{S35}$$

$$\times (2\pi)^4\delta^{(4)}(p_h - p_{l^-} - p_{l^+})|\bar{\mathcal{M}}_{h\to e^+e^-}|^2. \tag{S36}$$

By reusing results from problem set 5, we immediately find

$$\Gamma_{h\to l^+l^-} = \frac{1}{16\pi m_h^2}\sqrt{m_h^2 - 4m_e^2}|\bar{\mathcal{M}}_{h\to e^+e^-}|^2 = \frac{m_e^2 m_h}{8\pi v^2}\left(1 - 4m_e^2/m_h^2\right)^{3/2} = 0.021\text{eV}. \tag{S37}$$

b) Find the ratios $\Gamma_{H\to\mu^+\mu^-}/\Gamma_{H\to e^+e^-}$ and $\Gamma_{H\to\tau^+\tau^-}/\Gamma_{H\to e^+e^-}$.

Solution: Note that since m_e, m_μ, and m_τ are all much less than m_h, we have

$$\Gamma_{h\to l^+l^-} \approx \frac{m_l^2 m_h}{8\pi v^2} \tag{S38}$$

for each charged lepton. That is, the decay rate is proportional to the mass squared, so the Higgs boson is much more likely to decay to heavier leptons,

$$\frac{\Gamma_{H\to\mu^+\mu^-}}{\Gamma_{H\to e^+e^-}} \approx \frac{m_\mu^2}{m_e^2} = 4.3\times 10^4, \quad \frac{\Gamma_{H\to\tau^+\tau^-}}{\Gamma_{H\to e^+e^-}} \approx \frac{m_\tau^2}{m_e^2} = 1.2\times 10^7. \tag{S39}$$

c) Find the probability a Higgs boson decays to $\tau^+\tau^-$, also known as the branching ratio

$$\text{BR}_{H\to\tau^+\tau^-} = \frac{\Gamma_{H\to\tau^+\tau^-}}{\sum_X \Gamma_{H\to X}}. \tag{8}$$

The denominator is the total decay rate of the Higgs boson, equal to 4.1 MeV. (To check your answer, you can consult the so-called Yellow Report.)

Solution: Plugging in the numbers gives $\text{BR}_{H\to\tau^+\tau^-} = 6.3\%$. (The Yellow Report gives a slightly different value because it accounts for higher-order corrections.)

The decay of Higgs bosons to taus was confirmed experimentally only recently [1,2], and first evidence for the Higgs coupling to muons has been detected [3,4]. Establishing the coupling of the Higgs to electrons remains a monumental experimental challenge.

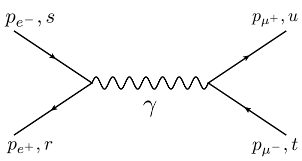
3. Electron-positron annihilation to muons. (15 points)

One of the key successes of quantum electrodynamics is its description of particle creation and annihilation processes at relativistic energies. In this exercise we will consider the process $e^+e^- \to \mu^+\mu^-$, where the electron and muon are Dirac fields of charge e, and mass m_e and m_μ respectively.

a) Let the initial momenta be $p_{e^+}^\mu$ and $p_{e^-}^\mu$ and the final momenta be $p_{\mu^+}^\mu$ and $p_{\mu_-}^\mu$. Find the scattering matrix element $\mathcal{M}$ for this process, to leading nonvanishing order in e.

Solution: There is one relevant Feynman diagram, which gives

$$\mathcal{M}_{e^+e^-\to\mu^+\mu^-} = \tag{S40}$$



(S41)

(S42)

$$= \bar{u}_t(p_\mu^-)(-ie\gamma^\mu_{tu})v_u(p_{\mu^+})\left(\frac{-i\eta_{\mu\nu}}{(p_{e^+}+p_{e^-})^2}\right)\bar{v}_r(p_{e^+})(-ie\gamma^\nu_{rs})u_s(p_{e^-}). \tag{S43}$$

b) Let $|\bar{\mathcal{M}}|^2$ be the square of the matrix element, summed over final spin states and averaged over initial spin states. Compute $|\bar{\mathcal{M}}|^2$ in terms of e, m_e, m_μ, and the Mandelstam variables s and t, where

$$s = (p_{e^+}+p_{e^-})^2, \quad t = (p_{e^-}-p_{\mu^-})^2, \quad u = (p_{e^-}-p_{\mu^+})^2. \tag{9}$$

Solution: Before starting, we note that the Mandelstam variables are simply related to inner products,

$$\frac{s}{2} = m_e^2 + p_{e^+} \cdot p_{e^-} = m_{\mu^2} + p_{\mu^+} \cdot p_{\mu^-} \tag{S44}$$

and

$$t = m_e^2 + m_\mu^2 - p_{e^-} \cdot p_{\mu^-} = m_e^2 + m_\mu^2 - p_{e^+} \cdot p_{\mu^+} \tag{S45}$$
$$u = m_e^2 + m_\mu^2 - p_{e^-} \cdot p_{\mu^+} = m_e^2 + m_\mu^2 - p_{e^+} \cdot p_{\mu^-} \tag{S46}$$

Now, squaring, summing over final spins and averaging over initial spins yields

$$|\bar{\mathcal{M}}|^2 = \frac{e^4}{s^2}\frac{1}{4}\sum_{\text{spins}} \bar{u}_t(p_\mu^-)\gamma^\mu_{tu} v_u(p_{\mu^+})\bar{v}_{u'}(p_\mu^+)\gamma^\nu_{u't'}u_{t'}(p_{\mu^-})$$
$$\times \bar{v}_r(p_{e^+})(\gamma_\mu)_{rs}u_s(p_{e^-})\bar{u}_{s'}(p_{e^-})(\gamma_\nu)_{s'r'}v_{r'}(p_{e^+}) \tag{S47}$$
$$= \frac{e^4}{s^2}\frac{1}{4}\operatorname{tr}\left[(\not{p}_{\mu^-} + m_\mu)\gamma^\mu(\not{p}_{\mu^+} - m_\mu)\gamma^\nu\right]\operatorname{tr}\left[(\not{p}_{e^-} + m_e)\gamma_\mu(\not{p}_{e^+} - m_e)\gamma_\nu\right]. \tag{S48}$$

Both traces have the same form, and can be evaluated using the results from problem set 6,

$$\operatorname{tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 - m)\gamma^\nu\right] = \operatorname{tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] - m^2\operatorname{tr}[\gamma^\mu\gamma^\nu] \tag{S49}$$
$$= 4(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1 \cdot p_2)\eta^{\mu\nu}) - 4m^2\eta^{\mu\nu} \tag{S50}$$
$$= 4(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{s}{2}\eta^{\mu\nu}) \tag{S51}$$

where in the last step we used (S44). Inserting this result twice above, we have

$$|\bar{\mathcal{M}}|^2 = \frac{4e^4}{s^2}\left(p_{e^+}^\mu p_{e^-}^\nu + p_{e^-}^\mu p_{e^+}^\nu - \frac{s}{2}\eta^{\mu\nu}\right)\left(p_{\mu^+,\mu}p_{\mu^-,\nu} + p_{\mu^-,\mu}p_{\mu^+,\nu} - \frac{s}{2}\eta_{\mu\nu}\right) \tag{S52}$$
$$= \frac{4e^4}{s^2}\left(s^2 - s(p_{\mu^+}\cdot p_{\mu^-} + p_{e^+}\cdot p_{e^-}) + 2(p_{e^+}\cdot p_{\mu^+})(p_{e^-}\cdot p_{\mu^-}) + 2(p_{e^+}\cdot p_{\mu^-})(p_{e^-}\cdot p_{\mu^+})\right). \tag{S53}$$

We can then write the inner products in terms of s, t, and u using (S44) and (S45). There are many possible forms for the answer, since

$$s + t + u = 2m_e^2 + 2m_\mu^2. \tag{S54}$$

If we eliminate u, then we get

$$|\bar{\mathcal{M}}|^2 = \frac{2e^4}{s^2}\left[(s+t)^2 + t^2 - 4t(m_e^2 + m_\mu^2) + 2(m_e^2 + m_\mu^2)^2\right]. \tag{S55}$$

Alternatively, if we eliminate s inside the brackets, then we get

$$|\bar{\mathcal{M}}|^2 = \frac{2e^4}{s^2}\left[(u - 2(m_e^2 + m_\mu^2))^2 + (t - 2(m_e^2 + m_\mu^2))^2 - 2(m_e^2 + m_\mu^2)^2\right]. \tag{S56}$$

For the rest of this problem, we specialize to the center of mass frame.

c) Rewrite $|\bar{\mathcal{M}}|^2$ in terms of e, m_e, m_μ, s, and the angle θ between $\mathbf{p}_{e^-}$ and $\mathbf{p}_{\mu^-}$.

Solution: In the centre-of-mass frame we have

$$E_{e^-} = E_{e^+}, \quad E_{\mu^-} = E_{\mu^+}, \quad \mathbf{p}_{e^-} = -\mathbf{p}_{e^+}, \quad \mathbf{p}_{\mu^-} = -\mathbf{p}_{\mu^+} \tag{S57}$$

which implies that

$$s = 4E_{e^-}^2 = 4E_{\mu^-}^2. \tag{S58}$$

Furthermore, we have $E_{e^\pm}^2 = |\mathbf{p}_{e^\pm}|^2 + m_e^2$ and $E_{\mu^\pm}^2 = |\mathbf{p}_{\mu^\pm}|^2 + m_\mu^2$.

Starting from (S55), we need to eliminate t in favor of θ. Starting from the definition of t,

$$t = m_e^2 + m_\mu^2 - 2E_{e^-}E_{\mu^-} + 2|\mathbf{p}_{e^-}||\mathbf{p}_{\mu^-}|\cos\theta \tag{S59}$$

$$= \frac{1}{2}\left(2m_e^2 + 2m_\mu^2 - s + \sqrt{s-4m_e^2}\sqrt{s-4m_\mu^2}\cos\theta\right). \tag{S60}$$

Plugging this into (S55) and simplifying gives

$$|\bar{\mathcal{M}}|^2 = e^4\left[1 + \frac{4(m_e^2+m_\mu^2)}{s} + (1-4m_e^2/s)(1-4m_\mu^2/s)\cos^2\theta\right] \tag{S61}$$

Also note that solving (S60) for $\cos\theta$ gives

$$\cos\theta = 2\,\frac{t - m_e^2 - m_\mu^2 + \frac{s}{2}}{\sqrt{s-4m_e^2}\sqrt{s-4m_\mu^2}}. \tag{S62}$$

d) Starting from equation (4.84) of Peskin and Schroeder, compute the differential cross section $d\sigma/d\Omega$ and the total cross section.

Solution: Equation (4.84) of Peskin and Schroeder is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_{e^-}\,2E_{e^+}\,|\mathbf{v}_{e^-} - \mathbf{v}_{e^+}|}\frac{|\mathbf{p}_{\mu^-}|}{(2\pi)^2\,4E_{\mathrm{cm}}}|\bar{\mathcal{M}}|^2. \tag{S63}$$

Our first task is to write the phase space factors in the same variables we've been using above. We note that $\sqrt{s} = E_{\mathrm{cm}} = 2E_{e^-} = 2E_{e^+}$ in the center of mass frame. Furthermore,

$$|\mathbf{v}_{e^-} - \mathbf{v}_{e^+}| = 2|\mathbf{v}_{e^-}| = 2\frac{|\mathbf{p}_{e^-}|}{E_{e^-}} = 2\frac{\sqrt{E_{e^-}^2 - m_{e^-}^2}}{E_{e^-}} = 2\sqrt{1-4m_e^2/s} \tag{S64}$$

and similarly

$$|\mathbf{p}_{\mu^-}| = \frac{\sqrt{s}}{2}\sqrt{1-4m_\mu^2/s}. \tag{S65}$$

Plugging these results in and simplifying gives

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{\sqrt{1-4m_\mu^2/s}}{\sqrt{1-4m_e^2/s}}|\bar{\mathcal{M}}|^2. \tag{S66}$$

To find the total cross section, we integrate over solid angle, noting that

$$\int d\Omega = 4\pi, \quad \int \cos^2\theta\, d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\cos^2\theta = \frac{4\pi}{3}. \tag{S67}$$

The final result is

$$\sigma = \frac{e^4}{16\pi s}\frac{\sqrt{1-4m_\mu^2/s}}{\sqrt{1-4m_e^2/s}}\left[1 + \frac{4(m_e^2+m_\mu^2)}{s} + \frac{(1-4m_e^2/s)(1-4m_\mu^2/s)}{3}\right] \tag{S68}$$

$$= \frac{e^4}{12\pi s}\frac{\sqrt{1-4m_\mu^2/s}}{\sqrt{1-4m_e^2/s}}(1+2m_e^2/s)(1+2m_\mu^2/s). \tag{S69}$$

As a check, we recover the expected result in the ultrarelativistic limit,

$$\sigma \approx \frac{e^4}{12\pi s} = \frac{4\pi\alpha^2}{3s} \tag{S70}$$

which matches the result in Peskin and Schroeder.

You can see section 5.1 of Peskin and Schroeder to get started or check your answer. But note that the book neglects the mass of the electron, while here we account for it.

4. ⋆ Nonminimal couplings. (5 points)

In problem 1, you found the charge and g-factor for a Dirac fermion minimally coupled to the electromagnetic field, i.e. via the simplest possible interaction (1). However, if the fermion is composite, or interacts with heavier particles, we might need additional terms to describe the coupling. For all parts of this problem, you should adopt the formalism of problem 1 and work in the nonrelativistic limit. Detailed calculations are not needed; qualitative final answers (with justification) are sufficient.

a) The g-factors of the proton and neutron are *not* given by the result you found in problem 1. When the Dirac equation was invented, physicists explained this by adding an additional term to $\mathcal{H}_I$, proportional to $i\bar{\Psi}[\gamma^\mu, \gamma^\nu]\Psi\, F_{\mu\nu}$. Show that the physical effect of such a term is indeed to shift the magnetic dipole moment.

