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SEPTEMBER/OCTOBER 1990

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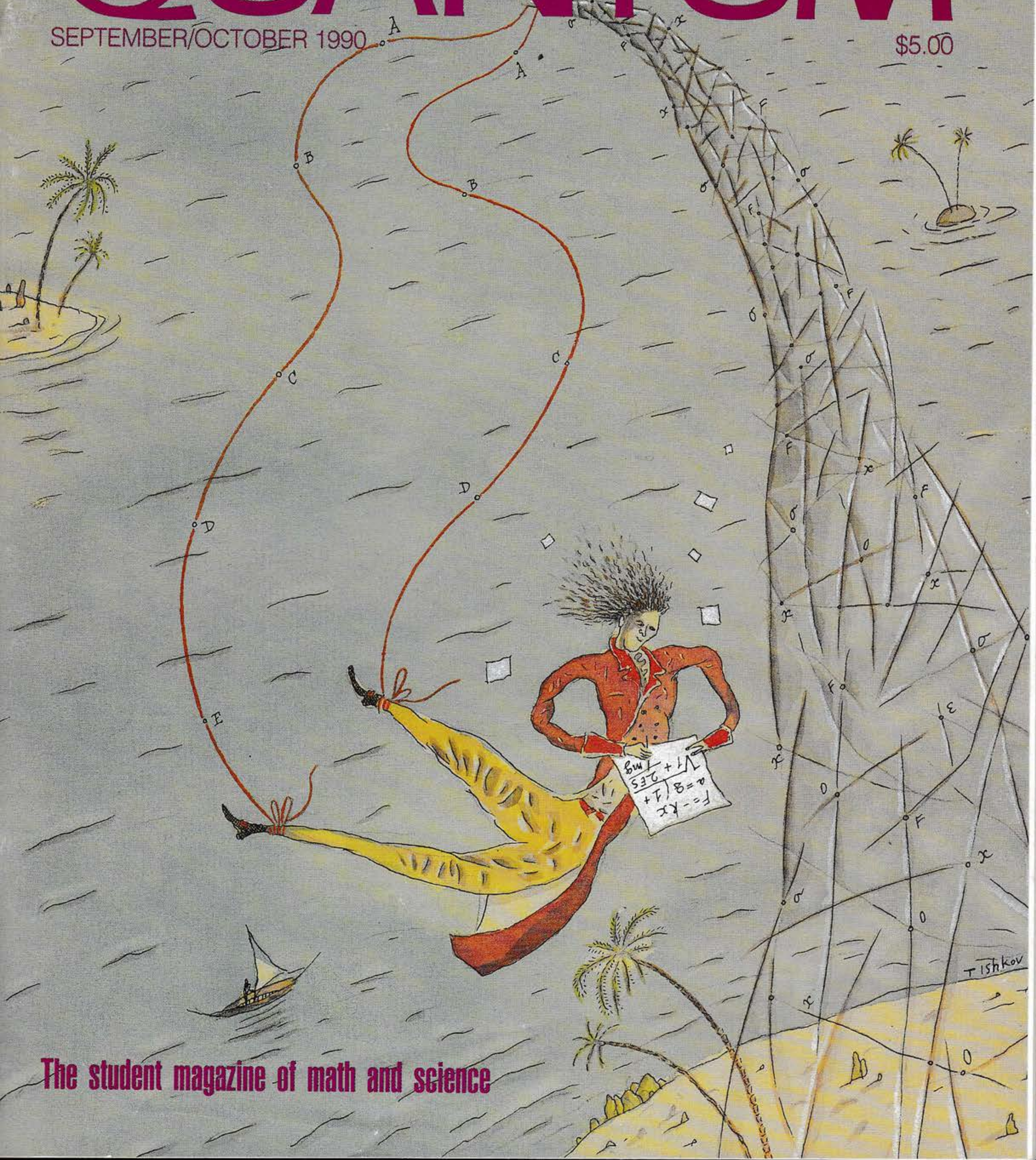
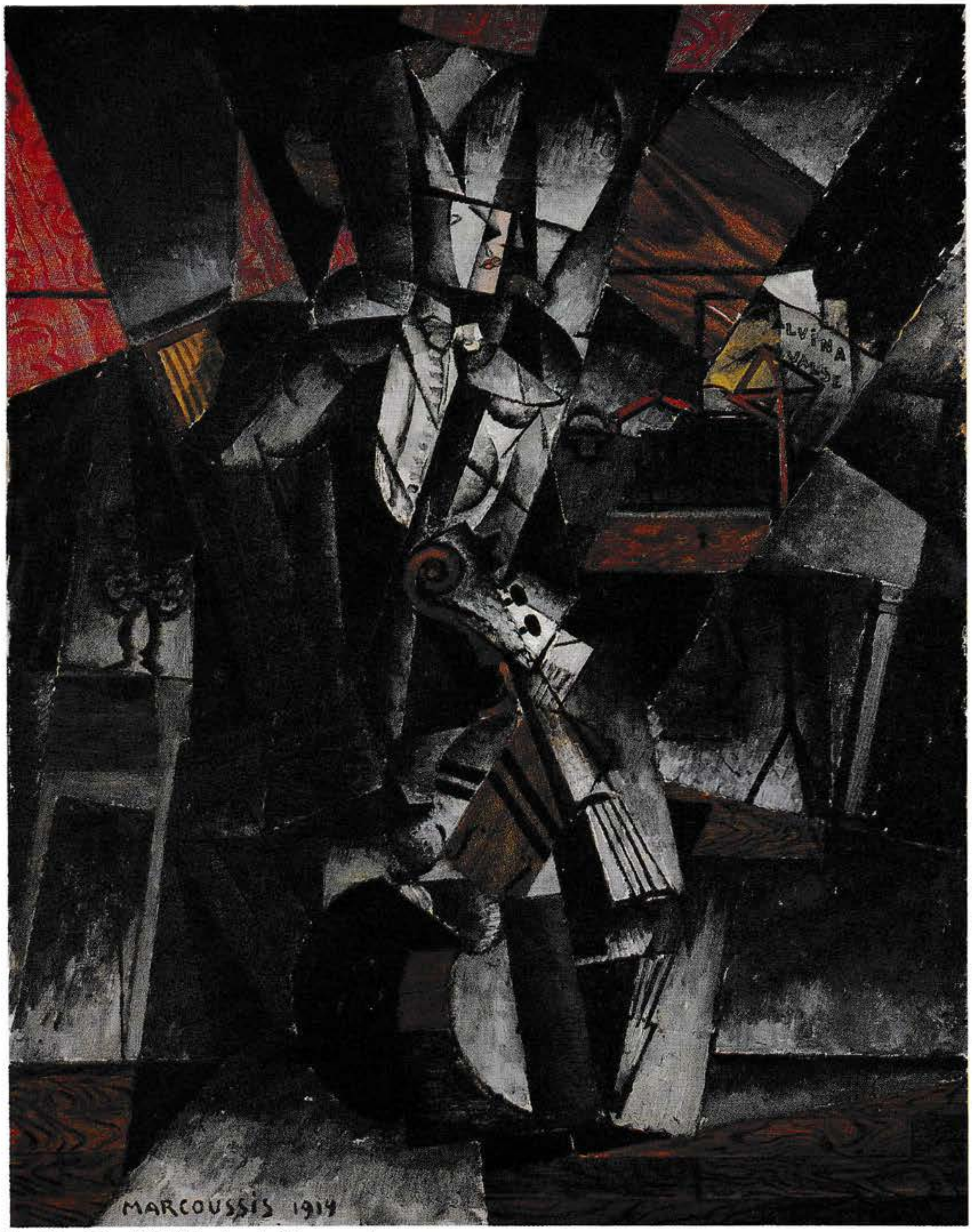


The student magazine of math and science



The Musician (1914) by Louis Marcoussis

The Polish painter Louis Casimir Ladislas Marcoussis (1883–1941) was one of the group of artists in Paris who were exploring new possibilities of spatial representation in the second decade of this century. The movement eventually came to be called "Cubism," and its practitioners included some of the most illustrious names of modern art. Some Cubists, Marcoussis among them, saw themselves as obeying a severe discipline, one inspired directly by mathematical laws. Others, like Pablo Picasso and Georges Braque, saw a greater role for the individual imagination and denied being "Cubists" at all. Cubists, they felt, simply adhered to a strict set of compositional rules.

Labels aside, it's apparent that Marcoussis wasn't content to produce an objectively "realistic" image of a musician playing his instrument. After all, that's what cameras are for. As a self-conscious painter, Marcoussis was able to use fractured planes, skewed angles, and jumbled shapes to create a sense of rhythm and motion. (Some Cubists would present multiple sides of a three-dimensional object simultaneously, mixing "points of view" in a single "snapshot," to suggest another dimension—time.) Other aspects of the painting, though, suggest restraint and discipline: the narrow range of colors used, the way much of the painting seems to radiate from a single point, the sense of architecture in the placement of the gray slabs.

Many artists have been profoundly affected by the new theories in physics that have bubbled up into the public consciousness throughout the 20th century. And many of them have been fascinated by music and frequently used musicians as their subjects. No doubt you've come across discussions of the more traditional physics and mathematics of music—columns of vibrating air, the relationship of string length to pitch, and so on. But what about the music of physicists? For a lighthearted peek into that relatively unexplored area, see page 54.

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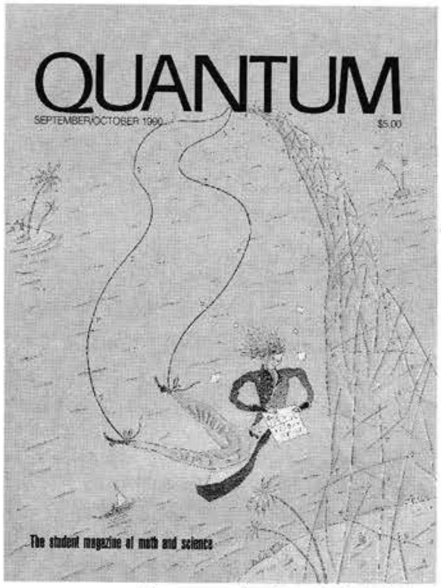

Cover art by Leonid Tishkov

Maybe you saw it on the news a while back: a perfectly sane-looking man diving headfirst off a bridge, elastic straps attached to his ankles and nothing but water to look at on the way down. From that height, at that speed, it looked about as inviting as a sheet of concrete. He was wired for sound (he carried a tiny video camera on his back as well), and the sounds he emitted during his flight were remarkable in their variety and intensity. When his fall was successfully broken by the ankle straps, his relief was apparent in his shouts and laughter.

This man was, in fact, an Australian television reporter, and he was investigating a new sport that has sprouted down under. New as a sport, anyway. It's actually based on a ceremonial practice that can be seen on the islands of Vanuatu, which is the subject of "Taking a Flying Leap" on page 10. (And the gentleman in the long coat on our cover, calmly making calculations—just who is *he*?!)

FEATURES

DEPARTMENTS

PUBLISHER'S PAGE

A good question

—it's worth a thousand routine answers

AT *QUANTUM* WE'RE CONTINUALLY questioning what we do. We're a rather new magazine, so that's to be expected, but I hope we keep this self-critical attitude as we grow older. If you don't mind, I'd like to draw you into our questioning mode . . .

Do you think you're intelligent? Do you know physics or math pretty well? Have you thought much about what it means to "know" something? I have a good friend who has carefully probed for genuine understanding of physics among young people who have studied it. His observations suggest that true knowledge and understanding are frequently elusive.

Knowing a fact or how to solve a particular class of problem isn't enough. How do you know the "fact"? Why do you believe it? And, as for the problem, do you merely know a procedure that can unquestionably (and unquestioningly) be followed to arrive at a solution? Or, instead, can you identify the relevant laws, or principles, and definitions and use them in a correct and *efficient* approach to the problem?

Can you understand the difference between facts, data, or observations and inferences? How is a theory or model created, and what makes one better than another?

The point, of course, is this. It's not what you profess to know that's important. *How* do you know something? Why do you believe it? How would you find out? *That's* what's really important. I regret that we, as teachers, haven't emphasized the ability to ask good questions and think things through. Too often we've taught and tested for facts and information. As my friend has observed, there almost seems to be a "destructive collusion between students and teachers—a collusion in which students agree to accept bad teaching provided they are given bad examinations." What do *you* think?

We publish *Quantum* for *you*. It is meant to be challenging, as well as entertaining, and should demonstrate what it means to think about a problem, think about a solution, think about assumptions, think about alternatives—in short, to think. It's dedicated to the proposition that thinking is enjoyable.

Can you read most of what is printed in *Quantum*, and do you understand what you are reading? You probably can't just say "yes" or "no" to that question. Is some of it easy? Does some of it require careful study while you fill in details left out of the article? Is some material simply too hard?

We aim *Quantum* mainly at young people who would answer "yes" to all of these questions. These young people would be among our nation's best and brightest. But we need to know for sure that we haven't aimed either too high or too low. Again, what do *you* think? We don't have a Letters to the Editor column, but we *do* read and think about every letter we get.

So write a letter and tell us if *Quantum* is interesting, challenging, and mostly comprehensible to you. If it isn't, tell us exactly what you think we could do to make it more nearly what you'd like.

—Bill G. Aldridge

QUANTUM

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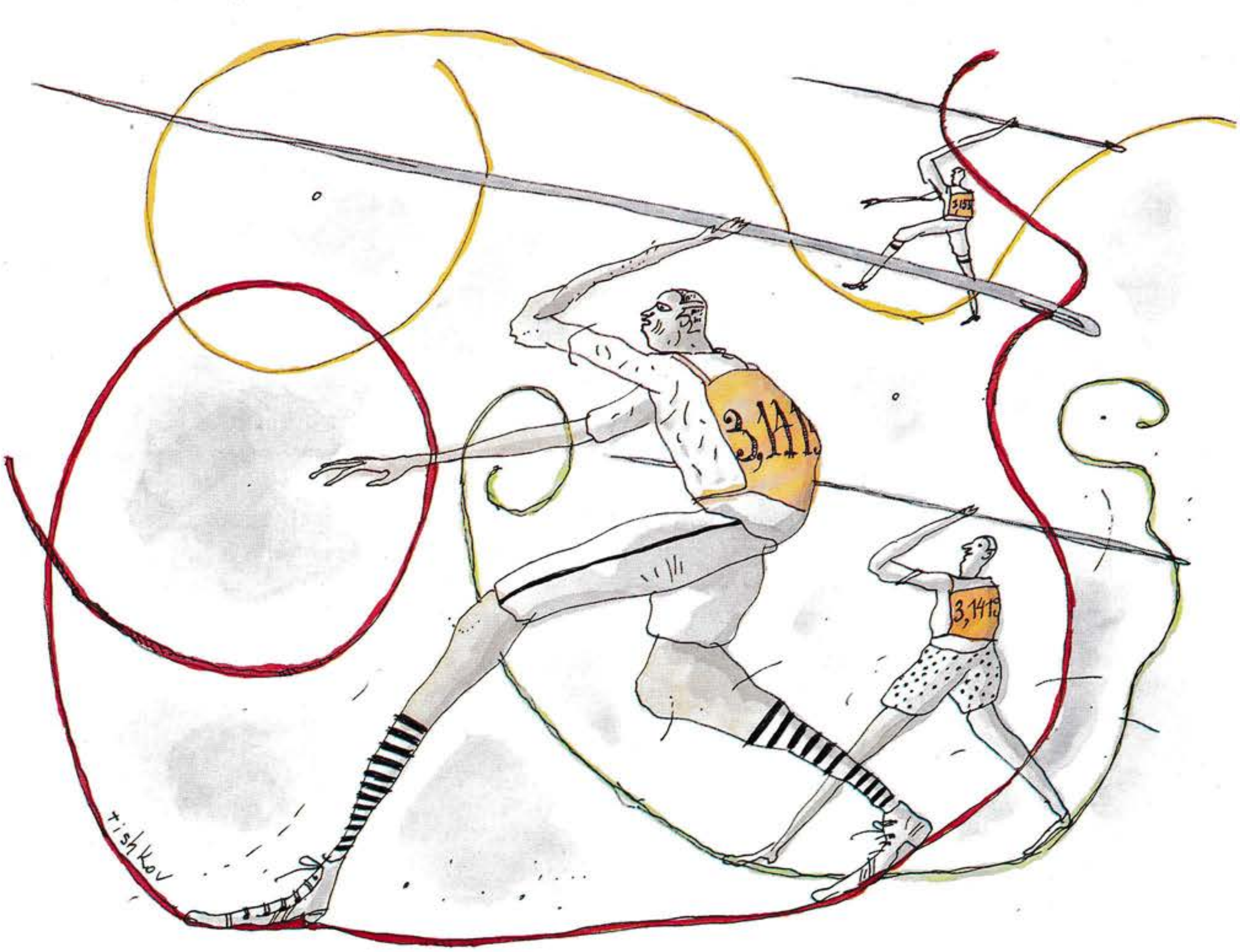


Art by Leonid Tishkov

EATING HUMBLE PI

Delusion or fraud?

Tossing a needle and counting crossings to calculate π—and the suspicious results of a certain Lazzarini

by A.N. Zaydel

IS THERE AN EXPERIMENTAL way to determine the value of π? Of course there is: take a thread, measure the circumference of a circle with a known diameter, and divide that amount by the diameter. There is also, however, a completely different way. We can get an approximate value of π by using a needle . . . and a touch of probability theory. This approach was invented by the French naturalist G.L.L. Buffon (1707–1788). Later Buffon's original experiment was repeated time and again to confirm or refute some conclusions of probability theory (or, to be more precise, their applicability). This resulted in an amusing incident, which I'm about to relate. But first I'll describe the experiment.

Buffon's experiment

To do Buffon's experiment you need a needle and a horizontal surface ruled by a grid of parallel equidistant lines. The distance h between the lines and the length l of the needle (fig. 1) must satisfy the relation $l \leq h$. Toss the needle above the surface, each time giving it a little flip so that it falls freely from a height of about 50–60 cm and lands at random angles relative to the lines. After each toss write down whether the needle does or does not intersect one of the parallel lines (fig. 1a and fig. 1b, respectively) and calculate the frequency of such "crossings"—that is, the ratio of the number of throws m resulting in intersection to the total number of throws n. You'll soon see that as the number of tosses in our experiment increases, the frequency scarcely changes. Not only that, if we perform many trials consisting of many tosses, the frequency of crossings is approximately the same for every trial.

This property of frequency "stabilization" (along with the unpredictability of the result of each individual throw) is a characteristic feature of all experiments in probability theory. A certain number p ($0 \le p \le 1$) is assigned to each outcome of such an experiment and is called its probability. This number simply expresses the likelihood of this outcome in the experiment. The probability is the value around which the frequency of the outcome oscillates in an experiment of sufficiently long duration. So the approximate value of a probability can be obtained empirically by calculating the corresponding frequency. (The precise mathematical formulation of how the experimental frequency tends to a certain limiting value, or probability, is given by the law of large numbers, proved in its simplest form by Jacob Bernoulli 300 years ago.) On the other hand, probability theory makes it possible in most cases to calculate a probability theoretically by examining experimental conditions.

Later we'll show how Buffon's problem—that is, how to find the probability p that the needle will intersect a line—is solved. The result is really quite amazing: $p = (2/\pi)(l/h)$! Since the frequency of intersections m/n is approximately equal to this probability, we can approximate π by using the equation

$$\pi \cong \frac{2l}{h} \cdot \frac{n}{m}. \qquad (1)$$

If $l = h$, we can simply say that π is approximately equal to twice the total number of tosses divided by the number of crossings.

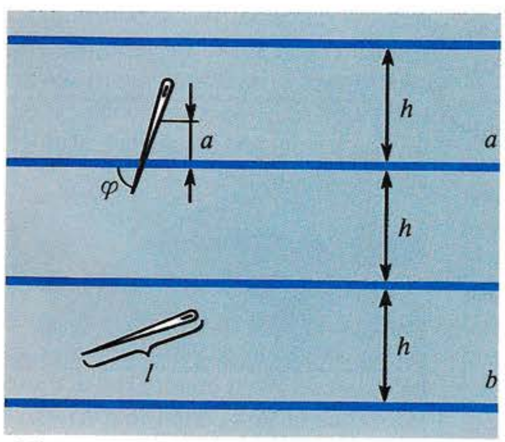


Figure 1

What the experiments showed

The accuracy of the approximation in equation (1) depends on the number of tosses n. At first glance it seems that by increasing the number of tosses we can obtain the value of π to any desired precision. Try to carry out such an experiment and you'll see that it's quite easy to obtain the value 3.1 for π. But the next decimal, 4, is much harder to get. In the 19th century, when probability theory was often regarded as a semiempirical science, such experiments were of great value and were scrupulously staged by many scientists. A table from B.V. Gnedenko's textbook on probability (which is well known and widely used in the Soviet Union) is given here by way of illustration. (The names of the scientists involved in the needle throwing are listed in the first column.)

Name	Year	Number of tosses	Experimental value of p
Wolf	1850	5000	3.1596
Smith	1855	3204	3.1553
Fox	1894	1120	3.1419
Lazzarini	1901	3408	3.1415929
True value of π to the seventh decimal place:			3.1415927

Compare the experimental results with the true value of π. The values in the first two lines differ from π by 0.01–0.02. The value obtained by Fox is greater than π by only 0.0003. This is an amazing result. But the value obtained by Lazzarini is only 0.0000002 over the true value. This is a miracle! (Or so it seems.)

The inevitability of error

What astonishes us and, to be frank, makes us skeptical of the result of the last experiment? Several things. First, the accuracy of the measurements. To obtain the precise value of π from equation (1), we must be able to precisely measure h and l (or rather, their ratio). This can be done only by actually measuring both values. Errors in measurement will surely affect the accuracy of π. Let's estimate the magnitude of the error. Suppose the needle is 50 mm long, which we'll take to be equal to the distance between the lines. Using ordinary measuring devices—for instance, a slide gage or vernier calipers—we can measure both lengths to an accuracy of 0.1 mm (0.2 %).[1]

Making use of more sophisticated instrumentation, we can measure all the lengths to an accuracy of 0.01 mm. At that point we've pretty much reached the practical limit—it's nearly impossible to reduce the error to 0.001 mm because a variation in the needle's or the surface's temperature of only 1 or 2 degrees results in a variation of about 0.001 mm in the measured distances. Deformation of the needle caused by its collision with the surface, wear at its tips, deformation of the surface itself—all these factors make it quite unreasonable to try to achieve a level of experimental error of 0.001 mm. Even without a more detailed evaluation, we can still say with assurance that the experiment can't determine the value of π to an accuracy better than 0.2–0.02%.

[1]For a small number of tosses (approximately 0.1–0.2% of the total number of tosses) the distance from the needle's tip to the line will be less than 0.1 mm. In this case the naked eye can't discern whether there is an intersection. This may also contribute to the resulting error, although this contribution is smaller than that caused by other sources of experimental error.

This means that with conventional measuring devices the best we can expect to obtain is $\pi = 3.141 + 0.006$. If the experiment is conducted with the utmost rigor, we might hope to get $\pi = 3.1416 + 0.0006$. Wanting to derive the value of π in this experiment to eight decimal places is like trying to weigh a match on a railroad scale. In both cases, the instruments are too crude.

In any measurement we should evaluate the accuracy first and then reduce the result of the calculation to the corresponding number of decimal places. If in measuring a value A we get $A = 2.474329$, and the accuracy of the measuring device is 1%, we should write $A = 2.47$, dropping all the subsequent digits, which merely reflect the accuracy of our computing device. (Even ordinary calculators now display 6 to 8 decimal places.) By writing all the decimal places obtained, we overrate the accuracy of the experiment and thereby report a misleading result.

There is, however, no use in trying to increase the accuracy of our measurements. It's quite sufficient to measure the needle's length to 0.1 mm. That's because the accuracy of determining π is already limited by a completely different, "probabilistic" circumstance—the impossibility of performing enough tosses to ensure that the approximating equation (1) holds with a relative error of less than 0.01! There's a way of estimating the rate at which the difference $|m/n - p|$ decreases as the number of tosses increases. It turns out that to increase the accuracy of the approximate equality $m/n \cong p$ by a factor of N, we have to increase the number of tosses by a factor of N^2. In other words, *the error is inversely proportional to the square root of the number of tosses.* A detailed examination of Buffon's experiment shows that for $h \cong l$, the accuracy α that can reliably be expected for π after n tosses is given by the formula

$$\alpha \cong \sqrt{\frac{5}{n}}. \qquad (2)$$

If we demand that α be less than 0.02, we should be prepared, according to equation (2), to toss the needle nearly 12,000 times. The experiments listed in the table (except for Fox's) involve only 3,000 to 5,000 throws. According to equation (2), the resulting α is somewhat higher then—about 0.025–0.030, which is in good agreement with the results of Wolf and Smith. About the same level of error occurs when the needle's length is measured to an accuracy of 0.1 mm. So there's no need to strive for greater accuracy in these measurements: it's useless to measure the length of the needle to an accuracy of 0.001% if the limitation imposed by the number of tosses results in an overall error of 0.2–0.3%.

Choosing a number of tosses between 3,000 and 5,000 makes good sense. One toss might take about 5 seconds (try to toss a needle faster). An experiment consisting of 10,000 tosses would take approximately 14 hours—two full working days. If we want to obtain results 10 times as precise, it'll take 100 times as long—200 days (according to the "$1/n^{1/2}$ law"), which is too long for such an experiment. In order to obtain the Lazzarini result, whose accuracy was 0.0000002, we would have to throw the needle for about 4 million years! (And this result could have been obtained only if the length measurements were made to an absolutely unreal level of accuracy.) So, starting his experiment in 1901 and throwing the needle until now, Lazzarini would have been as far from his published result as he was on the first day.

There's one more source of experimental error in Buffon's experiment. For the probability of intersections to equal the theoretical value of $p = 2l/\pi h$, we must ensure that all the needle's positions with respect to the lines upon the surface are absolutely equivalent—that is, that none of them has an intrinsic tendency to occur more often than the others. (A more precise formulation of this condition is given below.) In a real experiment it's very unlikely that this could be achieved.

So there are ample grounds for concluding that the number of decimal places given in the table is too high. To be credible, the first two entries in the last column should read 3.16, and the third should read 3.14.

And so—delusion or fraud?

But how can we explain the result obtained by Lazzarini? We can hardly suspect him of deliberate fraud. When Lazzarini was throwing his needle, the law of large numbers was already well known—our calculation could have been done by any mathematician of the time. Most probably the scientific community never took his result seriously, despite the fact that it has been republished a number of times. Before he began his experiment, Lazzarini should have known what he could expect from it. And even after obtaining a number as a result of certain arithmetic exercises, he should have refrained from publishing such a fantastic result until he was able to reproduce it in an independent series of tests. Personally, I would guess that he was too eager to outdo all his predecessors. Overweening ambition sometimes deludes researchers, leading them to find what they *want* to find.

Another possible explanation is that while throwing the needle Lazzarini calculated π after each throw and ended the experiment after 3,408 throws, having obtained the value given above by pure chance. Of course, even after 10,000 throws it's practically impossible to obtain a given value to an accuracy of $2 \cdot 10^{-7}$ even once—the probability is about 10^{-3}—but it could happen. If that was indeed the case, Lazzarini deceived himself rather than others. Unfortunately, this happens in science from time to time.

Although the result of Lazzarini's experiment didn't confirm the conclusions of probability theory (taken seriously, it would have contradicted them), it did serve to generate this cautionary tale. Once again we are reminded to be cautious with experimental results and the statistical analysis of these results.

Solving Buffon's problem

It remains for us to explain how the probability that the needle will intersect a line in Buffon's experiment is calculated. We'll denote the distance from the needle's center to the nearest line by a $(0 \le a \le h/2)$ and the angle between the needle and the line by φ $(0 \le \varphi \le \pi/2)$. Then, as figure 1a shows, the needle intersects a line if and only if

$$a \le \frac{l}{2}\sin\phi .$$

Each possible outcome in our experiment is described by a point in the plane having coordinates (a,φ) and lying inside the rectangle bounded by the coordinate axes and the straight lines $a = h/2$ and $\varphi = \pi/2$ (fig. 2). The points of the rectangle lying below the curve $a = (l/2)\sin\varphi$ represent the intersections, while the points above the curve correspond to the outcomes when the needle doesn't cross a line.

The problem can now be reformulated in the following way: *A point is chosen at random within the rectangle* $\{0 \le \varphi \le \pi/2; 0 \le a \le h/2\}$. *What is the probability that it lies below the sinusoidal curve* $a = (1/2)\sin\varphi$?

