QUANTUM

MARCH/APRIL 1991

$5.00

The student magazine of math and science

GALLERY Q

National Gallery of Art, Washington (Collection of Mr. and Mrs. Paul Mellon) © NGA

The Bicycle Race (1912) by Lyonel Feininger

LYONEL FEININGER (1871–1956) presents us with another case of a musician diverted into another field. The great physicist Max Planck seriously considered music as a career, as we learned in Quantum Smiles (Sept./Oct. 1990). Feininger was born in New York City and went to Germany in 1887 to study music, but ended up taking up painting instead. He studied in several European cities and worked as a cartoonist for German humor magazines and the *Chicago Tribune*. He came under the influence of the Cubists, as did many artists of the time, and was particularly affected by the work of Robert Delaunay (see Gallery Q in the same issue). After the first world war Feininger joined the Bauhaus, Germany's innovative school of design, whose aim was a synthesis of art, science, and technology. He returned to the United States in 1936 when the Nazis came to power.

Feininger's style is generally not as distorted as that of other Cubists. "The Bicycle Race" is characteristic of his work, with its faceted objects and vibrant colors. In this painting he also effectively transmits a sense of motion and freshness. It seems to propel us into spring—or gives us the energy to do the pedaling ourselves! After all, what's the point of all our mathematical training if not "problem racing"? See Gary Sherman's article on page 50.

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Cover art by Sergey Ivanov

The fellow doing a beautiful swan dive on the cover may be surprised to find out he's diving into a pool of paint! He dove off a perfectly ordinary infinite planar surface, but what a mess he's headed for. It turns out that the amount of paint in the pool is enough to cover the infinite surface that served as his diving platform. What's going on here? It's the "painter's paradox"—see if you can make heads or tails of it. The article begins on page 10.

FEATURES

DEPARTMENTS

Thanks for your support!

You helped us through our first year

WITH THIS ISSUE OF *Quantum* we come to the end of our first publishing year. We'll use the time between now and the September/October issue to work with our new copublisher, Springer-Verlag, to make a smooth transition to a new production arrangement. We'll also work with our Soviet colleagues at Quantum Bureau to plan upcoming issues and try to get the materials ready for the printer (and mailed to you) earlier. Speaking of the mail, we'll be applying for second-class mailing privileges, which should result in faster and more consistent delivery.

About your subscription . . .

Many aspects of *Quantum* evolved rather rapidly during this first year. For one thing, we started out the year as an academic-year quarterly and ended up a bimonthly, year-round magazine (with this catch: year-round publication will begin with the September/October 1991 issue). This may have created some confusion for our subscribers. I'd like to clarify the situation.

For those of you who subscribed early on, after receiving one of the pilot issues, your subscription runs out with this issue. We will be sending you a renewal notice, although you may certainly send in your renewal without waiting for the notice.

Our introductory rate of $9.95 (for four issues) expired on October 31, 1990. But because the pilot issues continued to circulate, we continued to receive subscription orders at $9.95, rather than the new prices (for six issues). *If you paid $9.95, you will receive four issues, regardless of when you subscribed. If you paid full price ($14 for students, $18 for nonstudents, and so on), you will receive six issues.* So, for instance, if you subscribed at full price in December and the January/February issue was the first you received, your subscription will expire with the *March/April 1992 issue*.

In short: *Quantum* will be published six times a year, year-round, beginning with Volume 2, Number 1 (September/October 1991). There will be no issues published during the summer of '91.

Looking back

It's been an exciting and instructive year for us, and for you as well, we hope. We're gratified by the positive response to *Quantum* by students and teachers, and we take seriously the comments, suggestions, and criticisms we've received. We realize we're trying to walk a fine line between challenge and banality, encouragement and discouragement, excitement and boredom. We're still feeling our way toward the proper balance. If you're a student who's used to understanding everything in your school work without too much effort, we hope you've been interested by things in *Quantum* that your textbooks may not have prepared you for, and even by things that may be just out of your reach. If you're a student who's attracted to math and physics but who has to work hard at it, we hope you've found topics that gave you unexpected insights and perhaps some new skills.

We'd like to thank our advisory board for their help during the past year. We haven't been able to do all that they've advised us to do, but they've made *Quantum* a better magazine, and their ideas may still bear fruit in the months to come. Whether reviewing potential articles or contributing pieces themselves, our advisory board members have helped increase *Quantum*'s American content and make it a true collaboration with our sister magazine *Kvant*.

We're also grateful to the National Science Foundation for their continued financial support. Without the foundation's seed money, *Quantum* would never have seen the light of day.

Looking ahead

To help make *Quantum* more responsive to the needs of our primary audience of American students, we plan to add associate editors whose task will be to solicit appropriate articles from their colleagues in the academic community. They will also help us adjust the translated Soviet articles to the math and physics background of students in US high schools, providing footnotes and explanatory material as needed. We hope to make the excitement of *Quantum* accessible to a broader range of students without watering it down.

There's still plenty of room for improvement, and you can help us by dropping a line with your comments. Let us know what you like, what you don't like, and what you'd like to see in *Quantum*.

See you in September!

—Bill G. Aldridge

QUANTUM

THE STUDENT MAGAZINE OF MATH AND SCIENCE

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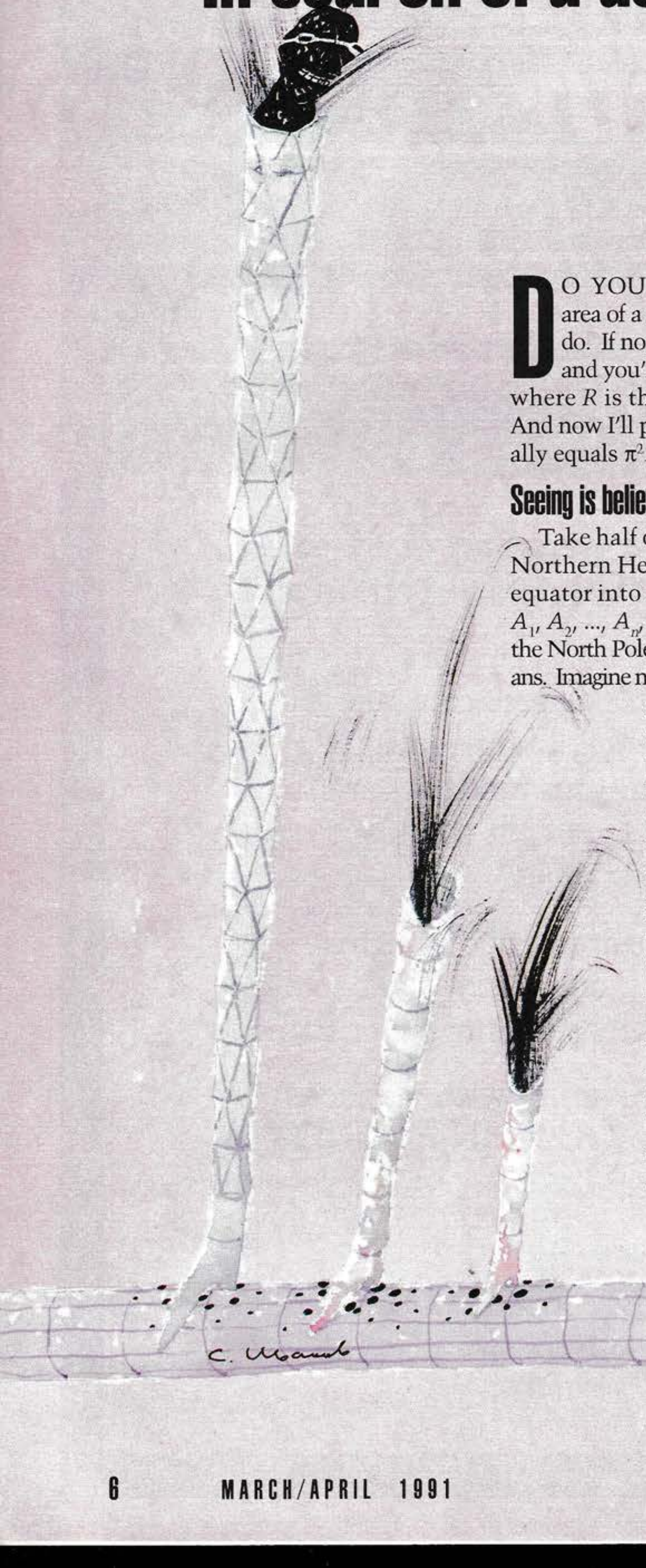
In search of a definition of surface area



Now you see it, now you don't

by Vladimir Dubrovsky

DO YOU KNOW WHAT THE area of a sphere is? You probably do. If not, just look in a textbook and you'll find the formula $4\pi R^2$, where R is the radius of the sphere. And now I'll prove that this area actually equals $\pi^2 R^2$. Pay attention, now!

Seeing is believing

Take half of a sphere (let it be the Northern Hemisphere). Divide the equator into n equal parts by points $A_1, A_2, \ldots, A_n$, and join these points to the North Pole N by the arcs of meridians. Imagine now that polygon $A_1A_2\ldots A_n$ starts to rise over the equatorial plane, staying parallel to it and contracting on its way so that its vertices slide along the meridians. Then its sides will cover the surface much like a closed bud (fig. 1a). If the bud opens, we get n triangles (fig. 1b). Let a_n be the base of any of the triangles ($a_n = A_1A_2 = A_2A_3 = \ldots$), and let h_n denote the height of the triangles. Then the total area of all the triangles equals $na_nh_n/2$. It's clear that as n increases, the area of the bud tends to the area of the hemisphere, while the polygon's perimeter na_n tends to the equator's length $2\pi R$ and the height tends to $\pi R/2$ (one quarter of a meridian's length). Thus, the area of the hemisphere, which is the limit of the bud area, equals $2\pi R \cdot (\pi R/2)/2 = \pi^2 R^2/2$, and so the surface area of the sphere equals $\pi^2 R^2$.

Something's wrong here. I mean, we can't disbelieve geometry books from all over the world! We'll approach this paradox from far off and begin with the deepest root of the problem: what is the area of a surface?

The first attempt: developing

Sir, I admit your general rule,
That every poet is a fool:
But you yourself may serve to show it,
That every fool is not a poet.
—*Matthew Prior*

The simplest task is to find the surface areas of a cylinder and a cone: it's possible to cut them along a linear ruling and unroll them—or, as mathematicians prefer to say, "develop" them (fig. 2). We get a rectangle and a circular sector, respectively. Their areas can easily be computed by means

Art by Sergey Ivanov

of the well-known formulas from plane geometry. By the way, the area of the petals of our bud have also been computed by developing. It might be the simplest way of determining the surface area, but, unfortunately, few surfaces are developable.[1] For example, anyone who ever tried to wrap an apple, an orange, or, say, a watermelon in a sheet of paper knows that it's impossible to get rid of folds or creases. So it's common knowledge that in practice it's impossible to develop even such a simple surface as a sphere. But how can we prove it?