Solution: If we follow the exact same logic as in problem 1, we'll end up with a term in $\langle p, s|H_I|p', s'\rangle$ proportional to

$$\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(p)[\gamma^\mu, \gamma^\nu]u_{s'}(p')F_{\mu\nu}(x). \tag{S71}$$

In the presence of a uniform magnetic field, this becomes

$$\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(p)[\gamma^i, \gamma^j]u_{s'}(p')F_{ij}(x) \propto \int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \epsilon^{ijk}\xi_s^\dagger \sigma^k \xi_{s'} F_{ij} \tag{S72}$$

$$\propto \delta\!\!\!/(\mathbf{p}-\mathbf{p}')\xi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{B})\xi_{s'} \tag{S73}$$

which, as we already saw, is precisely what you would get from a magnetic dipole moment.

You might be concerned that this term also causes an interaction with electric fields, as it contains

$$\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(p)[\gamma^0, \gamma^i]u_{s'}(p')F_{0i}(x). \tag{S74}$$

However, we have

$$[\gamma^0, \gamma^i] = -2\begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix} \tag{S75}$$

so that when we do the spinor contractions in the nonrelativistic limit, we get two terms that cancel in the nonrelativistic limit. So the only effect of this term is a magnetic dipole moment.

b) What is the physical effect of a term proportional to $\bar{\Psi}[\gamma^\mu, \gamma^\nu]\gamma^5\Psi\, F_{\mu\nu}$?

Solution: By the same logic as before, we'll get a term in the matrix element like

$$\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(p)[\gamma^\mu, \gamma^\nu]\gamma^5 u_{s'}(p')F_{\mu\nu}(x) \tag{S76}$$

where

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & \\ & \mathbb{1}_2 \end{pmatrix} \tag{S77}$$

in the Weyl representation. Recall that both $[\gamma^i, \gamma^j]$ and $[\gamma^i, \gamma^0]$ have Pauli matrices on the diagonal, but the latter has them with opposite signs. The presence of the γ^5 essentially exchanges the two. It is now the $[\gamma^i, \gamma^0]$ that contribute in the nonrelativistic limit, giving a term proportional to $\delta\!\!\!/(\mathbf{p}-\mathbf{p}')\xi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{E})\xi_{s'}$. Physically, this corresponds to giving the particle an electric dipole moment.

The presence of an electric dipole moment implies the violation of CP symmetry. Many past and ongoing experiments search for the electric dipole moment of the electron, which could arise from physics beyond the Standard Model, but so far none has been detected. The Standard Model generically predicts a sizable electric dipole moment for the neutron, which should have been easily detected decades ago, but mysteriously none has been found.

The above two are the only terms with dimension 5, and are therefore the simplest non-minimal couplings one could consider. Next, let's consider some dimension 6 terms.

c) What is the physical effect of a term proportional to $\bar{\Psi}\gamma^\mu\Psi\,\partial^\nu F_{\mu\nu}$?

Solution: This is just like the coupling in problem 1, but we have replaced A_μ with $\partial^\nu F_{\mu\nu} = -J_\mu$. In the nonrelativistic limit, the leading effect is the one we found in problem 1(b), as this is the only one not suppressed by powers of v. By the same logic, the result is a term in H_I^{nr} proportional to $J^0(\mathbf{x}) = \rho(\mathbf{x})$, where ρ is the external charge density. That is, it has *no* effect, regardless of the fields at the particle's location, unless the particle is sitting right on top of some other charge.

This might seem rather mysterious, but it's perfectly comprehensible within classical electromagnetism. Consider a tiny spherical capacitor with places of charge $\pm q$. This charge configuration produces exactly zero electromagnetic field everywhere outside it, which means that it does not interact with external charges until they are right inside the capacitor, in which case they feel the potential inside.

If the particle already has a net charge, then superposing the above configuration spreads the charge out in space. For this reason, this term is said to produce a "charge radius". The charge radius is one way to quantify the size of particles like the proton. (The "proton radius puzzle" is the fact that two independent measurements of this quantity seemed to give different results.) When we say the electron is a pointlike particle, we mean that it has no charge radius, as far as we've detected.

d) What is the physical effect of a term proportional to $\bar{\Psi}\gamma^\mu\gamma^5\Psi\,\partial^\nu F_{\mu\nu}$?

Solution: In this case the leading term in the nonrelativistic limit is that of 1(d), but with $\mathbf{B}$ replaced with $\mathbf{J}$, corresponding to an interaction of the form $\mathbf{S}\cdot\mathbf{J}(\mathbf{x})$. In other words, we have an interaction with currents directly on top of the particle.

To physically interpret this, imagine the particle is a tiny toroidal solenoid with its axis of symmetry aligned with $\mathbf{S}$. Such a configuration produces zero magnetic field outside itself, but the magnetic field affects currents that pass right over the toroid. When the current flows parallel to $\mathbf{S}$, it gets pushed directly towards (or away from) the toroid's axis, while a current flowing perpendicular to $\mathbf{S}$ experiences no net force; this therefore reproduces the $\mathbf{S}\cdot\mathbf{J}$ potential.

Depending on the field of physics, such a current configuration is called an anapole moment, or a polar toroidal dipole moment. You have probably not seen such a thing in electromagnetism classes, because it doesn't show up in the multipole expansion – it is a near field effect, not a far field effect. But anapole moments are important observables in nuclear physics.

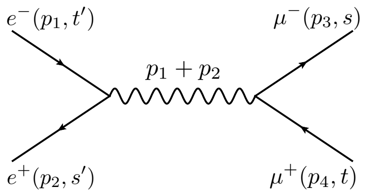
One can keep going. The anapole moment produces a localized parity-violating magnetic field, while the so-called Schiff moment, also important in nuclear physics, produces a localized parity-violating electric field. It corresponds to a term in H_I^{nr} proportional to $\mathbf{S}\cdot\nabla\rho$.

This problem set guides you through the calculation of loop diagrams and their application to the renormalization of QED. Unlike previous problem sets, it is ordered sequentially: each problem is easiest to approach after completing the previous problems. For reference, you may find it useful to consult sections 6.3 and 7.5 of Peskin and Schroeder, but this problem set is self-contained and its results are somewhat more general.

We will use dimensional regularization with $d = 4 - 2\epsilon$. To avoid confusion, we will rename the $i\epsilon$ in the Feynman propagator to $i0$ in this problem set.

1. $e^+e^- \to \mu^+\mu^-$ at one loop. (8 points)

In the previous problem set, you considered the leading order (or "tree level") contribution to the scattering matrix element for $e^+e^- \to \mu^+\mu^-$,

$\mathcal{M}^{\mathrm{LO}} =$  (1)

$$= \bar{u}_s(p_3)(-ie\gamma^\nu)v_t(p_4)\frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2 + i0}\bar{v}_{s'}(p_2)(-ie\gamma^\mu)u_{t'}(p_1) \tag{2}$$

which is proportional to e^2. The next-to-leading order contribution $\mathcal{M}^{\mathrm{NLO}}$ is of order e^4, and contains Feynman diagrams with one loop.

a) Draw all Feynman diagrams that contribute to $\mathcal{M}^{\mathrm{NLO}}$.

b) Write down $\mathcal{M}^{\mathrm{NLO}}$ using the Feynman rules. You don't have to evaluate the loop integrals; this is just to show you examples of their typical form.

2. Feynman parameters. (12 points)

As you saw in problem 1, loop integrals take a few generic forms. For simplicity, let's consider a scalar theory, which doesn't have nontrivial factors in the numerator of the loop integrand. Then some generic loop integrals are the "tadpole", "bubble", and "triangle",

$$I_T(a; m^2) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2 - m^2)^a} \tag{3}$$

$$I_B(a, b; p^2) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^a((k+p)^2)^b} \tag{4}$$

$$I_\triangle(a, b, c; p_1, p_2) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^a((k+p_1)^2)^b((k-p_2)^2)^c} \tag{5}$$

named after the kinds of Feynman diagrams in which they appear. Here we have suppressed $i0$ terms, but you should keep in mind that they're implicitly there, which will be important in the last subpart. We regard a, b, and c as general real exponents, though in practice they will usually be positive integers. The dimension d is also a real parameter.

To compute the bubble and triangle integrals, it is helpful to introduce Feynman parameters, which combine the factors into the denominator into a power of a single quantity, like the tadpole integral. We will first have to build up some mathematical machinery.

a) The gamma function is defined by

$$\Gamma(z) = \int_0^\infty dx\, x^{z-1} e^{-x} \tag{6}$$

and for positive integer n, satisfies $\Gamma(n) = (n-1)!$. Show that for real ν,

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty dx\, x^{\nu-1} e^{-xA}. \tag{7}$$

b) Show that for general ν_i,

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{\Gamma\left(\sum_{i=1}^n \nu_i\right)}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^1 \left(\prod_{i=1}^n dx_i\right) \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{\prod_{i=1}^n x_i^{\nu_i - 1}}{\left[\sum_{i=1}^n x_i A_i\right]^{\sum_{i=1}^n \nu_i}}. \tag{8}$$

Here, the x_i are called Feynman parameters. (Hint: you should *not* base your answer on the derivation in Peskin and Schroeder, which only works for integer ν. Instead, start from the result you proved in part (a).)

c) Apply (8) to $I_B(a, b; p^2)$ to get an integrand whose denominator contains a single quantity raised to the $a+b$ power.

d) Complete the evaluation of $I_B(a, b; p^2)$ by Wick rotating to Euclidean signature and performing all the remaining integrals. As a hint, the Euclidean integration measure is $d^d k_E = |k_E|^{d-1} d|k_E|\, d\Omega_d$, where the surface area of a unit sphere in d dimensions is

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{9}$$

As a second hint, you can use the result

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dx\, x^{a-1} (1-x)^{b-1}. \tag{10}$$

To check your answer, if you set $d = 4 - 2\epsilon$ then you should find

$$I_B(1, 1; p^2) = \left(\frac{-p^2 - i0}{4\pi}\right)^{-\epsilon} \frac{i\Gamma(1-\epsilon)^2 \Gamma(1+\epsilon)}{(4\pi)^2 (1-2\epsilon)\epsilon\Gamma(1-2\epsilon)}. \tag{11}$$

3. Passarino–Veltman reduction. (10 points)

More generally, loop integrands will have momenta and other factors in the numerator. However, we can often use symmetry properties and a technique called Passarino–Veltman reduction to reduce them to the simpler loop integrands considered in problem 2. For example, consider the rank 2 tadpole integral

$$I_T^{\mu\nu}(a; m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 - m^2)^a}. \tag{12}$$

Because the right-hand side is a Lorentz tensor, the left-hand side must be as well. As there are no momenta in the integral, the only option is the metric, so we must have

$$I_T^{\mu\nu}(a;m^2) = \eta^{\mu\nu} A \tag{13}$$

for some Lorentz scalar A. Contracting both sides with the metric, we find

$$\begin{aligned} A &= \frac{1}{d}\int \frac{d^dk}{(2\pi)^d}\frac{k^2}{(k^2-m^2)^a} && (14)\\ &= \frac{1}{d}\int \frac{d^dk}{(2\pi)^d}\frac{(k^2-m^2)+m^2}{(k^2-m^2)^a}. && (15)\end{aligned}$$

This is just two "scalar" tadpole integrals, as defined in (3), so we conclude

$$I_T^{\mu\nu}(a;m^2) = \frac{\eta^{\mu\nu}}{d}\left[I_T(a-1;m^2) + m^2 I_T(a;m^2)\right]. \tag{16}$$

a) Apply the same reasoning to the rank 3 and rank 4 tadpole integrals,

$$\begin{aligned} I_T^{\mu\nu\rho}(a;m^2) &= \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu k^\rho}{(k^2-m^2)^a}, && (17)\\ I_T^{\mu\nu\rho\sigma}(a;m^2) &= \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2-m^2)^a}. && (18)\end{aligned}$$

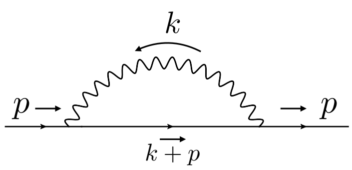
b) Similarly, the rank 1 and 2 bubble integrals can be written as

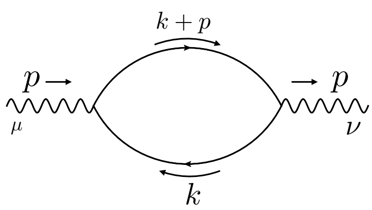
$$\begin{aligned} I_B^{\mu}(a,b;p^2) &= \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu}{(k^2)^a((k+p)^2)^b} = p^\mu C.\\ I_B^{\mu\nu}(a,b;p^2) &= \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{(k^2)^a((k+p)^2)^b} = \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)A_\perp + \frac{p^\mu p^\nu}{p^2}B_{||}. && (19)\end{aligned}$$

Express $A_\perp$ and $B_{||}$ and C in terms of p^2 and the scalar bubble integral (4).

4. Self-energy corrections in QED. (10 points)

Loop corrections to the two-point correlation function play a key role in renormalization, and are often referred to as self-energy corrections. The one-loop self-energy corrections for the electron and photon in QED are given by the diagrams below.

$\Sigma(p^2) =$ 

$\Pi^{\mu\nu}(p^2) =$ 

(20)

Throughout this problem we will work in massless QED, $m_e = 0$.

a) Using the Feynman rules, write down $\Sigma(p^2)$ and $\Pi^{\mu\nu}(p^2)$. Since we are viewing these quantities as corrections to the electron and photon propagators, you should *not* include factors for the external legs, such as $u_s(p)$ or ϵ_μ, to get a scalar matrix element. Instead, Σ should be a 4×4 spinor matrix and $\Pi^{\mu\nu}$ a rank 2 Lorentz tensor.

b) Evaluate the resulting loop integrals, expressing your final result in terms of p^2 and ϵ. (Hint: once you handle the gamma matrices, all of the integrals you get will be ones evaluated earlier in the problem set.)

c) The Euler–Mascheroni constant is defined as

$$\gamma_E = -\int_0^\infty dx\, e^{-x} \log x \approx 0.577. \tag{21}$$

Show that

$$\Gamma(1-\epsilon) = 1 + \gamma_E \epsilon + O(\epsilon^2). \tag{22}$$

d) Using (22), expand your results from part (b), dropping terms that vanish as $\epsilon \to 0$.