The term "at random" here replaces the requirement that all the positions of the fallen needle be equivalent. There are several ways of precisely defining this term. The simplest is to require that the probability that a point will fall into a square $\{\varphi_1 \le \varphi \le \varphi_2; a_1 \le a \le a_2\}$ $(\varphi_2 - \varphi_1 = a_2 - a_1 = d)$ with a given side doesn't depend on the position of the square inside the rectangle (although it obviously depends on the length d of its side). For instance, the probabilities of finding the point inside the squares K_1 and K_2 (fig. 2) are equal. From this we can easily deduce that the probability of landing in any figure inside our rectangle is proportional to its area and consequently equals the ratio of this area to the area of the entire rectangle.

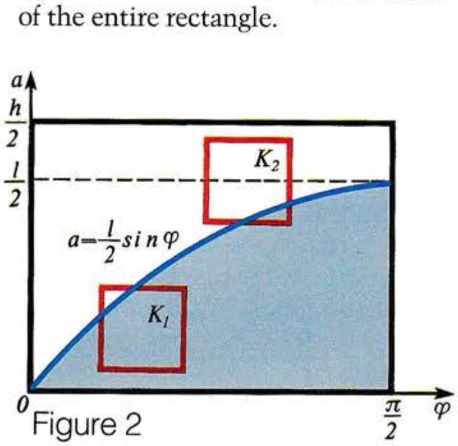


Figure 2

In particular, the area of the curvilinear trapezoid formed by the sinusoidal curve is equal to

$$S_1 = \int_0^{\pi/2} \frac{l}{2}\sin\phi\, d\phi$$

$$= \frac{l}{2}\left[\left(-\cos\frac{\pi}{2}\right) - (-\cos 0)\right]$$

$$= \frac{l}{2}.$$

Since the area S of the rectangle equals $\pi h/4$, we get the probability

$$p = \frac{S_1}{S} = \frac{2l}{\pi h}$$

—just as we predicted a few pages back! ◘

Taking a flying leap

Applying Hooke's law on a South Seas isle

by A.A. Dozorov

NATIVES ON ONE OF THE ISlands of Vanuatu (formerly the New Hebrides) celebrate their festivals in a rather bizarre way. A young man mounts a special tower, attaches vines to his ankles, and then, accompanied by the music of a ritual dance, throws himself off. The vines, tied at the other end to the top of the tower, break his fall in such a way as to dampen the shock, and the young man lands safely.

The height of the towers used on the island varies from 15 to 30 meters. You'd think the load on the jumper's legs would increase as the tower grows taller and the vines grow longer. As a result, there should be a maximum altitude for safe jumping. This straightforward and obvious conclusion, however, is absolutely wrong. The right answer is provided by Hooke's law.

Let the length of a freely suspended string be equal to l. When a force is applied to the string, it stretches to the length $l + x$. The value x is called the absolute deformation of the string, and the value $x/l = \varepsilon$ is called the relative deformation. This deformation is said to be "elastic" if, after the force is removed, the string returns to its original length. Generally, for small elastic deformations ($x \ll l$) the value of the absolute deformation is proportional to the force applied. The direction of the elastic force tending to return the string to its initial state and the direction of deformation are opposite. This situation is described by Hooke's law:

$$F = -kx. \qquad (1)$$

The factor k is the rigidity coefficient of the stretched body (for example, a string or spring).

The greater the cross section of the string, the greater the force that must be applied to achieve the same stretching of the string. In other words, the rigidity coefficient is a function of the string's cross section. In such a situation the term stress, $\sigma = F/S$, is often used. (If a rod of cross section S is compressed by a force F, then σ is the average pressure on its end.)

The absolute value of the relative stretching ε for small deformations of elastic bodies is proportional to the absolute value of the stress σ:

$$\varepsilon = \frac{1}{E}\sigma . \qquad (2)$$

This relationship reflects another formulation of Hooke's law. The coefficient E is called Young's modulus. Using equation (2) and recalling that $\varepsilon = x/l$, we can write the expression for the force in another way:

$$F = \frac{SE}{l}x . \qquad (3)$$

Comparing equations (1) and (3), we get the following relationship between Young's modulus and the rigidity coefficient: $k = SE/l$. Young's modulus is dependent only on the material from which the string is made, while the coefficient k depends also on the shape of the stretched sample.

According to the definition, Young's modulus is numerically equal to the force stretching a string with a unit cross section to double its length (the dimensionality of Young's modulus in SI units is N/m^2).

Let's return to Vanuatu now and try to estimate the maximum stretching of the vines. This amounts to solving the following problem.

A mass m = *72 kg is suspended on an elastic weightless string and dropped from the point where the other end is* *attached. Find the maximum force stretching the string and the maximum acceleration of the mass while its fall is being broken. Young's modulus for the string is $E = 10^7$ N/m², and its cross section $S = 9$ cm².*

The work expended in stretching the string by an amount Δx is equal to

$$\Delta A = F \cdot \Delta x.$$

The force F is proportional to x (fig. 1). The work expended over the interval Δx is numerically equal to the area of the trapezoid $ABCD$.

If the string stretches from length l to length $l + x$, the work expended (and consequently the potential energy acquired by the string) is determined by adding up all the components of the work. That is, the string's potential energy W is defined by the area of the triangle OCD:

$$W = \frac{1}{2}Fx = \frac{1}{2}kx^2 . \qquad (4)$$

The potential energy of the mass is transformed into the kinetic energy of its motion and then into the energy of the string's deformation. Since the total height from which the mass falls is equal to $l + x$, then

$$mg(l+x) = \frac{1}{2}kx^2 .$$

So the stretching of the string is equal to

$$x = \frac{mg + \sqrt{m^2g^2 + 2kmgl}}{k},$$

and the maximum stretching force (since $k = SE/l$) is given by

$$F = mg + \sqrt{(mg)^2 + 2mgES} .$$

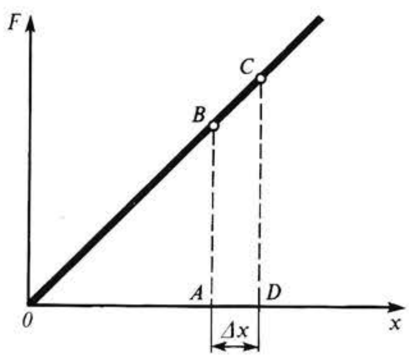


Figure 1

The overload is the difference between this force and the force of gravity mg. Therefore, the maximum overload acceleration is

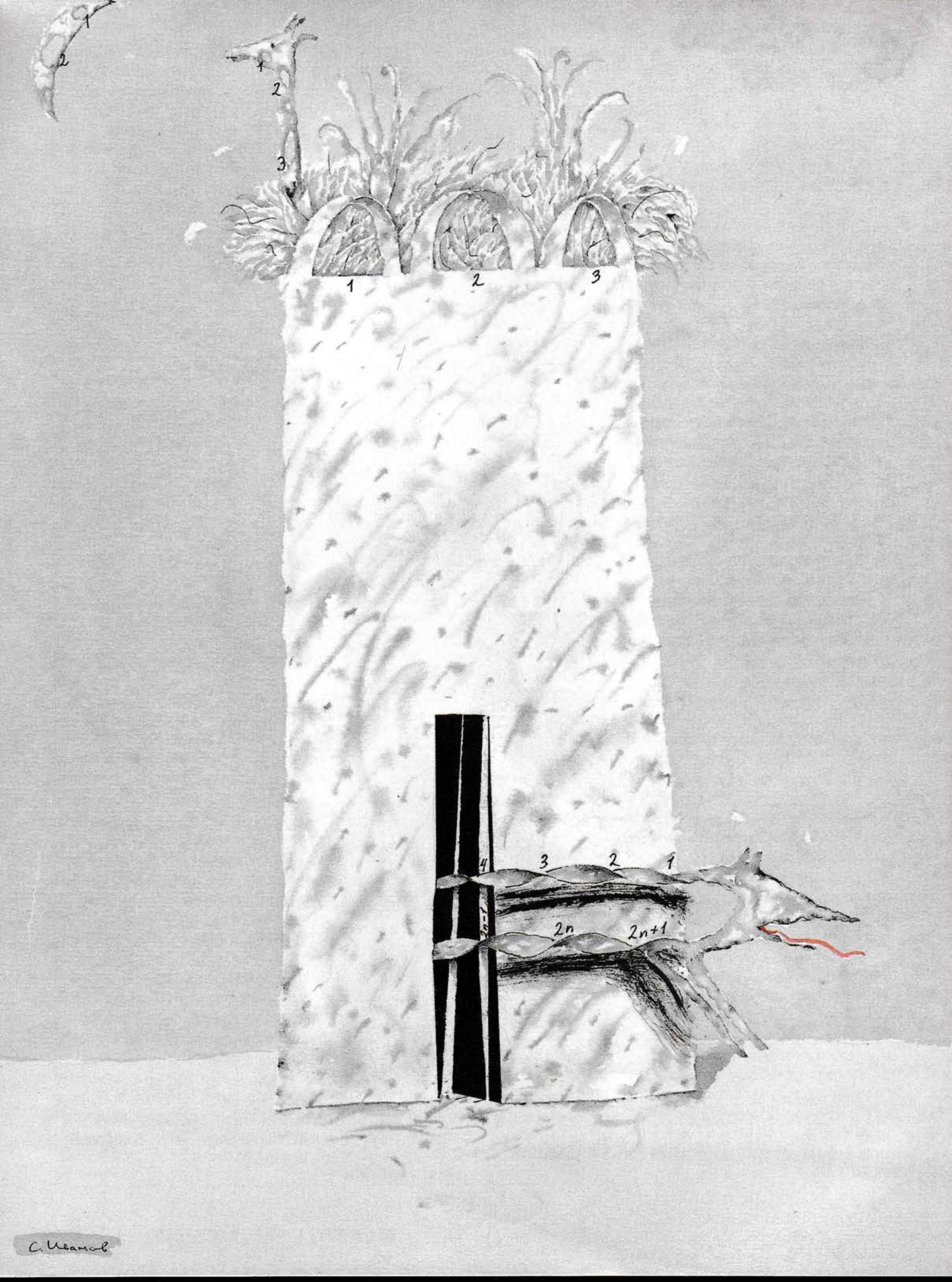
$$a = g\sqrt{1 + \frac{2ES}{mg}} .$$

(This obviously occurs at the lowest point.)

So neither the maximum stretching force nor the maximum overload acceleration depends on the string's length.

As for jumping from a tower, we can now conclude that there is no critical height beyond which the human body could not withstand the overload (that is, the additional g-forces). If the elastic properties of the vine are close to those of rubber (that is, $E \cong 10^7$ N/m²), the vine's cross section S = 9 cm², and the jumper's mass is 72 kg, we get $a = 5g$. The human body can withstand such overloads.[1]

[1]Here's something else to think about: is the overload more or less if we take the vine's weight into account? And on the same tack, who will experience more g's, a little person or a big person?—*Ed.*



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Flexible in the face of adversity

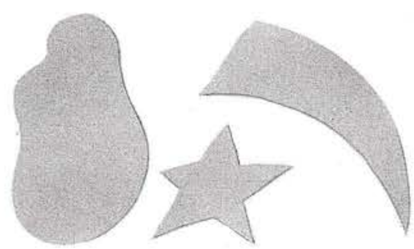
Identifying and classifying a certain breed of fanciful "creature"

by A.P. Vesyolov

THE HERO OF OUR STORY IS the "flexie"—an imaginary flexible organism living in our three-dimensional world and capable of marvelous transformations. You can think of a flexie as a flat amoebalike creature of finite size and zero thickness having a single mobile closed line as its boundary. In the vicinity of each inner (that is, nonboundary) point a flexie resembles a tiny piece of plane, possibly a bent one. Its life consists not only of continuous deformations (that is, bending, stretching, and shrinking without tearing and gluing) but sometimes of more dramatic transformations of the "cut-and-heal" type.

Here's how that happens. First, a self-inflicted wound appears on the flexie's surface—a cut that goes along an arc that starts and ends on the boundary. Then the flexie experiences convulsions—stretching, shrinking, or twisting in any possible way—until, finally, the torn edges rejoin (each point joining precisely the point from which it had been torn away), and the wound heals again. This cut-and-heal act doesn't alter the inner structure of the flexie, but it may thoroughly change its position in space (resulting in entanglement or disentanglement of the flexie). If you've ever tried to untangle a fishing line you'll certainly appreciate the extent to which your task would be simplified if the fishing line had the same properties. We'll come back to the transformations of flexies a little later, but first let's get to know some of the

Figure 1



members of the flexie family.

The simplest flexie (fig. 1) looks like a genuine amoeba, although it can take the form of a triangle, a square, or even a star. A special term, "disk," has been coined to designate such objects.

The flexie shown in figure 2 is called (more or less understandably) a "handle," although it resembles a punctured inner tube more than the broken handle of a cup. Its shape, however, can change beyond recognition (fig. 3).

The next flexie (fig. 4) merits a bit more discussion. We'll call it "amoebius" after its discoverer, A.F. Moebius (though it's usually known as a "Moebius strip").

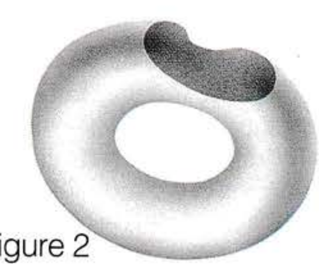


Figure 2

August Ferdinand Moebius (1790–1868), a leading geometer of his time, was the director of a German observatory and devoted much of his time to astronomy. But for many years he also contemplated the properties of objects similar to our flexies, in particular those of the Moebius strip shown in figure 4. Only after many years did Moebius venture to submit the results of his work to the French Academy in Paris, but the subject of his "Memorandum on Single-sided Surfaces" was so unusual his manuscript gathered dust on the Academy's shelves until the author finally decided to publish it as a separate book at his own expense. Around the same time the German astronomer I.B. Listing (1808–1882) independently obtained and published results similar to those of Moebius.

It's easy to construct a model of the amoebius—in fact, you should do that before we go any further. Take a strip

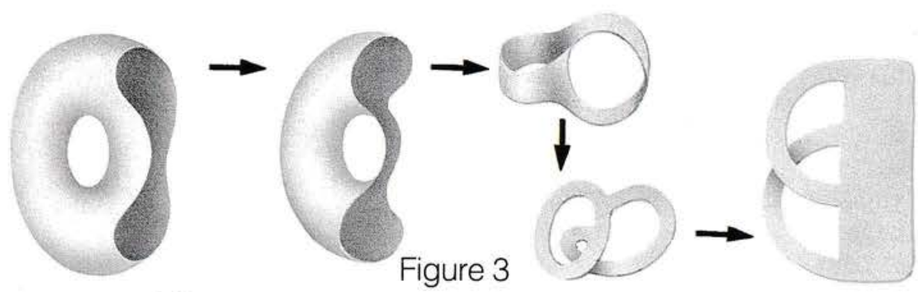


Figure 3

Art by Sergey Ivanov

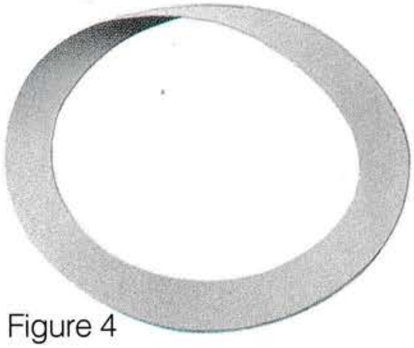


Figure 4

of paper (approximately 20 cm by 4 cm), draw a blue line down the middle along its entire length on one side and a red line on the other, and then glue the ends of the strip together after giving one of them a twist of 180° (fig. 5). Where the ends meet, the blue line runs into the red one, so that now both lines turn out to be on one and the same side of the surface. This means the amoebius has only one side! You can now run your finger along the entire surface of the amoebius without going over the edge, which you couldn't have done with the original strip of paper that had two sides (with blue and red lines).

Let's perform an act of pure barbarism: take a pair of scissors and cut the amoebius along the red–blue line. How do you like the result? Be honest—did

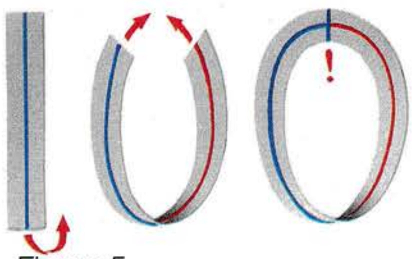


Figure 5

you expect such an outcome (unless you had already heard about it)? Although the amoebius didn't split into two pieces, the cut still had a disastrous effect: the boundary of the new surface consists of two closed curves. (Check to make sure.) So, according to our definition, it's no longer a flexie! Not only that, it has acquired a second side! The next cut won't produce such a striking result, although I doubt you'll be able to predict what will happen.

Let's quit our brutal behavior and take a look at what goes on with an amoebius under natural conditions. First of all, owing to its cut-and-heal ability it can acquire the shape of a strip twisted by any odd number of half turns (fig. 6). (Notice that a strip twisted by a whole number of turns isn't a flexie at all since it has *two* closed curves for its boundaries.) In fact, after cutting itself apart the amoebius can untwist itself by any whole number of turns and then heal itself along the same edge again. (Check this with a strip of paper.) Similarly, the amoebius can tie itself into any knot and disentangle itself again (fig. 6).

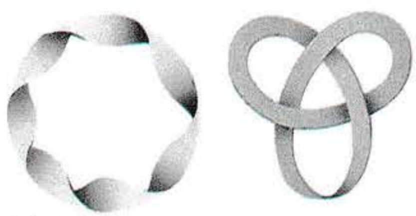


Figure 6

No temporary self-inflicted wounds like these would have been necessary if the amoebius lived in four-dimensional space. In addition to providing a convenient escape from a completely sealed room (an achievement well known and widely exploited in science fiction), the fourth dimension enables one to freely undo knots and disentangle flexies. In the course of evolution the cut-and-heal ability of our flexies arose precisely because of the "limited dimensionality" of our space.

We can identify an entire (infinite) family of flexies, which we'll call the "amoebius family." After the amoebius itself, the next representative of this family is obtained by "fusing" two amoebii together along a section of their boundaries (fig. 7). The third representative is obtained by fusing three amoebii, and so on. The shapes of these flexies can be quite diverse, so they're not easy to recognize. Figure 8 shows the entire family in one of its

Figure 7

most symmetric states.

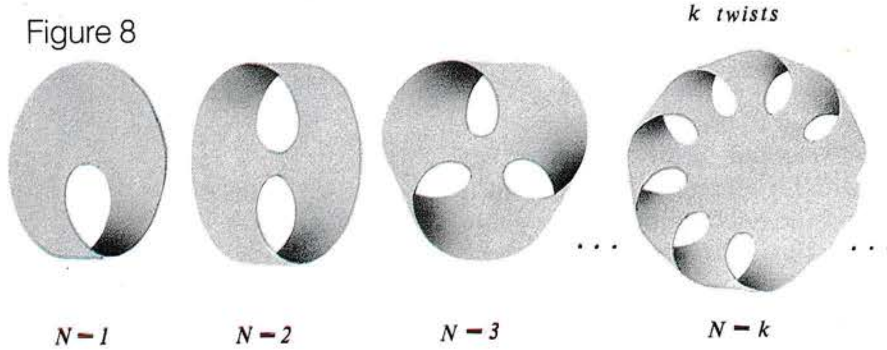
In exactly the same way the handle generates a family of two-sided flexies obtained by sequentially fusing a number of handles to one another (fig. 9). For lack of a better name, we'll call this the "bilateral family." The bilateral family likewise consists of an infinite number of species described by the number of handles making up a particular flexie. Since they are two-sided, disks naturally belong to this family as well. (In this case the number of handles $N = 0$.)

I just can't decide which of these many shapes is the most beautiful, so the bilateral family is represented twice, in figures 10a and 10b. Take the double handle ($N = 2$), for example, and try to see for yourself that you actually have two different forms of the same flexie. It's quite a challenge, but it will bring true pleasure to a genuine geometrician. Maybe you'll discover even more elegant forms of these same flexies.

By now you're probably thinking that, various and rich as the class of our flexies is, their external appearance is just as complex. So the following result, which is the cornerstone of "flexiology," may catch you by surprise:

> *The class of all flexies consists of two infinite families—the amoebius family (fig. 8) and the bilateral family (fig. 10).*

Figure 8



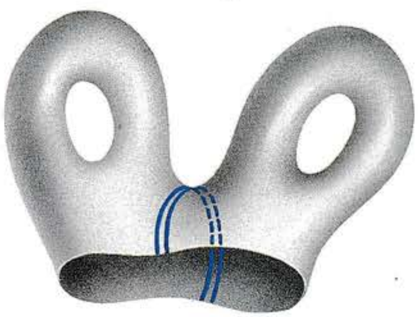


Figure 9

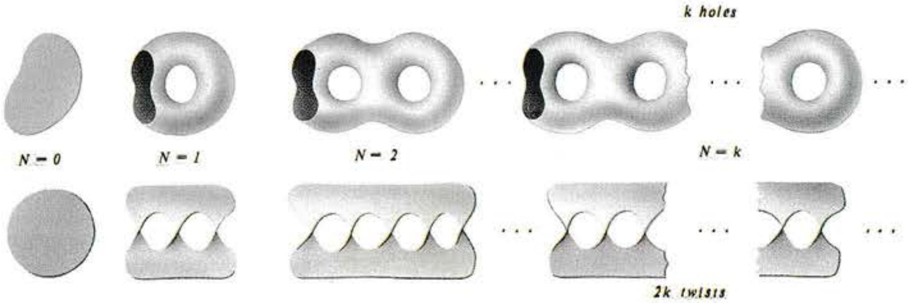


Figure 10

This is a typical classification theorem. It states that any representative of a certain class (in our case, any flexie) can be classified as belonging to one species in a certain list (in our case, the amoebius and bilateral species).

"No way!" you may exclaim after thinking about it for a while. "To begin with, you forgot about the flexie we get by fusing an amoebius with a handle" (fig. 11). But have I really overlooked it? Let's analyze the situation more closely by taking, for the sake of convenience, the handle in the

Figure 11

form presented at the extreme right in figure 3 (transferred now into figure 12). The amoebius "turns the handle inside out," transforming it into two amoebii so that our flexie becomes nothing other than a triple amoebius, shown in figure 8 (marked "$N = 3$"). This is really an amazing result, since we could have achieved it by replacing the handle with a double amoebius—an absolutely different flexie! Now it's easy to understand that the fusion of k amoebii ($k \geq 0$) and l handles ($l \geq 0$) yields a $(k + 2l)$-fold amoebius. So combining handles with amoebii gets us nothing new. (Notice that from a biologist's point of view two-sidedness is a recessive trait and amoebiusness is dominant. See, for example,

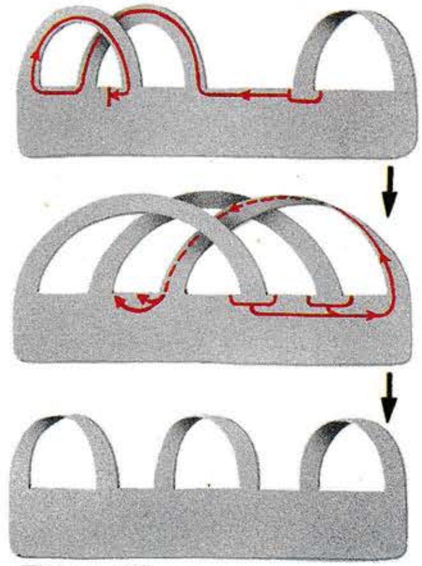


Figure 12

The amazing transformation of a handle fused with an amoebius into a triple amoebius. First, one of the bases of the amoebius is dragged along the edge of the handle (a) until it takes a position between the two neighboring bases of the handle (b). Then the two right bases of the handle are dragged one after the other along the edge of the twisted amoebius strip (see figure b). Moving along the edge of the amoebius, the two strips of the handle are twisted, which results in two new twisted strips, one of them embracing the other. The strip that is embraced can be released by dragging both its bases along the edge of the embracing strip. This results in a triple amoebius!

(Here we have dragged the end of a strip along the flexie's edge. Any flexie can be subjected to such an operation: the section along the flexie's boundary is exceptionally elastic, whereas the base of the strip becomes rigid; the strip moving along its edge stretches the flexie's elastic edge behind it and compresses it in front. To an outside observer it looks as if the base of the strip simply slides along the flexie's edge.)

Quantum, May 1990, "The Geometry of Population Genetics" by I.M. Yaglom.)

The time has come, however, to prove our main result. We'll make use of the intuitively obvious fact that an appropriate number of cuts can transform any flexie into a disk. Two cuts are necessary for a handle, whereas one cut is sufficient for an amoebius. The boundary of the disk obviously includes the traces of the cuts (fig. 13). To restore the flexie we should perform the inverse operation and "heal" them. Here's how to do it. For each of the cuts, take a strip of paper and glue one end to one edge of the cut. Then stretch the strip over the disk in an arc and glue the other end to the other edge. Notice that the strip obtained in this way can be one of two types. We'll call them type 1 (fig. 14a) or type 2 (fig. 14b), depending on the relative orientation of the cut's edges on the boundary of the disk.

The resulting flexie looks like the one in figure 15.

Any flexie can be deformed into such a shape in two stages. Stage 1 is shown in figure 16. You take a type 1 strip and drag the ends of all other strips from under the first one, separating it from the rest. For type 2 strips the situation is different. We can't

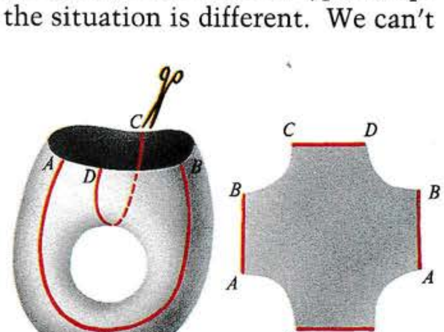


Figure 13

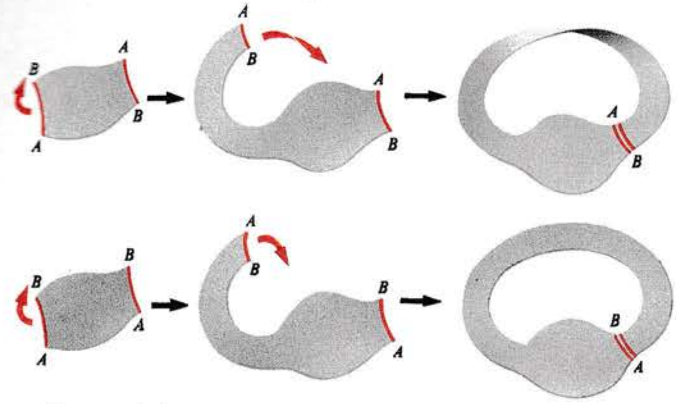


Figure 14

remove all the other ends from under a type 2 strip (otherwise the boundary of the whole flexie would consist of at least *two* curves). One end of a strip must remain. The remaining strip is necessarily also of type 2. It's logical to call this pair of strips a handle (recall figure 3).

In stage 2 a similar operation moves everything from under each handle (fig. 17) and the flexie acquires the shape of a disk with separate "pure" handles and strips of type 1 or type 2 attached to it. (No doubt you recognized the latter as amoebiuslike.) If there is at least one amoebius, it will turn all the handles into double amoebii (see figure 12), yielding a representative of the amoebius family (fig. 8). Otherwise, the flexie consists only of handles and is therefore bilateral, which completes our proof.

But is this really the end of the proof? We've established that any flexie belongs to one of the species in the list. Can it belong to several species at once? In other words, can the two species actually be the same (that is, be transformed into one another according to the natural laws governing the behavior of flexies)? The answer is no, but I won't give you the proof here. Check the books mentioned at the end of the article if you're curious.

Notice that our proof provides an effective method of determining to which of the species a given flexie belongs. In particular, it makes it possible to judge when two given shapes are actually different forms of one and the same flexie.

Problem. Determine the species of flexie depicted in figure 18. Also, show that the shapes given in figure 10 are in fact different forms of the same bilateral flexie.

To avoid any possible misunderstanding, before I finish I'd like to emphasize that the subject here has not really been biology but rather topology—an area of mathematics dealing with the properties of bodies that retain their shape after being arbitrarily shrunk and stretched. So I'll give you a short dictionary for translating our main ideas from the language of biology used here into the language of mathematics used elsewhere. "Flexies" are *compact connected two-dimensional manifolds with a connected boundary*. "Bilateral" and "amoebiuslike" flexies correspond to *oriented* and *nonoriented* two-dimensional manifolds with a connected boundary, respectively. "Amoebius," as I mentioned earlier, is the *Moebius strip*. The terms "disk" and "handle" are borrowed from topology and need no translation. So what we have is actually a topological classification of two-dimensional manifolds with a connected boundary. It yields, in turn, a topological classification of such connected manifolds as the sphere or torus (the surface of a doughnut), since cutting a small hole in such a manifold turns it into a flexie.

A more detailed treatment of these problems can be found in an exceptionally interesting book by V.G. Boltyansky and V.A. Efremovich, *Topology in Pictures* (Moscow, Nauka Publishers, 1980, in Russian). I also recommend you take a look at *Experiments in Topology* by Stephen Barr (New York, Thomas Y. Crowell Company, 1964).

