Problem Solving I: Mathematical Techniques

For the basics of dimensional analysis and limiting cases, see chapter 1 of Morin or chapter 2 of Order of Magnitude Physics. Many more examples are featured in The Art of Insight; some particularly relevant sections are 2.1, 5.5, 6.3, 8.2, and 8.3. Other sections will be mentioned throughout the course. There is a total of **83** points.

1 Dimensional Analysis

Idea 1

Dimensional analysis is simply the statement that the dimensions of physical equations should match on both sides. This simple idea can sometimes solve whole problems by itself.

Dimensional analysis is also a valuable consistency check. For example, if you're trying to derive the surface area of a sphere and find $4\pi r^3$, you can instantly know you made a mistake. As another example, if a problem says the speed of an object is "small", this technically doesn't obey dimensional analysis unless we compare it to another speed. Thus, the problem might really mean you should assume the speed is small compared to the speed of light, $v \ll c$, which tells you something important.

To be precise, we should distinguish dimensions and units. The dimensions of a physical quantity determine what kind of quantity it is, while a unit is a measure of a dimension. Thus, for example, somebody's height h can be measured in units of feet or meters, but both have dimensions of length; this can be written as $[h] = [\text{ft}] = [\text{m}] = L$, where the brackets indicate dimensions. Another example is that angles are dimensionless, but can be measured in units of degrees or radians. These distinctions are not that important for our purposes, so we will be sloppy and conflate dimensions with units, writing the equivalent of $[h] = \text{m}$.

Example 1: $F = ma$ 2018 B11

A circle of rope is spinning in outer space with an angular velocity ω_0. Transverse waves on the rope have speed v_0, as measured in a rotating reference frame where the rope is at rest. If the angular velocity of the rope is doubled, what is the new speed of transverse waves?

Solution

To solve this problem by dimensional analysis, we reason about what could possibly affect the speed of transverse waves. The result could definitely depend on the rope's length L, mass per length λ, and angular velocity ω_0. It could also depend on the tension, but since this tension balances the centrifugal force, it is determined by all of the other quantities. Thus the quantities we have are

$$[L] = \text{m}, \quad [\lambda] = \text{kg/m}, \quad [\omega_0] = 1/\text{s}.$$

Since λ is the only thing with dimensions of mass, it can't affect the speed, because there is nothing that could cancel out the mass dimension. So the only possible answer is

$$v_0 \sim L\omega_0$$

where the $\sim$ indicates equality up to a dimensionless constant, which cannot be found by dimensional analysis alone. In practice, the constant usually won't be too big or too small, so $L\omega_0$ is a decent estimate of v_0. But even if it isn't, the dimensional analysis tells us the scaling: if ω_0 is doubled, the new speed is $2v_0$.

Example 2

Find the dimensions of the magnetic field.

Solution

To do this, we just think of some simple equation involving B, then solve for its dimensions. For example, we know that $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$, so

$$[B] = \frac{[F]}{[q][v]} = \frac{\mathrm{kg} \cdot \mathrm{m}}{\mathrm{s}^2} \frac{1}{\mathrm{C}} \frac{1}{\mathrm{m/s}} = \frac{\mathrm{kg}}{\mathrm{C} \cdot \mathrm{s}}.$$

[2] **Problem 1.** Find the dimensions of power, the gravitational constant G, the permittivity of free space ϵ_0, and the ideal gas constant R.

[1] **Problem 2.** Derive Kepler's third law for circular orbits, using only dimensional analysis. (Why do you think people didn't figure out this argument 2000 years ago?)

[2] **Problem 3.** Some questions about vibrations.

(a) The typical frequency f of a vibrating star depends only on its radius R, density ρ, and the gravitational constant G. Use dimensional analysis to find an expression for f, up to a dimensionless constant. Then estimate f for the Sun, looking up any numbers you need.

(b) The typical frequency f of a small water droplet freely vibrating in zero gravity could depends on its radius R, density ρ, surface tension γ, and the gravitational constant G. Argue that at least one of these parameters doesn't matter, and find an expression for f up to a dimensionless constant.

[3] **Problem 4.** Some questions about the speed of waves, to be covered in greater detail in **W3**. For all estimates, you can look up any numbers you need.

(a) The speed of sound in an ideal gas depends on its pressure p and density ρ. Explain why we don't have to use the temperature T or ideal gas constant R in the dimensional analysis, and then estimate the speed of sound in air.

(b) The speed of sound in a fluid depends only on its density ρ and bulk modulus $B = -V\, dP/dV$. Estimate the speed of sound in water, which has $B = 2.1\,\mathrm{GPa}$.

The speed of waves on top of the surface of water can depend on the water depth h, the wavelength λ, the density ρ, the surface tension γ, and the gravitational acceleration g.

(c) Find the speed of capillary waves, i.e. water waves of very short wavelength, up to a dimensionless constant.

(d) Find the speed of long-wavelength waves in very deep water, up to a dimensionless constant.

[3] **Problem 5** (Morin 1.5)**.** A particle with mass m and initial speed v is subject to a velocity-dependent damping force of the form bv^n.

(a) For $n = 0, 1, 2, \ldots$, find how the stopping time and stopping distance depend on m, v, and b.

(b) Check that these results actually make sense as m, v, and b are changed, for a few values of n. You should find something puzzling going on. (Hint: to resolve the problem, it may be useful to find the stopping time explicitly in a few examples.)

Idea 2

Dimensional analysis applies everywhere. The argument of any function that is not a monomial, such as $\sin x$, must have no dimensions. The derivative d/dx has the opposite dimensions to x, and the dx in an integral has the same dimensions as x. When you perform an integral, your first step should usually be to "nondimensionalize" it, i.e. to separate out dimensionful factors to leave a dimensionless integral.

Example 3

Evaluate the indefinite integral

$$I = \int_0^a \frac{dx}{bx^2 + c}$$

where b and c are both positive.

Solution

This isn't a hard integral by any means, but it's a simple way to demonstrate what we mean by "nondimensionalizing". If you do the integral directly, you'll get lots of intermediate expressions with a, b, and c in them, which produces clutter and more opportunities for error. Instead, start by substituting $u = \sqrt{b/c}\,x$, to get

$$I = \frac{1}{\sqrt{bc}} \int_0^{u_0} \frac{du}{u^2 + 1} = \frac{1}{\sqrt{bc}} \tan^{-1}(u_0), \quad u_0 = a\sqrt{b/c}.$$

We can now check this by dimensional analysis. Let's suppose this integral arose from a problem where x had dimensions of length, $[x] = \text{m}$, and b was dimensionless. Then we must have $[a] = \text{m}$, $[c] = \text{m}^2$. Our answer makes sense if $[I] = \text{m}^{-1}$ and $[u_0] = 1$, which both hold.

[2] **Problem 6.** We are given the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

For positive a, find the value of the integral

$$\int_{-\infty}^{\infty} e^{-ax^2 + bx + c}\, dx$$

and verify that your answer makes dimensional sense.

Remark

Consider the value of the definite integral

$$\int_{-\infty}^{y} e^{-x^2}\,dx.$$

You can try all day to compute the value of this integral, using all the integration tricks you know, but nothing will work. The function e^{-x^2} simply doesn't have an antiderivative in terms of the functions you already know, i.e. in terms of polynomials, exponents and logarithms, and trigonometric functions (for more discussion, see here).

If you ask a computer algebra system like Mathematica, it'll spit out something like $\text{erf}(x)$, which is defined by being an antiderivative of e^{-x^2}. But is this really an "analytic" solution? Isn't that just saying "the integral of e^{-x^2} is equal to the integral of e^{-x^2}"? Well, like many things in math, it depends on what the meaning of the word "is" is.

The fact is, the set of functions we regard as "elementary" is arbitrary; we just choose a set that's big enough to solve most of the problems we want, and small enough to attain fluency with. (Back in the days before calculators, it just meant all the functions whose values were tabulated in the references on hand.) If you're uncomfortable with $\text{erf}(x)$, note that a similar thing would happen if a little kid asked you what the ratio of the opposite to adjacent sides of a right triangle is. You'd say $\tan(x)$, but they could say it's tautological, because the only way to define $\tan(x)$ at their level is as the ratio of opposite to adjacent sides. Similarly, $1/x$ has no elementary antiderivative – unless you count $\log(x)$ as elementary, but ultimately $\log(x)$ is simply *defined* to be such an antiderivative. It's all tautology, but it's still useful.

[2] **Problem 7.** In particle physics it is conventional to work in "natural units", where the numeric values of $\hbar$ and c are equal to 1. For example, if we take the second as the unit of time, then we can take the light-second as the unit of length, so that $c = 1\,\text{light-second/second}$. This is usually sloppily written as "$\hbar = c = 1$" so that factors of $\hbar$ and c can be suppressed. However, you can always restore these factors by dimensional analysis.

According to standard references, the mass of the Higgs boson is about $125\,\text{GeV}$, where $1\,\text{eV}$ is the energy gained by an electron accelerated through a voltage difference of $1\,\text{V}$. Fix the dimensions of this statement and find the mass of the Higgs boson in kilograms.

[3] **Problem 8.** USAPhO 2002, problem A3.

Example 4

The Schrodinger equation for an electron in the electric field of a proton is

$$-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi.$$

Estimate the size of the hydrogen atom.

Solution

This is yet another dimensional analysis problem: there is only one way to form a length using the quantities given above. We have

$$[m] = \text{kg}, \quad [\hbar] = \text{J} \cdot \text{s} = \text{kg}\,\text{m}^2\,\text{s}^{-1}, \quad [e^2/4\pi\epsilon_0] = \text{J} \cdot \text{m} = \text{kg}\,\text{m}^3\,\text{s}^{-2}.$$

Doing dimensional analysis, the only length scale is the Bohr radius,

$$a_0 = \frac{4\pi\hbar^2\epsilon_0}{me^2} \sim 10^{-10}\,\text{m}.$$

I've thrown in a 4π above because ϵ_0 always appears in the equations as $4\pi\epsilon_0$. The dimensional analysis would be valid without this factor, but as you'll see in problem 11, if you don't include it then annoying compensating factors of 4π will appear elsewhere.

Classically (i.e. without $\hbar$), there is no way to form a length, and hence there should be no classically stable radius for the atom. (This was one of the arguments used by Bohr to motivate quantum mechanics; it appears in the beginning of his paper introducing the Bohr model.) Once we introduce $\hbar$, there are three dimensionful parameters in the problem, as listed above. And there are exactly three fundamental dimensions. So there is only one way to create a length, which we found above, one way to create a time, one way to create an energy, and so on. This means that the solutions to the Schrodinger equation above look qualitatively the same no matter what these parameters are; all that changes are the overall length, time, and energy scales. In problem 11, you'll investigate how this conclusion changes when we add more dimensionful parameters.

Dimensional analysis is especially helpful with scaling relations. For example, a question might ask you how the radius of the hydrogen atom would change in a world where the electron mass was twice as large. You would solve this problem in the exact same way as the example above, using dimensional analysis to show that $a_0 \propto 1/m$.

[3] **Problem 9.** In this problem we'll continue the dimensional analysis of the Schrodinger equation.

(a) Estimate the typical energy scale of quantum states of the hydrogen atom, as well as the typical "velocity" of the electron, using dimensional analysis.

(b) Do the same for one-electron helium, the system consisting of a helium nucleus (containing two protons) and one electron.

(c) Estimate the electric field needed to rip the electron off the hydrogen atom.

Idea 3: Buckingham Pi Theorem

Dimensional analysis can't always pin down the form of the answer. If one has N quantities with D independent dimensions, then one can form $N - D$ independent dimensionless quantities. Dimensional analysis can't say how the answer depends on them.

A familiar but somewhat trivial example is the pendulum: its period depends on L, g, and the amplitude θ_0, three quantities which contain two dimensions (length and time). Hence we can form one dimensionless group, which is clearly just θ_0 itself. The period of a pendulum is $T = f(\theta_0)\sqrt{L/g}$.

Example 5: $F = ma$ 2014 12

A paper helicopter with rotor radius r and weight W is dropped from a height h in air with a density of ρ. Assuming the helicopter quickly reaches terminal velocity, use dimensional analysis to analyze the total flight time T.

Solution

The answer can only depend on the parameters r, W, h, and ρ. There are four quantities in total, but three dimensions (mass, length, and time), so by the Buckingham Pi theorem we can form one independent dimensionless quantity. In this case, it's clearly r/h. Continuing with routine dimensional analysis, we find

$$T = f(r/h)\, h^2 \sqrt{\frac{\rho}{W}}.$$

The form of this expression is a bit arbitrary; for instance, we could also have written $f(r/h)r^2$ in front, or even $f(r/h)r^{37}h^{-35}$. These adjustments just correspond to pulling factors of r/h out of f, not to changing the actual result.

This is as far as we can get with dimensional analysis alone, but we can go further using physical reasoning. If the helicopter quickly reaches terminal velocity, then it travels at a constant speed. So we must have $T \propto h$, which means that $f(x) \propto x$, and

$$T \propto rh\sqrt{\frac{\rho}{W}}.$$

Example 6

An hourglass is constructed with sand of density ρ and an orifice of diameter d. When the sand level above the orifice is h, what is the mass flow rate μ?

Solution

The answer can only depend on ρ, d, h, and g. The Buckingham Pi theorem gives

$$\mu = f(h/d)\rho\sqrt{gd^5}.$$

That's as far as we can get with dimensional analysis; to go further we need to know more about sand. If we were dealing with an ideal fluid, then the flow speed would be $v = \sqrt{2gh}$ by Torricelli's law, which means the flow rate has to be proportional to $\sqrt{h}$. Then $f(x) \propto \sqrt{x}$, giving the result $\mu \propto \rho d^2\sqrt{gh}$. This is a good estimate as long as the orifice isn't so small that viscosity starts to dominate.

But this isn't how sand works: measurements show that the pressure at the orifice doesn't actually depend on the height of the sand, an empirical result known as Janssen's law. That's because sand is a granular material whose motion is dominated by the friction between sand grains, and this friction prevents the additional pressure from propagating downward. The resulting flow rate is independent of h, as can be confirmed by watching an hourglass run. Then $f(x)$ is a constant, giving $\mu \propto \rho\sqrt{gd^5}$. This neat, experimentally verified result is called Beverloo's law, and it's essential in industry to design grain hoppers and corn silos.

Remark

One has to be a little careful with the Buckingham Pi theorem. For example, if all we had were 3 speeds v_i, we can form two dimensionless quantities: v_1/v_2 and v_1/v_3. (The quantity v_2/v_3 is not independent, since it is the quotient of these two.) But there are 3 quantities with 2 dimensions (length and time), so we expect only 1 dimensionless quantity.

The problem is that the two dimensions really aren't independent: for any quantity built from the v_i, a power of length always comes with an inverse power of time, so there's only one independent dimension. These considerations can be put on a more rigorous footing in linear algebra, where the Buckingham Pi theorem is merely a special case of the rank-nullity theorem. If you're ever in doubt, you can just forget about the theorem and play with the equations directly.

Remark

Dimensional analysis is an incredibly common tool in Olympiad physics because it lets you say a lot even without much advanced knowledge. If a problem ever says to find some quantity "up to a constant/dimensionless factor", or how that quantity scales as another quantity changes, or what that quantity is proportional to, it's almost certainly asking you to do dimensional analysis. Another giveaway is if the problem *looks* extremely technical and advanced, because they can't actually be.

[3] **Problem 10** (Insight)**.** In this problem we'll do one of the most famous dimensional analyses of all time: estimating the yield of the first atomic bomb blast. Such a blast will create a shockwave of air, which reaches a radius R at time t after the blast. The air density is ρ, and we want to estimate the blast energy E.

(a) Declassified photographs of the blast indicate that $R \approx 100\,\text{m}$ at time $t \approx 15\,\text{ms}$. The density of air is $\rho \approx 1\,\text{kg/m}^3$. Estimate the blast energy E.

(b) How much mass-energy (in grams) was used up in this blast?

(c) If we measure the entire function $R(t)$, what general form would we expect it to have, if this dimensional analysis argument is correct?

Remark

The British physicist G. I. Taylor performed the dimensional analysis in problem 10 upon seeing a picture of the first atomic blast in a magazine. The result was so good that the physicists at the Manhattan project thought their security had been breached!

During World War II, the exact value of the critical mass needed to set off a nuclear explosion was important and nontrivial information. The Nazi effort to make a bomb had been stopped by Werner Heisenberg's huge overestimation of this quantity, and after the war, the specific value was kept a closely guarded secret. That is, it was until 1947, when a Chinese physicist got the answer using a rough estimate that took four lines of algebra.

[5] **Problem 11.** We now consider the Schrodinger equation for the hydrogen atom in greater depth. We begin by switching to dimensionless variables, which is useful for the same reason that writing integrals in terms of dimensionless variables is: it highlights what is independent of unit choices.

(a) Define a dimensionless length variable $\tilde{r} = r/a_0$, where a_0 is the length scale found in example 4. The ∇^2 term in the Schrodinger equation is a second derivative, the 3D generalization of d^2/dx^2. Using the chain rule, argue that

$$\tilde{\nabla}^2 = a_0^2 \nabla^2$$

where $\tilde{\nabla}$ is the gradient with respect to $\tilde{r}$.

(b) Similarly define a dimensionless energy $\tilde{E} = E/E_0$, using the energy scale E_0 found in problem 9. Show that the Schrodinger equation can be written in a form like

$$-\tilde{\nabla}^2 \psi - \frac{1}{\tilde{r}} \psi = \tilde{E} \psi$$

Here I've suppressed all dimensionless constants, like factors of 2, because they depend on how you choose to define E_0 and don't really matter at this level of precision.

The result of this part confirms what we concluded above: solutions to the Schrodinger equation don't qualitatively depend on the values of the parameters, because they all come from scaling a solution to this one dimensionless equation appropriately.

(c) This is no longer true in relativity, where the total energy is

$$E = \sqrt{p^2 c^2 + m^2 c^4}.$$

Assuming $p \ll mc$, perform a Taylor expansion to show that the next term is Ap^4, and find the coefficient A. (If you don't know how to do this, work through the next section first.)

(d) In quantum mechanics, the momentum is represented by a gradient, $p \to -i\hbar\nabla$. (We will see why in **X1**.) Show that the Schrodinger equation with the first relativistic correction is

$$-\frac{\hbar^2}{2m} \nabla^2 \psi - \frac{e^2}{4\pi\epsilon_0 r} \psi + \hbar^4 A \nabla^4 \psi = E\psi.$$

(e) Since there is now one more dimensionful quantity in the game, it is possible to combine the quantities to form a dimensionless one. Create a dimensionless quantity α that is proportional to $e^2/4\pi$, then numerically evaluate it. This is called the fine structure constant. It serves as an objective measure of the strength of the electromagnetic force, because it is dimensionless, and hence its value doesn't depend on an arbitrary unit system.

(f) As the number of protons in the nucleus increases, the relativistic correction becomes more important. Estimate the atomic number Z where the correction becomes very important.

You probably won't see any differential equations as complex as the ones in the above problem anywhere in Olympiad physics, but the key idea of using dimensionless quantities to simplify and clarify the physics can be used everywhere.

[5] **Problem 12.** IPhO 2007, problem "blue". This problem applies thermodynamics and dimensional analysis in some exotic contexts.

Example 7

Estimate the Young's modulus for a material with interatomic separation a and typical atomic bond energy E_b. Use this to estimate the spring constant of a rod of area A and length L, as well as the speed of sound, if each atom has mass m.

Solution

This example is to get you comfortable with the Young's modulus Y, which occasionally comes up. It is defined in terms of how much a material stretches as it is pulled apart,

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{\text{restoring force/cross-sectional area}}{\text{change in length/length}}.$$

The Young's modulus is a useful way to characterize materials, because unlike the spring constant, it doesn't depend on the shape of the material. For example, putting two identical springs side-by-side doubles the spring constant, because they both contribute to the force. But since the stress is the force per area, it's unchanged. Similarly, putting two identical springs end-to-end halves the spring constants, because they both stretch, but since the strain is change in length per length, it's unchanged. So you would quote a material's Young's modulus instead of its spring constant, for the same reason you would quote a material's resistivity instead of its resistance.

We note that Y has the dimensions of energy per length cubed, so

$$Y \sim \frac{E_b}{a^3}$$

solely by dimensional analysis. (Of course, for this dimensional analysis to work, one has to understand why E_b and a are the only relevant quantities. It's because Y, or equivalently the spring constant k, determines the energy stored in a stretched spring. But microscopically this comes from the energy stored in interatomic bonds when they're stretched. So the relevant energy scale is the bond energy E_b, and the relevant distance scale is a, because that determines how many bonds get stretched, and by how much.)

To relate Y to the spring constant of a rod, note that

$$Y = \frac{F/A}{\Delta L/L} = \frac{L}{A}\frac{F}{\Delta L} = k\frac{L}{A}$$

for a rod, giving the estimate $k \sim AE_b/La^3$. This is correct to within an order of magnitude!

To relate Y to the speed of sound, note that the sound speed, like most wave speeds, depends on the material's inertia and its restoring force against distortions. Since the speed of sound doesn't depend on the extrinsic features of a metal object, such as a length, both of these should be measured intrinsically. The intrinsic measure of inertia is the mass density $\rho \sim m/a^3$, while the intrinsic measure of restoring force is just Y. By dimensional analysis,

$$v \sim \sqrt{\frac{Y}{\rho}} \sim \sqrt{\frac{E_b/a^3}{m/a^3}} \sim \sqrt{\frac{E_b}{m}}.$$

This is also reasonably accurate. For example, in diamond, $E_b \sim 1\,\text{eV}$ (a typical atomic energy scale), while a carbon nucleus contains 12 nucleons, so to the nearest order of magnitude, $m \sim 10 m_p$, where a useful fact is $m_p \sim 1\,\text{GeV/c}^2$. Thus,

$$v \sim \sqrt{\frac{1\,\text{eV}}{10^{10}\,\text{eV}}}\, c \sim 10^{-5}\, c \sim 3\,\text{km/s}$$

which is the right order of magnitude. (The true answer is $12\,\text{km/s}$.)

Amazingly, we can get an even rougher estimate of v for any solid in terms of nothing besides fundamental constants. To be very rough, the binding energy is on the order of that of hydrogen. As you found in problem 9, this is, by dimensional analysis,

$$E_b \sim \frac{1}{4\pi\epsilon_0}\frac{e^2}{a_0} \sim m_e \left(\frac{e^2}{4\pi\epsilon_0 \hbar}\right)^2.$$

We take the nuclear mass to be very roughly the proton mass m_p, which gives

$$\frac{v}{c} \sim \sqrt{\frac{m_e}{m_p}\left(\frac{e^2}{4\pi\epsilon_0 \hbar c}\right)^2} \sim \alpha \sqrt{\frac{m_e}{m_p}}$$

where α is as found in problem 11. This expresses the speed of sound in terms of the dimensionless strength of electromagnetism α, the electron to proton mass ratio, and the speed of light. Of course, the approximations we have made here have been so rough that now the answer is off by *two* orders of magnitude, but now we know how the answer would change if the fundamental constants did.

Estimates as simple as these can be surprising to even seasoned physicists: in 2020, the simple estimate above was rediscovered and published in one of the top journals in science. If you want to learn how to do more of these estimates, this paper is a good starting point.

Remark

A warning: from these examples, you could get the idea that dimensional analysis gives you nearly godlike powers, and the ability to write down the answer to most physics problems instantly. In reality, it only works if you're pretty sure your physical system depends on only about 3 or 4 variables – and the hard part is often finding *which* variables matter. For example, as we saw above, you can't get Kepler's third law for free because that requires knowing the dimensions of G, which require knowing that gravity is an inverse square law in the first place, a luxury Kepler didn't have. And as another example, we couldn't have figured out $E = mc^2$ long before Einstein, as who would have thought that the speed of light had anything to do with the energy of a lump of matter? Without the framework of relativity, it seems as irrelevant as the speed of sound or the speed of water waves.

Fortunately, carrying out dimensional analysis in practice is usually fairly straightforward. Often, on exams, you'll simply be told which variables matter. And in general, you should get into the habit of doing it constantly, to check your work.

Example 8

Cutting-edge archeological research has found that the famed T. Rex was essentially a gigantic chicken. Suppose a T. Rex is about $N = 20$ times larger in scale than a chicken. How much larger is its weight, cross-sectional area of bone, and walking speed?

Solution

These kinds of biological scaling arguments are fun to think about, though the reliability of the results is somewhat questionable – the data is extremely noisy, and if any given scaling law doesn't quite match it, you can always think a bit more, and come up with a new argument yielding a different scaling. But here are a few simple examples:

- Since the densities should match, the weight should scale with the volume, so as N^3.
- Since the maximum compressive pressure that bone can take should be the same, the bone area should scale with the weight, so also as N^3. That is, the width of the bones scales as $N^{3/2}$, while their length L scales only as N. This is the reason small animals are strong relative to their weight, while large ones need to be very bony to even stand. The largest animals today are whales, as they don't need to support their own weight.
- As a very crude model of walking, we can think of the legs as swinging like a free pendulum. The length of one step is proportional to L, while the period of the steps is proportional to $\sqrt{L}$. Thus, the walking speed scales as $\sqrt{L} \propto \sqrt{N}$.

There's an entire literature on these arguments. For instance, this delightful paper discusses how furry mammals shake to dry themselves off. This is an increasingly severe problem for smaller mammals, since a relatively larger amount of water will cling to them after getting wet, which can cause hypothermia. Using elementary fluid mechanics, the paper argues that the optimal frequency the mammal will shake to dry itself off scales as $f \propto m^{-3/16}$.

Example 9

How does the maximum jump height of an animal depend on its length scale L? How about the gravitational acceleration g?

Solution

The maximum jump height h satisfies $E = mgh$ where E is the energy supplied by the muscles. But both the total mass m and the animal's muscle mass (which determines E) scale as L^3, so we have $h \propto L^0$. So the jump height doesn't scale with size: a dinosaur can't jump much higher than a human – and indeed, *we* can't jump much higher than fleas can!

The other half of the problem seems very simple: we must have $h \propto 1/g$ because neither E or m depend on g. But this is completely wrong! In gravity $10g$, a person wouldn't be able to jump at all; they'd be so crushed by their own weight that they wouldn't even be able to stand. Mathematically, the dimensional analysis argument fails here because the answer depends on the detailed biomechanics of muscle and bone, which involve many more dimensionful quantities. By contrast, the result $h \propto L^0$ works well because the animals we were comparing all evolved so that their muscles would work reasonably well in Earth's gravity, releasing a decent fraction of their stored energy in the short time required for a jump. So, as remarked above, you can't solve every problem by just listing a few quantities and doing dimensional analysis – you really have to understand the system each time.

2 Approximations

Idea 4: Taylor Series

For small x, a function $f(x)$ may be approximated as

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + O(x^{n+1})$$

where $O(x^{n+1})$ stands for an error term which grows at most as fast as x^{n+1}.

There are a few Taylor series that are essential to know. The most important are

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4), \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - O(x^4)$$

and the small angle approximations

$$\sin x = x - \frac{x^3}{6} + O(x^5), \quad \cos x = 1 - \frac{x^2}{2} + O(x^4).$$

Another Taylor series you learned long before calculus class is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + O(x^4).$$

Usually you'll only need the first one or two terms, but for practice we'll do examples with more. If any of these results aren't familiar, you should rederive them!

Example 10

Find the Taylor series for $\tan x$ up to, and including the fourth order term.

Solution

By the fourth order term, we mean the term proportional to x^4. (Not the fourth nonzero term, which would be $O(x^7)$.) Of course, $\tan x$ is an odd function, so the $O(x^4)$ term is zero, which means we only need to expand up to $O(x^3)$. That means we can neglect $O(x^4)$ terms and higher everywhere in the computation, subject to some caveats we'll point out later.

By definition, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - x^3/6 + O(x^5)}{1 - x^2/2 + O(x^4)}.$$

However, it's a little tricky because we have a Taylor series in a denominator. There are two ways to deal with this. We could multiply both sides by $\cos x$, and expand $\tan x$ in a Taylor series with unknown coefficients. Then we would get a system of equations that will allow us to solve for the coefficients recursively, a technique known as "reversion of series".

A faster method is to use the Taylor series for $1/(1-x)$. We have

$$\frac{1}{1-u} = 1 + u + O(u^2)$$

and substituting $u = x^2/2 - O(x^4)$ gives

$$\frac{1}{\cos x} = 1 + \frac{x^2}{2} + O(x^4).$$

Therefore, we conclude

$$\tan x = (x - x^3/6 + O(x^5))(1 + x^2/2 + O(x^4)) = x + x^3/3 + O(x^5).$$

Here I was fairly careful with writing out all the error terms and intermediate steps, but as you get better at this process, you'll be able to do it faster. (Of course, one could also have done this example by just directly computing the Taylor series of $\tan x$ from its derivatives. This is possible, but for more complicated situations it's generally not a good idea, because computing high derivatives of a complex expression tends to get very messy. It's better to just Taylor expand the individual pieces and combine the results, as we did here.)

Remark

Finding series up to a given order can be subtle. For example, if you want to compute an $O(x^4)$ term, it is *not* always enough to expand everything up to $O(x^4)$, because powers of x might cancel. To illustrate this, the last step here is wrong:

$$\tan x = \frac{x^3 \sin x}{x^3 \cos x} = \frac{x^4 + O(x^6)}{x^3 + O(x^5)} \neq x + O(x^5).$$

[2] **Problem 13.** Find the Taylor series for $1/\cos x$ up to and including the fourth order ($O(x^4)$) term.

[2] **Problem 14.** Extend the computation above to get the x^5 term in the Taylor series for $\tan x$.

[3] **Problem 15.** For small x, approximate the quantity

$$\frac{x^2 e^x}{(e^x - 1)^2} - 1$$

to lowest order. That is, find the first nonzero term in the Taylor series. (Hint: if you don't take enough terms in the Taylor series to begin with, you'll get an answer of zero, indicating you approximated too loosely. But if you take too many, the computation will get extremely messy.)

[3] **Problem 16.** The function $\cos^{-1}(1-x)$ does not have a Taylor series about $x = 0$. However, it does have a series expansion about $x = 0$ in a different variable.

(a) What is this variable, and what's the first term in the series?

(b) $\star$ What's the next nontrivial term in the series?

Idea 5: Binomial Theorem

When the quantity xn is small, it is useful to use the binomial theorem,

$$(1+x)^n = 1 + xn + O(x^2 n^2).$$

It applies even when n is not an integer. In particular, n can be very large, very small, or even negative. The extra terms will be small as long as xn is small. If desired, one can find higher terms using binomial coefficients,

$$(1+x)^n = \sum_{m=0}^{\infty} \binom{n}{m} x^m$$

where the definition of the binomial coefficient is formally extended to arbitrary real n.

The binomial theorem is one of the most common approximations in physics. It's really just taking the first two terms in the Taylor series of $(1+x)^n$, but we give it a name because it's so useful.

[1] **Problem 17.** Suppose the period of a pendulum is one second, and recall that

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

If the length is increased by 3% and g is increased by 1%, use the binomial theorem to estimate how much the period changes. This kind of thinking is extremely useful when doing experimental physics, and you should be able to do it in your head.

[1] **Problem 18.** Consider an electric charge q placed at $x = 0$ and a charge $-q$ placed at $x = d$. The electric field along the x axis is then

$$E(x) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{x^2} - \frac{1}{(x-d)^2}\right).$$

For large x, use the binomial theorem to approximate the field.

[3] **Problem 19.** Some exercises involving square roots.

(a) Manually find the Taylor series for $\sqrt{1+x}$ up to second order, and verify they agree with the binomial theorem.

(b) Approximate $\sqrt{1+2x+x^2}$ for small x using the binomial theorem. Does the result match what you expect? If not, how can you correct it?

Example 11: Birthday Paradox

If you have n people in a room, around how large does n have to be for there to be at least a 50% chance of two people sharing the same birthday?

Solution

Imagine adding people one at a time. The second person has a $1/365$ chance of sharing a birthday with the first. If they don't share a birthday, the third person has a $2/365$ chance of sharing a birthday with either, and so on. So a decent estimate for n is the n where

$$\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\ldots\left(1-\frac{n}{365}\right)\approx\frac{1}{2}.$$

The surprising point of the birthday paradox is that $n \ll 365$. So we can use the binomial theorem in reverse, approximating the left-hand side as

$$\left(1-\frac{1}{365}\right)\left(1-\frac{1}{365}\right)^2\ldots\left(1-\frac{1}{365}\right)^n\approx\left(1-\frac{1}{365}\right)^{n^2/2}$$

which is valid since $n/365$ is small. It's tempting to use the binomial theorem again to write

$$\left(1-\frac{1}{365}\right)^{n^2/2}\approx 1-\frac{n^2}{2\cdot 365}=\frac{1}{2}$$

which gives $n = 19$. However, this is a bad approximation, because the binomial theorem only works if $(n^2/2)(1/365)$ is very small, but here we've set it to $1/2$, which isn't particularly small. Since the series expansion variable is $1/2$, each term in the series expansion is *roughly* $1/2$ as big as the last (ignoring numerical coefficients), so we expect to be off by about $(1/2)^2 = 25\%$.

The binomial theorem is an expansion for $(1+x)^y$ which works when xy is small. Here xy isn't small, and we instead want an approximation that works when only x is small. One trick to dealing with an annoying exponent is to take the logarithm, since that just turns it into a multiplicative factor. Note that

$$\log((1+x)^y) = y\log(1+x)\approx yx$$

by Taylor series, which implies that

$$(1+x)^y \approx e^{yx}$$

when x is small, an important fact which you should remember. So we have

$$\left(1-\frac{1}{365}\right)^{n^2/2}\approx e^{-n^2/2(365)}=\frac{1}{2}$$

and solving gives $n = 22.5$. We should round up since n is actually an integer, giving $n = 23$, which is indeed the exact answer.

Remark

Precisely how accurate is the approximation $(1+x)^y \approx e^{yx}$? Note that the only approximate step used to derive it was taking $\log(1+x) \approx x$, which means we can get the corrections by expanding to higher order. If we take the next term, $\log(1+x) \approx x - x^2/2$, then we find

$$(1+x)^y \approx e^{yx} e^{-x^2 y/2}.$$

Note that because we are approximating the logarithm of the quantity we want, the next correction is multiplicative rather than additive; we'll see a similar situation with Stirling's approximation in **T2**. Our approximation has good fractional precision as long as $x^2 y \ll 1$. In the previous example, $x^2 y/2 = (22.5/365)^2/4 = 0.1\%$, so our answer was quite accurate.

[2] **Problem 20.** Find a series approximation for x^y, given that y is small and x is neither small nor exponentially huge. (Hint: to check if you have it right, you can try concrete numbers, such as $y = 0.01$ and $x = 10$. The series expansion variable may look a bit unusual.)

Remark

If these questions seem complicated, rest assured that 90% of approximations on the USAPhO and IPhO boil down to using

$$\sin x \approx x, \quad \cos x \approx 1 - x^2/2, \quad (1+x)^n \approx 1 + xn, \quad e^x \approx 1 + x, \quad \log(1+x) \approx x.$$

I've given you a lot of subtle situations above, but it's these that you have to know by heart. Almost all situations where you will use these will look like problem 17 or problem 18.

Remark

Just for fun, here are some more examples of tough series expansions. First, in the "Lindhard" theory of electrical shielding in metals, one has to perform the small x expansion

$$\frac{1}{2} + \frac{x^2 - 1}{4x} \log\left(\frac{1+x}{1-x}\right) \approx \frac{x^2}{3} + O(x^4).$$

Since there's a $1/x$ in front of the second term, you have to expand the logarithm to *third* order to get the right prefactor, like in problem 15, but the algebra's a lot messier.

The Soviet mathematician V.I. Arnold used to say that math has gone downhill since Newton, because people in that time could supposedly quickly evaluate the limit

$$\lim_{x \to 0} \frac{\sin(\tan(x)) - \tan(\sin(x))}{\arcsin(\arctan(x)) - \arctan(\arcsin(x))}.$$

The first terms that don't cancel are $O(x^7)$, giving

$$\lim_{x \to 0} \frac{-\frac{55x^7}{1008} + \frac{107x^7}{5040} + O(x^9)}{-\frac{341x^7}{1008} + \frac{173x^7}{5040} + O(x^9)} = 1.$$

This amazingly simple answer can be found with a very tricky geometric argument, of the style common in Newton's Principia, though it's hard to make it totally rigorous.

Finally, as we will see in **E2**, the capacitance of two spheres of radius a separated by $r \gg a$ can be written as an infinite series in a/r. In his *Treatise on Electricity and Magnetism* (1891, section 146), Maxwell manually evaluated this series out to order $(a/r)^{22}$! I don't even know what the point of that was, but it illustrates why physicists took decades to fully comprehend the implications of Maxwell's Treatise. Today, we teach the conceptual essentials of Newton and Maxwell's physics, but the reason they're giants is because they put their theories to *work*, in tough calculations we barely hear about today.

3 Numeric Solutions

Idea 6

In Olympiads, you may have to find numeric solutions for equations that can't be solved analytically. A simple but reliable method is to "guess and check", starting with a reasonable first guess (e.g. derived by solving an approximated version of the equation, or sketching the graphs of both sides), plugging it into both sides, then proceeding with binary search.

[3] **Problem 21.** Sometimes, you can get an accurate numeric answer very quickly on a basic calculator by using the method of iteration, which solves equations of the form $x = f(x)$.

(a) Take a scientific calculator (in radians), put in any number, and press the "cos" button many times. Convince yourself that the final number you get is the unique solution to $x = \cos x$.