5. ⋆ The scalar triangle integral. (5 points)

In this optional problem, we consider a somewhat more difficult loop integral.

a) Evaluate the scalar triangle integral $I_\triangle(a, b, c; p_1, p_2)$ when $p_1^2 = p_2^2 = 0$, but for general a, b, c, and d, and give your answer in terms of $s = (p_1 + p_2)^2$.

b) It turns out there is a simple relation between the triangle and bubble integrals,

$$I_\triangle(1, 1, 1; p_1, p_2)\Big|_{p_1^2 = p_2^2 = 0} = C I_B(1, 1; (p_1 + p_2)^2). \tag{23}$$

Find the coefficient C in terms of d and s.

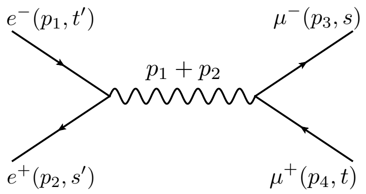
Stanford University, 2022 — Bernhard Mistlberger, Kevin Zhou

This problem set guides you through the calculation of loop diagrams and their application to the renormalization of QED. Unlike previous problem sets, it is ordered sequentially: each problem is easiest to approach after completing the previous problems. For reference, you may find it useful to consult sections 6.3 and 7.5 of Peskin and Schroeder, but this problem set is self-contained and its results are somewhat more general.

We will use dimensional regularization with $d = 4 - 2\epsilon$. To avoid confusion, we will rename the $i\epsilon$ in the Feynman propagator to $i0$ in this problem set.

1. $e^+e^- \to \mu^+\mu^-$ at one loop. (8 points)

In the previous problem set, you considered the leading order (or "tree level") contribution to the scattering matrix element for $e^+e^- \to \mu^+\mu^-$,

$\mathcal{M}^{\rm LO} =$ 

(1)

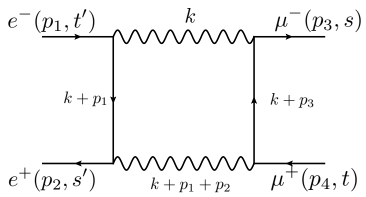
$$= \bar{u}_s(p_3)(-ie\gamma^\nu)v_t(p_4)\frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2 + i0}\bar{v}_{s'}(p_2)(-ie\gamma^\mu)u_{t'}(p_1) \tag{2}$$

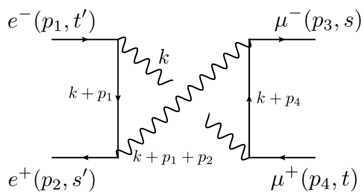
which is proportional to e^2. The next-to-leading order contribution $\mathcal{M}^{\rm NLO}$ is of order e^4, and contains Feynman diagrams with one loop.

a) Draw all Feynman diagrams that contribute to $\mathcal{M}^{\rm NLO}$.

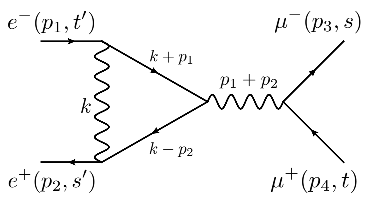
Solution: The matrix element at next-to-leading order (NLO) is given by

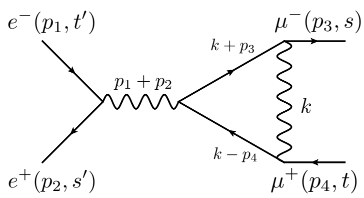
$$M^{\sf NLO} = D_1 + D_2 + D_3 + D_4 + D_5 + D_6, \tag{S1}$$

$D_1 =$ 

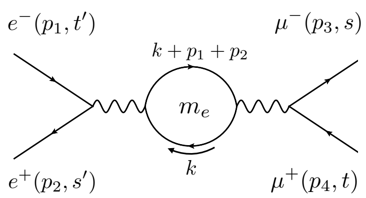
$D_2 =$ 

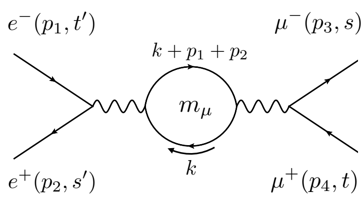
(S2)

$D_3 =$ 

$D_4 =$ 

(S3)

$D_5 =$ 

$D_6 =$ 

(S4)

b) Write down $\mathcal{M}^{\text{NLO}}$ using the Feynman rules. You don't have to evaluate the loop integrals; this is just to show you examples of their typical form.

Solution:

$$\begin{aligned} D_1 = \int \frac{d^dk}{(2\pi)^d} &\bar{u}_s(p_3)(-ie\gamma^\mu)\frac{i(\not{k}+\not{p}_3+m_\mu)}{(k+p_3)^2-m_\mu^2+i0}(-ie\gamma^\nu)v_t(p_4)\frac{-i\eta_{\mu\sigma}}{k^2+i0} \\ &\times \bar{v}_{s'}(p_2)(-ie\gamma^\rho)\frac{i(\not{k}+\not{p}_1+m_e)}{(k+p_1)^2-m_e^2+i0}(-ie\gamma^\sigma)u_{t'}(p_1)\frac{-i\eta_{\nu\rho}}{(k+p_1+p_2)^2+i0}. \end{aligned} \tag{S5}$$

$$\begin{aligned} D_2 = \int \frac{d^dk}{(2\pi)^d} &\bar{u}_s(p_3)(-ie\gamma^\mu)\frac{i(\not{k}+\not{p}_4+m_\mu)}{(k+p_4)^2-m_\mu^2+i0}(-ie\gamma^\nu)v_t(p_4)\frac{-i\eta_{\mu\sigma}}{k^2+i0} \\ &\times \bar{v}_{s'}(p_2)(-ie\gamma^\sigma)\frac{i(\not{k}+\not{p}_1+m_e)}{(k+p_1)^2-m_e^2+i0}(-ie\gamma^\rho)u_{t'}(p_1)\frac{-i\eta_{\nu\rho}}{(k+p_1+p_2)^2+i0}. \end{aligned} \tag{S6}$$

$$\begin{aligned} D_3 = \int \frac{d^dk}{(2\pi)^d} &\bar{u}_s(p_3)(-ie\gamma^\mu)v_t(p_4)\frac{-i\eta_{\mu\nu}}{(p_3+p_4)^2+i0}\frac{-i\eta_{\sigma\rho}}{k^2+i0} \\ &\times \bar{v}_{s'}(p_2)(-ie\gamma^\rho)\frac{i(\not{k}-\not{p}_2+m_e)}{(k-p_2)^2-m_e^2+i0}(-ie\gamma^\nu)\frac{i(\not{k}+\not{p}_1+m_e)}{(k+p_1)^2-m_e^2+i0}(-ie\gamma^\sigma)u_{t'}(p_1). \end{aligned} \tag{S7}$$

$$\begin{aligned} D_4 = \int \frac{d^dk}{(2\pi)^d} &\bar{v}_{s'}(p_2)(-ie\gamma^\mu)u_{t'}(p_1)\frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2+i0}\frac{-i\eta_{\sigma\rho}}{k^2+i0} \\ &\times \bar{u}_s(p_3)(-ie\gamma^\rho)\frac{i(\not{k}+\not{p}_3+m_\mu)}{(k+p_3)^2-m_\mu^2+i0}(-ie\gamma^\nu)\frac{i(\not{k}-\not{p}_4+m_\mu)}{(k-p_4)^2-m_\mu^2+i0}(-ie\gamma^\sigma)v_t(p_4). \end{aligned} \tag{S8}$$

$$\begin{aligned} D_5 = \int \frac{d^dk}{(2\pi)^d} &\bar{u}_s(p_3)(-ie\gamma^\nu)v_t(p_4)\bar{v}_{s'}(p_2)(-ie\gamma^\mu)u_{t'}(p_1) \\ &\times \frac{-i\eta_{\mu\rho}}{(p_1+p_2)^2+i0}\frac{\operatorname{tr}\left[(-ie\gamma^\rho)i(\not{k}+m_e)(-ie\gamma^\sigma)i(\not{k}+\not{p}_1+\not{p}_2+m_e)\right]}{(k^2-m_e^2+i0)(k+p_1+p_2)^2-m_e^2+i0}\frac{-i\eta_{\sigma\nu}}{(p_1+p_2)^2+i0}. \end{aligned} \tag{S9}$$

$$\begin{aligned} D_6 = \int \frac{d^dk}{(2\pi)^d} &\bar{u}_s(p_3)(-ie\gamma^\nu)v_t(p_4)\bar{v}_{s'}(p_2)(-ie\gamma^\mu)u_{t'}(p_1) \\ &\times \frac{-i\eta_{\mu\rho}}{(p_1+p_2)^2+i0}\frac{\operatorname{tr}\left[(-ie\gamma^\rho)i(\not{k}+m_\mu)(-ie\gamma^\sigma)i(\not{k}+\not{p}_1+\not{p}_2+m_\mu)\right]}{(k^2-m_\mu^2+i0)(k+p_1+p_2)^2-m_\mu^2+i0}\frac{-i\eta_{\sigma\nu}}{(p_1+p_2)^2+i0}. \end{aligned} \tag{S10}$$

If you collect all factors of i, e and (-1), the common pre-factor is $i^8(-1)^6e^4 = (4\pi\alpha)^2$.

2. Feynman parameters. (12 points)

As you saw in problem 1, loop integrals take a few generic forms. For simplicity, let's consider a scalar theory, which doesn't have nontrivial factors in the numerator of the loop integrand. Then some generic loop integrals are the "tadpole", "bubble", and "triangle",

$$I_T(a;m^2) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2)^a} \tag{3}$$

$$I_B(a,b;p^2) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^a((k+p)^2)^b} \tag{4}$$

$$I_\triangle(a,b,c;p_1,p_2) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^a((k+p_1)^2)^b((k-p_2)^2)^c} \tag{5}$$

named after the kinds of Feynman diagrams in which they appear. Here we have suppressed $i0$ terms, but you should keep in mind that they're implicitly there, which will be important in the last subpart. We regard a, b, and c as general real exponents, though in practice they will usually be positive integers. The dimension d is also a real parameter.

To compute the bubble and triangle integrals, it is helpful to introduce Feynman parameters, which combine the factors into the denominator into a power of a single quantity, like the tadpole integral. We will first have to build up some mathematical machinery.

a) The gamma function is defined by

$$\Gamma(z) = \int_0^\infty dx\, x^{z-1} e^{-x} \tag{6}$$

and for positive integer n, satisfies $\Gamma(n) = (n-1)!$. Show that for real ν,

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty dx\, x^{\nu-1} e^{-xA}. \tag{7}$$

b) Show that for general ν_i,

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{\Gamma\left(\sum_{i=1}^n \nu_i\right)}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^1 \left(\prod_{i=1}^n dx_i\right) \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{\prod_{i=1}^n x_i^{\nu_i - 1}}{\left[\sum_{i=1}^n x_i A_i\right]^{\sum_{i=1}^n \nu_i}}. \tag{8}$$

Here, the x_i are called Feynman parameters. (Hint: you should *not* base your answer on the derivation in Peskin and Schroeder, which only works for integer ν. Instead, start from the result you proved in part (a).)

Solution: Using the result of part (a) n times yields

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{1}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^\infty \prod_{i=1}^n dx_i\, x_i^{\nu_i-1} e^{-\sum_{i=1}^n x_i A_i}. \tag{S11}$$

This is already pretty close to the desired answer, so we now introduce an identity

$$1 = \int_0^\infty dy\, \delta\left(y - \sum_{i=1}^n x_i\right) \tag{S12}$$

which converts our result to

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{1}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^\infty dy \prod_{i=1}^n dx_i\, x_i^{\nu_i-1} \delta\left(y - \sum_{i=1}^n x_i\right) e^{-\sum_{i=1}^n x_i A_i}. \tag{S13}$$

Let us rescale all the x_i by y so they sum up to 1, as required in the final answer. The result is

$$\begin{aligned}
\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} &= \frac{1}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^\infty dy\, y^{\sum_{i=1}^n \nu_i} \prod_{i=1}^n dx_i\, x_i^{\nu_i-1} \delta\left(y - y\sum_{i=1}^n x_i\right) e^{-y\sum_{i=1}^n x_i A_i} \\
&= \frac{1}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^\infty dy\, y^{-1+\sum_{i=1}^n \nu_i} \prod_{i=1}^n dx_i\, x_i^{\nu_i-1} \delta\left(1 - \sum_{i=1}^n x_i\right) e^{-y\sum_{i=1}^n x_i A_i} \\
&= \frac{\Gamma(\sum_{i=1}^n \nu_i)}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^1 \prod_{i=1}^n dx_i\, x_i^{\nu_i-1} \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{1}{\left[\sum_{i=1}^n x_i A_i\right]^{\sum_{i=1}^n \nu_i}}
\end{aligned} \tag{S14}$$

as desired, where we performed the y integral using (7).

c) Apply (8) to $I_B(a,b;p^2)$ to get an integrand whose denominator contains a single quantity raised to the $a+b$ power.

