Cosmic Inflation

A. Expansion of Universe

Question A.1

Answer	Marks
For any test mass m on the boundary of the sphere, $$m\ddot{R}(t) = -GmM_s/R^2(t) \qquad \text{(A.1.1)}$$ where M_s is mass portion inside the sphere	**0.2**
Multiplying equation (A.1.1) with $\dot{R}$ and integrating it gives $$\int \dot{R}\frac{d\dot{R}}{dt}\,dt = \frac{1}{2}\,\dot{R}^2 = \frac{GM_s}{R} + A$$ where A is a integration constant	**0.6**
Taking $M_s = \frac{4}{3}\pi R^3(t)\rho(t)$, and $\dot{R} = \dot{a}\,R_s$	**0.2**
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho(t) + \frac{2A}{R_s^2 a^2(t)}$$	**0.2**
Therefore, we have $A_1 = \frac{8\pi G}{3}$	**0.1**
Total	**1.3**

Question A.2

Answer	Marks

The 2nd Friedmann equation can be obtained from the 1st law of thermodynamics : $$dE = -pdV + dQ.$$	**0.1**
For adiabatic processes $dE + pdV = 0$ and its time derivative is $\dot{E} + p\,\dot{V} = 0$.	**0.1**
For the sphere $\dot{V} = V\,(3\,\dot{a}/a)$	**0.1**
Its total energy is $E = \rho(t)V(t)\,c^2$	**0.2**
Therefore $\dot{E} = \left(\dot{\rho} + 3\,\frac{\dot{a}}{a}\right)Vc^2$	**0.1**
It yields $$\dot{\rho} + 3\,\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0$$	**0.2**
Therefore, we have $A_2 = 3$.	**0.1**
Total	**0.9**

Question A.3

Answer	Marks
Interpreting $\rho(t)c^2$ as total energy density, and substituting $\frac{p(t)}{c^2} = w\,\rho(t)$ in to the 2nd Friedmann equation yields: $$\dot{\rho} + 3\,\rho(1+w)\frac{\dot{a}}{a} = 0$$	**0.1**
$\rho \propto a^{-3(w+1)}$	**0.2**
(i) In case of radiation, photon as example, the energy is given by $E_r = h\nu = hc/\lambda$ then its energy density $\rho_r = \frac{E_r}{V} \propto a^{-4}$ so that $w_r = \frac{1}{3}$	**0.3**
(ii) In case of nonrelativistic matter, its energy density nearly $\rho_m \simeq \frac{m_0 c^2}{V} \propto a^{-3}$ since dominant energy comes from its rest energy $m_0 c^2$, so that $w_m = 0$	**0.3**
(iii) For a constant energy density, let say $\epsilon_\Lambda =$ constant, $\epsilon_\Lambda \propto a^0$ so that $w_\Lambda = -1$.	**0.3**
Total	**1.2**

Question A.4

Answer	Marks
(i) In case of $k=0$, for radiation we have $\rho_r a^4 =$ constant. So by comparing the parameters values with their present value, $\rho_r(t)a^4(t) = \rho_{r0}a_0^4$, $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{r0}\left(\frac{a_0}{a}\right)^4.$$ $$\int a\,da = \frac{1}{2}a^2 + K = \left(\frac{8\pi G}{3}\rho_{r0}a_0^4\right)^{\frac{1}{2}} t.$$	**0.2**
Because $a(t=0)=0, K=0$, then $$a(t) = (2)^{\frac{1}{2}}\left(\frac{8\pi G}{3}\rho_{r0}a_0^4\right)^{\frac{1}{4}} t^{\frac{1}{2}} = (2H_0)^{\frac{1}{2}} t^{\frac{1}{2}}.$$ where $H_0 = \left(\frac{8\pi G}{3}\rho_{r0}\right)^{\frac{1}{2}}$ after taking $a_0 = 1$.	**0.2**
(ii) for non-relativistic matter domination, using $\rho_m(t)a^3(t) = \rho_{m0}a_0^3$, and similar way we will get $$a(t) = \left(\frac{3}{2}\right)^{\frac{2}{3}}\left(\frac{8\pi G}{3}\rho_{m0}a_0^4\right)^{\frac{1}{3}} t^{\frac{2}{3}} = \left(\frac{3H_0}{2}\right)^{\frac{2}{3}} t^{\frac{2}{3}}.$$ where $H_0 = \left(\frac{8\pi G}{3}\rho_{m0}\right)^{\frac{1}{2}}$.	**0.4**
(iii) for constant energy density, $$\ln a = H_0 t + K'$$ Where K' is integration constant and $H_0 = \left(\frac{8\pi G}{3}\rho_\Lambda\right)^{\frac{1}{2}}$. Taking condition $a_0 = 1$, $$\ln\left(\frac{a}{a_0}\right) = H_0(t - t_0)$$ $$a(t) = e^{H_0(t-t_0)}$$	**0.4**
Total	**1.2**

Question A.5

Answer	Marks
Condition for critical energy condition: $$\rho_c(t) = \frac{3H^2}{8\pi G}$$ Friedmann equation can be written as $$H^2(t) = H^2(\text{t})\Omega(\text{t}) - \frac{kc^2}{R_0^2 a^2(t)}$$ $$\left(\frac{R_0^2}{\text{c}^2}\right) a^2H^2(\Omega - 1) = k \qquad \text{(A.5.1)}$$	**0.1**
Total	**0.1**

Question A.6

Answer	Marks
Because $\left(\frac{R_0^2}{\text{c}^2}\right) a^2H^2 > 0$, then $k = +1$ corresponds to $\Omega > 1$, $k = -1$ corresponds to $\Omega < 1$ and $k = 0$ corresponds to $\Omega = 1$	**0.3**
Total	**0.3**