(b) What are the key features of the graphs of x and $\cos x$ that made this work? For example, why doesn't pressing $\cos^{-1}$ repeatedly give the same result? As another example, since $x = \sin x$ has a unique solution, why does repeatedly pressing sin not work so well?

(c) Find a nonzero solution for $x = \tan(x/2)$.

(d) Find a nonzero solution for $e^x - 1 = 2x$.

(e) Find a positive solution for $x^x = e$.

[2] **Problem 22.** [A] Newton's method is a more sophisticated method for solving equations, which converges substantially faster than binary search. Suppose we want to solve the equation $f(x) = 0$. Starting with a nearby guess x_0, we evaluate $f(x_0)$ and $f'(x_0)$, then find our next guess by applying the tangent line approximation at this point,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

The process repeats until we get a suitably accurate answer.

(a) Use Newton's method to solve $x = \cos x$.

(b) Newton's method converges quadratically, in the sense that for typical functions, if your current guess is ϵ away from the answer, the next guess will be $O(\epsilon^2)$ away. (This implies that the number of correct digits in the answer roughly doubles with each iteration!) Explain why, and then find an example where Newton's method *doesn't* converge this fast.

Newton's method is very important in general, but it's not that useful on Olympiads. It takes a while to set up, especially if the derivative f' is complicated, and you usually don't need that many significant figures in your answer anyway. (There are alternatives to Newton's method, such as Halley's method, that converge even faster, but the tradeoff is the same: each iteration takes more effort to calculate, as higher derivatives of f must be computed.)

Remark

You've seen several numeric methods above, and going forward, you should feel free to use whichever looks best in each situation. However, if you're solving problems using the same calculator you use for schoolwork, you should make sure to not rely on its more advanced features. In Olympiads, you're generally only allowed to use an extremely basic scientific calculator, with a tiny display and no memory except for the "Ans" key.

Example 12

In units where $c = 1$, the Lorentz factor is defined as

$$\gamma = \frac{1}{\sqrt{1 - v^2}}.$$

Suppose that a particle traveling very close to the speed of light has $\gamma = 10^8$. Find the difference Δv between its speed and the speed of light.

Solution

This problem looks easy; by some trivial algebra we find

$$\Delta v = 1 - \sqrt{1 - 1/\gamma^2}.$$

But when you plug this into a cheap scientific calculator, you get *zero*, or something that's quite far from the right result. The problem is that we are trying to find a small quantity Δv by subtracting two nearby, much larger quantities. But the calculator has limited precision, and it ends up rounding $1 - 1/\gamma^2 = 1 - 10^{-16}$ a bit, giving a completely wrong answer!

Instead, we can apply the binomial theorem to find

$$\Delta v \approx \frac{1}{2\gamma^2} + O(1/\gamma^4).$$

This is no longer the exact answer, but it's a great approximation, because the error term is around $1/\gamma^2 \sim 10^{-16}$ times as small as the answer, and it's easy for a calculator to evaluate. The lesson is that it's better to be accurate in practice than to be precise in theory.

[1] **Problem 23.** Find the solutions of the equation $x^2 - 10^{20}x + 1 = 0$ to reasonable accuracy.

[4] **Problem 24.** [A] Consider the equation $\epsilon x^3 - x^2 + 1 = 0$, where ϵ is small. Find approximate expressions for all three roots of this equation, up to and including terms of order ϵ.

4 Limiting Cases

Idea 7

Limiting cases can be used to infer how the answer to a physical problem depends on its parameters. It is primarily useful for remembering the forms of formulas, but can also be powerful enough to solve multiple choice questions by itself.

Example 13

What is the horizontal range of a rock thrown with speed v at an angle θ to the horizontal?

Solution

This result is easy to derive, but dimensional analysis and extreme cases can be used to recover the answer too. It can only depend on v, g, and θ, so by dimensional analysis it is proportional to v^2/g. This is sensible, since the range increases with v and decreases with g. Now, the range is zero in the extreme cases $\theta = 0$ and $\theta = \pi/2$, but not anywhere in between, so if we remember the range contains a simple trigonometric function, it must be $\sin(2\theta)$, so

$$R \propto \frac{v^2}{g} \sin(2\theta).$$

We can also get the prefactor by a simple limiting case, the case $\theta \ll 1$. In this case, by the small angle approximation,

$$v_x \approx v, \quad v_y \approx v\theta.$$

The time taken is $t = 2v_y/g$, so the range is

$$R \approx v_x t = \frac{2v^2}{g}\theta.$$

Thus there is no proportionality constant; the answer is

$$R = \frac{v^2}{g} \sin(2\theta).$$

In reality, it's probably faster to go through the full derivation than all of this reasoning, but if you're just not sure about whether it's a sine or a cosine, or what the prefactor is, then limiting cases can be quickly used to recover that piece. Also note that the approximations we used above are frequently useful for evaluating limiting cases.

Example 14

Consider an Atwood's machine with masses m and M, and a massless pulley. Find the tension in the string.

Solution

Since the equations involved are all linear equations, we expect the answer should also be simple. It can only depend on g, m, and M, so by dimensional analysis, it must be proportional to g. By dimensional analysis, this must be multiplied by something with one net power of mass. Since the answer remains the same if we switch the masses, it should be symmetric in m and M.

Given all of this, the simplest possible answer would be

$$T \propto g(m+M).$$

To test this, we consider some limiting cases. If $M \gg m$, the mass M is essentially in freefall, so the mass m accelerates upward with acceleration g. Then the tension is approximately $2mg$. Similarly, in the case $M \ll m$, the tension is approximately $2Mg$. These can't be satisfied by the form above.

The next simplest option is a quadratic divided by a linear expression. Both of these must be symmetric, so the most general possibility is

$$T = g\frac{A(m^2+M^2)+BmM}{m+M}.$$

Then the limiting cases can be satisfied if $A=0$ and $B=2$, giving

$$T = \frac{2gmM}{m+M}.$$

[1] **Problem 25.** Find the perimeter of a regular N-gon, if L is the distance from the center to any of the vertices. By considering a limiting case, use this to derive the circumference of a circle.

[1] **Problem 26.** Use similar reasoning to find the acceleration of the Atwood's machine. (We will show an even easier way to do this, using "generalized coordinates", in **M4**.)

[2] **Problem 27** (Morin 1.6)**.** A person throws a ball (at an angle of her choosing, to achieve the maximum distance) with speed v from the edge of a cliff of height h. Which of the below could be an expression for the maximal range?

$$\frac{gh^2}{v^2}, \quad \frac{v^2}{g}, \quad \sqrt{\frac{v^2h}{g}}, \quad \frac{v^2}{g}\sqrt{1+\frac{2gh}{v^2}}, \quad \frac{v^2}{g}\left(1+\frac{2gh}{v^2}\right), \quad \frac{v^2/g}{1-2gh/v^2}.$$

If desired, try Morin problems 1.13, 1.14, and 1.15 for additional practice.

[2] **Problem 28.** Consider a triangle with side lengths a, b, and c. It turns out the area of its incircle can be expressed purely by multiplying and dividing combinations of these lengths. Moreover, the answer is the simplest possible one consistent with limiting cases, dimensional analysis, and symmetry. Guess it!

While we won't have more questions that are explicitly about dimensional analysis or limiting cases, these are not techniques but ways of life. For all future problems you solve, you should be constantly checking the dimensions and limiting cases to make sure everything makes sense.

5 Manipulating Differentials

You might have been taught in math class that manipulating differentials like they're just small, finite quantities, and treating derivatives like fractions is "illegal". But it's also very useful.

Idea 8

Derivatives can be treated like fractions, if all functions have a single argument.

The reason is simply the chain rule. The motion of a single particle only depends on a single parameter, so the chain rule is just the same as fraction cancellation. For example,

$$\frac{dv}{dt} = \frac{d}{dt}v(x(t)) = \frac{dv}{dx}\frac{dx}{dt}$$

which show that "canceling a dx" is valid. Similarly, you can show that

$$\frac{dy}{dx}\frac{dx}{dy} = 1$$

by considering the derivative with respect to x of the function $x(y(x)) = x$.

As a warning, for functions of multiple arguments, the idea above breaks down. For example, for a function $f(x(t), y(t))$, the chain rule says

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

where there are two terms, representing the change in f from changes only in x, and only in y. Therefore, when we start studying thermodynamics, where multivariable functions are common, we will treat differentials more carefully. But for now the basic rules will do.

Remark: Rigorous Notation

Math students tend to get extremely upset about the above idea: they say we shouldn't use convenient notation if it hides what's "really" going on. And they're right, if your goal is to put calculus on a rigorous footing. But in physics we have no time to luxuriate in such rigor, because we want to figure out how specific things work. The point of notation is to help us do that by suppressing mathematical clutter. A good notation suppresses *as much as possible* while still giving correct results in the context it's used.

To illustrate the point, note that elementary school arithmetic is itself an "unrigorous" notation that hides implementation details. If we wanted to be rigorous about, say, defining the number 2, we would write it as $S(1)$ where S is the successor function, obeying properties specified by the Peano axioms. And 4 is just a shorthand for $S(S(S(1)))$, so $2 + 2 = 4$ means

$$S(1) + S(1) = S(S(S(1))).$$

Even this is not "rigorous", because the Peano axioms don't specify how the numbers or the successor function are defined, just what properties they have to obey. To go deeper, we could define the integers as sets, and operations like + in terms of set operations. For example, in one formulation, we start with nothing but the empty set $\varnothing$ and define

$$4 = S(S(S(1))) = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}.$$

People have seriously advocated for 1st grade math to be taught this way, which has always struck me as insane. You can *always* add more arbitrary layers of structure underneath the current foundation, so such layers should only be added when absolutely necessary.

Here's another example, inspired by the physics education research literature. For uniformly accelerated motion starting from rest, $v(t) = at$, what is $v(x)$? Physics students would say that $v(x) = \sqrt{2ax}$ by the kinematic equations, while math students would say $v(x) = ax$ by the definition of a function. Who is correct? The point is that basic physics and math courses use functions differently. In introductory physics, we often denote several distinct mathematical functions with the same symbol, if they all represent the same physical quantity. (Otherwise, the simplest projectile motion problem would need half the alphabet.) By contrast, basic math courses carefully distinguish functions, but then denote distinct physical quantities with the same symbol: 1 m, 1 cm, and 1 s are all written as 1.

The crucial point is that nobody is wrong. There is no One True Definition of notation, which is ultimately just squiggly marks people make by dragging graphite cylinders against sheets of wood pulp. Every community makes its own notation for its own needs. And any notation system has to forget about something, or else it would be too clunky to do anything.

Remark: Advanced Notation

As an addendum to the previous remark, it turns out that as you get deeper into math and physics, notation tends to converge. For example:

- The physicist's "wrong" use of $v(t)$ and $v(x)$ can be formalized by differential geometry: here v is a scalar field defined on the particle's path, which is a one-dimensional manifold, and $v(t)$ and $v(x)$ are parametrizations of it in different coordinate charts.
- In math classes, vectors are anything you can take linear combinations of, but in physics classes we also require that they specify a direction in physical space, which math students often criticize as wrong, or meaningless. But the physicist is actually using more advanced math, which the math student doesn't know yet: the physicist's vector is a element of a vector space carrying the fundamental representation of $SO(3)$.
- Most vectors flip sign under an inversion of space, $\mathbf{r} \to -\mathbf{r}$ and $\mathbf{p} \to -\mathbf{p}$, but "axial vectors" such as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ don't. This also strikes many math students as a blatant inconsistency, but the reality is again that an axial vector is just a more advanced mathematical object they haven't met yet, specifically a rank 2 differential form.
- More generally, the "unrigorous" manipulations of differentials above, which we showed give you the right answer anyway, gain a rigorous footing in terms of differential forms. In fact, they become the *preferred* way to denote integration on general manifolds.

Arguments about notation are mostly raised by beginning students, who see the one way they know as the only possible way. Professionals know it both ways, and adjust as needed.

Example 15

Derive the work-kinetic energy theorem, $dW = F\,dx$.

Solution

Canceling the mass from both sides, we wish to show

$$\frac{1}{2}d(v^2) = a\,dx.$$

To do this, note that

$$\frac{1}{2}d(v^2) = v\,dv = \frac{dx}{dt}\,dv = \frac{dv}{dt}\,dx = a\,dx$$

as desired. If you're not satisfied with this derivation, because of the bare differentials floating around, we can equivalently prove that $F = dW/dx$, by noting

$$\frac{dW}{dx} = mv\frac{dv}{dx} = mv\frac{dv}{dt}\frac{dt}{dx} = m\frac{dv}{dt} = F.$$

[2] **Problem 29.** Some more about power.

(a) Use similar reasoning to derive $P = Fv$.

(b) An electric train has a power line that can deliver power $P(x)$, where x is the distance along the track. If the train starts at rest at $x = 0$, find its speed at point x_0 in terms of an integral of $P(x)$. (Hint: try to get rid of the dt's to avoid having to think about the time dependence.)

Example 16

A particle is initially at rest, at a distance r_0 from a star of mass M. Write the time it takes the particle to collide with the star, due to gravitational attraction, as a single integral.

Solution

If the particle has mass m, then conservation of energy gives

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{r_0}$$

so that its radial velocity is

$$\frac{dr}{dt} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}.$$

We can write the time taken to collide as

$$T = \int dt = \int_{r_0}^{0} dr\,\frac{dt}{dr} = \frac{1}{\sqrt{2GM}}\int_0^{r_0}\frac{dr}{\sqrt{1/r - 1/r_0}}.$$

It's good practice to write the integral in dimensionless form, so that the dependence of the answer on the dimensionful quantities is manifest. To do this, substitute $x = r/r_0$ to get

$$T = \sqrt{\frac{r_0^3}{2GM}} \int_0^1 \sqrt{\frac{x}{1-x}}\, dx.$$

This lets us read off $T^2 \propto r_0^3$, in accordance with Kepler's third law, and required by dimensional analysis as shown in problem 2. In case you're wondering, the value of the remaining integral is $\pi/2$, as can be shown by substituting $x = \sin^2\theta$.

[2] **Problem 30** (Kalda)**.** The deceleration of a boat in water due to drag is given by a function $a(v)$. Given an initial velocity v_0, write the total distance the boat travels as a single integral.

[5] **Problem 31.** A particle in a potential well.

(a) Consider a particle of mass m and energy E with potential energy $V(x)$, which performs periodic motion. Write the period of the motion in terms of a single integral over x.

(b) Suppose the potential well has the form $V(x) = V_0(x/a)^n$ for $n > 0$. If the period of the motion is T_0 when it has amplitude A_0, find the period when the amplitude is A, by considering how the integral you found in part (a) scales with A.

(c) Find a special case where you can check your answer to part (b). (In fact, there are two more special cases you can check, one which requires negative n and negative V_0, and one which requires $V(x)$ to be replaced with its absolute value.)

(d) Using a similar method to part (a), write down an integral over θ giving the period of a pendulum with length L in gravity g, without the small angle approximation. Using this, compute the period of the pendulum with amplitude θ_0, up to order θ_0^2. (This result was first published by Bernoulli, in 1749.)

(e) $\star$ Part (d) is the kind of involved computation you might see in a graduate mechanics course. But if you think you're *really* tough, you can go one step further. Consider a mass m oscillating on a spring of spring constant k with amplitude A. Calculate its period of oscillation up to order A^2, accounting for special relativity. (Concretely, assume that the spring force doesn't change the rest mass m, and has a potential $U = kx^2/2$. In relativity, the force $F = -dU/dx$ still obeys $F = dp/dt$, but now $E = \gamma mc^2$ and $p = \gamma mv$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$.)

6 Multiple Integrals

It's also useful to know how to set up multiple integrals. This is fairly straightforward, though technically an "advanced" topic, so we'll demonstrate it by example. For further examples, see chapter 2 of Wang and Ricardo, volume 1, or MIT OCW 18.02, lectures 16, 17, 25, and 26.

Idea 9

In most Olympiad problems, multiple integrals can be reduced to single integrals by symmetry.

Example 17

Calculate the area of a circle of radius R.

Solution

The area A is the integral of dA, i.e. the sum of the infinitesimal areas of pieces we break the circle into. As a first example, let's consider using Cartesian coordinates. Then the pieces will be the rectangular regions centered at (x, y) with sides (dx, dy), which have area $dx\,dy$. The area is thus

$$A = \int dA = \int dx \int dy.$$

The only tricky thing about setting up the integral is writing down the bounds. The inner integral is done first, so its bounds depend on the value of x. Since the boundary of the circle is $x^2 + y^2 = R^2$, the bounds are $y = \pm\sqrt{R^2 - x^2}$. Thus we have

$$A = \int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy.$$

We then just do the integrals one at a time, from the inside out, like regular integrals,

$$A = \int_{-R}^{R} 2\sqrt{R^2 - x^2}\,dx = 2R^2 \int_{-1}^{1} \sqrt{1 - u^2}\,du = 2R^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = \pi R^2$$

where we nondimensionalized the integral by letting $u = x/R$, and then did the trigonometric substitution $u = \sin\theta$. (To do the final integral trivially, notice that the average value of $\cos^2\theta$ along any of its periods is $1/2$.)

We can also use polar coordinates. We break the circle into regions bounded by radii r and $r + dr$, and angles θ and $\theta + d\theta$. These regions are rectangular, with side lengths of dr and $r\,d\theta$, so the area element is $dA = r\,dr\,d\theta$. Then we have

$$A = \int_0^R r\,dr \int_0^{2\pi} d\theta = 2\pi \int_0^R r\,dr = \pi R^2$$

which is quite a bit easier. In fact, it's so much easier that we didn't even need to use double integrals at all. We could have decomposed the circle into a bunch of thin circular shells, argued that each shell contributed area $(2\pi r)\,dr$, then integrated over them,

$$A = \int_0^R 2\pi r\,dr = \pi R^2.$$

In Olympiad physics, there's usually a method like this, that allows you to get the answer without explicitly writing down any multiple integrals.

Example 18

Calculate the moment of inertia of the circle above, about the y axis, if it has total mass M and uniform density.

Solution

The moment of inertia of a small piece of the circle is

$$dI = x^2\,dm = x^2\sigma\,dA = \frac{x^2 M}{\pi R^2}\,dA$$

where x^2 appears because x is the distance to the rotation axis, and σ is the mass density per unit area. Using Cartesian coordinates, we have

$$I = \frac{M}{\pi R^2}\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} x^2\,dy.$$

The inner integral is still trivial; the x^2 doesn't change anything, because from the perspective of the dy integral, x is just some constant. However, the remaining integral becomes a bit nasty. In general, when this happens, we can try flipping the order of integration, giving

$$I = \frac{M}{\pi R^2}\int_{-R}^{R} dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} x^2\,dx.$$

Unfortunately, this is equally difficult. Both of these integrals can be done with trigonometric substitutions, as you'll check below, but there's also a clever symmetry argument.

Notice that I is also equal to the moment of inertia about the x axis, by symmetry. So if we add them together, we get

$$2I = \int (x^2 + y^2)\,dm = \int r^2\,dm.$$

The r^2 factor has no dependence on θ at all, so the angular integral in polar coordinates is trivial. We end up with

$$2I = \frac{M}{\pi R^2}\int_0^R 2\pi r\, r^2\,dr = \frac{1}{2}MR^2$$

which gives an answer of $I = MR^2/4$, as expected.

[2] **Problem 32.** Calculate I in the previous example by explicitly performing either Cartesian integral.

[3] **Problem 33.** In this problem we'll generalize some of the ideas above to three dimensions, where we need triple integrals. Consider a ball of radius R.

(a) In Cartesian coordinates, the volume element is $dV = dx\,dy\,dz$. Set up an appropriate triple integral for the volume.

(b) The inner two integrals might look a bit nasty, but we already have essentially done them. Using the result we already know, perform the inner two integrals in a single step, and then perform the remaining integral to derive the volume of a sphere.

(c) In cylindrical coordinates, the volume element is $dV = r\,dr\,d\theta\,dz$. Set up a triple integral for the volume, and perform it. (Hint: this can either be hard, or a trivial extension of part (b), depending on what order of integration you choose.)

(d) In spherical coordinates, the volume element is $dV = r^2\,dr\,\sin\phi\,d\phi\,d\theta$. Set up a triple integral for the volume, and perform it.

(e) Let the ball have uniform density and total mass M. Compute its moment of inertia about the z-axis. (Hint: this can be reduced to a single integral if you use an appropriate trick.)

[2] **Problem 34.** Consider a spherical cap that is formed by slicing a sphere of radius R by a plane, so that the altitude from the vertex to the base is h. Find the area of its curved surface using an appropriate integral.

Remark

You might be wondering how good you have to be at integration to do Olympiad physics. The answer is: not at all! You need to understand how to set up integrals, but you almost never have to *perform* a nontrivial integral. There will almost always be a way to solve the problem without doing explicit integration at all, or an approximation you can do to render the integral trivial, or the integral will be given to you in the problem statement. The Asian Physics Olympiad takes this really far: despite having some of the hardest problems ever written, they often provide information like "$\int x^n\,dx = x^{n+1}/(n+1) + C$" as a hint! This is because physics competitions are generally written to make students think hard about physical systems, and the integrals are just viewed as baggage.

In fact, plain old AP Calculus probably has harder integrals than Olympiad physics. For example, in those classes everybody has to learn the integral

$$\int \sec x\,dx = \log|\sec x + \tan x| + C$$

which has a long history. When I was in high school, I was shocked by how the trick for doing this integral came out of nowhere; it seemed miles harder than anything else taught in the class. And it is! Historically, it arose in 1569 from Mercator's projection, where it gives the vertical distance on the map from the equator to a given latitude. For decades, cartographers simply looked up the numeric value of the integral in tables, where the Riemann sums had been done by hand. (They had no chance of solving it analytically anyway, since Napier only invented logarithms in 1614.) Gradually, tabulated values of the logarithms of trigonometric functions became available, and in 1645, Bond conjectured the correct result by noticing the close agreement of tabulated values of each side of the equation. Finally, Gregory proved the result in 1668, using what Halley called "a long train of Consequences and Complications of Proportions." So it took almost a hundred years for this integral to be sorted out! (Though to their credit, they had the handicap of not knowing about differentiation or the fundamental theorem of calculus; they were finding the area under the curve with just Euclidean geometry.)

Even though Olympiad physics tries to avoid tough integrals, doing more advanced physics tends to produce them, so physicists often get quite good at integration. By contrast, Spivak's calculus textbook for math majors only covers integration techniques in a single chapter towards the end of the book. He justifies the inclusion of this material by saying:

> Every once in a while you might actually need to evaluate an integral [...] For example, you might take a physics course [...] Even if you intend to forget how to

integrate (and you probably will forget some details the first time through), you must never forget the basic methods.

That attitude is why physics students frequently win the MIT Integration Bee.

Problem Solving I: Mathematical Techniques

For the basics of dimensional analysis and limiting cases, see chapter 1 of Morin or chapter 2 of Order of Magnitude Physics. Many more examples are featured in The Art of Insight; some particularly relevant sections are 2.1, 5.5, 6.3, 8.2, and 8.3. Other sections will be mentioned throughout the course. There is a total of **83** points.

1 Dimensional Analysis

Idea 1

Dimensional analysis is simply the statement that the dimensions of physical equations should match on both sides. This simple idea can sometimes solve whole problems by itself.

Dimensional analysis is also a valuable consistency check. For example, if you're trying to derive the surface area of a sphere and find $4\pi r^3$, you can instantly know you made a mistake. As another example, if a problem says the speed of an object is "small", this technically doesn't obey dimensional analysis unless we compare it to another speed. Thus, the problem might really mean you should assume the speed is small compared to the speed of light, $v \ll c$, which tells you something important.

To be precise, we should distinguish dimensions and units. The dimensions of a physical quantity determine what kind of quantity it is, while a unit is a measure of a dimension. Thus, for example, somebody's height h can be measured in units of feet or meters, but both have dimensions of length; this can be written as $[h] = [\text{ft}] = [\text{m}] = L$, where the brackets indicate dimensions. Another example is that angles are dimensionless, but can be measured in units of degrees or radians. These distinctions are not that important for our purposes, so we will be sloppy and conflate dimensions with units, writing the equivalent of $[h] = \text{m}$.

Example 1: $F = ma$ 2018 B11

A circle of rope is spinning in outer space with an angular velocity ω_0. Transverse waves on the rope have speed v_0, as measured in a rotating reference frame where the rope is at rest. If the angular velocity of the rope is doubled, what is the new speed of transverse waves?

Solution

To solve this problem by dimensional analysis, we reason about what could possibly affect the speed of transverse waves. The result could definitely depend on the rope's length L, mass per length λ, and angular velocity ω_0. It could also depend on the tension, but since this tension balances the centrifugal force, it is determined by all of the other quantities. Thus the quantities we have are

$$[L] = \text{m}, \quad [\lambda] = \text{kg/m}, \quad [\omega_0] = 1/\text{s}.$$

Since λ is the only thing with dimensions of mass, it can't affect the speed, because there is nothing that could cancel out the mass dimension. So the only possible answer is

$$v_0 \sim L\omega_0$$

where the $\sim$ indicates equality up to a dimensionless constant, which cannot be found by dimensional analysis alone. In practice, the constant usually won't be too big or too small, so $L\omega_0$ is a decent estimate of v_0. But even if it isn't, the dimensional analysis tells us the scaling: if ω_0 is doubled, the new speed is $2v_0$.

Example 2

Find the dimensions of the magnetic field.

Solution

To do this, we just think of some simple equation involving B, then solve for its dimensions. For example, we know that $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$, so

$$[B] = \frac{[F]}{[q][v]} = \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \frac{1}{\text{C}} \frac{1}{\text{m/s}} = \frac{\text{kg}}{\text{C} \cdot \text{s}}.$$

[2] **Problem 1.** Find the dimensions of power, the gravitational constant G, the permittivity of free space ϵ_0, and the ideal gas constant R.

Solution. The dimensions are

$$[P] = \frac{\text{kg}\,\text{m}^2}{\text{s}^3}, \quad [G] = \frac{\text{m}^3}{\text{kg}\,\text{s}^2}, \quad [\epsilon_0] = \frac{\text{C}^2\,\text{s}^2}{\text{kg}\,\text{m}^3}, \quad [R] = \frac{\text{J}}{\text{mol}\,\text{K}} = \frac{\text{kg}\,\text{m}^2}{\text{mol}\,\text{K}\,\text{s}^2}.$$

The easiest way to get these results is to use formulas containing the desired quantity, such as $P = Fv$, $F = GMm/r^2$, $F = q^2/(4\pi\epsilon_0 r^2)$, and $PV = nRT$, where the dimensions of the other quantities are already known.

[1] **Problem 2.** Derive Kepler's third law for circular orbits, using only dimensional analysis. (Why do you think people didn't figure out this argument 2000 years ago?)

Solution. The answer should only depend on G, M, and the radius r. By dimensional analysis, we have the equality of dimensions

$$[r] = [(GM)^{1/3} T^{2/3}]$$

which implies we must have $T^2 \propto r^3$. Of course, this doesn't mean Kepler's third law is trivial. The dimensions of G follow from the inverse square law for gravity, and you need to know which quantities are allowed in the dimensional analysis in the first place. In other words, you need the whole structure of Newtonian mechanics to be set up already to run this argument.

[2] **Problem 3.** Some questions about vibrations.

(a) The typical frequency f of a vibrating star depends only on its radius R, density ρ, and the gravitational constant G. Use dimensional analysis to find an expression for f, up to a dimensionless constant. Then estimate f for the Sun, looking up any numbers you need.

(b) The typical frequency f of a small water droplet freely vibrating in zero gravity could depends on its radius R, density ρ, surface tension γ, and the gravitational constant G. Argue that at least one of these parameters doesn't matter, and find an expression for f up to a dimensionless constant.

Solution. (a) We just do the usual dimensional analysis,

$$[f] = s^{-1} \qquad [R] = m \qquad [\rho] = kg/m^3 \qquad [G] = \frac{m^3}{kg \cdot s^2}$$

To cancel out kg, multiplying G and ρ will yield $[\rho G] = s^{-2}$. Then to get $[f] = s^{-1}$,

$$f \sim \sqrt{G\rho} \sim 3 \times 10^{-4}\,\text{Hz}$$

which is in the right range. These oscillations are measured in the field of helioseismology.

Another application of this result is that the time needed for a ball of gas of density ρ to collapse is of order $1/\sqrt{G\rho}$, called the free fall time. This timescale plays an important role in structure formation in the early universe.

(b) In this case the surface tension force dominates; the gravitational forces of the droplet on itself are negligible, so we can drop G. Performing dimensional analysis with R, ρ, and γ gives

$$f \sim \sqrt{\frac{\gamma}{\rho R^3}}.$$

Of course, part (a) is equivalent to starting with the same set of four parameters and dropping γ, which makes sense since the objects considered are huge.

[3] **Problem 4.** Some questions about the speed of waves, to be covered in greater detail in **W3**. For all estimates, you can look up any numbers you need.

(a) The speed of sound in an ideal gas depends on its pressure p and density ρ. Explain why we don't have to use the temperature T or ideal gas constant R in the dimensional analysis, and then estimate the speed of sound in air.

(b) The speed of sound in a fluid depends only on its density ρ and bulk modulus $B = -V\,dP/dV$. Estimate the speed of sound in water, which has $B = 2.1\,\text{GPa}$.

The speed of waves on top of the surface of water can depend on the water depth h, the wavelength λ, the density ρ, the surface tension γ, and the gravitational acceleration g.

(c) Find the speed of capillary waves, i.e. water waves of very short wavelength, up to a dimensionless constant.

(d) Find the speed of long-wavelength waves in very deep water, up to a dimensionless constant.

Solution. (a) We don't have to use R or T because all that matters is the restoring force, determined by p, and the inertia, determined by ρ. So we have

$$[p] = \frac{kg}{m\,s^2}, \quad [\rho] = \frac{kg}{m^3}$$

and a routine dimensional analysis gives

$$v \sim \sqrt{\frac{p}{\rho}} \sim \sqrt{\frac{10^5\,\text{Pa}}{1\,\text{kg/m}^3}} \sim 300\,\text{m/s}$$

which is reasonably close. (Actually, the exact answer is $v = \sqrt{\gamma p/\rho}$, as we'll derive in **T3** and **W3**, so thermodynamics actually does play a role through the dimensionless constant.)

(b) We have

$$[B] = \frac{kg}{m \cdot s^2} \qquad [\rho] = \frac{kg}{m^3}.$$

A routine dimensional analysis gives

$$v \sim \sqrt{\frac{B}{\rho}} \sim 1500\,\text{m/s}.$$

This is actually very close to the true answer; here there is no dimensionless constant.

(c) In this case, the surface tension force dominates, just as it did for a small water droplet in a previous problem, which also means that g doesn't matter. The wavelength is so short that the waves can't "see" the depth of the water, so h doesn't matter. Doing dimensional analysis with the remaining three parameters gives

$$v \sim \sqrt{\frac{\gamma}{\rho\lambda}}.$$

(d) In this case, the wave is big enough for surface tension not to matter; the restoring force is gravity, so we keep g and toss out γ. Since the water is even deeper than the wavelength, we again toss out h. Doing dimensional analysis with the remaining parameters gives

$$v \sim \sqrt{g\lambda}.$$

We will derive this in **W3**. The fact that ρ also dropped out makes sense: when gravity is the only force, ρ usually doesn't matter because scaling it up scales all the forces and all the masses up the same way, keeping accelerations the same.

[3] **Problem 5** (Morin 1.5)**.** A particle with mass m and initial speed v is subject to a velocity-dependent damping force of the form bv^n.

(a) For $n = 0, 1, 2, \ldots$, find how the stopping time and stopping distance depend on m, v, and b.

(b) Check that these results actually make sense as m, v, and b are changed, for a few values of n. You should find something puzzling going on. (Hint: to resolve the problem, it may be useful to find the stopping time explicitly in a few examples.)

Solution. (a) The dimensions of b can be found with $[b] = [F/v^n] = kg \cdot m^{1-n} \cdot s^{-2+n}$. To get a stopping time or distance, the mass term must be canceled out. So we're working with

$$\left[\frac{b}{m}\right] = m^{1-n}s^{-2+n} \qquad [v] = \frac{m}{s}$$

The stopping time t can be found by canceling out the length dimension. If $t \propto (b/m)^\alpha v^\beta$, then:

$$\alpha(1-n)+\beta = 0 \qquad \alpha(-2+n)-\beta = 1$$

Solving yields

$$\alpha = -1 \qquad \beta = 1-n, \quad t \propto \frac{mv^{1-n}}{b}.$$

The distance x traveled has dimensions of vt, so

$$x \propto \frac{mv^{2-n}}{b}.$$

(b) The results don't seem to make sense. At $n = 1$, it appears that the time it takes to stop no longer depends on v, which doesn't seem correct since the stopping time should always increase with velocity. And for $n > 1$, the stopping time *decreases* with velocity, which is even worse. Similar issues happen for the stopping distance for $n \geq 2$.

The resolution is that in these cases, the stopping time/distance are actually infinite, as you can check explicitly. In other words, dimensional analysis worked, but the hidden dimensionless prefactor was *infinity*.

Idea 2

Dimensional analysis applies everywhere. The argument of any function that is not a monomial, such as $\sin x$, must have no dimensions. The derivative d/dx has the opposite dimensions to x, and the dx in an integral has the same dimensions as x. When you perform an integral, your first step should usually be to "nondimensionalize" it, i.e. to separate out dimensionful factors to leave a dimensionless integral.

Example 3

Evaluate the indefinite integral

$$I = \int_0^a \frac{dx}{bx^2+c}$$

where b and c are both positive.

Solution

This isn't a hard integral by any means, but it's a simple way to demonstrate what we mean by "nondimensionalizing". If you do the integral directly, you'll get lots of intermediate expressions with a, b, and c in them, which produces clutter and more opportunities for error. Instead, start by substituting $u = \sqrt{b/c}\,x$, to get

$$I = \frac{1}{\sqrt{bc}} \int_0^{u_0} \frac{du}{u^2+1} = \frac{1}{\sqrt{bc}} \tan^{-1}(u_0), \quad u_0 = a\sqrt{b/c}.$$

We can now check this by dimensional analysis. Let's suppose this integral arose from a problem where x had dimensions of length, $[x] = \mathrm{m}$, and b was dimensionless. Then we must have $[a] = \mathrm{m}$, $[c] = \mathrm{m}^2$. Our answer makes sense if $[I] = \mathrm{m}^{-1}$ and $[u_0] = 1$, which both hold.

[2] **Problem 6.** We are given the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

For positive a, find the value of the integral

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\,dx$$

and verify that your answer makes dimensional sense.

Solution. We can simply factor out the dependence on c, and get rid of the a in the exponent by taking $u = \sqrt{a}\,x$, to get

$$\frac{e^c}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-u^2+(b/\sqrt{a})u}\,du.$$

To get rid of the pesky linear term in the exponent, we note that

$$-v^2 = -\left(u - \frac{b}{2\sqrt{a}}\right)^2 = -u^2 + \frac{b}{\sqrt{a}}\,u - \frac{b^2}{4a}$$

so that we can "complete the square" in the exponent by working in terms of v, for

$$\frac{e^{c+b^2/4a}}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-v^2}\,dv = e^{c+b^2/4a}\sqrt{\frac{\pi}{a}}.$$

To check this makes sense, let's again suppose that $[x] = \mathrm{m}$, which implies $[a] = \mathrm{m}^{-2}$, $[b] = \mathrm{m}^{-1}$, and $[c] = 1$. The overall integral must have dimensions of m, which it does, and the argument of the exponent must be dimensionless, which it is.

Remark

Consider the value of the definite integral

$$\int_{-\infty}^{y} e^{-x^2}\,dx.$$

You can try all day to compute the value of this integral, using all the integration tricks you know, but nothing will work. The function e^{-x^2} simply doesn't have an antiderivative in terms of the functions you already know, i.e. in terms of polynomials, exponents and logarithms, and trigonometric functions (for more discussion, see here).

If you ask a computer algebra system like Mathematica, it'll spit out something like $\mathrm{erf}(x)$, which is defined by being an antiderivative of e^{-x^2}. But is this really an "analytic" solution? Isn't that just saying "the integral of e^{-x^2} is equal to the integral of e^{-x^2}"? Well, like many things in math, it depends on what the meaning of the word "is" is.

The fact is, the set of functions we regard as "elementary" is arbitrary; we just choose a set that's big enough to solve most of the problems we want, and small enough to attain fluency with. (Back in the days before calculators, it just meant all the functions whose values were tabulated in the references on hand.) If you're uncomfortable with $\mathrm{erf}(x)$, note that a similar thing would happen if a little kid asked you what the ratio of the opposite to adjacent sides of a right triangle is. You'd say $\tan(x)$, but they could say it's tautological, because the only way to define $\tan(x)$ at their level is as the ratio of opposite to adjacent sides. Similarly, $1/x$ has no elementary antiderivative – unless you count $\log(x)$ as elementary, but ultimately $\log(x)$ is simply *defined* to be such an antiderivative. It's all tautology, but it's still useful.

[2] **Problem 7.** In particle physics it is conventional to work in "natural units", where the numeric values of $\hbar$ and c are equal to 1. For example, if we take the second as the unit of time, then we can take the light-second as the unit of length, so that $c = 1$ light-second/second. This is usually sloppily written as "$\hbar = c = 1$" so that factors of $\hbar$ and c can be suppressed. However, you can always restore these factors by dimensional analysis.

According to standard references, the mass of the Higgs boson is about 125 GeV, where 1 eV is the energy gained by an electron accelerated through a voltage difference of 1 V. Fix the dimensions of this statement and find the mass of the Higgs boson in kilograms.