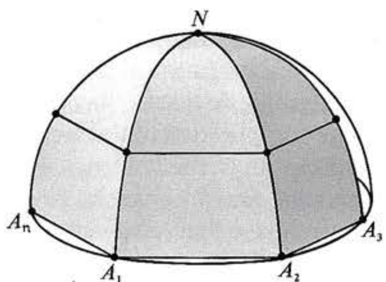


Figure 1a

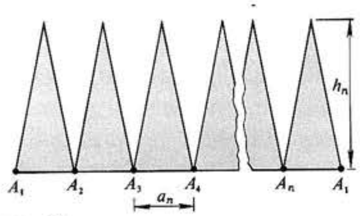


Figure 1b

To start with, it's necessary to understand what it means to develop. What's the difference between a sheet of rubber, which can wrap anything without folds being created, and a sheet of paper? Well, rubber can stretch and shrink, while paper can't. In other words, the length of a curve drawn on paper is constant for all bendings. Preserving the lengths of all curves is the main property of the process of developing.

Now imagine that we've managed to cut a sphere into segments that can be flattened on the plane so that this property holds. Mark a point A inside one of the segments and draw a circle C with center A that lies inside this

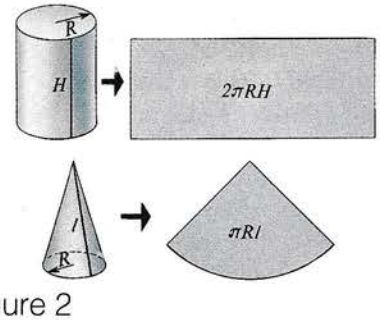


Figure 2

segment. It follows by symmetry that this circle is a locus of points for which the distance $d(X, A)$ to point A measured along the sphere is constant. (The distance along the sphere is the minimal length of a curve on the sphere joining the two points. It can be proved that such a curve is an arc of a "great circle"—that is, the intersection of the sphere and a plane through its center, but we won't need to do it.) Our spherical circle can also, of course, be considered a planar one. Then its center is the point O, the projection of any point X of the circle onto the diameter of the sphere passing through A (fig. 3), and its "planar" radius r equals OX. So the length of the circle equals $2\pi r$. After being unrolled the circle C transforms into the locus of points whose (planar) distance from A is R—that is, into an ordinary circle of radius R. The length of this circle equals $2\pi R$. But $r = OX < AX < d(A, X) = R$, since the length of the line segment AX is smaller than the length of any curve joining A and X. Thus, the length of the circle increases after developing, which contradicts the main property of developing.

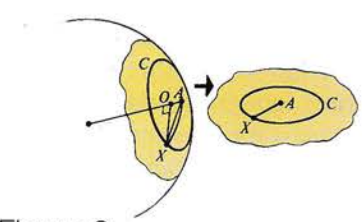


Figure 3

Exercise 1. Prove that despite the fact that the sphere is undevelopable, it's possible to construct (a) a mapping of a hemisphere onto a plane that transforms the shortest paths on the sphere into the shortest paths on the plane; (b) an area-preserving mapping of a sphere onto a plane. (Hint: see exercise 6.)

To refute a general rule it's sufficient to produce a single counterexample (see the epigraph). So we have proved that developing is of no use in our search for a general definition of the area of a surface.

The second attempt: approximation by polyhedra

Reread the Prior quotation!

Nevertheless, we have no problems with some types of surfaces. It's especially easy to find the area of a polyhedral surface—that is, a surface consisting of planar polygons.

So it's natural to try to approximate the area of a given surface S by the area of a polyhedral surface P close to it. The closer P is to S, the better the approximation, and in the limit we would get the precise value of the area of S. The curve length is defined in a similar way by broken-line approximations. But we can't afford to be careless, because taking arbitrary broken lines close to a given curve may result in an unpleasant surprise, as the exercise below demonstrates.

Exercise 2. Figure 4 shows a sequence of broken lines of length $2^{1/2}$ each converging to the line segment of length 1. The distance between points of the nth broken line and the line segment doesn't exceed $1/2^n$. Give examples of broken line sequences converging to the unit segment, whose lengths (a) tend to an arbitrary given number $l > 1$; (b) are unbounded; (c) are bounded but don't converge anywhere.

Everything is okay, though, with the length of the curve if we require that the approximating broken lines be inscribed in it—that is, that their vertices lie on the curve. Let's impose a similar condition on polyhedra approximating a curved surface and try to find the area of the cylinder.

Divide the height H of the cylinder into k equal parts, draw circles through

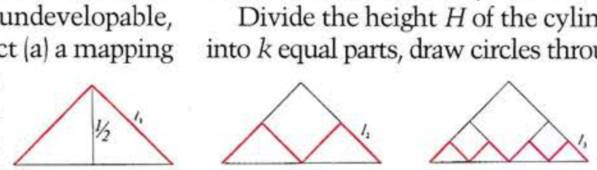


Figure 4

[1]Unrollable surfaces are examined in the article "Bend This Sheet" by Dmitry Fuchs in the the very first issue of *Quantum*, where they are called "developable."

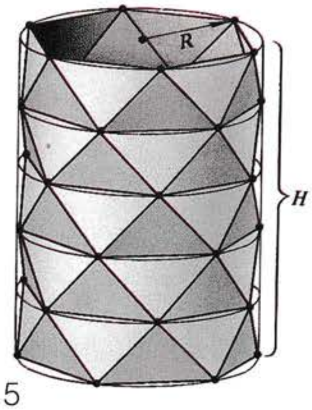


Figure 5

the dividing points (fig. 5), and inscribe regular n-gons in the circles so that each of them is rotated by the angle π/n with respect to the neighboring ones. Join the vertices of these n-gons by their edges as shown in the figure. We get a polyhedron $P(n, k)$ inscribed in a cylinder and consisting of $2nk$ congruent triangles. If this construction is viewed from above (fig. 6), we would see that the margin d_k between our cylinder and the cylinder touching horizontal edges of the polyhedron tends to 0 as n increases. Thus, polyhedra $P(n, k)$ "tends" to the cylinder as $n \longrightarrow \infty$ irrespective of the number of layers k. But what happens to their areas $A(n, k)$?

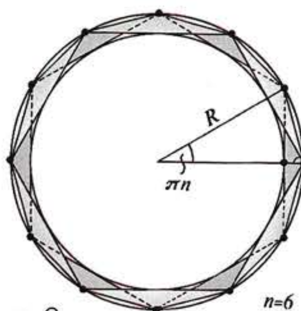


Figure 6

Let a_n be the base and $h_{n,k}$ the altitude of an arbitrary triangle forming $P(n, k)$ (the base length doesn't depend on k). Then

$$A(n,k) = 2nk \cdot a_n \cdot \frac{h_{n,k}}{2} = p_n \cdot kh_{n,k},$$

where p_n is the base perimeter of our polyhedron. Clearly, as n increases, the perimeters p_n tend to the length $2\pi R$ of the base circle of the cylinder (where R is its radius). But the behavior of the second term $kh_{n,k}$ depends on k. If we fix $k = 1$, then the value $kh_{n,k} = h_{n,1}$ evidently tends to H as $n \longrightarrow \infty$, and, consequently, $A(n, 1) \longrightarrow 2\pi RH$. This is the correct value of the cylinder's side area. The same happens for any fixed k. But if k increases with n (the number of faces in a layer), then the limit of the areas $A(n, k)$ can be different. You'll know the details from exercise 3, so now let me give you only one example. Choose a sequence of numbers k_n such that $k_n d_n$ increases indefinitely—for instance, in such a way that $k_n > n/d_n$ (recall that d_n is the margin between the inscribed and the circumscribed cylinders for the polyhedron $P(n, k)$; it follows from figure 6 that $d_n = R(1 - \cos \pi/n)$). Then

$$A(n,k_n) = p_n \cdot k_n d_n \cdot \frac{h_{n,k_n}}{d_n} > p_n \cdot k_n d_n$$

(evidently $h_{n,k} > d_n$ for any n and k). So the area $A(n, k_n)$ increases indefinitely despite the fact that the polyhedra $P(n, k_n)$ converge to the surface of the cylinder! The reason for this amazing phenomenon is that when n increases, the faces of $P(n, k_n)$ make increasing angles with the surface of the cylinder, and the area of the polyhedra increases because of the multiple folds.

This construction was invented in 1890 by the German mathematician H. A. Schwarz (1843–1921). In mathematical folklore it's called Schwarz's boot.[2] It shows that our new definition of surface area doesn't work. More precisely, the definition needs refining: the notion of the "closeness" of a polyhedron to a surface should take into account not only the distance between their points but also the angles between the faces of the polyhedron and the surface. But this makes the definition too complicated. In any case, to use it for calculating the area of a sphere is unreasonable, to say the least.

Exercise 3. Show that the area of Schwarz's boot is given by the formula

$$A(n,k) = p_n H \sqrt{1 + \left(\frac{4R^2}{H^2}\right) k^2 \sin^4\left(\frac{\pi}{2n}\right)}.$$

If you're familiar with the equality

$$\lim_{x \to 0} \sin x/x = 1,$$

try to choose for any given number $A \geq 2\pi RH$ a sequence k_n such that the areas $A(n, k_n)$ tend to this number.

The Minkowski definition

The road length equals its area divided by its width.

—From a lecture for high school students

About a hundred years ago Hermann Minkowski (1864–1909), an outstanding German mathematician and physicist, suggested a new approach to the definition of the area of a surface. He devised a way of reducing the computation of the area to the computation of the volume.

Imagine that you have to paint the roof of a house that has a very complicated shape. How much paint will you need? The answer is evident: the paint volume V approximately equals Ah, where A equals the roof area and h is the thickness of the paint layer. Thus, the roof's area approximately equals V/h. And the thinner the paint layer, the more precise this equality. Since not every surface has two sides,[3] it's desirable to "paint" it all and to divide the volume of the "paint" used by twice the layer's thickness.

The mathematical equivalent of a layer of constant thickness h is the h-neighborhood S_h of a surface S. It's the set of points in space no more than h units away from the surface. In other words, the h-neighborhood of a figure consists of points X such that any sphere of radius greater than h with its center at X intersects this figure. A planar neighborhood is defined similarly.

Exercise 4. Find the planar and spatial h-neighborhoods of a line segment. Prove that the area of the former and the volume of the latter equal $2hd + \pi h^2$ and $\pi h^2 d + (4/3)\pi h^3$, respectively, where d is the length of the segment.

[2]In the *Quantum* Toy Store at the end of this issue, we show how to make a paper model of Schwarz's boot.

[3]One-sided surfaces are discussed in the article "Flexible in the Face of Adversity" by A. Vesyolov in the Sept./Oct. 1990 issue of *Quantum*.