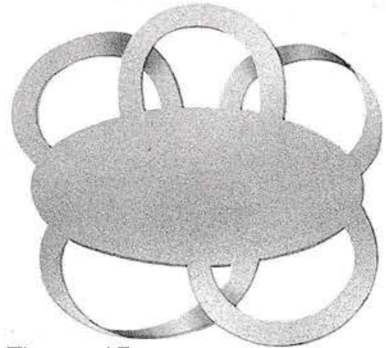


Figure 15

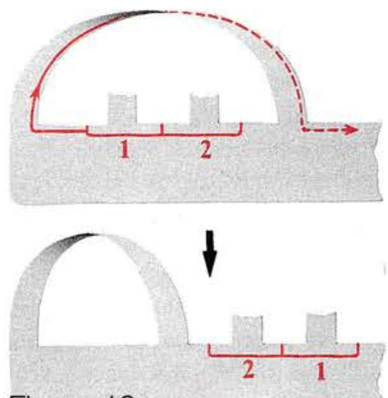


Figure 16

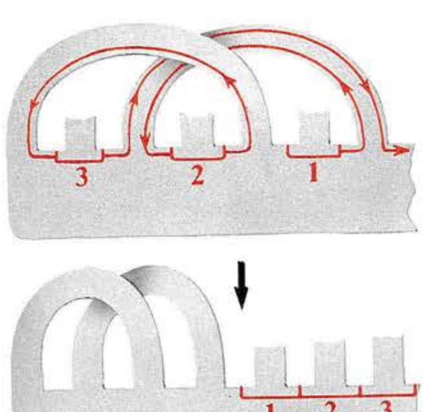


Figure 17

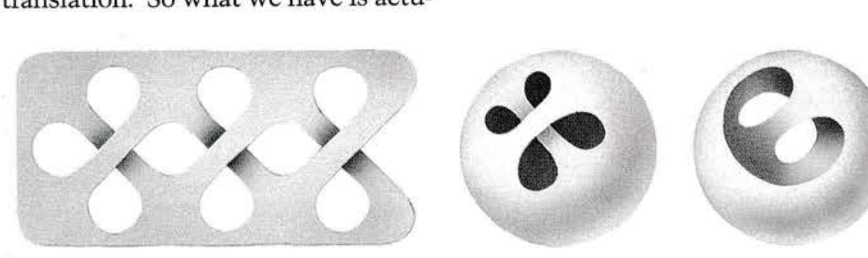


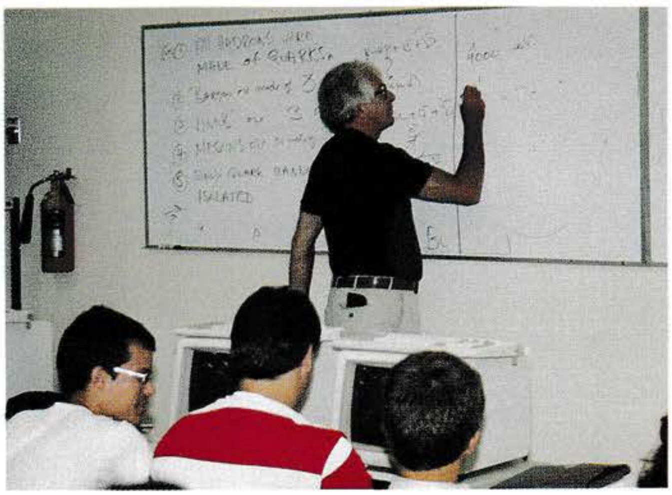
Figure 18

The Sky's *Not* the Limit!

The year 1992 has been declared the International Space Year (ISY) by the United Nations. Scientists from many countries will meet at various conferences, seminars, and symposiums to discuss the future of international cooperation in space. We hope there will be many new agreements on joint projects, including perhaps one about a joint Mars mission. All these projects will need many new researchers. Many of them will be among those who are presently going to high school. For this reason work with youth has been an important part of the ISY. One of the projects under development is the 1992 International Space Olympiad in Washington, DC.

Summer study in the USSR and US

To prepare for this olympiad, several American and Soviet organizations, including the magazines *Kvant* and *Quantum*, the US International Space Year Association, the Soviet Aerospace Society "Union," the National Science Teachers Association, and the International Educational Network, have decided to organize an International Summer Institute in the summer of 1991 in the United States and the Soviet Union. The program will feature **advanced classes** in mathematics, physics, biology, and other space-related subjects; lectures by **prominent scientists**; trips to major **scientific laboratories**; **sports and recreation**; and many **cultural activities**.



Nobel Laureate Sheldon Glashow of Harvard University instructs participants in a previous International Educational Network summer camp.

Three-stage competition

Sixty students from the US and 60 from the USSR will be selected, and we expect that students from other countries will also be interested in participating. The selection process will be based on the results of a three-stage competition. The questions for the first round are printed below. The second round will also be by correspondence and will include two math and two physics problems related to space. A total of 300 students will be invited to participate in the third round, which will be given at local universities or schools in the presence of the organizers' representatives.

Three-week program

The winners will participate in either the American or the Soviet part of the program, which will each last three weeks. The American session will take place July 1–21, 1991, while the Soviet session will take place August 1–21, 1991. Each session will feature two weeks of study and one week of travel in the host country. The winners of the competition, depending on their total score, will receive **scholarship prizes and awards** that will cover all or part of the program costs.

To enter the competition, please fill out the form and mail it, along with your answers to the questions printed below, postmarked no later than December 31, 1990, to:

Dr. Edward Lozansky, President
International Educational Network
3001 Veazey Terrace, NW
Washington, DC 20008
(Telephone: 202 362-7855)

Yes, I am interested in the 1991 International Summer Institute!

Last name ________________ First name ________________

Home address ________________

City ________________ State ________ Zip ________ Birthdate ________ Sex ______

Home phone (______) ________________ Parent's office phone (______) ________________

School name ________________

School address ________________

________________ Phone (______) ________________

Name of math or science teacher who can recommend you ________________ (print first and last name)

Please answer the following questions:

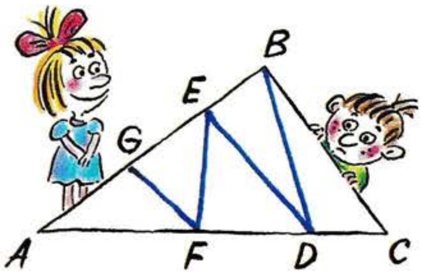
1. *When was the first manned space ship launched? Who piloted this ship?*
2. *Who was the first man on the moon?*
3. *Name all American and Soviet women who have been in space.*
4. *Write a short essay explaining why you would like to participate in this program.*
5. *Could you write this essay with a ball point pen while orbiting the Earth? Explain.*

Teachers are encouraged to copy this page and distribute it to potential participants.

Just for the fun of it

Problems offered for your enjoyment by A. Savin, G. Galperin, M. Lobak, Y. Kurlyandchik, and S. Sefibekov

B11
How can a polygonal line BDEFG be drawn in a triangle ABC so that the five triangles obtained have the same area?



B12
The product of a billion natural numbers is equal to a billion. What's the greatest value the sum of these numbers can have?

B13
A glass flask of an irregular shape contains a certain amount of liquid. Is it possible to tell (without any measuring devices or other containers) whether the flask is more or less than half full?

B14
Is it possible to add four digits to the right of the number 9999 so that the eight-digit number obtained becomes the square of an integer?

B15
Winnie-the-Pooh and Piglet went to visit each other. They started at the same time and walked along the same road. But since Winnie-the-Pooh was absorbed in composing a new "hum" and Piglet was trying to

Art by Edward Nazarov

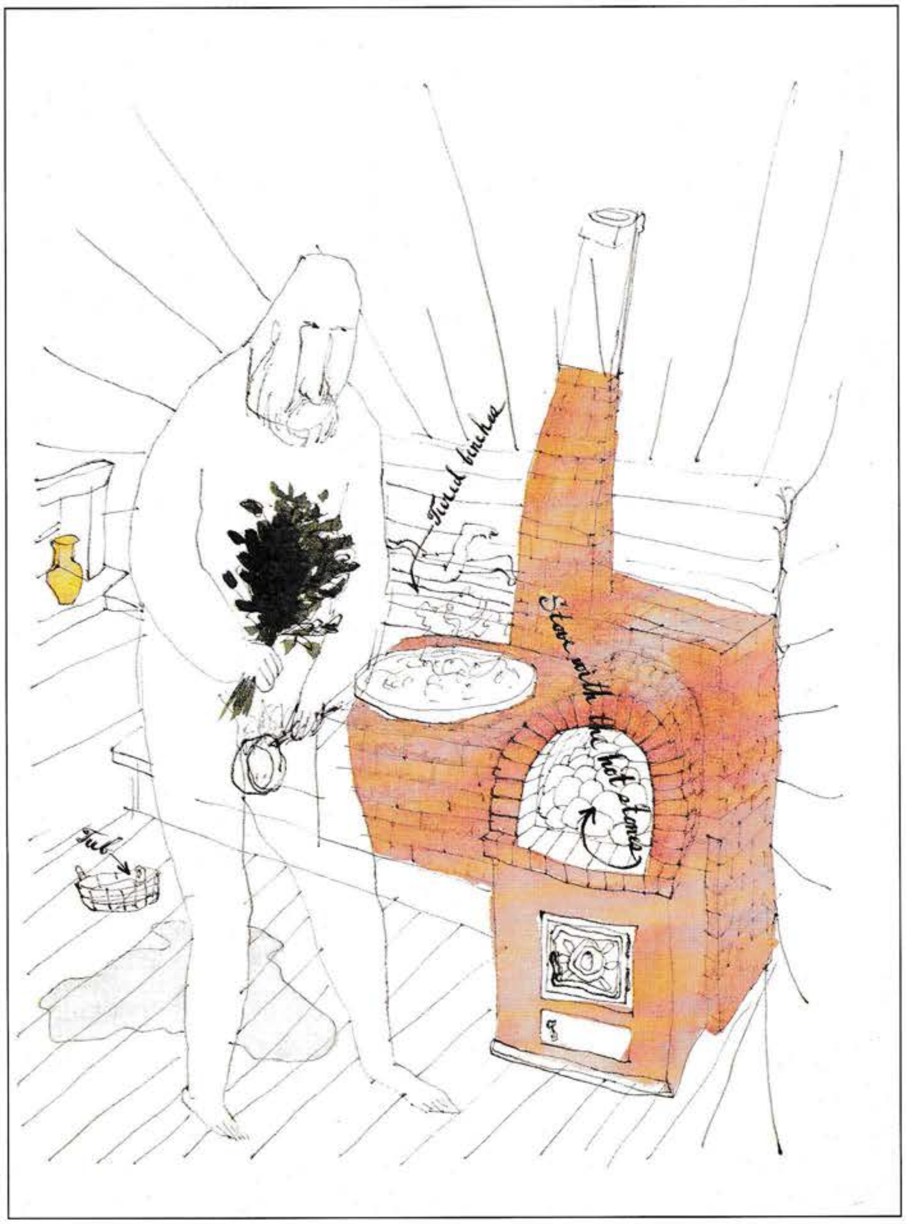
count up all the birds overhead, they didn't notice one another when they met. A minute after the meeting Winnie-the-Pooh was at Piglet's, and four minutes after the meeting Piglet was at Winnie-the-Pooh's. How long had each of them walked?

SOLUTIONS ON PAGE 61

An invitation to the bathhouse

Imagine you're the proprietor of a Russian bathhouse—how do you keep the air so pleasantly dry?

by I.I. Mazin



Art by Pavel Chernusky

"IN MY TRAVELS I SAW A marvelous thing in the Slavic lands," wrote an eleventh-century Russian chronicler. "I saw wooden bathhouses, and the people heat them until they are red-hot, and they undress, and they are naked, and they pour kvass[1] over themselves, and they take green twigs, and they whip themselves, and they beat themselves until they are barely alive when they leave the place, and they pour cold water over themselves, and in this way they come back to life."

You don't have to be a time-traveler, however, to see Russians enthusiastically "torture" themselves in the bathhouses. Why do they do it? Well, it's thought that the body benefits when the skin is exposed to high temperatures (in this case, hot air) for a short period of time. In humid air, though, the high temperature can't be withstood even briefly. So good Russian bathhouses feature not only heat but also a dry atmosphere. It's not an easy thing to build a good *banya*, and it's not obvious how you are to "prepare the steam" (or, to speak in more scientific language, provide the optimum microclimate). Our predecessors were able to perform this task back in the 11th century, but they hardly would have been able to explain why things are done this way, and not that way, in the *banya*. Nowadays any teenager has enough knowledge of physics to answer the questions that arise in an ordinary bathhouse. And this is exactly what I'm now inviting you to do. Let's analyze a session in the bathhouse step by step, formulating relevant physical problems and then trying to answer them.

And so—to the *banya*! To orient ourselves, let's use the picture on the left.

We've entered the steam room of the bathhouse. Wow, is it hot in here! Let's stay down below where it's a little bit cooler, at least for the time being.

Now that we've gotten used to the heat, we can go up on one of the tiered

[1]A slightly fermented drink made from bread and raisins.

benches. Hot? No, it's not so hot today. Some days you can't walk barefoot on the wooden floor let alone sit on the benches. But if the bench has iron nails in it, you'll do well to steer clear of it even if it's not too hot in the steam room—the head of the nail will still give your skin a nice little burn.

Question 1. *Why is it cooler down below in the steam room than it is on the tiered benches?*

Question 2. *Why is it possible to sit on heated wood but not on an iron object at the same temperature?*

Gradually we're getting used to the steam room. The air doesn't seem so hot now. Still, we feel that it's rather humid—there are wet spots here and there on the benches and on the floor. We can fix that. We need to "add the steam"—which is to say, we should throw boiling water in small portions onto the glowing stones in the stove. Immediately a hot wave rushes upward from the stove. It's getting hotter up on the benches, and the heat dries up the wet spots there and down below.

Question 3. *Why does it get drier when water is thrown on the hot stones in the stove?*

Question 4. *Why must the water be thrown in small portions? Why can't we just throw a whole tub of water into the stove?*

Question 5. *Why must we use boiling water?*

... Over an hour has passed. Many people have been to the steam room. The air has lost its freshness. It's too humid. Leaves from the twigs are scattered here and there. The time has come to "clean" the steam room. Here's how it's done. The room is vacated for about 10 minutes. In that time we have to sweep the floor, hose it down, open the steam room door, and throw several tubs of cold water on the floor in front of the door. Then we start "adding the steam." The new steam forces out the stale air. Now everything is ready again, and visitors can return and begin their enjoyment anew.

Question 6. *What's the purpose of the puddle of cold water at the entrance to the steam room?*

Question 7. *Why is the stale air in the steam room forced out by the steam?*

Now that you've mastered the rules of the *banya*, let's try to answer all the questions.

The first question is so easy all of you probably answered it right away. So let's skip to the second one.

What happens when you step or sit on a hot bench? Your body temperature doesn't exceed 40°C whereas in a good steam room the temperature of the air, and consequently that of the benches, varies from 80° to 120°C. A process of heat transfer from the hot body (the bench) to the cold one (you) begins when such contact is made. What's the rate of this process? It depends on the thermal conductivity of the hot body. The higher its thermal conductivity, the faster heat is transferred from its hotter areas to its cooler areas. Along with other metals, iron's thermal conductivity is significantly higher than that of wood (by a factor of approximately 300). When you touch a hot wooden bench you cool an area adjacent to the area of contact, drawing heat from just a small volume of the bench. The situation is quite different if there is a nail in the area of contact—the heat is pulled from the whole length of the nail and quickly gathers at the area of contact. Also, iron's specific thermal capacity (that is, the thermal capacity calculated for a unit volume) is about 40 times that of wood; so under similar cooling conditions, you get much more heat from an iron object than from a wooden one of the same volume.

Now I think you're capable of formulating the answer to question 2. (You'll notice I didn't mention either the thermal capacity or the thermal conductivity of the human body. Try to analyze the role of these parameters in the process on your own.)

In order to answer the remaining questions, let's recall several concepts of molecular physics, in particular those related to water vapor.

As we all know, water exists in three different states: solid (ice), liquid (what we usually mean when we say "water"), and gas (vapor or steam). We'll leave ice out of our discussion, since it's not directly related to the bathhouse, and concentrate on water in its liquid and gaseous states.

The process by which a liquid changes into a vapor is called evaporation; conversely, the process by which a liquid forms from its vapor is known as condensation. During evaporation heat is absorbed, while condensation of the same mass of vapor releases the same amount of heat. The atmosphere always contains a certain amount of water vapor. For instance, in a living room there is approximately 10 g of vapor per cubic meter. The density of water vapor present in a unit volume of air is called the absolute humidity.

Let's bring a saucer of water into the room. The water evaporates and changes into vapor. This causes the absolute humidity in the room to increase, although not significantly—the volume of the room usually amounts to several dozen cubic meters. If the saucer contains 10 g of water, the humidity increases by no more than 1 g/m^3. What happens if the same saucer is placed in a sealed flask with a volume of only 1 *l* (10^{-3} m^3)? The amount of water in the saucer decreases until the vapor in the flask becomes saturated. This happens when the number of molecules leaving the water per unit of time and the number of molecules entering the water become equal. From this time on, the absolute humidity of the air in the flask doesn't change. (It's assumed, of course, that the temperature of the flask is kept constant.)

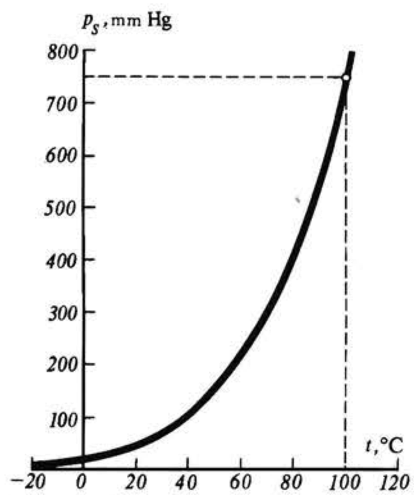


Figure 1

So for every constant temperature there is a maximum absolute humidity equal to the density of the saturated water vapor at that temperature. The higher the temperature, the greater the density of saturated vapor.

Here's another relationship that comes into play in the bathhouse: the lower the absolute humidity relative to the maximum humidity attainable at a given temperature, the more intense the evaporation. The ratio of the absolute humidity to the density of the saturated vapor for a given temperature is called the relative humidity, expressed as a percentage. Since the pressure exerted by the vapor is proportional to its density, relative humidity can be defined in another way—namely, as the ratio (expressed as a percentage) of the partial pressure of the water vapor to the pressure of the saturated vapor at a given temperature.

Both the pressure and the density of the saturated vapor increase as the temperature increases. The graph in figure 1 demonstrates this dependence.

An increase in the absolute humidity at a given air temperature causes an increase in the relative humidity. The same result can be achieved in another way—by decreasing the air temperature while keeping the absolute humidity at a fixed level. The relative humidity will again increase. At a certain temperature it reaches the 100% level—the vapor becomes saturated, which results in condensation and the creation of fog and dew. The temperature at which this happens is called the dew point.

But what happens if the absolute humidity *and* the temperature increase? In this case, the relative humidity depends on whichever increases more quickly: the density of the vapor in the air or the pressure of the saturated vapor.

Now we can get back to our questions.

We've made it clear that the evaporation rate depends on the relative (not absolute) humidity. If the steam room gets drier after boiling water is tossed in the stove, this means that the relative humidity decreases (whereas the absolute humidity obviously increases). Why does this happen? When a small amount of water is thrown vigorously into the stove, it turns into tiny droplets. Landing on stones heated to hundreds of degrees, the droplets immediately evaporate, and the temperature of the steam produced is close to that of the stones. The steam bursts out of the stove, and the overall temperature in the steam room increases. The higher the temperature in the room, the greater the density of saturated vapor. So despite an increase in the absolute humidity, the relative humidity should decrease. And, in fact, that's what happens.

Now it's easy understand why the water should be thrown in small portions. A large amount of water would plop on the stones in the form of a huge "drop." Such a "drop" can't evaporate as quickly as a small one. It starts to boil, which creates steam at a temperature of 100°C or a bit higher. This is just what we're trying to avoid! The secret of the *banya* lies in the rapidity of the process. For the same reason we mustn't use cold water. After all, the stones have a rather low thermal conductivity—even a small droplet being heated to 100°C cools the portion of the stone it lands on, and this lowers the temperature of the steam produced.

Now that we've answered questions 3, 4, and 5, question 6 can be dealt with easily enough. Near the puddle at the entrance, the temperature is below the dew point, so the cooler "waste" vapor leaving the steam room quickly condenses, or "precipitates," on the puddle. In well-designed steam rooms the entrance and the stove are located at opposite ends so that the steam created in the stove passes through the entire steam room, cools along the way, and precipitates at the exit.

Finally, the last question: why does the fresh steam force the stale air out of the steam room? When we cleaned the steam room, we threw boiling water (no less than 10 kilograms) into the stove. The temperature of the steam created by this water is about 300°C; at this temperature, 10 kilograms of steam occupy a volume of about 25 m^3 at a pressure of 1 bar. We can deduce that in a short period of time we have generated enough steam to fill about one third of the room. The steam is hot, so it rises and forces the stale air down and out the door.

Well, we're finished with the physics of that venerable Russian institution, the *banya*. With this introduction, I hope you'll make a point of visiting one of our bathhouses if you ever get the chance!

Challenges in physics and math

Math

M11

Bisector estimated. The lengths of two sides of a triangle are 10 and 15. Prove that the bisector of the angle between them is no greater than 12. (N. Vasilyev)

M12

Counting pairs of integers. Prove that any nonnegative integer n can be represented in the form $n = [(x + y)^2 + 3x + y]/2$ with nonnegative integers x and y, and that such a representation is unique.

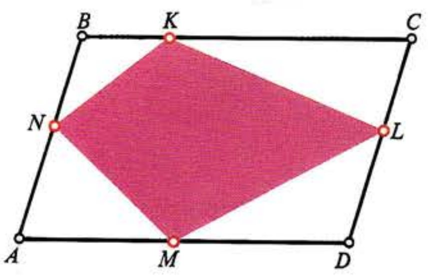


Figure 1

M13

A quadrangle in a parallelogram. A quadrangle is inscribed in a parallelogram whose area is twice that of the quadrangle, as shown in figure 1. Prove that at least one of the quadrangle's diagonals is parallel to one of the parallelogram's sides. (E. Sallinen)

M14

Leapfrog. Three frogs are playing—what else?—leapfrog. When frog A jumps over frog B, it lands at the same distance from B as it was before the jump (and, naturally, on the same line AB—see figure 2). Initially the frogs are located at three vertices of a square.

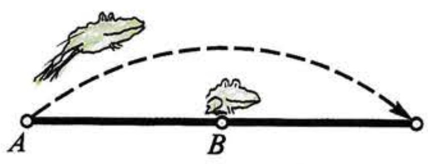


Figure 2

Can any them get to the fourth vertex after several jumps? (Y. Ionin)

M15

Counting the sides of a polyhedron's faces. Prove that any convex polyhedron has two faces with the same number of sides. (A. Gruntal)

Physics

P11

Wheel balancing. Automobile wheels have to be accurately balanced in order to position the wheel's center of mass exactly on the rotation axis. What's the purpose of this operation? (S. Semenchinsky)

P12

Breaking a string. A weight is suspended from an elastic string. An increasing force (whose initial value is zero) is applied to the weight until the string breaks under a force F_1. What's the minimum force that must be applied to break the string if the force could reach a constant value instantaneously? (G. Baronov)

P13

The bell. Water is poured through an orifice into a hemispheric bell lying on a table, tightly pressed against its surface (fig. 3). When the water level reaches the orifice the bell lifts up and the water begins to flow from under-

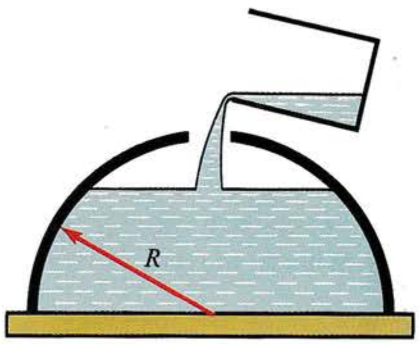


Figure 3

neath. Find the bell's mass if its radius is R and the density of water is ρ.

P14

Hot walls or cold walls? The temperature of the walls of a vessel containing a gas is T. The temperature of the gas is T_1. When is the gas pressure on the walls greater—when the vessel's temperature is lower than that of the gas ($T < T_1$) or vice versa ($T > T_1$)? (V. Myakishev)

P15

Unknown resistances. Figure 4 shows part of an electric circuit consisting of unknown resistances. Is it possible to find the value of one of the resistances using an ammeter, voltmeter, battery, and connecting wires without breaking any contact? (A. Zilberman)

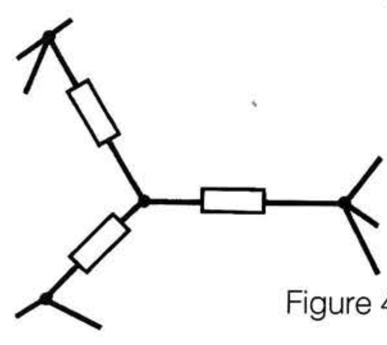


Figure 4

SOLUTIONS ON PAGE 58

An old fact and some new ones

Another excursion into the region of shape-numbers, number-shapes—mysterious numbers that may put a spell on you!

by John Conway

ADD UP THE FIRST n odd numbers, starting from 1. What do you get?

$$\begin{array}{rcl} 1 & = & 1 \\ 1+3 & = & 4 \\ 1+3+5 & = & 9 \\ 1+3+5+7 & = & 16 \\ 1+3+5+7+9 & = & 25. \end{array}$$

Do you recognize these numbers? You probably do. They're the square numbers

$$1 = 1 \cdot 1,\ 4 = 2 \cdot 2,\ 9 = 3 \cdot 3,\ 16 = 4 \cdot 4,\ 25 = 5 \cdot 5, \ldots .$$

This is our old fact. Can we explain it? Generalize it? Can we get the cube numbers

$$1 = 1 \cdot 1 \cdot 1,\ 8 = 2 \cdot 2 \cdot 2,\ 27 = 3 \cdot 3 \cdot 3,\ 64 = 4 \cdot 4 \cdot 4, \ldots$$

in a similar way?

Well, there are lots of ways to explain this fact, and they lead to lots of different new facts. Let's try.

From algebra

What we have to prove is that the differences between adjacent square numbers 0, 1, 4, 9, 16, 25, ... are just the odd numbers 1, 3, 5, 7, 9, 11, ... But this is easy—the typical difference is

$$(n+1)^2 - n^2 = (n^2 + 2n + 1) - (n^2) = 2n + 1$$

by some easy algebra.

You can see in the same way that the differences between adjacent values of any polynomial are the values of a polynomial function of the next lower degree. So, for instance, adjacent cubes differ by the numbers of the form $3n^2 + 3n + 1$—namely, the mysterious numbers

$$1, 7, 19, 37, 61, \ldots .$$

The cubes are just the sums of the first few of these:

$$1 = 1,\ 1 + 7 = 8,\ 1 + 7 + 19 = 27, \ldots .$$

*Bor*ing! Why should anyone be interested in these mysterious numbers?

Let's try again.

From arithmetic

The sum of several numbers is just the number of them multiplied by their average. So, for instance, the sum $1 + 3 + 5 + 7 + 9$ is just 5 times 5 (the middle number). The averages for the sums

$$1,\ 1+3,\ 1+3+5,\ 1+3+5+7, \ldots$$

are indeed

$$1, 2, 3, 4, \ldots,$$

since they can be found as half the sum of the first and last terms.

This does give us a nice way to get the cubes. Rather than always taking the sum of the first n odd numbers, we take the sum of the next n, starting from where we left off.

So, instead of

$$\begin{array}{rcl} 1 & = & 1^2, \\ 1+3 & = & 2^2, \\ 1+3+5 & = & 3^2, \\ & \vdots & \end{array}$$

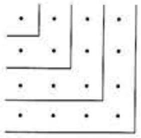
we have

$$\begin{array}{rcl} 1 & = & 1^3, \\ 3+5 & = & 2^3, \\ 7+9+11 & = & 3^3, \\ & \vdots & \end{array}$$

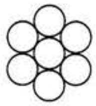
That's a bit nicer, but let's try yet another idea.

From geometry

Just look at this pattern:

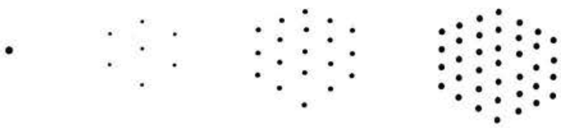


It shows very clearly how a typical square number can be broken up into consecutive odd numbers. Is there a way to see our mysterious numbers 1, 7, 19, 37, 61 as patterns of dots? The way that seven pennies naturally arrange themselves into a neat figure



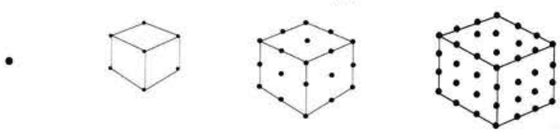
might suggest something... Yes! The mysterious numbers

1 7 19 37



are what combinatorialists call the "hex" numbers. (Beware—they use the term "hexagonal numbers" for something else!)