Solution. One easy way to start out dimensional analysis is with famous equations: $E = mc^2$, or $E = \frac{1}{2}mv^2$ to get $m \sim E/c^2$. Thus the mass of the Higgs boson is $m = 125\,\mathrm{GeV/c^2} = 2.22 \times 10^{-25}\,\mathrm{kg}$.

[3] **Problem 8.** USAPhO 2002, problem A3.

Example 4

The Schrodinger equation for an electron in the electric field of a proton is

$$-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi.$$

Estimate the size of the hydrogen atom.

Solution

This is yet another dimensional analysis problem: there is only one way to form a length using the quantities given above. We have

$$[m] = \mathrm{kg}, \quad [\hbar] = \mathrm{J \cdot s} = \mathrm{kg\,m^2\,s^{-1}}, \quad [e^2/4\pi\epsilon_0] = \mathrm{J \cdot m} = \mathrm{kg\,m^3\,s^{-2}}.$$

Doing dimensional analysis, the only length scale is the Bohr radius,

$$a_0 = \frac{4\pi\hbar^2\epsilon_0}{me^2} \sim 10^{-10}\,\mathrm{m}.$$

I've thrown in a 4π above because ϵ_0 always appears in the equations as $4\pi\epsilon_0$. The dimensional analysis would be valid without this factor, but as you'll see in problem 11, if you don't include it then annoying compensating factors of 4π will appear elsewhere.

Classically (i.e. without $\hbar$), there is no way to form a length, and hence there should be no classically stable radius for the atom. (This was one of the arguments used by Bohr to motivate quantum mechanics; it appears in the beginning of his paper introducing the Bohr model.) Once we introduce $\hbar$, there are three dimensionful parameters in the problem, as listed above. And there are exactly three fundamental dimensions. So there is only one way to create a length, which we found above, one way to create a time, one way to create an energy, and so on. This means that the solutions to the Schrodinger equation above look qualitatively the same no matter what these parameters are; all that changes are the overall length, time, and energy scales. In problem 11, you'll investigate how this conclusion changes when we add more dimensionful parameters.

Dimensional analysis is especially helpful with scaling relations. For example, a question might ask you how the radius of the hydrogen atom would change in a world where the electron mass was twice as large. You would solve this problem in the exact same way as the example above, using dimensional analysis to show that $a_0 \propto 1/m$.

[3] **Problem 9.** In this problem we'll continue the dimensional analysis of the Schrodinger equation.

(a) Estimate the typical energy scale of quantum states of the hydrogen atom, as well as the typical "velocity" of the electron, using dimensional analysis.

(b) Do the same for one-electron helium, the system consisting of a helium nucleus (containing two protons) and one electron.

(c) Estimate the electric field needed to rip the electron off the hydrogen atom.

Solution. (a) Recall the electrostatic potential energy formula, $E = kq^2/r$. We have a length scale, a_0 to replace r. For velocity, we use $E \sim mv^2$, giving

$$E \sim \frac{me^4}{(4\pi\epsilon_0)^2\hbar^2}, \quad v \sim \frac{e^2}{4\pi\epsilon_0\hbar}.$$

In fact, the binding energy of the hydrogen atom in its ground state is

$$E = \frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2} = 13.6\,\mathrm{eV}$$

which is a constant known as the Rydberg. So the dimensional argument (keeping the factors of 4π) gets the answer right to a factor of 2.

(b) Adding the second proton would double the charge inside the nucleus, so the expressions for energy and velocity should stay the same except e^2 would be replaced with $2e^2$ (not 4, since the electron charge stays the same) and thus the energy would be $4e^4$. In general, with Z as the atomic number,

$$E \sim \frac{mZ^2e^4}{(4\pi\epsilon_0)^2\hbar^2}, \quad v \sim \frac{Ze^2}{4\pi\epsilon_0\hbar}.$$

(c) Physically, the work the electric field does by moving the electron across the radius of its orbit should be enough to overcome its binding energy to the proton. This also tells us how to set up the dimensional analysis; we have electric field

$$|\mathbf{E}| \sim \frac{E}{ea_0} \sim \frac{m_e^2e^5}{(4\pi\epsilon_0)^3\hbar^4} \sim 10^{12}\,\mathrm{V/m}.$$

This is a tremendously large electric field!

All of the results above are not that accurate, but they become much more accurate if we replace ϵ_0 with $4\pi\epsilon_0$. That in turn makes sense because these factors always appear together in electromagnetism.

Idea 3: Buckingham Pi Theorem

Dimensional analysis can't always pin down the form of the answer. If one has N quantities with D independent dimensions, then one can form $N - D$ independent dimensionless quantities. Dimensional analysis can't say how the answer depends on them.

A familiar but somewhat trivial example is the pendulum: its period depends on L, g, and the amplitude θ_0, three quantities which contain two dimensions (length and time). Hence we can form one dimensionless group, which is clearly just θ_0 itself. The period of a pendulum is $T = f(\theta_0)\sqrt{L/g}$.

Example 5: $F = ma$ 2014 12

A paper helicopter with rotor radius r and weight W is dropped from a height h in air with a density of ρ. Assuming the helicopter quickly reaches terminal velocity, use dimensional analysis to analyze the total flight time T.

Solution

The answer can only depend on the parameters r, W, h, and ρ. There are four quantities in total, but three dimensions (mass, length, and time), so by the Buckingham Pi theorem we can form one independent dimensionless quantity. In this case, it's clearly r/h. Continuing with routine dimensional analysis, we find

$$T = f(r/h)\, h^2 \sqrt{\frac{\rho}{W}}.$$

The form of this expression is a bit arbitrary; for instance, we could also have written $f(r/h)r^2$ in front, or even $f(r/h)r^{37}h^{-35}$. These adjustments just correspond to pulling factors of r/h out of f, not to changing the actual result.

This is as far as we can get with dimensional analysis alone, but we can go further using physical reasoning. If the helicopter quickly reaches terminal velocity, then it travels at a constant speed. So we must have $T \propto h$, which means that $f(x) \propto x$, and

$$T \propto rh\sqrt{\frac{\rho}{W}}.$$

Example 6

An hourglass is constructed with sand of density ρ and an orifice of diameter d. When the sand level above the orifice is h, what is the mass flow rate μ?

Solution

The answer can only depend on ρ, d, h, and g. The Buckingham Pi theorem gives

$$\mu = f(h/d)\rho\sqrt{gd^5}.$$

That's as far as we can get with dimensional analysis; to go further we need to know more about sand. If we were dealing with an ideal fluid, then the flow speed would be $v = \sqrt{2gh}$ by Torricelli's law, which means the flow rate has to be proportional to $\sqrt{h}$. Then $f(x) \propto \sqrt{x}$, giving the result $\mu \propto \rho d^2 \sqrt{gh}$. This is a good estimate as long as the orifice isn't so small that viscosity starts to dominate.

But this isn't how sand works: measurements show that the pressure at the orifice doesn't actually depend on the height of the sand, an empirical result known as Janssen's law. That's because sand is a granular material whose motion is dominated by the friction between sand grains, and this friction prevents the additional pressure from propagating downward. The resulting flow rate is independent of h, as can be confirmed by watching an hourglass run. Then $f(x)$ is a constant, giving $\mu \propto \rho\sqrt{gd^5}$. This neat, experimentally verified result is called Beverloo's law, and it's essential in industry to design grain hoppers and corn silos.

Remark

One has to be a little careful with the Buckingham Pi theorem. For example, if all we had were 3 speeds v_i, we can form two dimensionless quantities: v_1/v_2 and v_1/v_3. (The quantity v_2/v_3 is not independent, since it is the quotient of these two.) But there are 3 quantities with 2 dimensions (length and time), so we expect only 1 dimensionless quantity.

The problem is that the two dimensions really aren't independent: for any quantity built from the v_i, a power of length always comes with an inverse power of time, so there's only one independent dimension. These considerations can be put on a more rigorous footing in linear algebra, where the Buckingham Pi theorem is merely a special case of the rank-nullity theorem. If you're ever in doubt, you can just forget about the theorem and play with the equations directly.

Remark

Dimensional analysis is an incredibly common tool in Olympiad physics because it lets you say a lot even without much advanced knowledge. If a problem ever says to find some quantity "up to a constant/dimensionless factor", or how that quantity scales as another quantity changes, or what that quantity is proportional to, it's almost certainly asking you to do dimensional analysis. Another giveaway is if the problem *looks* extremely technical and advanced, because they can't actually be.

[3] **Problem 10** (Insight)**.** In this problem we'll do one of the most famous dimensional analyses of all time: estimating the yield of the first atomic bomb blast. Such a blast will create a shockwave of air, which reaches a radius R at time t after the blast. The air density is ρ, and we want to estimate the blast energy E.

(a) Declassified photographs of the blast indicate that $R \approx 100\,\text{m}$ at time $t \approx 15\,\text{ms}$. The density of air is $\rho \approx 1\,\text{kg/m}^3$. Estimate the blast energy E.

(b) How much mass-energy (in grams) was used up in this blast?

(c) If we measure the entire function $R(t)$, what general form would we expect it to have, if this dimensional analysis argument is correct?

Solution. (a) The only way to write an expression with the right dimensions is

$$E \sim \frac{R^5 \rho}{t^2}.$$

Plugging in the numbers gives $E \sim 4 \times 10^{13}$ J.

(b) The mass-energy equivalent is $m = E/c^2 \sim 0.5$ g. This is quite reasonable, as fission can only release a small fraction of the mass-energy (about 0.1%) of a sample, and the critical mass is typically on the order of a few pounds.

(c) Let's do the dimensional analysis in reverse: we know E is fixed, so the only way to write an expression with the right dimensions for R is

$$R \sim (Et^2/\rho)^{1/5} \sim t^{2/5}.$$

So $R(t)$ must have this power-law dependence. If it doesn't, then it means some other quantity with dimensions is intervening, so our dimensional analysis is suspect. Luckily, around this range of time the relation above is true, and indeed the answer of part (a) is pretty close.

Remark

The British physicist G. I. Taylor performed the dimensional analysis in problem 10 upon seeing a picture of the first atomic blast in a magazine. The result was so good that the physicists at the Manhattan project thought their security had been breached!

During World War II, the exact value of the critical mass needed to set off a nuclear explosion was important and nontrivial information. The Nazi effort to make a bomb had been stopped by Werner Heisenberg's huge overestimation of this quantity, and after the war, the specific value was kept a closely guarded secret. That is, it was until 1947, when a Chinese physicist got the answer using a rough estimate that took four lines of algebra.

[5] **Problem 11.** We now consider the Schrodinger equation for the hydrogen atom in greater depth. We begin by switching to dimensionless variables, which is useful for the same reason that writing integrals in terms of dimensionless variables is: it highlights what is independent of unit choices.

(a) Define a dimensionless length variable $\tilde{r} = r/a_0$, where a_0 is the length scale found in example 4. The ∇^2 term in the Schrodinger equation is a second derivative, the 3D generalization of d^2/dx^2. Using the chain rule, argue that

$$\tilde{\nabla}^2 = a_0^2 \nabla^2$$

where $\tilde{\nabla}$ is the gradient with respect to $\tilde{r}$.

(b) Similarly define a dimensionless energy $\tilde{E} = E/E_0$, using the energy scale E_0 found in problem 9. Show that the Schrodinger equation can be written in a form like

$$-\tilde{\nabla}^2 \psi - \frac{1}{\tilde{r}} \psi = \tilde{E} \psi$$

Here I've suppressed all dimensionless constants, like factors of 2, because they depend on how you choose to define E_0 and don't really matter at this level of precision.

The result of this part confirms what we concluded above: solutions to the Schrodinger equation don't qualitatively depend on the values of the parameters, because they all come from scaling a solution to this one dimensionless equation appropriately.

(c) This is no longer true in relativity, where the total energy is

$$E = \sqrt{p^2c^2 + m^2c^4}.$$

Assuming $p \ll mc$, perform a Taylor expansion to show that the next term is Ap^4, and find the coefficient A. (If you don't know how to do this, work through the next section first.)

(d) In quantum mechanics, the momentum is represented by a gradient, $p \to -i\hbar\nabla$. (We will see why in **X1**.) Show that the Schrodinger equation with the first relativistic correction is

$$-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi + \hbar^4 A\nabla^4\psi = E\psi.$$

(e) Since there is now one more dimensionful quantity in the game, it is possible to combine the quantities to form a dimensionless one. Create a dimensionless quantity α that is proportional to $e^2/4\pi$, then numerically evaluate it. This is called the fine structure constant. It serves as an objective measure of the strength of the electromagnetic force, because it is dimensionless, and hence its value doesn't depend on an arbitrary unit system.

(f) As the number of protons in the nucleus increases, the relativistic correction becomes more important. Estimate the atomic number Z where the correction becomes very important.

Solution. (a) For the first derivative,

$$\frac{d\psi}{dr_x} = \frac{d\psi}{dx}\frac{dx}{dr_x}.$$

With the length scale, $dx/dr_x = a_0$ which is a constant. The second derivative does the same, which gives two factors of a_0. This holds true for all the other dimensions, so

$$\tilde{\nabla}^2 = a_0^2\nabla^2.$$

(b) Ignoring all numerical factors and dividing by $E_0 = e^2/\epsilon_0 a_0$, we get

$$-\frac{\hbar^2\epsilon_0 a_0}{me^2}\left(\frac{1}{a_0^2}\tilde{\nabla}^2\right)\psi - \frac{a_0}{r}\psi = (E/E_0)\psi$$

which simplifies to

$$-\tilde{\nabla}^2\psi - \frac{1}{\tilde{r}}\psi = \tilde{E}\psi.$$

(c) Since $\sqrt{1+x} \approx 1 + x/2 + (1/2)(-1/4)x^2$,

$$E = mc^2\sqrt{1 + \frac{p^2c^2}{m^2c^4}} \approx mc^2 + \frac{p^2}{2m} - \frac{1}{8}\frac{p^4c^4}{m^3c^6}$$

which implies

$$A = -\frac{1}{8m^3c^2}.$$

(d) With $p^4 = \hbar^4\nabla^4$, this is simply added to the left hand side of the equation as a correction of the first order momentum term $p^2/2m = -\hbar^2/2m$,

$$-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi + \hbar^4 A\nabla^4\psi = E\psi.$$

(e) Just like in part (b), divide both sides by E_0. The dimensionless quantity in the added term should be
$$\frac{\hbar^4}{m^3c^2a_0^4}\frac{\epsilon_0 a_0}{e^2} = \frac{e^4}{\hbar^2c^2\epsilon_0^2}.$$
To make it proportional to e^2, take the square root to get
$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}.$$

(f) The relativistic correction is important when the above term is of order 1, and since there's an electron charge e and a nucleus with charge $+Ze$, replace e^2 with Ze^2. It's order one when
$$Z\alpha \approx 1.$$
So the atomic number when the correction becomes very important is around 137. Actually, even for moderately heavy elements, the corrections are already noticeable and must be accounted for. As a concrete example, if you don't account for relativistic effects, you would predict the color of gold to be silver instead. For more about the relativistic chemistry of gold, see this paper.

You probably won't see any differential equations as complex as the ones in the above problem anywhere in Olympiad physics, but the key idea of using dimensionless quantities to simplify and clarify the physics can be used everywhere.

[5] **Problem 12.** IPhO 2007, problem "blue". This problem applies thermodynamics and dimensional analysis in some exotic contexts.

Example 7

Estimate the Young's modulus for a material with interatomic separation a and typical atomic bond energy E_b. Use this to estimate the spring constant of a rod of area A and length L, as well as the speed of sound, if each atom has mass m.

Solution

This example is to get you comfortable with the Young's modulus Y, which occasionally comes up. It is defined in terms of how much a material stretches as it is pulled apart,
$$Y = \frac{\text{stress}}{\text{strain}} = \frac{\text{restoring force/cross-sectional area}}{\text{change in length/length}}.$$
The Young's modulus is a useful way to characterize materials, because unlike the spring constant, it doesn't depend on the shape of the material. For example, putting two identical springs side-by-side doubles the spring constant, because they both contribute to the force. But since the stress is the force per area, it's unchanged. Similarly, putting two identical springs end-to-end halves the spring constants, because they both stretch, but since the strain is change in length per length, it's unchanged. So you would quote a material's Young's modulus instead of its spring constant, for the same reason you would quote a material's resistivity instead of its resistance.

We note that Y has the dimensions of energy per length cubed, so

$$Y \sim \frac{E_b}{a^3}$$

solely by dimensional analysis. (Of course, for this dimensional analysis to work, one has to understand why E_b and a are the only relevant quantities. It's because Y, or equivalently the spring constant k, determines the energy stored in a stretched spring. But microscopically this comes from the energy stored in interatomic bonds when they're stretched. So the relevant energy scale is the bond energy E_b, and the relevant distance scale is a, because that determines how many bonds get stretched, and by how much.)

To relate Y to the spring constant of a rod, note that

$$Y = \frac{F/A}{\Delta L/L} = \frac{L}{A}\frac{F}{\Delta L} = k\frac{L}{A}$$

for a rod, giving the estimate $k \sim AE_b/La^3$. This is correct to within an order of magnitude!

To relate Y to the speed of sound, note that the sound speed, like most wave speeds, depends on the material's inertia and its restoring force against distortions. Since the speed of sound doesn't depend on the extrinsic features of a metal object, such as a length, both of these should be measured intrinsically. The intrinsic measure of inertia is the mass density $\rho \sim m/a^3$, while the intrinsic measure of restoring force is just Y. By dimensional analysis,

$$v \sim \sqrt{\frac{Y}{\rho}} \sim \sqrt{\frac{E_b/a^3}{m/a^3}} \sim \sqrt{\frac{E_b}{m}}.$$

This is also reasonably accurate. For example, in diamond, $E_b \sim 1\,\text{eV}$ (a typical atomic energy scale), while a carbon nucleus contains 12 nucleons, so to the nearest order of magnitude, $m \sim 10m_p$, where a useful fact is $m_p \sim 1\,\text{GeV/c}^2$. Thus,

$$v \sim \sqrt{\frac{1\,\text{eV}}{10^{10}\,\text{eV}}}\,c \sim 10^{-5}\,c \sim 3\,\text{km/s}$$

which is the right order of magnitude. (The true answer is $12\,\text{km/s}$.)

Amazingly, we can get an even rougher estimate of v for any solid in terms of nothing besides fundamental constants. To be very rough, the binding energy is on the order of that of hydrogen. As you found in problem 9, this is, by dimensional analysis,

$$E_b \sim \frac{1}{4\pi\epsilon_0}\frac{e^2}{a_0} \sim m_e\left(\frac{e^2}{4\pi\epsilon_0\hbar}\right)^2.$$

We take the nuclear mass to be very roughly the proton mass m_p, which gives

$$\frac{v}{c} \sim \sqrt{\frac{m_e}{m_p}\left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)^2} \sim \alpha\sqrt{\frac{m_e}{m_p}}$$

where α is as found in problem 11. This expresses the speed of sound in terms of the dimensionless strength of electromagnetism α, the electron to proton mass ratio, and the speed of light. Of course, the approximations we have made here have been so rough that now the answer is off by *two* orders of magnitude, but now we know how the answer would change if the fundamental constants did.

Estimates as simple as these can be surprising to even seasoned physicists: in 2020, the simple estimate above was rediscovered and published in one of the top journals in science. If you want to learn how to do more of these estimates, this paper is a good starting point.

Remark

A warning: from these examples, you could get the idea that dimensional analysis gives you nearly godlike powers, and the ability to write down the answer to most physics problems instantly. In reality, it only works if you're pretty sure your physical system depends on only about 3 or 4 variables – and the hard part is often finding *which* variables matter. For example, as we saw above, you can't get Kepler's third law for free because that requires knowing the dimensions of G, which require knowing that gravity is an inverse square law in the first place, a luxury Kepler didn't have. And as another example, we couldn't have figured out $E = mc^2$ long before Einstein, as who would have thought that the speed of light had anything to do with the energy of a lump of matter? Without the framework of relativity, it seems as irrelevant as the speed of sound or the speed of water waves.

Fortunately, carrying out dimensional analysis in practice is usually fairly straightforward. Often, on exams, you'll simply be told which variables matter. And in general, you should get into the habit of doing it constantly, to check your work.

Example 8

Cutting-edge archeological research has found that the famed T. Rex was essentially a gigantic chicken. Suppose a T. Rex is about $N = 20$ times larger in scale than a chicken. How much larger is its weight, cross-sectional area of bone, and walking speed?

Solution

These kinds of biological scaling arguments are fun to think about, though the reliability of the results is somewhat questionable – the data is extremely noisy, and if any given scaling law doesn't quite match it, you can always think a bit more, and come up with a new argument yielding a different scaling. But here are a few simple examples:

- Since the densities should match, the weight should scale with the volume, so as N^3.
- Since the maximum compressive pressure that bone can take should be the same, the bone area should scale with the weight, so also as N^3. That is, the width of the bones scales as $N^{3/2}$, while their length L scales only as N. This is the reason small animals are strong relative to their weight, while large ones need to be very bony to even stand. The largest animals today are whales, as they don't need to support their own weight.

- As a very crude model of walking, we can think of the legs as swinging like a free pendulum. The length of one step is proportional to L, while the period of the steps is proportional to $\sqrt{L}$. Thus, the walking speed scales as $\sqrt{L} \propto \sqrt{N}$.

There's an entire literature on these arguments. For instance, this delightful paper discusses how furry mammals shake to dry themselves off. This is an increasingly severe problem for smaller mammals, since a relatively larger amount of water will cling to them after getting wet, which can cause hypothermia. Using elementary fluid mechanics, the paper argues that the optimal frequency the mammal will shake to dry itself off scales as $f \propto m^{-3/16}$.

Example 9

How does the maximum jump height of an animal depend on its length scale L? How about the gravitational acceleration g?

Solution

The maximum jump height h satisfies $E = mgh$ where E is the energy supplied by the muscles. But both the total mass m and the animal's muscle mass (which determines E) scale as L^3, so we have $h \propto L^0$. So the jump height doesn't scale with size: a dinosaur can't jump much higher than a human – and indeed, *we* can't jump much higher than fleas can!

The other half of the problem seems very simple: we must have $h \propto 1/g$ because neither E or m depend on g. But this is completely wrong! In gravity $10g$, a person wouldn't be able to jump at all; they'd be so crushed by their own weight that they wouldn't even be able to stand. Mathematically, the dimensional analysis argument fails here because the answer depends on the detailed biomechanics of muscle and bone, which involve many more dimensionful quantities. By contrast, the result $h \propto L^0$ works well because the animals we were comparing all evolved so that their muscles would work reasonably well in Earth's gravity, releasing a decent fraction of their stored energy in the short time required for a jump. So, as remarked above, you can't solve every problem by just listing a few quantities and doing dimensional analysis – you really have to understand the system each time.

2 Approximations

Idea 4: Taylor Series

For small x, a function $f(x)$ may be approximated as

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + O(x^{n+1})$$

where $O(x^{n+1})$ stands for an error term which grows at most as fast as x^{n+1}.

There are a few Taylor series that are essential to know. The most important are

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4), \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - O(x^4)$$

and the small angle approximations

$$\sin x = x - \frac{x^3}{6} + O(x^5), \quad \cos x = 1 - \frac{x^2}{2} + O(x^4).$$

Another Taylor series you learned long before calculus class is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + O(x^4).$$

Usually you'll only need the first one or two terms, but for practice we'll do examples with more. If any of these results aren't familiar, you should rederive them!

Example 10

Find the Taylor series for $\tan x$ up to, and including the fourth order term.

Solution

By the fourth order term, we mean the term proportional to x^4. (Not the fourth nonzero term, which would be $O(x^7)$.) Of course, $\tan x$ is an odd function, so the $O(x^4)$ term is zero, which means we only need to expand up to $O(x^3)$. That means we can neglect $O(x^4)$ terms and higher everywhere in the computation, subject to some caveats we'll point out later.

By definition, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - x^3/6 + O(x^5)}{1 - x^2/2 + O(x^4)}.$$

However, it's a little tricky because we have a Taylor series in a denominator. There are two ways to deal with this. We could multiply both sides by $\cos x$, and expand $\tan x$ in a Taylor series with unknown coefficients. Then we would get a system of equations that will allow us to solve for the coefficients recursively, a technique known as "reversion of series".

A faster method is to use the Taylor series for $1/(1-x)$. We have

$$\frac{1}{1-u} = 1 + u + O(u^2)$$

and substituting $u = x^2/2 - O(x^4)$ gives

$$\frac{1}{\cos x} = 1 + \frac{x^2}{2} + O(x^4).$$

Therefore, we conclude

$$\tan x = (x - x^3/6 + O(x^5))(1 + x^2/2 + O(x^4)) = x + x^3/3 + O(x^5).$$

Here I was fairly careful with writing out all the error terms and intermediate steps, but as you get better at this process, you'll be able to do it faster. (Of course, one could also have done this example by just directly computing the Taylor series of $\tan x$ from its derivatives. This is possible, but for more complicated situations it's generally not a good idea, because computing high derivatives of a complex expression tends to get very messy. It's better to just Taylor expand the individual pieces and combine the results, as we did here.)

Remark

Finding series up to a given order can be subtle. For example, if you want to compute an $O(x^4)$ term, it is *not* always enough to expand everything up to $O(x^4)$, because powers of x might cancel. To illustrate this, the last step here is wrong:

$$\tan x = \frac{x^3 \sin x}{x^3 \cos x} = \frac{x^4 + O(x^6)}{x^3 + O(x^5)} \neq x + O(x^5).$$

[2] **Problem 13.** Find the Taylor series for $1/\cos x$ up to and including the fourth order ($O(x^4)$) term.

Solution. The derivatives of $\cos(x)$ at $x = 0$ are 0, -1, 0, 1, so

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + O(x^6).$$

To expand the inverse, note that

$$\frac{1}{1-u} = 1 + u + u^2 + O(u^3)$$

where in our case, $u = x^2/2 - x^4/24$. Plugging this in gives

$$\frac{1}{\cos x} = 1 + (x^2/2 - x^4/24) + (x^2/2 - x^4/24)^2 + O(x^6) = 1 + \frac{x^2}{2} + \frac{5x^4}{24}.$$

[2] **Problem 14.** Extend the computation above to get the x^5 term in the Taylor series for $\tan x$.

Solution. From this point on we will start omitting the explicit $O(x^n)$ error terms. We have

$$(x - x^3/6 + x^5/120)(1 + x^2/2 + 5x^4/24) = x + x^3/2 + 5x^5/24 - x^3/6 - x^5/12 + x^5/120$$

giving the answer,

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15}.$$

[3] **Problem 15.** For small x, approximate the quantity

$$\frac{x^2 e^x}{(e^x - 1)^2} - 1$$

to lowest order. That is, find the first nonzero term in the Taylor series. (Hint: if you don't take enough terms in the Taylor series to begin with, you'll get an answer of zero, indicating you approximated too loosely. But if you take too many, the computation will get extremely messy.)

Solution. After some trial and error, you'll find that the constant and linear terms vanish. Next, we try the quadratic term. For this term, the e^x in the numerator has to be expanded out to quadratic order. However, the denominator itself is proportional to x^2, cancelling with the power of x^2 in the numerator, which means that we have to expand the e^x in the denominator to *third* order. If we don't do this, we'll still get a quadratic term, but it won't hve the right prefactor.

Again suppressing the error terms, we have

$$\begin{aligned}\frac{x^2(1+x+x^2/2)}{(x+x^2/2+x^3/6)^2}-1&=\frac{1+x+x^2/2}{1+x+7x^2/12}-1\\&=(1+x+x^2/2)(1-x+5x^2/12)-1\\&=-\frac{x^2}{12}.\end{aligned}$$

Note that we have been careful to keep the manipulations as simple as possible, e.g. by canceling the x/x as early as possible. If you don't do this, everything gets very messy and it's unclear what is contributing at what order, because of the subtlety pointed out in the above remark. Now, the factor of $-1/12$ in the final answer is actually the *same* factor as in the classic result $1+2+3+\ldots=-1/12$. The reason will be explained in an example in **X1**.

[3] **Problem 16.** The function $\cos^{-1}(1-x)$ does not have a Taylor series about $x=0$. However, it does have a series expansion about $x=0$ in a different variable.

(a) What is this variable, and what's the first term in the series?

(b) $\star$ What's the next nontrivial term in the series?

Solution. (a) We have

$$\frac{d}{dx}\arccos(1-x)=\frac{1}{\sqrt{1-(1-x)^2}}$$

which is unfortunately undefined at $x=0$, so there is no Taylor series. But note that if we let $y=\cos^{-1}(1-x)$ and take the cosine of both sides, we have

$$\cos y=1-x.$$

Now y does have a good Taylor series near $y=0$, which corresponds to where $x=0$. At lowest order, we have

$$1-y^2/2\approx 1-x$$

which implies that

$$y\approx\sqrt{2x}.$$

More generally, the answer is a series in $\sqrt{x}$. Since cosine is even, the next term is $O(x^{3/2})$.

(b) In order to get higher order terms, we can write

$$\cos^{-1}(1-x)=\cos^{-1}(1-u^2)$$

where $u=\sqrt{x}$, and directly compute a Taylor series in u, using the usual rule for a derivative of an inverse function. There is also an alternative route that uses the Taylor series for cosine directly. Here we just want the $O(x^{3/2})$ term, so let

$$y=\sqrt{2}x^{1/2}+Ax^{3/2}+O(x^{5/2}).$$

We also know that
$$1 - x = \cos y = 1 - \frac{y^2}{2} + \frac{y^4}{24} + O(y^6).$$
Now, the lowest order term in the series found above is what matches to $1 - x$. The next term in the series can be found by demanding that the right-hand side contain x^2 with zero coefficient. Thus, we are only interested in expanding up to $O(x^2)$, and since $y^6 = O(x^3)$ we can drop it, so
$$\begin{aligned} 1 - x &= 1 - \frac{1}{2}\left(\sqrt{2}x^{1/2} + Ax^{3/2}\right)^2 + \frac{1}{24}\left(\sqrt{2}x^{1/2} + Ax^{3/2}\right)^4 + O(x^3) \\ &= 1 - \frac{1}{2}(2x + 2\sqrt{2}Ax^2) + \frac{1}{24}\left(\sqrt{2}x^{1/2}\right)^4 + O(x^3) \\ &= 1 - x - \sqrt{2}Ax^2 + \frac{x^2}{6} + O(x^3) \end{aligned}$$
from which we conclude $A = 1/(6\sqrt{2})$, so we conclude
$$\cos^{-1}(1 - x) = \sqrt{2x} + \frac{x^{3/2}}{6\sqrt{2}} + O(x^{5/2}).$$
This is an example of reversion of series.

Idea 5: Binomial Theorem

When the quantity xn is small, it is useful to use the binomial theorem,
$$(1 + x)^n = 1 + xn + O(x^2n^2).$$
It applies even when n is not an integer. In particular, n can be very large, very small, or even negative. The extra terms will be small as long as xn is small. If desired, one can find higher terms using binomial coefficients,
$$(1 + x)^n = \sum_{m=0}^{\infty} \binom{n}{m} x^m$$
where the definition of the binomial coefficient is formally extended to arbitrary real n.

The binomial theorem is one of the most common approximations in physics. It's really just taking the first two terms in the Taylor series of $(1 + x)^n$, but we give it a name because it's so useful.

[1] **Problem 17.** Suppose the period of a pendulum is one second, and recall that
$$T = 2\pi\sqrt{\frac{L}{g}}.$$
If the length is increased by 3% and g is increased by 1%, use the binomial theorem to estimate how much the period changes. This kind of thinking is extremely useful when doing experimental physics, and you should be able to do it in your head.

Solution. The change in L increases the period by $3\%/2 = 1.5\%$ and the change in g decreases the period by $1\%/2 = 0.5\%$. So the net change is an increase of 1%.

[1] **Problem 18.** Consider an electric charge q placed at $x = 0$ and a charge $-q$ placed at $x = d$. The electric field along the x axis is then

$$E(x) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{x^2} - \frac{1}{(x-d)^2}\right).$$

For large x, use the binomial theorem to approximate the field.

Solution. Use the binomial theorem with $d/x \ll 1$ to get

$$\frac{1}{(x-d)^2} = \frac{1}{x^2}\left(1 + \frac{2d}{x}\right).$$

Then

$$E(x) = -\frac{2qd}{4\pi\epsilon_0 x^3} = -\frac{qd}{2\pi\epsilon_0 x^3}.$$

This is the on-axis field of an electric dipole.

[3] **Problem 19.** Some exercises involving square roots.

(a) Manually find the Taylor series for $\sqrt{1+x}$ up to second order, and verify they agree with the binomial theorem.

(b) Approximate $\sqrt{1+2x+x^2}$ for small x using the binomial theorem. Does the result match what you expect? If not, how can you correct it?

Solution. (a) The binomial theorem gives $1 + x/2$. By differentiating, we get $1/(2\sqrt{1+x})$ and $-1/(4(1+x)^{3/2})$. Then

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3).$$

The first two terms agree with the usual form of the binomial theorem. For the third term, note that the coefficient should be

$$\binom{1/2}{2} = \frac{(1/2)(-1/2)}{2} = -\frac{1}{8}$$

which is indeed what we find.

(b) Of course, the result is $1 + x$, so we want the $O(x^2)$ term to vanish. On the other hand, applying the binomial theorem gives

$$\sqrt{1+2x+x^2} \approx 1 + \frac{1}{2}(2x + x^2) = 1 + x + \frac{x^2}{2}$$

which is wrong! The reason is that the first order binomial theorem isn't good enough, because the second order term in the binomial theorem will also contribute a second order term to the answer. Using the result of part (a),

$$\begin{aligned}\sqrt{1+2x+x^2} &= 1 + \frac{1}{2}(2x+x^2) - \frac{1}{8}(2x+x^2)^2 + O((2x+x^2)^3)\\ &= 1 + x + \frac{x^2}{2} - \frac{1}{8}(2x+x^2)^2 + O(x^3)\\ &= 1 + x + \frac{x^2}{2} - \frac{1}{8}(2x)^2 + O(x^3)\\ &= 1 + x + O(x^3)\end{aligned}$$

as desired.

Example 11: Birthday Paradox

If you have n people in a room, around how large does n have to be for there to be at least a 50% chance of two people sharing the same birthday?

Solution

Imagine adding people one at a time. The second person has a $1/365$ chance of sharing a birthday with the first. If they don't share a birthday, the third person has a $2/365$ chance of sharing a birthday with either, and so on. So a decent estimate for n is the n where

$$\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\ldots\left(1-\frac{n}{365}\right)\approx\frac{1}{2}.$$

The surprising point of the birthday paradox is that $n \ll 365$. So we can use the binomial theorem in reverse, approximating the left-hand side as

$$\left(1-\frac{1}{365}\right)\left(1-\frac{1}{365}\right)^2\ldots\left(1-\frac{1}{365}\right)^n\approx\left(1-\frac{1}{365}\right)^{n^2/2}$$

which is valid since $n/365$ is small. It's tempting to use the binomial theorem again to write

$$\left(1-\frac{1}{365}\right)^{n^2/2}\approx 1-\frac{n^2}{2\cdot 365}=\frac{1}{2}$$

which gives $n = 19$. However, this is a bad approximation, because the binomial theorem only works if $(n^2/2)(1/365)$ is very small, but here we've set it to $1/2$, which isn't particularly small. Since the series expansion variable is $1/2$, each term in the series expansion is *roughly* $1/2$ as big as the last (ignoring numerical coefficients), so we expect to be off by about $(1/2)^2 = 25\%$.

The binomial theorem is an expansion for $(1 + x)^y$ which works when xy is small. Here xy isn't small, and we instead want an approximation that works when only x is small. One trick to dealing with an annoying exponent is to take the logarithm, since that just turns it into a multiplicative factor. Note that

$$\log((1+x)^y)=y\log(1+x)\approx yx$$

by Taylor series, which implies that

$$(1+x)^y\approx e^{yx}$$

when x is small, an important fact which you should remember. So we have

$$\left(1-\frac{1}{365}\right)^{n^2/2}\approx e^{-n^2/2(365)}=\frac{1}{2}$$

and solving gives $n = 22.5$. We should round up since n is actually an integer, giving $n = 23$, which is indeed the exact answer.

Remark

Precisely how accurate is the approximation $(1+x)^y \approx e^{yx}$? Note that the only approximate step used to derive it was taking $\log(1+x) \approx x$, which means we can get the corrections by expanding to higher order. If we take the next term, $\log(1+x) \approx x - x^2/2$, then we find

$$(1+x)^y \approx e^{yx} e^{-x^2 y/2}.$$

Note that because we are approximating the logarithm of the quantity we want, the next correction is multiplicative rather than additive; we'll see a similar situation with Stirling's approximation in **T2**. Our approximation has good fractional precision as long as $x^2 y \ll 1$. In the previous example, $x^2 y/2 = (22.5/365)^2/4 = 0.1\%$, so our answer was quite accurate.

[2] **Problem 20.** Find a series approximation for x^y, given that y is small and x is neither small nor exponentially huge. (Hint: to check if you have it right, you can try concrete numbers, such as $y = 0.01$ and $x = 10$. The series expansion variable may look a bit unusual.)

Solution. Let's write

$$x^y = e^{y \log(x)}.$$

If y is small, then for any reasonable x (i.e. x not exponentially huge), $y \log(x)$ is also small. So we can use the Taylor series for the exponential to get

$$x^y \approx 1 + y \log(x) + O((y \log(x))^2)$$

with further terms easily computed.

By the way, this shows you what logarithms really are: they are the limits of "zeroth" powers,

$$\log x = \lim_{\epsilon \to 0} \frac{x^\epsilon - 1}{\epsilon}.$$

The power rule for integration, $\int x^n \, dx = x^{n+1}/(n+1) + C$, breaks down for $n = -1$, where it would give $x^0/0$. Given the above result, it's not surprising that in this case the integral is a logarithm.