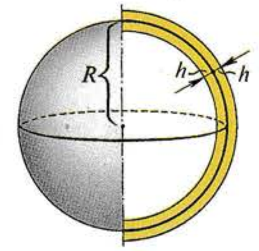


Figure 7

Exercise 5. (a) Find the spatial h-neighborhood of a regular hexagon. Denote its volume by $V(h)$ and prove that $V(h)/2h$ tends to the area of the hexagon as $h \longrightarrow 0$.

(b) Prove that the volume of the h-neighborhood of a convex polygon equals $2hA + \pi h^2 p + (4/3)\pi h^3$, where A is its area and p is half of its perimeter.

Now we can get rigorous.

Definition. *Let $V(S_h)$ be the volume of the h-neighborhood of a surface S. The area of this surface is the limit of the ratio $V(S_h)/2h$ as $h \longrightarrow 0$.*

Of course, a direct computation of surface area with this definition is possible only in the simplest cases. And it requires at least some familiarity with calculus—namely, the ability to calculate the simplest limits (you'll see that these calculations are almost self-evident). For more complex cases, there are special integral formulas that can be derived from the Minkowski definition or other definitions. Here are some examples.

Sphere (fig. 7). The points whose distance from a sphere S of radius R does not exceed h ($h < R$) fill the space between two spheres having radii $R - h$ and $R + h$ and the same center. So the volume of the h-neighborhood S_h equals the difference between the volumes of balls with radii $R + h$ and $R - h$. (The volume of a ball of radius r equals $4\pi r^3/3$.) So

$$\frac{V(S_h)}{2h} = \frac{4\pi}{3}\frac{(R+h)^3 - (R-h)^3}{2h}$$
$$= \frac{4\pi}{3}\left(3R^2 + h^2\right)$$
$$= 4\pi R^2 + \frac{4\pi h^2}{3}.$$

As h tends to zero, the second term drops out and we get the answer $4\pi R^2$. ("And what about the formula $\pi^2 R^2$, which you tried to pass off as the right one?" a credulous reader is now asking. Just hold on, I'll come back to it before the article ends.)

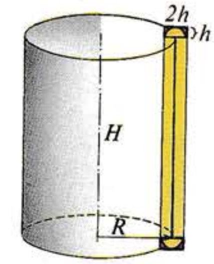


Figure 8a

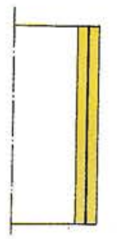


Figure 8b

Cylinder. On the right side of figure 8a you see an axial section of the h-neighborhood of the side surface of a cylinder C, which has a radius R and height H (more precisely, you see half of the section). It's a rectangle with curved angles. The curvatures correspond to points whose distance from the cylinder edges is less than h, and they complicate the exact computation of the volume of the h-neighborhood. But fortunately, as we'll soon see, the volume of these curvatures is so small that it doesn't affect the answer, and so we can just cut them off (fig. 8b). Now we have to find the volume of the body obtained by rotating the rectangle of size $2h \cdot H$ about the axis whose distance from its longer midline is R. This equals the difference between the volumes of two cylinders of height H and with radii of $R + h$ and $R - h$, respectively:

$$\pi H[(R+h)^2 - (R-h)^2] = 2\pi RH \cdot 2h. \quad (1)$$

Dividing by $2h$ we see that the area of the side surface of a cylinder equals $2\pi Rh$.

Now about the curvatures. To evaluate their volume let's replace them with rings whose rectangular section equals $2h \cdot h$ (fig. 8a) and compute the volume of the rings using formula (1). It turns out that the total volume of the curvatures divided by $2h$ does not exceed $4\pi Rh$ and tends to zero as $h \longrightarrow 0$.

Exercise 6. The volume of a ball sector (that is, the body cut out of a ball by a cone whose vertex is at the ball's center) equals $2\pi R^3(1 - \cos\alpha)/3$, where R is the ball's radius and α is the angle between the axis and the ruling of the cone. Using this formula, show that the area of the part of a sphere sandwiched between two parallel planes intersecting this sphere depends only on the distance H between these planes and equals $2\pi RH$.

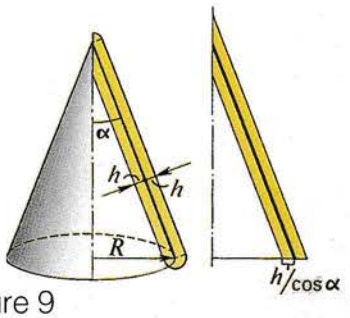


Figure 9

Exercise 7. Using figure 9, show that the area of the side surface of a cone with base radius R and slant height l equals πRl.

Now for one more example.

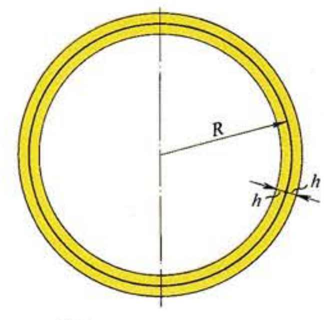


Figure 10

Circle. "But a circle isn't a surface," you say. Of course not, but Minkowski's idea can also be used to compute lengths. To find the length of a planar curve L, we should replace the volume of the h-neighborhood of a surface in this definition with the area of the planar h-neighborhood of the curve. In particular, the h-neighborhood of a circle with radius R (for $h < R$) is a ring between two concentric circles with radii $R - h$ and $R + h$, respectively (fig. 10). Its area equals $\pi[(R + h)^2 - (R - h)^2] = 2\pi R \cdot 2h$. Dividing by $2h$ we get the well-known formula for the length of the circle $2\pi R$. Another approach is to leave the numerator in the Minkowski definition as it is and replace the denominator with πh^2: the spatial h-neighborhood of a curve is a thin curved pipe whose volume approximately equals the product of

CONTINUED ON PAGE 44

The painter's paradox

Is this why some artists starve?

by A. A. Panov

ANYONE WHO HAS EVER HELD a brush knows that the greater the surface area, the more paint is used up. In other words, the amount of paint needed is proportional to the painted area. (One of the definitions of surface area is based on this observation—see Vladimir Dubrovsky's article in this issue.)


Figure 1

1. Let's consider, by way of example, a flat plate composed of an infinite number of rectangles, as shown in figure 1. Here the first rectangle is a square with sides of length 1 cm. Each subsequent rectangle is twice as long as the previous one, but its width is one half that of the previous one. Clearly the area of each rectangle equals 1 cm^2. So the area of the whole figure S is infinite:

$$S = (1 + 1 + 1 \ldots)\ \text{cm}^2,$$

and requires an infinite amount of paint.

2. Now let's think about it another way. Rotate the plate about the line ray bounding it. The resulting surface of revolution consists of an infinite number of cylinders (fig. 2). Its internal volume equals the sum of the volumes of all the cylinders:

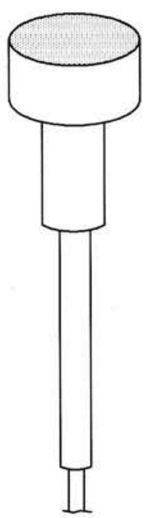


Figure 2

$$V = V_1 + V_2 + V_3 + \ldots .$$

We know that the volume of a cylinder with a base radius r and height h equals $\pi r^2 h$. For the nth cylinder (counting from the top in figure 2) we have

$$r = \frac{1}{2^{n-1}}\text{cm},$$

$$h = 2^{n-1}\ \text{cm}^3 .$$

Thus

$$V_n = \frac{\pi}{2^{n-1}}\text{cm}^3$$

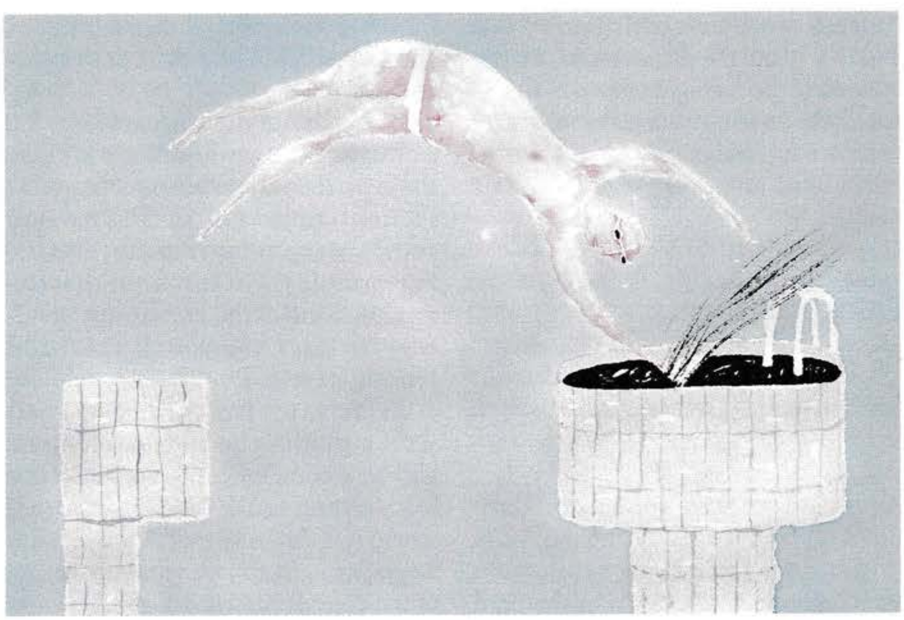
and, consequently,

$$V = \pi\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right)\text{cm}^3 .$$

In the parentheses we have the well-known sum of a geometric sequence. This sum equals 2, and so $V = 2\pi$ cm^3.

Imagine that our body of revolution is hollow. Fill it to the brim with 2π cm^3 of paint, and then immerse our plate in it. After you take it out it will surely be painted—on both sides, even!

So we have two examples of impeccable reasoning that lead to contradictory conclusions. Following the first line of thought, we conclude that we need an infinite amount of paint, while the second one shows that a mere 2π cm^3 of paint is sufficient. This is the painter's paradox. I think you'll enjoy pondering it!



Short takes

Sayings

"The law isn't written for fools," says a Russian proverb.[1] So is it worthwhile to improve your intellect?

Every action has an equal and opposite reaction. But what about inaction? (O. Donskoy)

Art by Pavel Chernusky

[1]Compare the English saying: "Fools rush in where angels fear to tread."

From the scientific folklore

Two people are traveling in a balloon over unknown territory.

"Hey!" they call out to a passerby, "where are we?"

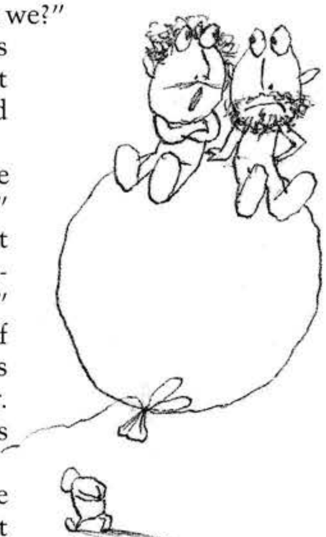
He looks carefully at them and yells back:

"You're in a balloon!"