Is it really true that the sum of the first few hex numbers is a cube number? Yes! To see this, just draw in some lines



and those patterns of dots become cubical shells.

My daughter had some hexagonal blocks like the ones pictured below, which she used to stack into hexagonal pyramids. How many blocks did she need to build a pyramid n layers high?

It seems there are lots of ways to generalize our basic fact, and this is nice, because it means that no matter how long you live, you'll always see some new ones. Here's a striking one that was discovered only recently.

Moessner's magic

First, the squares. We write down the first few whole numbers, circle every second one, and then add up the others:

1	2	3	4	5	6	7	8	9	...
1		4		9		16		25	...

Of course, we get the squares.

Now, instead, circle every third number in the top row, get the next row as before, circle every second number in it, and add the uncircled numbers in this row to get the last row:

1	2	3	4	5	6	7	8	9	...
1	3		7	12		19	27		...
1			8			27			...

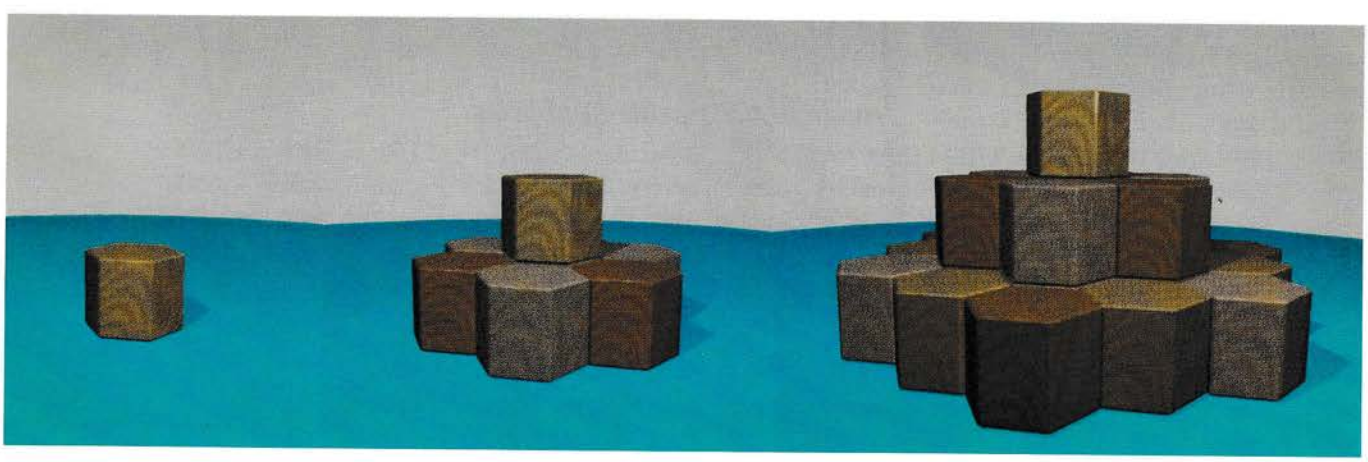
Surprise! There are the cubes! Can you show that this continues forever?

This seems to work for all powers. For instance, if we circle every fifth number in the first row, every fourth number in the second, and so on, we at least get the first three fifth powers correctly:

1	2	3	4	5	6	7	8	9	10	11...
1	3	6	10		16	23	31	40		51
1	4	10			26	49	80			131
1	5				31	80				211
1					32					243

Can you verify that this, too, will continue forever?

In general, you can circle any selection of numbers in the top row, thus dividing the remaining numbers into blocks; then in subsequent rows you circle the last number



These pictures were rendered by Toby Orloff at the Geometry Supercomputer Project and printed on a color laser imager developed at 3M.

in each block, sum the uncircled numbers to get the next row, and so on.

We've seen that if in the top row you circle the numbers

$$1+1,\ 2+2,\ 3+3,\ 4+4, \ldots$$

you get the squares—namely, the numbers

$$1 \cdot 1,\ 2 \cdot 2,\ 3 \cdot 3,\ 4 \cdot 4, \ldots,$$

while circling

$$1+1+1,\ 2+2+2,\ 3+3+3,\ 4+4+4, \ldots$$

gives the cubes

$$1 \cdot 1 \cdot 1,\ 2 \cdot 2 \cdot 2,\ 3 \cdot 3 \cdot 3,\ 4 \cdot 4 \cdot 4, \ldots .$$

What happens when you circle the triangular numbers

$$1,\ 1+2,\ 1+2+3,\ 1+2+3+4,\ 1+2+3+4+5, \ldots ?$$

Answers

Of course, my daughter used exactly n^3 blocks to make her pyramid of n layers.

Yes, those cubes do continue forever! It's quite easy to prove this by working out the general form of the pattern:

$$\begin{array}{ccccccc} \ldots & 3n-1 & 3n & 3n+1 & & 3n+2 & 3n+3 \\ & 3n^2 & & 3n^2+3n+1 & & 3(n+1)^2 & \\ n^3 & & & (n+1)^3 & & & \end{array}$$

It's a bit harder to prove that the kth power rule works for all k. We'll publish the best proof we receive.

From the triangular numbers

$$1,\ 1+2,\ 1+2+3,\ 1+2+3+4,\ 1+2+3+4+5, \ldots$$

the Moessner magic leads (of course!) to the factorials:

$$1,\ 1 \cdot 2,\ 1 \cdot 2 \cdot 3,\ 1 \cdot 2 \cdot 3 \cdot 4,\ 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5, \ldots .$$

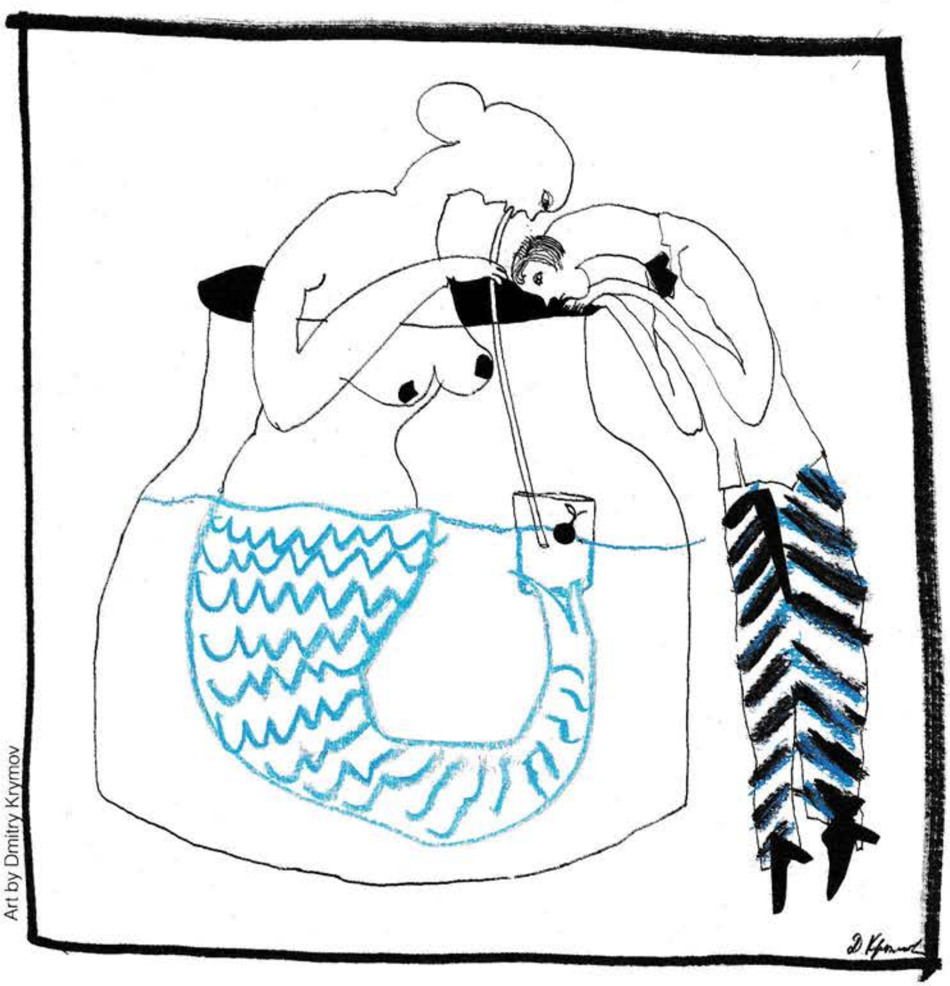
Boy-oh-buoyancy!

We hope you won't get dunked by these problems from Soviet university entrance exams

by Alexander Buzdin and Sergey Krotov

FIRST, WE'LL REFRESH YOUR memory by reviewing the basic laws of fluid statics.

A liquid or gas, when it moves as a unit, constitutes a mechanical system in which different parts interact with each other only through pressure. If a liquid (and when we say liquid we'll also mean gas) is at rest—that is, in static equilibrium—viscosity doesn't appear because the liquid friction emerges only when layers of liquid move relatively to each other or a solid body in contact with the liquid.



Art by Dmitry Krymov

The pressure in a liquid is governed by Pascal's law: pressure applied to a liquid is transmitted without change to every part of the liquid. If only the force of gravity is applied to a liquid, the pressure p increases with depth h according to the law $p = \rho gh$, where ρ is the density of the liquid. Therefore, different forces of pressure are applied to various parts of a body immersed in the liquid so that there is upward (lift) force, which is called the buoyancy force. This phenomenon is described by Archimedes's law: a body totally immersed in a fluid is pushed upwards by a buoyancy force equal to the weight of the fluid displaced by the body—that is, the weight of the fluid contained in the volume of the body.

We should notice that Archimedes's law can't be applied in the situation when a body is tightly pressed to the walls or the bottom of a container. For example, a submarine lying on silt is taken to be pressing against the ground and not being pushed upward.

Now let's turn to some specific problems.

Problem 1. *One pan of a balance holds a glass of water and a stand from which a weight is suspended (fig. 1). What will happen to the balance's equilibrium if we lengthen the string so as to immerse the weight in the water?*

The most common answer, and the wrong one, is that the equilibrium will be disrupted. Some students say that the buoyancy force acts on the weight in accordance with Archimedes's law and decreases the tension of the string so that the pressure of the stand on the balance pan is decreased as well. Others say that because the weight has been immersed in the water, the water level will increase so that the pressure on the bottom of the glass will increase as well and the left pan will drop down.

To get the right answer we need to notice that the contents of the pan don't depend on the position of the weight, inside or outside the water; consequently, the equilibrium of the balance will be preserved. Now let's find the errors in the arguments the students offered.

We'll take into account the fact that when the weight is lowered into the water the tension of the string is decreased owing to the buoyancy force acting on the weight. So the force of the pressure of the stand on the pan also decreases. But according to Newton's third law, the force acting on the water and the bottom of the vessel will be increased by the amount of the buoyancy force. Therefore, the pressure of the glass on the pan increases. We see that the decrease in the pressure due to the stand will be compensated by the increase in the pressure of the glass on the pan. From this we get the correct answer: the equilibrium will be preserved.

Can you figure out what will happen with balance if you put your finger in the glass of water without touching the walls and the bottom of the glass, or if the stand were on the other pan of the balance?

Problem 2. *A glass containing a small ball floats in a vessel of water (fig. 2). How does the level of the water change if the ball, which is made of either steel or wood, is transfered from the glass to the vessel?*

The force of pressure on the bottom of the vessel equals the weight of the water, the glass, and the ball. If we put the vessel on a balance, which for simplicity's sake we'll consider weightless, it will indicate the weight of the total contents. It's important that its reading doesn't depend on whether the ball is inside the glass or in the vessel of water. On the other side, the balance must indicate the force that acts on the bottom of the vessel, and initially the force was determined only by the level of the water in the vessel.

If we transfer the wooden ball, it will float on the surface of the water, and the force acting on the bottom of the vessel will be determined by the level of the water. Since the force doesn't change, the water level must also remain the same.

The result will be different if the ball is made of steel. Such a ball drops to the bottom of the vessel, and the total force of pressure on the bottom comprises the force of pressure of the water and the force of pressure of the ball. Therefore, the level of the water in this case will decrease.

A similar problem involves a glass in which a piece of ice floats with (1) a piece of cork, (2) a small lead pellet, or (3) a bubble of air embedded in it. How does the water level change in the glass after the ice melts?

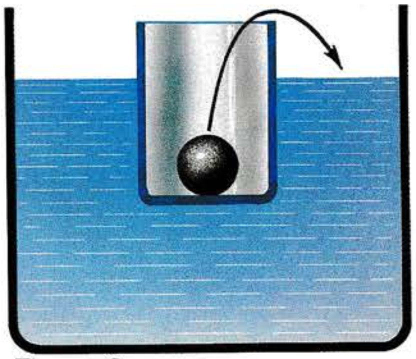


Figure 2

Problem 3. *A steel ball floats in mercury. Water is added so as to cover the ball with water (fig. 3). How does the depth of immersion of the ball in the mercury change?*

You're tempted to apply Archimedes's law right away, aren't you? But the difficulty is that different parts of the ball are in different liquids so that we can't consider the ball as a whole while applying this law.

Let's choose a small area of the ball's surface inside the mercury and find the force of pressure acting on it. It's not hard to see that it equals

$$f = (\rho_1 g h_1 + \rho_2 g h_2)\Delta S,$$

where ρ_1 is the density of water, ρ_2 is the density of mercury, and ΔS is the area. Let's cast this equation in the form

$$f = [\rho_1 g(h_1 + h_2) + (\rho_2 - \rho_1) g h_2]\Delta S = f_1 + f_2.$$

Next, we'll sum the forces of pressure that act on all areas of the ball's surface that touch both the water and

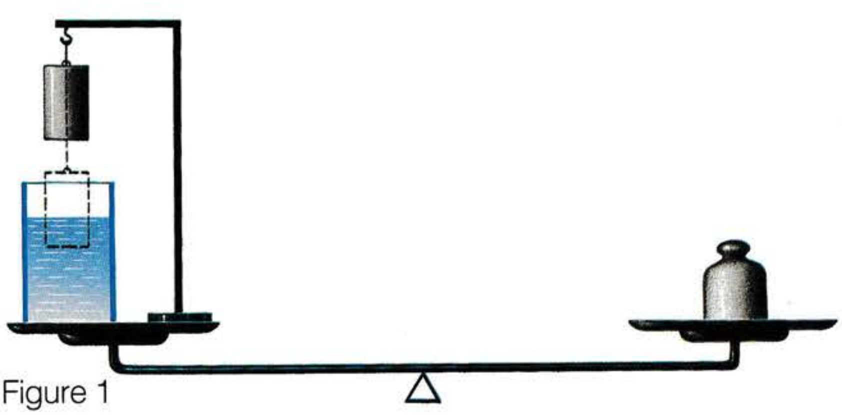


Figure 1

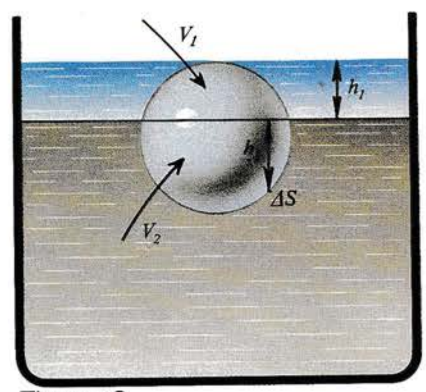


Figure 3

the mercury, obtaining the two forces F_1 and F_2. The first force,

$$F_1 = \rho_1 g(V_1 + V_2),$$

is the buoyancy force that acts on the ball when it's immersed only in water. The second force,

$$F_2 = (\rho_1 - \rho_2)gV_2,$$

is the buoyancy force if the ball is immersed down to the level occupied by the mercury in liquid with a density of $\rho_2 - \rho_1$. The resultant buoyancy force equals

$$f = F_1 + F_2 = \rho_1 g V_1 + \rho_2 g V_2.$$

We see that the force is resolved into two terms—the first for the water, the second for the mercury—corresponding to the parts of the ball immersed in either liquid. We may say there is a principle of independence of the buoyancy forces so that either liquid makes a contribution to the resultant force, even though we might think that mercury pushes the ball out while the water presses it against the mercury.

Thus, the water appears to help the mercury hold the ball up so that it emerges a little from the mercury, and the depth of the ball's immersion in the mercury decreases.

Problem 4. *A thin plank of length* l *is propped up at its higher end by a stone that emerges above the water's surface to a height* H *(fig. 4). What is the minimal coefficient of friction between the stone and the plank that is needed for the plank to remain at rest? (We'll let the densities of water and wood be ρ_0 and ρ, respectively.)*

Four forces act on the plank: (1) the force of gravity $M\mathbf{g}$, (2) the force of reaction of the prop $\mathbf{N}$, (3) the force of friction $\mathbf{F}_f$, and (4) the buoyancy force $\mathbf{F}_b$. The first force is applied to the center of the plank, the second and third at the point where the plank touches the stone. The force $\mathbf{N}$ is directed along the normal to the plank, the force $\mathbf{F}_f$ along the plank.

Until now we had to know only the magnitude of the buoyancy force. Now we need to know where the force is applied. Let's imagine a certain region in the liquid. In the state of equilibrium the force of gravity that acts on the region is balanced by the buoyancy force. The angular momentum of the force of gravity with respect to the center of mass of the region is taken to be equal to zero. Consequently, the sum of angular moments of the forces of pressure is also equal to zero. If we replace the liquid with a rigid body of the same shape, we can convince ourselves that the forces acting on it from the surrounding medium don't change. So we can infer that the forces of pressure are equivalent to a force that acts vertically through the center of gravity of the displaced liquid. It should be noticed that we've found only the line of action of the buoyancy force, but we can't say anything about the point at which it's applied.

Therefore, in our case the buoyancy force has an upward direction and passes through the center of the immersed part of the plank (the center of mass of the displaced water). Let the area of the section of the plank be S, the length of the immersed part of the plank $2x$, and the angle that the plank forms with the horizon (the surface of the water) α. Then

$$\mathbf{F}_b = 2x\rho_0 S\mathbf{g}$$

and

$$M\mathbf{g} = \rho S l\mathbf{g}.$$

Since we're interested only in the minimal value of the friction coefficient μ, we can assume that

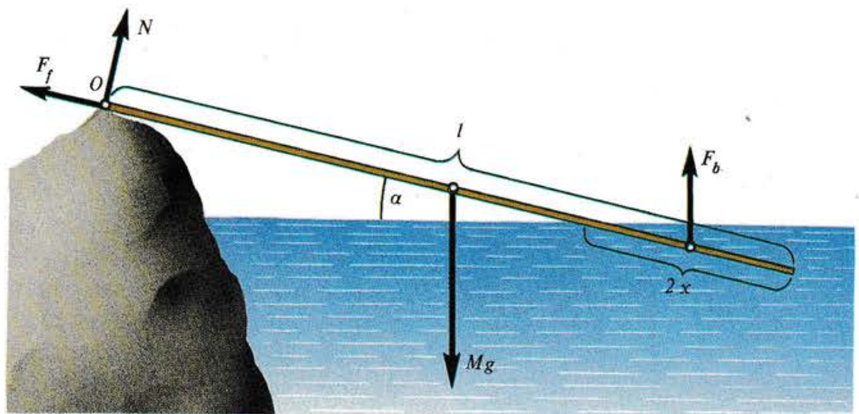


Figure 4

$$\mathbf{F}_f = \mu\,\mathbf{N}.$$

Because the plank is in a state of equilibrium, the sum of all forces acting on it is equal to zero. Let's write all these forces by using the projections on directions tangent and perpendicular to the plank:

$$2x\rho_0 S\mathbf{g}\sin\alpha - \mathbf{g}l S\mathbf{g}\sin\alpha + \mu\mathbf{N} = 0,$$
$$2x\rho_0 S\mathbf{g}\cos\alpha - \mathbf{g}l S\mathbf{g}\cos\alpha + \mathbf{N} = 0.$$

From the equations given above we infer that

$$\mu = tg\alpha.$$

On the other hand, from figure 4 we infer that

$$tg\alpha = H\sqrt{(l-2x)^2 - H^2}.$$

The quantity x can be found from the requirement that the sum of the angular moments of all the forces acting on the plank be equal to zero. It's convenient to consider the angular moments of the forces with respect to the point at which the plank touches the stone (point O) because the angular moments of the friction force, and the reaction of the prop, are equal to zero at this point.

The buoyancy force passes through the center of mass of the immersed part of the plank so that its arm with respect to point O equals $(l - x)\cos\alpha$. The arm for the force of gravity equals $\frac{1}{2}l\cos\alpha$. The equation for the total angular momentum reads

$$\rho l S\frac{1}{2}l\cos\alpha - \rho_0 2xS(l-x)\cos\alpha = 0,$$

or

$$x^2 - lx + \frac{gl^2}{4\rho_0} = 0.$$

From this we get

$$x = \frac{1}{2}\left(1 - \sqrt{\frac{\rho}{\rho_0}}\right).$$

We've dropped the second root because it doesn't satisfy the constraint $2x < l$. Finally, we obtain

$$\mu = \frac{H}{\sqrt{l^2\left(1 - \frac{\rho}{\rho_0}\right) - H^2}}.$$

The problem of the angular momentum of the buoyancy force is of primary importance for studying the equilibrium of floating bodies. In fact, an important concept used in shipbuilding is the "metacenter," which is where the line of action of the buoyancy force of a ship in a slanting position intersects the plane of the ship's symmetry (fig. 5). The metacenter (point M) isn't allowed to descend below the ship's center of gravity (point O); if it did, the angular momentum of the buoyancy force couldn't return the ship to its upright position.

Problem 5. *An aquarium of rectangular cross section is filled with water (whose density* $\rho = 10^3\ kg/m^3$*) to the height* $H = 0.5$ *m. Find the force acting on the aquarium wall (whose length* $l = 1$ *m) and the angular momentum of the pressure on the wall with respect to its lowest edge.*

In this case the pressure changes with the depth of immersion h according to the linear law $p = \rho gh$. The resultant force of pressure is directed horizontally, and the problem is to find its magnitude. To solve it, students usually introduce the formula

$$F = p_{av}S,$$

with $S = lH$ (the area of the wall in contact with water) and p_{av} being the average pressure, which is equal to the pressure at the middle depth. The answer is the right one, but we need to know why this formula was chosen for the average pressure.

Let's consider a rectangular prism made of a material of density ρ and height l that has an isosceles right triangle with a side H as its base. Place the prism on a horizontal surface (fig. 6). It's easy to convince ourselves that the force of pressure of the prism on the surface equals the pressure of the water on the side surface of the aquarium because of the fact that pressures are equally distributed on the contact surface. But the force of pressure for the prism is its weight; so we have

$$F = \frac{\rho gHH}{2l} = \frac{\rho gh}{2}lH = 1{,}250\ \text{N}.$$

Consequently, the mean pressure of water is to be taken as the pressure at the middle depth.

The second question of the problem is more difficult because both the pressure and the arms of the corresponding forces depend on the depth. Sometimes, drawing an analogy from the preceding result, students propose to use the average force of pressure and the average arm, equal to $H/2$, for finding the angular momentum of the forces of pressure. But this is quite wrong. To get the right answer we'll use a different analogy, one based on the prism mentioned above. The corresponding angular momentum for the prism M_p equals the product of the force of gravity on the prism and the arm taken with respect to the straight line AA' (fig. 6). Since the center of mass for a homogeneous triangle is the point of intersection of its medians, the line of action of the force of gravity is at a distance of $(1/3)H$ from the edge AA'. Therefore, we have

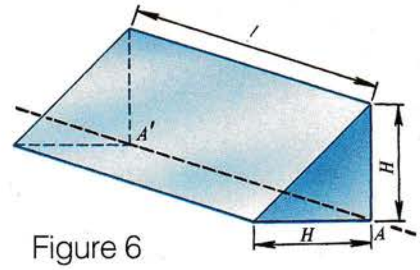


Figure 6

$$M_p = \frac{\rho gH}{2}lH\frac{H}{3} \cong 208.3\ \text{N}\cdot\text{m}.$$

You'll notice that in this solution we haven't allowed for atmospheric pressure. Can it change the answer? One more question is whether the force of pressure on the wall changes if the aquarium wall is made of rubber.

Problem 6. *What happens to the depth of immersion of a ball floating in a glass of water if the glass begins to move with acceleration upward?*

Let's consider a system comprising the water and the ball floating in it, and let's suppose that it moves upward with an acceleration a. The acceleration is caused by the interplay of the force N_a, which is the pressure exerted by the bottom of the glass, and the force of gravity of the system $(M + m)g$, where M is the mass of the water, m is the mass of the ball, and

$$N_a - (M + m)g = (M + m)a.$$

When the system was at rest, the pressure N_0 at the bottom of the glass is determined by the equation

$$N_0 - (M + m)g = 0.$$

Comparing the forces N_a and N_0, we get

$$\frac{N_a}{N_0} = \frac{a + g}{g}.$$

Let's show that the pressure at any point of the liquid has increased by the same ratio. Imagine a water cylinder of section ΔS; one of its bases is at the surface of the water and the other is at a depth h. We can write the following equations of motion for the cylinder in the upward direction:

$$P_h\,\Delta S - \rho\Delta Shg = \rho\Delta Sha,$$

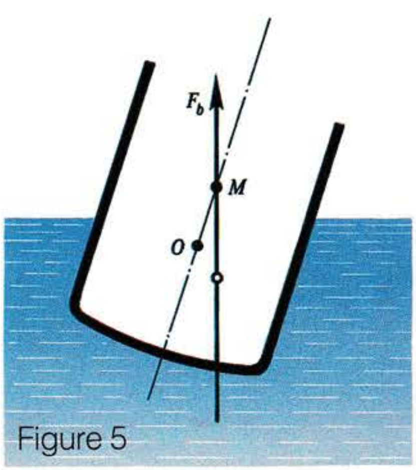


Figure 5

where P_h is the pressure of water at a depth h and ρ is the density of water. We see that

$$P_h = \rho(g + a)h.$$

That is, it has increased by a factor of $(g + a)/g$ compared to the static case. Consequently, the buoyancy force has increased by the same ratio. Now we write the equation for vertical motion of the ball:

$$\rho V_a g\frac{a+g}{g} - mg = ma.$$

From this we infer that the volume V_a of the immersed part of the ball during the accelerated motion of the glass doesn't depend on the system's acceleration and equals $V_a = m/\rho$. Consequently, the depth of the part of the ball immersed in water doesn't change.

Problem 7. *An aquarium in the shape of a cube with edge* L *is half filled with water. Find the shape of the surface of the water in the aquarium and the pressure at point* M *if the aquarium moves in the horizontal direction with an acceleration* a $(a < g)$ *(fig. 7).*

We'll show that the surface of the water is an area in a plane that forms an angle α with the horizon.

Imagine a small region of the liquid of mass m close to a point A on the surface of the water. The resultant of forces of pressure from all other parts of the water is normal to the surface at that point. Let it be equal to N and form an angle α with the vertical. The fixed area of the surface, which we can consider flat because it's small, then forms the same angle with horizon. (Can you explain why?)

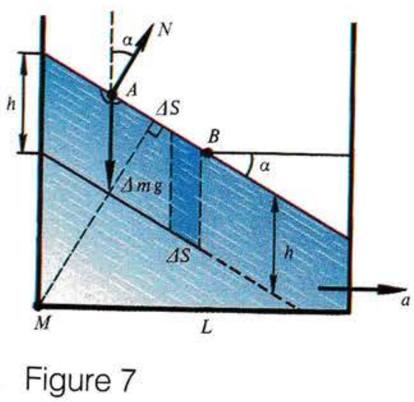


Figure 7

Let's write the equations of motion for this region by using projections of the force and acceleration in the vertical and horizontal directions:

$$N\cos\alpha - \Delta mg = 0,$$
$$N\sin\alpha = \Delta ma.$$

From this we infer that

$$tg\alpha = \frac{a}{g},$$

so that the angle of the slope doesn't depend on the choice of point A.

Let's choose a plane inside the water that is parallel to the surface of the water and at a distance h from it in the vertical direction. Let's show that the pressure of the water at all its points is given by the formula

$$P_h = \rho gh.$$

To this end let's imagine a slant cylinder of slant height h and base ΔS. Since it doesn't move in the vertical direction, the sum of all the vertical projections of forces acting on the cylinder is equal to zero:

$$\rho gh\,\Delta S\cos\alpha - P_h\,\Delta S\cos\alpha = 0.$$

Here the first term is the force of gravity for the cylinder, the second the vertical projection of the force of pressure on the lower base. From this we get

$$P_h = \rho gh.$$

Therefore, the surfaces of constant pressure are planes parallel to the free surface of the water.