Remark

If these questions seem complicated, rest assured that 90% of approximations on the USAPhO and IPhO boil down to using

$$\sin x \approx x, \quad \cos x \approx 1 - x^2/2, \quad (1+x)^n \approx 1 + xn, \quad e^x \approx 1 + x, \quad \log(1+x) \approx x.$$

I've given you a lot of subtle situations above, but it's these that you have to know by heart. Almost all situations where you will use these will look like problem 17 or problem 18.

Remark

Just for fun, here are some more examples of tough series expansions. First, in the "Lindhard" theory of electrical shielding in metals, one has to perform the small x expansion

$$\frac{1}{2} + \frac{x^2 - 1}{4x} \log\left(\frac{1+x}{1-x}\right) \approx \frac{x^2}{3} + O(x^4).$$

Since there's a $1/x$ in front of the second term, you have to expand the logarithm to *third* order to get the right prefactor, like in problem 15, but the algebra's a lot messier.

The Soviet mathematician V.I. Arnold used to say that math has gone downhill since Newton, because people in that time could supposedly quickly evaluate the limit

$$\lim_{x\to 0} \frac{\sin(\tan(x)) - \tan(\sin(x))}{\arcsin(\arctan(x)) - \arctan(\arcsin(x))}.$$

The first terms that don't cancel are $O(x^7)$, giving

$$\lim_{x\to 0} \frac{-\frac{55x^7}{1008} + \frac{107x^7}{5040} + O(x^9)}{-\frac{341x^7}{1008} + \frac{173x^7}{5040} + O(x^9)} = 1.$$

This amazingly simple answer can be found with a very tricky geometric argument, of the style common in Newton's Principia, though it's hard to make it totally rigorous.

Finally, as we will see in **E2**, the capacitance of two spheres of radius a separated by $r \gg a$ can be written as an infinite series in a/r. In his *Treatise on Electricity and Magnetism* (1891, section 146), Maxwell manually evaluated this series out to order $(a/r)^{22}$! I don't even know what the point of that was, but it illustrates why physicists took decades to fully comprehend the implications of Maxwell's Treatise. Today, we teach the conceptual essentials of Newton and Maxwell's physics, but the reason they're giants is because they put their theories to *work*, in tough calculations we barely hear about today.

3 Numeric Solutions

Idea 6

In Olympiads, you may have to find numeric solutions for equations that can't be solved analytically. A simple but reliable method is to "guess and check", starting with a reasonable first guess (e.g. derived by solving an approximated version of the equation, or sketching the graphs of both sides), plugging it into both sides, then proceeding with binary search.

[3] **Problem 21.** Sometimes, you can get an accurate numeric answer very quickly on a basic calculator by using the method of iteration, which solves equations of the form $x = f(x)$.

(a) Take a scientific calculator (in radians), put in any number, and press the "cos" button many times. Convince yourself that the final number you get is the unique solution to $x = \cos x$.

(b) What are the key features of the graphs of x and $\cos x$ that made this work? For example, why doesn't pressing $\cos^{-1}$ repeatedly give the same result? As another example, since $x = \sin x$ has a unique solution, why does repeatedly pressing sin not work so well?

(c) Find a nonzero solution for $x = \tan(x/2)$.

(d) Find a nonzero solution for $e^x - 1 = 2x$.

(e) Find a positive solution for $x^x = e$.

Solution. (a) Well, just try it!

(b) What makes $\cos x$ work and $\arccos x$ fail is that near the Dottie number (the solution to $x = \cos x$), the slope of $\cos(x)$ (call it m_1) is greater than -1, and the slope of $\arccos(x)$ (m_2) is less than -1. This means that when one starts with $x + \epsilon$, to the first order $\cos(x)$ will map it to $x + m_1\epsilon$, and for $\arccos x$ it will be $x + m_2\epsilon$. The repeated factors of m_1 with absolute value less than 1 will make iterations of $\cos x$ converge to the Dottie number exponentially, but with $|m_2| > 1$, the iterations will result in numbers getting farther away from the Dottie number.

Another, more global reason that $\cos x$ works so well is that it's bounded. So whatever your initial guess is, at the next stage it'll be mapped to within $[-1, 1]$, and from then on it'll close into the answer. For general functions, you usually have to choose the initial guess more carefully, or else you'll get the wrong solution, or diverge to infinity.

The solution to $\sin x = x$ is $x = 0$, but near zero, the slope of sine is gets closer and closer to 1. This makes convergence excruciatingly slow! If you play around a bit with series, you can show that after n iterations, your answer starts shrinking as $1/\sqrt{n}$, which is much worse than the exponential convergence. This is a pretty weird case though; you probably won't see it in practice.

In general, iteration can "go wrong" in far weirder ways. For example, suppose you tried to iterate $x \to rx(1-x)$ for a constant r. This is called the logistic map, and it turns out that if r is in the right range, the result is chaotic! The result bounces around in an unpredictable way, never repeating itself, and you get a completely different result after a few iterations if you start with a very slightly different number.

(c) Note that iterating $\tan(x/2)$ will lead to $x = 0$. In this case, the solution $x = 0$ is stable, while the solution we actually want is unstable. Thus to get the other solution, try the inverse: $x = 2\arctan(x)$. So type in a guess in your calculator like 3, and then enter $2\arctan(\text{Ans})$, and keep pressing "=". Eventually you'll get $x = 2.331$. There's also a negative solution, $x = -2.331$, and which one you get depends on your initial guess.

(d) Iterating $(e^{\text{Ans}} - 1)/2$ will also yield $x = 0$, so iterate $x = \ln(1 + 2x)$. Type in a guess like 2, and type in $\ln(1 + 2\text{Ans})$. Eventually you'll get to $x = 1.256$.

By the way, here I'm writing ln because that's what the button for natural logarithm says on most calculators, but for the entire rest of the problem sets, I'll always denote the natural logarithm with log, which is the standard for all advanced physics courses.

(e) Taking the log of both sides gives $x \log x = 1$. The iteration from $x = 1/\log(x)$ is unstable, so instead iterate $x = e^{1/x}$. That is, type in a guess close to 1.8 or so, and iterate $e^{1/\text{Ans}}$. You'll get $x = 1.7632$.

[2] **Problem 22.** [A] Newton's method is a more sophisticated method for solving equations, which converges substantially faster than binary search. Suppose we want to solve the equation $f(x) = 0$. Starting with a nearby guess x_0, we evaluate $f(x_0)$ and $f'(x_0)$, then find our next guess by applying the tangent line approximation at this point,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

The process repeats until we get a suitably accurate answer.

(a) Use Newton's method to solve $x = \cos x$.

(b) Newton's method converges quadratically, in the sense that for typical functions, if your current guess is ϵ away from the answer, the next guess will be $O(\epsilon^2)$ away. (This implies that the number of correct digits in the answer roughly doubles with each iteration!) Explain why, and then find an example where Newton's method *doesn't* converge this fast.

Newton's method is very important in general, but it's not that useful on Olympiads. It takes a while to set up, especially if the derivative f' is complicated, and you usually don't need that many significant figures in your answer anyway. (There are alternatives to Newton's method, such as Halley's method, that converge even faster, but the tradeoff is the same: each iteration takes more effort to calculate, as higher derivatives of f must be computed.)

Solution. (a) We want to solve $f(x) = \cos x - x = 0$, which means we iterate

$$x - \frac{\cos x - x}{\sin x + 1}.$$

Starting from a reasonable guess $x_0 = 0.5$, we find

$$x_1 = 0.755222, \quad x_2 = 0.739142, \quad x_3 = 0.739085.$$

The next iteration gives the same thing for the first six decimal places, so after just three iterations, we already have six significant digits in the answer.

(b) If the tangent line approximation was exact, then Newton's method would converge to the answer in one iteration, $f(x_1) = 0$. So if you're already close to the answer, the leading source of inaccuracy is the second-order term in the Taylor expansion of f, giving $f(x_1) \approx \epsilon^2 f''(x_0)/2$. Applying the tangent line approximation again, this implies we are roughly a distance $\epsilon^2 f''(x_0)/2f'(x_1) \propto \epsilon^2$ from the answer.

Convergence will be slower if $f'(x_1)$ happens to be small. For example, for finding roots of polynomials, this will occur for double roots, as the first derivative vanishes at the root itself. In this case $f'(x_1) \propto \epsilon$, so the error after an iteration is still order ϵ, not ϵ^2.

The simplest example where this happens is $f(x) = x^2$, where

$$x_1 = x_0 - \frac{x_0^2}{2x_0} = \frac{x_0}{2}.$$

This is no longer quadratically convergent; instead the error goes down by the same factor in each iteration, so the number of significant figures correct goes up linearly.

It's interesting to compare this to iteration. When the method of iteration works, we typically have exponential convergence, which means the number of significant figures goes up linearly. However, in cases like $f(x) = x^2$ where $f'(x)$ vanishes at the solution, the error is squared in each iteration, so the method of iteration instead converges quadratically! In other words, for these exceptional cases, the convergence rates of iteration and Newton's method swap.

Remark

You've seen several numeric methods above, and going forward, you should feel free to use whichever looks best in each situation. However, if you're solving problems using the same calculator you use for schoolwork, you should make sure to not rely on its more advanced features. In Olympiads, you're generally only allowed to use an extremely basic scientific calculator, with a tiny display and no memory except for the "Ans" key.

Example 12

In units where $c = 1$, the Lorentz factor is defined as

$$\gamma = \frac{1}{\sqrt{1 - v^2}}.$$

Suppose that a particle traveling very close to the speed of light has $\gamma = 10^8$. Find the difference Δv between its speed and the speed of light.

Solution

This problem looks easy; by some trivial algebra we find

$$\Delta v = 1 - \sqrt{1 - 1/\gamma^2}.$$

But when you plug this into a cheap scientific calculator, you get *zero*, or something that's quite far from the right result. The problem is that we are trying to find a small quantity Δv by subtracting two nearby, much larger quantities. But the calculator has limited precision, and it ends up rounding $1 - 1/\gamma^2 = 1 - 10^{-16}$ a bit, giving a completely wrong answer!

Instead, we can apply the binomial theorem to find

$$\Delta v \approx \frac{1}{2\gamma^2} + O(1/\gamma^4).$$

This is no longer the exact answer, but it's a great approximation, because the error term is around $1/\gamma^2 \sim 10^{-16}$ times as small as the answer, and it's easy for a calculator to evaluate. The lesson is that it's better to be accurate in practice than to be precise in theory.

[1] **Problem 23.** Find the solutions of the equation $x^2 - 10^{20}x + 1 = 0$ to reasonable accuracy.

Solution. Applying the quadratic formula, the solutions are

$$x = \frac{10^{20} \pm \sqrt{10^{40} - 4}}{2}.$$

Of course you can't just plug this into a calculator and expect a reasonable result. Instead, we need to approximate. For the larger root, an excellent approximation is

$$x \approx \frac{10^{20} + \sqrt{10^{40}}}{2} = 10^{20}.$$

Then by Vieta's formula, an excellent approximation for the other root is 10^{-20}.

[4] **Problem 24.** [A] Consider the equation $\epsilon x^3 - x^2 + 1 = 0$, where ϵ is small. Find approximate expressions for all three roots of this equation, up to and including terms of order ϵ.

Solution. If we set $\epsilon = 0$, then the roots of the resulting quadratic equation are ± 1. Thus, two of the roots should be near ± 1. To calculate the $O(\epsilon)$ correction, let $x = 1 + A\epsilon + O(\epsilon^2)$. Then plugging this into the equation gives

$$\epsilon(1 + A\epsilon)^3 - (1 + A\epsilon)^2 + 1 = \epsilon - 2A\epsilon + O(\epsilon)^2 = 0.$$

Thus, we find $A = 1/2$. A similar calculation can be done for the root near $x = -1$, giving roots

$$x = 1 + \frac{\epsilon}{2} + O(\epsilon^2), \quad x = -1 + \frac{\epsilon}{2} + O(\epsilon^2).$$

However, the third root is nowhere to be found in this analysis, because the quadratic only has two roots. Upon graphing the function, you can see that the third root is at very large x, once the cubic term catches up in size to the quadratic term. This happens when $x \approx 1/\epsilon$. This appearance of an inverse power of ϵ makes this a "singular perturbation series".

Here's a general way to conceptualize what's going on here. The equation in this problem has three terms, and it's easy to find a root if any one of the terms is negligible compared to the others. For example, for the first two roots, we assumed the ϵx^3 term was negligible, and then found $x = \pm 1$. Then, adding on the ϵx^3 term produces $O(\epsilon)$ and higher corrections to the left-hand side, which can be used to compute $O(\epsilon)$ and higher corrections to the root itself. Now, this third root we've just found occurs when the 1 term is negligible. In this case, both of the first two terms are of order $1/\epsilon^2$, and the 1 creates small corrections to the root (relative to its huge size).

Since 1 is two orders in ϵ smaller than $1/\epsilon^2$, we expect these terms only appear two orders down in the root. That is, we expect the root has the form

$$x = \frac{1}{\epsilon}\left(1 + A\epsilon^2 + O(\epsilon^3)\right)$$

with no $O(\epsilon)$ term in parentheses. (If you don't believe this, check this term vanishes for yourself!) Plugging this into the equation gives

$$\frac{1}{\epsilon^2}(1 + A\epsilon^2 + O(\epsilon^3))^3 - \frac{1}{\epsilon^2}(1 + A\epsilon^2 + O(\epsilon^3))^2 + 1 = 0$$

which is equivalent to

$$3A - 2A + 1 + O(\epsilon) = 0$$

from which we conclude $A = -1$, and hence the third root is

$$x = \frac{1}{\epsilon} - \epsilon + O(\epsilon^2).$$

Finally, you might be wondering what happens if the x^2 term is the negligible one. However, this never happens. If we assume it's negligible, then we need $x \approx -\epsilon^{-1/3}$, so that both the other terms are about 1. But then the x^2 term is $1/\epsilon^{2/3} \gg 1$. So we can't assume the x^2 term is negligible self-consistently, so it doesn't give any new roots. The idea used above, of supposing two of the terms are large, using that to solve a simpler equation, and then checking for consistency, is known as the method of dominant balance.

4 Limiting Cases

Idea 7

Limiting cases can be used to infer how the answer to a physical problem depends on its parameters. It is primarily useful for remembering the forms of formulas, but can also be powerful enough to solve multiple choice questions by itself.

Example 13

What is the horizontal range of a rock thrown with speed v at an angle θ to the horizontal?

Solution

This result is easy to derive, but dimensional analysis and extreme cases can be used to recover the answer too. It can only depend on v, g, and θ, so by dimensional analysis it is proportional to v^2/g. This is sensible, since the range increases with v and decreases with g. Now, the range is zero in the extreme cases $\theta = 0$ and $\theta = \pi/2$, but not anywhere in between, so if we remember the range contains a simple trigonometric function, it must be $\sin(2\theta)$, so

$$R \propto \frac{v^2}{g} \sin(2\theta).$$

We can also get the prefactor by a simple limiting case, the case $\theta \ll 1$. In this case, by the small angle approximation,

$$v_x \approx v, \quad v_y \approx v\theta.$$

The time taken is $t = 2v_y/g$, so the range is

$$R \approx v_x t = \frac{2v^2}{g}\theta.$$

Thus there is no proportionality constant; the answer is

$$R = \frac{v^2}{g} \sin(2\theta).$$

In reality, it's probably faster to go through the full derivation than all of this reasoning, but if you're just not sure about whether it's a sine or a cosine, or what the prefactor is, then limiting cases can be quickly used to recover that piece. Also note that the approximations we used above are frequently useful for evaluating limiting cases.

Example 14

Consider an Atwood's machine with masses m and M, and a massless pulley. Find the tension in the string.

Solution

Since the equations involved are all linear equations, we expect the answer should also be simple. It can only depend on g, m, and M, so by dimensional analysis, it must be proportional to g. By dimensional analysis, this must be multiplied by something with one net power of mass. Since the answer remains the same if we switch the masses, it should be symmetric in m and M.

Given all of this, the simplest possible answer would be

$$T \propto g(m+M).$$

To test this, we consider some limiting cases. If $M \gg m$, the mass M is essentially in freefall, so the mass m accelerates upward with acceleration g. Then the tension is approximately $2mg$. Similarly, in the case $M \ll m$, the tension is approximately $2Mg$. These can't be satisfied by the form above.

The next simplest option is a quadratic divided by a linear expression. Both of these must be symmetric, so the most general possibility is

$$T = g\frac{A(m^2+M^2)+BmM}{m+M}.$$

Then the limiting cases can be satisfied if $A=0$ and $B=2$, giving

$$T = \frac{2gmM}{m+M}.$$

[1] **Problem 25.** Find the perimeter of a regular N-gon, if L is the distance from the center to any of the vertices. By considering a limiting case, use this to derive the circumference of a circle.

Solution. By basic trigonometry, the perimeter is $2NL\sin(\pi/N)$. Then the circumference of a circle is

$$\lim_{N\to\infty} 2NL\sin(\pi/N) = 2NL\,\frac{\pi}{N} = 2\pi L$$

as expected. We can see that the limit of $N\sin(\pi/N)$ is π through the small angle approximation. If you want more rigor, you could also say that this is an indeterminate form $\infty \times 0$, and use l'Hospital's rule.

[1] **Problem 26.** Use similar reasoning to find the acceleration of the Atwood's machine. (We will show an even easier way to do this, using "generalized coordinates", in **M4**.)

Solution. We know from dimensional analysis that the acceleration is $gf(m,M)$ where $f(m,M)$ is dimensionless. Thus it should be a fraction.

If either of the masses is much more massive than the other mass, then the acceleration should be g. Thus the coefficients of m, M should be ± 1. If the masses are equal, then the acceleration should be 0. This leads to a $M-m$ term in the numerator. Since the denominator should be different but still have factors of ± 1, a reasonable answer is

$$a = \frac{m-M}{m+M}g.$$

which is indeed the real answer.

[2] **Problem 27** (Morin 1.6). A person throws a ball (at an angle of her choosing, to achieve the maximum distance) with speed v from the edge of a cliff of height h. Which of the below could be an expression for the maximal range?

$$\frac{gh^2}{v^2}, \quad \frac{v^2}{g}, \quad \sqrt{\frac{v^2 h}{g}}, \quad \frac{v^2}{g}\sqrt{1+\frac{2gh}{v^2}}, \quad \frac{v^2}{g}\left(1+\frac{2gh}{v^2}\right), \quad \frac{v^2/g}{1-2gh/v^2}.$$

If desired, try Morin problems 1.13, 1.14, and 1.15 for additional practice.

Solution. First check if they're all dimensionally correct (they are). When $h = 0$, the maximum range as found above with $\sin(2\theta) = 1$ is v^2/g. Also the maximum range obviously depends on the height of the edge of the cliff, and there shouldn't be a case of a finite height or velocity where the range becomes infinite. This leaves 2 options:

$$\frac{v^2}{g}\sqrt{1+\frac{2gh}{v^2}}, \quad \frac{v^2}{g}\left(1+\frac{2gh}{v^2}\right)$$

When h is small, the extra distance at the end of the trajectory from dipping down a vertical distance h can be found with binomial theorem: h, and $2h$ respectively. Since the trajectory is symmetric, when $h \approx 0$ (to be more concise, $h \ll v^2/g$) the optimal launch angle is 45 deg, so by geometry the extra distance should also be h. Thus the correct formula is

$$\frac{v^2}{g}\sqrt{1+\frac{2gh}{v^2}}.$$

[2] **Problem 28.** Consider a triangle with side lengths a, b, and c. It turns out the area of its incircle can be expressed purely by multiplying and dividing combinations of these lengths. Moreover, the answer is the simplest possible one consistent with limiting cases, dimensional analysis, and symmetry. Guess it!

Solution. In the limiting case $a = b + c$, the triangle collapses and the area must be zero, which means the answer must be proportional to $b + c - a$. But the answer should also be symmetric between exchanging a, b, and c, so it must be proportional to $(b + c - a)(c + a - b)(a + b - c)$. The dimension of this quantity is one too high, so we need to divide by a length, and the only possibility consistent with symmetry is $a + b + c$. Finally, the overall constant can be fixed using the special case of an equilateral triangle, giving the result

$$A = \frac{\pi}{4}\frac{(a+b-c)(b+c-a)(c+a-b)}{a+b+c}.$$

Incidentally, the area of the excircle is $\pi(abc)^2/((a + b + c)(a + b - c)(b + c - a)(c + a - b))$. While most of the denominator makes sense from limiting cases, the overall expression is certainly harder to guess, since powers of abc and $a + b + c$ could cancel while preserving all the limiting cases and symmetry. That just goes to show that limiting cases can only get you so far. In some sense, "real" math starts once all the easy information accessible to methods like these has been accounted for.

While we won't have more questions that are explicitly about dimensional analysis or limiting cases, these are not techniques but ways of life. For all future problems you solve, you should be constantly checking the dimensions and limiting cases to make sure everything makes sense.

5 Manipulating Differentials

You might have been taught in math class that manipulating differentials like they're just small, finite quantities, and treating derivatives like fractions is "illegal". But it's also very useful.

Idea 8

Derivatives can be treated like fractions, if all functions have a single argument.

The reason is simply the chain rule. The motion of a single particle only depends on a single parameter, so the chain rule is just the same as fraction cancellation. For example,

$$\frac{dv}{dt} = \frac{d}{dt}v(x(t)) = \frac{dv}{dx}\frac{dx}{dt}$$

which show that "canceling a dx" is valid. Similarly, you can show that

$$\frac{dy}{dx}\frac{dx}{dy} = 1$$

by considering the derivative with respect to x of the function $x(y(x)) = x$.

As a warning, for functions of multiple arguments, the idea above breaks down. For example, for a function $f(x(t), y(t))$, the chain rule says

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

where there are two terms, representing the change in f from changes only in x, and only in y. Therefore, when we start studying thermodynamics, where multivariable functions are common, we will treat differentials more carefully. But for now the basic rules will do.

Remark: Rigorous Notation

Math students tend to get extremely upset about the above idea: they say we shouldn't use convenient notation if it hides what's "really" going on. And they're right, if your goal is to put calculus on a rigorous footing. But in physics we have no time to luxuriate in such rigor, because we want to figure out how specific things work. The point of notation is to help us do that by suppressing mathematical clutter. A good notation suppresses *as much as possible* while still giving correct results in the context it's used.

To illustrate the point, note that elementary school arithmetic is itself an "unrigorous" notation that hides implementation details. If we wanted to be rigorous about, say, defining the number 2, we would write it as $S(1)$ where S is the successor function, obeying properties specified by the Peano axioms. And 4 is just a shorthand for $S(S(S(1)))$, so $2 + 2 = 4$ means

$$S(1) + S(1) = S(S(S(1))).$$

Even this is not "rigorous", because the Peano axioms don't specify how the numbers or the successor function are defined, just what properties they have to obey. To go deeper, we could define the integers as sets, and operations like + in terms of set operations. For example, in one formulation, we start with nothing but the empty set $\varnothing$ and define

$$4 = S(S(S(1))) = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}.$$

People have seriously advocated for 1^{st} grade math to be taught this way, which has always struck me as insane. You can *always* add more arbitrary layers of structure underneath the current foundation, so such layers should only be added when absolutely necessary.

Here's another example, inspired by the physics education research literature. For uniformly accelerated motion starting from rest, $v(t) = at$, what is $v(x)$? Physics students would say that $v(x) = \sqrt{2ax}$ by the kinematic equations, while math students would say $v(x) = ax$ by the definition of a function. Who is correct? The point is that basic physics and math courses use functions differently. In introductory physics, we often denote several distinct mathematical functions with the same symbol, if they all represent the same physical quantity. (Otherwise, the simplest projectile motion problem would need half the alphabet.) By contrast, basic math courses carefully distinguish functions, but then denote distinct physical quantities with the same symbol: $1\,\text{m}$, $1\,\text{cm}$, and $1\,\text{s}$ are all written as 1.

The crucial point is that nobody is wrong. There is no One True Definition of notation, which is ultimately just squiggly marks people make by dragging graphite cylinders against sheets of wood pulp. Every community makes its own notation for its own needs. And any notation system has to forget about something, or else it would be too clunky to do anything.

Remark: Advanced Notation

As an addendum to the previous remark, it turns out that as you get deeper into math and physics, notation tends to converge. For example:

- The physicist's "wrong" use of $v(t)$ and $v(x)$ can be formalized by differential geometry: here v is a scalar field defined on the particle's path, which is a one-dimensional manifold, and $v(t)$ and $v(x)$ are parametrizations of it in different coordinate charts.

- In math classes, vectors are anything you can take linear combinations of, but in physics classes we also require that they specify a direction in physical space, which math students often criticize as wrong, or meaningless. But the physicist is actually using more advanced math, which the math student doesn't know yet: the physicist's vector is a element of a vector space carrying the fundamental representation of $SO(3)$.

- Most vectors flip sign under an inversion of space, $\mathbf{r} \to -\mathbf{r}$ and $\mathbf{p} \to -\mathbf{p}$, but "axial vectors" such as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ don't. This also strikes many math students as a blatant inconsistency, but the reality is again that an axial vector is just a more advanced mathematical object they haven't met yet, specifically a rank 2 differential form.

- More generally, the "unrigorous" manipulations of differentials above, which we showed give you the right answer anyway, gain a rigorous footing in terms of differential forms. In fact, they become the *preferred* way to denote integration on general manifolds.

Arguments about notation are mostly raised by beginning students, who see the one way they know as the only possible way. Professionals know it both ways, and adjust as needed.

Example 15

Derive the work-kinetic energy theorem, $dW = F\,dx$.

Solution

Canceling the mass from both sides, we wish to show

$$\frac{1}{2}d(v^2) = a\,dx.$$

To do this, note that

$$\frac{1}{2}d(v^2) = v\,dv = \frac{dx}{dt}\,dv = \frac{dv}{dt}\,dx = a\,dx$$

as desired. If you're not satisfied with this derivation, because of the bare differentials floating around, we can equivalently prove that $F = dW/dx$, by noting

$$\frac{dW}{dx} = mv\frac{dv}{dx} = mv\frac{dv}{dt}\frac{dt}{dx} = m\frac{dv}{dt} = F.$$

[2] **Problem 29.** Some more about power.

(a) Use similar reasoning to derive $P = Fv$.

(b) An electric train has a power line that can deliver power $P(x)$, where x is the distance along the track. If the train starts at rest at $x = 0$, find its speed at point x_0 in terms of an integral of $P(x)$. (Hint: try to get rid of the dt's to avoid having to think about the time dependence.)

Solution. (a) First, let's use differentials. Since $P = dW/dt$, we have

$$dW = Fv\,dt.$$

Using the same reasoning as before, $dW = md(v^2)/2 = mv\,dv$, so

$$mv\,dv = mav\,dt.$$

Canceling on both sides, this simplifies to $dv = a\,dt$, which is clearly true. Alternatively, we can use derivatives directly. We have

$$P = \frac{dW}{dt} = mv\frac{dv}{dt} = mva = Fv$$

as desired.

(b) We note that

$$dW = mv\,dv$$

but we also have

$$dW = P\,dt = P\frac{dt}{dx}\,dx = \frac{P}{v}\,dx$$

where we introduced the power of v to convert dt (which we don't want to deal with) to dx. Doing some rearrangement,

$$\int mv^2\,dv = \int P\,dx.$$

Performing the integral, we have

$$v = \left(\frac{3}{m}\int P\,dx\right)^{1/3}.$$

Example 16

A particle is initially at rest, at a distance r_0 from a star of mass M. Write the time it takes the particle to collide with the star, due to gravitational attraction, as a single integral.

Solution

If the particle has mass m, then conservation of energy gives

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{r_0}$$

so that its radial velocity is

$$\frac{dr}{dt} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}.$$

We can write the time taken to collide as

$$T = \int dt = \int_{r_0}^{0} dr\,\frac{dt}{dr} = \frac{1}{\sqrt{2GM}}\int_0^{r_0}\frac{dr}{\sqrt{1/r - 1/r_0}}.$$

It's good practice to write the integral in dimensionless form, so that the dependence of the answer on the dimensionful quantities is manifest. To do this, substitute $x = r/r_0$ to get

$$T = \sqrt{\frac{r_0^3}{2GM}}\int_0^1\sqrt{\frac{x}{1-x}}\,dx.$$

This lets us read off $T^2 \propto r_0^3$, in accordance with Kepler's third law, and required by dimensional analysis as shown in problem 2. In case you're wondering, the value of the remaining integral is $\pi/2$, as can be shown by substituting $x = \sin^2\theta$.

[2] **Problem 30** (Kalda)**.** The deceleration of a boat in water due to drag is given by a function $a(v)$. Given an initial velocity v_0, write the total distance the boat travels as a single integral.

Solution. We have

$$\int dx = \int dv\,\frac{dx}{dv} = \int dv\,\frac{dx}{dt}\frac{dt}{dv} = \int\frac{v\,dv}{a(v)}$$

which is a single integral in terms of the function $a(v)$, as desired. Putting the bounds in, the total distance is

$$\Delta x = \int_{v_0}^{0}\frac{v\,dv}{a(v)}.$$

The signs are correct here, since both dv and $a(v)$ are negative.

[5] **Problem 31.** A particle in a potential well.

(a) Consider a particle of mass m and energy E with potential energy $V(x)$, which performs periodic motion. Write the period of the motion in terms of a single integral over x.

(b) Suppose the potential well has the form $V(x) = V_0(x/a)^n$ for $n > 0$. If the period of the motion is T_0 when it has amplitude A_0, find the period when the amplitude is A, by considering how the integral you found in part (a) scales with A.

(c) Find a special case where you can check your answer to part (b). (In fact, there are two more special cases you can check, one which requires negative n and negative V_0, and one which requires $V(x)$ to be replaced with its absolute value.)

(d) Using a similar method to part (a), write down an integral over θ giving the period of a pendulum with length L in gravity g, without the small angle approximation. Using this, compute the period of the pendulum with amplitude θ_0, up to order θ_0^2. (This result was first published by Bernoulli, in 1749.)

(e) $\star$ Part (d) is the kind of involved computation you might see in a graduate mechanics course. But if you think you're *really* tough, you can go one step further. Consider a mass m oscillating on a spring of spring constant k with amplitude A. Calculate its period of oscillation up to order A^2, accounting for special relativity. (Concretely, assume that the spring force doesn't change the rest mass m, and has a potential $U = kx^2/2$. In relativity, the force $F = -dU/dx$ still obeys $F = dp/dt$, but now $E = \gamma mc^2$ and $p = \gamma mv$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$.)

Solution. (a) The statement of conservation of energy is

$$E = \frac{1}{2}mv^2 + V(x), \quad v = \sqrt{\frac{2(E - V(x))}{m}}.$$

Therefore, the period is

$$T = \int dt = \int \frac{dt}{dx}\, dx = \int \frac{dx}{\sqrt{2(E - V(x))/m}}.$$

To be more precise, we should put the bounds of integration back in. If the lowest and highest values of x are x_{min} and x_{max}, then

$$T = 2\int_{x_{\text{min}}}^{x_{\text{max}}} \frac{dx}{\sqrt{2(E - V(x))/m}}$$

where the factor of two is because this is just half of the oscillation.

(b) The particle can perform periodic motion if at $x = \pm A$, $v = 0$ so $V_0(A/a)^n = E$. Thus

$$T = 2\int_{-A}^{A} \frac{dx}{\sqrt{2(V_0(A/a)^n - V_0(x/a)^n))/m}} \propto \int_{-A}^{A} \frac{dx}{\sqrt{A^n - x^n}}$$

By dimensional analysis, the integral (a function of A) is proportional to $A^{1-n/2}$, so

$$T = T_0\left(\frac{A}{A_0}\right)^{1-n/2}$$

Incidentally, you can also do this problem by dimensional analysis directly on the parameters. At first glance, this is impossible because there are too many dimensionful quantities: E, m, a, V_0, and T, which permit $5-3=2$ dimensionless groups. (Recall from an earlier problem that one can usually get a scaling relation only if there's only 1 dimensionless group.) However, the situation can be saved by noting that V_0 and a only ever appear together in the combination V_0/a^n. So there are only 4 independent dimensionful parameters, and a standard dimensional analysis yields the same result.

(c) The three analytically tractable example are:

- For $n=2$ we have simple harmonic motion, and indeed here the period is independent of amplitude. (Incidentally, can you think of any potentials that aren't simple harmonic, but also have this property?)
- For $n=-1$ we have an inverse square force and $T \propto A^{3/2}$. This makes sense, because it matches the form of Kepler's third law, which gives the general scaling of orbits in inverse square forces. (Here we're considering the degenerate case of a straight-line orbit.)
- For $n=1$ we have a constant force, which doesn't yield oscillations. But the scaling argument of part (b) would still work if we used the potential $V(x)=V_0|x/a|$, which does have oscillations. In this case we predict $T \propto \sqrt{A}$, which makes sense; it corresponds to the usual time-dependence $\Delta x = gt^2/2 \propto t^2$ of uniformly accelerated motion.

That's basically as far as you can go with the functions you learn in high school and college. There are analytic solutions for other n, but they tend to be in terms of exotic special functions. For instance, for $n=4$ the solutions can be written in terms of Jacobi elliptic functions, as you can see here. Of course, since we're not living in the 19^{th} century, you don't need to know about them to do Olympiads, or even most fields of physics research.

(d) Conservation of energy states

$$\frac{1}{2}I\omega^2 = mgL(\cos\theta - \cos\theta_0), \quad I = mL^2$$

which means

$$T = 4\int_0^{\theta_0} \frac{d\theta}{\omega} = 4\int_0^{\theta_0} \frac{d\theta}{\sqrt{(2g/L)(\cos\theta - \cos\theta_0)}}.$$

This form is a bit annoying, because the cosines are both extremely close to 1. To fix this issue, we use the half-angle formula $(1-\cos\theta)/2 = \sin^2(\theta/2)$, to get

$$T = 2\sqrt{\frac{L}{g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}}.$$

This newer form still has the problem that as θ_0 goes to zero, the integration range goes to zero while the integrand can get large, making it hard to see how big the result is. To fix this, change variables from θ to ϕ, where $\sin\phi = \sin(\theta/2)/\sin(\theta_0/2)$, which gives

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\sin^2\phi\,\sin^2(\theta_0/2)}}.$$

This is the easiest form to handle, because it makes the $\theta_0 \to 0$ limit obvious, and it makes it manifest that the first correction is of order θ_0^2. To extract that term, we expand the sine to first order, giving

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} d\phi \left(1 + \frac{\theta_0^2}{8} \sin^2 \phi\right) = 2\pi\sqrt{\frac{L}{g}} \left(1 + \frac{\theta_0^2}{16}\right).$$

There are many other ways to get this result, but this form is probably the best if you're interested in even higher-order terms. For instance, the coefficient of the θ_0^4 term is 11/3072.

(e) This is a taste of the kind of problem you'll see in **R2**. It can get quite messy, but it's not too bad if you work in the right variables. First, note that since $F = -dU/dx$, we still have energy conservation, but with the relativistic energy expression,

$$\gamma mc^2 + \frac{1}{2} m\omega_0^2 x^2 = mc^2 + \frac{1}{2} m\omega_0^2 A^2$$

where $\omega_0^2 = k/m$ as usual. Solving for γ, we find

$$\gamma = 1 + \frac{\omega_0^2}{2c^2}(A^2 - x^2).$$

Next, using the definition of γ, we have

$$T = 4\int_0^A \frac{dx}{v} = \frac{4}{c} \int_0^A \frac{\gamma}{\sqrt{\gamma^2 - 1}}\, dx.$$

At this point we can perform a quick check to make sure we're on the right track. Note that in the ultrarelativistic limit, where the spring is so strong that the mass is always moving at nearly the speed of light, we have $\gamma \to \infty$, so that the integrand just reduces to 1. Then $T \approx 4A/c$, which is exactly as expected.

Anyway, in this problem we're considering small relativistic corrections, $\gamma - 1 \ll 1$. The easiest way to make this manifest is to eliminate γ in favor of A, using our result above. There we found that $\gamma - 1 = O((\omega_0 A/c)^2)$, so we can expand in the small quantity $\omega_0 A/c$, giving

$$T = \frac{4}{c} \int_0^A \frac{c}{\omega_0} \frac{1}{\sqrt{A^2 - x^2}} + \frac{3}{8} \frac{\omega_0}{c} \sqrt{A^2 - x^2} + O((\omega_0 A/c)^4)\, dx.$$

The first term simply recovers the nonrelativistic result $T = 2\pi/\omega_0$, and the second term is straightforward to integrate, yielding

$$T = \frac{2\pi}{\omega_0} \left(1 + \frac{3}{16} \frac{\omega_0^2 A^2}{c^2} + O((\omega_0 A/c)^4)\right).$$

Since the peak speed v_0 is approximately $\omega_0 A$ in the nonrelativistic limit, this result is therefore accurate up to corrections of order $(v_0/c)^4$.

6 Multiple Integrals

It's also useful to know how to set up multiple integrals. This is fairly straightforward, though technically an "advanced" topic, so we'll demonstrate it by example. For further examples, see chapter 2 of Wang and Ricardo, volume 1, or MIT OCW 18.02, lectures 16, 17, 25, and 26.

Idea 9

In most Olympiad problems, multiple integrals can be reduced to single integrals by symmetry.

Example 17

Calculate the area of a circle of radius R.