"He must be a mathematician," says one of the travelers to the other.

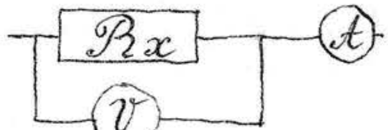
"Why is that?"

"First, he thought awhile before answering. Second, his answer is absolutely precise. And third, it's utterly useless."

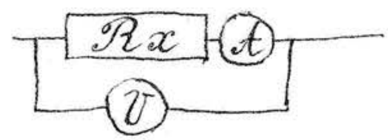


Clever!

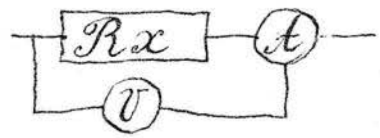
If the resistance measured in the circuit



is less than expected, and in the circuit

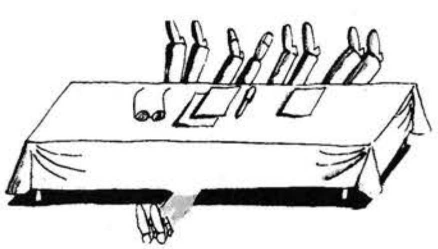


it's more than expected, then measure it in the circuit



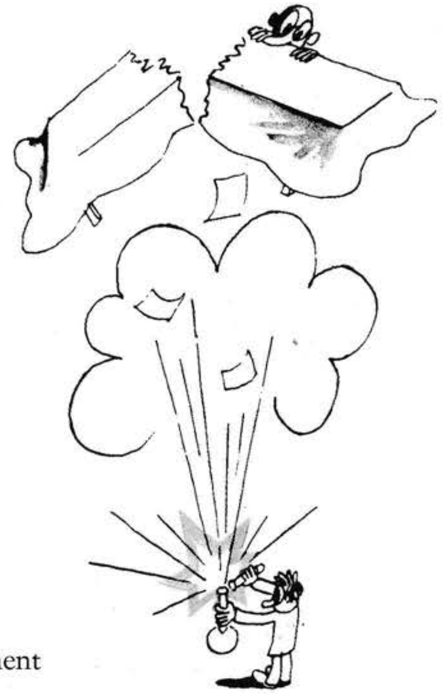
and it'll be right on. (G. Zadov)

Sshhhhh! Quiet! Examinations are in progress. . .



. . . at the Auto Repair Academy

. . . at the Mining Institute



. . . in the Chemistry Department

Art by A. I. Semyonov

Atmospherics

A look at the Earth's airy shell

by A. V. Byalko

LOOK UP AT THE HEAVENLY azure—this is the Earth's atmosphere. Breathe in the air—this, too, is the Earth's atmosphere. But what is atmosphere from the point of view of physics? What accounts for its composition, pressure, and temperature at different altitudes? Let's try to answer these questions briefly.

The composition of the atmosphere

You probably know that the Earth's atmosphere consists of nitrogen (78%), oxygen (21%), and argon (1%). There are also small admixtures of carbon dioxide and water vapor as well as negligible quantities of neon, helium, krypton, and hydrogen. Now let's try to understand why our planet's diaphanous shell consists of these particular gases and water.

The composition of the atmosphere is determined, first of all, by the geological history of the planet. The Earth, along with the Moon and the other planets of the solar system, is thought to have been formed by the collision and merging of small solid celestial bodies. The primary substance of the planet was compressed by the force of gravity—under its influence the Earth took the form of a sphere (flattened slightly at the poles because of rotation). Compression caused the center of the Earth to heat up. Under the action of high temperatures and pressure, chemical reactions took place in the primary substance. Heavier reaction products descended to form the Earth's core; lighter ones formed the Earth's crust; and gases were separated from the solid part to form the atmosphere. Then the abundant water vapor in the Earth's atmosphere condensed to form the ocean.

Don't think that all this happened once and for all in the Earth's remote past. This process is going on even now, although not so intensively as in the beginning of evolution. At present the Earth's crust is still being renewed, and volcanoes are expelling considerable quantities of water vapor, carbon dioxide, and nitrogen. Sulfur dioxide, hydrogen chloride, and other unpleasant gases are also being released into the atmosphere. But why are they absent in the normal atmosphere?

The answer is pretty obvious. All the gases in the atmosphere must be in chemical equilibrium with each other, with the ocean, and with the material of terrestrial rocks. So acid oxides thrown out by volcanoes dissolve rapidly in the ocean to form acids. Interacting with the basic oxides of the Earth's crust, these acids form salts. Some of the salts are dissolved in the ocean, while the insoluble salts form sedimentary rocks.

The attentive reader has probably noticed a weak point in this theory. Oxygen! There's no oxygen in volcanic gases, and there's no oxygen in the atmospheres of other planets.

The main source of the Earth's oxygen is vegetation. The chlorophyll of plants, under the action of the Sun's rays, process carbon dioxide. The carbon is absorbed in organic compounds and the oxygen is released into the atmosphere. There is, however, another source of oxygen on our planet. To understand how it works, we must first answer the following questions: what restrains the atmospheres of planets? Why don't the atmospheric gases fly off into outer space?

The atmospheric pressure at the Earth's surface is known to equal $p_0 = 1.013 \cdot 10^5$ Pa. This means that the force with which the atmosphere acts on each square meter of the Earth's surface equals $1.013 \cdot 10^5$ N. This force comes from the Earth's gravity and equals the weight of the air column over the Earth's surface with a cross section of 1 m^2. Since the height of the atmosphere is small in comparison with the Earth's radius, we can consider that within the atmosphere the free-fall acceleration is constant and equal to $g \cong 9.8$ m/s^2. The weight of an atmospheric column with a cross section of 1m^2 is equal to $p_0 = m_1 g$, where m_1 (kg/m^2) is the mass of the air. So over the entire Earth's surface, whose area is $4\pi R_E^2$, the mass of the air is equal to $m_1 \cdot 4\pi R_E^2$—that is,

$$m_A = \frac{p_0}{g} 4\pi R_E^2 \cong 5.3 \cdot 10^{18}\,\text{kg}\,.$$

As you can see, the mass of the atmosphere constitutes approximately one millionth of the whole mass of the Earth $m_E \cong 6 \cdot 10^{24}$ kg. It's also interesting to compare the mass of the atmosphere with the mass of the water on our planet: the mass of the water on the Earth is $1.4 \cdot 10^{21}$ kg. So the atmosphere's mass is 1/266th that of water.

The force of the Earth's gravitation produces not only atmospheric pressure near the surface, it also prevents atmospheric gases from dissipating into outer space. You know that gas molecules are in random thermal motion. The root-mean-square velocity of molecules at a temperature T is equal to

$$v = \sqrt{\frac{3RT}{\mu}}\,,$$

where R is the universal gas constant and μ is the gas's mass in moles. Let's

Art by Leonid Tishkov

compare the velocities of various gas molecules at a temperature of, say, 300 K with an escape velocity

$$v_e = \sqrt{2gR_E} = 11.2\,\text{km/s}\,.$$

For hydrogen, $v_H \cong 1.3$ km/s; for helium, $v_{He} \cong 1$ km/s; for oxygen and nitrogen, the mean molecular velocity is about 0.5 km/s. At first glance everything is fine. The velocities of the gases are less than the escape velocity. This means that the Earth can keep any of these gases in its atmosphere. Nevertheless, atmospheric gases do volatilize very slowly into outer space. This is because the temperature in the upper layers of the atmosphere is much higher than the temperature near the Earth's surface (as we shall see), reaching 1,000 K; so the velocities of the molecules are about two times those near the surface. Not only that, we evaluated the mean velocities of molecules. In thermal equilibrium the overwhelming majority of molecules move with velocities close to the mean velocity. But there is always a small portion of the molecules whose velocities exceed the mean velocity and reach the value of the escape velocity. These fast molecules can escape the Earth for good.

At a given temperature, hydrogen and helium molecules have the greatest mean velocities. It's easier for them than for other gases to escape into outer space, and it's easier for them to rise to great altitudes. One would think that the quantity of these gases in the atmosphere must continually decrease. But it doesn't. Why not?

It turns out there are processes that maintain the amount of the light gases in the atmosphere. Helium is formed in the Earth's crust by the decay of heavy radioactive elements, and hydrogen in the upper atmosphere is formed from water. At altitudes above 50 km, molecules of H_2O break down into hydrogen and oxygen under the action of the Sun's ultraviolet radiation. So hydrogen loss into outer space results in a decrease in water on the Earth and an increase in the atmosphere's oxygen content.

Every second about 1 kg of hydrogen escapes from the atmosphere. Is that a lot or a little? Let's figure out whether the water in the oceans and glaciers of the planet will be enough to last a while. (You'll recall that the mass of the Earth's water is $1.4 \cdot 10^{21}$ kg.) Nine kilograms of water contain 1 kg of hydrogen. Consequently, the Earth's water will suffice for $1.5 \cdot 10^{20}$ s, which is 5,000 billion years. The ocean can be considered inexhaustible, since the Earth's age is "only" 4.5 billion years.

So the Earth's water is one more source of atmospheric oxygen. Now let's estimate how much oxygen has been generated over the time the Earth has existed. This will be a very rough estimate: in reality, we can't assume that the Sun is shining the same way now as it has all that time. But let's try. Eight kilograms per second for 4 billion years gives us 10^{18} kg of oxygen—just the amount our atmosphere contains: one fifth of the mass of the atmosphere.

Don't overestimate the coincidence. After all, much more oxygen than is now present in the atmosphere was needed to establish the present chemical equilibrium of the Earth—to oxidize methane and ammonia in the primary atmosphere, to oxidize all the rocks of the Earth's crust. This would be impossible without vegetation. Plants produce on the order of 10^{14} kg of oxygen per year—$3 \cdot 10^6$ kg per second. This is much more than that produced by the dissipation of hydrogen into outer space. But at present the oxygen content of the atmosphere doesn't increase. All oxygen generated by vegetation is consumed by the breathing of animals, oxidation of volcanic gases, combustion, and the decaying of dead plants.

Near the Earth's surface, especially at altitudes of 20 to 50 km, an oxygen molecule can decompose into atoms: $O_2 \longrightarrow 2O$. This reaction is induced by the Sun's ultraviolet radiation, lightning discharges, and certain atmospheric impurities that act as catalysts. Active monatomic oxygen rapidly reacts with other gases, including oxygen, to form ozone: $O + O_2 \longrightarrow O_3$. Although ozone is a strong oxidizer, it's rather stable—near the Earth's surface, there is one ozone molecule on average for every 10^7 oxygen molecules. This relation, however, varies considerably, depending on the time of the day, the geographic latitude, and the presence of other impurities in the atmosphere.