To find the pressure at point M, let's notice that the middle point B remains at rest because of the incompressibility of water. So we have

$$P_m = \rho g\left(\frac{L}{2} + \frac{L}{2}tg\alpha\right) = \rho\frac{L}{2}(a+g).$$

We'll leave a question for you to answer: have we employed the condition $a < g$? Also, find the forces of pressure of the water on the walls and bottom of the aquarium when it moves with acceleration.

Exercises

1. A mercury manometer (fig. 8) consists of two tubes with cross sections S_1 and S_2 such that $S_1/S_2 = 2$. Find the change in measured pressure if the level of mercury in the first tube increases by $\Delta h = 10$ mm.

2. A funnel of mass M, which has the shape of a truncated cone with a base of radius R, stands on the table. The edges of the funnel are tightly pressed against the surface of the table. How much water must be poured into the funnel if, at the moment it breaks away from the table, the water level in the funnel is equal to h?

3. A cylindrical weight suspended from a spring balance is lowered into a vessel of water until the water level is changed by $\Delta h = 8$ cm (fig. 9). The reading of the spring balance is changed by $\Delta F = 0.5$ newton. Determine the vessel's cross section.

4. Where does a gas burn better, on the ground floor or the top floor of a fourteen-story building?

5. A wooden ball floats in a glass that is filled with water up to the brim and closed on top. How does the pressure of the ball on the cover change if the glass is moved upward with an acceleration of **a**?

SOLUTIONS ON PAGE 61

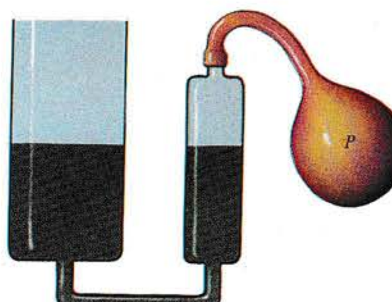
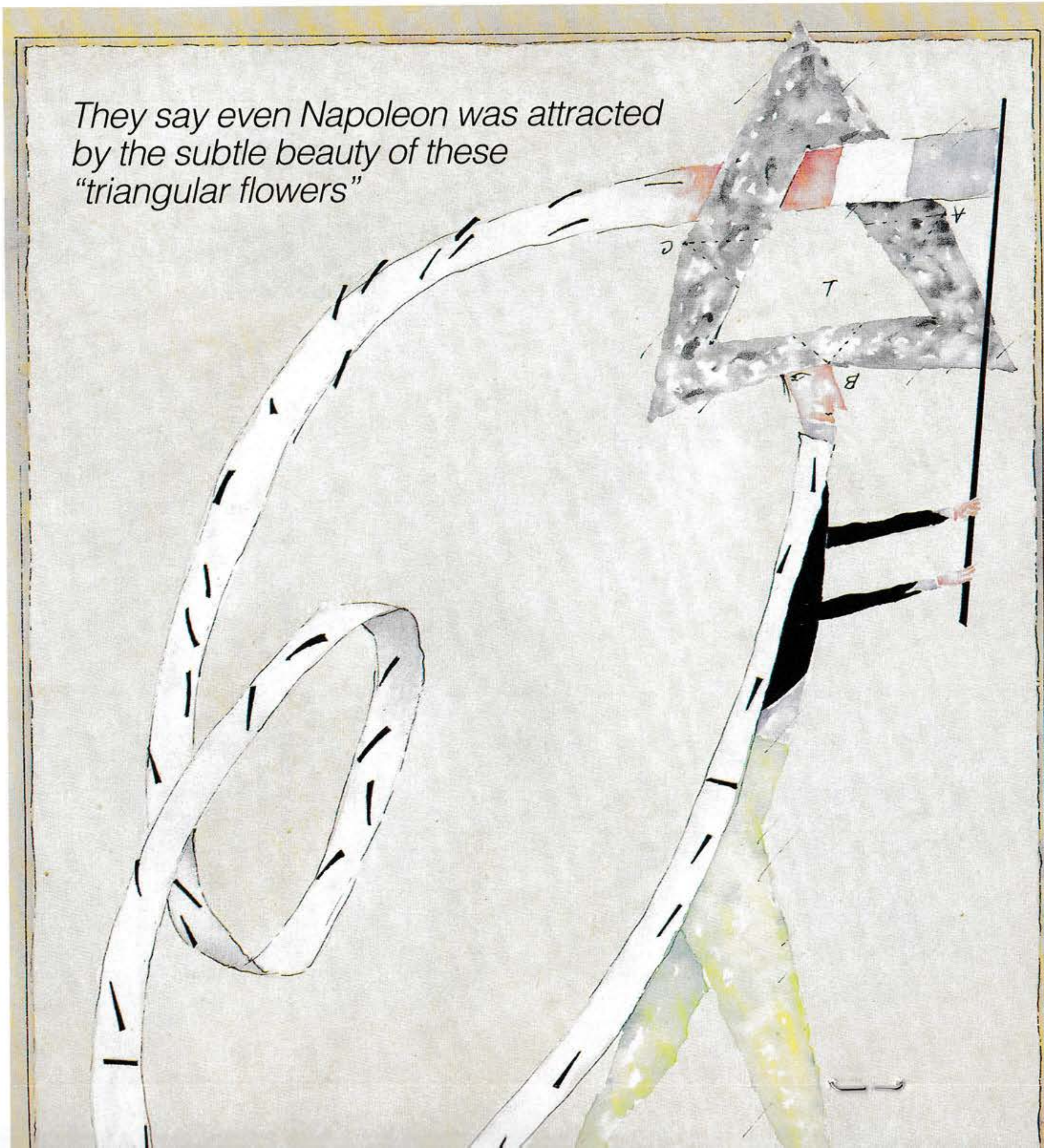


Figure 8

Figure 9

Botanical geometry



They say even Napoleon was attracted by the subtle beauty of these "triangular flowers"

AS IT HAS DEVELOPED OVER the course of more than two millenia, elementary geometry has come to be chock full of beautiful and sometimes unexpected facts. A theorem may seem to lie on the surface, but dig around just two inches deeper and you'll simply be dazzled by real pearls of wit and elegance, here and there labelled with glorious names from the history of science. In this edition of the *Quantum* Kaleidoscope we'll examine the remarkable properties of a sort of "triangular flower." This figure consists of four triangles—an arbitrary triangle with three equilateral triangles constructed externally on its sides.

We'll begin with a simple observation made by Evangelista Torricelli (1608–1647), a famous physicist (you may have heard about the Torricellian vacuum) but also an excellent mathematician, one of the pioneers of calculus. He noticed that the circumcircles of our equilateral triangles meet at one point, T (fig. 1). Both the circles and the point took on his name.

Art by Sergey Ivanov

of our equilateral triangles meet at one point, T (fig. 1). Both the circles and the point took on his name.

Problem 1. *Show that this property of our "flower" will remain valid even if we replace equilateral triangles with arbitrary triangles, preserving only the condition that the sum of their "remote" angles is 180°.*

Now join each vertex of the central triangle of the "flower" to the tip of the opposite equilateral "petal" (fig. 2). You'll see that the three lines thus drawn pass through one point. Not only that, it's the same Torricelli point T!

Problem 2. *Demonstrate this by using Torricelli circles.*

All this was discovered by the great French mathematician Pierre Fermat while solving the following problem: In an acute-angled triangle, find the point for which the sum of the distances to the vertices is the smallest.

Of course, the answer is point T once again (fig. 3). This remarkable property of T makes it clear why it's often called the Fermat point.

We can construct a visual, if not quite strict, solution to Fermat's problem by mechanical means. Imagine we've made holes at the vertices of the given triangle, passed three strings with equal weights attached through the holes, tied the ends of the strings at the top, and let the weights hang freely below (fig. 4). The point K where the knot will stop moving is the one we're looking for, because the center of gravity of the weights must sink as low as possible. On the other hand, the sum of three equal forces of tension acting on the stationary knot is zero, and so the angles between the strings at K must be equal (120° each). Therefore, K lies on each of the Torricelli circles.

A geometrical solution involving rotation reveals new details in our figure. Let's perform the rotation about any vertex of the given triangle ABC through 60°. One of the above three segments intersecting at T will be transformed into another one (fig. 2). So the segments are all of equal length l.

Problem 3. *Use rotation to show that* $l = AT + BT + CT$ *if the angles of triangle* ABC *don't exceed 120° and* $l = BT + CT - AT$ *if angle* $A > 120°$*, and use rotation again to complete the solution to Fermat's problem.*

Problem 4. *Find the point in a convex quadrangle with the smallest sum of distances from the vertices. (It's easy!)*

A legend readily repeated in books on geometry says that the "triangular flower" once became a matter of interest to Napoleon Bonaparte. It's said he discovered a hidden fourth equilateral triangle in the figure, whose vertices are the centers of the "petals" (fig. 5). It turns out that perpendiculars dropped onto the sides of the Napoleon triangle from the corresponding vertices of the central triangle meet at one point when they're extended. Can you guess what point that is?

Problem 5. *Prove that the sides of the Napoleon triangle of a triangle* ABC *are perpendicular bisectors of segments* TA, TB, TC. *(Use Torricelli circles!)*

SOLUTIONS ON PAGE 62

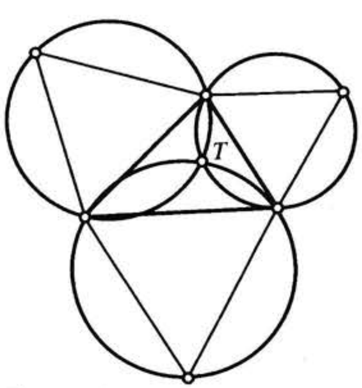


Figure 1

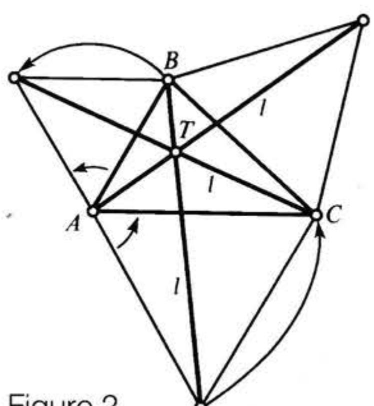


Figure 2

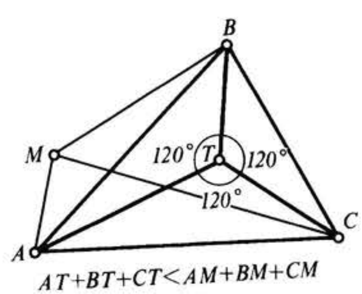


Figure 3

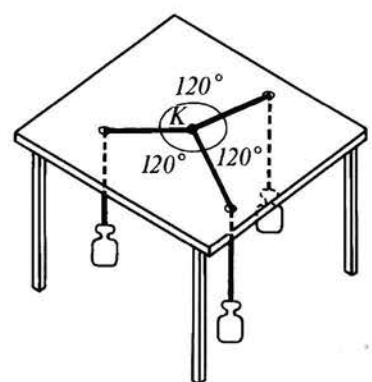


Figure 4

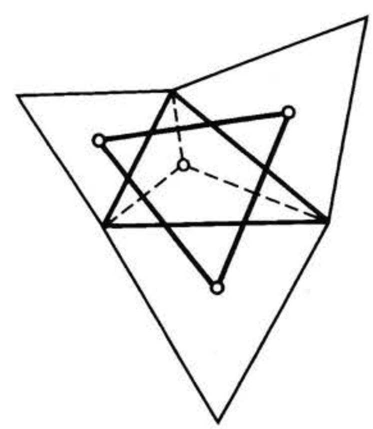


Figure 5

Through a glass brightly

There's more to green glass than meets the eye

by B. Fabrikant

WHAT COLOR IS GREEN GLASS?

This question may bother you. You'll probably answer that green glass is called green because it's . . . But don't be in a hurry to offer condescending explanations. A simple experiment will show you that the question of the color of green glass is not as simple as it seems.

If you have a piece of green glass, break it carefully so as to get several pieces (not too small). Then look through one of them at the incandescent filament of a clear light bulb. As expected, the filament appears green (fig. 1). Place this piece of glass on another one and look at the filament again.

You probably won't see any change in the color of the filament—it'll appear green, as before. But if you lay a third piece of glass on the first two and look at the filament through all three, you'll see that it's discolored and whitish. The filament will appear reddish when seen through four pieces, and ruby through five pieces.

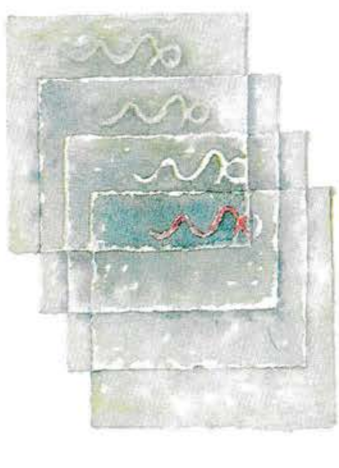


Figure 1
Alteration of the visible color of an incandescent filament from green to red for different numbers of pieces of green glass.

This result is totally unexpected and instructive. It turns out that the color of glass depends on its thickness, and glass that is green when it is not so thick changes its color to red when it is quite thick. Not every kind of green glass has this property, but most common types of cheap green glass do. It's interesting that this property is characteristic of the most common dye on Earth, chlorophyll. It's known that chlorophyll gives the leaves of plants their green color. By putting some leaves in alcohol, you can get a chlorophyll solution and perform the following experiment.

Put a glass on a sheet of white paper and slowly pour in the chlorophyll solution. The bottom of the glass will appear green at first; then, as the layer of solution increases in thickness, it will take on a deep red color.

Let's get back to the green glass. We can muddy the problem of its color even more if, after the filament, we look at the end of a red-hot poker. With only three pieces of glass it will already appear ruby-red. So here we have our second unexpected result: the visible color of glass depends not only on its thickness but also on the properties of the object we're observing. Three pieces of green glass layered together look discolored when we look at the filament but red when we're looking at the red-hot poker.

We can perform another experiment that has a practical outcome. When taken out of the fireplace, the poker cools very quickly. Try observing the poker as it cools down. As we noticed earlier, the end of the poker appears red through three pieces of green glass. After cooling a bit, though, it appears red through only two pieces. If you wait a little, you'll see the poker as red through a single piece of glass. Our experiment shows that the higher the temperature of a red-hot object, the more layers of glass needed for its color to change. So we can estimate the temperature of a red-hot object by the thickness of the glass needed to change its color.

This experiment with the poker helps explain the design of a simple yet ingenious instrument for determining the temperature of red-hot objects—the optical pyrometer (fig. 2). It consists of a wedge of green glass whose thickness gradually increases from one end to the other. The wedge can be moved in a metal holder with an opening for observing red-hot objects. A temperature scale runs along the edge of the glass wedge.

The opening is aimed at the object under consideration and the wedge is moved inside the holder until the color of the object seen through the opening changes. Then a reading is taken from the scale at the point opposite the opening, and in this way the temperature of the object is

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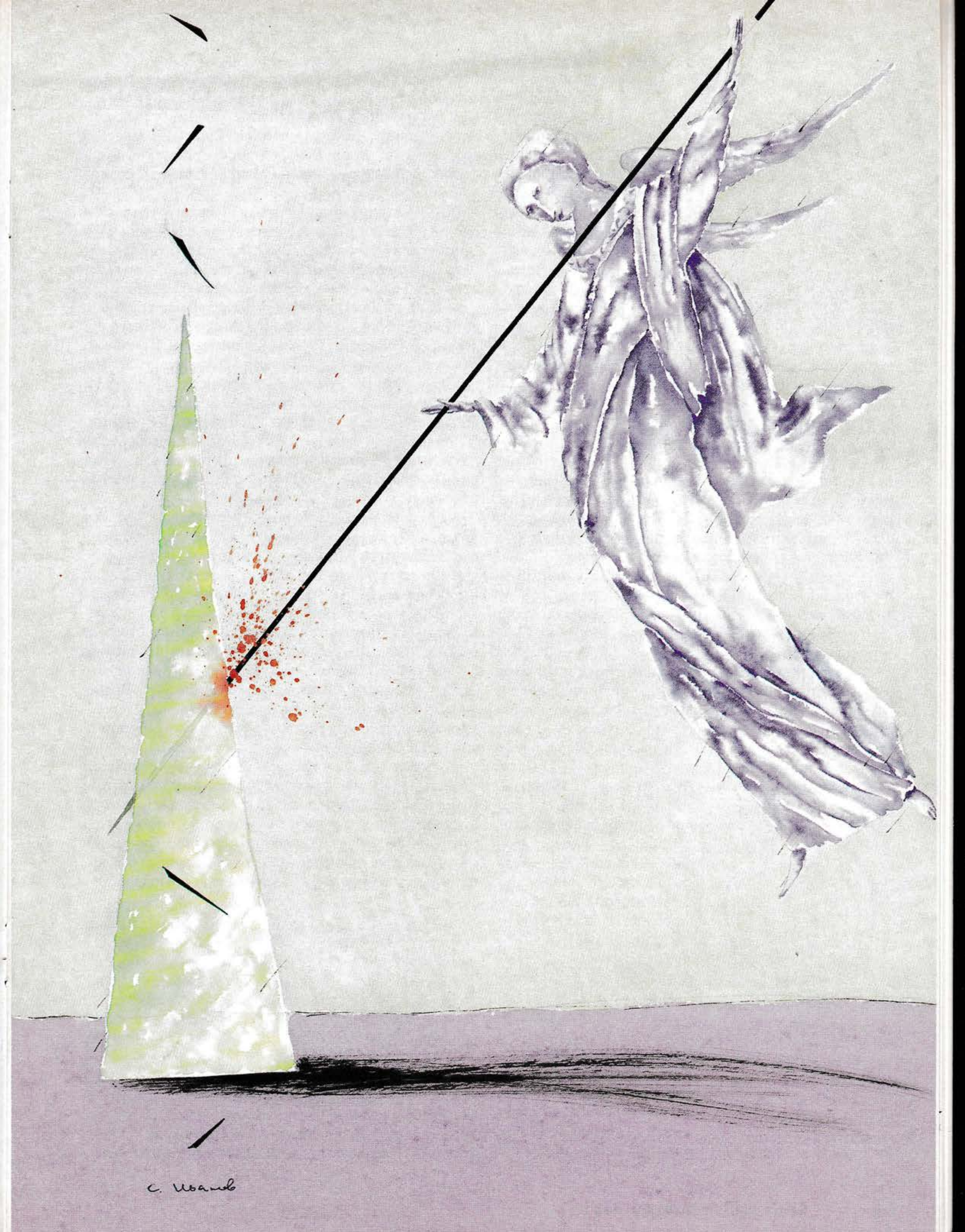

С. Иванов

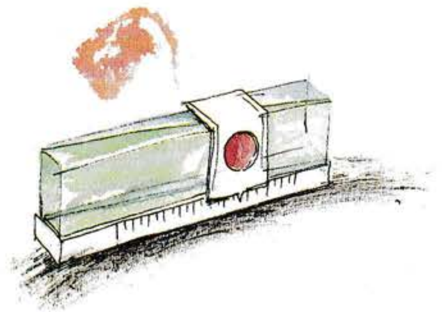


Figure 2
Optical pyrometer for determining the temperature of red-hot objects.

ascertained. The optical pyrometer is widely used to determine the temperature of molten metals (for example, in open-hearth furnaces). Despite its simplicity, in experienced hands it provides a high degree of accuracy.

So now you know about a clever application of the strange properties of green glass, but the mystery of the green glass itself remains unsolved.

An experiment Newton didn't perform and a look at landscape painting

I'm sure many of you remember Newton's famous experiment in which he split a beam of sunlight into a rainbow-colored band (the visible spectrum) by means of a glass prism. The experiment showed that sunlight is a mixture of beams of different colors: red, orange, yellow, green, blue, dark blue (indigo), and violet. For some reason, Newton didn't perform a more sophisticated experiment by putting colored glass or a beaker of colored liquid in the path of the sunbeam. At least, he never mentioned any such experiment.

As it turns out, if red glass is used, the experiment gives us nothing new. Instead of a multicolored spectral band, we get only a narrow band corresponding to red beams. This result could have been predicted beforehand: the red glass is red precisely because it allows only red light to pass through it and absorbs all the others.

An experiment with green glass or a beaker filled with chlorophyll solution is much more interesting. Instead of one band, two bands remain—green and dark red. This means the green glass and the chlorophyll allow not only green but also red beams to pass through.

Figure 3
A green landscape seen through dark-blue glass.

The famous Russian scientist K.A. Temerezyaev made the following very interesting observation about chlorophyll: "It's very easy to convince yourself that chlorophyll allows only red beams to pass—it's enough to look at a sunny landscape through a piece of special dark-blue glass (fig. 3) that allows red and dark-blue beams to pass but absorbs the green ones. Before your amazed eyes the whole of nature is completely transformed, and under the usual dark-blue sky we see dark-red vegetation. Do the troubles that landscape painters continually have to overcome lie in this specific property of chlorophyll? No doubt the painter's palette doesn't include the green hues peculiar to brightly colored green plants."

But let's leave painting for now and return to the optical pyrometer, making a few changes in the experiment with the wedge of green glass. We'll use an incandescent filament as the light source and put the optical pyrometer between it and a prism (fig. 4). Again two bands will be thrown on the wall, green and red; the relative brightness of these bands will depend on the thickness of the wedge at the point where the light passes through it. If the beam passes through the thin part of the wedge, the green band is brighter than the red one. As the thickness of the wedge increases, the brightness of the green band diminishes and, after a certain point, the red band will be brighter. When the green band is brighter, the filament is seen as green; when the reverse is true, the filament is seen as red. If the two bands are equally bright, the filament appears discolored.

So the mystery of the green glass seems to be solved. But it remains to be explained why the ratio of the brightness of the red and green bands is inverse as the thickness of the glass increases. To get an answer, we need to look at an important optical law discovered by a French scientist about 200 years ago.

Pierre Bouguer was the first to focus on the problem of measuring the intensity of light and illumination. He devised the first instruments for measuring the intensity of light, discovered that the intensity of the Sun's light is 300 times that of the Moon, and in his *Optical Treatise* formulated the important law describing how the intensity of light diminishes in absorbing media.

To understand the meaning of this law, which we'll call the Bouguer law, we'll employ an analogy borrowed from sports—not particularly accurate but conveniently graphic. Imagine that we're watching a seven-kilometer race. It turns out the race is rather poorly organized. The participants' lack of training becomes apparent right at the outset, and the observers soon discover the following interest-

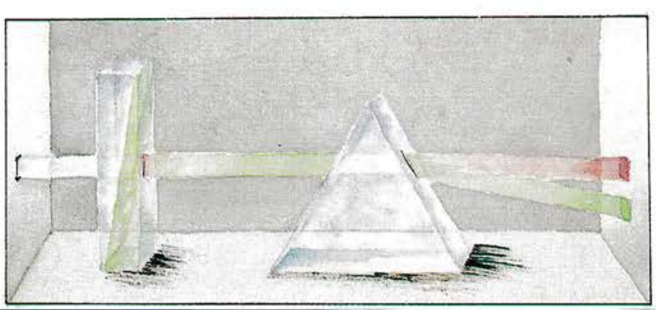


Figure 4
Newton's experiment: the optical pyrometer allows only the green and red components of white light to pass through.

ing law: only one third of the runners that begin a given kilometer make it to the end of it. There were 2,187 participants at the start of the race; at the end of the first kilometer, 729 remain; at the end of the second, 243; at the end of the third, 81; at the end of the fourth, 27; at the end of the fifth, 9; at the end of the sixth, 3. Only one runner makes it across the finish line at the end of the seventh kilometer and is naturally declared the winner.

Let's write out the numbers of runners that ran the different distances: 2,187, 729, 243, 81, 27, 9, 3, 1. It's easy to see that these numbers form a geometric progression in which every number after the first is one third its predecessor (standing to the left of it).

Now let's get back to optics. Take a piece of a colored glass. Suppose it allows one third of the incident light to pass through. Add another piece of glass similar to the first. It transmits one third of the light that passes through the first piece—that is, one ninth of the light that falls on the first piece. Adding another piece, we get one twenty-seventh part, and so on. Obviously, the same result is obtained by doubling, tripling, and so on, the thickness of the glass.

We can conclude that if the thickness of the glass increases, the ratio of transmitted light decreases in a geometric progression. This is the law Bouguer discovered. As we saw in the example of the race, numbers in a geometric progression can decrease very quickly.

A little more sports

Equipped with the Bouguer law we can attack the mystery of the green glass, but first let's revisit the seven-kilometer race. Suppose the novices who ran so poorly were conceited enough to challenge experienced athletes to a race. These well-trained athletes accept the challenge and suggest very generous conditions: 2,187 novices and only 512 experienced athletes will take part in the race. Whichever team has the most members who cross the finish line after 7 kilometers wins.

Both teams arrive at the event dressed in colored T-shirts: the novices wear green, the experienced athletes red. After the first kilometer the supporters of the novices are encouraged by the results: as in the previous competition, 729 novice runners remain in the race, while only 256 experienced athletes remain. The numerical superiority remains with the novices. The supporters of the experienced athletes are disappointed by the fact that half the runners on their team have quit. But one of the fans, after doing some simple calculations on a race program, states firmly that if the race continues the way it's going, the athletes will win.

After the second kilometer 243 greens and 128 reds are left; after the third, 81 greens and 64 reds; after the fourth, 27 greens and 32 reds. Everybody looks respectfully at the person who made the calculations. The remaining three kilometers only worsen the defeat of the greens: 9 greens and 16 reds remain after the fifth kilometer; 3 greens and 8 reds after the sixth kilometer. Finally, 1 green and 4 reds arrive at the finish line at the end of the seventh kilometer.

Let's jot down the numbers of runners at each stage of the race, the greens above and the reds below:

2,187	729	243	81	27	9	3	1
512	256	128	64	32	16	8	4

In the second line each number after the first is one half the number on its left; in the first line, as previously, one third. It turns out that this small difference in the divisors is enough to compensate for the great numerical advantage of the green team and lead the red team to victory. The race just needs to be long enough (at least 4 kilometers).

The behavior of green and red beams is analogous to the performance of the green and red teams (fig. 5). The green glass transmits dark-red beams better than it does green ones, and according to the Bouguer law the difference in the transmission of these beams increases rapidly with the thickness of the glass layer (the "long-distance effect" in the race).

The question arises: why does the glass appear green for a thin layer if it transmits red beams better than green? The explanation lies in the spectral composition of light from the incandescent filament used in the experiment—its green band is much brighter than its dark-red band. In a thin layer (a short race, according to our analogy), the difference in the absorption of dark-red and green beams isn't large enough to compensate for the initial advantage of the brighter green beam, so that the green beam plays the major role and provides the dominant color.

Now we need only explain the part played by the temperature of a red-hot object viewed through the glass. It's widely known that the higher the temperature of a heated object, the whiter the light radiated. For example, at low voltages an incandescent filament gives off a reddish light; at normal voltage its light is much whiter. This effect is explained by the fact that the brightness of the green and blue beams increases with temperature faster than that of the red beams. So at high temperatures the difference in brightness between the green and red bands of the spectrum is much greater, and it's difficult to compensate for it with the glass. This is why at higher temperatures we need a thicker glass to change the color of the heated object under observation.

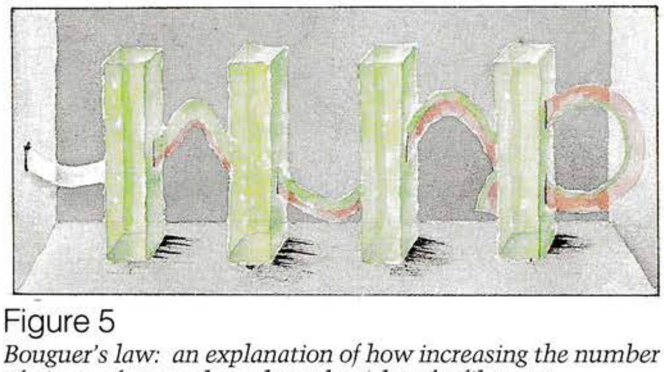


Figure 5

Bouguer's law: an explanation of how increasing the number of pieces of green glass alters the color of a filament.

Ancient Russian icons and an observation by Leonardo da Vinci

Some ancient Russian icons are particularly striking because of the peculiar way the clothing of the saints is painted. Folds are painted with colors that are in sharp contrast to the smooth parts of the garment. For example, we may see red folds on a green coat (fig. 6) or orange folds on a blue garment. The experienced eye of the Russian painters of antiquity noticed that some fabrics have the property of being double colored—that is, their folds take on a color different from that of their flat surfaces. We can say that the cause of this phenomenon is the same as the one we discovered in our experiments with the optical pyrometer.