Solution

The area A is the integral of dA, i.e. the sum of the infinitesimal areas of pieces we break the circle into. As a first example, let's consider using Cartesian coordinates. Then the pieces will be the rectangular regions centered at (x, y) with sides (dx, dy), which have area $dx\,dy$. The area is thus

$$A = \int dA = \int dx \int dy.$$

The only tricky thing about setting up the integral is writing down the bounds. The inner integral is done first, so its bounds depend on the value of x. Since the boundary of the circle is $x^2 + y^2 = R^2$, the bounds are $y = \pm\sqrt{R^2 - x^2}$. Thus we have

$$A = \int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy.$$

We then just do the integrals one at a time, from the inside out, like regular integrals,

$$A = \int_{-R}^{R} 2\sqrt{R^2 - x^2}\, dx = 2R^2 \int_{-1}^{1} \sqrt{1-u^2}\, du = 2R^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = \pi R^2$$

where we nondimensionalized the integral by letting $u = x/R$, and then did the trigonometric substitution $u = \sin\theta$. (To do the final integral trivially, notice that the average value of $\cos^2\theta$ along any of its periods is $1/2$.)

We can also use polar coordinates. We break the circle into regions bounded by radii r and $r + dr$, and angles θ and $\theta + d\theta$. These regions are rectangular, with side lengths of dr and $r\,d\theta$, so the area element is $dA = r\,dr\,d\theta$. Then we have

$$A = \int_0^R r\,dr \int_0^{2\pi} d\theta = 2\pi \int_0^R r\,dr = \pi R^2$$

which is quite a bit easier. In fact, it's so much easier that we didn't even need to use double integrals at all. We could have decomposed the circle into a bunch of thin circular shells, argued that each shell contributed area $(2\pi r)\,dr$, then integrated over them,

$$A = \int_0^R 2\pi r\,dr = \pi R^2.$$

In Olympiad physics, there's usually a method like this, that allows you to get the answer without explicitly writing down any multiple integrals.

Example 18

Calculate the moment of inertia of the circle above, about the y axis, if it has total mass M and uniform density.

Solution

The moment of inertia of a small piece of the circle is

$$dI = x^2 \, dm = x^2 \sigma \, dA = \frac{x^2 M}{\pi R^2} \, dA$$

where x^2 appears because x is the distance to the rotation axis, and σ is the mass density per unit area. Using Cartesian coordinates, we have

$$I = \frac{M}{\pi R^2} \int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} x^2 \, dy.$$

The inner integral is still trivial; the x^2 doesn't change anything, because from the perspective of the dy integral, x is just some constant. However, the remaining integral becomes a bit nasty. In general, when this happens, we can try flipping the order of integration, giving

$$I = \frac{M}{\pi R^2} \int_{-R}^{R} dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} x^2 \, dx.$$

Unfortunately, this is equally difficult. Both of these integrals can be done with trigonometric substitutions, as you'll check below, but there's also a clever symmetry argument.

Notice that I is also equal to the moment of inertia about the x axis, by symmetry. So if we add them together, we get

$$2I = \int (x^2 + y^2) \, dm = \int r^2 \, dm.$$

The r^2 factor has no dependence on θ at all, so the angular integral in polar coordinates is trivial. We end up with

$$2I = \frac{M}{\pi R^2} \int_0^R 2\pi r \, r^2 \, dr = \frac{1}{2} M R^2$$

which gives an answer of $I = MR^2/4$, as expected.

[2] **Problem 32.** Calculate I in the previous example by explicitly performing either Cartesian integral.

Solution. Starting from the second expression in the example,

$$I = \frac{M}{\pi R^2} \int_{-R}^{R} dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} x^2 \, dx = \frac{M}{3\pi R^2} \int_{-R}^{R} 2(R^2 - y^2)^{3/2} \, dy.$$

Let $y = R \sin\theta$. Then we have

$$I = \frac{2MR^2}{3\pi} \int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta.$$

This integral can be done by repeatedly using the double angle formula,

$$\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = \int \left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta = \int_{-\pi/2}^{\pi/2} \left(\frac{1}{4} + \cos(2\theta) + \frac{1}{8} + \frac{1}{8}\cos(4\theta)\right) d\theta = \frac{3\pi}{8}.$$

Personally, I can never remember all the trigonometric formulas, and I usually just expand everything in complex exponentials. Here that method gives a slick solution, as

$$\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = \frac{1}{16}\int_{-\pi/2}^{\pi/2} (e^{i\theta} + e^{-i\theta})^4\,d\theta.$$

Now note that expanding with the binomial theorem gives terms of the form $e^{2in\theta}$ for integers n, which integrate to zero unless $n = 0$. So the only term that matters gives

$$\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = \frac{1}{16}\int_{-\pi/2}^{\pi/2} \binom{4}{2}\,d\theta = \frac{3\pi}{8}.$$

Whichever method you used, we conclude the answer is $I = MR^2/4$, as expected.

[3] **Problem 33.** In this problem we'll generalize some of the ideas above to three dimensions, where we need triple integrals. Consider a ball of radius R.

(a) In Cartesian coordinates, the volume element is $dV = dx\,dy\,dz$. Set up an appropriate triple integral for the volume.

(b) The inner two integrals might look a bit nasty, but we already have essentially done them. Using the result we already know, perform the inner two integrals in a single step, and then perform the remaining integral to derive the volume of a sphere.

(c) In cylindrical coordinates, the volume element is $dV = r\,dr\,d\theta\,dz$. Set up a triple integral for the volume, and perform it. (Hint: this can either be hard, or a trivial extension of part (b), depending on what order of integration you choose.)

(d) In spherical coordinates, the volume element is $dV = r^2\,dr\,\sin\phi\,d\phi\,d\theta$. Set up a triple integral for the volume, and perform it.

(e) Let the ball have uniform density and total mass M. Compute its moment of inertia about the z-axis. (Hint: this can be reduced to a single integral if you use an appropriate trick.)

Solution. (a) By analogy to the two-dimensional case,

$$V = \int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dz.$$

(b) The inner two integrals just represent the area of a circle, formed by slicing the ball along a plane of constant x. Thus, the answer has to be πr^2 where r is the radius of that circle (as we derived explicitly in the example), and in this case $r^2 = R^2 - x^2$. Thus, we have

$$V = \int_{-R}^{R} \pi(R^2 - x^2)\,dx = \pi R^3 \int_{-1}^{1} 1 - x^2\,dx = \frac{4}{3}\pi R^3.$$

(c) By analogy to the two-dimensional case,

$$V = \int_{-R}^{R} dz \int_0^{\sqrt{R^2-z^2}} r\,dr \int_0^{2\pi} d\theta.$$

Again, the inner two integrals look a bit nasty, but they represent nothing more than the area of a circle of radius r, leaving

$$V = \int_{-R}^{R} \pi(R^2 - x^2)\,dx$$

upon which the solution continues just as in part (b).

(d) The first task is to decide what order the integrals appear in. It's probably best to have the dr integral be the outermost one, because surfaces of constant dr are spheres, which are simple; thus the final integral is just an integral over spherical slices, which we know are simple. By comparison, if the last integral were $d\theta$ we would have hemispherical slices, while if it were $d\phi$ we would have slices with a really weird shape. We thus have

$$V = \int_0^R r^2\,dr \int_0^{\pi} \sin\phi\,d\phi \int_0^{2\pi} d\theta.$$

The inner two integrals can be done easily, giving

$$V = 4\pi \int_0^R r^2\,dr = \frac{4}{3}\pi R^3.$$

(e) We are looking for

$$I = \int x^2 + y^2\,dm.$$

By spherical symmetry, the integrals of $x^2\,dm$, $y^2\,dm$, and $z^2\,dm$ are all equal. Thus,

$$I = \frac{2}{3}\int x^2 + y^2 + z^2\,dm$$

but this integral is now easy to do because it has spherical symmetry. We have

$$I = \frac{2}{3}\frac{M}{\frac{4}{3}\pi R^3}\int_0^R 4\pi r^2\,r^2\,dr = \frac{2}{5}MR^2$$

as expected. The same trick can be used to show that the moment of inertia of a spherical shell is $(2/3)MR^2$.

[2] **Problem 34.** Consider a spherical cap that is formed by slicing a sphere of radius R by a plane, so that the altitude from the vertex to the base is h. Find the area of its curved surface using an appropriate integral.

Solution. This is a double integral, where it's best to use spherical coordinates. Recall that the volume element in spherical coordinates was $dV = r^2\,dr\,\sin\phi\,d\phi\,d\theta$. Thus, the area element for a part of this sphere is $dA = R^2 \sin\phi\,d\phi\,d\theta$. The area integral is

$$A = R^2 \int_0^{\cos^{-1}((R-h)/R)} \sin\phi\,d\phi \int_0^{2\pi} d\theta = 2\pi hR.$$

After doing the trivial inner integral, this approach is just slicing the surface by $d\phi$. You can also equivalently solve it by slicing it in dz. In that case the integrand is a bit more complicated, but the bounds are simpler.

Remark

You might be wondering how good you have to be at integration to do Olympiad physics. The answer is: not at all! You need to understand how to set up integrals, but you almost never have to *perform* a nontrivial integral. There will almost always be a way to solve the problem without doing explicit integration at all, or an approximation you can do to render the integral trivial, or the integral will be given to you in the problem statement. The Asian Physics Olympiad takes this really far: despite having some of the hardest problems ever written, they often provide information like "$\int x^n\,dx = x^{n+1}/(n+1) + C$" as a hint! This is because physics competitions are generally written to make students think hard about physical systems, and the integrals are just viewed as baggage.

In fact, plain old AP Calculus probably has harder integrals than Olympiad physics. For example, in those classes everybody has to learn the integral

$$\int \sec x\,dx = \log|\sec x + \tan x| + C$$

which has a long history. When I was in high school, I was shocked by how the trick for doing this integral came out of nowhere; it seemed miles harder than anything else taught in the class. And it is! Historically, it arose in 1569 from Mercator's projection, where it gives the vertical distance on the map from the equator to a given latitude. For decades, cartographers simply looked up the numeric value of the integral in tables, where the Riemann sums had been done by hand. (They had no chance of solving it analytically anyway, since Napier only invented logarithms in 1614.) Gradually, tabulated values of the logarithms of trigonometric functions became available, and in 1645, Bond conjectured the correct result by noticing the close agreement of tabulated values of each side of the equation. Finally, Gregory proved the result in 1668, using what Halley called "a long train of Consequences and Complications of Proportions." So it took almost a hundred years for this integral to be sorted out! (Though to their credit, they had the handicap of not knowing about differentiation or the fundamental theorem of calculus; they were finding the area under the curve with just Euclidean geometry.)

Even though Olympiad physics tries to avoid tough integrals, doing more advanced physics tends to produce them, so physicists often get quite good at integration. By contrast, Spivak's calculus textbook for math majors only covers integration techniques in a single chapter towards the end of the book. He justifies the inclusion of this material by saying:

> Every once in a while you might actually need to evaluate an integral [...] For example, you might take a physics course [...] Even if you intend to forget how to integrate (and you probably will forget some details the first time through), you must never forget the basic methods.

That attitude is why physics students frequently win the MIT Integration Bee.

Problem Solving II: Data and Uncertainty

For more about the theory of uncertainty analysis, see this handout, or for a more introductory take, this handout and this comic. For some entertaining general discussion, see chapters I-5 and I-6 of the Feynman lectures. There is a total of **78** points.

1 Basic Probability

Idea 1

If a quantity X has the probability distribution $p(x)$, that means

$$\text{the probability that } a \leq X \leq b \text{ is } \int_a^b p(x)\, dx.$$

In particular, the total probability has to sum to one, so

$$\int_{-\infty}^{\infty} p(x)\, dx = 1.$$

Using the probability distribution, we can calculate expectation values, i.e. averages. For example, the expectation value of X, also called the mean, is

$$\langle X \rangle = \int_{-\infty}^{\infty} x p(x)\, dx$$

while the expectation value of an arbitrary function of X is

$$\langle f(X) \rangle = \int_{-\infty}^{\infty} f(x) p(x)\, dx.$$

One especially important quantity is the variance of X, defined as

$$\operatorname{var} X = \langle X^2 \rangle - \langle X \rangle^2.$$

The standard deviation is defined by $\sigma_X = \sqrt{\operatorname{var} X}$. It describes how "spread out" the distribution of X is, and it will play an important role in uncertainty analysis.

[1] **Problem 1.** Suppose that x is a length. What are the dimensions of $p(x)$, $\langle X \rangle$, $\operatorname{var} X$, and σ?

Example 1

Trains arrive at a train station every 10 minutes. If I arrive at a random time, and X is the number of minutes I have to wait, what is the standard deviation of X?

Solution

We see that X can be anywhere between 0 and 10, with all possibilities equally likely, so

$$p(x) = \begin{cases} 1/(10\,\mathrm{min}) & 0 \leq x \leq 10, \\ 0 & \text{otherwise} \end{cases}$$

where the denominator guarantees the total probability is 1. For the rest of this example, we'll suppress the units. We have

$$\langle X \rangle = \int_{-\infty}^{\infty} x p(x)\, dx = \int_0^{10} \frac{x}{10}\, dx = 5$$

which makes sense, as I should have to wait half the maximum time on average, and

$$\langle X^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x)\, dx = \int_0^{10} \frac{x^2}{10}\, dx = \frac{100}{3}.$$

Then the standard deviation is

$$\sigma_X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \frac{5}{\sqrt{3}}\,\mathrm{min}.$$

[3] **Problem 2.** Consider an exponentially distributed quantity,

$$p(x) = \begin{cases} a e^{-ax} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Verify that the total probability is 1, and compute the mean and standard deviation. To perform the integrals, you will have to integrate by parts.

[2] **Problem 3.** The purpose of subtracting $\langle X \rangle^2$ in the variance is to make sure it doesn't change when a constant is added to x, since shifting something left or right on the number line shouldn't change its spread. Verify that for any constant c, $\operatorname{var} X = \operatorname{var}(X + c)$.

[3] **Problem 4.** We say X is normally distributed if

$$p(x) \propto e^{-a(x-b)^2}.$$

For simplicity, let's shift X so that it's centered about $x = 0$, so

$$p(x) \propto e^{-ax^2}.$$

You may use the result given in **P1**,

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Find the constant of proportionality in $p(x)$, the mean, and the standard deviation.

[2] **Problem 5.** If two random variables X_1 and X_2 are independent, then

$$\langle X_1 X_2 \rangle = \langle X_1 \rangle \langle X_2 \rangle.$$

Use this result to show that

$$\mathrm{var}(X_1 + X_2) = \mathrm{var}(X_1) + \mathrm{var}(X_2)$$

which implies that the standard deviation "adds in quadrature",

$$\sigma_{X_1+X_2} = \sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2}.$$

This is an important result we'll use many times below.

2 Uncertainty Propagation

In this section, we'll establish the fundamental results needed to compute uncertainties.

Idea 2

When a physical quantity is measured in an experiment and reported as $x \pm \Delta x$, it is uncertain what the true value of the quantity is. If the quantity has a probability distribution $p(x)$, then the reported uncertainty Δx is essentially the standard deviation of $p(x)$.

Remark

In practice, you'll have to use intuition and experience to assign uncertainties for real measurements. For example, if you're using a clock that times only to the nearest second, you might take $\Delta t = 0.5\,\mathrm{s}$. If you're using a good ruler, which has millimeter markings, you might take $\Delta x = 0.5\,\mathrm{mm}$. Of course, the ultimate test is the results: if you assigned the uncertainties right, your final uncertainty should encompass the true result most (but not all) of the time.

[2] **Problem 6.** Suppose x has uncertainty Δx and y has uncertainty Δy, where x and y are independent. Explain why the uncertainty of $x + y$ is

$$\Delta(x + y) = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

This is called "addition in quadrature". What is the uncertainty of $x - y$? How about $x + x$?

Remark

Note how this differs from "high school" uncertainty analysis. In school, you might be told to show uncertainty using significant figures, and when adding two things, to keep only the figures that are significant in both of them. That corresponds to

$$\Delta(x + y) = \max(\Delta x, \Delta y)$$

which is an underestimate. Or, you might be told that the uncertainty needs to encapsulate all the possible values, which implies that

$$\Delta(x + y) = \Delta x + \Delta y$$

which is an overestimate, since the uncertainties could cancel.

Example 2: $F = ma$ 2016 25

Three students make measurements of the length of a 1.50 m rod. Each reports an uncertainty estimate representing an independent random error applicable to the measurement.

- Alice performs a single measurement using a 2.0 m tape measure, to within 2 mm.
- Bob performs two measurements using a wooden meter stick, each to within 2 mm, which he adds together.
- Christina performs two measurements using a machinist's meter rule, each to within 1 mm, which she adds together.

Rank the measurements in order of their uncertainty.

Solution

The uncertainty in Alice's measurement is 2 mm. The uncertainty in Bob's is $2\sqrt{2}$ mm by quadrature, while the uncertainty in Christina's is $\sqrt{2}$ mm by quadrature. So the lowest uncertainty is Christina's, followed by Alice's, followed by Bob's.

[2] **Problem 7.** Given N independent measurements of the same quantity with the same uncertainty, $x_i \pm \Delta x$, find the uncertainty of their sum. Hence show the uncertainty of their average is $\Delta x/\sqrt{N}$.

This result is extremely important, since repeating trials is one of the main ways to reduce uncertainty. But it's important to remember that the results derived above hold only for independent measurements. For example, taking a single measurement, then averaging that single number with itself 100 times certainly wouldn't reduce the uncertainty at all!

Idea 3

If x has uncertainty Δx, and $f(x)$ can be approximated by its tangent line, $f(x') \approx f(x) + (x' - x)f'(x)$ within the region $x \pm \Delta x$, then $f(x)$ has approximate uncertainty $f'(x)\,\Delta x$.

[2] **Problem 8.** If x has uncertainty Δx, find the uncertainties of x^2, $\sqrt{x}$, $1/x$, $1/x^4$, $\log x$, and e^x.

[2] **Problem 9.** The tangent line approximation doesn't always make sense. For example, suppose that x is measured to be zero, up to uncertainty Δx. Show that the above results for the uncertainties of x^2 and $\sqrt{x}$ give nonsensical results. What would be a more reasonable uncertainty to report?

[2] **Problem 10.** Consider two quantities with independent uncertainties, $x \pm \Delta x$ and $y \pm \Delta y$.

(a) Show that the uncertainty of xy is

$$\Delta(xy) = xy\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}.$$

To do this, start by writing xy as $\exp(\log x + \log y)$.

(b) If we set $x = y$, then we find

$$\Delta(x^2) = x^2\sqrt{2\left(\frac{\Delta x}{x}\right)^2} = \sqrt{2}x\Delta x.$$

On the other hand, in a previous problem we found $\Delta(x^2) = 2x\Delta x$. Which result is correct?

(c) Find the uncertainty of x/y.

[2] **Problem 11.** A student launches a projectile with speed $v = 5 \pm 0.1\,\mathrm{m/s}$ in gravitational acceleration $g = 9.81 \pm 0.01\,\mathrm{m/s^2}$. The resulting range is $d = 1.5 \pm 0.02\,\mathrm{m}$. Given that the launch angle was less than 45°, find the launch angle, with uncertainty, assuming all uncertainties are independent.

[2] **Problem 12.** Two physical quantities are related by $y = xe^x$.

(a) If x is measured to be 1.0 ± 0.1, find the resulting value of y, with uncertainty.

(b) If y is measured to be 2.0 ± 0.1, find the resulting value of x, with uncertainty.

Idea 4

For practical computations, it is often useful to use relative uncertainties. The relative uncertainty of x is $\Delta x/x$, and can be expressed as a percentage.

[1] **Problem 13.** Some basic relative uncertainty results.

(a) Show that the relative uncertainty of the product or quotient of two quantities with independent uncertainties is the square root of the sum of the squares of their relative uncertainties.

(b) Show that averaging N independent trials as in problem 7 reduces the relative uncertainty by a factor of $\sqrt{N}$.

Remark

There are many situations where the rules above can't be used. For example, consider the uncertainty of $x+y^2/x$, where x and y have independent uncertainties. You can calculate the uncertainty of either term with the standard rules, but you can't calculate the uncertainty of their sum, because the terms are not independent (both contain x).

In these cases, you can use the multivariable equivalent of the tangent line approximation,

$$f(x', y') \approx f(x, y) + (x' - x)\frac{\partial f}{\partial x} + (y' - y)\frac{\partial f}{\partial y}.$$

Adding the two contributions to the uncertainty in quadrature gives

$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial x}\,\Delta x\right)^2 + \left(\frac{\partial f}{\partial y}\,\Delta y\right)^2}.$$

This is the general rule that includes the rules you derived above as special cases. However, it shouldn't be necessary in Olympiad problems. If you run into such situations in an experiment, often one of the uncertainties is much smaller, and can be neglected entirely.

Remark

As you saw in problem 9, the tangent line approximation can sometimes fail. The proper way to handle situations like these would be to find the full probability distribution of the desired quantity, rather than just describing it crudely with its standard deviation. However, this can't be done analytically except in the simplest of cases. So when professional physicists run into situations like these, which are quite common, they often just numerically compute a few million or billion values, starting with randomly drawn inputs each time, and use that to infer the probability distribution. This technique is called Monte Carlo. It's very powerful, but certainly not needed for Olympiads! On Olympiads, you should just fall back to something reasonable, such as taking the minimum and maximum possible values.

3 Using Uncertainties

Example 3: $F = ma$ 2022 B21

Amora and Bronko are given a long, thin rectangle of sheet metal. (It has been machined very precisely, so they can assume it is perfectly rectangular.) Using calipers, Amora measures the width of the rectangle as 1 cm with 1% uncertainty. Using a tape measure, Bronko independently measures its length as 100 cm with 0.1% uncertainty. What are the relative uncertainties they should report for the area and the perimeter of the rectangle?

Solution

To compute the area, we multiply the two measurements, which means we add the relative uncertainties in quadrature,

$$\frac{\Delta A}{A} = \sqrt{(1\%)^2 + (0.1\%)^2} \approx 1\%.$$

Note that in this case, the relative uncertainty of Bronko's measurement is negligible; the relative uncertainty of the area is approximately the relative uncertainty of Alice's measurement.

Computing the perimeter involves adding the measurements, which means the absolute uncertainties are added in quadrature instead. These are 0.01 cm and 0.1 cm for Alice and Bronko's measurements, respectively, so the absolute uncertainty of Alice's measurement is negligible. Thus, the relative uncertainty of the perimeter is approximately the relative uncertainty of Bronko's measurement, 0.1%.

In simple Olympiad experiments, often only one uncertainty will really matter. This can dramatically simplify calculations, but it might take a little thought to tell which one.

[3] **Problem 14.** Solve $F = ma$ 2018 problems A12, A25, B19, and B25, and $F = ma$ 2019 problems A16, B18, and B25. Make sure to strictly adhere to the total time. Since these are $F = ma$ problems, you don't have to produce a writeup. If you find these questions difficult to finish in the allotted time, go back and review the earlier material!

[2] **Problem 15.** Suppose the goal of an experiment is to measure the ratio T_1/T_2 of the durations of

two physical processes, where T_1 is about 15 seconds, and T_2 is about 3 seconds. Also suppose your stopwatch is only accurate to the nearest second. You have two minutes to perform measurements. Assume each measurement is independent.

(a) Using your instinct, figure out whether it's better to spend more total time measuring T_1, more total time measuring T_2, or an equal amount of time on both.

(b) To confirm this, qualitatively sketch the relative uncertainty of T_1/T_2 as a function of the fraction of time x spent measuring T_1, using explicit numeric examples if necessary.

Calculations of this sort are common when doing Olympiad experimental physics. You should be able to do them instinctively, getting the ballpark right answer without explicit calculation.

[3] **Problem 16.** In the preliminary problem set, you measured g using a pendulum. If you didn't do uncertainty analysis for it, as we covered above, then you should go back and estimate uncertainties more precisely. In this problem you'll do a different experiment: you will estimate g by finding the time needed for an object to roll down a ramp, with everything again made of household materials.

(a) Before starting, think about what the dominant sources of uncertainty will be, and how you can design the experiment to minimize them. In particular, do you think the result will be more or less precise than your pendulum experiment?

(b) Perform the experiment, taking at least ten independent measurements, and report the data and results with uncertainty.

[3] **Problem 17.** [A] Consider N independent measurements of the same quantity, with results $x_i \pm \Delta x_i$. They can be combined into a single result by taking a weighted average. What is the optimal weighted average, which minimizes the uncertainty?

All of the examples above involve combining continuous quantities, so we'll close this section with some applications to "counting" experiments, which work slightly differently.

Remark

In this problem set, we have given rules for calculating the mean and standard deviation of derived quantities. But in general, probability distributions can have all kinds of weird features, which aren't captured by those two numbers. The reason we focus on them anyway is because of the central limit theorem, which roughly states that if we have many independent random variables, the distribution of the sum will approach a normal distribution. As you saw in problem 4, normal distributions are characterized entirely by their mean and standard deviation, so we don't lose any information by reporting only those two quantities.

Example 4

A fair coin is tossed 1000 times, and the number of heads is counted. If this process is repeated many times, what is the standard deviation of the number of heads?

Solution

Consider one trial of 1000 tosses. The number of heads is $X = X_1 + X_2 + \ldots + X_{1000}$, where

$$X_i = \begin{cases} 1 & \text{heads on toss } i \\ 0 & \text{tails on toss } i \end{cases}.$$

Of course, the mean of each of these variables is $\langle X_i \rangle = 0.5$, so that the mean of X is 500. In addition, the X_i are independent of each other, so the variances add. The variance of each one of them is

$$\operatorname{var} X_i = \langle X_i^2 \rangle - \langle X_i \rangle^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Thus, the standard deviation of the number of heads is

$$\sqrt{\operatorname{var} X} = \sqrt{1000/4} \approx 16.$$

So getting 520 heads would not be surprising, but if you got 550, you might be justified in suspecting the coin isn't fair. (Also, the number of heads is very close to normally distributed, by the central limit theorem mentioned above.) To check whether you understand this, you can redo it with a general probability p of getting heads, where you should get $\sqrt{1000\, p(1-p)}$.

[3] **Problem 18.** At any moment, a Geiger counter can click, indicating that it has detected a particle of radiation. Suppose that there is an independent probability $\alpha\, dt$ of clicking at each infinitesimal time interval dt. Let the number of clicks observed in a total time T be X.

(a) Find the expected value and standard deviation of X, and thereby compute its relative uncertainty. (Hint: split the total time into many tiny time intervals, and let X_i be the number of clicks in interval i, so $X = \sum_i X_i$.)

(b) Using a Geiger counter on a sample, you hear 197 clicks in 5 minutes of operation. Estimate the activity α of the sample (i.e. the expected clicks per second), with uncertainty. If you measure for longer, how does the uncertainty reduce over time?

(c) Now suppose that for a different sample, $N = 0$ after 5 minutes. Estimate the activity α of the sample (i.e. the expected clicks per second), with a reasonable uncertainty. If you measure for longer, and continue to hear no clicks, how does the uncertainty reduce over time?

[4] **Problem 19.** [A] This problem extends problem 18 to derive some canonical results.

(a) Let $\lambda = \alpha T$. Find the probability $p(X = k)$ of hearing exactly k clicks in terms of λ and k.

(b) To check your result, show that the sum of the $p(X = k)$ is equal to one.

(c) $\star$ In the limit $\lambda \gg 1$, show that the probabilities $p(X = k)$ approach that of a normal distribution with the mean and standard deviation calculated in problem 18, thereby providing an example of the central limit theorem at work. This is a rather involved calculation, which will use many of the techniques from **P1**. It will also require Stirling's approximation,

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

for $n \gg 1$, which will be important in **T2**. (Hint: because the relative uncertainty falls as λ increases, start by writing $k = \lambda(1+\delta)$ for $|\delta| \ll 1$, and expand in powers of δ. Be careful not to drop too many terms, as δ is small, but $\lambda\delta$ isn't.)

[3] **Problem 20.** IPhO 2023, problem 1, parts A, B, and D.3. A short derivation of the key features of Brownian motion. It requires only the ideas of this problem set, and some basic mechanics.

4 Data Analysis

Idea 5

All data analysis for the USAPhO and IPhO can be done using extremely basic methods. Sometimes, it suffices to just calculate a value based on a single data point, or by cleverly using a pair of data points. When this isn't enough, you'll have to do graphical data analysis, which will usually correspond to drawing a line and measuring its slope and intercept. This is quite limited compared to modern statistical tools, but also can be surprisingly powerful.

Example 5

The activity of a radioactive substance obeys $A(t) = A_0 e^{-t/\tau}$. Using measurements of t and $A(t)$, plot a line to find A_0 and τ.

Solution

To handle exponential relationships, take the logarithm of both sides for

$$\log A(t) = \log A_0 - t/\tau.$$

Then a plot of $\log A(t)$ vs. t has slope $-1/\tau$ and y-intercept $\log A_0$.

[1] **Problem 21.** For a power law $y = \alpha x^n$ where y and x are measured, what line can be plotted to find α and n?

[2] **Problem 22.** The rate R of electron emission from a solid in an electric field E is

$$R = \beta e^{-E/E_0}$$

for some constants β and E_0. The particular form is because the effect is due to quantum tunneling, and you will derive it in **X2**.

(a) If E and R are measured, what line can be plotted to find β and E_0?

(b) Your answer for part (a) should have formally incorrect dimensions, by the standards of **P1**. This often happens when one takes logarithms. What's going on? If the dimensions are wrong, how can the result be right?

(c) Suppose both β and E_0 have 1% uncertainty. For small E, which is more important for the uncertainty of R? What about for large E? Around where is the crossover point?

Example 6

Suppose that y and x are related nonlinearly, as

$$y = bx + ax^2.$$

For example, this could model the force due to a non-Hookean spring. Using measurements of x and y, plot a line to find a and b.

Solution

If we divide by x, we find

$$\frac{y}{x} = ax + b.$$

Therefore, we can plot y/x versus x, which gives a line with slope a and intercept b. More generally, we can plot a line whenever we can rearrange a given relation into the form

$$(\text{known}) = (\text{unknown})(\text{known}) + (\text{unknown})$$

where all four terms can be arbitrarily complicated. In this way, it is possible to turn a lot of very nonlinear relations into lines.

[3] **Problem 23.** Some more examples of finding lines to plot.

(a) Suppose that you are given points (x, y) that lie on a circle centered at $(a, 0)$ with radius r. What line can be plotted to find a and r?

(b) Consider an Atwood's machine with masses m and $M > m$. The acceleration of the machine is measured as a function of M. However, since the pulley has mass, it slows the acceleration of the Atwood's machine, so that

$$a = \frac{M - m}{M + m + \delta m} g.$$

Find a line that can be plotted to find g and δm, assuming m, M, and a are known. This is an example of how plotting a line can separate out a systematic error, i.e. the value of δm, which would be impossible if only one value of M were used.

(c) Suppose an object is undergoing simple harmonic motion with amplitude A and angular frequency ω. Given measurements of the position x and velocity v, what line can be plotted to find A and ω?

[3] **Problem 24.** USAPhO 2012, problem A2. (This one requires basic thermodynamics.)

[3] **Problem 25.** USAPhO 2011, problem A2.

[3] **Problem 26.** INPhO 2018, problem 7. (This one requires basic fluid dynamics.)

[3] **Problem 27** (USAPhO 2024)**.** An experimentalist drives a series RLC circuit with an sinusoidal voltage $V(t) = V_0 \cos \omega t$. In **E6**, you will learn how to show that the voltage across the capacitor, in the steady state, oscillates with amplitude

$$V_c = \frac{V_0}{\sqrt{(1 - \omega^2/\omega_0^2)^2 + (\omega/\omega_0 Q)^2}}$$

where ω_0 is the resonant angular frequency and Q is the circuit's quality factor. The experimentalist takes the following data near the resonance, for a fixed value of V_0:

ω (rad/s)	133.0	133.5	134.0	134.5	135.0	135.5	136.0	136.5	137.0
V_c (Volts)	3.64	4.76	6.52	8.53	8.18	6.06	4.44	3.42	2.75

Find the values of ω_0 and Q as accurately as possible. Uncertainty analysis is not required. (Hint: this is the trickiest data analysis problem in the history of the USAPhO. It *can* be solved by drawing lines, but such a method is relatively inefficient. It is better to carefully approximate the given formula, and to consider just a few data points at a time.)

5 Estimation

Estimation is a useful skill for checking the answers to real-world problems.

Example 7

Estimate the circumference of the Earth.

Solution

If you know that the United States is 3,000 miles wide, and there is a time zone difference of three hours between California and New York, then a reasonable estimate is 24,000 miles. Or, if you know the factoid that light can go about seven times around the Earth in a second, then a reasonable estimate is $(3/7) \times 10^8\,\text{m} \approx 4 \times 10^7\,\text{m}$.

Let's check these results are compatible. There are about 5 miles in 8 kilometers, a fact you can get by remembering how your car's speedometer looks, or by noting that 3 feet are about 1 meter. Then $4 \times 10^4\,\text{km} \approx (5/8) \times 4 \times 10^4\,\text{mi} = 2.5 \times 10^4\,\text{mi}$, so the two results are compatible. There are probably at least a hundred more ways to perform this estimation.

Example 8

Estimate the density of air, and compare this to the density of water.

Solution

We can directly use the ideal gas law, $PV = nRT$. The density is $\rho = \mu n/V$ where μ is the mass of one mole of air, so

$$\rho = \frac{\mu P}{RT}.$$

Atmospheric pressure is about $10^5\,\text{Pa}$, typical temperatures are about $300\,\text{K}$, and air is mostly N_2, which has a molar mass of $\mu = 28\,\text{g/mol}$, so

$$\rho = \frac{(0.028)(10^5)}{(8.3)(300)} \frac{\text{kg}}{\text{m}^3} \approx 1 \frac{\text{kg}}{\text{m}^3}.$$

The density of water is, almost by definition,

$$\rho_w \approx 10^3 \frac{\text{kg}}{\text{m}^3}.$$

Most liquids and solids have densities within an order of magnitude of this, since in all cases the atoms are packed close together. Evidently, air molecules are about a factor of $(10^3)^{1/3} = 10$ times further apart than typical water molecules.

Example 9

Estimate how much useful power you can produce in a short burst.

Solution

This is a bit tricky to test, because most exercises just burn energy against air resistance or friction, which is hard to estimate. However, a task that directly performs work is useful. I weigh about 75 kg and can run up a 3 m high staircase in around 3 s, so

$$P = mgv = (75)(10)(3/3)\,\text{W} \approx 750\,\text{W}.$$

This is a typical max power output, while typical steady state power outputs are several times smaller, and the corresponding numbers for elite athletes are several times larger.

For the below questions, feel free to look up specific numbers if you're stuck. In all cases, an answer to the nearest order of magnitude is good enough.

[3] **Problem 28.** Some questions about light energy.

(a) Estimate the number of photons emitted per second by a standard light bulb. (The energy of a photon is $E = hf$, and the frequency of a photon is related to the wavelength by $c = f\lambda$.)

(b) The Sun supplies power of intensity $1400\,\text{W/m}^2$ to the Earth. The nearest star is about 4 light years away. Assuming this star is similar to the Sun, about how many of its photons enter your eye per second?

[2] **Problem 29.** Estimate the radius of the largest asteroid you could jump off of, and never return.

[4] **Problem 30.** Some questions about energy.

(a) Estimate the digestible energy content of a stick of butter. (A calorie is about 4000 J, and is also the energy needed to raise the temperature of a kilogram of water by 1 K.)

(b) Estimate the rate at which your body burns energy when at rest.

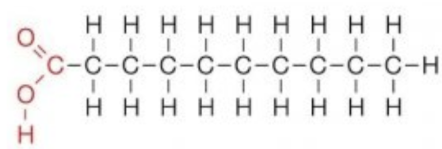
(c) Estimate the rate at which a human being radiates energy. (The Stefan–Boltzmann law states that the radiation power per unit area from a blackbody is σT^4, where $\sigma = 5.7\times10^{-8}\,\text{W/m}^2\text{K}^4$.) Is radiation a significant source of energy loss for a human being, or is it negligible?

(d) A human being develops hypothermia, with their core body temperature dropping by 5 °F. Neglecting any heat transfer with the environment, estimate the number of calories required to raise their temperature back to normal.

Now let's verify the energy content of the butter microscopically. This will be a very rough estimate, so expect answers to be only within two orders of magnitude.

(e) A chemical bond typically involves two electrons, and a characteristic atomic separation distance of one angstrom, $r \sim 10^{-10}\,\text{m}$. Estimate the binding energy of one chemical bond.

(f) The fats in butter are digested by inputting energy to break the bonds in the molecules, then harvesting energy by combining the atoms into CO_2 and H_2O, which have somewhat more stable bonds.



Estimate the energy content of a kilogram of butter. How close is this to the true result?

[2] **Problem 31** (Povey). When human beings lose weight, most of it is by exhalation of carbon. About 20% of the air in the atmosphere is oxygen. When we breathe in and then out, about 25% of the oxygen is converted to carbon dioxide.

(a) Estimate the mass of air contained in a single breath.

(b) Estimate the amount of weight we lose every day by breathing alone.

[2] **Problem 32** (Insight). How long a line can you write with a pencil?

Problem Solving II: Data and Uncertainty

For more about the theory of uncertainty analysis, see this handout, or for a more introductory take, this handout and this comic. For some entertaining general discussion, see chapters I-5 and I-6 of the Feynman lectures. There is a total of **78** points.