Ozone concentration increases with altitude—at a height of 30 km, there is one molecule of O_3 for every 10^5 molecules of O_2; beyond that it falls off sharply. The presence of the ozone layer is of inestimable importance for the very existence of life on Earth.

It must be obvious to you from what's been said so far that the com-

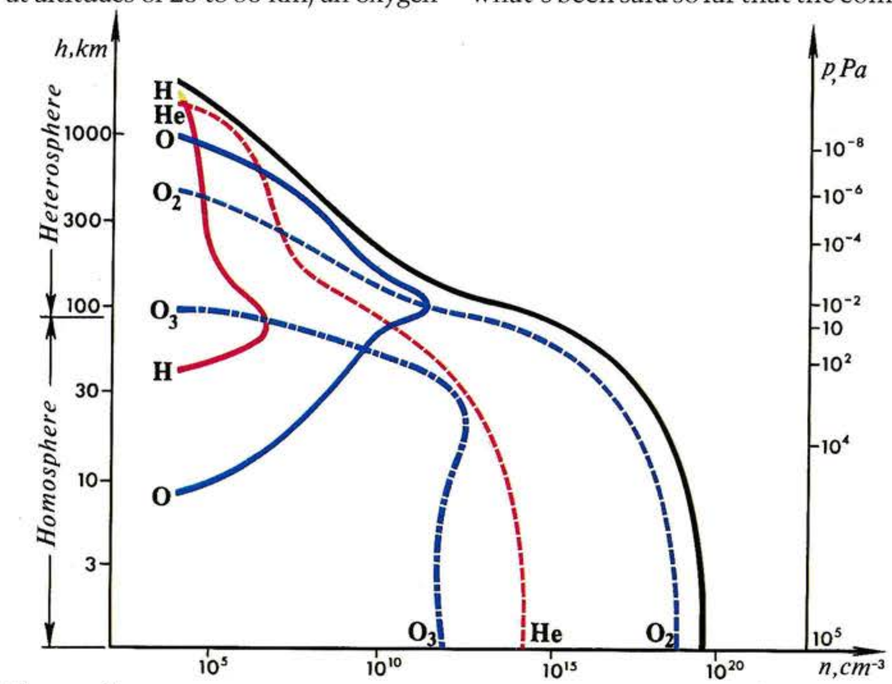


Figure 1

The change in concentration of n *molecules of certain gases (colored curves) and the change in total concentration of molecules (the black curve) in the atmosphere with altitude.*

In the homosphere the concentration of most gases decreases equally. Above 90 km the gas content essentially depends on altitude. Under the influence of the Sun's ultraviolet radiation, oxygen decomposes into atoms, and at altitudes of 200 to 600 km monatomic oxygen is the most abundant gas in the atmosphere. At higher altitudes the Earth's atmosphere mainly consists of helium. Finally, at the altitudes where gases escape into outer space, hydrogen is the most abundant gas in the atmosphere.

Make note of the shape of the ozone (O_3) curve at altitudes of 20 to 50 km. Why this is important for us is explained later in the article.

position of the atmosphere depends on altitude. In fact, the Earth's atmosphere has a uniform composition (that is, it's well mixed) only below 90 km. This part of the atmosphere is called the homosphere. The heterosphere is the portion of the Earth's atmosphere whose composition varies with altitude. In fact, above 700 km the Earth's atmosphere consists only of helium and hydrogen. It is basically these gases that are dissipated into outer space.

Figure 1 illustrates how the concentration of individual gases and the total concentration of molecules in the atmosphere vary with altitude.

The planet's thermal equilibrium

We've seen that the flow of gases that are lost into outer space depends on temperature. But as inhabitants of the Earth, atmospheric temperature interests us primarily from the practical point of view: our living conditions depend on it to such a great extent.

The main source of the Earth's thermal energy is the Sun. Humankind has always realized this. The Russian poet Alexander Sumarokov wrote in 1760:

O Sun, you are the life and beauty of Nature,
The source of eternity, the image of divinity!
You give life to earth, air, and water,
You are the soul of Time and Matter!

Let's try to arrive at a quantitative evaluation of these eternal truths. The Sun is a sphere of radius $R_S = 6.96 \cdot 10^8$ m that is heated by thermonuclear reactions; the temperature of the Sun's surface $T_S = 5{,}800$ K.

It's well known that hot bodies shine and radiate energy. At the end of the last century the Austrian physicists Stefan and Boltzmann discovered the law of radiation: the flow of energy—that is, energy radiated per unit of time from a unit surface area of a body in thermal equilibrium (having a constant temperature)—is proportional to the fourth power of the temperature: $\varepsilon = \sigma T^4$, where $\sigma = 5.67 \cdot 10^{-8}$ W/(m$^2 \cdot$ K^4) is the coefficient of proportionality, which is called the Boltzmann constant.

The Sun isn't in full thermal equilibrium, but the Stefan–Boltzmann law is approximately true for it. According to this law, the power emitted by the Sun is equal to

$$P_S = 4\pi R_S^2 \sigma T_S^4 \cong 3.8 \cdot 10^{26}\ \text{W}.$$

This power is emitted uniformly in all directions. It isn't a difficult task to calculate what portion of that power reaches the Earth. At a distance $R = 1.5 \cdot 10^{11}$ m, which is the radius of the Earth's orbit, the power reaching every square meter of the surface perpendicular to the rays is equal to $P_S/4\pi R^2$. The surface area of the Earth that blocks these rays is equal to πR_E^2. So the power reaching the Earth from the Sun is equal to

$$P = P_S \frac{\pi R_E^2}{4\pi R^2} = 1.75 \cdot 10^{17}\ \text{W}.$$

What is this power spent on? Some is reflected by the Earth back into outer space. As you well know, the planets and the Moon are visible in the starry sky precisely because of reflected sunlight. In the same way we can see the light reflected by the Earth as we travel into outer space. The portion of the reflected radiant power is called the albedo (from the Latin *albus*, "white"), which implies a kind of whiteness factor. The accuracy with which we know the albedo of our planet is quite small. The value of the Earth's albedo A is taken to be within the range of 28 to 36 percent. What is the rest of the energy $P(1 - A)$ expended on?

Clearly this portion of the energy is responsible for the warm climate on our planet. But the Earth is continually exposed to solar radiation, and if no heat were removed, the Earth's temperature would constantly increase. So heat removal exists. It is performed by the same physical process as the solar radiation itself. Just imagine, the Earth and the other planets are also sources of radiation. But the spectrum of this radiation is in an area that the eye can't see—the infrared range.

Let's calculate the temperature of the Earth's thermal radiation, taking its albedo to be equal to 0.28 (in accordance with the latest satellite measurements). We should equate the solar radiation power $P(1 - A)$ absorbed by the Earth with the power of the Earth's thermal radiation. If the Earth's temperature is taken to be T_E, the power radiated from its surface is equal to $4\pi R_E^2 \sigma T_E^4$. So

$$P(1 - A) = 4\pi R_E^2 \sigma T_E^4,$$

from which we get

$$T_E = \left[\frac{P(1-A)}{4\pi R_E^2 \sigma}\right]^{\frac{1}{4}} = 258\ \text{K} = -15°\text{C}.$$

Not a bad frost! But we know that the average annual temperature of the Earth's moderate latitudes is above zero; and the tropics occupy a considerable portion of the Earth's surface, where in both summer and winter the temperature seldom drops below +25°C. What's going on here? Maybe the Earth has its own source of thermal energy.

Well, such a source actually does exist. It's the Earth's core. With every kilometer of descent into the Earth's crust, the temperature increases 30 degrees on average. This heating is due to the energy released by the decay of heavy radioactive elements. Calculations show, however, that the thermal flow reaching the atmosphere from the core is less than that of solar radiation by a factor of 6,000. So the heat of the Earth's center doesn't influence the climate of our planet.

Why does the average temperature of the Earth's surface remain constant? Why doesn't it fall to –15°C, which corresponds to the thermal equilibrium?

The "layer cake" of the atmosphere

It turns out that heat isn't radiated into outer space by the surface of the Earth itself but by the air enveloping it—the atmosphere. At first glance this seems strange—after all, the air is transparent. But the radiation spectrum at a temperature of about 300 K is in the far infrared region. Depending on our senses only, we can't say anything about the ability of substances to transmit radiation of such wavelengths. Measurements made by infrared spectrometers, however, show that the main gases of the air—nitrogen, oxygen, and inert gases—are also transparent in the infrared range, while carbon dioxide and water vapor, present in the atmosphere in small quantities, absorb infrared radiation so strongly that they determine the radiative properties of the Earth's atmosphere and the Earth as a whole.

The Earth's thermal radiation is emitted in atmospheric layers at altitudes of 6 to 12 km. This is where the average temperature is equal to approximately 258 K. Imagine a newcomer to our solar system who can see only infrared light. To this space traveler the Earth would look like a luminous sphere with a radius just a bit larger than the true one. But only seldom could the creature see the real surface of the Earth: near the South Pole during the polar night or in the Northern Hemisphere (in eastern Siberia) during the winter. This is because only in a severe frost does the air becomes so dry that the atmosphere is transparent in infrared light also.

The part of the atmosphere that's below the surface of the infrared radiation is called the troposphere. The troposphere contains about 80% of the entire mass of the Earth's atmosphere. You know that pressure decreases with altitude. At the height of Mt. Everest, which is about as high as the radiating surface, the air pressure is only a fifth of normal pressure. This means that one fifth of the atmospheric mass is situated above this peak. The troposphere is the part of the atmosphere that has the most practical significance for us. Its motion determines all meteorological phenomena. All the ordinary clouds are also situated in the troposphere—there is very little water vapor above it. So the state of the troposphere accounts for precipitation: rain, snow, and hail.

Strictly speaking, the troposphere extends into the region located 2 to 3 km above the surface of the infrared radiation, where the air temperature continues to decrease with height.

But above 15 km the atmospheric temperature starts rising! And it increases to 270 K—that is, it again reaches approximately 0°C—at an altitude of 50 km. This atmospheric layer is called the stratosphere. Why does the temperature rise in the stratosphere?

It turns out that the heat energy of this layer results from chemical reactions taking place because of the action of the Sun's ultraviolet radiation. These reactions are the decomposition of oxygen into atoms and the creation of ozone molecules (O_3). The layer of increased ozone concentration in the stratosphere (see figure 1) screens the Earth's surface from solar ultraviolet radiation, which is harmful to all living things.

The ozone layer is necessary for the Earth, but it's unstable. Ozone is partially broken down because of the formation of nitrous oxides in the stratosphere, which end up there after atmospheric nuclear explosions, powerful volcanic eruptions, meteor showers, and even rocket launches. Organic gases containing chlorine and fluorine, which are used in aerosol spray cans and refrigerator heat exchangers, are also harmful to the ozone layer. Human activities account for some of the destruction of the ozone layer, and their negative consequences are, unfortunately, hard to predict.