If we allow a beam reflected from a double-colored fabric to pass through a prism, two colored bands will remain in the spectrum. For green double-colored fabric, the scenario will be the same as for green glass: the green and red bands remain, and all other beams are absorbed.

In fact, double-colored green fabric reflects red beams better than green ones, but the greater intensity of the green beams is decisive when light is reflected from a smooth surface. We conclude that green beams are predominant in light reflected only once from the green fabric.

When there are folds in the fabric, however, the beams of light are reflected more than once. At the second reflection, the red beams are reflected more strongly than the green beams, so the double reflection results in the same effect as that for thick green glass—the red beams become more intense than the green ones, and the fabric changes color. Further reflection simply increases this effect.

Most ordinary fabrics have properties that are just the opposite of those discussed above. In their folds we merely see a color that's darker than that on a smooth surface. The cause? Again, multiple reflection.

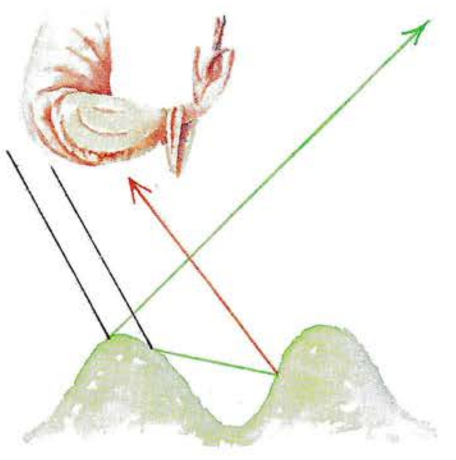


Figure 6
An ancient Russian icon: alteration of color in the fabric folds because of multiple reflection.

When light reflected from such fabric is split by a prism, there is only one band in its spectrum instead of the two bands for double-colored fabrics. For example, light reflected once from yellow velvet produces a broad band in the spectrum with the greatest intensity in the middle of the yellow range. There are also green and blue beams in the spectrum in addition to the yellow ones. This band in the spectrum becomes narrower for double reflections because the blue beams disappear almost completely and the green ones become substantially weaker. This is a result of the law of geometric progressions discussed above. Consequently, the yellow-orange color becomes darker.

Leonardo da Vinci, a multitalented genius who was not only a painter but a sculptor, architect, engineer, writer, musician, and anatomist, noticed the peculiarity of folds in fabric and came up with the correct explanation for this phenomenon. In his *Treatise on Painting* he wrote that reflected colors are more beautiful than the original ones and that this fact is particularly noticeable with folds in golden fabrics when one surface is reflected in another and then reflected back, over and over again.

Maybe I've given you a good reason to take another look at the work of the old masters. Perhaps they can pass their powers of observation on to us, and we can share in their profound understanding of the properties of light. Once you've learned to notice the subtleties of multiple reflection and the transmutation of colors, no doubt the world will never look quite the same . . .

Why all the Russian names in Quantum?

Because *Quantum* is the product of a joint venture with the Soviet magazine *Kvant*. Much of what you read in *Quantum* is translated from Russian. Our goal is to increase the content contributed from this side of the globe to the fifty-percent level.

Quantum's Soviet and American staffs work very hard to make the translations read well. How are we doing? If you don't look at the names of the authors, can you tell which articles were translated and which were written originally in English?

We hope you enjoy the glimpses of Soviet life offered in *Quantum*. And we hope you enjoy the wonderful articles from our sister magazine *Kvant*, written in the international language of math and science!

A pigeonhole for every pigeon

An attempt to provide private accommodations for that special mathematical bird

by George Berzsenyi

In Martin Gardner's August 1980 Mathematical Games column in *Scientific American*, the famous mathematics expositor Ross Honsberger showed the following: If S is a subset of $\{1, 2, ..., 99\}$ and if S has 10 elements, then S must have two disjoint subsets A and B such that the sum of the elements of A is the same as the sum of the elements of B. Thus, for example, if $S = \{3, 9, 14, 21, 27, 35, 42, 59, 63, 76\}$, then $A = \{14, 63\}$ and $B = \{35, 42\}$ yield the same sum (77), and so do $A = \{3, 9, 14\}$ and $B = \{26\}$.

To prove his assertion, Honsberger applied the pigeonhole principle, which was the topic of a beautiful article by Alexander Soifer and Edward Lozansky in the premier issue of *Quantum*.[1] More specifically, he observed that the sum of the elements of S is at most $90 + 91 + ... + 99$, or 945, and so the subsets of S can be sorted according to the sum of their elements into pigeonholes numbered 1, 2, ..., 945. For the pigeons, he chose the nonempty subsets of S, of which there are $2^{10} - 1$, or 1,023. Thus, there must be a pigeonhole with more than one pigeon—that is, there must be at least two subsets of S whose elements have the same sums. Upon discarding common elements, the reduced sums will remain equal for the resulting disjoint subsets of S.

As he was reading Honsberger's arguments, Andy Liu, another Canadian mathematician, became interested in the maximum size of S that will allow for the sum of the elements of each pair of its disjoint subsets to be distinct. In other words, he was distressed by the overcrowded conditions of the pigeonholes and wanted to ensure private accommodations for each of the pigeons. To make the arithmetic more manageable, he switched from 99 to 25 and submitted the following problem to the subcommittee of the Mathematical Association of America in charge of the American Invitational Mathematics Examination (AIME): Let S be a subset of $\{1, 2, ..., 25\}$ such that for every two disjoint subsets of S, the sum of the elements of one subset is different from that of the other subset. Find the maximum value of the sum of the elements of S.

In view of the time limitations of the AIME, it was felt that even this problem was a bit too ambitious. Since I was chairing the AIME Subcommittee at that time, I replaced 25 with 15 and posed the revised problem on the 1986 AIME, saving Andy Liu's original problem for this year's USA Mathematical Talent Search (see Happenings).

Here I'd like to reopen Andy Liu's investigations in a more general setting, searching for maximal subsets of S_n of $\{1, 2, ..., n\}$. As it turns out, for $n = 15$, $S_{15} = \{15, 14, 13, 11, 8\}$, with five elements whose sum is 61; while for $n = 25$, $S_{25} = \{25, 24, 23, 21, 18, 12\}$, with six elements whose sum is 123. In both cases (as well as for other values of n investigated so far), S_n is uniquely determined by the Greedy Algorithm, according to which we always pick the largest avaliable numbers for S_n that don't lead to contradictions.

I invite you to explore this problem more systematically, possibly by first gathering some more data via clever computer programs. Here are some

CONTINUED ON PAGE 42

[1] See answer M15 on page 59 for a restatement of this principle.

Click, click, click . . .

"The causes of events are ever more interesting than the events themselves."—Cicero, Ad Atticum, *Book IX, Section 5*

by Arthur Eisenkraft and Larry Kirkpatrick

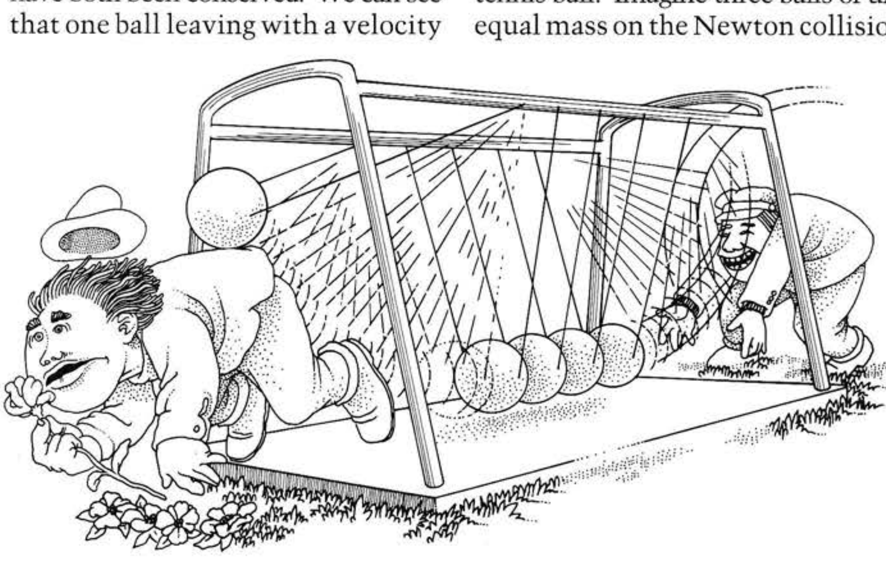
NO DOUBT MANY OF YOU have played delightedly with Newton's collision toy (though not one so large as in the picture!). In this toy five identical balls are suspended by strings so that they lie along a line. If you pull one ball back and release it, the three middle balls remain stationary and the last ball flies off the other end. If two balls are raised on one side, two balls fly off the other end. The pleasure we take in the toy comes from the repetitive motion of the colliding balls and the click, click, clicking sound of the collisions.

The physics of the toy is both interesting and informative. The toy propels the inquisitive idler into an examination of the conservation laws in nature. Conservation of momentum states that, in a system free of outside forces, the momentum (mass times velocity) before a collision must be equal to the momentum after the collision. In our collision toy, one ball with velocity v collides with the hanging balls and a single ball leaves with velocity v. Momentum is conserved. Two balls in, two balls out—momentum is conserved. Three balls in, three balls out—momentum is conserved. The toy certainly obeys the law of conservation of momentum.

If momentum conservation were the only law governing the collisions, then other results would be plausible. Couldn't two balls enter with velocity v and one ball leave with velocity $2v$? Momentum would be conserved. Couldn't two balls enter with velocity v and four balls leave with velocity $v/2$? Although momentum conservation allows this, these events *never* occur. Nature is warning us that we don't know the whole story. There must be another restriction on the motion of the balls that forbids these other events.

The second restriction is the conservation of kinetic energy ($K = \frac{1}{2}mv^2$). The sum of the kinetic energies before a collision must be equal to the sum of the kinetic energies after the collision. When two balls of mass m enter with velocity v, the momentum is $2mv$ and the kinetic energy is mv^2. If two balls leave with velocity v, the momentum and the kinetic energy have both been conserved. We can see that one ball leaving with a velocity $2v$ would conserve momentum ($2mv$) but would have a kinetic energy of $2v^2$! Similarly, all other possible collision scenarios that conserve momentum do not conserve kinetic energy except for the one that we really observe. The collision toy has yielded some important physics.

The collision toy leads us to wonder what would happen if the balls didn't have the same mass. Take any two balls of different mass (a basketball and a table tennis ball would work well). Drop each one separately onto the ground and observe the height each reaches. Now place the table tennis ball atop the basketball and release the basketball. Watch your eyes! The table tennis ball goes "sky high." Here, the collision is between the Earth, the basketball, and the table tennis ball. Imagine three balls of unequal mass on the Newton collision

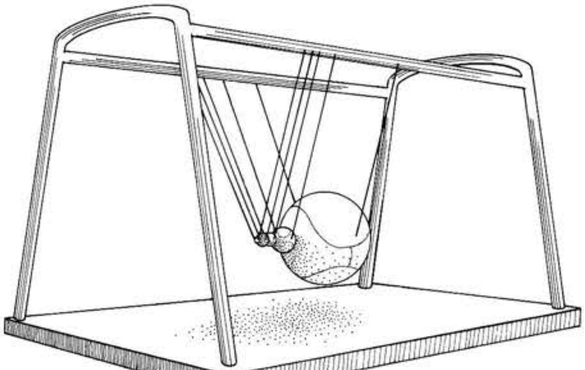


Art by Max-Karl Winkler

toy. The large mass ball is pulled back and released. The collision occurs. The big ball hits the middle ball, the middle ball hits the small ball, and the small ball flies off the other end.

Here's part A of our contest problem: **Find the mass of the middle ball, in terms of the masses of the big and small balls, such that the velocity of the small ball will be greatest.**

Part B: **Extend your proof to find the relationship between the masses of the balls if three balls of intermediate mass are involved in the collision.**



In our discussion so far we've assumed there's no energy loss in the system. We know that the kinetic energy must continually decrease, since the Newton toy eventually stops. We also know that some of the kinetic energy becomes sound energy—that's the click, click, click. Since collisions with macroscopic objects aren't perfectly elastic, we need a means by which we can quantify the loss of kinetic energy. Newton defined the coefficient of restitution e of a collision as the ratio of the final relative velocities to the ratio of the initial relative velocities of two balls. Newton then discovered, experimentally, that this number stays relatively constant for balls of a given material.

Finally, part C: **Determine the middle mass in a three-mass collision, given a coefficient of restitution *e* for each collision.**

This interesting physics has some interesting applications. When one wishes to hammer a small nail, an intermediate mass called a punch is placed between the massive hammer and the less massive nail. In some gravitational wave detectors, the tiny signal hits a succession of masses in just the ratio discovered in part B of this month's problem. Can you think of other applications for maximizing collisions of unequal masses?

Those of you who are just beginning your study of physics may attempt part A alone. When submitting your solution, indicate your physics background so that we can reward our younger readers for their excellent attempts. Here's our address: *Quantum*, 1742 Connecticut Avenue NW, Washington, DC 20009.

We're still getting some interesting answers to our first contest problem, so we'll hold off selecting the best ones until the next issue.

CONTINUED FROM PAGE 40

questions to answer: **Is it true that n, $n-1$, $n-2$ must be elements of S_n for $n > 3$? Is the number of elements of S_n always maximized when the sum of its elements is largest? Is there some n for which there are several choices for S_n?** Send your findings to *Quantum*, 1742 Connecticut Avenue NW, Washington, DC 20009. The best results will be acknowledged, and their creators will receive a free one-year subscription to *Quantum*.

We've received some very impressive answers to our first contest problem. In fact, we're still receiving answers as we go to press, so I'll wait until the November/December issue to discuss the best ones.

Jules Verne's cryptogram

"Everybody realized that if the attempts to decipher the document failed, the convict would stand no chance."

by G.A. Gurevich

SECRET WRITING AND THE mystery it conceals . . . puzzling out the code that finally gives the clue to hidden treasures or saves a human life . . . there's quite an attraction in a mystery story that's not only well written but contains intriguing coded messages—think of "The Gold Bug" by Edgar Allan Poe or "The Adventure of the Dancing Men" by Sir Arthur Conan Doyle.

In his works Jules Verne also made a good use of mysterious documents whose secret is disclosed at the very end of the story—*Captain Grant's Children* and *A Journey to the Center of the Earth* are good examples. Another novel, *Jangada, or Eight Hundred Miles Down the Amazon*, isn't as well known. It's full of detailed geographic, historical, and ethnographic descriptions as well as pages devoted to the curious peculiarities of plant and animal life in the Amazon basin. But the most gripping chapters of the book are devoted to the deciphering of a document containing the confession of a criminal who took part in a diamond robbery 23 years before the action in the novel.

The story has it that Joao da Costa, through a fatal coincidence, finds himself on trial, facing a charge of theft and murder. The crime was committed a long time ago and he's unable to produce any evidence of his innocence. A message with the confession of the real culprit is his only hope, but it's written in some unknown code. Here's the text to be deciphered:

WFDPFWQQUMWYMPVXRNJE
LEPQSEVEQFPJOPHFLVDV
GUTOWLHPNHKXRHOBQYDT
DUKIWYJOWCVGHPQHHKLI
WIHPYXHRWAXJAMVPTUMS
DQIBFSVVFVQNXUYMBGRP
IFPRHFYPGIDVMCXXPGFN
LWHTFCOIFNJSNSIVMFGM
DOTOGHLUYSLGWCINLRLU
YSDXLQSZHWPGFMRRHYMJ
FLLJJSHFBUNHQMQODPZR
QCRFRVWGLB.

Judge Jarriques volunteers to unravel the knot. "The first thing we need," he declares, "is a system. System means logic, and logic means success." The judge doesn't have a shadow of a doubt about success. He decides to employ the method brilliantly described by Poe, which is based on a comparison of the frequency with which different symbols, in ciphertext, and letters, in ordinary plaintext, occur: "I arranged all the letters of the alphabet in numerical order, starting with the most preponderant, and replaced the letters in the document with new ones according to the procedure described by our immortal analyst Edgar Allan Poe, and then I tried to read the message . . . but, alas, I failed!"

After a thorough analysis of the text, the judge comes to the conclusion that the key to the code is a number. He explains to the convict's son Manuel how the document was enciphered.

"Let's take a phrase, any phrase. This one, for example: 'Judge Jarriques is cute.' And now I take any number at random to make a cryptogram. Let's assume that it's a three-digit number: 423, for example. I write '423' underneath the words so that each letter corresponds to one of the digits, and I repeat this process until I reach the end of the sentence:

```
JUDGE JARRIQUES IS CUTE
42342 342342342 34 2342
```

Then we replace each letter in the sentence with the one that follows it in the alphabet by the number of places indicated by the corresponding number. For example, if the number '3' stands under the letter 'D,' you count off three letters and replace it with the letter 'G.' If the letter is at the end of the alphabet and there aren't enough letters after it, we continue counting from the beginning of the alphabet.

"So let's complete our cryptogram based on the key number 423—which, mind you, was chosen at random. Instead of our plaintext message, we'll end up with the following coded one: `NWGKGMETUMSXIULWEXXG`."

Art by Dmitry Krymov


WFDPFWQQUMWYMP

After the judge arrives at the conclusion that the cryptogram has a numerical key, his certitude gives way to the darkest pessimism. Calculations carried out by Jarriques show that a random search for the key, by going through all possible combinations of numbers containing not more than 10 digits, will take over 300 years! Eventually, he gets bogged down in guesswork and turns into a gambler who is trying to hit upon the right number.

Meanwhile, the day of the execution is approaching. Joao da Costa is going to the gallows.

But all ends well. Luckily, Joao's friend comes to know that the name of the man who had written the cryptogram was Ortega. The judge places the letters O, R, T, E, G, A over the last six letters of the message, determines the amounts of the shift, and obtains the key to the code:

```
ORTEGA
343251
RVWGLB
```

Jules Verne is a great writer, and he easily leads the reader to believe that except for a happy coincidence, it's impossible to guess the number 343251.[1]

Now it's time to tell you that Jarriques could actually have deciphered the cryptogram without waiting for a lucky break. The most amazing thing about it is that the judge was on the right track and had practically solved the puzzle. He had the key right in his pocket.

[1]An interesting fact is mentioned in the commentary appended to the novel: "The author . . . received a letter from his friend Professor Maurice d'Ocagne informing him that a student at a polytechnic school had managed to read the cryptogram lying at the core of *Jangada*. At the time, the novel was still being published serially in a magazine. So it was not too late to correct the regrettable inadvertence. Before the book appeared as a separate edition, Jules Verne had time to think up a more complex code—one that precluded premature deciphering of the document. . . .

"One will undoubtedly not find as intricate a cryptogram in any other of his works."

Let's get back to the text of the novel. Jarriques's line of thinking was as follows: "I am sure that the name of Joao da Costa is mentioned in the document. Had the lines of the message been separated into words, we could have picked out the pairs of words that could stand for "da Costa"—that is, 'two letters–space–five letters' combinations. Trying them one by one, we could possibly find the key to the cryptogram."

It's not quite clear why the absence of spaces between the words seems an insurmountable barrier to the judge. In fact, it merely increases the range of the exhaustive search. That's why Manuel, who has a better grasp of the problem, disagrees with Jarriques: "What of it? If we assume the name of da Costa is mentioned and take each letter in turn to be the first letter of his name, we'll eventually find it the key."

That's it! The direct way to the solution has been found. Not only that, the range of the search is not so wide. The text consists of 230 letters, which means that the number of possible combinations doesn't exceed 223. Eventually, having written the words "da Costa" over the IBFSVVF fragment, we'd determine the following sequence of figures: 5134325. It would be natural to assume that the last number opens up the following numerical pattern:

```
      ...DACOSTA...
...5134325134325134..
...TUMSDQIBFSVVFVQN..
```

So instead of the key 343251, we have found its cyclic permutation 513432, which in no way prevents us from deciphering the text. (And, by chance, it's the very combination that opens the coding line of numbers.)

Finally, let's consider the following problem. In the case described above we knew what kind of document it was and so were able to guess one of the words, which gave us a clue to the solution. But what do we do when the content of the document is completely obscure?

There are several possible paths to pursue. Just as with our cryptogram, we can try to guess a word (or its component). Some words ("the," "which," "that," as well as suffixes like -tion, -ing, -able, and so on) occur quite frequently in all kinds of text. Naturally, in this case the scope of the search is substantially increased, but the chances would still be good enough. It seems, though, that a more rational way is to analyze the frequency with which different letters occur in the cryptogram. According to Jarriques, however, this approach will be unacceptable if the key to the code is a number: "Consequently, the meaning of each letter is determined by the underlying number chosen at random, and the same letters in the cryptogram never correspond to a particular letter in the plaintext."

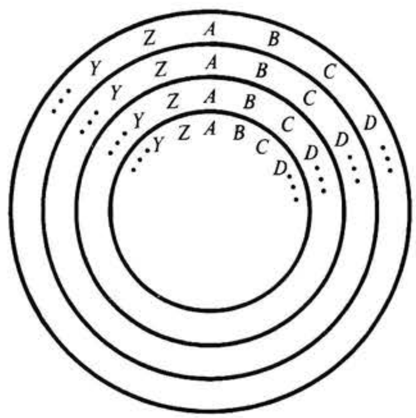


Figure 1

The special coding device shown in figure 1 will help us see that the judge was wrong. To simplify the matter we'll assume that the key is a three-digit number. The letters of the alphabet are written in order around the outside edges of four concentric disks. Three of them can rotate, while the inner disk is stationary. The inner disk can be considered the "plaintext" disk, the three rotating disks the "ciphertext" disks.

Suppose the key number is 259. Turn the first ciphertext disk counterclockwise through two letters; the next disk, through five letters; and the third disk, through nine letters (fig. 2). Now we're ready for coding. Find the first letter of the plaintext on the innermost disk and replace it with the letter across from it on the first rotating disk; replace the second letter with the corresponding letter on the second

Table 1: The number of instances of each letter in sets selected from a cryptogram with a 6-digit key

Set No.	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	0	0	0	2	0	4	0	1	3	4	0	3	4	2	1	1	1	1	2	3	0	0	1	1	5	0
2	0	3	2	0	2	4	0	2	1	4	0	0	2	2	3	4	1	0	4	0	2	1	0	1	0	1
3	1	1	0	3	0	2	3	3	0	1	2	4	0	1	2	1	4	2	0	0	1	2	3	1	0	1
4	0	0	1	0	2	1	2	2	4	0	0	2	4	1	0	2	0	5	3	0	0	2	2	4	1	0
5	0	1	0	4	0	2	1	7	1	0	0	4	0	0	0	2	4	3	0	0	0	3	5	1	0	0
6	1	0	2	0	1	3	3	1	1	1	0	0	0	3	2	3	3	0	0	2	5	3	1	0	3	0

The columns under letters obtained from the letter E by shifting no more than 9 places are tinted yellow. In each line, red indicates the number under the letter that actually denotes an E in the corresponding set; green—the maximum number of instances in the yellow (that is, permissible) range; blue—the maximum values that fall outside the permissible range.

rotating disk, and replace the third letter with the corresponding letter on the outer disk. In other words, the respective letter of the ith rotating disk replaces the $(3k + i)$th letter of the text (where $i = 1, 2, 3$ and $k = 0, 1, 2, \ldots$).

The very process of coding suggests that if the difference between the numbers of any two letters in the text is a multiple of 3, the same number—that is, the same disk—is used to encipher them. So Jarriques was mistaken in maintaining that the same letters in the ciphertext never denote the same letters in the plaintext.

Now, let's start deciphering. Suppose we know that the key is a three-digit number. To determine its first digit we should analyze the 1st, 4th, 7th, ... letters of the cryptogram. If the first digit of the key is 1, all these letters should be replaced by those immediately preceding them in the alphabet; if it's 2, the letters should be shifted back two places; and so on. But how can we determine the actual magnitude of the shift? The trick is that in the set of correctly shifted letters, the frequency of each letter's occurrence is approximately the same as in the language as a whole. That's the gist of the matter! By comparing frequencies at different shifts, we'll determine the most probable first digit. Then similar analysis of the set containing the 2nd, 5th, 8th, ... letters of the cryptogram will give the second digit of the key, while the third set (the 3rd, 6th, 9th, ... letters) will suggest the third digit. Finally, there might be several sufficiently probable keys at our disposal—we just have to choose the one that gives a coherent text.

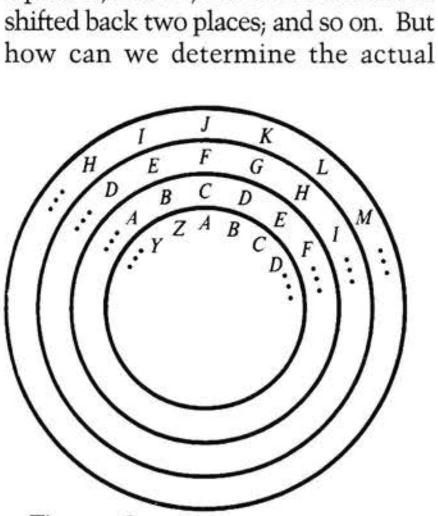


Figure 2

All that remains is to clarify how to approach the problem when the number of digits in the key is unknown. This case, too, requires a good deal of sequential searching. First, we assume that the key is a two-digit number; then, a three-digit number; and so on, until the text has been deciphered.

Omitting the intermediate variants, let's make use of the method to decipher our cryptogram, whose key, as we already know, is a six-digit number. In this case, the text of the cryptogram is divided into six sets of letters according to the pattern described above (first set: the letters 1, 7, 13, ..., 229; second set: the letters 2, 8, 14, ..., 230; ...; sixth set: the letters 6, 12, 18, ..., 228). The first two sets have 39 letters in each; the rest, 38 letters.

To begin with, we count up all the instances of each letter in each set. The results are listed in table 1. And that's all the information we need for decoding. Now we can try to puzzle out the key number right away, comparing the frequencies of the different letters. Because the most frequent letter in English is E, we can assume that its counterparts are the most frequent in their respective code sets. For example, in the first set the prime suspect is Y (5 entries—see the first line of table 1). But Y is 20 places away from E in the alphabet, whereas the maximum shift is 9 places. So Y falls away. For the same reason we can generally confine ourselves to the 10 yellow columns in table 1 (from E to N). The predominant "yellow letters" in the first line are F, J, and M, which means the first digit of the key should be 1, 5, or 8. The second line gives us F and J as plausible letters (or 1 and 5 as digits); the third line suggests L (or 7); and so on. Although we've already seen that the actual key number is 513432, to play fair we should check all the $3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 3 = 36$ possibilities. It looks to be a little boring, and we still wouldn't be all that sure we didn't miss the right digit (in fact, we did miss it in the third line). But there's nothing to be surprised at: our sets are too small for us to draw reliable conclusions about the frequencies of individual letters. And yet, if we take a group of the most common letters right off the bat, statistical laws will inevitably take over.

To make the superiority in frequency significant, a group of four letters, E, T, A, O, will suffice. This time we'll be clever right from the start and restrict ourselves to this group and its shifts:

Table 2: The number of instances of the letters E, T, A, and O in each cryptogram set

	Magnitude of Shift									
Set No.	0	1	2	3	4	5	6	7	8	9
1	4	5	1	5	6	16	2	2	5	6
2	5	13	4	2	8	8	3	3	6	9
3	3	5	9	11	1	3	7	10	4	3
4	2	3	5	9	13	2	2	6	10	6
5	0	5	8	19	2	2	1	14	7	1
6	6	11	11	2	2	9	8	5	2	6

In each line red denotes the number corresponding to the actual magnitude of the shift for the set. It's always the greatest number in the line, and it's almost always unique. (The exception is the sixth line, where there is another number as large, which is marked green.)

F, U, B, P; G, V, C, Q; ...; N, C, J, X (that is, the "yellow portion" of a giant "table of letter-quadruplets"). The number of letters of each group in each set is easily calculated by adding up the corresponding four numbers in table 1. This gives us table 2, by means of which we can more or less definitely conclude that the first five key digits are 5, 1, 3, 4, 3. Although the last digit remains uncertain—it's a choice between 1 and 2—full decoding of the cryptogram now poses no problem.

I leave it to you to figure out what's written in the ciphertext!

GETTING TO KNOW...

The elementary particles

Fishing for the Higgs boson, stalking the top quark. . .