Solution:

$$\begin{aligned}
I_B(a,b;p^2) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int\frac{d^dk}{(2\pi)^d}\int_0^\infty dx_1x_1^{a-1}dx_2x_2^{b-1}\frac{\delta(1-x_1-x_2)}{[k^2x_1+(k+p)^2x_2]^{a+b}} \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int\frac{d^dk}{(2\pi)^d}\int_0^\infty dx_1x_1^{a-1}dx_2x_2^{b-1}\frac{\delta(1-x_1-x_2)}{[k^2(x_1+x_2)+(2kp+p^2)x_2]^{a+b}} \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int\frac{d^dk}{(2\pi)^d}\int_0^\infty dx_1x_1^{a-1}(1-x_1)^{b-1}\frac{1}{[k^2+(2kp+p^2)(1-x_1)]^{a+b}}.
\end{aligned} \tag{S15}$$

d) Complete the evaluation of $I_B(a,b;p^2)$ by Wick rotating to Euclidean signature and performing all the remaining integrals. As a hint, the Euclidean integration measure is $d^dk_E = |k_E|^{d-1}d|k_E|\,d\Omega_d$, where the surface area of a unit sphere in d dimensions is

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{9}$$

As a second hint, you can use the result

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dx\,x^{a-1}(1-x)^{b-1}. \tag{10}$$

To check your answer, if you set $d = 4-2\epsilon$ then you should find

$$I_B(1,1;p^2) = \left(\frac{-p^2-i0}{4\pi}\right)^{-\epsilon}\frac{i\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{(4\pi)^2(1-2\epsilon)\epsilon\Gamma(1-2\epsilon)}. \tag{11}$$

Solution: Next, we perform the linear shift $k \to k - p(1-x_1)$ and find

$$I_B(a,b;p^2) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int\frac{d^dk}{(2\pi)^d}\int_0^\infty dx_1x_1^{a-1}(1-x_1)^{b-1}\frac{1}{[k^2+p^2x_1(1-x_1)]^{a+b}}. \tag{S16}$$

Notice, that we dropped the "$+i0$" Feynman prescription from the integral at the very beginning. We could have kept an infinitesimal mass term throughout. To track the infinitesimal imaginary part, we can simply associate it with the momentum, p^2+i0. Next, we perform a Wick rotation $k^0 = ik_E^0$ and $\vec{k} = \vec{k}_E$.

$$\begin{aligned}
I_B(a,b;p^2) &= i\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int\frac{d^dk_E}{(2\pi)^d}\int_0^\infty dx_1x_1^{a-1}(1-x_1)^{b-1}\frac{1}{[-k_E^2+p^2x_1(1-x_1)]^{a+b}} \\
&= \frac{i(-1)^{a+b}}{2(2\pi)^d}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^\infty d(k_E^2)\int d\Omega_d(k_E^2)^{\frac{d}{2}-1}\int_0^\infty dx_1x_1^{a-1}(1-x_1)^{b-1}\frac{1}{[k_E^2-p^2x_1(1-x_1)]^{a+b}} \\
&= \frac{i(-1)^{a+b}}{2(2\pi)^d}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(-a-b-\frac{d}{2}\right)}{\Gamma(a)\Gamma(b)}\Omega_d\int_0^1 dx_1x_1^{a-1}(1-x_1)^{b-1}\left(-(p^2+i0)x_1(1-x_1)\right)^{\frac{d}{2}-a-b} \\
&= \frac{i(-1)^{a+b}\left(-(p^2+i0)\right)^{\frac{d}{2}-a-b}}{2(2\pi)^d}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(a+b-\frac{d}{2}\right)}{\Gamma(a)\Gamma(b)}\Omega_d\int_0^1 dx_1x_1^{\frac{d}{2}-b-1}(1-x_1)^{\frac{d}{2}-a-1} \\
&= \frac{i(-1)^{a+b}\left(-(p^2+i0)\right)^{\frac{d}{2}-a-b}}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(a+b-\frac{d}{2}\right)}{\Gamma(a)\Gamma(b)}\frac{\Gamma\left(\frac{d}{2}-a\right)\Gamma\left(\frac{d}{2}-b\right)}{\Gamma(d-a-b)}.
\end{aligned} \tag{S17}$$

Above we used the Euler β-function integral

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dx x^{a-1}(1-x)^{b-1}. \tag{S18}$$

3. Passarino–Veltman reduction. (10 points)

More generally, loop integrands will have momenta and other factors in the numerator. However, we can often use symmetry properties and a technique called Passarino–Veltman reduction to reduce them to the simpler loop integrands considered in problem 2. For example, consider the rank 2 tadpole integral

$$I_T^{\mu\nu}(a;m^2) = \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2-m^2)^a}. \tag{12}$$

Because the right-hand side is a Lorentz tensor, the left-hand side must be as well. As there are no momenta in the integral, the only option is the metric, so we must have

$$I_T^{\mu\nu}(a;m^2) = \eta^{\mu\nu} A \tag{13}$$

for some Lorentz scalar A. Contracting both sides with the metric, we find

$$A = \frac{1}{d}\int \frac{d^dk}{(2\pi)^d} \frac{k^2}{(k^2-m^2)^a} \tag{14}$$

$$= \frac{1}{d}\int \frac{d^dk}{(2\pi)^d} \frac{(k^2-m^2)+m^2}{(k^2-m^2)^a}. \tag{15}$$

This is just two "scalar" tadpole integrals, as defined in (3), so we conclude

$$I_T^{\mu\nu}(a;m^2) = \frac{\eta^{\mu\nu}}{d}\left[I_T(a-1;m^2) + m^2 I_T(a;m^2)\right]. \tag{16}$$

a) Apply the same reasoning to the rank 3 and rank 4 tadpole integrals,

$$I_T^{\mu\nu\rho}(a;m^2) = \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu k^\rho}{(k^2-m^2)^a}, \tag{17}$$

$$I_T^{\mu\nu\rho\sigma}(a;m^2) = \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2-m^2)^a}. \tag{18}$$

Solution: Again, only combinations of the metric tensor are available to us. There is no combination of metric tensors that would allow us to write down a tensor with 5 indices, so the first integral vanishes. (Another way of seeing this is that the integrand is odd under the transformation $k \to -k$ and the integral is consequently equal to minus itself and thus zero.)

The integrand with four Lorentz indices is symmetric under the exchange of any pair of indices, so we must have

$$I_T^{\mu\nu\rho\sigma}(a) = A\left(\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right). \tag{S19}$$

Contracting with $\eta_{\mu\nu}\eta_{\rho\sigma}$ we find

$$I_T^{\mu\nu\rho\sigma}\eta_{\mu\nu}\eta_{\rho\sigma}(a) = A(d^2+2d) \tag{S20}$$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{(k^2)^2}{(k^2-m^2)^a} \tag{S21}$$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{(k^2-m^2+m^2)^2}{(k^2-m^2)^a} \tag{S22}$$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-m^2)^{a-2}} + \int \frac{d^dk}{(2\pi)^d} \frac{2m^2}{(k^2-m^2)^{a-1}} + \int \frac{d^dk}{(2\pi)^d} \frac{m^4}{(k^2-m^2)^a} \tag{S23}$$

from which we conclude

$$I_T^{\mu\nu\rho\sigma}(a;m^2) = \frac{\eta^{\mu\nu}\eta^{\rho\sigma}+\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}}{d^2+2d}\left(I_T(a-2;m^2)+2m^2 I_T(a-1;m^2)+m^4 I_T(a;m^2)\right). \quad \text{(S24)}$$

b) Similarly, the rank 1 and 2 bubble integrals can be written as

$$I_B^{\mu}(a,b;p^2) = \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu}{(k^2)^a((k+p)^2)^b} = p^\mu C.$$
$$I_B^{\mu\nu}(a,b;p^2) = \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{(k^2)^a((k+p)^2)^b} = \left(\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)A_\perp + \frac{p^\mu p^\nu}{p^2}B_{||}. \quad (19)$$

Express $A_\perp$ and $B_{||}$ and C in terms of p^2 and the scalar bubble integral (4).

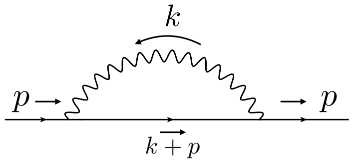
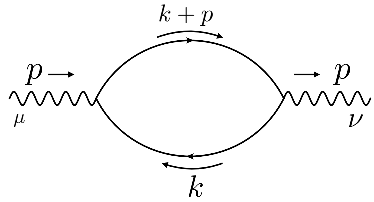
Solution:

$$\begin{aligned} I_B^{\mu}(a,b;p^2)p_\mu = p^2 C &= \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{2kp}{(k^2)^a((k+p)^2)^b} \\ C &= \frac{1}{2p^2}\int \frac{d^dk}{(2\pi)^d}\frac{(k+p)^2-k^2-p^2}{(k^2)^a((k+p)^2)^b} \\ &= \frac{1}{2p^2}\left[I_B(a,b-1;p^2)-I_B(a-1,b;p^2)-p^2 I_B(a,b;p^2)\right]. \end{aligned} \quad \text{(S25)}$$

$$\begin{aligned} I_B^{\mu\nu}(a,b;p^2)\eta_{\mu\nu} &= (d-1)A_\perp + B_{||} = I_B(a-1,b). \\ I_B^{\mu\nu}(a,b;p^2)\frac{p^\mu p^\nu}{p^2} &= B_{||} = \frac{1}{4p^2}\int \frac{d^dk}{(2\pi)^d}\frac{(2kp)^2}{(k^2)^a((k+p)^2)^b} \\ &= \frac{1}{4p^2}\int \frac{d^dk}{(2\pi)^d}\frac{((k+p)^2-k^2-p^2)^2}{(k^2)^a((k+p)^2)^b} \\ &= \frac{1}{4p^2}\big[I_B(a-2,b;p^2)+I_B(a,b-2;p^2)+(p^2)^2 I_B(a,b;p^2) \\ &\quad -2\,I_B(a-1,b-1;p^2)-2p^2 I_B(a-1,b;p^2)-2p^2 I_B(a,b-1;p^2)\big] \\ A_\perp &= \frac{1}{d-1}\left[I_B(a-1,b;p^2)-B_{||}\right]. \end{aligned} \quad \text{(S26)}$$

4. Self-energy corrections in QED. (10 points)

Loop corrections to the two-point correlation function play a key role in renormalization, and are often referred to as self-energy corrections. The one-loop self-energy corrections for the electron and photon in QED are given by the diagrams below.

$\Sigma(p^2) =$  $\Pi^{\mu\nu}(p^2) =$  (20)

Throughout this problem we will work in massless QED, $m_e = 0$.

a) Using the Feynman rules, write down $\Sigma(p^2)$ and $\Pi^{\mu\nu}(p^2)$. Since we are viewing these quantities as corrections to the electron and photon propagators, you should *not* include factors for the external legs, such as $u_s(p)$ or ϵ_μ, to get a scalar matrix element. Instead, Σ should be a 4×4 spinor matrix and $\Pi^{\mu\nu}$ a rank 2 Lorentz tensor.

Solution: Using the Feynman rules, we simply read off the answers,

$$\Sigma(p^2) = \int \frac{d^dk}{(2\pi)^d}(-ie)\gamma^\mu \frac{i(\not{k}+\not{p})}{(k+p)^2+i0}(-ie)\gamma^\nu \frac{-ig_{\mu\nu}}{k^2+i0} \tag{S27}$$

$$= -4\pi\alpha \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu(\not{k}+\not{p})\gamma_\mu}{(k^2+i0)((k+p)^2+i0)}. \tag{S28}$$

and

$$\Pi^{\mu\nu}(p^2) = -\int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{tr}\left[(-ie\gamma^\mu)i(\not{k}+\not{p})(-ie\gamma^\nu)i\not{k}\right]}{(k^2+i0)((k+p)^2+i0)} \tag{S29}$$

$$= -4\pi\alpha \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{tr}\left[\gamma^\mu(\not{k}+\not{p})\gamma^\nu\not{k}\right]}{(k^2+i0)((k+p)^2+i0)}. \tag{S30}$$

b) Evaluate the resulting loop integrals, expressing your final result in terms of p^2 and ϵ. (Hint: once you handle the gamma matrices, all of the integrals you get will be ones evaluated earlier in the problem set.)

Solution: To compute Σ, we first note that $\gamma^\mu(\not{k}+\not{p})\gamma_\mu = (2-d)(\not{k}+\not{p})$. We then find that

$$\begin{aligned}\Sigma(p^2) &= -4\pi\alpha(2-d)\left[\not{p}I_B(1,1)+\gamma_\mu I_B^\mu(1,1)\right] \\ &= -2\pi\alpha(2-d)\not{p}I_B(1,1) \\ &= \frac{\alpha}{\pi}\left(\frac{-p^2-i0}{4\pi}\right)^{-\epsilon}\not{p}\frac{i(1-\epsilon)\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{4(1-2\epsilon)\epsilon\Gamma(1-2\epsilon)}.\end{aligned} \tag{S31}$$

To compute $\Pi^{\mu\nu}$ we first compute the trace in the numerator.

$$\begin{aligned}\mathrm{tr}\left[\gamma^\mu(\not{k}+\not{p})\gamma^\nu\not{k}\right] &= 4(2k^\mu k^\nu + p^\mu k^\nu + k^\mu p^\nu - \eta^{\mu\nu}(k^2+k.p)) \\ &= 4(2k^\mu k^\nu + p^\mu k^\nu + k^\mu p^\nu - \eta^{\mu\nu}\left(k^2+\frac{1}{2}\left((k+p)^2-k^2-p^2\right)\right).\end{aligned} \tag{S32}$$

With this we find

$$\begin{aligned}\Pi^{\mu\nu}(p^2) &= -16\pi\alpha\left[2I_B^{\mu\nu}(1,1;p^2)+p^\mu I_B^\nu(1,1;p^2)+p^\nu I_B^\mu(1,1;p^2)+\eta^{\mu\nu}\frac{p^2}{2}I_B(1,1;p^2)\right] \\ &= -8\pi\alpha p^2\frac{d-2}{d-1}I_B(1,1;p^2)\left(\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right) \\ &= -\frac{\alpha}{\pi}\left(\frac{-p^2-i0}{4\pi}\right)^{-\epsilon}\frac{ip^2(1-\epsilon)\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{(1-2\epsilon)(3-2\epsilon)\epsilon\Gamma(1-2\epsilon)}\left(\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right).\end{aligned} \tag{S33}$$

c) The Euler–Mascheroni constant is defined as

$$\gamma_E = -\int_0^\infty dx\, e^{-x}\log x \approx 0.577. \tag{21}$$

Show that

$$\Gamma(1-\epsilon) = 1+\gamma_E\epsilon+O(\epsilon^2). \tag{22}$$

Solution: Using the definition,

$$\Gamma(1-\epsilon) = \int_0^\infty dx\, x^{-\epsilon}e^{-x} \tag{S34}$$

$$= \int_0^\infty dx\, e^{-\epsilon\log x}e^{-x} \tag{S35}$$

$$= \int_0^\infty dx\,(1-\epsilon\log x+O(\epsilon^2))e^{-x} \tag{S36}$$

$$= 1+\gamma_E\epsilon+O(\epsilon^2) \tag{S37}$$

as desired.

d) Using (22), expand your results from part (b), dropping terms that vanish as $\epsilon \to 0$.