Since the temperature in the stratosphere increases with altitude, this portion of the atmosphere is extremely resistant to mixing. Once chemical impurities and fine dust reach the stratospheric layer, they can remain in it for several years, and above 20 km they can stay for decades. As these impurities slowly propagate upward, they intensify the destruction of the ozone. But ozone is formed by the action of the Sun's rays, and so the maximum ozone content is at a height of 40 km above the tropics. The distribution of the ozone layer across latitudes is determined by these two flows: (1) ozone from above and from the tropics and (2) impurities from below. But the time it takes the ozone layer to react to the new sources of impurities is measured in decades.

In the 1960s the average thickness of the ozone screen was decreasing. From the middle of the seventies it began increasing, which may have been a result of the fact that most atmospheric nuclear testing stopped after an international treaty was signed in 1963. Now the average ozone concentration throughout all the latitudes causes no worries, but in the early eighties a new phenomenon emerged: a hole began to appear in the ozone

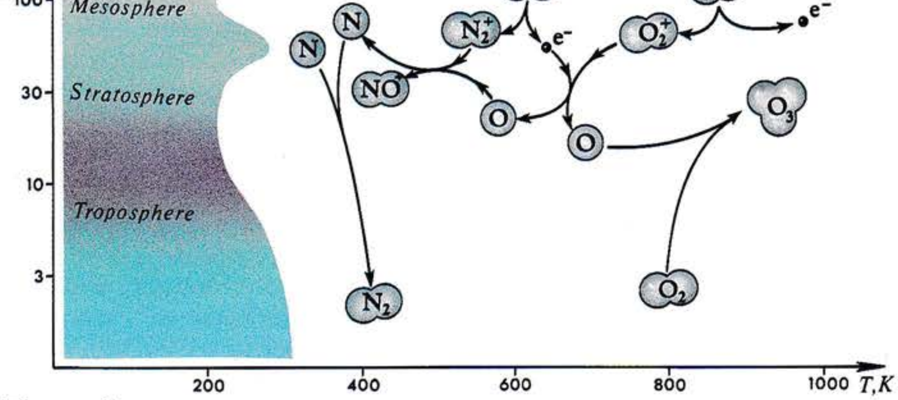


Figure 2

over Antarctica every year at the end of the polar night. It's observed in the Southern Hemisphere's spring months—September and October every year. The total concentration of ozone in the entire column of atmosphere is minimal over the center of Antarctica and is decreasing from year to year.

It's awful to think what might happen . . . All the more so in that we still don't know the culprit—the process that is unambiguously, directly responsible for this phenomenon. But I think we mustn't exaggerate the danger. The ozone hole will surely not migrate from the region of polar night to the lower latitudes. It's more difficult to say whether a similar phenomenon will arise near the North Pole.

Above 55 km the air temperature falls. It drops to 187 K at a height of 90 km above the Earth's surface. This layer of the atmosphere is called the mesosphere. The temperature falls in the mesosphere because the air in it absorbs sunlight weakly. The thermosphere and exosphere are situated above the mesosphere. In the thermosphere the temperature begins to rise sharply and increases to 1,000 K at altitudes of 350 to 400 km.[1] This results from absorption of the solar ultraviolet radiation by the main atmospheric gases—oxygen and nitrogen. Above the thermosphere the temperature doesn't change with altitude. This is the exosphere, a transitional region between the Earth's atmosphere and outer space; it's the part of the atmosphere where hydrogen, helium, and negligible quantities of other gases escape from the Earth.

And that's how the "layer cake" of the atmosphere is arranged (fig. 2).

Energy streams in the atmosphere

Solar energy falling on the Earth is redistributed by the various layers of the Earth's atmosphere and by its solid or liquid surface. How this happens is apparent from figure 3, which follows the paths of 100 arbitrary units of solar power falling on the Earth.

Notice that in this illustration the heat flow coming from the Earth's surface is equal to 145 units, whereas the original flow of solar energy was equal to only 100 units. This schematically illustrates the fact that the average temperature of the Earth's surface is +15°C, or 288 K, and the heat flow emitted by the Earth's surface is 1.45 times that of the original 100 units given to us by the Sun. But only some of this radiation goes off into outer space. The rest—97 units—circulates continuously between the Earth's surface and the troposphere.

This heating of the Earth's surface and lower layers of the atmosphere is sometimes called the greenhouse effect, and it's really an apt description. The Sun's rays easily penetrate the transparent roof of a greenhouse, warming up the soil and air. But the heat has a hard time escaping—the glass or plastic film of the greenhouse doesn't allow either the warm air or the infrared radiation to get out. So the greenhouse cools rather slowly.

In closing I should point out that our current knowledge of the heat flows shown in figure 3 is not very precise.

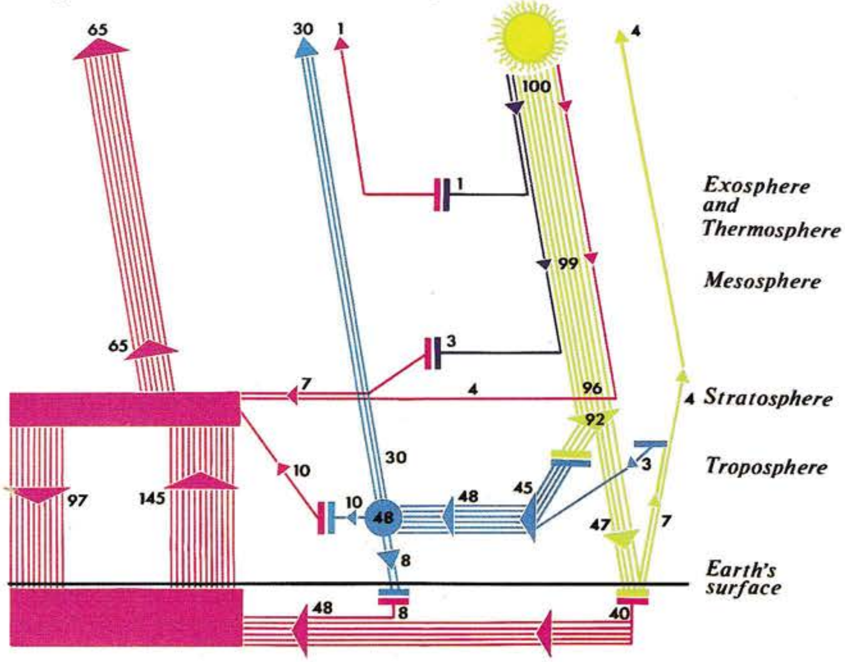


Figure 3

Thermal equilibrium in the atmosphere. One percent of the solar power is absorbed by molecules in the exosphere and thermosphere. Another 3% is absorbed by ozone in the stratosphere. The upper layers of the troposphere, which contain water vapor, absorb energy in the infrared range of the solar spectrum. This leaves 92 units of the original power.

This power, the bulk of which lies in the visible range, penetrates the dense layers of air near the Earth's surface. A considerable portion of it—about 45 units—dissipates in the air. Direct sunlight—the remaining 47 units of the original stream of light—makes it all the way to the Earth's surface. About 7 units are reflected upward. The remaining 40 units are absorbed by the Earth's surface, heating up the land and seas.

The solar power dissipated in the atmosphere (48 units altogether) is partially absorbed (10 units); the rest is distributed between the Earth's surface and outer space. The amount of this energy going off into space (30% of the original flow) is greater than that reaching the surface.

There are 65 units of power left, which are absorbed and transformed into heat in the atmosphere and on the Earth's surface: ozone accounts for 3 units; water vapor in the upper troposphere accounts for 4; another 10 units are absorbed in the main thick layer of the atmosphere; and, finally, 48 units are transformed into heat in the soil and water. These 65 units of power are ultimately transformed into heat and are radiated into space, not in the visible or ultraviolet ranges but as thermal radiation.

[1]This temperature can't be measured with a thermometer—the gas density of the thermosphere and exosphere is so low that thermal equilibrium between the gas and the thermometer is never established. Here temperature is measured by the average kinetic energy of the gas molecules.

Latin rectangles

A mathematical housing project

by V. Shevelyov

WHAT ON EARTH IS THIS article about?

It's about rectangles "inhabited" by various positive integer numbers. If you divide a rectangle by lines parallel to its sides into $m \times n$ squares and "populate" this $m \times n$ "apartment house" having m "floors" and n "entrances" by positive integers $1, ..., n$ in such a way that the numbers on each floor and in each entrance are different, you get what mathematicians call a *Latin rectangle.* Each of the numbers $1, 2, ..., n$ in it is repeated m times, and the "families" of ones, twos, threes, and so on, inhabit m apartments situated on different floors and in different entrances. In times past this house used to be inhabited by letters of the Latin alphabet, and this is the reason for its name.

There is a branch of mathematics that has to do with counting various sets and configurations. It's called combinatorics. The problem of enumerating Latin rectangles is a matter for combinatorics. But in its general form this problem is extremely difficult. Despite the efforts of some of the world's very best mathematicians, more than two hundred years passed between the enumeration of "two-story" and "three-story" rectangles. The two-story rectangles were enumerated by P. R. de Monmort in France back in 1713, and the three-story rectangles by the American mathematician W. J. Riordan just 40 years ago. In addition to the beautiful formulas of de Monmort and Riordan, which we'll discuss below, there are useful recurrent formulas found by the great Euler (we'll prove his formula) and Keravala from India. It's curious that Keravala had actually disproved the erroneous recurrent formulas for the number of three-story Latin rectangles that were proposed by the English mathematician Jacob, which held for 12 years (from 1930 to 1941). Well, this happens in mathematics when difficult problems are being solved.

Existence of Latin rectangles

THEOREM 1. *For any pair of numbers $m \leq n$ there exists a Latin $m \times n$ rectangle.*

PROOF. We'll populate our m-story house by starting from the top floor. On the mth floor the numbers $1, ..., n$ are settled in their natural order. On the $(m-1)$th floor we begin with the two: $2, 3, ..., n, 1$; on the $(m-2)$th floor we begin with the three: $3, 4, ..., n, 1, 2$; and so on. Finally, we populate the first floor by starting with m: $m, m+1, ..., n, 1, 2, ..., m-1$. Then the house will be inhabited as shown in the table. It's clear that this procedure doesn't allow two identical "tenants" to live on the same floor or in the same entrance. So we've got an m-story Latin rectangle of length n. Thus, $m \times n$ Latin rectangles do exist.

Table

1	2	3	...	$n-1$	n
2	3	4	...	n	1
...	...	...	...	...	...
m	$m+1$	$m+2$	...	$m-2$	$m-1$

Two stories

Now let's count Latin rectangles. The problem of counting one-story rectangles is solved very easily.