1 Basic Probability

Idea 1

If a quantity X has the probability distribution $p(x)$, that means

$$\text{the probability that } a \leq X \leq b \text{ is } \int_a^b p(x)\,dx.$$

In particular, the total probability has to sum to one, so

$$\int_{-\infty}^{\infty} p(x)\,dx = 1.$$

Using the probability distribution, we can calculate expectation values, i.e. averages. For example, the expectation value of X, also called the mean, is

$$\langle X \rangle = \int_{-\infty}^{\infty} x p(x)\,dx$$

while the expectation value of an arbitrary function of X is

$$\langle f(X) \rangle = \int_{-\infty}^{\infty} f(x) p(x)\,dx.$$

One especially important quantity is the variance of X, defined as

$$\operatorname{var} X = \langle X^2 \rangle - \langle X \rangle^2.$$

The standard deviation is defined by $\sigma_X = \sqrt{\operatorname{var} X}$. It describes how "spread out" the distribution of X is, and it will play an important role in uncertainty analysis.

[1] **Problem 1.** Suppose that x is a length. What are the dimensions of $p(x)$, $\langle X \rangle$, $\operatorname{var} X$, and σ?

Solution. Since $p(x)\,dx$ is dimensionless, we have

$$[p(x)] = L^{-1}$$

where L denotes length. Similarly,

$$[\langle X \rangle] = L, \quad [\operatorname{var} X] = L^2, \quad [\sigma_X] = L.$$

Example 1

Trains arrive at a train station every 10 minutes. If I arrive at a random time, and X is the number of minutes I have to wait, what is the standard deviation of X?

Solution

We see that X can be anywhere between 0 and 10, with all possibilities equally likely, so

$$p(x) = \begin{cases} 1/(10\,\text{min}) & 0 \leq x \leq 10, \\ 0 & \text{otherwise} \end{cases}$$

where the denominator guarantees the total probability is 1. For the rest of this example, we'll suppress the units. We have

$$\langle X \rangle = \int_{-\infty}^{\infty} x p(x)\, dx = \int_0^{10} \frac{x}{10}\, dx = 5$$

which makes sense, as I should have to wait half the maximum time on average, and

$$\langle X^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x)\, dx = \int_0^{10} \frac{x^2}{10}\, dx = \frac{100}{3}.$$

Then the standard deviation is

$$\sigma_X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \frac{5}{\sqrt{3}}\,\text{min}.$$

[3] **Problem 2.** Consider an exponentially distributed quantity,

$$p(x) = \begin{cases} a e^{-ax} & x \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Verify that the total probability is 1, and compute the mean and standard deviation. To perform the integrals, you will have to integrate by parts.

Solution. First, to check normalization,

$$\int_{-\infty}^{\infty} p(x) dx = \int_0^{\infty} a e^{-ax} dx = \int_0^{\infty} e^{-u} du = 1 - 0 = 1.$$

Now, the mean can be evaluated using integration by parts,

$$\langle x \rangle = \int_0^{\infty} x a e^{-ax}\, dx = -x e^{-ax}\Big|_0^{\infty} + \int_0^{\infty} e^{-ax}\, dx = 0 - \frac{1}{a} e^{-ax}\Big|_0^{\infty} = \frac{1}{a}.$$

To calculate the standard deviation, we must evaluate

$$\langle x^2 \rangle = \int_0^{\infty} x^2 a e^{-ax} dx = 0 + \int_0^{\infty} (2x) e^{-ax} dx = \frac{2}{a} \langle x \rangle = \frac{2}{a^2}.$$

We thus conclude

$$\sigma_X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \sqrt{\frac{2}{a^2} - \frac{1}{a^2}} = \frac{1}{a}.$$

[2] **Problem 3.** The purpose of subtracting $\langle X\rangle^2$ in the variance is to make sure it doesn't change when a constant is added to x, since shifting something left or right on the number line shouldn't change its spread. Verify that for any constant c, $\operatorname{var} X = \operatorname{var}(X+c)$.

Solution. We have

$$\operatorname{var}(X+c) = \langle (X+c)^2\rangle - \langle X+c\rangle^2.$$

By the definition of the expectation value, we have

$$\langle A+B\rangle = \langle A\rangle + \langle B\rangle, \quad \langle cA\rangle = c\langle A\rangle$$

for any quantities A and B and any constant c. Thus,

$$\operatorname{var}(X+c) = \langle X^2\rangle + \langle 2Xc\rangle + \langle c^2\rangle - \langle X\rangle^2 - 2\langle X\rangle\langle c\rangle - \langle c\rangle^2 = \operatorname{var} X$$

as desired.

[3] **Problem 4.** We say X is normally distributed if

$$p(x) \propto e^{-a(x-b)^2}.$$

For simplicity, let's shift X so that it's centered about $x=0$, so

$$p(x) \propto e^{-ax^2}.$$

You may use the result given in **P1**,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Find the constant of proportionality in $p(x)$, the mean, and the standard deviation.

Solution. Let $p(x) = ke^{-ax^2}$. We fix the constant k by demanding normalization,

$$\int_{-\infty}^{\infty} ke^{-ax^2}\,dx = \int_{-\infty}^{\infty} \frac{k}{\sqrt{a}} e^{-u^2}\,du = 1.$$

Using the provided integral, we conclude

$$k = \sqrt{\frac{a}{\pi}}.$$

The mean is clearly zero, since the distribution is symmetric about that point. Thus, we have

$$\operatorname{var} X = \langle X^2\rangle = \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-ax^2}\,dx = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2}\,du.$$

This remaining integral can be evaluated using integration by parts,

$$\int u^2 e^{-u^2}\,du = -\frac{1}{2} u e^{-u^2}\Big|_{-\infty}^{\infty} + \frac{1}{2}\int e^{-u^2}\,du = 0 + \frac{\sqrt{\pi}}{2}$$

from which we conclude

$$\operatorname{var} X = \frac{1}{a\sqrt{\pi}}\frac{\sqrt{\pi}}{2}, \quad \sigma = \frac{1}{\sqrt{2a}}.$$

[2] **Problem 5.** If two random variables X_1 and X_2 are independent, then

$$\langle X_1 X_2 \rangle = \langle X_1 \rangle \langle X_2 \rangle.$$

Use this result to show that

$$\text{var}(X_1 + X_2) = \text{var}(X_1) + \text{var}(X_2)$$

which implies that the standard deviation "adds in quadrature",

$$\sigma_{X_1+X_2} = \sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2}.$$

This is an important result we'll use many times below.

Solution. By definition, we have

$$\text{var}(X_1 + X_2) = \langle (X_1 + X_2)^2 \rangle - \langle X_1 + X_2 \rangle^2$$

Using the properties listed in problem 3,

$$\begin{aligned}\text{var}(X_1 + X_2) &= \langle X_1^2 \rangle + 2\langle X_1 X_2 \rangle + \langle X_2 \rangle^2 - \langle X_1 \rangle^2 - 2\langle X_1 \rangle \langle X_2 \rangle - \langle X_2 \rangle^2 \\ &= \text{var}(X_1) + \text{var}(X_2) + 2(\langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle)\end{aligned}$$

When X_1 and X_2 are independent, the last term vanishes, giving

$$\text{var}(X_1 + X_2) = \text{var}(X_1) + \text{var}(X_2).$$

2 Uncertainty Propagation

In this section, we'll establish the fundamental results needed to compute uncertainties.

Idea 2

When a physical quantity is measured in an experiment and reported as $x \pm \Delta x$, it is uncertain what the true value of the quantity is. If the quantity has a probability distribution $p(x)$, then the reported uncertainty Δx is essentially the standard deviation of $p(x)$.

Remark

In practice, you'll have to use intuition and experience to assign uncertainties for real measurements. For example, if you're using a clock that times only to the nearest second, you might take $\Delta t = 0.5\,\text{s}$. If you're using a good ruler, which has millimeter markings, you might take $\Delta x = 0.5\,\text{mm}$. Of course, the ultimate test is the results: if you assigned the uncertainties right, your final uncertainty should encompass the true result most (but not all) of the time.

[2] **Problem 6.** Suppose x has uncertainty Δx and y has uncertainty Δy, where x and y are independent. Explain why the uncertainty of $x + y$ is

$$\Delta(x + y) = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

This is called "addition in quadrature". What is the uncertainty of $x - y$? How about $x + x$?

Solution. For independent variables, $\text{var}(X_1 + X_2) = \text{var}(X_1) + \text{var}(X_2)$. Since our uncertainties represent the standard deviation, $\sigma_X = \sqrt{\text{var}(X)}$, we have

$$\Delta(x+y) = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

For $x - y = x + (-y)$, and $\Delta(-y) = \Delta y$, we get that $\Delta(x-y) = \Delta(x+y)$. Finally, by linearity we clearly have $\Delta(x+x) = 2\Delta x$. (The formula above doesn't apply, because x isn't independent of x.)

Remark

Note how this differs from "high school" uncertainty analysis. In school, you might be told to show uncertainty using significant figures, and when adding two things, to keep only the figures that are significant in both of them. That corresponds to

$$\Delta(x+y) = \max(\Delta x, \Delta y)$$

which is an underestimate. Or, you might be told that the uncertainty needs to encapsulate all the possible values, which implies that

$$\Delta(x+y) = \Delta x + \Delta y$$

which is an overestimate, since the uncertainties could cancel.

Example 2: $F = ma$ 2016 25

Three students make measurements of the length of a 1.50 m rod. Each reports an uncertainty estimate representing an independent random error applicable to the measurement.

- Alice performs a single measurement using a 2.0 m tape measure, to within 2 mm.
- Bob performs two measurements using a wooden meter stick, each to within 2 mm, which he adds together.
- Christina performs two measurements using a machinist's meter rule, each to within 1 mm, which she adds together.

Rank the measurements in order of their uncertainty.

Solution

The uncertainty in Alice's measurement is 2 mm. The uncertainty in Bob's is $2\sqrt{2}$ mm by quadrature, while the uncertainty in Christina's is $\sqrt{2}$ mm by quadrature. So the lowest uncertainty is Christina's, followed by Alice's, followed by Bob's.

[2] **Problem 7.** Given N independent measurements of the same quantity with the same uncertainty, $x_i \pm \Delta x$, find the uncertainty of their sum. Hence show the uncertainty of their average is $\Delta x/\sqrt{N}$.

This result is extremely important, since repeating trials is one of the main ways to reduce uncertainty. But it's important to remember that the results derived above hold only for independent measurements. For example, taking a single measurement, then averaging that single number with itself 100 times certainly wouldn't reduce the uncertainty at all!

Solution. The uncertainty of their sum ΔX can be found by adding in quadrature,

$$\Delta X^2 = \sum_i \Delta x^2 = N \Delta x^2$$

which implies $\Delta X = \Delta x / \sqrt{N}$.

Idea 3

If x has uncertainty Δx, and $f(x)$ can be approximated by its tangent line, $f(x') \approx f(x) + (x' - x) f'(x)$ within the region $x \pm \Delta x$, then $f(x)$ has approximate uncertainty $f'(x)\,\Delta x$.

[2] **Problem 8.** If x has uncertainty Δx, find the uncertainties of x^2, $\sqrt{x}$, $1/x$, $1/x^4$, $\log x$, and e^x.

Solution. Differentiate the functions and multiply by Δx to find the uncertainties. The sign isn't important, since uncertainties are always positive. The results are:

$$\Delta(x^2) = 2x\Delta x \qquad \Delta(\sqrt{x}) = \frac{\Delta x}{2\sqrt{x}} \qquad \Delta(1/x) = \frac{\Delta x}{x^2}$$

$$\Delta(1/x^4) = \frac{4\Delta x}{x^5} \qquad \Delta(\log(x)) = \frac{\Delta x}{x} \qquad \Delta(e^x) = e^x \Delta x$$

[2] **Problem 9.** The tangent line approximation doesn't always make sense. For example, suppose that x is measured to be zero, up to uncertainty Δx. Show that the above results for the uncertainties of x^2 and $\sqrt{x}$ give nonsensical results. What would be a more reasonable uncertainty to report?

Solution. The above uncertainties give 0, ∞ for the uncertainties of x^2 and $\sqrt{x}$ respectively. Since the uncertainties were found with $\Delta x \ll x$, now with $x \ll \Delta x$, we can get $(x + \Delta x)^2 - x^2 = 2x\Delta x + \Delta x^2 \approx \Delta x^2$ and $\sqrt{x + \Delta x} - \sqrt{x} \approx \sqrt{\Delta x}$. Thus, more reasonable uncertainties are $(\Delta x)^2$ and $\sqrt{\Delta x}$. There are numerical factors of order 1 because the shapes of the probability distributions will be distorted, but we won't worry about those, because we're just looking to get a reasonable result. (Of course, a professional would keep track of all these details.)

[2] **Problem 10.** Consider two quantities with independent uncertainties, $x \pm \Delta x$ and $y \pm \Delta y$.

(a) Show that the uncertainty of xy is

$$\Delta(xy) = xy \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}.$$

To do this, start by writing xy as $\exp(\log x + \log y)$.

(b) If we set $x = y$, then we find

$$\Delta(x^2) = x^2 \sqrt{2\left(\frac{\Delta x}{x}\right)^2} = \sqrt{2}x\Delta x.$$

On the other hand, in a previous problem we found $\Delta(x^2) = 2x\Delta x$. Which result is correct?

(c) Find the uncertainty of x/y.

Solution. (a) We can write

$$xy = \exp(\log x + \log y)$$

which implies

$$\Delta(xy) = \exp(\log x + \log y)\Delta(\log x + \log y) = xy\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}.$$

(b) The result that $\Delta(x^2) = 2x\Delta x$ is correct, since the formula for $\Delta(xy)$ assumes x, y are independent, which fails when we set $y = x$.

(c) We have

$$\frac{x}{y} = \exp(\log x - \log y)$$

and by a very similar calculation to part (a), we conclude

$$\Delta(x/y) = \frac{x}{y}\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}.$$

[2] **Problem 11.** A student launches a projectile with speed $v = 5 \pm 0.1\,\text{m/s}$ in gravitational acceleration $g = 9.81 \pm 0.01\,\text{m/s}^2$. The resulting range is $d = 1.5 \pm 0.02\,\text{m}$. Given that the launch angle was less than $45°$, find the launch angle, with uncertainty, assuming all uncertainties are independent.

Solution. From the projectile range equation $d = v^2 \sin(2\theta)/g$, we get

$$\theta = \frac{1}{2}\arcsin\left(\frac{dg}{v^2}\right) = 18.03°.$$

To find the uncertainty, we write $\sin(2\theta) = gd/v^2$. The left-hand side is

$$2\cos(2\theta)\Delta\theta$$

by the tangent line approximation. By the result of problem 10, the right-hand side is

$$\frac{dg}{v^2}\sqrt{\left(\frac{\Delta d}{d}\right)^2 + \left(\frac{\Delta g}{g}\right)^2 + \left(\frac{2\Delta v}{v}\right)^2} = 0.0248$$

Combining the results, we have

$$\Delta\theta = 0.015\ \text{rad} = 0.9°$$

which means the final result should be written as

$$\theta = 18.0° \pm 0.9°$$

where we removed a superfluous significant figure.

[2] **Problem 12.** Two physical quantities are related by $y = xe^x$.

(a) If x is measured to be 1.0 ± 0.1, find the resulting value of y, with uncertainty.

(b) If y is measured to be 2.0 ± 0.1, find the resulting value of x, with uncertainty.

Solution. (a) To find the central value of y, we plug in to get $y = e = 2.7183$. To find the error, we use the tangent line approximation,

$$\frac{dy}{dx} = e^x(x+1)$$

which gives us

$$\Delta y \approx e^x(x+1)\Delta x = 0.54.$$

Thus, rounding to a reasonable number of significant figures, we have

$$y = 2.7 \pm 0.5.$$

Note that it would be incorrect to apply the "addition in quadrature" rule for products,

$$\Delta y = xe^x \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta(e^x)}{e^x}\right)^2}$$

because x and e^x aren't independent.

(b) To find the central value of x, we solve the equation $2 = xe^{-x}$ numerically. This can be done using the method of iteration introduced in **P1**. That is, we have $x = 2e^{-x}$, so by repeatedly plugging $2e^{-\mathrm{Ans}}$ into the calculator, we get $x = 0.8526$.

Under the tangent line approximation,

$$\Delta x \approx \frac{\Delta y}{e^x(x+1)} = 0.023.$$

Rounding to a reasonable number of significant figures, we conclude

$$x = 0.85 \pm 0.02.$$

Idea 4

For practical computations, it is often useful to use relative uncertainties. The relative uncertainty of x is $\Delta x/x$, and can be expressed as a percentage.

[1] **Problem 13.** Some basic relative uncertainty results.

(a) Show that the relative uncertainty of the product or quotient of two quantities with independent uncertainties is the square root of the sum of the squares of their relative uncertainties.

(b) Show that averaging N independent trials as in problem 7 reduces the relative uncertainty by a factor of $\sqrt{N}$.

Solution. (a) Above we found that

$$\Delta(xy) = xy\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

Dividing both sides by xy gives

$$\frac{\Delta(xy)}{xy} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

which is the desired result.

(b) We have $\Delta x_N = \Delta(\Sigma x)/N = \Delta x\sqrt{N}/N = \Delta x/\sqrt{N}$. Then

$$\frac{\Delta x_N}{x} = \frac{\Delta x}{x}\frac{1}{\sqrt{N}}$$

as expected.

Remark

There are many situations where the rules above can't be used. For example, consider the uncertainty of $x + y^2/x$, where x and y have independent uncertainties. You can calculate the uncertainty of either term with the standard rules, but you can't calculate the uncertainty of their sum, because the terms are not independent (both contain x).

In these cases, you can use the multivariable equivalent of the tangent line approximation,

$$f(x', y') \approx f(x, y) + (x' - x)\frac{\partial f}{\partial x} + (y' - y)\frac{\partial f}{\partial y}.$$

Adding the two contributions to the uncertainty in quadrature gives

$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial x}\,\Delta x\right)^2 + \left(\frac{\partial f}{\partial y}\,\Delta y\right)^2}.$$

This is the general rule that includes the rules you derived above as special cases. However, it shouldn't be necessary in Olympiad problems. If you run into such situations in an experiment, often one of the uncertainties is much smaller, and can be neglected entirely.

Remark

As you saw in problem 9, the tangent line approximation can sometimes fail. The proper way to handle situations like these would be to find the full probability distribution of the desired quantity, rather than just describing it crudely with its standard deviation. However, this can't be done analytically except in the simplest of cases. So when professional physicists run into situations like these, which are quite common, they often just numerically compute a few million or billion values, starting with randomly drawn inputs each time, and use that to infer the probability distribution. This technique is called Monte Carlo. It's very powerful, but certainly not needed for Olympiads! On Olympiads, you should just fall back to something reasonable, such as taking the minimum and maximum possible values.

3 Using Uncertainties

Example 3: $F = ma$ 2022 B21

Amora and Bronko are given a long, thin rectangle of sheet metal. (It has been machined very precisely, so they can assume it is perfectly rectangular.) Using calipers, Amora measures the width of the rectangle as 1 cm with 1% uncertainty. Using a tape measure, Bronko independently measures its length as 100 cm with 0.1% uncertainty. What are the relative uncertainties they should report for the area and the perimeter of the rectangle?

Solution

To compute the area, we multiply the two measurements, which means we add the relative uncertainties in quadrature,

$$\frac{\Delta A}{A} = \sqrt{(1\%)^2 + (0.1\%)^2} \approx 1\%.$$

Note that in this case, the relative uncertainty of Bronko's measurement is negligible; the relative uncertainty of the area is approximately the relative uncertainty of Alice's measurement.

Computing the perimeter involves adding the measurements, which means the absolute uncertainties are added in quadrature instead. These are 0.01 cm and 0.1 cm for Alice and Bronko's measurements, respectively, so the absolute uncertainty of Alice's measurement is negligible. Thus, the relative uncertainty of the perimeter is approximately the relative uncertainty of Bronko's measurement, 0.1%.

In simple Olympiad experiments, often only one uncertainty will really matter. This can dramatically simplify calculations, but it might take a little thought to tell which one.

[3] **Problem 14.** Solve $F = ma$ 2018 problems A12, A25, B19, and B25, and $F = ma$ 2019 problems A16, B18, and B25. Make sure to strictly adhere to the total time. Since these are $F = ma$ problems, you don't have to produce a writeup. If you find these questions difficult to finish in the allotted time, go back and review the earlier material!

[2] **Problem 15.** Suppose the goal of an experiment is to measure the ratio T_1/T_2 of the durations of two physical processes, where T_1 is about 15 seconds, and T_2 is about 3 seconds. Also suppose your stopwatch is only accurate to the nearest second. You have two minutes to perform measurements. Assume each measurement is independent.

(a) Using your instinct, figure out whether it's better to spend more total time measuring T_1, more total time measuring T_2, or an equal amount of time on both.

(b) To confirm this, qualitatively sketch the relative uncertainty of T_1/T_2 as a function of the fraction of time x spent measuring T_1, using explicit numeric examples if necessary.

Calculations of this sort are common when doing Olympiad experimental physics. You should be able to do them instinctively, getting the ballpark right answer without explicit calculation.

Solution. (a) Since T_2 is smaller, a single measurement of T_2 has a much higher relative uncertainty. Furthermore, T_2 takes less time to measure. This means we definitely want more distinct measurements of T_2 than of T_1. As for how we split up the time, this is a bit harder to judge, but intuitively because uncertainty adds in quadrature, taking a single measurement of each makes T_2's uncertainty not 5 times as bad, but 25 times as bad. So T_2 really completely dominates the uncertainty here, and we should spent most of our time getting its uncertainty down.

(b) We have $\Delta T \approx 1$ s and $\Delta T_i = \Delta T/\sqrt{N_i}$, giving

$$\Delta(T_1/T_2) = \frac{T_1}{T_2}\sqrt{\left(\frac{\Delta T}{T_1\sqrt{N_1}}\right)^2 + \left(\frac{\Delta T}{T_2\sqrt{N_2}}\right)^2}$$

The total time T_t is constant, $N_1 T_1 + N_2 T_2 = T_t$ where $N_1 T_1 / T_t = x$. We want to minimize

$$f(x) = \frac{1}{T_1 x} + \frac{1}{(1-x)T_2}.$$

The derivative is

$$f'(x) = -\frac{1}{T_1 x^2} + \frac{1}{T_2(1-x)^2}$$

and setting this to zero gives

$$x^2(1 - T_1/T_2) - 2x + 1 = 0.$$

The smaller root is the desired one since $x < 1$, giving

$$x = \frac{1 - \sqrt{T_1/T_2}}{1 - T_1/T_2} = \frac{1}{1 + \sqrt{T_1/T_2}}$$

so we should spend 30% of our time measuring T_1. The graph of the uncertainty as a function of x is concave up, with vertical asymptotes at $x = 0$ and $x = 1$.

[3] **Problem 16.** In the preliminary problem set, you measured g using a pendulum. If you didn't do uncertainty analysis for it, as we covered above, then you should go back and estimate uncertainties more precisely. In this problem you'll do a different experiment: you will estimate g by finding the time needed for an object to roll down a ramp, with everything again made of household materials.

(a) Before starting, think about what the dominant sources of uncertainty will be, and how you can design the experiment to minimize them. In particular, do you think the result will be more or less precise than your pendulum experiment?

(b) Perform the experiment, taking at least ten independent measurements, and report the data and results with uncertainty.

Solution. Our formula for g is

$$g = \frac{2\ell(1+\beta)}{t^2 \sin(\theta)}$$

where $\beta = I/MR^2$ of the rolling object, and ℓ, t are the distance and time for the path. Let's assume you found a nice object, like a hollow can or a fully filled one, so that β is known relatively precisely. Then the uncertainty is

$$\Delta g = g\sqrt{\left(\frac{2\Delta t}{t}\right)^2 + \left(\frac{\Delta \ell}{\ell}\right)^2 + \left(\frac{\cos(\theta)\Delta\theta}{\sin(\theta)}\right)^2}.$$

Given the above, you definitely want a ramp as long as possible, and there's a tradeoff with the angle: if the angle is very large, t will be small so that the relative error on t will be large, while if the angle is very small, the relative error on θ will be large. So in practice you want to choose a moderately small, but not too small value of θ.

Some reasonable ballpark numbers are $\theta = (10 \pm 1)^\circ$, and $t = (3 \pm 0.3)\,\mathrm{s}$, so you probably can't easily get an uncertainty smaller than a few percent. The overall result will be less precise than the pendulum experiment, because for the pendulum there is no $\Delta\theta$ term, and you can measure $N \gg 1$ periods in a single trial so that the relative error on t falls as $1/N$. With the ramp, you can partially compensate by doing N separate trials, so that the relative error at best falls as $1/\sqrt{N}$, which isn't as good. It might not even be as good as $1/\sqrt{N}$, because your uncertainties may not be independent: you might systematically overestimate or underestimate the time or angle.

[3] **Problem 17.** [A] Consider N independent measurements of the same quantity, with results $x_i \pm \Delta x_i$. They can be combined into a single result by taking a weighted average. What is the optimal weighted average, which minimizes the uncertainty?

Solution. Let the weights be w_i, so we report the value

$$\overline{x} = \sum_i w_i x_i.$$

The uncertainty obeys

$$(\Delta \overline{x})^2 = \sum_i w_i^2 (\Delta x_i)^2.$$

A tempting but incorrect way to minimize this quantity to set the derivative with respect to w_i equal to zero. This doesn't work because the solution is just $w_1 = \ldots = w_N = 0$, which isn't a weighted average at all. To actually have a weighted average, we need the weights to sum to one,

$$\sum_i w_i = 1.$$

This is an optimization problem with a constraint, which can be solved with Lagrange multipliers.

However, for this particular problem, the constraint is simple enough to handle manually. Because of the constraint, if one increases some weight, then one must decrease others. At the minimum, the effect of increasing any weight infinitesimally and decreasing another the same amount must be zero, as if it weren't, we could just adjust those two weights to get a lower uncertainty. Setting the change in the uncertainty due to adjusting w_i and w_j in this way to zero gives

$$0 = d(w_i^2)(\Delta x_i)^2 + d(w_j^2)(\Delta x_j)^2 = (2\, dw)(-w_i(\Delta x_i)^2 + w_j(\Delta x_j)^2).$$

This tells us that $w_i \propto 1/\Delta x_i^2$, which means

$$w_i = \frac{1/(\Delta x_i)^2}{\sum_j 1/(\Delta x_j)^2}.$$

Note that all measurements are included in the optimal average, no matter how bad they may be.

All of the examples above involve combining continuous quantities, so we'll close this section with some applications to "counting" experiments, which work slightly differently.

Remark

In this problem set, we have given rules for calculating the mean and standard deviation of derived quantities. But in general, probability distributions can have all kinds of weird features, which aren't captured by those two numbers. The reason we focus on them anyway is because of the central limit theorem, which roughly states that if we have many independent random variables, the distribution of the sum will approach a normal distribution. As you saw in problem 4, normal distributions are characterized entirely by their mean and standard deviation, so we don't lose any information by reporting only those two quantities.

Example 4

A fair coin is tossed 1000 times, and the number of heads is counted. If this process is repeated many times, what is the standard deviation of the number of heads?

Solution

Consider one trial of 1000 tosses. The number of heads is $X = X_1 + X_2 + \ldots + X_{1000}$, where

$$X_i = \begin{cases} 1 & \text{heads on toss } i \\ 0 & \text{tails on toss } i \end{cases}.$$

Of course, the mean of each of these variables is $\langle X_i \rangle = 0.5$, so that the mean of X is 500. In addition, the X_i are independent of each other, so the variances add. The variance of each one of them is

$$\operatorname{var} X_i = \langle X_i^2 \rangle - \langle X_i \rangle^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Thus, the standard deviation of the number of heads is

$$\sqrt{\operatorname{var} X} = \sqrt{1000/4} \approx 16.$$

So getting 520 heads would not be surprising, but if you got 550, you might be justified in suspecting the coin isn't fair. (Also, the number of heads is very close to normally distributed, by the central limit theorem mentioned above.) To check whether you understand this, you can redo it with a general probability p of getting heads, where you should get $\sqrt{1000\,p(1-p)}$.

[3] **Problem 18.** At any moment, a Geiger counter can click, indicating that it has detected a particle of radiation. Suppose that there is an independent probability $\alpha\, dt$ of clicking at each infinitesimal time interval dt. Let the number of clicks observed in a total time T be X.

(a) Find the expected value and standard deviation of X, and thereby compute its relative uncertainty. (Hint: split the total time into many tiny time intervals, and let X_i be the number of clicks in interval i, so $X = \sum_i X_i$.)

(b) Using a Geiger counter on a sample, you hear 197 clicks in 5 minutes of operation. Estimate the activity α of the sample (i.e. the expected clicks per second), with uncertainty. If you measure for longer, how does the uncertainty reduce over time?

(c) Now suppose that for a different sample, $N = 0$ after 5 minutes. Estimate the activity α of the sample (i.e. the expected clicks per second), with a reasonable uncertainty. If you measure for longer, and continue to hear no clicks, how does the uncertainty reduce over time?

Solution. (a) There are $N = T/dt$ time intervals. Using the hint and applying linearity of expectation,

$$\langle X \rangle = \sum_i \langle X_i \rangle = N(\alpha\, dt) = \alpha T.$$

Since the X_i are independent, their variances add. The variance of X_i is

$$\langle X_i^2 \rangle - \langle X_i \rangle^2 = \alpha\, dt - (\alpha\, dt)^2 = \alpha\, dt.$$

Thus, by adding the variances, we have

$$\operatorname{var} X = \alpha T$$

so the standard deviation is $\Delta X = \sqrt{\alpha T}$. The relative uncertainty is $\Delta X / \langle X \rangle = 1/\sqrt{\alpha T}$.

(b) Applying the formulas above, we estimate

$$\alpha = \frac{197}{T} = 0.66\,\mathrm{s}^{-1}$$

with an uncertainty of

$$\Delta\alpha = \frac{\alpha}{\sqrt{\alpha T}} = \sqrt{\frac{\alpha}{T}} = 0.05\,\mathrm{s}^{-1}.$$

The uncertainty falls as $1/\sqrt{T}$. Note that this is very similar to previous results we've found, where the uncertainty falls as $1/\sqrt{n}$ where n is the number of trials. In some sense, each instant of time we wait is another trial here.

(c) Of course, we estimate $\alpha = 0$, but then the formulas above imply $\Delta\alpha = 0$ and hence that we are absolutely certain $\alpha = 0$, which is absurd. (If you don't think that's absurd, note that the same result would have occurred if we had heard zero clicks in an *arbitrarily short* time interval, such as a nanosecond.)

This is a case where the basic rules of uncertainty propagation break down, and we need to think. The point of giving an uncertainty is to indicate the range of parameter values compatible with the data we observed. Now, the probability of having no clicks in time T is $e^{-\alpha T}$. If $\alpha T \gg 1$, then it would be very unlikely to have no clicks, so we can rule out $\alpha \gg 1/T$. But if $\alpha T \lesssim 1$, this isn't unlikely at all. Thus, your uncertainty window should be $\alpha \in [0, c/T]$ where c is an order-one number, whose value depends on the specific statistical procedure you use. (Note that the upper bound falls as $1/T$, not $1/\sqrt{T}$.)

[4] **Problem 19.** [A] This problem extends problem 18 to derive some canonical results.

(a) Let $\lambda = \alpha T$. Find the probability $p(X = k)$ of hearing exactly k clicks in terms of λ and k.

(b) To check your result, show that the sum of the $p(X = k)$ is equal to one.

(c) $\star$ In the limit $\lambda \gg 1$, show that the probabilities $p(X = k)$ approach that of a normal distribution with the mean and standard deviation calculated in problem 18, thereby providing an example of the central limit theorem at work. This is a rather involved calculation, which will use many of the techniques from **P1**. It will also require Stirling's approximation,

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

for $n \gg 1$, which will be important in **T2**. (Hint: because the relative uncertainty falls as λ increases, start by writing $k = \lambda(1 + \delta)$ for $|\delta| \ll 1$, and expand in powers of δ. Be careful not to drop too many terms, as δ is small, but $\lambda\delta$ isn't.)

Solution. (a) Following the notation of problem 18, we have $X = \sum_i X_i$, and we get k clicks if precisely k of the X_i are equal to 1. Thus,

$$p(X = k) = \binom{N}{k} (\alpha\, dt)^k \, (1 - \alpha\, dt)^{N-k} \approx \frac{N^k}{k!} (\alpha\, dt)^k \, (1 - \alpha\, dt)^N = \frac{\lambda^k}{k!} e^{-\lambda}.$$

This is known as the Poisson distribution.

(b) This follows from the Taylor series of the exponential,

$$\sum_{k=0}^{\infty} p(X=k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1.$$

(c) Using Stirling's approximation, we have

$$\begin{aligned} p(X=k) &= \frac{1}{\sqrt{2\pi k}} \left(\frac{\lambda e}{k}\right)^k e^{-\lambda} \\ &= \frac{1}{\sqrt{2\pi\lambda(1+\delta)}} \left(\frac{e}{1+\delta}\right)^{\lambda(1+\delta)} e^{-\lambda} \\ &\approx \frac{1}{\sqrt{2\pi\lambda}}\, e^{\delta\lambda}(1+\delta)^{-\lambda(1+\delta)} \end{aligned}$$

where we used the fact that $\delta \ll 1$.

Now we need to use a technique from **P1**. Letting the final term be equal to $1/y$, we have

$$\log y = \lambda(1+\delta)\log(1+\delta) = \lambda(1+\delta)\left(\delta - \frac{\delta^2}{2} + O(\delta^3)\right) = \delta\lambda + \frac{\delta^2\lambda}{2} + O(\delta^3\lambda).$$

In **P1**, we only expanded up to the first term, but here we need to keep the order δ^2 term. The reason is we want an approximation that works for the whole peak of the probability distribution, and we know it has relative uncertainty $1/\sqrt{\lambda}$, which means we need to take $\delta \sim 1/\sqrt{\lambda}$. That implies that $\delta^2\lambda$ is of order one and cannot be dropped, but $\delta^3\lambda$ is small and can be dropped. Anyway, plugging this in, we find

$$p(X=k) \approx \frac{1}{\sqrt{2\pi\lambda}}\, e^{-\delta^2\lambda/2} = \frac{1}{\sqrt{2\pi\lambda}}\, e^{-(k-\lambda)^2/2\lambda}$$

which is precisely a normal distribution with the appropriate mean and standard deviation.

[3] **Problem 20.** IPhO 2023, problem 1, parts A, B, and D.3. A short derivation of the key features of Brownian motion. It requires only the ideas of this problem set, and some basic mechanics.

4 Data Analysis

Idea 5

All data analysis for the USAPhO and IPhO can be done using extremely basic methods. Sometimes, it suffices to just calculate a value based on a single data point, or by cleverly using a pair of data points. When this isn't enough, you'll have to do graphical data analysis, which will usually correspond to drawing a line and measuring its slope and intercept. This is quite limited compared to modern statistical tools, but also can be surprisingly powerful.

Example 5

The activity of a radioactive substance obeys $A(t) = A_0 e^{-t/\tau}$. Using measurements of t and $A(t)$, plot a line to find A_0 and τ.

Solution

To handle exponential relationships, take the logarithm of both sides for

$$\log A(t) = \log A_0 - t/\tau.$$

Then a plot of $\log A(t)$ vs. t has slope $-1/\tau$ and y-intercept $\log A_0$.

[1] **Problem 21.** For a power law $y = \alpha x^n$ where y and x are measured, what line can be plotted to find α and n?

Solution. We have

$$\log(y) = \log(\alpha x^n) = \log(\alpha) + n \log(x).$$

Thus, if we plot $\log y$ against $\log x$, the slope will be n and the y-intercept will be $\log(\alpha)$.

[2] **Problem 22.** The rate R of electron emission from a solid in an electric field E is

$$R = \beta e^{-E/E_0}$$

for some constants β and E_0. The particular form is because the effect is due to quantum tunneling, and you will derive it in **X2**.

(a) If E and R are measured, what line can be plotted to find β and E_0?

(b) Your answer for part (a) should have formally incorrect dimensions, by the standards of **P1**. This often happens when one takes logarithms. What's going on? If the dimensions are wrong, how can the result be right?

(c) Suppose both β and E_0 have 1% uncertainty. For small E, which is more important for the uncertainty of R? What about for large E? Around where is the crossover point?

Solution. (a) Take the natural log of the equation to get

$$\log R = -\frac{E}{E_0} + \log \beta.$$

Plotting E on the x-axis and $\log R$ on the y-axis will give a line with slope $-\frac{1}{E_0}$ and y-intercept $\log \beta$.

(b) This gets into the details of what it even means to plot data. As a simpler case, consider the relationship $y = kx$ where y and x both have units of energy. We can plot y versus x to find the slope k, but in reality, you can't actually plot a dimensionful quantity: what would it even mean to move your pencil a distance of "3.7 J" on a page? Instead, we write y and x as dimensionless multiples of a standard unit of energy. That is, we are actually plotting

$$\frac{y}{E_0} = k \, \frac{x}{E_0}$$

where E_0 is some unit of energy, which is typically 1 J. But we don't bother to write E_0 explicitly because this step is kind of obvious.

Exactly the same thing is going on in this problem, but it looks strange because logarithms have the property $\log(xy) = \log x + \log y$. Both R and β have units of rate, so define a unit of rate R_0 and subtract $\log R_0$ from both sides to get an equation with correct dimensions,

$$\log \frac{R}{R_0} = -\frac{E}{E_0} + \log \frac{\beta}{R_0}.$$

This reflects what we actually do when constructing a log plot, though it is usually left implicit.

(c) The uncertainty in β alone always gives a 1% uncertainty in R. But the uncertainty in R due to the uncertainty in E_0 depends on the value of E. For $E \ll E_0$, we can expand the exponential as $(1 - E/E_0)$, and in this case the uncertainty in E_0 does almost nothing at all, so the uncertainty in β dominates. For $E \gg E_0$, the reverse is true. By dimensional analysis, the crossover must be around $E \sim E_0$.