Solution: Plugging in the above result and simplifying gives

$$\Sigma = i\frac{\alpha}{\pi}\left(\frac{-p^2 - i0}{4\pi}\right)^{-\epsilon} e^{-\epsilon\gamma_E} \not{p} \left(\frac{1}{4\epsilon} + \frac{1}{4} + \mathcal{O}(\epsilon)\right) \tag{S38}$$

and

$$\Pi^{\mu\nu}(p^2) = i\frac{\alpha}{\pi}\left(\frac{-p^2 - i0}{4\pi}\right)^{-\epsilon} e^{-\epsilon\gamma_E}\left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\left(-\frac{1}{3\epsilon} - \frac{5}{9} + \mathcal{O}(\epsilon)\right). \tag{S39}$$

5. ⋆ The scalar triangle integral. (5 points)

In this optional problem, we consider a somewhat more difficult loop integral.

a) Evaluate the scalar triangle integral $I_\triangle(a, b, c; p_1, p_2)$ when $p_1^2 = p_2^2 = 0$, but for general a, b, c, and d, and give your answer in terms of $s = (p_1 + p_2)^2$.

Solution: Introducing $s = (p_1 + p_2)^2$, Feynman parameters and shifting the loop momentum $k \to k + (1 - x_1 - x_2)p_2 - x_2 p_1$ we find

$$I_T(a, b, c; s) = \frac{\Gamma(a + b + c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx_1 x_1^{a-1} dx_2 x_2^{b-1} (1 - x_1 - x_2)^{c-1} \frac{1}{[k^2 + s x_2(1 - x_1 - x_2)]^{a+b+c}}. \tag{S40}$$

Next, we perform the Wick rotation $k^0 = i k_E^0$ and integrate out k_E. We find

$$\begin{aligned} I_T(a, b, c; s) = {} & i(-1)^{a+b+c}\pi^{d/2}(-s - i0)^{-a-b-c+\frac{d}{2}} \frac{\Gamma\left(a + b + c - \frac{d}{2}\right)}{(2\pi)^d \Gamma(a)\Gamma(b)\Gamma(c)} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \\ & \times x_1^{a-1} x_2^{-a-c+\frac{d}{2}-1} (1 - x_1 - x_2)^{-a-b+\frac{d}{2}-1}. \end{aligned} \tag{S41}$$

Next, we perform the transformation $x_2 = (1 - x_1)x_2$ and subsequently integrate out the parameter integrals over x_1 and x_2. We find

$$I_T(a, b, c; s) = i(4\pi)^{-\frac{d}{2}}(-1)^{a+b+c}(-s - i0)^{-a-b-c+\frac{d}{2}} \frac{\Gamma\left(-a - b + \frac{d}{2}\right)\Gamma\left(-a - c + \frac{d}{2}\right)\Gamma\left(a + b + c - \frac{d}{2}\right)}{\Gamma(b)\Gamma(c)\Gamma(-a - b - c + d)}. \tag{S42}$$

b) It turns out there is a simple relation between the triangle and bubble integrals,

$$I_\triangle(1, 1, 1; p_1, p_2)\Big|_{p_1^2 = p_2^2 = 0} = C I_B(1, 1; (p_1 + p_2)^2). \tag{23}$$

Find the coefficient C in terms of d and s.

Solution: By comparing our previous results, we conclude

$$C = -\frac{2(d - 3)}{(d - 4)s}. \tag{S43}$$

Quantum Field Theory I

Final Examination

Stanford University, Autumn 2022

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- If you have questions about the problems, email both of us. If we find there is an error or omission in the problems, we will send an announcement to the entire class.
- You are allowed to use the course textbook, by Peskin and Schroeder, and all course materials, including lecture notes, section notes, problem sets, and their solutions.
- You may use Mathematica or other computer programs to perform algebra, but you cannot use external packages and must include a copy of your source code.
- Collaboration with other students, or use of other sources or the internet in general, is prohibited. By submitting this exam, you affirm that you have received no unauthorized aid and engaged in no academic dishonesty.

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> Not only God knows, I know, and by the end of the semester, you will know.
> — Sidney Coleman, in a Harvard QFT I lecture

1. Field uncertainties. (10 points)

Quantum field theories have states with definite numbers of particles, in analogy with the number states $|n\rangle$ of a quantum harmonic oscillator. However, there also exist states with definite field profiles, in analogy with the position states $|x\rangle$. A general state is a superposition of such field eigenstates, and thus can have an uncertain field value.

a) Consider a free real scalar field ϕ of mass m in the vacuum state. The variance of the field value at $\mathbf{x}$ is

$$\sigma^2_{\phi(\mathbf{x})} = \langle 0|\phi(\mathbf{x})^2|0\rangle - \left(\langle 0|\phi(\mathbf{x})|0\rangle\right)^2. \tag{1}$$

Show that this quantity is infinite.

b) There's nothing wrong with the result in part (a), because in practice we can never measure the field value at a point. We only measure "smeared" field values averaged over some length scale a, such as

$$\phi_a(\mathbf{x}) = \frac{1}{a^3\pi^{3/2}} \int d^3y\, \phi(\mathbf{y}) e^{-|\mathbf{x}-\mathbf{y}|^2/a^2}. \tag{2}$$

Write $\sigma^2_{\phi_a(\mathbf{x})}$ in the form of an integral over a single variable.

c) The standard deviation of the smeared field is finite. Find expressions for $\sigma^2_{\phi_a(\mathbf{x})}$ in the limits $a \ll 1/m$ and $a \gg 1/m$. (Hint: in each regime, your answer should have the form $\alpha\, a^\beta$ for some values of α and β. If you're having trouble doing the final integral in one of these cases, you can start from the result $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.)

d) Consider a cubical copper microwave cavity of volume $V = 1\,\mathrm{m}^3$. The electric field in the cavity does not have a definite value, due to the quantization of the electromagnetic field. Find a rough, order of magnitude estimate for the standard deviation of the average electric field in the cavity in its ground state, and evaluate it in volts per meter. (Hint: you will need to go from natural units back to SI units using $\hbar = 10^{-34}\,\mathrm{J\,s}$, $c = 3 \times 10^8\,\mathrm{m/s}$, and $e = 1.6 \times 10^{-19}\,\mathrm{C}$.)

2. Forces on external currents. (5 points)

In problem set 7, you considered the "improved" electromagnetic stress-energy tensor,

$$\hat{T}^{\mu\nu} = -\eta_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma} + \frac{1}{4}\eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{3}$$

and showed that it was conserved, $\partial_\mu \hat{T}^{\mu\nu} = 0$, when no current was present.

a) When there is a nonzero current $J^\mu = (\rho, \mathbf{J})$, the stress-energy tensor defined in (3) has divergence $\partial_\mu \hat{T}^{\mu\nu} = -K^\nu$. Find K^ν.

b) The vector $K^\mu = (K^0, \mathbf{K})$ is the four-momentum delivered to the current by the field, per unit space and unit time. Explicitly write K^0 and $\mathbf{K}$ in terms of ρ, $\mathbf{J}$, $\mathbf{E}$, and $\mathbf{B}$.

3. Pseudoscalar and vector decays. (10 points)

In this problem we will consider decays of particles to pairs of electrons. As usual, the electron is described by a Dirac field Ψ of mass m_e. All decay rates should be computed in the rest frame of the decaying particle, and you can reuse results from problem sets.

a) Consider a pseudoscalar particle, described by a real scalar field ϕ of mass M, where $M > 2m_e$. Write down the Feynman rule for the interaction

$$\mathcal{L}_{\text{int}} = ig\phi\,\bar{\Psi}\gamma^5\Psi. \tag{4}$$

Then find the decay rate for $\phi \to e^+e^-$ to leading nontrivial order in g.

b) Now consider a vector particle, described by a massive vector field A_μ of mass M, where $M > 2m_e$. Write down the Feynman rule for the interaction

$$\mathcal{L}_{\text{int}} = g\,\bar{\Psi}\gamma^\mu A_\mu \Psi. \tag{5}$$

Then find the decay rate for $A_\mu \to e^+e^-$ to leading nontrivial order in g. (Hint: the Feynman rule for an incoming massive vector is $\epsilon_\mu(p)$, where ϵ_μ is its polarization. The polarization satisfies $\epsilon \cdot \epsilon = -1$ and $p \cdot \epsilon = 0$, since $\partial_\mu A^\mu = 0$. By rotational symmetry, the decay rate doesn't depend on the vector's polarization, so you can compute it by either averaging over possible polarizations using

$$\sum_\lambda \epsilon^\mu(p,\lambda)\epsilon^{*\nu}(p,\lambda) = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{M^2} \tag{6}$$

or by picking any specific polarization.)

4. Higgs boson production at a muon collider. (15 points)

The high energy physics community is currently debating how to best study the Higgs boson with future particle accelerators. One idea is to build a machine that collides muons and anti-muons at very high energies and produces Higgs bosons in their annihilation.

You should treat the muon, Higgs, and photon in the same way as in problem set 8, and neglect any other particles or interactions. That is, consider the free Lagrangian

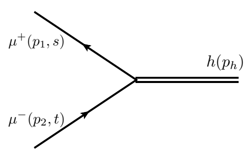
$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}m_h^2\,h^2 + \bar{\Psi}(i\partial\!\!\!/ - m_\mu)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{7}$$

with interactions

$$\mathcal{L}_{\text{int}} = -\frac{m_\mu}{v}\,h\bar{\Psi}\Psi - e\bar{\Psi}A\!\!\!/\Psi. \tag{8}$$

We first consider the case where a muon and anti-muon annihilate to produce a Higgs boson, $\mu^+(p_1)\mu^-(p_2) \to h(p_h)$.

a) Compute the scattering amplitude $\mathcal{M}_{p_1\,p_2\to p_h}$ for this process at lowest nontrivial order in perturbation theory, by explicitly using Wick's theorem. Then show that you get the same answer if you simply apply the Feynman rules, where the vertex is



$$= -\frac{im_\mu}{v}. \tag{9}$$

The scattering cross section of particles a and b with masses m_a and m_b to produce N final state particles is

$$\sigma_{p_a\, p_b \to q_1 \ldots q_N} = \frac{1}{2\sqrt{(2p_a p_b)^2 - 4m_a^2 m_b^2}} \int \prod_{i=1}^{N} \left(\frac{d^4 q_i}{(2\pi)^4} (2\pi)\delta(q_i^2 - m_i^2)\theta(q_i^0 - m_i) \right)$$
$$\times (2\pi)^4 \delta^{(4)}(p_a + p_b - q_1 - \cdots - q_N) |\bar{\mathcal{M}}_{p_a\, p_b \to q_1 \ldots q_N}|^2 \quad (10)$$

where $|\bar{\mathcal{M}}_{p_a\, p_b \to q_1 \ldots q_N}|^2$ is the modulus squared of the scattering amplitude, summed over final state spins and averaged over initial state spins.

b) Compute $|\bar{\mathcal{M}}_{p_1\, p_2 \to p_h}|^2$ and write it in terms of the constants defined in (7) and (8).

c) Using this result and (10), compute the total cross section for this process. Your result should contain distributions – what is their interpretation?

It is also possible to produce additional particles in the final state. For the rest of this problem, we consider the process $\mu^+(p_1)\, \mu^-(p_2) \to \gamma(p_3)\, h(p_h)$ at leading nontrivial order in perturbation theory.

d) Draw all the relevant Feynman diagrams and use the Feynman rules to write down the corresponding amplitude $\mathcal{M}_{p_1 p_2 \to p_3 p_h}$.

e) Since a muon collider would operate at a center-of-mass energy much higher than the muon mass, we can treat m_μ as a small quantity. Compute $|\bar{\mathcal{M}}_{p_1\, p_2 \to p_3\, p_h}|^2$ to leading nontrivial order in m_μ, and write it in terms of the Mandelstam invariants

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_h)^2, \qquad u = (p_1 - p_3)^2 \quad (11)$$

and the constants defined in (7) and (8). (Hint: your expression will contain a sum over photon polarizations of the form

$$\sum_\lambda \epsilon^\mu(p_3, \lambda)\epsilon^{*\nu}(p_3, \lambda) \quad (12)$$

which you can simply replace with $-\eta^{\mu\nu}$.)

f) Using this result and (10), compute the differential cross section $d\sigma/d\cos\theta$ for this process in the center-of-mass frame, where θ is the angle between the three-momenta of the muon and the photon. You may again work to leading nontrivial order in m_μ. Express your result in terms of s, θ, and the constants defined in (7) and (8).

5. ⋆ Kaluza–Klein theory. (5 points)

Some theories of physics beyond the Standard Model involve compactified extra spacetime dimensions. In this problem, you will see why this generically gives rise to many new particles, and why the extra dimensions are hard to detect when they are very small.

Consider a massless real scalar field on a five-dimensional spacetime, where the extra dimension is compactified on a circle of radius R. That is, points are labeled by (x, w) where $x = (t, x, y, z)$ is the usual four-dimensional spacetime coordinate, and $w \in [0, 2\pi R]$, with the points $w = 0$ and $w = 2\pi R$ identified. The metric on this spacetime is still mostly negative, so $\eta^{00} = 1$ with the other diagonal elements negative. The action is

$$S = \int d^4x \int_0^{2\pi R} dw\, \frac{1}{2}(\partial_M \phi(x, w))(\partial^M \phi(x, w)) \quad (13)$$

where the index M ranges from 0 to 4.

a) Using Fourier series, the w-dependence of the field can be written as

$$\phi(x,w) = \sum_n \phi^{(n)}(x)\, e^{ik_n w} \tag{14}$$

where the sum is over integer n. What are the k_n, and how is $\phi^{(n)}$ related to $\phi^{(-n)}$?

b) Plug this decomposition into the action and perform the w integral to yield an ordinary four-dimensional action, written in terms of the fields $\phi^{(n)}(x)$ for $n \geq 0$. What are the physical masses m_n of these fields? (Hint: you should rescale the fields to give the kinetic terms the usual normalizations.)