B. Motivation To Introduce Inflation Phase and Its General Conditions

Question B.1

Answer	Marks
Equation (A.5.1) shows that $(\Omega - 1) = \frac{kc^2}{R_0^2}\frac{1}{\dot{a}^2}$.	**0.1**
In a universe dominated by non-relativistic matter or radiation, scale factor can be written as a function of time as $a = a_0\left(\frac{t}{t_0}\right)^p$ where $p < 1$ ($p = \frac{1}{2}$ for radiation and $p = \frac{2}{3}$ for non-relativistic matter)	**0.2**
$(\Omega - 1) = \tilde{k}\, t^{2(1-p)}$	**0.2**
Total	**0.5**

Question B.2

Answer	Marks
For a period dominated by constant energy provides the solution $a(t) = e^{Ht}$ so that $\dot{a} = He^{Ht}$	**0.1**
$(\Omega - 1) = \frac{k}{H^2}\, t^{-2Ht}$	**0.2**
Total	**0.3**

Question B.3

Answer	Marks
Inflation period can be generated by constant energy period, therefore it is a phase where $w = -1$ so that $p = w\rho c^2 = -\rho c^2$ (negative pressure).	**0.2**
Differentiating Friedmann equation leads to $$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0^2}$$ $$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^2 + 2\rho a\,\dot{a}) = \frac{8\pi G}{3}\left(-3\left(\rho + \frac{p}{c^2}\right)a\dot{a} + 2\rho a\dot{a}\right).$$ $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$	**0.4**
So that because during inflation $p = -\rho c^2$, it is equivalent with condition $\ddot{a} > 0$ (accelerated expansion)	**0.1**
As a result, $\ddot{a} = d(\dot{a})/dt = d(Ha)/dt > 0$ or $d(Ha)^{-1}/dt < 0$ (shrinking Hubble radius).	**0.2**
Total	**0.9**

Question B.4

Answer	Marks
Inflation condition can be written as $\frac{d(aH)^{-1}}{dt} < 0$, with $H = \dot{a}/a$ as such $$\frac{d(aH)^{-1}}{dt} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\epsilon) < 0 \Rightarrow \epsilon < 1$$	**0.2**
Total	**0.2**

C. Inflation Generated by Homogenously Distributed Matter

Question C.1

Answer	Marks
Differentiating equations (4) and employing equation 4 we can get $$2H\dot{H} = \frac{1}{3M_{pl}^2}\left[\dot{\phi}\ddot{\phi} + \left(\frac{\partial V}{\partial \phi}\right)\dot{\phi}\right] = \frac{1}{3M_{pl}^2}\left[-3H\,\dot{\phi}^2\right]$$ $$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{pl}^2}$$	**0.3**
Therefore $\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{M_{pl}^2 H^2}$	**0.1**
The inflation can occur when the potential energy dominates the particle's energy ($\dot{\phi}^2 \ll V$) such that $H^2 \approx V/(3M_{pl}^2)$.	**0.2**
Slow-roll approximation: $3H\dot{\phi} \approx -V'$	**0.1**
Implies $$\epsilon \approx \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2 \qquad \text{(C.1.1)}$$	**0.3**
we also have $$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -V''\dot{\phi}$$ $\delta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{V''}{3H^2} - \epsilon$ Therefore $$\eta_V \approx M_{Pl}^2\frac{V''}{V} \qquad \text{(C.1.2)}$$	**0.4**
$$\mathrm{d}N = H\,dt = \left(\frac{H}{\dot{\phi}}\right)d\phi \approx -\frac{1}{M_{pl}^2}(V/V')\,d\phi \qquad \text{(C.1.3)}$$ $$\frac{dN}{d\phi} \approx -\frac{1}{M_{pl}^2}(V/V')$$	**0.3**
Total	**1.7**

D. **Inflation with A Simple Potential**

Question D.1

Answer	Marks
Inflation ends at $\epsilon = 1$. Using $V(\phi) = \Lambda^4 \left(\phi/M_{pl}\right)^n$ yields $$\epsilon = \frac{M_{pl}^2}{2}\left[\frac{n}{\phi_{\text{end}}}\right]^2 = 1 \Rightarrow \phi_{end} = \frac{n}{\sqrt{2}}M_{pl}$$	0.5
Total	0.5

Question D.2

Answer	Marks
From equations (C.1.1), (C.1.2) and (C.1.3) we can obtain $$N = -\left[\frac{\phi}{M_{pl}}\right]^2 \frac{1}{2n} + \beta$$ where β is a integration constant. As $N = 0$ at ϕ_{end} then $\beta = \frac{n}{4}$. $$N = -\left[\frac{\phi}{M_{pl}}\right]^2 \frac{1}{2n} + \frac{n}{4}$$	0.2
$$\eta_V = n(n-1)\left[\frac{M_{pl}}{\phi}\right]^2 = \frac{2(n-1)}{n-4N}$$	0.2
$$\varepsilon = \frac{n^2}{2}\left[\frac{M_{pl}}{\phi}\right]^2 = \frac{n}{n-4N}$$	0.2
so that $$r = 16\varepsilon = \frac{16n}{n-4N}$$	0.1

$$n_s = 1 + 2\eta_V - 6\epsilon = 1 - \frac{2(n+2)}{(n-4N)}$$	**0.1**
To obtain the observational constraint $n_s = 0.968$ we need $n = -5.93$ which is inconsistent with the condition $r < 0.12$. There is no a closest integer n that can obtains $r < 0.12$. As example, for $n = -6$ leads a contradiction $0 < (-0.27)$ and for $n = -5$ leads a contradiction $0 < (-0.2)$.	**0.1**
Total	**0.9**