Example 6

Suppose that y and x are related nonlinearly, as

$$y = bx + ax^2.$$

For example, this could model the force due to a non-Hookean spring. Using measurements of x and y, plot a line to find a and b.

Solution

If we divide by x, we find

$$\frac{y}{x} = ax + b.$$

Therefore, we can plot y/x versus x, which gives a line with slope a and intercept b. More generally, we can plot a line whenever we can rearrange a given relation into the form

$$(\text{known}) = (\text{unknown})(\text{known}) + (\text{unknown})$$

where all four terms can be arbitrarily complicated. In this way, it is possible to turn a lot of very nonlinear relations into lines.

[3] **Problem 23.** Some more examples of finding lines to plot.

(a) Suppose that you are given points (x, y) that lie on a circle centered at $(a, 0)$ with radius r. What line can be plotted to find a and r?

(b) Consider an Atwood's machine with masses m and $M > m$. The acceleration of the machine is measured as a function of M. However, since the pulley has mass, it slows the acceleration of the Atwood's machine, so that

$$a = \frac{M - m}{M + m + \delta m} g.$$

Find a line that can be plotted to find g and δm, assuming m, M, and a are known. This is an example of how plotting a line can separate out a systematic error, i.e. the value of δm, which would be impossible if only one value of M were used.

(c) Suppose an object is undergoing simple harmonic motion with amplitude A and angular frequency ω. Given measurements of the position x and velocity v, what line can be plotted to find A and ω?

Solution. (a) The equation of the circle is

$$(x-a)^2 + y^2 = r^2, \quad y^2 + x^2 = 2ax + r^2 - a^2$$

Plotting $y^2 + x^2$ vs. x will give a slope of $2a$ and a y-intercept of $r^2 - a^2$. Combining the two pieces of information yields a and r.

(b) The equation can be slightly rearranged to give

$$\frac{M-m}{a} = \frac{M+m}{g} + \frac{\delta m}{g}.$$

Therefore, a plot of $(M-m)/a$ vs. $M+m$ has slope $1/g$ and y-intercept $\delta m/g$.

(c) By conservation of energy, $A^2 = x^2 + v^2/\omega^2$, so

$$x^2 = A^2 - v^2/\omega^2.$$

Thus, a plot of x^2 vs. v^2 has y-intercept A^2 and slope $-1/\omega^2$.

[3] **Problem 24.** USAPhO 2012, problem A2. (This one requires basic thermodynamics.)

[3] **Problem 25.** USAPhO 2011, problem A2.

[3] **Problem 26.** INPhO 2018, problem 7. (This one requires basic fluid dynamics.)

Solution. See the official solutions here.

[3] **Problem 27** (USAPhO 2024)**.** An experimentalist drives a series RLC circuit with an sinusoidal voltage $V(t) = V_0 \cos \omega t$. In **E6**, you will learn how to show that the voltage across the capacitor, in the steady state, oscillates with amplitude

$$V_c = \frac{V_0}{\sqrt{(1-\omega^2/\omega_0^2)^2 + (\omega/\omega_0 Q)^2}}$$

where ω_0 is the resonant angular frequency and Q is the circuit's quality factor. The experimentalist takes the following data near the resonance, for a fixed value of V_0:

ω (rad/s)	133.0	133.5	134.0	134.5	135.0	135.5	136.0	136.5	137.0
V_c (Volts)	3.64	4.76	6.52	8.53	8.18	6.06	4.44	3.42	2.75

Find the values of ω_0 and Q as accurately as possible. Uncertainty analysis is not required. (Hint: this is the trickiest data analysis problem in the history of the USAPhO. It *can* be solved by drawing lines, but such a method is relatively inefficient. It is better to carefully approximate the given formula, and to consider just a few data points at a time.)

5 Estimation

Estimation is a useful skill for checking the answers to real-world problems.

Example 7

Estimate the circumference of the Earth.

Solution

If you know that the United States is 3,000 miles wide, and there is a time zone difference of three hours between California and New York, then a reasonable estimate is 24,000 miles. Or, if you know the factoid that light can go about seven times around the Earth in a second, then a reasonable estimate is $(3/7) \times 10^8\,\text{m} \approx 4 \times 10^7\,\text{m}$.

Let's check these results are compatible. There are about 5 miles in 8 kilometers, a fact you can get by remembering how your car's speedometer looks, or by noting that 3 feet are about 1 meter. Then $4 \times 10^4\,\text{km} \approx (5/8) \times 4 \times 10^4\,\text{mi} = 2.5 \times 10^4\,\text{mi}$, so the two results are compatible. There are probably at least a hundred more ways to perform this estimation.

Example 8

Estimate the density of air, and compare this to the density of water.

Solution

We can directly use the ideal gas law, $PV = nRT$. The density is $\rho = \mu n/V$ where μ is the mass of one mole of air, so

$$\rho = \frac{\mu P}{RT}.$$

Atmospheric pressure is about 10^5 Pa, typical temperatures are about 300 K, and air is mostly N_2, which has a molar mass of $\mu = 28\,\text{g/mol}$, so

$$\rho = \frac{(0.028)(10^5)}{(8.3)(300)} \frac{\text{kg}}{\text{m}^3} \approx 1\,\frac{\text{kg}}{\text{m}^3}.$$

The density of water is, almost by definition,

$$\rho_w \approx 10^3\,\frac{\text{kg}}{\text{m}^3}.$$

Most liquids and solids have densities within an order of magnitude of this, since in all cases the atoms are packed close together. Evidently, air molecules are about a factor of $(10^3)^{1/3} = 10$ times further apart than typical water molecules.

Example 9

Estimate how much useful power you can produce in a short burst.

Solution

This is a bit tricky to test, because most exercises just burn energy against air resistance or friction, which is hard to estimate. However, a task that directly performs work is useful. I weigh about 75 kg and can run up a 3 m high staircase in around 3 s, so

$$P = mgv = (75)(10)(3/3)\,\mathrm{W} \approx 750\,\mathrm{W}.$$

This is a typical max power output, while typical steady state power outputs are several times smaller, and the corresponding numbers for elite athletes are several times larger.

For the below questions, feel free to look up specific numbers if you're stuck. In all cases, an answer to the nearest order of magnitude is good enough.

[3] **Problem 28.** Some questions about light energy.

(a) Estimate the number of photons emitted per second by a standard light bulb. (The energy of a photon is $E = hf$, and the frequency of a photon is related to the wavelength by $c = f\lambda$.)

(b) The Sun supplies power of intensity $1400\,\mathrm{W/m^2}$ to the Earth. The nearest star is about 4 light years away. Assuming this star is similar to the Sun, about how many of its photons enter your eye per second?

Solution. Before we continue, it's important to note that for estimation questions, one should only expect an answer to within an order of magnitude. Some teachers tweak their example calculations until they give almost exactly the right answer. This makes them look brilliant, but it's deceptive, because then when the student tries to do the same, their results will be much further off. So to combat this, in all solutions here, we've just presented our very first, simplest guesses. They can be up to an order of magnitude off from the real numbers, so if your numbers are within *two* orders of magnitude of ours, you're fine!

(a) We can estimate a standard light bulb to have around 50 W of power. The power $P = NE$ where N is the number of photons emitted per second, and the wavelength of visible light is from $400 - 700$ nm (we can use 500). Then

$$N = \frac{P\lambda}{hc} \approx 10^{20} \text{ photons/s}$$

(b) 1 AU is about 1.5×10^{11} m. (If you forget, you can use something like $GM_S/r^2 = (2\pi/T)^2 r$, where T is one year and $M_S \approx 2 \times 10^{30}$ kg). 1 light year is $c \times$ 1 year $\approx 9.5 \times 10^{15}$ m. Then the intensity from the star is reduced by a factor of $(1 \text{ AU}/4 \text{ ly})^2$ due to the inverse square law, so $I \approx 3.5 \times 10^{-7}$ W/m^2. The area of a human pupil depends on the light conditions, but is roughly $\pi r^2 = \pi(5\,\mathrm{mm})^2$. Then the number of photons that enter it per second is $P\lambda/hc$, which gives $N \approx 10^7$ photons/s. That's plenty, so it's very easy to see such a star at night, while it might be difficult during the day because of the background light from the Sun.

[2] **Problem 29.** Estimate the radius of the largest asteroid you could jump off of, and never return.

Solution. The escape velocity is $v = \sqrt{2GM/R}$, and we will assume a uniform spherical asteroid with density ρ. Rock is probably a few times denser than water, so $\rho \approx 3 \times 10^3$ kg/m^3 and $M \approx \frac{4}{3}\pi\rho R^3$. Humans can jump around half a meter, which determines $v = \sqrt{2gh}$. Thus

$$2gh = \frac{2G}{R}\frac{4}{3}\pi\rho R^3.$$

Since $g \approx \pi^2$ in SI units, this simplifies to

$$R \approx \sqrt{\frac{3\pi h}{4G\rho}} \approx 2 \text{ km}.$$

[4] **Problem 30.** Some questions about energy.

(a) Estimate the digestible energy content of a stick of butter. (A calorie is about 4000 J, and is also the energy needed to raise the temperature of a kilogram of water by 1 K.)

(b) Estimate the rate at which your body burns energy when at rest.

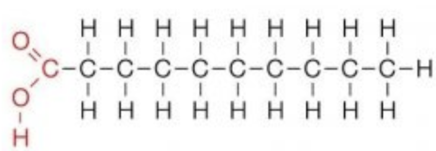
(c) Estimate the rate at which a human being radiates energy. (The Stefan–Boltzmann law states that the radiation power per unit area from a blackbody is σT^4, where $\sigma = 5.7 \times 10^{-8}\,\text{W/m}^2\text{K}^4$.) Is radiation a significant source of energy loss for a human being, or is it negligible?

(d) A human being develops hypothermia, with their core body temperature dropping by 5 °F. Neglecting any heat transfer with the environment, estimate the number of calories required to raise their temperature back to normal.

Now let's verify the energy content of the butter microscopically. This will be a very rough estimate, so expect answers to be only within two orders of magnitude.

(e) A chemical bond typically involves two electrons, and a characteristic atomic separation distance of one angstrom, $r \sim 10^{-10}$ m. Estimate the binding energy of one chemical bond.

(f) The fats in butter are digested by inputting energy to break the bonds in the molecules, then harvesting energy by combining the atoms into CO_2 and H_2O, which have somewhat more stable bonds.



Estimate the energy content of a kilogram of butter. How close is this to the true result?

Solution. (a) Recall the usual "2000 calories per day diet" you see on the nutrition facts for food. Note that those calories are referring to kilocalories ($\sim$ 4000 J). Eating a few sticks of butter will probably make me feel quite full for a day (and disgusted), so a stick of butter probably has around 500-1000 kcal of digestive energy content (let's use 800, which is close to the actual value). Then $E = 800 \text{ kcal} \times 4000 \text{ J/kcal} \approx 3 \times 10^6$ J.

(b) Again, we will use what we see on the nutrition facts: 2000 cal/day$\approx$ 100 W. This energy is used to maintain homeostasis in your body, and it eventually gets exhausted as heat.

(c) First we approximate the surface area of a human, then assume a spherical human that's a perfect blackbody. Our height is about 1.7 m, and our width is around 0.25 m and negligible thickness. Then the surface area is around $2 * 1.7 * 0.25 \approx 1 \text{ m}^2$ (rounding up makes more sense for thickness and limbs). Then using the Stefan–Boltzmann law, $P = A\sigma T^4$. Humans skin is on the order of 300 K, so $P \approx 500$ W.

This is much too high, as it can't possible be higher than (b). The main difference is that the radiation output by the human body is almost completely cancelled by the radiation input by the environment, which is at almost the same temperature (in absolute terms). For example, in typical room-temperature conditions, the environment is at 70° F and human skin is at 90° F, for a difference of about 10 K. So the power is smaller by a factor of $1 - (290/300)^4 = 0.13$, giving a reasonable 65 W. It's still a significant contribution, but not unreasonably large. Of course, in colder environments one can reduce this contribution by, e.g. wearing clothes.

(d) 5°F is 10/9°C ≈ 1°C. Now we use $Q = mcT$, and since humans are mostly water, we'll approximate the specific heat to be the same as water. The mass of humans is usually around 60 kg. Since the "food calorie" is a kilocalorie (amount of energy needed to raise 1 kg by 1° C), we need 60 food calories to raise our temperature back to normal.

(e) A basic estimate for the binding energy is

$$E \sim \frac{e^2}{4\pi\epsilon_0 r} \sim 2 \times 10^{-18}\,\mathrm{J}.$$

As a check, this is about 10 eV, and the binding energy of hydrogen is about 13.6 eV (one of those classic numbers you should remember), so this is in the right ballpark. Of course, the energy is actually *negative*, even though electrons repel, because it's due to how the electrons are attracted to the nuclei. We can, however, very roughly estimate this negative energy using the positive energy of repulsion $e^2/4\pi\epsilon_0 r$ because all energy scales in the problem should be roughly similar.

Actually, in reality the answer should be about an order of magnitude lower, for two reasons. The first is simply that atomic separations are a bit bigger, but this is cancelled by the fact that the nuclei have charge $Z_i > 1$. The main issue is that covalent bonds are a bit more subtle.

Naively, you could say that a covalent bond is attractive because the electrons in one atom are attracted to the nuclei of the other. But this is too naive, because at least parametrically, it's cancelled out by the repulsion of the nuclei with each other, and the repulsion of the electrons with each other, as all four of these terms are of order $\pm e^2/4\pi\epsilon_0 r$. Covalent bonds are stable because the electron orbitals can deform a bit, so that the negative contributions end up a bit bigger than the positive ones. So $e^2/4\pi\epsilon_0 r$ isn't really an estimate for the binding energy, but for the sizes of terms which *mostly* cancel out to give the binding energy, which is why the real answer is about 10 times smaller.

(f) Fats are mostly carbon. As a very rough estimate let's say that the carbon atoms end up in bonds that are twice as stable as before, so the energy released per carbon atom is on the order of magnitude of what we found in part (e). Then

$$\frac{\text{energy}}{\text{kilogram}} = \frac{\text{energy}}{\text{C atom}} \frac{\text{C atoms}}{\text{mole}} \left(\frac{\text{kilograms}}{\text{mole}}\right)^{-1} \sim (2 \times 10^{-18}\,\mathrm{J})\, N_A \left(\frac{12\,\mathrm{g}}{\text{mole}}\right)^{-1} = 10^8\,\mathrm{J/kg}.$$

For comparison, the energy of one gram of fat is 9 calories, so the true answer is

$$(9)(4000)(1000)\frac{\mathrm{J}}{\mathrm{kg}} = 3.6 \times 10^7\,\mathrm{J/kg}$$

which is not too far off!

[2] **Problem 31** (Povey)**.** When human beings lose weight, most of it is by exhalation of carbon. About 20% of the air in the atmosphere is oxygen. When we breathe in and then out, about 25% of the oxygen is converted to carbon dioxide.

(a) Estimate the mass of air contained in a single breath.

(b) Estimate the amount of weight we lose every day by breathing alone.

Solution. (a) If I don't take a deep breath, I can barely blow up a crushed plastic water bottle (holds half a liter of volume), so I would estimate the volume in a single breath to be around 0.5 L. From chemistry class (or ideal gas law: $n = PV/RT$), we know that mole of gas takes up 22.4 liters of volume at STP (our body temperature, 310 K isn't that much more than 273 K but we can just use $22.4 \times 310/273 \approx 25$ L). Most of the air is nitrogen (N_2) with molecular mass 28 g/mol (oxygen, O_2, is 32 which is pretty close). Then one breath should have a mass of 0.5 L/25 L/mol $\times$ 28 g/mol ≈ 0.6 g.

(b) By counting, we can estimate humans to breathe around 10 to 15 times a minute, so let's use 12.5, giving around 20,000 breaths in a day. In each breath, 20% of the air is oxygen, and 25% of the oxygen is convert to carbon dioxide, for a net fraction of 5%. Carbon dioxide (CO_2) has a molecular mass of 48 g/mol, and oxygen is 32. Thus we lose a proportion of $(48/32 - 1) \times 0.05 = 0.5 = 0.025$ of the mass of the air we breathe in every day, which is about 0.3 kg. Most of the (non-water) mass of the food we eat leaves this way.

[2] **Problem 32** (Insight)**.** How long a line can you write with a pencil?

Solution. Graphene, a layer of carbons arranged in a hexagonal way, famously can be made from using scotch tape to extract a few layers of graphite from pencil markings. It'll take plenty of tries to erase pencil from paper with tape probably (but progress is definitely noticeable), so we can estimate there to be around 100 layers of the hexagonal carbon from graphite.

We will assume that the line is drawn with the pencil perfectly vertical and the lead not sharpened. The diameter of the lead is around 2 mm, and the mass of a pencil should be around 2-10 grams, so the mass of the lead is on the order of 1 g. Assuming that the lead is almost all made out of carbon, we can estimate how many carbon atoms it has, and the surface density of carbon atoms.

The carbons are spread apart in a hexagonal fashion with a characteristic distance of $r \approx 10^{-10}$ m, and the centers of 3 adjacent hexagons will have a carbon atom at its center, giving a spacing of 1 carbon atom every r^2 square meters. There should be

$$6.022 \times 10^{23} \text{atoms/mole} \times 1 \text{ g} \times \frac{1}{12 \text{ g/mol}} \approx 5 \times 10^{22} \text{ atoms of C}$$

Thus that gives around 500 m^2 of a single layer of carbon, so around 5 m^2 of lead usage. The line will be approximately a rectangle with area $d\ell$, where d is the diameter of 2 mm.

Thus the pencil line should be around 2.5 km long. One can find other estimates of the order 50 km, i.e. a spread of an order of magnitude. The precise result within this order of magnitude of course depends on the details of the pencil.

Preliminary Problems

These basic problems should be approachable if you understand the core material in Halliday, Resnick, and Krane. If you can solve at least 75% of these completely and correctly, you're ready to start the main problem sets. Work carefully: many of the problems are more subtle than they look. Answers are provided for most questions, so you can check your work; solutions are deliberately not provided, so that you have the chance to work them out for yourself.

1 Mechanics

These problems can be solved using the material in chapters 1 through 17 of Halliday and Resnick.

[2] **Problem 1.** As a warmup, take the Force Concept Inventory test. This should take no longer than 30 minutes. You do not have to justify your answers; just list them.

[1] **Problem 2.** A projectile is thrown up and passes point A, then point B a height h above A. Let T_A and T_B be the time intervals between the two times the projectile passes A and B, respectively.

(a) Show that an experimentalist can measure g by computing $g = 8h/(T_A^2 - T_B^2)$.

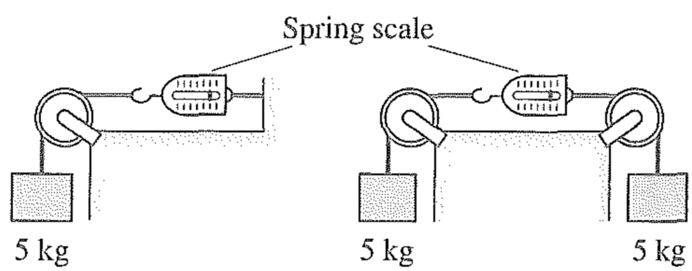
(b) This procedure probably looks a little contrived. Why is it better than doing something simpler, such as just dropping the ball and using $\Delta y = gt^2/2$?

[2] **Problem 3.** A projectile is launched with speed v at an angle θ from the horizontal on a flat plane.

(a) Find $y(x)$ and the ratio of the range to the maximum height.

(b) What is the maximum θ for which the projectile always increases its distance from the thrower?

[1] **Problem 4.** Consider the two following setups involving pulleys and spring scales. Treat the ropes and spring scales as massless, and the pulleys as frictionless. Model the pulleys as uniform discs with masses of 5 kg which are fixed by a rigid support.



(a) Draw a free-body diagram for the second setup, showing all external forces on the spring scale.

(b) What are the readings on the two spring scales?

(c) Draw a free-body diagram for the first setup, showing and naming all external forces on the pulley. In particular, what is the magnitude of the force between the pulley and the rope?

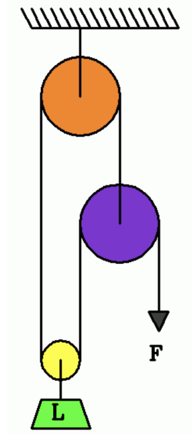
(d) What is the magnitude of the force that must be provided by the support?

[1] **Problem 5.** A sewage worker is using a ladder inside a large, frictionless, horizontal circular aqueduct. The ladder is of the same length as the diameter of the aqueduct.

(a) First the ladder is placed perfectly vertically and the worker climbs to the midpoint. Draw a free body diagram indicating all forces on the ladder, and their names. Do the forces balance?

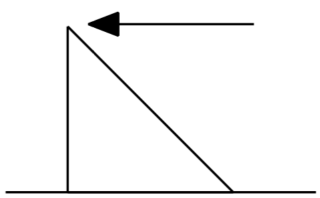
(b) Now suppose the ladder is placed perfectly horizontally and the worker hangs statically from the midpoint. Draw a free body diagram indicating all forces on the ladder, and their names. Do the forces balance? If so, show this explicitly. If not, what happens next?

[1] **Problem 6.** A student proposes to build this set of pulleys and string, called a "fool's tackle".



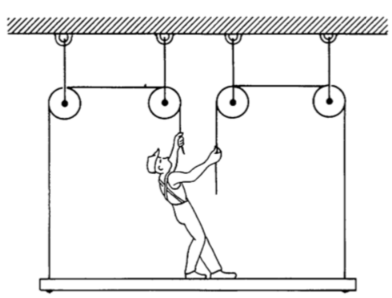
If the load L has mass m, and the pulleys and strings are frictionless and have negligible mass, find the force F needed to keep the system static.

[2] **Problem 7.** A wooden isosceles right triangle with uniform mass density is placed on a table, and a force is applied as shown.



The force is gradually increased until the triangle begins to tip over without sliding. The force is then removed. Next, the surface is inclined with angle θ. For what range of θ can you be certain the triangle will not slide down the incline?

[1] **Problem 8.** A painter of mass M stands on a platform of mass m as shown.



He pulls each rope down with force F, and accelerates upward with acceleration a. Find a.

[1] **Problem 9.** A wheel of radius R is rolling without slipping with angular velocity ω. What is the average speed of a point on the rim?

[3] **Problem 10.** A small block lies at the bottom of a spherical bowl of radius R.

(a) Find the period of small oscillations, assuming no friction. Can you give an intuitive explanation of the simplicity of your answer?

(b) What is the period if the block is replaced with a small uniform ball, with sufficient friction to roll without slipping?

(c) Now suppose the block moves in a circle, staying a constant height $h \ll R$ above the bottom of the bowl. Find the period of the motion, assuming no friction.

(d) Again, find the period if the block is replaced with a small uniform ball, with sufficient friction to roll without slipping.

(e) Now consider part (c) again. Suppose that at some moment, the speed of the block is instantaneously increased by a small amount. Qualitatively describe the subsequent motion, e.g. sketch what a top-down view would look like. What if h is not small?

[2] **Problem 11.** A car accelerates uniformly from rest. Initially, its door is slightly ajar. Calculate how far the car travels before the door slams shut. Assume the door has a frictionless hinge, a uniform mass distribution, and a length L from front to back.

[2] **Problem 12.** Two diametrically opposite points on a ring of mass M and radius R are marked out. The ring is placed at rest on a frictionless floor. An ant of mass m starts at one point, then walks horizontally along the ring's edge to the other. Through what total angle does the ring turn?

[2] **Problem 13.** A baseball player holds a bat, modeled as a uniform rigid rod, horizontally at one of its ends. Usually, when the baseball hits the bat, the player will feel a sharp jolt in their hands as the bat recoils. This can be avoided if the baseball hits the "sweet spot". Where is it?

[3] **Problem 14.** Maxwell's wheel is a toy which demonstrates conservation of energy. It consists of a uniform disc of mass M and radius R, with a massless axle of radius $r \ll R$.

If the wheel is released from rest, it falls downward, in such a way so that the strings supporting it are always approximately vertical.

(a) Find the approximate acceleration during descent.

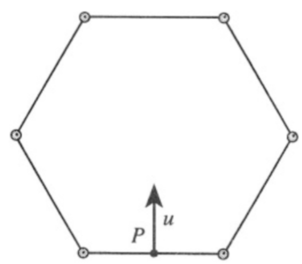
(b) Suppose each string has length $L \gg R^2/r$. When the string completely unravels, the wheel turns around and starts climbing back up. During the turn around, what is the maximum magnitude of the acceleration of the wheel?

[3] **Problem 15.** A cue ball is a uniform sphere of radius R.

(a) Find the height at which the cue ball must be hit horizontally so that it immediately begins rolling without slipping.

(b) Skillful players can hit the cue ball so that it begins moving forwards, but then ends up moving backwards. Model the hit as an instantaneous impulse applied at an arbitrary point on the back half of the cue ball, in an arbitrary direction. For what impulses will this trick work? You can treat the situation as two-dimensional; justify your answer carefully.

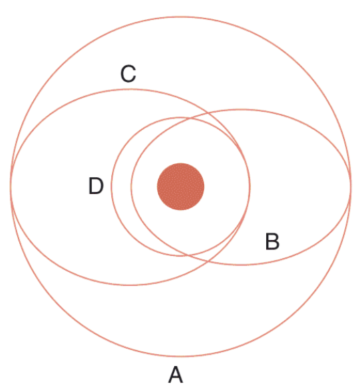
[4] **Problem 16.** Six identical uniform rods, fastened at their ends by frictionless massless pivots, form a regular hexagon and lie on a frictionless surface.



A blow is given at a right angle to the midpoint of the bottom rod. Immediately afterward, the bottom rod has velocity u, as shown. Find the speed of the opposite rod at this moment.

This is the hardest question in this problem set. It is intended to give you a taste of the trickier problems in the main problem sets, but it's definitely not a prequisite to start them. The shortest solution I'm aware of uses just symmetry and elementary Newtonian mechanics.

[2] **Problem 17.** Several possible elliptical orbits of a satellite are shown below.



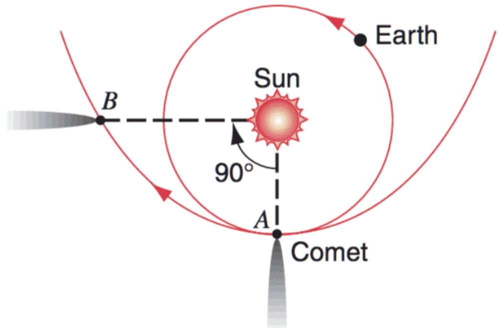
(a) Which orbit has the most angular momentum?

(b) Which orbit has the highest total energy?

(c) On which orbit is the largest speed acquired?

In all cases, justify your answer carefully.

[2] **Problem 18.** A comet passes by Sun as shown, in a parabolic path.



How long, in years, does the comet take to get from point A to point B? (Hint: if you apply Kepler's laws and properties of conics, this problem can be done with almost no computation.)

[2] **Problem 19.** Because of the rotation of the Earth, the line of a plumb bob will not align with the local gravitational field. Find the (small) angle of deviation between them as a function of the latitude θ, the gravitational acceleration g, the radius R of the Earth, and its angular velocity ω.

[2] **Problem 20.** You should be comfortable with setting up multiple integrals. Consider a cylindrical shell whose axis of symmetry is the z-axis. It has non-uniform mass per unit area $\sigma(\phi, z)$ in cylindrical coordinates, and the shell has radius a and height h, with the bottom edge at $z = 0$.

(a) Write down an integral that gives the total mass of the shell.

(b) Write down an integral that gives the moment of inertia of the shell about the z-axis.

(c) How would these results change if we had a solid cylinder with mass per unit volume $\rho(\phi, z, r)$?

[2] **Problem 21.** An entrepreneur proposes to propel the Earth through space by attaching many balloons to one side of it with ropes. The balloons will experience a buoyant force, which will create a tension in the ropes. Now consider the forces on the solid Earth. Because the atmospheric pressure on the surface is uniform, the only net force on the Earth is from the tension, so the Earth will get propelled.

Is this correct? If you think it is, explain why momentum conservation isn't violated. If you think it isn't, identify the specific force acting on the solid Earth that cancels the tension force.

2 Problem Solving Skills

You should be able to start these questions without any background reading. However, for some fun background on estimation, see *Guesstimation: Solving the World's Problems on the Back of a Napkin*. For practical tips for real experiments, see chapter 7 of *Physics Olympiad: Basic to Advanced Exercises*.

[2] **Problem 22.** Argon atoms are special because they stay in the atmosphere for a very long time. They are not recycled like oxygen and nitrogen. An average breath inhales around 0.5 L of air and people breath on average around once every five seconds. Air is about 1% argon and has density $1.2\,\mathrm{kg/m^3}$. Assume all air particles have a mass of approximately 5×10^{-26} kg. Take the atmosphere to have constant density and be around 20 km thick. The radius of the Earth is 6.4×10^6 m.

(a) Estimate the total number of distinct argon atoms inhaled by Galileo throughout their life.

(b) Assuming the atmosphere has been uniform mixed since then, estimate the number of argon atoms in each of your breaths that were once in Galileo's lungs.

[3] **Problem 23.** The acceleration due to gravity can be measured by measuring the time period of a simple pendulum. However, it can be challenging to get an accurate result.

(a) Suppose you constructed a pendulum using regular household materials. Name at least five sources of possible experimental error in your calculated value of g. How would you make the pendulum and perform the measurements to minimize these sources of error?

(b) Make an actual pendulum yourself and carry out the measurement. Describe your experimental procedure and show your data. Estimate as many of the sources of experimental error identified in part (a) as you can, and using them, give a value of g with a reasonable uncertainty.

(c) If you had \$1,000 and a week to do plenty of measurements, how would you go about it? How precise a result do you think you could get? What would be the dominant sources of uncertainty remaining?

[2] **Problem 24.** Blackbody radiation is an electromagnetic phenomenon, so the radiation intensity depends on the speed of light c. It is also a thermal phenomenon, so it depends on the thermal energy $k_B T$, where T is the object's temperature and k_B is Boltzmann's constant. And it is a quantum phenomenon, so it depends on Planck's constant h.

(a) Using the relation $E = hf$, find the dimensions of h.

(b) Using dimensional analysis, show that the power emitted by the blackbody per unit area, called the radiation intensity I, obeys $I \propto T^4$, and find the constant of proportionality up to a dimensionless constant.

(c) How would the result change in a world with d spatial dimensions?

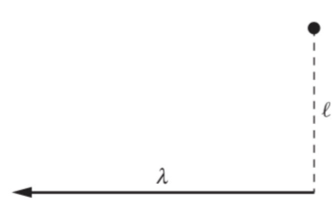
(d) The result you derived in part (b) is known as the Stefan–Boltzmann law. But if it can be derived with pure dimensional analysis, without needing any detailed calculations or experimental data at all, then why is it considered a law at all? Isn't it obvious?

3 Electromagnetism

These problems can be solved using the material in chapters 25 through 38 of Halliday and Resnick.

[2] **Problem 25.** As a warmup, take the Conceptual Survey of Electricity and Magnetism test. This should take no longer than 1 hour. You do not have to justify your answers; just list them.

[2] **Problem 26.** A half-infinite line has linear charge density λ.



(a) Find the electric field at a point that is "even" with the end, a distance ℓ from it, as shown.

(b) You should find the direction of the field is independent of ℓ. Explain why.

(c) Sketch the electric field lines everywhere.

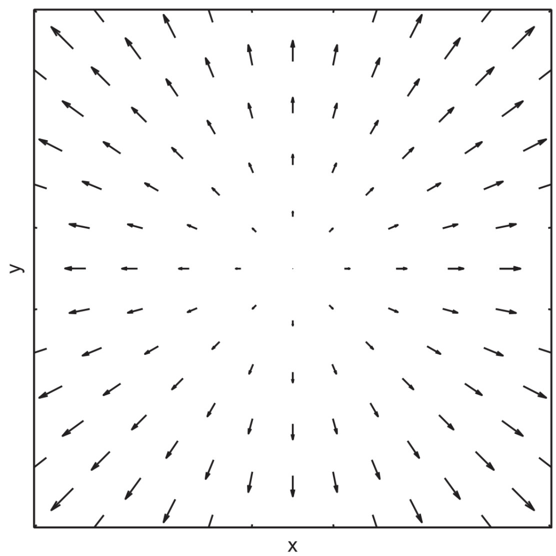
[2] **Problem 27.** Some basic tasks involving intuition for vector fields.

(a) Consider the vector field

$$\mathbf{v} = 2\hat{\mathbf{x}} + x\hat{\mathbf{y}}.$$

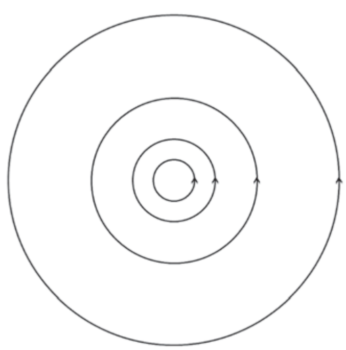
Sketch some field vectors at regular points. Then, on a separate sketch, draw some field lines.

(b) Some electric field vectors in a certain situation are shown below.



Sketch a corresponding field line diagram. Then, give a mathematical expression that could describe this field, and a physical situation which could produce it.

(c) The electric field lines in another situation are shown below.



Sketch a corresponding set of field vectors at regular points. Then, give a mathematical expression that could describe this field, and a physical situation which could produce it.

[3] **Problem 28.** A parallel plate capacitor of capacitance C is placed in a region of zero electric field. The first plate is given total charge Q_1 and the second plate is given total charge Q_2. Let the plates have area A and separation d, where $A \gg d^2$.

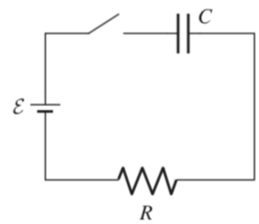
(a) Each of the plates has an inner and outer surface. Using Gauss's law, find the total charge on each of these four surfaces.

(b) Find the potential difference between the plates.

(c) Find the force between the plates.

(d) In addition to the force between the plates found in part (c), there is a contribution to the internal stress (force per unit area) within each plate due to the charges on its two surfaces. Find this part of the stress for each plate, and assuming $Q_1 > Q_2 > 0$, indicate whether it is tension or compression.

[3] **Problem 29.** A battery is connected to an RC circuit as shown.



Initially, the switch is open and the charge on the capacitor is zero. The switch is closed at $t = 0$.

(a) Solve for the charge on the capacitor as a function of time.

(b) Solve for the power dissipated in the resistor as a function of time.

(c) What is the total energy dissipated in the resistor over all time? Can you find a simple way to derive this result?

(d) Suppose that the emf $\mathcal{E}(t)$ supplied by the battery can be adjusted freely over time, and the capacitor must be given a total charge Q by the time $t = T$. Sketch the profile $\mathcal{E}(t)$ that maximizes the efficiency of this process, i.e. the ratio of the energy stored in the capacitor to the energy output by the battery, and find this efficiency.

[2] **Problem 30.** In this problem we estimate the maximum firing speed of a human neuron. Model a human cell simply as a sphere of radius 10^{-6} m.

(a) It has been measured that $1\,\text{cm}^2$ of cell membrane has a resistance of $1000\,\Omega$. Estimate the resistance of a single human cell.

(b) Estimate the capacitance of a single human cell, treating the two sides of the membrane as capacitor plates. You will have to estimate the thickness of the cell membrane.

(c) By modeling the cell as an RC circuit, estimate the maximum firing speed of a human neuron. Is this a reasonable result? If yes, how do you know? If not, how could this model be refined?

[2] **Problem 31.** An infinite solenoid with radius b has n turns per unit length. The current varies in time according to $I(t) = I_0 \cos \omega t$. A ring with radius $r < b$ and resistance R is centered on the solenoid's axis, with its plane perpendicular to the axis.

(a) What is the induced current in the ring?

(b) A given little piece of the ring will feel a magnetic force. For what values of t is this force maximum? At this moment, sketch the electric field everywhere.

(c) What is the effect of the force on the ring? That is, does the force cause the ring to translate, spin, etc.?

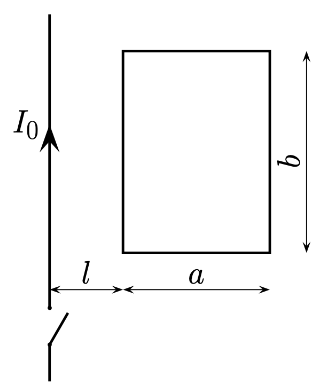
(d) If the current is driven by the AC power from a wall outlet, which has frequency 60 Hz in America, the ring will emit a humming sound. What is the frequency of this sound?

[2] **Problem 32.** A long, insulating cylinder with radius r and uniform surface charge density σ on its outer surface rotates about its symmetry axis with angular velocity ω.

(a) Find the magnetic field everywhere.

(b) A wire is connected to a point on the cylinder, with the other end on the axis of rotation. The wire rotates along with the cylinder. Show that the emf across the wire does not depend on the shape of the wire's path, and find this emf.

[2] **Problem 33.** A rectangular loop of wire with dimensions a and b is placed with one side parallel to a long, straight wire carrying current I_0, at a distance l.



The resistance of the loop is R. The current in the long wire is quickly switched off.

(a) What is the net momentum p acquired by the loop?

(b) How is momentum conserved in this setup?

[1] **Problem 34.** A 120 V rms, 60 Hz line provides power to a 40 W light bulb. By what factor will the brightness change if a 10 μF capacitor is connected in series with the light bulb?

4 Thermodynamics

These problems can be solved using the material in chapters 21 through 24 of Halliday and Resnick.

[2] **Problem 35.** Two moles of a monatomic ideal gas are taken through the following cycle.