Now consider the case of a massless vector field, which can be decomposed as

$$A_M(x,w) = \sum_n A_M^{(n)}(x) e^{ik_n w}. \tag{15}$$

The action is the five-dimensional analogue of the usual electromagnetic action,

$$S = -\frac{1}{4}\int d^4x \int_0^{2\pi R} dw\, F_{MN}F^{MN} \tag{16}$$

and we have the five-dimensional analogue of the usual gauge symmetry, $A_M \to A_M + \partial_M \alpha$.

c) Show that we can use a gauge transformation to set $A_4^{(n)} = 0$ when $n \neq 0$, and explain why this is not possible when $n = 0$.

d) Using this gauge, find all the resulting four-dimensional fields and their masses.

Quantum Field Theory I

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$$\sigma^2_{\phi(\mathbf{x})} = \langle 0|\phi(\mathbf{x})^2|0\rangle - \left(\langle 0|\phi(\mathbf{x})|0\rangle\right)^2. \tag{1}$$

Show that this quantity is infinite.

Solution: The expectation value is

$$\langle 0|\phi(\mathbf{x})|0\rangle = 0 \tag{S1}$$

while the expectation value of the square is

$$\begin{aligned}
\langle 0|\phi(\mathbf{x})^2|0\rangle &= \int \frac{d\mathbf{p}}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3}\frac{1}{\sqrt{2E(\mathbf{p})}\sqrt{2E(\mathbf{q})}} \\
&\quad \times \langle 0|\left(a(\mathbf{p})e^{i\mathbf{x}\cdot\mathbf{p}} + a^\dagger(\mathbf{p})e^{-i\mathbf{x}\cdot\mathbf{p}}\right)\left(a(\mathbf{q})e^{i\mathbf{x}\cdot\mathbf{q}} + a^\dagger(\mathbf{q})e^{-i\mathbf{x}\cdot\mathbf{q}}\right) && \text{(S2)}\\
&= \int \frac{d\mathbf{p}}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3}\frac{1}{\sqrt{2E(\mathbf{p})}\sqrt{2E(\mathbf{q})}}\langle 0|a(\mathbf{p})a^\dagger(\mathbf{q})e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{q})}|0\rangle && \text{(S3)}\\
&= \int \frac{d\mathbf{p}}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3}\frac{1}{\sqrt{2E(\mathbf{p})}\sqrt{2E(\mathbf{q})}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{q})} && \text{(S4)}\\
&= \int \frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{2E(\mathbf{p})} && \text{(S5)}\\
&= \frac{1}{2\pi^2}\int_0^\infty d|\mathbf{p}|\frac{|\mathbf{p}|^2}{\sqrt{|\mathbf{p}|^2+m^2}} && \text{(S6)}
\end{aligned}$$

which is infinite.

b) There's nothing wrong with the result in part (a), because in practice we can never measure the field value at a point. We only measure "smeared" field values averaged over some length scale a, such as

$$\phi_a(\mathbf{x}) = \frac{1}{a^3\pi^{3/2}}\int d^3y\, \phi(\mathbf{y})e^{-|\mathbf{x}-\mathbf{y}|^2/a^2}. \tag{2}$$

Write $\sigma^2_{\phi_a(\mathbf{x})}$ in the form of an integral over a single variable.

Solution: The expectation value still vanishes, but now we have

$$\begin{aligned}
\langle 0|\phi_a(\mathbf{x})^2|0\rangle &= \frac{1}{a^6\pi^3}\int d\mathbf{y}d\mathbf{z}\, e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{a^2}}e^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{a^2}}\langle 0|\phi(\mathbf{y})\phi(\mathbf{z})|0\rangle && \text{(S7)}\\
&= \frac{1}{a^6\pi^3}\int d\mathbf{y}d\mathbf{z}\,\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{2E(\vec{p})}e^{i\mathbf{p}(\mathbf{y}-\mathbf{z})}e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{a^2}}e^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{a^2}}. && \text{(S8)}
\end{aligned}$$

Next, we perform the shifts $\mathbf{z}\to\mathbf{z}+\mathbf{x}$ and $\mathbf{y}\to\mathbf{y}+\mathbf{x}$. Then the integral

$$\begin{aligned}
\int d\mathbf{y}\, e^{i\mathbf{p}(\mathbf{y})}e^{-\frac{|\mathbf{y}|^2}{a^2}} &= \int d\mathbf{y}\, e^{i|\mathbf{p}||\mathbf{y}|\cos\theta}e^{-\frac{|\mathbf{y}|^2}{a^2}} && \text{(S9)}\\
&= (2\pi)\int_0^\infty d|\mathbf{y}||\mathbf{y}|^2\int_{-1}^1 d\cos\theta|\mathbf{y}|e^{i|\mathbf{p}|\cos\theta}e^{-\frac{|\mathbf{y}|^2}{a^2}} && \text{(S10)}\\
&= \frac{(4\pi)}{|\mathbf{p}|}\int_0^\infty d|\mathbf{y}||\mathbf{y}|\sin(|\mathbf{y}||\mathbf{p}|)e^{-\frac{|\mathbf{y}|^2}{a^2}} && \text{(S11)}\\
&= a^3\pi^{3/2}e^{-\frac{|\mathbf{p}|^2a^2}{4}}. && \text{(S12)}
\end{aligned}$$

Inserting this above, we find

$$\langle 0|\phi_a(\mathbf{x})^2|0\rangle = \int \frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{2E(\vec{p})}e^{-\frac{|\mathbf{p}|^2a^2}{2}}. \tag{S13}$$

Simplifying, we thus conclude that

$$\sigma^2_{\phi_a(\mathbf{x})} = \langle 0|\phi_a(\mathbf{x})^2|0\rangle = \frac{1}{4\pi^2}\int_0^\infty d|\mathbf{p}|\frac{|\mathbf{p}|^2}{\sqrt{|\mathbf{p}|^2+m^2}}e^{-\frac{|\mathbf{p}|^2a^2}{2}}. \tag{S14}$$

c) The standard deviation of the smeared field is finite. Find expressions for $\sigma^2_{\phi_a(\mathbf{x})}$ in the limits $a \ll 1/m$ and $a \gg 1/m$. (Hint: in each regime, your answer should have the form $\alpha\, a^\beta$ for some values of α and β. If you're having trouble doing the final integral in one of these cases, you can start from the result $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$.)

Solution: Changing to the variable $t = a|\mathbf{p}|$, we have

$$\sigma^2_{\phi_a(\mathbf{x})} = \frac{1}{4\pi^2a^2}\int_0^\infty dt\frac{t^2}{\sqrt{t^2+a^2m^2}}e^{-\frac{t^2}{2}}. \tag{S15}$$

Expanding around $ma \ll 1$, we find

$$\lim_{ma\to 0}\sigma^2_{\phi_a(\mathbf{x})} = \frac{1}{4\pi^2a^2}\int_0^\infty dt\, te^{-\frac{t^2}{2}} = \frac{1}{4\pi^2a^2}. \tag{S16}$$

Expanding around $ma \gg 1$, we find

$$\lim_{1/(ma)\to 0}\sigma^2_{\phi_a(\mathbf{x})} = \frac{1}{4\pi^2a^3m}\int_0^\infty dt\, t^2e^{-\frac{t^2}{2}} = \frac{1}{4\pi^{3/2}a^3m\sqrt{2}}. \tag{S17}$$

d) Consider a cubical copper microwave cavity of volume $V = 1\,\mathrm{m}^3$. The electric field in the cavity does not have a definite value, due to the quantization of the electromagnetic field. Find a rough, order of magnitude estimate for the standard deviation of the average electric field in the cavity in its ground state, and evaluate it in volts per meter. (Hint: you will need to go from natural units back to SI units using $\hbar = 10^{-34}\,\mathrm{J\,s}$, $c = 3\times 10^8\,\mathrm{m/s}$, and $e = 1.6\times 10^{-19}\,\mathrm{C}$.)

Solution: In part (c), we showed that for a massless vector field, the standard deviation of the field average over a region of size a is $\sigma \sim 1/a$. The same logic applies to a massless vector field A_μ. The electric field in the cavity is related to A_μ by one spatial or temporal derivative, which by dimensional analysis also gives a factor of $1/a$. Thus, we expect $\sigma_E \sim 1/a^2 \sim 1/V^{2/3}$. (We could also have jumped directly to this result. We already know $[E] = 2$ from problem set 1, and part (c) tells us the answer is neither zero nor infinity, so by dimensional analysis, $1/a^2$ is the only possible answer.)

Going from natural units back to SI units, we have

$$\sigma_E \sim \frac{\hbar c}{ea^2} \sim 10^{-7}\,\mathrm{V/m}. \tag{S18}$$

This might be surprisingly large if you haven't seen it before, but it lines up with typical "quantum noise" scales in precision electronics. Any answer within a few orders of magnitude of this is acceptable.

2. Forces on external currents. (5 points)

In problem set 7, you considered the "improved" electromagnetic stress-energy tensor,

$$\hat{T}^{\mu\nu} = -\eta_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{3}$$

and showed that it was conserved, $\partial_\mu \hat{T}^{\mu\nu} = 0$, when no current was present.

a) When there is a nonzero current $J^\mu = (\rho, \mathbf{J})$, the stress-energy tensor defined in (3) has divergence $\partial_\mu \hat{T}^{\mu\nu} = -K^\nu$. Find K^ν.

Solution: The only difference is that the equation of motion is now $\partial_\mu F^{\mu\nu} = J^\nu$. We already know that when $\partial_\mu F^{\mu\nu} = 0$, the stress-energy is conserved, so to compute K^ν we only need to consider what terms appear which are proportional to J^μ. This can only occur when the derivative ∂_μ hits the factor of $F^{\mu\rho}$ in the first term, so

$$\partial_\mu \hat{T}^{\mu\nu} = -\eta_{\rho\sigma}(\partial_\mu F^{\mu\rho})F^{\nu\sigma} = -\eta_{\rho\sigma} J^\rho F^{\nu\sigma}. \tag{S19}$$

Thus, the answer is $K^\nu = J_\sigma F^{\nu\sigma}$.

You might be wondering why we are allowed to use the original expression for the stress-energy tensor, as rerunning the Noether/Belinfante procedure will give additional terms when J^μ is nonzero. The point is that we're trying to find the change in the energy-momentum of the electromagnetic field alone, so we shouldn't include those extra terms, which represent something like potential energy-momentum.

b) The vector $K^\mu = (K^0, \mathbf{K})$ is the four-momentum delivered to the current by the field, per unit space and unit time. Explicitly write K^0 and $\mathbf{K}$ in terms of ρ, $\mathbf{J}$, $\mathbf{E}$, and $\mathbf{B}$.

Solution: Expanding the components out, we have

$$K^i = J_\sigma F^{i\sigma} = J_0 F^{i0} + J_j F^{ij} = J_0 E^i - J_j \epsilon^{ijk} B^k. \tag{S20}$$

Noting that $J_0 = J^0 = \rho$ and $J_i = -J^i$, we conclude

$$\mathbf{K} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{S21}$$

which of course is just the continuous version of the Lorentz force law. Similarly, we have

$$K^0 = J_i F^{0i} = (-J^i)(-F^{i0}) = \mathbf{J} \cdot \mathbf{E} \tag{S22}$$

which is the rate at which the electric field does work on the charges.

3. Pseudoscalar and vector decays. (10 points)

In this problem we will consider decays of particles to pairs of electrons. As usual, the electron is described by a Dirac field Ψ of mass m_e. All decay rates should be computed in the rest frame of the decaying particle, and you can reuse results from problem sets.

a) Consider a pseudoscalar particle, described by a real scalar field ϕ of mass M, where $M > 2m_e$. Write down the Feynman rule for the interaction

$$\mathcal{L}_{\text{int}} = ig\phi \, \bar{\Psi}\gamma^5\Psi. \tag{4}$$

Then find the decay rate for $\phi \to e^+e^-$ to leading nontrivial order in g.

Solution: As usual, the Feynman rule comes from $-iH_{\text{int}} = iL_{\text{int}}$, so the vertex factor is $-g\gamma^5$. (This looks unusual, but it's correct: there had to be an i in L_{int} to ensure the action is real when g is.) Then

$$\mathcal{M} = -g\bar{u}_r(p_1)\gamma^5 v_s(p_2). \tag{S23}$$

Squaring and summing over final spin states gives

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 &= g^2 \sum_{r,s} \bar{u}_r(p_1)\gamma^5 v_s(p_2) v_s^\dagger(p_2)(\gamma^5)^\dagger(\gamma^0)^\dagger u_r(p_1) && \text{(S24)}\\
&= -g^2 \operatorname{tr}((\not{p}_1 + m_e)\gamma^5(\not{p}_2 - m_e)\gamma^5) && \text{(S25)}\\
&= g^2 \operatorname{tr}((\not{p}_1 + m_2)(\not{p}_2 + m_e)\gamma^5\gamma^5) && \text{(S26)}\\
&= 4g^2(p_1 \cdot p_2 + m_e^2) && \text{(S27)}\\
&= 2g^2 M^2. && \text{(S28)}
\end{aligned}$$

Plugging this into the decay rate result derived in problem set 5 gives

$$\Gamma = \frac{g^2}{8\pi}\sqrt{M^2 - 4m_e^2}. \tag{S29}$$

b) Now consider a vector particle, described by a massive vector field A_μ of mass M, where $M > 2m_e$. Write down the Feynman rule for the interaction

$$\mathcal{L}_{\text{int}} = g\,\bar{\Psi}\gamma^\mu A_\mu \Psi. \tag{5}$$

Then find the decay rate for $A_\mu \to e^+e^-$ to leading nontrivial order in g. (Hint: the Feynman rule for an incoming massive vector is $\epsilon_\mu(p)$, where ϵ_μ is its polarization. The polarization satisfies $\epsilon \cdot \epsilon = -1$ and $p \cdot \epsilon = 0$, since $\partial_\mu A^\mu = 0$. By rotational symmetry, the decay rate doesn't depend on the vector's polarization, so you can compute it by either averaging over possible polarizations using

$$\sum_\lambda \epsilon^\mu(p,\lambda)\epsilon^{*\nu}(p,\lambda) = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{M^2} \tag{6}$$

or by picking any specific polarization.)