THEOREM 2. *The number of Latin $1 \times n$ rectangles equals $n! = 1 \cdot 2 \cdot ... \cdot n$.*

PROOF. A Latin $1 \times n$ rectangle is simply an arbitrary permutation of n numbers. There are $n!$ such permutations (the first position can be occupied by any of the n numbers; for the second position there are only $n-1$ numbers left; and so on).

Consider a $2 \times n$ rectangle. The top line of such a rectangle is an arbitrary permutation. The lower line is a permutation in which the number in each position does not coincide with the number in this position in the first permutation. If we permute the columns of our rectangle arbitrarily, it will still be a Latin one, so the top permutation can be made to coincide with any given one. So for any given permutation the number of Latin rectangles coinciding with this permutation in their top line is the same. A Latin $2 \times n$ rectangle is said to be *normalized* if its top line is $1, 2, ..., n-1, n$. It follows from the reasoning above that the number $L(2, n)$ of Latin $2 \times n$ rectangles equals the number D_n of normalized Latin $2 \times n$ rectangles multiplied by the number of permutations of

n numbers—that is,

$$L(2, n) = n! \cdot D_n.$$

There are several elegant formulas for the number D_n of normalized Latin $2 \times n$ rectangles.

THEOREM 3 *(the Euler formula).* $D_n = (n-1)(D_{n-1} + D_{n-2})$.

This is called a recurrent formula since, knowing D_1 and D_2 (evidently $D_1 = 0$, $D_2 = 1$), we can compute D_n for all n:

$$D_3 = 2 \cdot (1 + 0) = 2,$$
$$D_4 = 3 \cdot (2 + 1) = 9,$$
$$D_5 = 4 \cdot (9 + 2) = 44,$$
$$D_6 = 5 \cdot (44 + 9) = 265,$$

and so on.

PROOF. Each permutation can be written as a system of cycles. Here's how it's done. Let the 1st position be occupied by k_1. Then we write $1 \longrightarrow k_1$. If the k_1th position is occupied by k_2, we write $1 \longrightarrow k_1 \longrightarrow k_2$. Proceeding in the same way we write $1 \longrightarrow k_1 \longrightarrow k_2 \longrightarrow k_3$ and so on until we come back to 1 (fig. 1). No other number already written turns up again before 1. For instance, if we get $\ldots \longrightarrow k_9 \longrightarrow k_{10} \longrightarrow k_5$, then k_5 occupies both the k_{10}th position and the k_4th position, so that $k_{10} = k_4$. Similarly, $k_9 = k_3, \ldots, k_6 = 1$). Having constructed the first cycle, we take the smallest of the remaining numbers and construct a cycle starting with that number. Ultimately all the n numbers will be arranged in cycles (fig. 2).

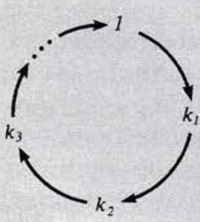


Figure 1

The number of cycles can take any integer value between 1 and n, and the length of a cycle is any number from 1 to n. The condition of "Latin rectangularity" is that no number stays in its position, so this forbids cycles of length 1.

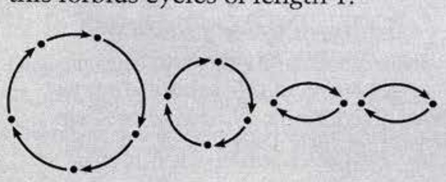


Figure 2

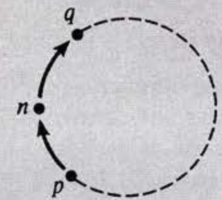


Figure 3

Now let's go back to our permutation and construct another permutation of length $n - 1$ or $n - 2$ that also does not contain cycles of length 1. One of the cycles includes the number n (fig. 3). If the length of this cycle exceeds 2, we just throw n out and join p and q (fig. 4). If n is part of a cycle of length 2 (for instance, as shown in figure 5), we throw this cycle out and subtract 1 from all the numbers between $k + 1$ and $n - 1$: $k + 1 \longrightarrow k$, $k + 2 \longrightarrow k + 1, \ldots, n - 1 \longrightarrow n - 2$. In the first case we get a permutation of $n - 1$ numbers, while in the second case we get a permutation of $n - 2$ numbers. How many permutations of n numbers can give a certain permutation of $n - 1$ numbers? Clearly $n - 1$: if we want to reconstruct a permutation of n numbers, we must break an arbitrary arrow $p \longrightarrow q$ (of which there are $n - 1$) and insert n in between p and q, like this: $p \longrightarrow n \longrightarrow q$. How many ways are there to obtain a permutation of $n - 2$ numbers? Again, $n - 1$: we add a cycle $n \longleftrightarrow k$, where k is an arbitrary number between 1 and $n - 1$,

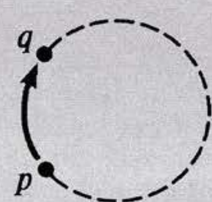


Figure 4

Figure 5

Art by Sergey Ivanov

and increase the numbers $k, k+1, \ldots, n-2$ by 1. Thus

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2},$$

and the proof is complete.

Here is another recurrent formula for D_n that is much simpler.

THEOREM 4. $D_n = nD_{n-1} + (-1)^n$.

PROOF. Let $D_n - nD_{n-1} = E_n$. From Euler's formula

$$\begin{aligned} E_n &= D_n - nD_{n-1} \\ &= (n-1)D_{n-1} + (n-1)D_{n-2} - nD_{n-1} \\ &= -D_{n-1} + (n-1)D_{n-2} = -E_{n-1}. \end{aligned}$$

Thus

$$E_n = -E_{n-1} = E_{n-2} = \ldots = (-1)^n E_2.$$

But $E_2 = 1 - 2 \cdot 0 = 1$, so $E_n = (-1)^n$, and $D_n = nD_{n-1} + (-1)^n$.

The last formula easily yields a formula for D_n that is not a recurrent but an explicit one.

THEOREM 5 *(the de Monmort formula).*

$$D_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

PROOF. See the box below.

The expression in brackets may seem familiar to those of you who know basic calculus. It does to me, but I'll postpone "unmasking" it until the concluding section of the article.

As for the de Monmort formula, it was proved before the two previous formulas were stated. Its direct proof is based on the inclusion–exclusion formula,[1] and readers familiar with it can easily derive the de Monmort formula. Now we'll turn our attention to three-story Latin rectangles.

$$\begin{aligned} D_n &= nD_{n-1} + (-1)^n \\ &= n\left[(n-1)D_{n-2} + (-1)^{n-1}\right] + (-1)^n \\ &= n(n-1)D_{n-2} + (-1)^{n-1}n + (-1)^n \\ &= n(n-1)(n-2)D_{n-3} + (-1)^{n-2}n(n-1) + (-1)^{n-1}n + (-1)^n \\ &= \ldots \\ &= n(n-1)(n-2)\cdot\ldots\cdot 3\cdot D_2 - n(n-1)\cdot\ldots \\ &\quad \cdot 4 + n(n-1)\cdot\ldots\cdot 5 - \ldots + (-1)^{n-1}n + (-1)^n \\ &= n!\left[\frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right] \end{aligned}$$

Proof of the de Monmort formula

Three stories

Attempts to find the number $L(3, n)$ of Latin $3 \times n$ rectangles were crowned with success in 1944 when W. J. Riordan, basing his effort on results of his numerous predecessors, at last wrote out the final formula. I won't give you its derivation (which falls well outside any high school course), or even the formula itself, which is rather cumbersome. I'll just give you the recurrent Keravala–Riordan formula: if

$$K_n = \frac{1}{n!}L(3,n),$$

then for $n \geq 4$

$$\begin{aligned} K_n = {} & n^2 K_{n-1} + n(n-1)K_{n-2} \\ & + 2n(n-1)(n-2)K_{n-3} \\ & + (-1)^n\left(e_n + 2ne_{n-1}\right), \end{aligned}$$

where e_n is found from its own recurrent formula (somewhat similar to the formula for D_n):

$$e_0 = 1,$$
$$e_n = ne_{n-1} + (-2)^n.$$

Using these formulas and the fact that $K_1 = K_2 = 0$, $K_3 = 2$, we can easily find K_n for all n. For instance,

$$K_5 = 552,$$
$$K_7 = 1{,}073{,}760.$$

It's clear from these formulas that the numbers K_n grow very rapidly. The numbers $L(3, n)$, of course, grow even more rapidly.

Approximate formulas

The expression

$$\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$

looks familiar because it reminds us of the power series for e^x, the natural logarithm:[2]

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

In particular,

$$\begin{aligned} &\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \\ &\cong 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots = e^{-1}, \end{aligned}$$

where the absolute error doesn't exceed

$$\frac{1}{(n+1)!}.$$

This means that

$$D_n \cong n!e^{-1} = \frac{n!}{e},$$

where the error doesn't exceed

$$\frac{1}{n+1}.$$

(Since D_n is an integer, the two conditions determine D_n uniquely.) In 1946, generalizing this formula, P. Erdos and I. Kaplansky derived an elegant approximate formula for the number of m-story Latin rectangles of length n:

$$L(m,n) \cong (n!)^m e^{-\frac{(m-1)m}{2}}.$$

The larger the n, the smaller the relative error in this formula. (Such formulas are called asymptotic.) Actually, Erdos and Kaplansky proved this statement under the condition that $m < (\ln n)^{1.5}$. But later it became clear that it holds for larger m as well. The Japanese mathematician K. Yamamoto proved that it is sufficient

CONTINUED ON PAGE 41

[1] A version of this formula for areas was given in the solution to problem M23 (Jan./Feb. 1991).

[2] See the Getting to Know department in the November/December issue.

A portrait of Poisson

A fish out of water who found his calling

by B. Geller and Y. Bruk

SIMEON DENIS POISSON,[1] the eminent French scientist, is one of the founders of modern mathematical physics. In the history of science he occupies a position alongside his great contemporaries Lagrange, Laplace, Fourier, and Cauchy. His name is frequently mentioned in textbooks on differential and integral calculus, probability theory, electromagnetism, acoustics, elasticity, and quantum mechanics.

Poisson was born on June 21, 1781. There is scant information about his parents. It's known that his father first chose a military career and joined the Hannoverian army, but resenting its strict discipline, he left the army and finally settled in the small French town of Pitiviers. By the time his son was born, he had the modest but respectable position of notary.

In his early childhood Poisson was a quite ordinary boy who didn't show any promise of becoming a great man. His parents even had some misgivings about his intellectual abilities. His father, of course, wanted his son to become a notary, but the "family council" decided he wasn't up to the job and should become a doctor instead. The decision of the family was a kind of law unto itself, so they sent Poisson to his uncle's place in the town of Fontainebleau to study the respectable and, in their judgment, simple art of the surgeon. But mastering the profession turned out to be a very difficult task; for instance, to learn how to do a venesection (bloodletting), a basic form of treatment at the time, one had to practice for long hours, puncturing the veins in cabbage leaves with a needle. Later, Poisson told his friends that even the largest veins would elude his needle at the last moment. These exercises, which Poisson hated so much, took a whole year, but his first attempt to treat a patient without supervision resulted in the death of the patient. The event so shattered the youth that he quit medicine right then and there and returned home to Pitiviers.