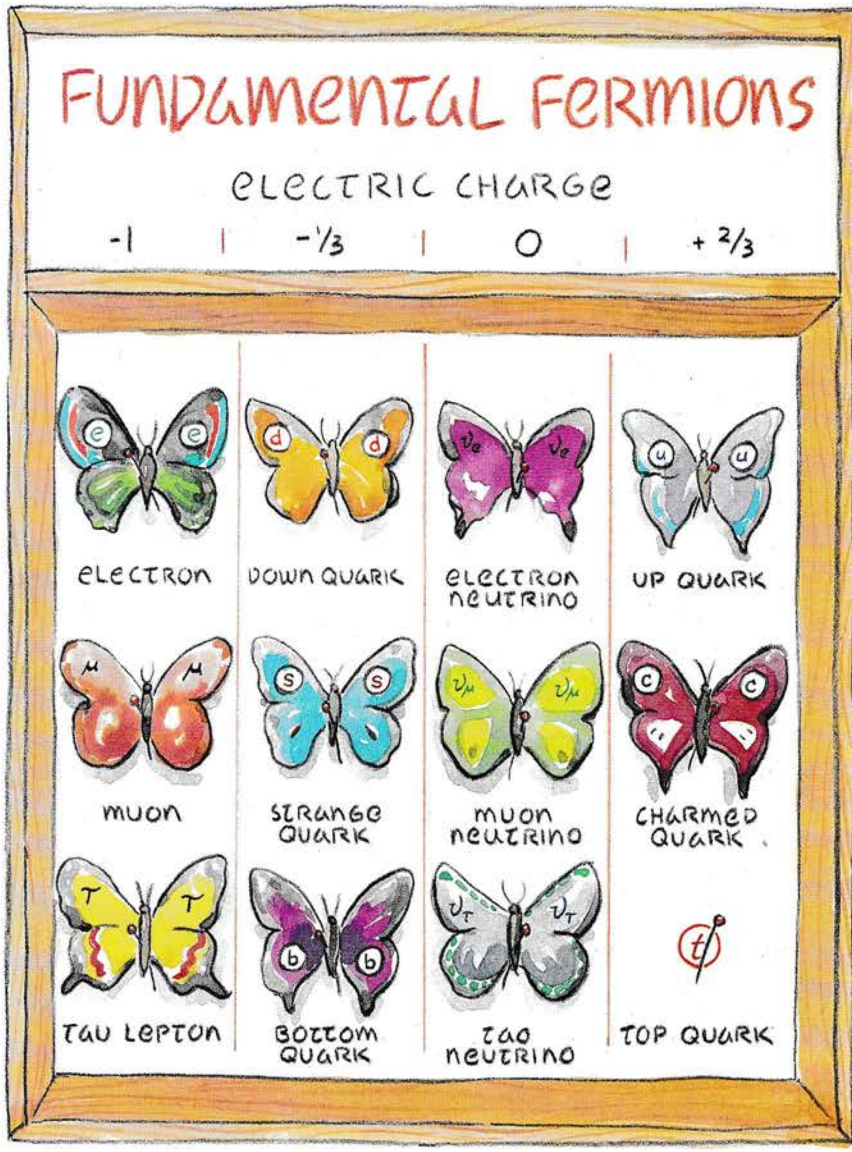
by Sheldon Lee Glashow

Mass, spin, and the antiparticle

Quantum mechanics and relativity tell us that every particle is characterized by a nonnegative *mass* and a nonnegative integer or half-integer *spin*, and that it has an *antiparticle* with the same mass and spin but opposite electric charge. Massive particles travel slower than light and can be brought to rest, while massless particles (like photons and gravitons) travel at light speed relative to all watchers. Spin is the measure of a particle's intrinsic angular momentum. A massive particle of spin s may be found in any one of $2s + 1$ different quantum spin states. The antiparticle of the electron, called the *positron*, was seen first in cosmic rays in 1932. *Antiprotons* were first produced and detected at the Berkeley Bevatron in 1955. Photons are their own antiparticles. Particles annihilate their antiparticles on contact. All earthly matter (and virtually all celestial matter) consists of particles, not antiparticles. Otherwise, we wouldn't be here to tell the tale.

Fermions and bosons

Particles with half-integer spin (like electrons with spin 1/2) satisfy Fermi–Dirac statistics, which means no two of them may be in the same quantum state at the same time (the Pauli Exclusion Principle). Such particles are called *fermions*. Particles with integer spin (like photons) satisfy Bose–Einstein statistics. Many of these

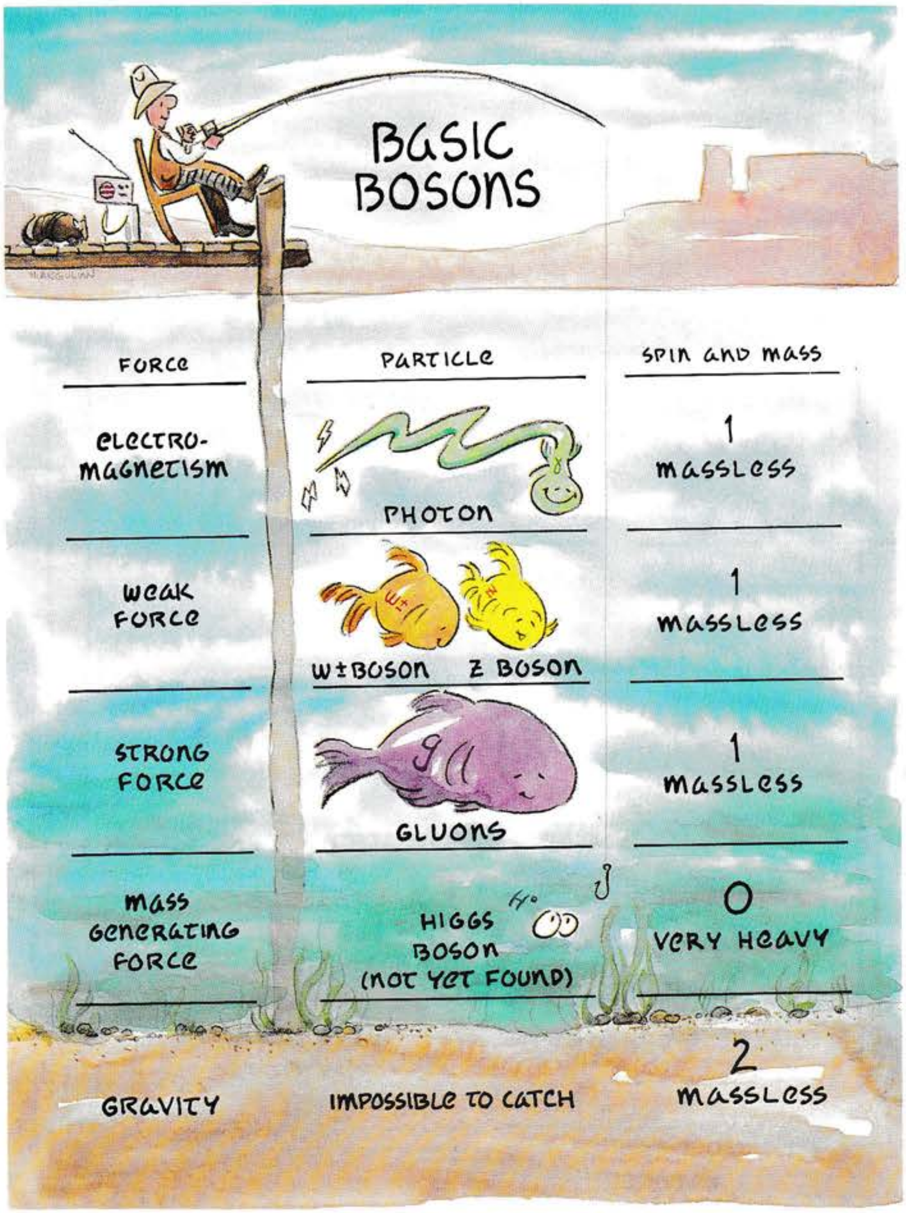


Art by Nishan Akgulian

bosons can (and, in a sense, like to) congregate in the same quantum state, which is the principle underlying the operation of lasers.

Fundamental fermions—quarks and leptons

Our particle directory lists twelve spin 1/2 particles: six *quarks* and six *leptons*. Quarks (rhyming with forks) were invented by M. Gell-Mann and G. Zweig in 1963. "Up," "charmed," and "top" quarks carry electric charges of 2/3, while "down," "strange," and "bottom" quarks carry charges of –1/3. An individual quark can't be isolated from the *hadron* of which it forms a part. Thus, quarks can't be seen as particles in their own right. The word lepton comes from the Greek *leptos* meaning "small" or "slight," and was coined by L. Rosenfeld in 1948 to mean any fermion of small mass, like the electron or neutrino. Today, leptons include any of six known fermions lacking strong nuclear interactions. Three are electrically charged: the electron, the muon (about 200 times heavier), and the tau-lepton (about 17 times heavier yet). Each one is associated with its own sort of neutrino, making six leptons in all. Neutrinos are very light, perhaps even massless. Recent experiments suggest that there are no more than three neutrino species. This implies that our list of fundamental fermions is complete. Whether it really is or not, we shall see!



Basic bosons

These particles mediate forces among the fundamental fermions. Electromagnetism results from the exchange between charged particles of massless *photons*, the particles of light. The strong nuclear force arises from the exchange between quarks of massless *gluons*. The weak nuclear force is produced by the exchange of massive W or Z bosons between any of the fundamental fermions. Gravity may be thought of as due to the exchange of massless *gravitons*. Gluons, like quarks, are "confined": they can't be seen as isolated particles. Charged W's and neutral Z's were discovered at the European Center for Particle Physics (CERN) in 1983. The last of the basic bosons in our bestiary is the *Higgs boson*, an elusive and still hypothetical particle responsible for generating all particle masses. It should show up at the Superconducting Super Collider, now a-building in Texas.

Hadrons

In 1962 the Soviet physicist L. Okun used the Greek word adros, meaning "thick and bulky," in choosing a name for any seemingly elementary particle that partakes in the strong nuclear force, like the proton but not the electron. Today, a hadron is any particle made up of quarks. Three quarks stick together to form a *baryon*; a quark binds to an antiquark to form a *meson*; and three antiquarks form an *antibaryon*. These are the only known ways in which quarks combine to form hadrons. Because they are made up of an odd number of fermions, baryons and antibaryons are themselves fermions. Mesons are bosons.

Nucleons

This is a word that has been used since 1941 to refer to neutrons or protons. An atomic nucleus with mass number Z contains A nucleons, Z of which are protons. Nuclei with the same Z but differing A are known as *isotopes*. Nucleons are fermions. They are the lightest baryons, consisting exclusively of up and down quarks:

two ups and a down make a proton, while two downs and an up make a neutron. About 99.98% by weight of all ordinary matter consists of nucleons. The rest is electrons.

Pions and muons

Hideki Yukawa suggested in the 1930s that the nuclear force results from the exchange of hypothetical elementary particles between nucleons. He called his particles mesotrons (soon truncated to mesons) because they had to be intermediate in mass between electrons and nucleons. Particles with such masses were observed in 1938, but they turned out to be muons. Yukawa's particles were finally discovered in 1947. Both pions and muons were first seen in cosmic rays. Many other kinds of meson have been discovered since. Yukawa's mesons became known as pi-mesons and eventually as pions. They aren't elementary: like all mesons, they're each made up of one quark and one antiquark.

The top quark

Our theory demands that such a particle exists and weighs no more than 200 protons. Experimenters have not yet found it. They are confident it must be heavier than 100 protons; otherwise, it would have shown up already. This window is rapidly being closed: I predict that "top" (the Last of the Quarks!) will be found by physicists working at the Fermilab proton–antiproton collider within two years.

Neutrinos

Neutrinos produced by a nuclear reactor were first observed in 1953. Since then, physicists have observed neutrinos produced at particle accelerators, by cosmic rays, by the nuclear furnace of the sun, and by the last "nearby" supernova in 1987 (which was a mere 160,000 light-years away). Some scientists believe that neutrinos have mass and that the mysterious dark matter of the universe consists of swarms of neutrinos left over from the Big Bang.

Physics limericks

Mysteries of the universe in rhyme, the riddle of time in verse

by Robert Resnick

PUTTING PHYSICS in limerick form has become popular recently. It's really an old game, though. Back in Albert Einstein's day, when there was a writer named Gertrude Stein ("a rose is a rose is a rose") and a modernist sculptor Louis Epstein, a fashionable limerick went

> There's a curious family named Stein—
> There's Gertrude, there's Ep, and there's Ein.
> Gert's verses are bunk,
> Ep's statues are junk,
> And nobody understands Ein.

I work humor into my physics classes and often cite a relevant limerick. For example, in warning students what could happen if they try to defy the laws of physics, I would say

> There was a young woman named Bright
> Whose speed was much faster than light.
> She eloped one fine day
> In a relative way
> And conceived on the previous night.

Of course, today that could be considered a sociological limerick.

Or, in another vein,

> A mathematician named Haines,
> After infinite racking of brains,
> Now says he has found
> A new kind of sound
> That travels much faster than planes.

Back in 1958 I gave the students in a modern physics class an examination in limerick form. They had to complete the limerick I started by adding the couplet or the last line. One test item, for example, was

> An electron quite debonair
> Spied a positron up on the stair.
> ....
> ....
> And finished him off in mid-air.

I should tell you the sort of legitimate answers I had in mind for this one. Remembering that the electron and the positron circle about their common center of mass, forming a short-lived positronium atom before mutual annihilation, the student might have used a couplet like

> ...
> She put him in a trance
> With her infamous dance
> ...

A somewhat incorrect answer, for which I'd give part credit, might be

> ...
> She meant him no harm,
> But turned on her charm,
> ...

A recent correspondent[1] who heard of my test offered this answer:

> ...
> She needed no tact,
> For unlikes attract,
> ...

So you see, you really can find lots of meaningful solutions.

Another test item was

> There once was a hard gamma ray
> And a nucleus it forced to decay.
> A resultant bambino
> Was called the neutrino
> ....

The same correspondent met the challenge with this last line:

> You ask if it had mass? No weigh!

Now I'll give you some of the other test items and let

[1]Barbara Levi, *Physics Today.*

you try your hand at filling in the blanks. Remember, you must make physical sense, be clever, and keep the galloping rhythm (or at least try)![2]

Said a slow little neutron ere fission,
"Don't speak of me with such derision.
I may have no charge,
And be not so large,
....

An atom that came from the tap
Had electrons all over her map,
But in her interstices
Lurked a much worse disease
....

A meson descending in flight
Was veering first left and then right.
So brisk was its action,
The Lorentz attraction
....

An X ray shot out like a tear
Took off for a crystal quite bare.
It wasn't the plasticity,
But that darn periodicity
....

The mesons are nuclear glue—
Just listen to what they can do:
....
....
The pi ones, that is, not mu.

[2]There was a young man from San Fran
Whose verses never would scan.
When asked why this thing
Never went with a swing,
He said, "I try to get as many words into the last line as I possibly can."

An electron was spinning around
And moving quite close to the ground,
Exotic its rapture
At the thought of K-capture
....

That exam spread far and wide since 1958. I do think it had something to do with popularizing physics limericks. Since then physics journals and others have held physics limerick contests.[3] Once you open it up to classical physics, as well as modern, and allow complete ones instead of just test items, the sky's the limit. I must have filed hundreds of them away—not all the greatest, of course.

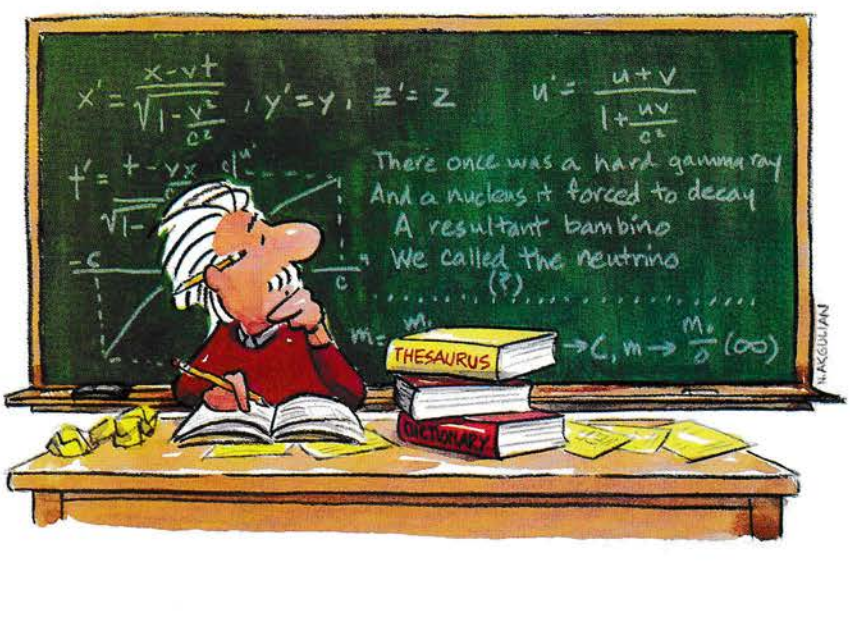
A recent article[4] in *The Physics Teacher* reprinted a story of mine from the Rensselaer alumni magazine that told of my experiences with student limericks. Already I'm getting mail with complete physics limericks. One of the best sets came from a high school student.[5] Here are a few of his creations:

Is a quark a thing or a wave?
This question spurs many to rave.
Please, don't you jeer,
I'm being sincere—
A quark can as either behave.

Where is it? 'Tis really uncertain,
Like trying to peer through a curtain.
Heisenberg had no doubt
You can never find out
Both position and momentum for certain.

And one for skeptics:

They thought they discovered cold fusion,
And in general caused quite some confusion.
Fleischman and Pons
May unfortunately be cons—
Their research might just be illusion.



Art by Nishan Akgulian

So maybe we should have a contest among students for the best complete physics limericks. What do you think? (For our foreign readers, this may push your English as well as your physics to the limit!) Meanwhile, send in your lines to complete the protolimericks I've given.

Next time—palindromes. That's a lot tougher!

Robert Resnick *is Professor of Physics at Rensselaer Polytechnic Institute, Troy, NY.*

[3]See *The Physics Teacher,* September 1986 and September 1987.

[4]"There Was a Professor from Troy," *The Physics Teacher*, January 1990.

[5]Patrick T. Baker of Rockville, Maryland.

The music of physicists

Short tales (not "from the Vienna woods") that show the human side of some of the giants of modern science

Albert Einstein (1879–1955) liked to play the violin, and he especially liked being accompanied by the great pianist Artur Schnabel. One particular day, Einstein played a wrong note—after all, he was a physicist, not a professional musician! He and Schnabel tried the passage again, and again Einstein missed the note. Schnabel lost his cool. "Wrong, Albert, wrong! Just listen to me play it: one, two, three . . . Dear me, how could it be you can't count?!"

Robert Bunsen (1811–1899) invented many things, including the carbon zinc electric cell and the ice calorimeter, but he played only a minor role in developing the ubiquitous burner that bears his name. Once he went to the local music conservatory for a concert and a bulb went on in his head. During the intermission he turned to the person sitting next to him: "Tell me, are all the violins over there playing the same thing?" His neighbor told him yes, they all play the same notes. Bunsen shook his head. "Well," he said, "that's pretty uneconomical. They ought to exchange them for one big violin and have just one person play it!"

Max Planck (1858–1947), the instinctively conservative scientist who revolutionized physics with his work on the quantum theory and relativity, was strongly drawn to music in his youth. In fact, he seriously considered making it his career. Planck had such a good ear that, as he used to tell his friends, no concert could be completely pleasurable for him because he always noticed even the slightest mistakes the musicians made. Only after many years did he—to his great joy—lose this "supersensitivity."

The great nuclear physicist Ernest Rutherford (1871–1937), on the other hand, didn't have a very good ear for music. But he did have a pretty loud voice. His repertoire consisted of just two things, which his lab assistants reliably used to ascertain his mood. If Rutherford was walking down the hall bellowing "Onward, Christian soldiers" (recognizable only by the words, not the tune), work was going well. But if he was carefully fitting words to the doleful strains of a ponderous dirge, his coworkers began mentally preparing themselves: Watch out, Rutherford's in a rotten mood!

Art by Pavel Chernusky

The USA Mathematical Talent Search

Where else do you get an entire month to solve five problems?

by George Berzsenyi

Prove that an integer can be expressed as the arithmetic average of two perfect squares if and only if it is the sum of two perfect squares.

For what values of n is it possible to partition the set {1, 2, ..., n} into five disjoint subsets so that within each subset the sum of the elements is the same?

IN THE FALL OF 1989, VIA A column of the same name in *Consortium*, I initiated the USA Mathematical Talent Search (USAMTS), broadening the well-known Wisconsin Mathematical Talent Search to the national level. This endeavor was supported by Rose-Hulman Institute of Technology, the Consortium for Mathematics and Its Applications (COMAP), and the Exxon Education Foundation. It was aimed at talented high school students to attract them to the fields of science and engineering. Four sets of five problems were published in the quarterly *Consortium* during the 1989–90 school year. The participants were given one month to submit their solutions to each set of problems, and their work was carefully evaluated by a team of faculty members at Rose-Hulman. These evaluations, along with complete sets of solutions with insightful commentary, were sent to the students, who could gather 5 points for each perfect solution and thus a total of 100 points over the year. On the basis of the outcome, several winners were declared in each of grades 9 through 12 and were awarded valuable book prizes by COMAP.

During the first year of the USAMTS, nearly 300 students took advantage of this unique opportunity. The states of New York, Illinois, and Texas provided the largest numbers of competitors, but most of the other states were also well represented. It was particularly gratifying that more than half of the participants of the US Mathematics Olympiad were active in the USAMTS and that five of the six members of this year's International Mathematics Olympiad team took part in it. Many of the participants commented on their preference for this type of competition, which doesn't impose stringent time constraints but instead encourages careful exposition of mathematical thought. That takes time, and the beautiful work submitted by most of the contestants is most rewarding. Unfortunately, year-round problem solving is not yet a national pastime in America, so we must redouble our efforts to attract even more of you to the USAMTS.

To this end, I offer two problems from year 2, round 1, in the hope that they will whet the appetites of thousands of *Quantum*'s student readers. For the complete set of five problems, see the Fall 1990 issue of *Consortium* or write to me at the USA Mathematical Talent Search, Box 121, Rose-Hulman Institute of Technology, Terre Haute, IN 47803.

Bulletin Board

Supercomputing for high school students

The Cornell Theory Center, one of four national supercomputing centers, offers a summer program called "SuperQuest." Open to all of the 23,000 high schools in the United States, SuperQuest is the only program to offer advanced supercomputing specifically for high schools. Four teams, consisting of 3-4 students and one teacher-coach, are selected to come to Cornell for one month in the summer to learn about supercomputing research and its applications. The students take classes in supercomputing techniques, meet with supercomputer researchers such as Carl Sagan, and work with Cornell's technical staff to develop their own programs.

Sponsored by IBM and the National Science Foundation, SuperQuest's goal is to foster creativity in devising computational solutions to scientific problems, and no area of scientific endeavor is out of bounds. For an application booklet and more information on SuperQuest, write to SuperQuest, P.O. Box 6345, Princeton, NJ 08541, or call 607 255-4859.

National Science Olympiad results

Approximately 2,000 students representing 94 schools in 35 states gathered at Clarion University of Pennsylvania in May to take part in the sixth annual National Science Olympiad. The students competed in 32 science events testing their knowledge of biology, earth science, chemistry, physics, computers, and technology. Winners received medals, trophies, or scholarships for their efforts. This year Irmo, South Carolina, bested competitors in both the high school and middle/junior high school divisions.

The Science Olympiad, a nonprofit organization headquartered in Rochester, Michigan, seeks to improve the quality of science education, increase student interest in science, and recognize outstanding achievement in science education. The teams advancing to the finals at Clarion University were the survivors of regional and state Science Olympiad tournaments.

Next year's National Science Olympiad will be held at Penn Valley Community College in Kansas City, Missouri. For more information, write to National Science Olympiad, 5955 Little Pine Lane, Rochester, MI 48604.

Duracell scholarships

Each year the Duracell/NSTA Scholarship Competition awards 41 students in grades 9 through 12 over $30,000 in scholarships and cash prizes for battery-powered devices they have designed and built. Judges evaluate devices on the basis of originality, creativity, and practicality, and all entrants receive an award certificate and a special gift.

This year's top winners will also win an expense-paid trip to Houston for the awards ceremony. All entrants must have a teacher/sponsor, so ask your science teacher for rules and applications. For more information, write to Duracell/NSTA Scholarship Competition, National Science Teachers Association, 1742 Connecticut Avenue NW, Washington, DC 20009.

Introducing Network Earth

The Turner Broadcasting System recently premiered "Network Earth," an informative and entertaining weekly program that offers a provocative look at current environmental problems and their solutions. Presented in a fast-paced magazine format, Network Earth's topics range from the well-publicized and widely discussed to the obscure and unexplored issues of the environment. The series will also include interviews with environmentally active celebrities and local heroes, and pointers toward leading an environmentally positive lifestyle.

Network Earth invites viewers to participate with its staff, environmental experts and organizations, and each other via computer. Through CompuServe Information Service, viewers with access to a personal computer and modem will be able to log onto the system, read about current environmental activities, ask questions about what they have seen on the show, access material that could help them become more environmentally aware, and participate in live computer conferences.

Network Earth runs on Sundays at 11:00 P.M. on TBS. For more information, write to Network Earth, One CNN Center, Box 105366, Atlanta, GA 30348.

What's happening?

Summer study ... competitions ... new books ... ongoing activities ... clubs and associations ... free samples ... contests ... whatever it is, if you think it's of interest to *Quantum* readers, let us know about it! Help us fill Happenings and the Bulletin Board with short news items, firsthand reports, and announcements of upcoming events.

What's on your mind?

Write to us! We want to know what you think of *Quantum*. What do you like the most? What would you like to see more of? And, yes—what don't you like about *Quantum*? We want to make it even better, but we need your help.

What's our address?

Quantum
1742 Connecticut Avenue NW
Washington, DC 20009

Be a factor in the QUANTUM equation!

In the next issue of Quantum . . .

What are the prospects for using light instead of electricity for information transfer in electronic devices? Y.R. Nosov gives us a guided tour of this area of solid-state physics in "Lightning in a Crystal."

What does it mean to "take the extreme case," as we were told to do in the answer to Math Challenge 15? A.L. Rosenthal plays variations on that rich theme in "Math to the Max."

Have you ever wondered why the holes in Swiss cheese are round? In his article Sergey Krotov reminds us not to be afraid of asking "childish" questions.

What do Euclid's greatest common divisor, Pythagorean triplets, and geneological trees have in common? A.A. Panov shows us what in his mathematically arboreal article.

How did Native Americans throw a tomahawk so that it stuck instead of bouncing? V.A. Davydov's love of James Fennimore Cooper's novels led him to investigate, and he offers his results in "Tomahawk Throwing Made Easy."

Plus . . .

- A glitch while inventing the steam engine
- Physics for dummies
- A discomforting incident in a railway tunnel
- New problems from the Tournament of Towns

. . . and our regular features!

Math

M11

From a great variety of solutions to this problem we've chosen two of the most instructive and elegant, in our view.

In a triangle ABC let $CD = l$ be the bisector of angle ACB, $AC = b$, $BC = a$. We'll prove that

$$l < \frac{2ab}{a+b}.$$

For $a = 10$, $b = 15$, this gives $l < (2 \cdot 10 \cdot 15)/(10 + 15) = 12$.

Solution 1. Draw a line through D parallel to BC and intersecting AC at E (fig. 1). Obviously, $CD < CE + ED$, and the angles marked in the figure are equal. Therefore, triangle CDE is isosceles, $CE = ED = x$, and because of the similarity of triangles ADE and ABC

$$\frac{CE}{EA} = \frac{ED}{EA} = \frac{CB}{CA}, \qquad (1)$$

or $x/(b - x) = a/b$. So $x = ab/(a + b)$ and $l < 2x = 2ab/(a + b)$.

Solution 2. If we fix the vertices A and C, the locus of B will be the circle with center C and radius a (fig. 2). By the well-known property of a bisector, which can be obtained easily from figure 1 and equation (1), $BD/DA = BC/CA = a/b$. Thus $AD/AB = b/(a + b)$; therefore, D is the image of B after a dilation with center A and scale factor $b/(a + b)$. This dilation maps the locus of B onto the locus of D. So the latter is a circle of radius $ab/(a + b)$ passing through C (see figure 2), and $l = CD$ is always less than its diameter, $2ab/(a + b)$. Also, we see immediately that for any l from the interval $0 < l < 2ab/(a + b)$ there exists a triangle with a, b, and l as two sides and the bisector between them.