Solution: The Feynman rule for the vertex is $ig\gamma^\mu$, as in ordinary QED, so

$$\mathcal{M} = ig\epsilon_\mu^\lambda \bar{u}_r(p_1)\gamma^\mu v_s(p_2). \tag{S30}$$

Summing over final spin states and averaging over the initial three spin states, we have

$$|\bar{\mathcal{M}}|^2 = \frac{g^2}{3}\left(-\eta_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}\right)\sum_{rs}\bar{u}_r(p_1)\gamma^\mu v_s(p_2)\bar{v}_2(p_2)\gamma^\nu u_r(p_1). \tag{S31}$$

Note that the $p_\mu p_\nu$ terms won't contribute anything, as by the Dirac equation,

$$\bar{u}_r(p_1)\not{p}v_s(p_2) = \bar{u}_r(p_1)(\not{p}_1 + \not{p}_2)v_s(p_2) = \bar{u}_r(p_1)(m_e - m_e)v_s(p_2) = 0. \tag{S32}$$

Thus, keeping just the metric term and rewriting the spinor sum as a trace,

$$|\bar{\mathcal{M}}|^2 = -\frac{g^2}{3}\,\eta_{\mu\nu}\,\text{tr}((\not{p}_1 + m_e)\gamma^\mu(\not{p}_2 - m_e)\gamma^\nu) \tag{S33}$$

$$= -\frac{4g^2}{3}\,\eta_{\mu\nu}(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - (p_1\cdot p_2 - m_e^2)\eta^{\mu\nu}) \tag{S34}$$

$$= \frac{8g^2}{3}(p_1\cdot p_2 + 2m_e^2) \tag{S35}$$

$$= \frac{4g^2M^2}{3}\left(1 + \frac{2m_e^2}{M^2}\right). \tag{S36}$$

Plugging this into our decay rate result gives

$$\Gamma = \frac{g^2}{12\pi}\left(1 + \frac{2m_e^2}{M^2}\right)\sqrt{M^2 - 4m_e^2}. \tag{S37}$$

4. Higgs boson production at a muon collider. (15 points)

The high energy physics community is currently debating how to best study the Higgs boson with future particle accelerators. One idea is to build a machine that collides muons and anti-muons at very high energies and produces Higgs bosons in their annihilation.

You should treat the muon, Higgs, and photon in the same way as in problem set 8, and neglect any other particles or interactions. That is, consider the free Lagrangian

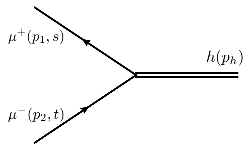
$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}m_h^2\, h^2 + \bar{\Psi}(i\partial\!\!\!/ - m_\mu)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{7}$$

with interactions

$$\mathcal{L}_{\rm int} = -\frac{m_\mu}{v}\, h\bar{\Psi}\Psi - e\bar{\Psi}A\!\!\!/\Psi. \tag{8}$$

We first consider the case where a muon and anti-muon annihilate to produce a Higgs boson, $\mu^+(p_1)\mu^-(p_2) \to h(p_h)$.

a) Compute the scattering amplitude $\mathcal{M}_{p_1\, p_2 \to p_h}$ for this process at lowest nontrivial order in perturbation theory, by explicitly using Wick's theorem. Then show that you get the same answer if you simply apply the Feynman rules, where the vertex is



$$= -\frac{im_\mu}{v}. \tag{9}$$

Solution: The initial and final states are

$$|\mu^+(p_1,s), \mu^-(p_2,t)\rangle = \sqrt{2E(\vec{p_1})}\sqrt{2E(\vec{p_2})}b_s^\dagger(p_1)a_t^\dagger(p_2)|0\rangle, \tag{S38}$$
$$|h\rangle = \sqrt{2E(\vec{p_h})}a^\dagger(p_h)|0\rangle. \tag{S39}$$

We can then compute for example the S-matrix element

$$\begin{aligned}\langle \mu^+(p_1,s), \mu^-(p_2,t)| \int d^4x i\mathcal{L}_{int}|h\rangle &= \langle \mu^+(p_1,s), \mu^-(p_2,t)| \int d^4x i\frac{m_h}{v}h(x)\bar{\Psi}_i(x)\Psi_i(x)|h\rangle \\ &= \frac{-im_\mu}{v}\sqrt{8E(p_h)E(p_1)E(p_2)}\langle 0| \int d^4x b_s(p_1)a_t(p_2)h(x)\bar{\Psi}_i(x)\Psi_i(x)a^\dagger(p_h)|0\rangle.\end{aligned} \tag{S40}$$

We find for example for

$$\begin{aligned} h(x)a^\dagger(p_h) &= \int \frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E(p)}}\left[a(p)e^{-ipx} + a^\dagger(p)e^{ipx}\right]a^\dagger(p_h) \\ &= a^\dagger(p_h)h(x) + \int \frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E(p)}}[a(p), a^\dagger(p_h)]e^{-ipx} \\ &= a^\dagger(p_h)h(x) + \int \frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E(p)}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}_h)e^{-ipx} \\ &= a^\dagger(p_h)h(x) + \frac{1}{\sqrt{2E(p_h)}}e^{-ip_hx} \end{aligned} \tag{S41}$$

In our vacuum expectation value of (S40), the creation operator of the scalar field then acts directly on the vacuum state on the left and annihilates it and only the commutator term remains. Similarly, we find

$$\begin{aligned} a_t(p_2)\Psi_i(x) &= u_i^t(p_2)\frac{e^{ip_2x}}{\sqrt{2E(p_2)}} - \Psi_i(x)a_t(p_2). \\ b_s(p_1)\bar{\Psi}_i(x) &= \bar{v}_i^s(p_1)\frac{e^{ip_1x}}{\sqrt{2E(p_1)}} - \bar{\Psi}_i(x)b_s(p_1). \end{aligned} \tag{S42}$$

As

$$\{b_s^\dagger(p_1), \Psi(x)\} = \{\bar{\Psi}_i(x), a_t^\dagger(p_2)\} = 0, \tag{S43}$$

only the anti-commutator terms survive. We find

$$\langle \mu^+(p_1, s), \mu^-(p_2, t)| \int d^4x i\mathcal{L}_{int}|h\rangle = -\frac{im_\mu}{v}\int d^4x e^{-ix(p_h - p_1 - p_2)}\bar{v}_i^s(p_1)\delta_{ij}u_j^t(p_2)$$
$$= (2\pi)^4\delta^{(4)}(p_h - p_1 - p_2)\bar{v}_i^s(p_1)\left(-\frac{im_\mu}{v}\delta_{ij}\right)u_j^t(p_2). \tag{S44}$$

The scattering matrix elements are defined by

$$S_{fi} = \delta_{fi} + (2\pi)^4\delta(p_f - p_i)\mathcal{M}, \tag{S45}$$

and we associate $u_i^t(p_2)$ with an ingoing muon and $\bar{v}_j^s(p_1)$ with an ingoing anti-muon. We thus conclude

$$\mathcal{M}_{p_1\,p_2\to p_h} = \bar{v}_i^s(p_1)\left(-i\frac{m_\mu}{v}\right)u_i^t(p_2) \tag{S46}$$

which is clearly also what we get from the Feynman rules.

The scattering cross section of particles a and b with masses m_a and m_b to produce N final state particles is

$$\sigma_{p_a\,p_b\to q_1\ldots q_N} = \frac{1}{2\sqrt{(2p_a p_b)^2 - 4m_a^2 m_b^2}}\int\prod_{i=1}^N\left(\frac{d^4q_i}{(2\pi)^4}(2\pi)\delta(q_i^2 - m_i^2)\theta(q_i^0 - m_i)\right)$$
$$\times (2\pi)^4\delta^{(4)}(p_a + p_b - q_1 - \cdots - q_N)|\bar{\mathcal{M}}_{p_a\,p_b\to q_1\ldots q_N}|^2 \tag{10}$$

where $|\bar{\mathcal{M}}_{p_a\,p_b\to q_1\ldots q_N}|^2$ is the modulus squared of the scattering amplitude, summed over final state spins and averaged over initial state spins.

b) Compute $|\bar{\mathcal{M}}_{p_1\,p_2\to p_h}|^2$ and write it in terms of the constants defined in (7) and (8).

Solution: Using the result of part (a), we have

$$|\bar{\mathcal{M}}_{p_1\,p_2\to p_h}|^2 = \frac{1}{4}\sum_{s,t}\bar{u}_i^t(p_2)\left(i\frac{m_\mu}{v}\right)v_i^s(p_1)\bar{v}_j^s(p_1)\left(-i\frac{m_\mu}{v}\right)u_j^t(p_2) \tag{S47}$$
$$= \frac{m_\mu^2}{4v^2}\,\mathrm{tr}[(\not{p}_1 - m_\mu)(\not{p}_2 + m_\mu)] \tag{S48}$$
$$= \frac{m_\mu^2}{4v^2}\,\mathrm{tr}[\not{p}_1\not{p}_2] - 4m_\mu^2 \tag{S49}$$
$$= \frac{m_\mu^2}{2v^2}(m_h^2 - 4m_\mu^2). \tag{S50}$$

The factor of $1/4$ arises due to averaging over 2 spin states per initial state fermion.

c) Using this result and (10), compute the total cross section for this process. Your result should contain distributions – what is their interpretation?

Solution: Introducing $s = (p_1 + p_2)^2$, we find

$$\sigma_{p_1\,p_2\to p_h} = \frac{\pi}{\sqrt{s(s - 4m_\mu^2)}}\int d^4p_h\delta(p_h^2 - m_h^2)\theta(p_h^0 - m_h)\delta^{(4)}(p_1 + p_2 - p_h)|\bar{\mathcal{M}}_{p_1\,p_2\to p_h}|^2 \tag{S51}$$
$$= \frac{\pi}{\sqrt{s(s - 4m_\mu^2)}}\delta(s - m_h^2)\theta(p_1^0 + p_2^0 - m_h)|\bar{\mathcal{M}}_{p_1\,p_2\to p_h}|^2 \tag{S52}$$
$$= \frac{\pi m_\mu^2}{2v^2}\sqrt{1 - \frac{4m_\mu^2}{m_h^2}}\delta(s - m_h^2)\theta(p_1^0 + p_2^0 - m_h). \tag{S53}$$

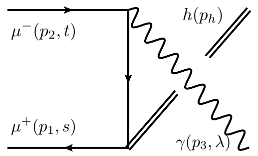
The distributions forces the center-of-mass energy s to be exactly equal to the Higgs boson mass, which makes sense by energy-momentum conservation. (There is something weird about this result, though: in physics you can never have two things be *exactly* equal to each other, so this would suggest Higgs bosons are never produced this way at all. The resolution is that all unstable particles actually have a slightly indefinite mass (heuristically due to the energy-time uncertainty principle applied to their finite decay time) so the delta function is replaced with a narrow peak.)

It is also possible to produce additional particles in the final state. For the rest of this problem, we consider the process $\mu^+(p_1)\,\mu^-(p_2) \to \gamma(p_3)\,h(p_h)$ at leading nontrivial order in perturbation theory.

d) Draw all the relevant Feynman diagrams and use the Feynman rules to write down the corresponding amplitude $\mathcal{M}_{p_1p_2\to p_3p_h}$.

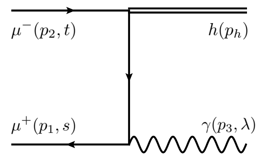
Solution: There are two relevant diagrams:

$D_1 =$



$$= \bar{v}_s(p_1)\left(-i\frac{m_\mu}{v}\right)\frac{i(\not{p}_1 - \not{p}_h + m_\mu)}{(p_1-p_h)^2 - m_\mu^2}(-ie\gamma^\mu)u_t(p_2)\epsilon_\mu(p_3,\lambda). \tag{S54}$$

$D_2 =$



$$= \bar{v}_s(p_1)(-ie\gamma^\mu)\frac{i(\not{p}_1 - \not{p}_3 + m_\mu)}{(p_1-p_3)^2 - m_\mu^2}\left(-i\frac{m_\mu}{v}\right)u_t(p_2)\epsilon_\mu(p_3,\lambda). \tag{S55}$$

The amplitude is $D_1 + D_2$. All other diagrams are suppressed by more powers of m_μ/v, or more powers of e, or both.

e) Since a muon collider would operate at a center-of-mass energy much higher than the muon mass, we can treat m_μ as a small quantity. Compute $|\bar{\mathcal{M}}_{p_1\,p_2\to p_3\,p_h}|^2$ to leading nontrivial order in m_μ, and write it in terms of the Mandelstam invariants

$$s = (p_1+p_2)^2, \qquad t = (p_1 - p_h)^2, \qquad u = (p_1 - p_3)^2 \tag{11}$$

and the constants defined in (7) and (8). (Hint: your expression will contain a sum over photon polarizations of the form

$$\sum_\lambda \epsilon^\mu(p_3,\lambda)\epsilon^{*\nu}(p_3,\lambda) \tag{12}$$

which you can simply replace with $-\eta^{\mu\nu}$.)