- The gas begins at point A with pressure P_0 and volume V_0.
- The gas is heated at constant volume until it doubles its pressure, reaching point B.
- The gas is expanded at constant pressure until it doubles its volume, reaching point C.

- The gas is cooled at constant volume until it halves its pressure, reaching point D.
- The gas is compressed at constant pressure until it halves its volume, returning to point A.

Assume that all processes are quasistatic and reversible.

(a) Draw the process on a PV diagram.

(b) Calculate the net work done by the gas during the cycle.

(c) Calculate the efficiency of the cycle.

(d) Calculate the change in entropy of the gas as the system goes from state A to state D.

[3] **Problem 36.** Deriving some basic results in thermodynamics.

(a) Starting from the first law of thermodynamics, derive the fact that PV^γ is constant in an adiabatic process.

(b) Using the ideal gas law, derive the total work done by a gas as it expands at constant temperature from volume V_1 to V_2, in terms of n, R, T, V_1, and V_2.

(c) Show that if a general gas, not necessarily ideal, satisfies the equation $PV = kU$, where U is the total internal energy, then PV^n is constant in an adiabatic process for some power n, and find n in terms of k.

(d) Does a monatomic ideal gas satisfy $PV = kU$? If so, what is k?

(e) Two Carnot engines operate with the same minimum and maximum pressures, temperatures, and volumes. One uses helium as its working substance, and the other uses air. (At the relevant temperatures, helium behaves like a monatomic gas.) Which one performs more work per cycle?

[1] **Problem 37.** A monatomic ideal gas is adiabatically compressed to 1/8 of its original volume. For each of the following quantities, indicate by what factor they change.

(a) The rms velocity v_{rms}.

(b) The mean free path λ.

(c) The average time between collisions τ for each gas molecule.

(d) The molar heat capacity C_v.

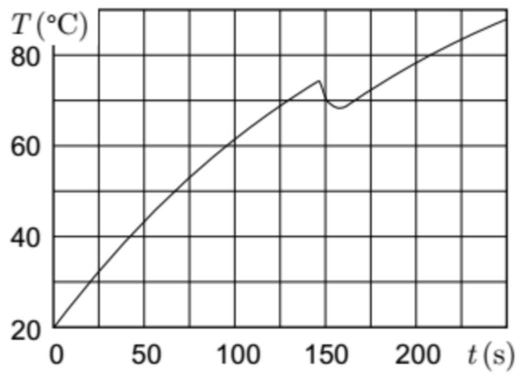
[2] **Problem 38.** A simple heat engine consists of a movable piston in a cylinder filled with an ideal monatomic gas. Initially the gas in the cylinder is at a pressure P_0 and volume V_0. The gas is slowly heated at constant volume until the pressure is $32P_0$. The gas is then adiabatically expanded until its pressure is P_0 again. Finally, the gas is cooled at constant pressure until its volume is V_0 again. Find the efficiency of the cycle.

[2] **Problem 39.** The total mass of a hot-air balloon (envelope, basket, and load) is 320 kg. Initially the air pressure inside and outside the envelope is 1.01×10^5 Pa and its density is $1.29\,\text{kg/m}^3$. In order to raise the hot-air balloon, a gas burner is used to heat the air inside the balloon. The volume of the envelope filled with hot air is $650\,\text{m}^3$. The molar mass of air is 29 g/mol. Treat the temperature of the air in the balloon as uniform.

(a) The balloon can either be tightly sealed, so that none of its air mixes with the outside air, or have a hole, so that its pressure equalizes with that of the outside air. For the purpose of generating lift, which is better?

(b) Assuming the better option has been taken, to what temperature must the air inside the balloon be heated to make the balloon begin to rise?

[3] **Problem 40.** Water is heated in an electric kettle. At a certain moment of time, a piece of ice at temperature $T_0 = 0\,°\mathrm{C}$ was put in the kettle. The figure below shows the water temperature as a function of time.



Find the mass of the ice if the heating power of the kettle is $P = 1\,\mathrm{kW}$. The latent heat of melting for ice is $L = 335\,\mathrm{kJ/kg}$, the heat capacity of water is $c = 4.2\,\mathrm{kJ/kg\,K}$, and the temperature of the room is $T_1 = 20\,°\mathrm{C}$. (Hint: it's very easy to get an answer that's off by up to 50% if you're careless.)

5 Relativity and Waves

These problems can be solved using the material in chapters 18 through 20, and 39 through 44 of Halliday and Resnick.

[2] **Problem 41.** Two bombs lie on a train platform, a distance L apart. As a train passes by at speed v, the bombs explode simultaneously (in the platform frame) and leave marks on the train. Due to the length contraction of the train, we know that the marks on the train are a distance γL apart when viewed in the train's frame, because this distance is what is length-contracted down to the given distance L in the platform frame. How would someone on the train quantitatively explain why the marks are a distance γL apart, considering that the bombs are a distance of only L/γ apart in the train frame?

[1] **Problem 42.** An atom with rest mass m radiates a photon with angular frequency ω. What is the rest mass of the atom afterward?

[3] **Problem 43.** A rope of linear mass density σ is hung between two poles, a distance L apart, and the middle of the wire sags a distance $d \ll L$ below the ends. Find the approximate frequencies of standing waves on the rope. (Hint: you do not have to solve for the shape of the rope. As an intermediate step, it will be useful to consider torque balance on half of the rope.)

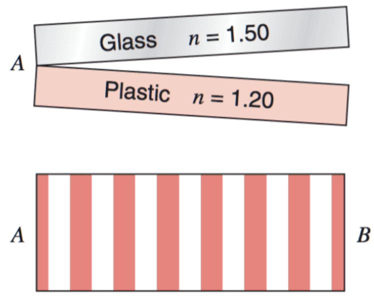
[3] **Problem 44.** A uniform string of length L and linear density ρ is stretched between two fixed supports. The tension in the string is T.

(a) Find the standing wave solutions and angular frequencies for the given boundary conditions.

(b) A very small mass m is now placed a distance ℓ from one end of the string. Find the change in the angular frequencies to first order in m, by using the fact that the average potential energy and average kinetic energy for each standing wave should remain equal.

(c) How would you go about finding the exact standing wave frequencies for this setup?

[2] **Problem 45.** A perfectly flat piece of glass is placed over a perfectly flat piece of black plastic.



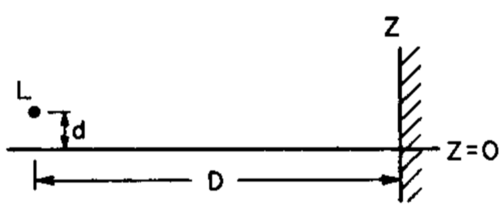
They touch at point A. Light of wavelength 600 nm is incident normally from above. The location of the dark fringes in the reflected light is shown above.

(a) How thick is the space between the glass and plastic at B?

(b) Water with $n = 1.33$ seeps into the region between the glass and the plastic. How many dark fringes are seen when all of the air has been displaced by water? The straightness and equal spacing of the fringes is an accurate test of the flatness of the glass.

(c) A setup like this one was used by the physicist Otto Wiener to measure the wavelength of light. When setting up the experiment, one must decide on the angle between the two objects. What are the advantages and disadvantages of making this angle smaller, for the purposes of measuring the wavelength?

[2] **Problem 46.** A point source of light L emitting a single wavelength λ is situated a distance d from an ideal mirror, at $z = 0$. A screen stands at the end of the mirror at distance $D \gg d$ from L.



Find the relative intensity of light on the screen as a function of z. (This setup is known as Lloyd's mirror.)

Preliminary Problems

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1 Mechanics

These problems can be solved using the material in chapters 1 through 17 of Halliday and Resnick.

[2] **Problem 1.** As a warmup, take the Force Concept Inventory test. This should take no longer than 30 minutes. You do not have to justify your answers; just list them.

Answer. 31352 22251 42441 12254 52213 53523

[1] **Problem 2.** A projectile is thrown up and passes point A, then point B a height h above A. Let T_A and T_B be the time intervals between the two times the projectile passes A and B, respectively.

(a) Show that an experimentalist can measure g by computing $g = 8h/(T_A^2 - T_B^2)$.

(b) This procedure probably looks a little contrived. Why is it better than doing something simpler, such as just dropping the ball and using $\Delta y = gt^2/2$?

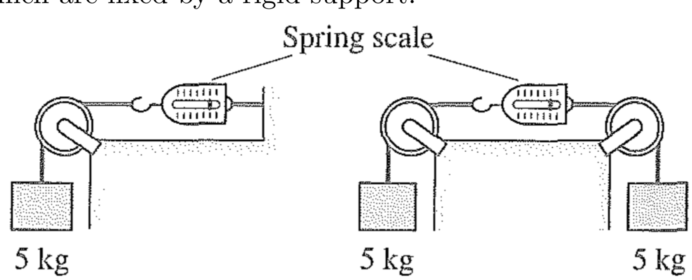
[2] **Problem 3.** A projectile is launched with speed v at an angle θ from the horizontal on a flat plane.

(a) Find $y(x)$ and the ratio of the range to the maximum height.

(b) What is the maximum θ for which the projectile always increases its distance from the thrower?

Answer. (b) $\sin^{-1}(2\sqrt{2}/3)$

[1] **Problem 4.** Consider the two following setups involving pulleys and spring scales. Treat the ropes and spring scales as massless, and the pulleys as frictionless. Model the pulleys as uniform discs with masses of 5 kg which are fixed by a rigid support.



(a) Draw a free-body diagram for the second setup, showing all external forces on the spring scale.

(b) What are the readings on the two spring scales?

(c) Draw a free-body diagram for the first setup, showing and naming all external forces on the pulley. In particular, what is the magnitude of the force between the pulley and the rope?

(d) What is the magnitude of the force that must be provided by the support?

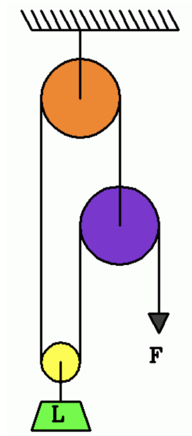
Answer. (c) normal force from the rope, weight, and the force from the support, (d) $50\sqrt{5}\,\text{N}$

[1] **Problem 5.** A sewage worker is using a ladder inside a large, frictionless, horizontal circular aqueduct. The ladder is of the same length as the diameter of the aqueduct.

(a) First the ladder is placed perfectly vertically and the worker climbs to the midpoint. Draw a free body diagram indicating all forces on the ladder, and their names. Do the forces balance?

(b) Now suppose the ladder is placed perfectly horizontally and the worker hangs statically from the midpoint. Draw a free body diagram indicating all forces on the ladder, and their names. Do the forces balance? If so, show this explicitly. If not, what happens next?

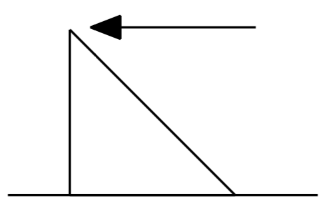
[1] **Problem 6.** Consider the following system of massless pulleys and string, called a "fool's tackle".



If the load L has mass m, find the force F needed to keep the system static.

Answer. Static equilibrium is impossible.

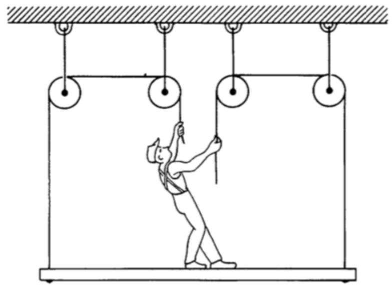
[2] **Problem 7.** A wooden isosceles right triangle with uniform mass density is placed on a table, and a force is applied as shown.



The force is gradually increased until the triangle begins to tip over without sliding. The force is then removed. Next, the surface is inclined with angle θ. For what range of θ can you be certain the triangle will not slide down the incline?

Answer. $\theta < \tan^{-1}(1/3)$

[1] **Problem 8.** A painter of mass M stands on a platform of mass m as shown.



He pulls each rope down with force F, and accelerates upward with acceleration a. Find a.

Answer. $-g + 4T/(M+m)$

[1] **Problem 9.** A wheel of radius R is rolling without slipping with angular velocity ω. What is the average speed of a point on the rim?

Answer. $(4/\pi)\omega R$

[3] **Problem 10.** A small block lies at the bottom of a spherical bowl of radius R.

(a) Find the period of small oscillations, assuming no friction. Can you give an intuitive explanation of the simplicity of your answer?

(b) What is the period if the block is replaced with a small uniform ball, with sufficient friction to roll without slipping?

(c) Now suppose the block moves in a circle, staying a constant height $h \ll R$ above the bottom of the bowl. Find the period of the motion, assuming no friction.

(d) Again, find the period if the block is replaced with a small uniform ball, with sufficient friction to roll without slipping.

(e) Now consider part (c) again. Suppose that at some moment, the speed of the block is instantaneously increased by a small amount. Qualitatively describe the subsequent motion, e.g. sketch what a top-down view would look like. What if h is not small?

Answer. (a, c) $2\pi\sqrt{R/g}$, (b, d) $2\pi\sqrt{7R/5g}$

[2] **Problem 11.** A car accelerates uniformly from rest. Initially, its door is slightly ajar. Calculate how far the car travels before the door slams shut. Assume the door has a frictionless hinge, a uniform mass distribution, and a length L from front to back.

Answer. $\pi^2 L/12$

[2] **Problem 12.** Two diametrically opposite points on a ring of mass M and radius R are marked out. The ring is placed at rest on a frictionless floor. An ant of mass m starts at one point, then walks horizontally along the ring's edge to the other. Through what total angle does the ring turn?

Answer. $\pi m/(M+m)$

[2] **Problem 13.** A baseball player holds a bat, modeled as a uniform rigid rod, horizontally at one of its ends. Usually, when the baseball hits the bat, the player will feel a sharp jolt in their hands as the bat recoils. This can be avoided if the baseball hits the "sweet spot". Where is it?

Answer. If the rod has length L, the center of percussion/sweet spot is $2L/3$ from the held end.

[3] **Problem 14.** Maxwell's wheel is a toy which demonstrates conservation of energy. It consists of a uniform disc of mass M and radius R, with a massless axle of radius $r \ll R$.

If the wheel is released from rest, it falls downward, in such a way so that the strings supporting it are always approximately vertical.

(a) Find the approximate acceleration during descent.

(b) Suppose each string has length $L \gg R^2/r$. When the string completely unravels, the wheel turns around and starts climbing back up. During the turn around, what is the maximum magnitude of the acceleration of the wheel?

Answer. (a) $a \approx -2r^2 g/R^2$

(b) $a_{\text{max}} \approx 4grL/R^2$

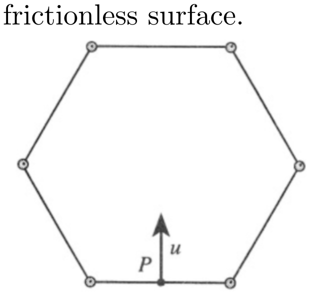
[3] **Problem 15.** A cue ball is a uniform sphere of radius R.

(a) Find the height at which the cue ball must be hit horizontally so that it immediately begins rolling without slipping.

(b) Skillful players can hit the cue ball so that it begins moving forwards, but then ends up moving backwards. Model the hit as an instantaneous impulse applied at an arbitrary point on the back half of the cue ball, in an arbitrary direction. For what impulses will this trick work? You can treat the situation as two-dimensional; justify your answer carefully.

Answer. (a) $2R/5$, (b) the line of the impulse passes under the contact point with the ground

[4] **Problem 16.** Six identical uniform rods, fastened at their ends by frictionless massless pivots, form a regular hexagon and lie on a frictionless surface.

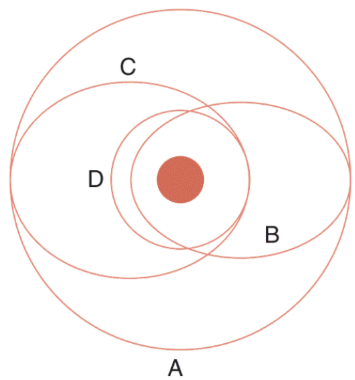


A blow is given at a right angle to the midpoint of the bottom rod. Immediately afterward, the bottom rod has velocity u, as shown. Find the speed of the opposite rod at this moment.

This is the hardest question in this problem set. It is intended to give you a taste of the trickier problems in the main problem sets, but it's definitely not a prerequisite to start them. The shortest solution I'm aware of uses just symmetry and elementary Newtonian mechanics.

Answer. $u/10$

[2] **Problem 17.** Several possible elliptical orbits of a satellite are shown below.



(a) Which orbit has the most angular momentum?

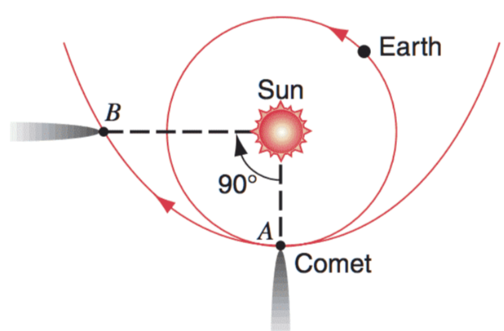
(b) Which orbit has the highest total energy?

(c) On which orbit is the largest speed acquired?

In all cases, justify your answer carefully.

Answer. (a) A, (b) A, (c) B

[2] **Problem 18.** A comet passes by Sun as shown, in a parabolic path.



How long, in years, does the comet take to get from point A to point B? (Hint: if you apply Kepler's laws and properties of conics, this problem can be done with almost no computation.)

Answer. $2\sqrt{2}/3\pi$ years

[2] **Problem 19.** Because of the rotation of the Earth, the line of a plumb bob will not align with the local gravitational field. Find the (small) angle of deviation between them as a function of the latitude θ, the gravitational acceleration g, the radius R of the Earth, and its angular velocity ω.

[2] **Problem 20.** You should be comfortable with setting up multiple integrals. Consider a cylindrical shell whose axis of symmetry is the z-axis. It has non-uniform mass per unit area $\sigma(\phi, z)$ in cylindrical coordinates, and the shell has radius a and height h, with the bottom edge at $z = 0$.

(a) Write down an integral that gives the total mass of the shell.

(b) Write down an integral that gives the moment of inertia of the shell about the z-axis.

(c) How would these results change if we had a solid cylinder with mass per unit volume $\rho(\phi, z, r)$?

[2] **Problem 21.** An entrepreneur proposes to propel the Earth through space by attaching many balloons to one side of it with ropes. The balloons will experience a buoyant force, which will create a tension in the ropes. Now consider the forces on the solid Earth. Because the atmospheric pressure on the surface is uniform, the only net force on the Earth is from the tension, so the Earth will get propelled.

Is this correct? If you think it is, explain why momentum conservation isn't violated. If you think it isn't, identify the specific force acting on the solid Earth that cancels the tension force.

Answer. The force is balanced by the *gravitational* force of the atmosphere on the Earth.

2 Problem Solving Skills

You should be able to start these questions without any background reading. However, for some fun background on estimation, see *Guesstimation: Solving the World's Problems on the Back of a Napkin*. For practical tips for real experiments, see chapter 7 of *Physics Olympiad: Basic to Advanced Exercises*.

[2] **Problem 22.** Argon atoms are special because they stay in the atmosphere for a very long time. They are not recycled like oxygen and nitrogen. An average breath inhales around 0.5 L of air and people breath on average around once every five seconds. Air is about 1% argon and has density $1.2\,\mathrm{kg/m^3}$. Assume all air particles have a mass of approximately 5×10^{-26} kg. Take the atmosphere to have constant density and be around 20 km thick. The radius of the Earth is 6.4×10^6 m.

(a) Estimate the total number of distinct argon atoms inhaled by Galileo throughout their life.

(b) Assuming the atmosphere has been uniform mixed since then, estimate the number of argon atoms in each of your breaths that were once in Galileo's lungs.

Answer. (a) about 10^{29}, (b) about 10^6

[3] **Problem 23.** The acceleration due to gravity can be measured by measuring the time period of a simple pendulum. However, it can be challenging to get an accurate result.

(a) Suppose you constructed a pendulum using regular household materials. Name at least five sources of possible experimental error in your calculated value of g. How would you make the pendulum and perform the measurements to minimize these sources of error?

(b) Make an actual pendulum yourself and carry out the measurement. Describe your experimental procedure and show your data. Estimate as many of the sources of experimental error identified in part (a) as you can, and using them, give a value of g with a reasonable uncertainty.

(c) If you had \$1,000 and a week to do plenty of measurements, how would you go about it? How precise a result do you think you could get? What would be the dominant sources of uncertainty remaining?

[2] **Problem 24.** Blackbody radiation is an electromagnetic phenomenon, so the radiation intensity depends on the speed of light c. It is also a thermal phenomenon, so it depends on the thermal energy $k_B T$, where T is the object's temperature and k_B is Boltzmann's constant. And it is a quantum phenomenon, so it depends on Planck's constant h.

(a) Using the relation $E = hf$, find the dimensions of h.

(b) Using dimensional analysis, show that the power emitted by the blackbody per unit area, called the radiation intensity I, obeys $I \propto T^4$, and find the constant of proportionality up to a dimensionless constant.

(c) How would the result change in a world with d spatial dimensions?

(d) The result you derived in part (b) is known as the Stefan–Boltzmann law. But if it can be derived with pure dimensional analysis, without needing any detailed calculations or experimental data at all, then why is it considered a law at all? Isn't it obvious?

Answer. (c) $I \propto T^{d+1}$

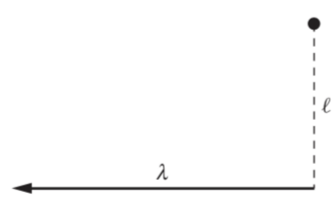
3 Electromagnetism

These problems can be solved using the material in chapters 25 through 38 of Halliday and Resnick.

[2] **Problem 25.** As a warmup, take the Conceptual Survey of Electricity and Magnetism test. This should take no longer than 1 hour. You do not have to justify your answers; just list them.

Answer. BABBC EBBBC EDEDA EEDAD EDACD AECCA ED

[2] **Problem 26.** A half-infinite line has linear charge density λ.



(a) Find the electric field at a point that is "even" with the end, a distance ℓ from it, as shown.

(b) You should find the direction of the field is independent of ℓ. Explain why.

(c) Sketch the electric field lines everywhere.

Answer. (b) scaling symmetry

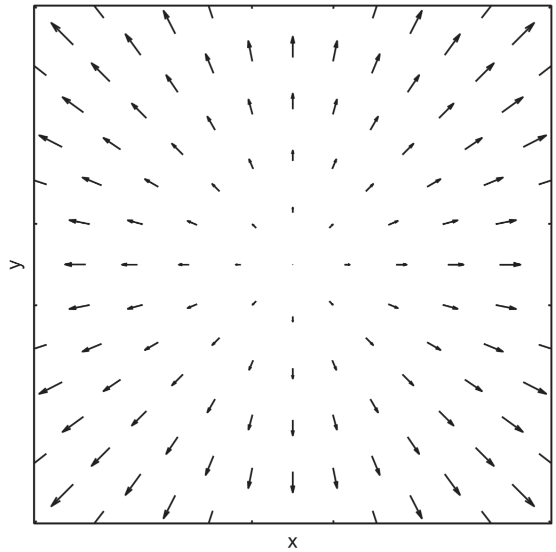
[2] **Problem 27.** Some basic tasks involving intuition for vector fields.

(a) Consider the vector field

$$\mathbf{v} = 2\hat{\mathbf{x}} + x\hat{\mathbf{y}}.$$

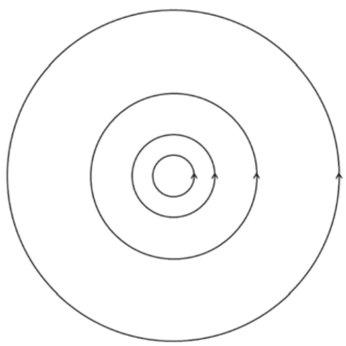
Sketch some field vectors at regular points. Then, on a separate sketch, draw some field lines.

(b) Some electric field vectors in a certain situation are shown below.



Sketch a corresponding field line diagram. Then, give a mathematical expression that could describe this field, and a physical situation which could produce it.

(c) The electric field lines in another situation are shown below.



Sketch a corresponding set of field vectors at regular points. Then, give a mathematical expression that could describe this field, and a physical situation which could produce it.

[3] **Problem 28.** A parallel plate capacitor of capacitance C is placed in a region of zero electric field. The first plate is given total charge Q_1 and the second plate is given total charge Q_2. Let the plates have area A and separation d, where $A \gg d^2$.

(a) Each of the plates has an inner and outer surface. Using Gauss's law, find the total charge on each of these four surfaces.

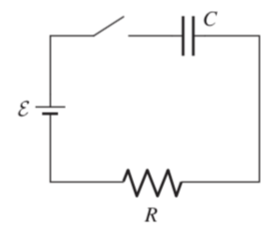
(b) Find the potential difference between the plates.

(c) Find the force between the plates.

(d) In addition to the force between the plates found in part (c), there is a contribution to the internal stress (force per unit area) within each plate due to the charges on its two surfaces. Find this part of the stress for each plate, and assuming $Q_1 > Q_2 > 0$, indicate whether it is tension or compression.

Answer. (c) $Q_1 Q_2/2A\epsilon_0$, (d) $(Q_1^2 - Q_2^2)/8A^2\epsilon_0$ for both, plate 1 under tension, plate 2 under compression

[3] **Problem 29.** A battery is connected to an RC circuit as shown.



Initially, the switch is open and the charge on the capacitor is zero. The switch is closed at $t = 0$.

(a) Solve for the charge on the capacitor as a function of time.

(b) Solve for the power dissipated in the resistor as a function of time.

(c) What is the total energy dissipated in the resistor over all time? Can you find a simple way to derive this result?

(d) Suppose that the emf $\mathcal{E}(t)$ supplied by the battery can be adjusted freely over time, and the capacitor must be given a total charge Q by the time $t = T$. Sketch the profile $\mathcal{E}(t)$ that maximizes the efficiency of this process, i.e. the ratio of the energy stored in the capacitor to the energy output by the battery, and find this efficiency.

Answer. (d) $\mathcal{E}(t)$ is linear, efficiency $1/(1 + (2RC/T))$.

[2] **Problem 30.** In this problem we estimate the maximum firing speed of a human neuron. Model a human cell simply as a sphere of radius 10^{-6} m.

(a) It has been measured that $1\,\text{cm}^2$ of cell membrane has a resistance of $1000\,\Omega$. Estimate the resistance of a single human cell.

(b) Estimate the capacitance of a single human cell, treating the two sides of the membrane as capacitor plates. You will have to estimate the thickness of the cell membrane.

(c) By modeling the cell as an RC circuit, estimate the maximum firing speed of a human neuron. Is this a reasonable result? If yes, how do you know? If not, how could this model be refined?

Answer. (c) The RC timescale is on the order of 10 ms, which is reasonable.

[2] **Problem 31.** An infinite solenoid with radius b has n turns per unit length. The current varies in time according to $I(t) = I_0 \cos \omega t$. A ring with radius $r < b$ and resistance R is centered on the solenoid's axis, with its plane perpendicular to the axis.

(a) What is the induced current in the ring?

(b) A given little piece of the ring will feel a magnetic force. For what values of t is this force maximum? At this moment, sketch the electric field everywhere.

(c) What is the effect of the force on the ring? That is, does the force cause the ring to translate, spin, etc.?

(d) If the current is driven by the AC power from a wall outlet, which has frequency 60 Hz in America, the ring will emit a humming sound. What is the frequency of this sound?

Answer. (c) grows and shrinks, (d) 120 Hz

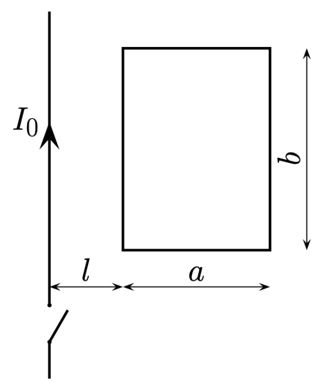
[2] **Problem 32.** A long, insulating cylinder with radius r and uniform surface charge density σ on its outer surface rotates about its symmetry axis with angular velocity ω.

(a) Find the magnetic field everywhere.

(b) A wire is connected to a point on the cylinder, with the other end on the axis of rotation. The wire rotates along with the cylinder. Show that the emf across the wire does not depend on the shape of the wire's path, and find this emf.

Answer. (b) $\mu_0 \sigma \omega^2 r^3/2$

[2] **Problem 33.** A rectangular loop of wire with dimensions a and b is placed with one side parallel to a long, straight wire carrying current I_0, at a distance l.



The resistance of the loop is R. The current in the long wire is quickly switched off.

(a) What is the net momentum p acquired by the loop?

(b) How is momentum conserved in this setup?

Answer.

$$\text{(a)}\ \frac{1}{8\pi^2}\frac{(b\mu_0 I_0)^2}{R}\log\left(\frac{a+l}{l}\right)\left(\frac{1}{l}-\frac{1}{a+l}\right)$$

[1] **Problem 34.** A 120 V rms, 60 Hz line provides power to a 40 W light bulb. By what factor will the brightness change if a 10 μF capacitor is connected in series with the light bulb?

Answer. 65%

4 Thermodynamics

These problems can be solved using the material in chapters 21 through 24 of Halliday and Resnick.

[2] **Problem 35.** Two moles of a monatomic ideal gas are taken through the following cycle.

- The gas begins at point A with pressure P_0 and volume V_0.
- The gas is heated at constant volume until it doubles its pressure, reaching point B.
- The gas is expanded at constant pressure until it doubles its volume, reaching point C.
- The gas is cooled at constant volume until it halves its pressure, reaching point D.
- The gas is compressed at constant pressure until it halves its volume, returning to point A.

Assume that all processes are quasistatic and reversible.

(a) Draw the process on a PV diagram.

(b) Calculate the net work done by the gas during the cycle.

(c) Calculate the efficiency of the cycle.

(d) Calculate the change in entropy of the gas as the system goes from state A to state D.

Answer. (b) P_0V_0, (c) $2/13$, (d) $5R\log 2$

[3] **Problem 36.** Deriving some basic results in thermodynamics.

(a) Starting from the first law of thermodynamics, derive the fact that PV^γ is constant in an adiabatic process.

(b) Using the ideal gas law, derive the total work done by a gas as it expands at constant temperature from volume V_1 to V_2, in terms of n, R, T, V_1, and V_2.

(c) Show that if a general gas, not necessarily ideal, satisfies the equation $PV = kU$, where U is the total internal energy, then PV^n is constant in an adiabatic process for some power n, and find n in terms of k.

(d) Does a monatomic ideal gas satisfy $PV = kU$? If so, what is k?

(e) Two Carnot engines operate with the same minimum and maximum pressures, temperatures, and volumes. One uses helium as its working substance, and the other uses air. (At the relevant temperatures, helium behaves like a monatomic gas.) Which one performs more work per cycle?

Answer. (c) $n = k + 1$, (d) $k = \gamma - 1$, (e) air

[1] **Problem 37.** A monatomic ideal gas is adiabatically compressed to 1/8 of its original volume. For each of the following quantities, indicate by what factor they change.

(a) The rms velocity v_{rms}.

(b) The mean free path λ.

(c) The average time between collisions τ for each gas molecule.

(d) The molar heat capacity C_v.

Answer. (a) 2 times larger, (b) 8 times smaller, (c) 16 times smaller, (d) same

[2] **Problem 38.** A simple heat engine consists of a movable piston in a cylinder filled with an ideal monatomic gas. Initially the gas in the cylinder is at a pressure P_0 and volume V_0. The gas is slowly heated at constant volume until the pressure is $32P_0$. The gas is then adiabatically expanded until its pressure is P_0 again. Finally, the gas is cooled at constant pressure until its volume is V_0 again. Find the efficiency of the cycle.

Answer. 58/93

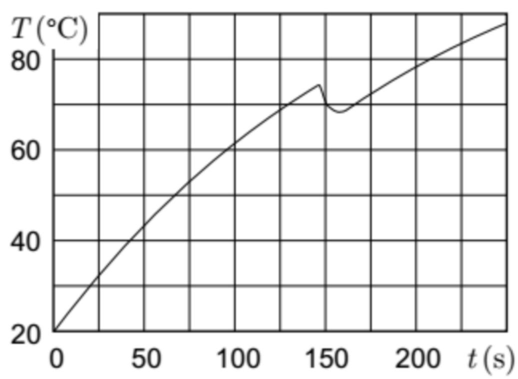
[2] **Problem 39.** The total mass of a hot-air balloon (envelope, basket, and load) is 320 kg. Initially the air pressure inside and outside the envelope is 1.01×10^5 Pa and its density is $1.29\,\mathrm{kg/m^3}$. In order to raise the hot-air balloon, a gas burner is used to heat the air inside the balloon. The volume of the envelope filled with hot air is $650\,\mathrm{m^3}$. The molar mass of air is 29 g/mol. Treat the temperature of the air in the balloon as uniform.

(a) The balloon can either be tightly sealed, so that none of its air mixes with the outside air, or have a hole, so that its pressure equalizes with that of the outside air. For the purpose of generating lift, which is better?

(b) Assuming the better option has been taken, to what temperature must the air inside the balloon be heated to make the balloon begin to rise?

Answer. (a) hole, (b) 440 K

[3] **Problem 40.** Water is heated in an electric kettle. At a certain moment of time, a piece of ice at temperature $T_0 = 0\,°\mathrm{C}$ was put in the kettle. The figure below shows the water temperature as a function of time.



Find the mass of the ice if the heating power of the kettle is $P = 1\,\mathrm{kW}$. The latent heat of melting for ice is $L = 335\,\mathrm{kJ/kg}$, the heat capacity of water is $c = 4.2\,\mathrm{kJ/kg\,K}$, and the temperature of the room is $T_1 = 20\,°\mathrm{C}$. (Hint: it's very easy to get an answer that's off by up to 50% if you're careless.)

Answer. 28 g

5 Relativity and Waves

These problems can be solved using the material in chapters 18 through 20, and 39 through 44 of Halliday and Resnick.

[2] **Problem 41.** Two bombs lie on a train platform, a distance L apart. As a train passes by at speed v, the bombs explode simultaneously (in the platform frame) and leave marks on the train. Due to the length contraction of the train, we know that the marks on the train are a distance γL apart when viewed in the train's frame, because this distance is what is length-contracted down to the given distance L in the platform frame. How would someone on the train quantitatively explain why the marks are a distance γL apart, considering that the bombs are a distance of only L/γ apart in the train frame?

[1] **Problem 42.** An atom with rest mass m radiates a photon with angular frequency ω. What is the rest mass of the atom afterward?

Answer. $\sqrt{m(m - 2\hbar\omega/c^2)}$

[3] **Problem 43.** A rope of linear mass density σ is hung between two poles, a distance L apart, and the middle of the wire sags a distance $d \ll L$ below the ends. Find the approximate frequencies of standing waves on the rope. (Hint: you do not have to solve for the shape of the rope. As an intermediate step, it will be useful to consider torque balance on half of the rope.)

Answer.

$$f_n = \frac{n}{4\sqrt{2}}\sqrt{\frac{g}{d}}$$

[3] **Problem 44.** A uniform string of length L and linear density ρ is stretched between two fixed supports. The tension in the string is T.

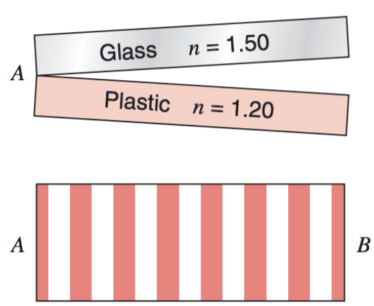
(a) Find the standing wave solutions and angular frequencies for the given boundary conditions.

(b) A very small mass m is now placed a distance ℓ from one end of the string. Find the change in the angular frequencies to first order in m, by using the fact that the average potential energy and average kinetic energy for each standing wave should remain equal.

(c) How would you go about finding the exact standing wave frequencies for this setup?

Answer. (a) $\omega_n = n\pi v/L$ where $v = \sqrt{T/\rho}$, (b) $\Delta\omega_n \approx (-m/\rho L)\,\omega_n \sin^2(n\pi\ell/L)$

[2] **Problem 45.** A perfectly flat piece of glass is placed over a perfectly flat piece of black plastic.



They touch at point A. Light of wavelength 600 nm is incident normally from above. The location of the dark fringes in the reflected light is shown above.

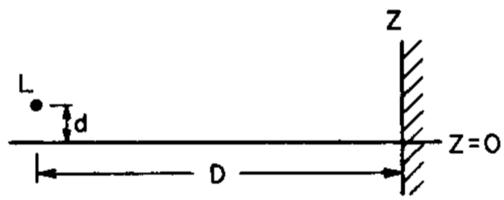
(a) How thick is the space between the glass and plastic at B?

(b) Water with $n = 1.33$ seeps into the region between the glass and the plastic. How many dark fringes are seen when all of the air has been displaced by water? The straightness and equal spacing of the fringes is an accurate test of the flatness of the glass.

(c) A setup like this one was used by the physicist Otto Wiener to measure the wavelength of light. When setting up the experiment, one must decide on the angle between the two objects. What are the advantages and disadvantages of making this angle smaller, for the purposes of measuring the wavelength?

Answer. (b) 10 dark fringes

[2] **Problem 46.** A point source of light L emitting a single wavelength λ is situated a distance d from an ideal mirror, at $z = 0$. A screen stands at the end of the mirror at distance $D \gg d$ from L.



Find the relative intensity of light on the screen as a function of z. (This setup is known as Lloyd's mirror.)

Answer. $I(z) \propto \sin^2(2\pi dz/D\lambda)$