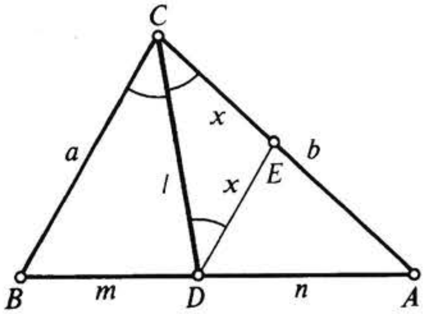


Figure 1

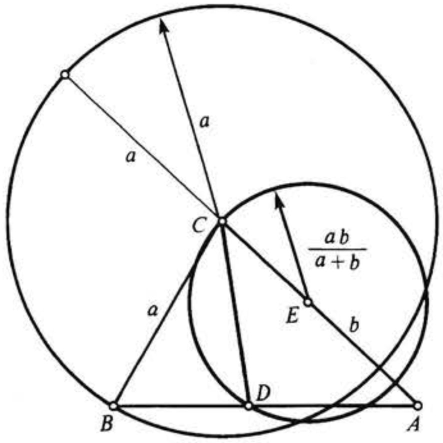


Figure 2

M12

Let's set the sum $s = x + y \geq 0$. Then the set $V(s)$ of values assumed by

$$(x+y)^2 + 3x + y = \frac{s^2+s}{2} + x,$$

when x varies from 0 to s, consists of all the integers from $(s^2 + s)/2$ to $(s^2 + s)/2 + s$, each of them assumed once.

Now let's notice that the last number of $V(s)$ and the first number of $V(s + 1)$ are consecutive:

$$\frac{s^2+1}{2} + s + 1 = \frac{(s+1)^2 + s + 1}{2}.$$

Therefore, the sets $V(s)$ cover all the nonnegative integers n without overlaps or gaps (fig. 3). Since any n gets into one and only one of the sets $V(s)$, it can be represented in the required form. Also, s, x, and $y = s - x$ are determined by n uniquely.

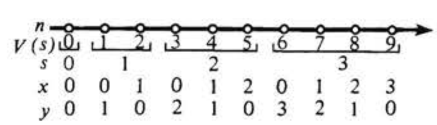


Figure 3

By the way, our formula shows that pairs (x, y) can be enumerated by the numbers n, as seen in figure 4. (N. Vasilyev)

M13

To expose the main idea of the proof let's consider a triangle XYZ with a fixed base YZ and vertex X moving along a line l (fig. 5). It's quite obvious that the area of XYZ is constant if l is parallel to YZ; otherwise, it varies monotonously as long as X doesn't cross YZ. (Actually, the area is proportional to the distance from X to YZ.)

Now let's denote the given quadrangles as in figure 6. If a diagonal of the inscribed quadrangle, say KM, is parallel to a side of the parallelogram (AB or CD), we're done. Otherwise we mark the point P on BC such that PM is parallel to AB. Using our moving method it's easy to transform $PLMN$ into the triangle ABC, which is just half of $ABCD$, so its area remains unchanged. Thus the quadrangles $KLMN$ and $PLMN$ have the same area, equal to half the area of $ABCD$. Subtracting the triangle LMN from both of these quadrangles, we get two triangles, LNK and LNP (fig. 7), with the com-

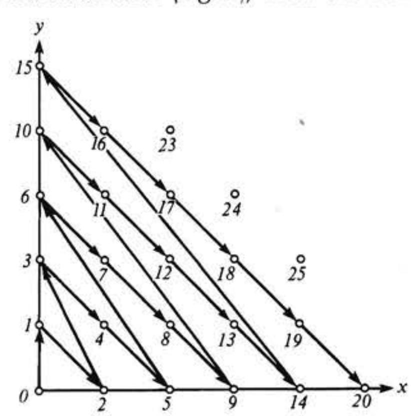


Figure 4

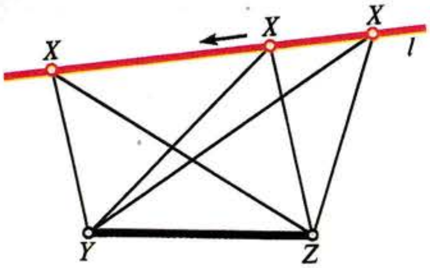


Figure 5

mon base LN and equal areas. From this it follows that KP (or the side BC) is parallel to the diagonal LN. (V. Dubrovsky)

M14

The answer is no. To prove it, let's introduce a coordinate system on the plane such that the initial positions of the frogs get the coordinates (0,0), (1,0), and (0,1) (fig. 8). It's easy to see that when a frog sitting at (x,y) jumps over a frog at (a,b), it lands at the point $(2a - x, 2b - y)$ (fig. 9). So the parities of a frog's coordinates don't change after a jump. At the start each frog had at least one even coordinate. Therefore, none of them can hit a point with two odd coordinates, in particular the point (1,1)—that is, the fourth vertex of the square.

This solution can be explained in a more visual way with the grid shown in figure 10. The grid contains three of the four vertices of the initial square at which our frogs start, and it's symmetrical about each of its points. Therefore, the frogs can't leave it to hit the fourth vertex.

Also, for a similar reason the "red frog" (starting at the red vertex in figure 10) can get only to red points

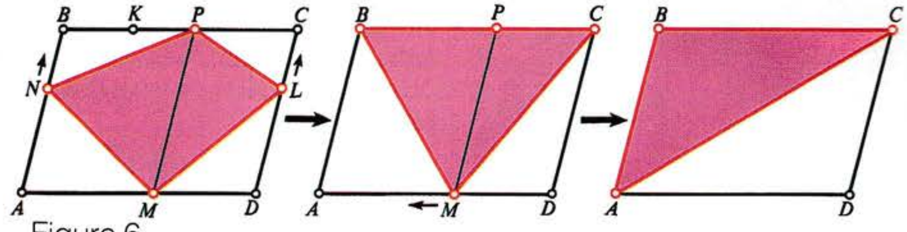


Figure 6

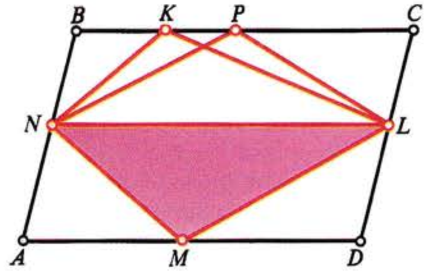


Figure 7

(the red "subgrid" is symmetrical about any point of the whole grid). Similarly, the "blue" and "black" frogs have to stay on blue and black points, respectively. We leave to you to prove that each frog can get to any point of its color. More difficult questions are these: (1) Can two frogs simultaneously get to any two given points of their respective colors? (2) What are the triplets of points accessible to three frogs at the same time?

The second question has a simple and beautiful answer, but we won't deprive you of the pleasure of finding it on your own. (N. Vasilyev)

M15

One of the most useful principles for solving olympiad problems says: "Take the extreme case." Following this recommendation, let's consider the face F of a given polyhedron with the greatest number of sides; let this number be m. Each side of the face F belongs to another face. This gives us at least m + 1 faces (including F), because a face of a convex polyhedron can't have more than one edge in common with any other face.

On the other hand, all the faces can have no more than m different numbers of sides (even less, since 1 and 2 are, in fact, impossible). So there are at least two faces having an equal number of sides.

This last conclusion is based on yet another useful principle, this time a very simple theorem known as the pigeonhole (or Dirichlet) principle: if more than m pigeons are placed in m pigeonholes, then at least one pigeonhole has more than one pigeon in it. (See the Contest Problem in this issue for more pigeon talk.)

Although our solution seems very much like many other pigeonhole solutions, it can't do without convexity, and so it's essentially geometrical. In fact, two adjacent faces of a nonconvex polyhedron can have more than one common edge (like a and b in figure 11). In this case our reasoning fails. Nevertheless, the statement of the problem remains valid for nonconvex polyhedrons, too. The only condition is that they shouldn't have a hole that goes all the way through, like the one shown in figure 12. The proof is based on Euler's famous formula

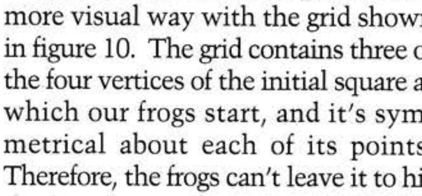


Figure 8

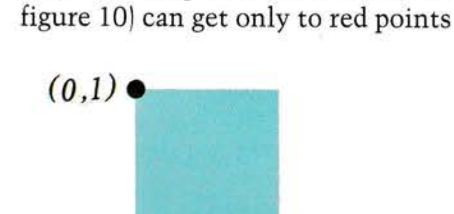


Figure 9

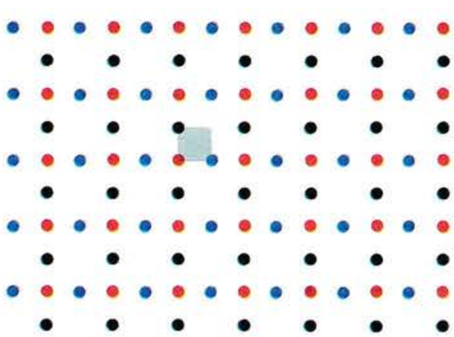


Figure 10

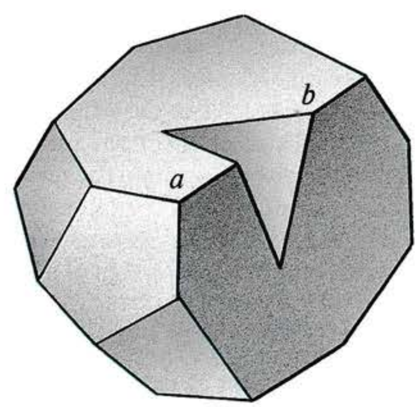


Figure 11

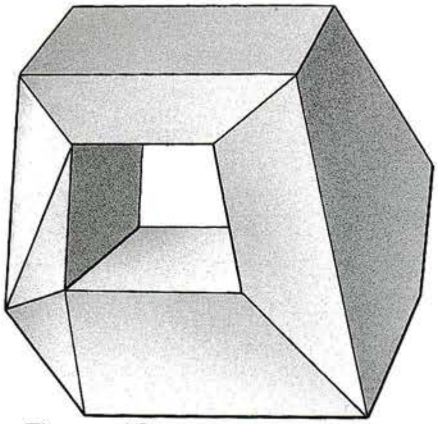


Figure 12

$$v - e + f = 2$$

(where v, e, and f stand for the number of vertices, edges, and faces in a polyhedron), but it's rather cumbersome and not elegant enough to present here. And yet another problem: does our statement remain true if holes are allowed? (G. Galperin)

Physics

P11

The wheel is subjected to the reaction force **N**. If the wheel is balanced, this force is equal in absolute value to 1/4 of the automobile's weight $M|\mathbf{g}|$ (assuming the car is traveling along a smooth, horizontal road).[1]

If the wheel is unbalanced, its center of mass moves along a circle of radius r, and the centrifugal acceleration is $\omega^2 r$ (r is the distance from the wheel's center of mass to the wheel's center, ω is the angular velocity of the wheel). When the center of mass reaches the lowest point, $|\mathbf{N}|$ exceeds ¼ $M|\mathbf{g}|$ by $m\omega^2 r$ (m is the wheel's mass). When the center of mass is at the upper point, $|\mathbf{N}|$ is less than ¼ $M|\mathbf{g}|$ by the same value. When the wheel slips on the road surface (a certain amount of slipping always occurs), the resulting force of friction between the wheel and road may have a different value. So with an unbalanced wheel some areas of the tire may wear more rapidly than others. The practical result? You may have to buy new tires sooner than you would if the wheels had been balanced.

[1]The exact fraction of the automobile's weight corresponding to each wheel depends on the position of the automobile's center of mass. In general, it's *not* equal to 1/4.

P12

The elastic string can be considered a spring. Let the spring's rigidity equal k. The string breaks when the full force reaches the value

$$F_f = mg + F_1, \qquad (1)$$

where m is the mass of the weight. Let's see what happens when the applied force reaches the value F immediately and then doesn't change.

At the initial moment (when the force is equal to zero), the string is stretched because of the weight of the hanging mass mg by the amount x, defined by the expression $kx_0 = mg$—that is, the coordinate of the string's end at equilibrium is equal to $x_0 = mg/k$ (the origin is assumed to be the position of the string's end when the weight isn't attached). After a force F is applied, the coordinate of the string's end at equilibrium is equal to $x' = mg/k + F/k$. But the system reaches the equilibrium state only after some oscillations. The initial amplitude of these oscillations depend on the initial deviation of the system from the equilibrium position—that is, $A = x' - x_0 = F/k$. The damping rate of these oscillations is quite low, so the full extension of the string after half of the period is

$$x_f = \frac{mg+F}{k} + A = \frac{mg}{k} + 2\frac{F}{k}.$$

This means that a total force of

$$F_f' = mg + 2F$$

is being applied to the string.

The string snaps when this force exceeds F_f—see equation (1); this defines the string's strength limit. So the minimum force F_{min} under which the string breaks can be determined from the condition $F_f' = F_f$—that is,

$$mg + 2F_{min} = mg + F_1,$$

which readily yields

$$F_{min} = \frac{F_1}{2}.$$

P13

The total pressure on the table is the total force of gravity acting on the bell and water:

$$F = Mg + \frac{4}{3}\pi R^3 \rho g .$$

But when the water lifts the bell and starts to leak out, the bell's weight itself no longer acts on the table. The pressure on the table then equals the pressure of the water multiplied by the area of the base. The water pressure is the same at all points ($\rho g R$), so $F = \rho g R \pi R^2$. This implies that

$$\pi R^3 \rho g = Mg + \frac{4}{3}\pi R^3 \rho g ,$$

which yields

$$M = \frac{1}{3}\pi R^3 \rho .$$

A *Kvant* reader proposed an interesting approach. Assume that the bell is placed in a cylindrical container whose radius and height are both equal to R. We'll fill the container with water and take the bell's mass to be negligible. We can easily see that since the inside and outside pressures on the bell are equal at all points (this follows from Pascal's law and from the assumption that the bell is very thin), the water's equilibrium won't be affected if the bell is removed. Neither does the pressure of the water on the table change. But this means the pressure of the water we poured into the cylindrical container acts exactly like the bell, so the bell's mass is equal to the mass of the poured water:

$$\begin{aligned} M &= \left(V_{\text{cylinder}} - V_{\text{hemisphere}}\right)\rho \\ &= \left(R\pi R^2 - \frac{2}{3}\pi R^3\right)\rho \\ &= \frac{1}{3}\pi R^3 \rho . \end{aligned}$$

P14

The gas temperature is defined by the average kinetic energy of its molecules

$$\frac{mv^2}{2} = \frac{3}{2}kT ,$$

where k is the Boltzmann constant. This means that the higher the gas temperature, the greater the average velocity of its molecules and, consequently, the greater the average molecular momentum.

If the temperature of the wall is the same as that of the gas, a molecule colliding with the wall changes its momentum from p_0 to $-p_0$. The change in momentum is $2p_0$. When $T > T_1$, the gas gets heated, which means the gas molecules move from the walls with a greater velocity than they had before the collision. So the resulting momentum is greater than the initial momentum (fig. 13a), and the change in momentum is greater than $2p_0$.

If $T < T_1$, the gas gets cooler, so after the collision a molecule's momentum is less than before (fig. 13b). In this case the change in momentum is evidently less than when $T > T_1$. According to Newton's second law, the change in momentum is proportional to the average force from the wall acting on the molecule; and according to Newton's third law, the average force exerted on the molecule is equal to the average force acting on the wall; therefore, the pressure of the gas on the wall is greater when $T > T_1$ than when $T < T_1$.

P15

The devices should be connected as shown in figure 14. Points O, A, and B have the same potentials. (The resistance of the ammeter is low, and we can ignore the corresponding voltage drop.) Consequently, there is no current through the resistances connecting point O with A and B. This means that the ammeter registers the current passing through the resistance between points O and C, while the voltmeter measures the voltage drop on this resistance. Dividing the voltmeter reading by that of the ammeter, we find the value of the resistance.

Brainteasers

B11

Point D should be positioned so that segment CD is equal to 1/5 of segment AC (fig. 15); then the area of triangle DBC will be 1/5 that of ABC. Similarly, point E is positioned so that $BE = AB/4$, point F so that $FD = AD/3$, and point G so that $EG = AE/2$.

B12

Answer: 1,999,999,999. If there are two numbers a and b greater than 1 among the given numbers, then, replacing one of them with ab and the other with 1, we'll retain the product of all the numbers and increase their sum because the inequality $(a-1)(b-1) > 0$ implies that $ab + 1 > a + b$. Thus, the sum will be greatest if one of the numbers is a billion and all the others are equal to 1.

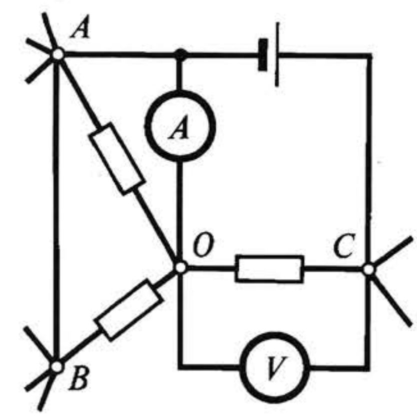


Figure 14

B13

Mark the level of the liquid and turn the flask upside down.

B14

No, since every such number is less than $10{,}000^2$ but greater than $9{,}999^2 = 99{,}980{,}001$.

B15

Winnie-the-Pooh had walked for 3 minutes and Piglet for 6 minutes. Suppose it took x minutes for Winnie-the-Pooh and Piglet to walk from their respective homes W and P to the meeting point M. Winnie-the-Pooh spent x minutes walking from W to M and 1 minute from M to P; therefore, $WM/MP = x$. The same reasoning for Piglet gives $PM/MW = x/4$. Since $WM/MP \cdot PM/MW = 1$, $x^2/4 = 1$, which gives us $x = 2$.

In tackling this kind of problem, you'll find it helpful to begin by plotting the motions in question. The graphs can then prompt you how to work out an equation or simply render the problem as pure geometry, as in figure 16.

Boy-oh-buoyancy!

1. $\Delta p = 3\rho_m g\Delta h \cong 4 kPa$ (ρ_m is the density of mercury).

2. $m = \rho\pi R^2 h - M$ (ρ is the density of water).

3. $S = \Delta F/(\rho g \Delta h) = 6.25\ \text{cm}^2$.

4. The intensity with which the gas burns is determined by the difference in the pressure of the gas and air. The pressure of the gas in the pipes of a building is usually low, and its density is lower than the density of air. The decrease in air pressure on the top floor of a fourteen-story building is greater than the decrease in gas pressure. As a result, the difference be-

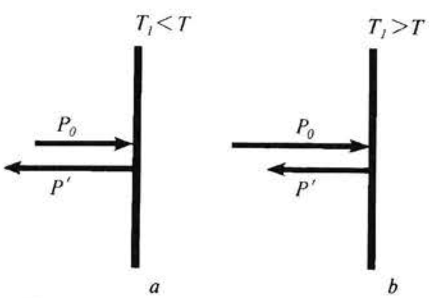


Figure 13

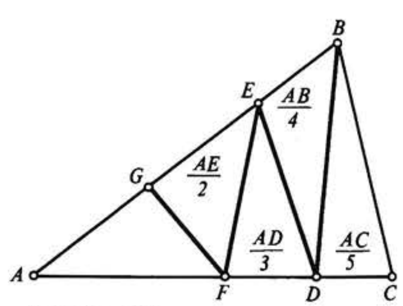


Figure 15

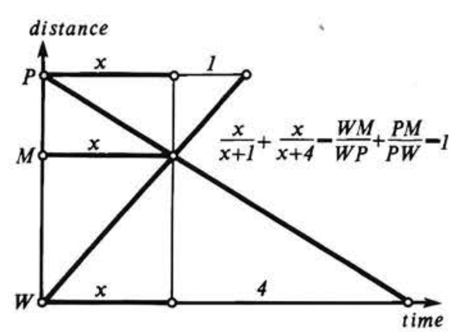


Figure 16

tween the pressures increases, and the gas burns better on the top floor.

5. The pressure the ball exerts on the cover must increase.

Kaleidoscope

1. In the notations for figure 17 let the circumcircles passing through A meet again at P. Then angle BPC = $360°$ – angle APB – angle $APC = 360° - (180° - \gamma) - (180° - \beta) = \gamma + \beta$. Thus, α + angle $BPC = \alpha + \gamma + \beta = 180°$, so the third circumcircle also passes through P. The case in which P lies outside triangle ABC is treated similarly.

2. In figure 18, angle ATC + angle $ATC_1 = 120° + 60° = 180°$; therefore, T lies on line CC_1. The case in which T lies outside the triangle requires obvious changes.

3. Let the rotation about A through $60°$ turn an arbitrary point M into M' (fig. 19). Then $AM = MM'$, $BM = C_1M'$. The segments CM, MM', and $M'C_1$ form one line if and only if $M = T$ and the angles of triangle ABC don't exceed $120°$. In this case $CM + AM + BM = CM + MM' + M'C_1 > CC_1 = l = CT + TT' + T'C_1 = CT + AT + BT$. If $A > 120°$ (fig. 20), $l = CC_1 = CT - TT' + T'C_1 = CT - AT + BT$. We leave it to you to prove that in the latter case the smallest value of $AM + BM + CM$ is attained for $M = A$.

4. The answer is the intersection of the diagonals.

5. Hint: each of the centers of the Torricelli circles passing through, say, vertex A is equidistant from the ends of segment TA.

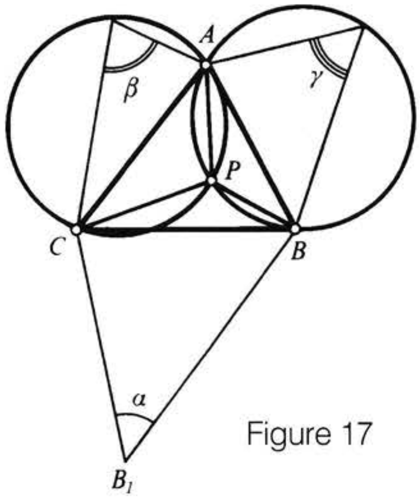


Figure 17

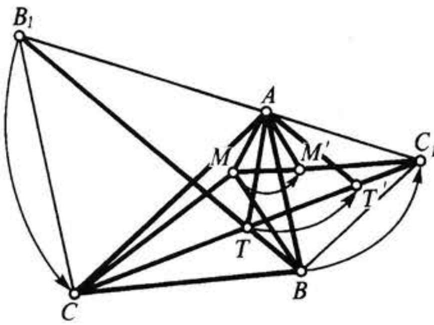


Figure 19

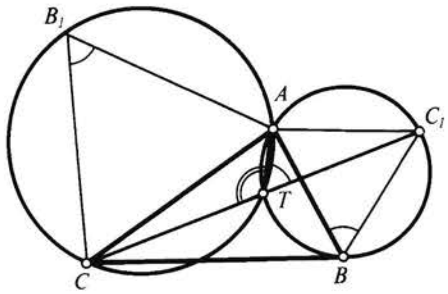


Figure 18

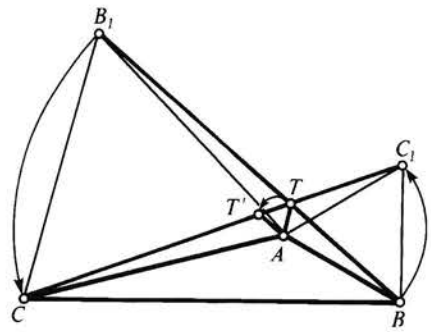


Figure 20

Index of Advertisers

Correction

In equation (1) on page 18 of the May issue, the plus sign should be a raised dot. (The equation is cited correctly on page 21.)

Fantasy chess

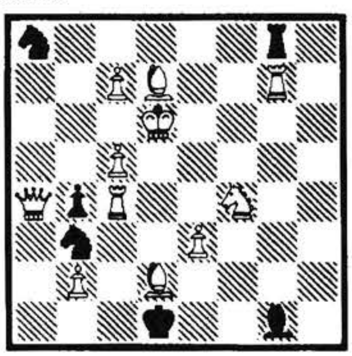
By introducing new parameters, you can add some interesting twists to the age-old game

by Yevgeny Gik

AMONG THE UNUSUAL PROBlems and brainteasers in the realm of "fantasy chess" (that is, chess with an extra rule or two), there are some rather rare but extremely witty variations. Let's take a look at three of these.

Circe

This highly original version of the fantasy genre differs from ordinary chess in the following way: after an enemy piece is captured, it isn't removed from the board—it's returned to the position it originally occupied at the start of the game. Rooks and knights return to the square of the same color as the one on which they were taken, and pawns go to the starting position in the row in which they were taken. If, however, the point of relocation is already occupied, the captured piece must leave the board, as usual.



N. Macleod, 1978
Mate in 2 moves (Circe)

It seems that white will achieve a checkmate in *one* move, and in two different ways: 1. Qa4xb3+ or 1. Bd7-g4+. But it's not so simple . . .

1. Qa4xb3 (the knight leaves the board because g8 is occupied by the black rook)—mate? The defense 1. ... Kd1xd2 is impossible because the captured bishop returns to c1 and the black king would thus be put in check with white yet to move. But black has another, more clever defense up its sleeve: 1. ... Na8xc7!, and the white pawn that appears on c2 blocks the diagonal a4–d1. It seems white will nevertheless achieve its goal: 2. c2-c3 mate, but after 2. ... b4xc3! a white pawn again appears on c2.

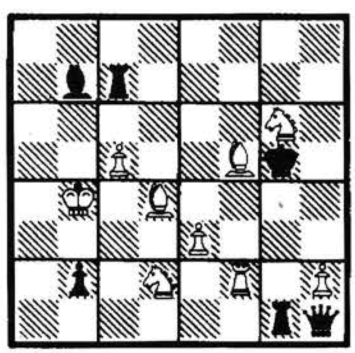
1. Bd7-g4 won't work either. That move can be repulsed by 1. ... Bg1xe3! The white pawn is restored to e2 and there's no mate after one move; there's none after two moves either because the e2 pawn can't move anywhere.

The quiet move 1. c7-c8R! will lead to success. This pawn disappears from the board and 2. Qa4xb3+ becomes a threat. Black answers with 1. ... Rg8xc8, whereby the newly made white rook appears on h1. The capture 2. Qa4xb3 is parried by 2. ... Rc8xc5!, and again a white pawn shows up on c2. But now we can get a checkmate by 2. Bd7-g4+, bringing the h1 rook unexpectedly into play: the black bishop is pinned and the response Bg1xe3 can't be made. Can't black be saved by 1. ... Bg1xe3 2. Qa4xb3 Be3xc5!? In this case 2. Rg7-g1+ will prove decisive: black can't take the g1 rook with the bishop or the rook because the white rook will appear on a1, putting the black king in check again with white yet to move. And opening with 1. Rg7xg1+ won't work because the bishop returns to f8 and the white king is already put in check.

Finally, 1. c7-c8Q(N) goes nowhere because of 1. ... Rg8xc8, and 1. c7-c8B? is a dead end because of 1. ... Bg1xe3!

Trellis board

In this interesting version of fantasy chess the board is broken into sixteen squares of four spaces each. The rules are simple: a piece is powerless in the square it sits in (it can't move or attack enemy pieces); it can become active only by moving into another square.



E. Wisserman, 1955
Mate in 2 moves on a trellis board

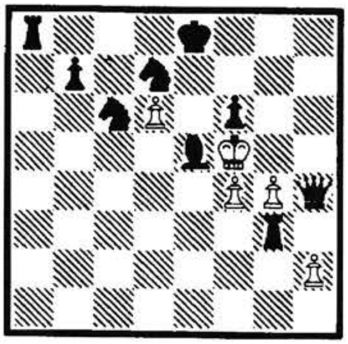
In this position the black king can't move to h5 or h6 (these spaces are in the same square as the king), while the pawn on b2 can transform itself into another piece only by capturing a piece on c1 (in the adjoining square)!

1. Bf5-d7! This threatens 2. Bd4-f6+, against which there's no defense. If black responds with 1. ... Bb7-f3, 2. Nd2-E4+! is decisive—the bishop on f3 finds itself in the same square as the knight and isn't allowed to capture it. In response to 1. ... Qh1-e4, which pins the bishop on d4, white's next move is 2. Nd2-f3+ (the queen protects the knight from the bishop on b7—an impracticable idea in an ordinary two-move problem).

Other variants: 1. ... Rc7-c6 2. Rf2-g2+!; 1. ... Qh1-f3(c6) 2. h2-h4+; 1. ... Rg1-g4 2. Rf2-f5+. In normal play the knight on g6 would be defenseless, but on a trellis board it's untouchable.

Frankfurt chess

In this kind of fantasy chess the capturing piece turns into the captured piece (without changing color).



N. Bakke, 1986
Cooperative mate in 2 moves
(Frankfurt chess)

As you know, in a cooperative problem black moves first and helps white achieve a checkmate. 1. 0-0-0! f4xe5 (now e5 is occupied by a white bishop). 2. Nc6-e7+ d6xe7+! Mate is effectively achieved by the new knight on e7.

Curiously enough, this problem has a very attractive twin. If the rook is taken off a8 and put on h8, castling again leads to a solution: 1. 0-0! h2xg3 (the pawn turns into a rook) 2. Qh4-h5+! g4xh5+ (the pawn turns into a queen), and the black king is checkmated. It's amusing that the white pawns in this diagram turn into all sorts of pieces and at a rather great distance from the last rank, where these kinds of things usually happen.