Solution: First, we find the leading term in m_μ for our diagrams:

$$D_1 = \frac{-iem_\mu}{vt}\bar{v}_s(p_1)(\not{p}_1 - \not{p}_h)\gamma^\mu u_t(p_2)\epsilon_\mu(p_3,\lambda), \tag{S56}$$

$$D_2 = \frac{-iem_\mu}{vu}\bar{v}_s(p_1)\gamma^\mu(\not{p}_h - \not{p}_2)u_t(p_2)\epsilon_\mu(p_3,\lambda). \tag{S57}$$

Since these quantities already have a factor of m_μ in front, and we only want the lowest order in m_μ, we can set m_μ to zero when evaluating what's left. The Dirac equation for the spinors becomes $\bar{v}_s(p_1)\not{p}_1 = 0 = \not{p}_2 u_t(p_2)$, from which we find

$$D_1 = \frac{iem_\mu}{vt}\bar{v}_s(p_1)\not{p}_h\gamma^\mu u_t(p_2)\epsilon_\mu(p_3,\lambda), \tag{S58}$$

$$D_2 = \frac{-iem_\mu}{vu}\bar{v}_s(p_1)\gamma^\mu\not{p}_h u_t(p_2)\epsilon_\mu(p_3,\lambda). \tag{S59}$$

Next, we compute the terms in the square of the total amplitude,

$$\frac{1}{4}\sum_{s,t,\lambda} D_1 D_1^\dagger = -\frac{e^2 m_\mu^2}{4v^2t^2}\,\mathrm{tr}[\not{p}_1\not{p}_4\gamma_\mu\not{p}_2\gamma^\mu\not{p}_4] = \frac{e^2m_\mu^2}{2v^2t^2}\,\mathrm{tr}[\not{p}_1\not{p}_4\not{p}_2\not{p}_4] = \frac{e^2m_\mu^2 u}{v^2t}. \tag{S60}$$

$$\frac{1}{4}\sum_{s,t,\lambda} D_1 D_2^\dagger = \frac{e^2 m_\mu^2}{4v^2tu}\,\mathrm{tr}[\not{p}_1\not{p}_4\gamma_\mu\not{p}_2\not{p}_4\gamma^\mu] = \frac{4e^2m_\mu^2}{v^2tu}(p_1p_4)(p_2p_4) = \frac{e^2m_\mu^2}{v^2tu}(m_h^2-t)(m_h^2-u). \tag{S61}$$

$$\frac{1}{4}\sum_{s,t,\lambda} D_2 D_2^\dagger = -\frac{e^2 m_\mu^2}{4v^2u^2}\,\mathrm{tr}[\not{p}_1\gamma_\mu\not{p}_4\not{p}_2\not{p}_4\gamma^\mu] = \frac{e^2m_\mu^2}{2v^2u^2}\,\mathrm{tr}[\not{p}_1\not{p}_4\not{p}_2\not{p}_4] = \frac{e^2m_\mu^2 t}{v^2u}. \tag{S62}$$

Above, we used the identities

$$\gamma^\mu\not{p}\gamma_\mu = -2\not{p}, \tag{S63}$$

$$\gamma^\mu\not{p}\not{q}\gamma_\mu = 4pq, \tag{S64}$$

$$\mathrm{tr}[\not{p}\not{q}] = 4pq, \tag{S65}$$

$$p_1p_4 = \frac{m_h^2-t}{2}, \tag{S66}$$

$$p_2p_4 = \frac{m_h^2-u}{2}, \tag{S67}$$

$$m_h^2 = s+t+u, \tag{S68}$$

$$\mathrm{tr}[\not{p}_1\not{p}_4\not{p}_2\not{p}_4] = 4\left[2(p_1p_4)(p_2p_4)-(p_1p_2)p_4^2\right] = 2tu \tag{S69}$$

several of which follow from neglecting m_μ. After some algebra, we find

$$|\bar{\mathcal{M}}_{p_1\,p_2\to p_3\,p_h}|^2 = \frac{1}{4}\sum_{s,t,\lambda}\left[D_1D_1^\dagger + 2\,\mathrm{Re}\left(D_1D_2^\dagger\right) + D_2D_2^\dagger\right] = \frac{e^2m_\mu^2}{v^2tu}\left(s^2+m_h^4\right). \tag{S70}$$

f) Using this result and (10), compute the differential cross section $d\sigma/d\cos\theta$ for this process in the center-of-mass frame, where θ is the angle between the three-momenta of the muon and the photon. You may again work to leading nontrivial order in m_μ. Express your result in terms of s, θ, and the constants defined in (7) and (8).

Solution: Starting from (10), we have

$$\sigma_{p_1\,p_2\to p_3\,p_h} = \frac{1}{2(2\pi)^2 s}\int d^4p_3 d^4p_h\,\delta(p_3^2)\delta(p_h^2-m_h^2)\theta(p_3^0)\theta(p_h^0-m_h) \times \delta^{(4)}(p_1+p_2-p_3-p_h)|\bar{\mathcal{M}}_{p_1\,p_2\to p_3\,p_h}|^2. \tag{S71}$$

We work in the center-of-mass frame where $s = E_{CM}^2$ and do the p_h integral, giving

$$\sigma_{p_1\,p_2\to p_3\,p_h} = \frac{1}{2(2\pi)^2 s}\int d^4p_3\,\delta(p_3^2)\delta(E_{CM}^2-2E_{CM}p_3^0-m_h^2)\theta(p_3^0)\theta(E_{CM}-p_3^0-m_h)|\bar{\mathcal{M}}_{p_1\,p_2\to p_3\,p_h}|^2. \tag{S72}$$

We parametrize the p_3 integral by writing

$$\int d^4p_3\,\delta(p_3^2) = \int_0^\infty dp_3^0\,\frac{d|\vec{p}_3|^2}{2}|\vec{p}_3|\int d\cos\theta\,d\phi\,\delta((p_3^0)^2-|\vec{p}_3|^2) \tag{S73}$$

$$= \pi\int_0^\infty dp_3^0\,p_3^0\int d\cos\theta. \tag{S74}$$

We can integrate over p_3^0 using the remaining delta function,

$$\delta(E_{CM}^2 - 2E_{CM}p_3^0 - m_h^2) = \frac{1}{2E_{CM}}\delta\left(p_3^0 - \frac{1}{2}(E_{CM} - \frac{m_h^2}{E_{CM}})\right). \tag{S75}$$

Since we want a differential cross section, we simply drop the last integral over $\cos\theta$. Using

$$t = (p_2 - p_3)^2 = -2p_2p_3 = -E_{CM}p_3^0(1 - \cos\theta) \tag{S76}$$

$$u = (p_1 - p_3)^2 = -2p_1p_3 = -E_{CM}p_3^0(1 + \cos\theta) \tag{S77}$$

we find

$$\frac{d\sigma_{p_1\,p_2\to p_3\,p_h}}{d\cos\theta} = \frac{s - m_h^2}{32s^2\pi}|\bar{\mathcal{M}}_{p_1\,p_2\to p_3\,p_h}|^2 = \frac{e^2m_\mu^2}{8\pi v^2 s^2}\frac{s^2 + m_h^4}{(s - m_h^2)(1 - \cos^2\theta)}. \tag{S78}$$

5. ⋆ Kaluza–Klein theory. (5 points)

Some theories of physics beyond the Standard Model involve compactified extra spacetime dimensions. In this problem, you will see why this generically gives rise to many new particles, and why the extra dimensions are hard to detect when they are very small.

Consider a massless real scalar field on a five-dimensional spacetime, where the extra dimension is compactified on a circle of radius R. That is, points are labeled by (x, w) where $x = (t, x, y, z)$ is the usual four-dimensional spacetime coordinate, and $w \in [0, 2\pi R]$, with the points $w = 0$ and $w = 2\pi R$ identified. The metric on this spacetime is still mostly negative, so $\eta^{00} = 1$ with the other diagonal elements negative. The action is

$$S = \int d^4x \int_0^{2\pi R} dw\, \frac{1}{2}(\partial_M\phi(x, w))(\partial^M\phi(x, w)) \tag{13}$$

where the index M ranges from 0 to 4.

a) Using Fourier series, the w-dependence of the field can be written as

$$\phi(x, w) = \sum_n \phi^{(n)}(x)\, e^{ik_n w} \tag{14}$$

where the sum is over integer n. What are the k_n, and how is $\phi^{(n)}$ related to $\phi^{(-n)}$?

Solution: Because the function is periodic in w with period $2\pi R$, we must have $k_n = n/R$. Since the field is real, we must have $(\phi^{(n)})^* = \phi^{(-n)}$.

b) Plug this decomposition into the action and perform the w integral to yield an ordinary four-dimensional action, written in terms of the fields $\phi^{(n)}(x)$ for $n \geq 0$. What are the physical masses m_n of these fields? (Hint: you should rescale the fields to give the kinetic terms the usual normalizations.)

Solution: The result is

$$S = \int d^4x \sum_{n,n'} \int_0^{2\pi R} dw\, \frac{1}{2}\partial_M(\phi^{(n)}e^{inw/R})\partial^M(\phi^{(n')}e^{in'w/R}) \tag{S79}$$

$$= \int d^4x \sum_{n,n'} \int_0^{2\pi R} dw\, \frac{1}{2}\left(\partial_\mu\phi^{(n)}\partial^\mu\phi^{(n')} - \frac{in}{R}\frac{in'}{R}\phi^{(n)}\phi^{(n')}\right)e^{inw/R}e^{in'w/R} \tag{S80}$$

$$= \int d^4x \sum_{n,n'} \frac{1}{2}\left(\partial_\mu\phi^{(n)}\partial^\mu\phi^{(n')} - \frac{in}{R}\frac{in'}{R}\phi^{(n)}\phi^{(n')}\right)2\pi R\delta_{n,-n'} \tag{S81}$$

$$= \int d^4x \sum_n \frac{1}{2}\left(\partial_\mu\phi^{(n)}\partial^\mu\phi^{(-n)} - \frac{n^2}{R^2}\phi^{(n)}\phi^{(-n)}\right)2\pi R. \tag{S82}$$

Pairing up terms with positive and negative n, we get

$$S = 2\pi R \int d^4x\, \frac{1}{2}\partial_\mu \phi^{(0)} \partial^\mu \phi^{(0)} + \sum_{n>0} \left(|\partial_\mu \phi^{(n)}|^2 - \frac{n^2}{R^2} |\phi^{(n)}|^2 \right). \tag{S83}$$

This is almost the action for a real scalar field and a bunch of complex scalar fields, but we need to rescale to get the right kinetic terms. Defining $\varphi^{(n)} = \sqrt{2\pi R}\, \phi^{(n)}$, we conclude

$$S = \int d^4x\, \frac{1}{2}\partial_\mu \varphi^{(0)} \partial^\mu \varphi^{(0)} + \sum_{n>0} \left(|\partial_\mu \varphi^{(n)}|^2 - \frac{n^2}{R^2} |\varphi^{(n)}|^2 \right) \tag{S84}$$

from which we see $m_0 = 0$ and $m_n = n/R$. We get an infinite tower of increasingly massive particles, whose masses are very high if R is very small.

Now consider the case of a massless vector field, which can be decomposed as

$$A_M(x, w) = \sum_n A_M^{(n)}(x) e^{ik_n w}. \tag{15}$$

The action is the five-dimensional analogue of the usual electromagnetic action,

$$S = -\frac{1}{4} \int d^4x \int_0^{2\pi R} dw\, F_{MN} F^{MN} \tag{16}$$

and we have the five-dimensional analogue of the usual gauge symmetry, $A_M \to A_M + \partial_M \alpha$.

c) Show that we can use a gauge transformation to set $A_4^{(n)} = 0$ when $n \neq 0$, and explain why this is not possible when $n = 0$.

Solution: We can set $A_4^{(n)} = 0$ for all $n \neq 0$ by taking

$$\alpha = -\sum_{n \neq 0} A_M^{(n)}(x) \frac{e^{ik_n w}}{ik_n}. \tag{S85}$$

This doesn't work for $n = 0$ because it would result in division by zero. Or, in gratuitously fancy language, $A_4^{(0)}$ corresponds to the constant function on the circle, which has zero derivative, yet itself is not the derivative of anything; such a function exists because of the nontrivial cohomology of the circle.

d) Using this gauge, find all the resulting four-dimensional fields and their masses.

Solution: Since we're going to have to rescale the fields later anyway, let's preemptively define

$$A_\mu(x, w) = \sqrt{2\pi R} \sum_n A_\mu^{(n)}(x) e^{ik_n w}, \qquad \phi = \sqrt{2\pi R}\, A_4^{(0)}. \tag{S86}$$

The action becomes

$$S = \int d^4x \int_0^{2\pi R} dw \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} F_{\mu 4} F^{\mu 4} \right). \tag{S87}$$

The first term is

$$S_1 = \int d^4x \int_0^{2\pi R} \frac{dw}{2\pi R} \left(-\frac{1}{4} \sum_{n,n'} e^{i(n+n')w/R} F_{\mu\nu}^{(n)} F^{\mu\nu\,(n')} \right) \tag{S88}$$

$$= -\frac{1}{4} \int d^4x\, F_{\mu\nu}^{(0)} F^{\mu\nu\,(0)} - \frac{1}{2} \sum_{n>0} \int d^4x\, (F_{\mu\nu}^{(n)})^* F^{\mu\nu\,(n)} \tag{S89}$$

which is just the kinetic term for a real vector field $A_\mu^{(0)}$ and a tower of *complex* vector fields $A_\mu^{(n)}$ for $n > 0$. Next, the second term is

$$S = -\frac{1}{2}\int d^4x \int_0^{2\pi R} \frac{dw}{2\pi R} \left(\partial_\mu \phi - \sum_n \partial_4 (A_\mu^{(n)} e^{inw/R})\right)\left(-\partial^\mu \phi - \sum_{n'} \partial^4 (A^{\mu\,(n')} e^{in'w/R})\right) \tag{S90}$$

$$= \int d^4x\, \frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi - \frac{1}{2}\int d^4x \int_0^{2\pi R} \frac{dw}{2\pi R} \sum_{n,n'} e^{i(n+n')w/R} \frac{nn'}{R^2} A_\mu^{(n)} A^{\mu\,(n')} \tag{S91}$$

$$= \int d^4x\, \frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi + \sum_{n>0} \int d^4x \frac{n^2}{R^2} (A_\mu^{(n)})^* A^{\mu\,(n)}. \tag{S92}$$

From this we conclude we have:

- One massless real scalar field ϕ.
- One massless real vector field $A_\mu^{(0)}$.
- A tower of massive complex vector fields $A_\mu^{(n)}$ with masses n/R.

By the way, if you do the analogous calculation starting with a five-dimensional metric, the massless degrees of freedom you'll get are a real scalar, a real vector, and a four-dimensional metric, suggesting a unification between electromagnetism and gravity; that's why Einstein was a big fan of this theory.