A great deal had changed while Poisson was away. His father had become a public figure, the mayor of Pitiviers. He had bought a new house, appropriate for his new position in society, and began to receive a lot of guests. He also subscribed to various periodicals, including the *Journal of the Polytechnical School.* Simeon enjoyed reading this one very much, and he especially liked to solve the mathematical problems proposed in it. Quite unexpectedly, solving the problems turned out to be very easy for the boy, who had never been taught how to do it but cracked them nevertheless, one after another. We must give credit where credit is due: Poisson's parents quickly changed their mind about their son's intelligence and sent him back to Fontainebleau, but this time—to school.

The famous Polytechnical School

Poisson was an excellent pupil. His talent and hard work enabled him to outpace the other students. Two years later Simeon, who was seventeen by that time, entered the Polytechnical School in Paris.

This school, one of the oldest and most unusual institutions of higher learning in France, was created on March 11, 1794, during the French revolution, by a decree of the National Assembly. Initially, its name was the Central School of Public Works; the name was changed to Polytechnical School a year later. Its purpose was to advance scientific knowledge and train engineers for the army. The Polytechnical School has remained, up to the present time, the school for military and civil engineers in France, and its graduates have been expected to occupy the highest governmental positions. The period of study at the Polytechnical School was relatively short (only two years) but intense. The outstanding role played by the Polytechnical School in the advance-

[1] Pronounced "pwah-SOHN."

ment of education in physics and mathematics is largely due to its excellent faculty, which included, in its early years, such eminent scientists as Monge, Laplace, Legendre, Fourier, and Carnot. Professors at the Polytechnical School created many courses and textbooks on differential and integral calculus, geometry, and analytical mechanics that shaped the development of mathematical education, and not only in France. Even now the Polytechnical School remains one of the leading French institutions of higher learning.

Poisson got a very good education at the Polytechnical School. The mathematicians Laplace and Lagrange noted his exceptional talent and spent a lot of time teaching him. Poisson also mastered the work of the previous generation of mathematicians, and studied the writings of Euler and d'Alembert in particular detail. Later, Poisson's friend and biographer, the eminent French physicist François Arago (who was also a graduate of the Polytechnical School) wrote: "Poisson never had to spend time and effort searching for things that had already been discovered." It's no accident, then, that his first mathematical papers, which he wrote in his early twenties, were mature enough to bring him instant fame. But it would be wrong to assume that as a student, and later in life, Poisson had no interests outside of mathematics. He was a sociable person who enjoyed the fine things in life. He loved the theater and went often—he knew the works of Molière, Corneille, and Racine by heart.

Poisson held many honorable positions in the French scientific hierarchy, including membership in the French Academy of Sciences, but his life was mainly connected with the Polytechnical School. He was made assistant professor at the school in 1802, and he became full professor at the age of 25 in 1806, taking the position vacated by the great Fourier. One of his important duties was administering examinations to pupils who wanted to enter the school and to students hoping to graduate from it. The position of examiner was in a sense somewhat higher than that of professor—in the examinations he tested both the students' knowledge and the professors' teaching.

All the French governments, which frequently changed during those troubled times, paid generous tribute to the service done by Poisson to his country. He received the title of baron, was awarded the order of the Legion of Honor (the highest honor in France), and became a peer of France. Poisson's achievements were recognized abroad as well—he was a member of all the scientific societies in Europe and America, and was an honorary member of the St. Petersburg Academy of Sciences (from 1826).

François Arago wrote in his memoirs that Poisson had another trait, one often found lacking even in people not so highly placed in academic life: he scrupulously fulfilled his obligations. For instance, every year he had to spend four weeks, nine hours a day, administering exams at the Polytechnical School. Only once did he refuse to participate in the examinations: when his own son was taking the tests. But the students at the school sent a delegation to him, declaring that they were sure of his impartiality and asking him to proceed with the examination. That Poisson liked teaching can be seen from his own words: "Life is made beautiful by two things—studying mathematics and teaching it." His lectures were noted for their clarity and depth.

During the last years of his life (he died in Paris in 1840), Poisson had set himself the task of writing a fundamental treatise on mathematical physics. Unfortunately, he did not live to complete the work.

From calculus to criminal law

The bulk of Poisson's scientific work, which comprises 350 papers, deals with problems in mathematical physics, so we're not able to discuss even his basic results in any great detail. We can only mention his most important and famous papers and also examine a few questions that can be understood with high school mathematics and physics.

The concept of electrical potential is one of the most important notions in physics. Potential always depends on the magnitude and location of electrical charges in space, and finding the potential is generally a difficult problem. In 1811 Poisson derived the differential equation that relates the potential to the distribution of charges. Of course, the simplest problems in electricity can be solved without the use of Poisson's equation. But when confronted with more complex problems in which there are many charges distributed randomly, we can calculate the dependence of the potential on the coordinates only with the aid of this equation. In fact, Poisson's equation, along with results obtained by Euler, Gauss, Laplace, Green, and Ostrogradsky, forms the foundation of the modern theory of potential, an important branch of mathematical physics.

The scope of Poisson's work is quite impressive. He made important contributions to theoretical and fluid mechanics, elasticity, heat conduction, the physics of gases, atmospheric electricity, geomagnetism, surface tension, and waves in deep water. He also investigated such practical problems as the deviation of an artillery shell from its intended trajectory. In astronomy he studied the stability of the solar system—a problem that continues to attract considerable attention even today. In the field of pure mathematics he obtained important results in differential and integral calculus and in the theory of differential equations.

Poisson's papers on probability theory are among his best known. Like Laplace, he paid considerable attention to the application of probability theory to criminal jurisprudence. One of his treatises is entitled "A Study of Verdict Probability in Criminal and Civil Cases." Today this approach is considered unsatisfactory as far as the legal aspect is concerned, but we must allow for the fact that Poisson solved a number of interesting mathematical problems while working in that field. Again going beyond the classical theory of probability, Poisson analyzed some problems of card games, and in that he can be considered one of forerunners of modern game theory.

The Poisson distribution

To give you a sense of Poisson's research and to illustrate how his ideas work, we'll look at several specific problems from probability theory and mechanics.

First, let's consider three problems that can be solved by using a formula called the Poisson distribution, which is encountered quite often in probability theory. We won't derive the formula—we'll just show how to use it.

The first problem has to do with typographical errors in books. To find some numerical characteristic of this troublesome phenomenon, we'll assume that the number of letters per page and the number of pages are so large that we can assume that there is a constant probability of the typesetter making a mistake, and that the probability is equal to the ratio of the number of typos to the total number of letters set in type. We'll also assume that all the pages in a book are similar in that the number and location of letters are approximately the same—that is, we assume that the conditions in which a typesetter works don't change and that the probability of making an error doesn't depend on the typesetter's previous work. Under these conditions, the probability of k misprints occurring on a page is approximately equal to

$$p(k,\lambda) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

The number λ in this formula, which is called Poisson's formula or Poisson's distribution, is a parameter characteristic of a printer's work—it equals the product of the probability of making a typo and the average number of letters per page.

We can "experimentally" test the result obtained with Poisson's formula. Here's how. We have to read carefully the pages set by the typesetter—the more the better—and find those pages that contain k misprints. Next, we have to divide the number of pages with k misprints by the number of pages read and compare the ratio with the value obtained by using Poisson's formula with the same k.

Here's the second problem. Let's say we'd like to know the probability that in a small town with a population of 1,991 citizens, k of them were born on the same day of the year as Poisson. The problem can be solved this way. Since all the days in the year are equal, we can assume that the probability of a person being born on the same day as Poisson equals about 1/365, so that the parameter $\lambda = (1/365) \cdot 1991 \cong 5.45$—that is, the product of the probability of an individual's being born on a particular day and the total number of people in the community. (The situation is similar to that in the previous problem, where instead of the total number of people we had the average number of letters per page.) Now we can find the probability by using Poisson's equation with $\lambda = 5.45$ and the necessary k.

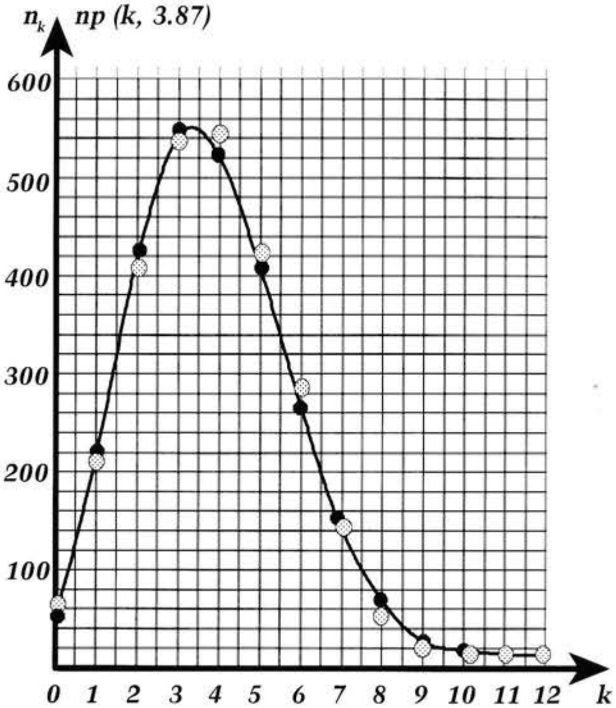


Figure 1

The third problem has to do with physics. In their classical paper on radioactive decay, Rutherford, Chadwick, and Ellis found that the probability of a radioactive sample's emitting k alpha particles in a unit of time is given by Poisson's formula. The problem was to find the constant λ from the experimental data. To be more specific, the paper dealt with radium. According to the theory worked out in the paper, the radioactive decay of radium is the transformation of the radium nucleus into a radon nucleus with the emission of an alpha particle. The transformation is a random process; it is assumed that the probability of a radium atom disintegrating per unit of time is constant and independent of the state of the other atoms.

k	n_k	$n \cdot p(k, 3.87)$
0	57	54.339
1	203	210.523
2	383	407.361
3	525	525.496
4	532	508.418
5	408	393.515
6	273	253.817
7	139	140.325
8	45	67.882
9	27	29.189
$k \geq 10$	16	17.075
Total	2,608	2,608.000

Rutherford and his coworkers used a detector to count the alpha particles emitted by a sample of radium during $n = 2{,}608$ intervals of time, each interval equal to 7.5 s. They found the number n_k of intervals during each of which exactly k particles were detected. The total number of particles detected in the experiment was equal to $\Sigma k n_k = 10{,}094$. If we divide this number by the number of time intervals $n = 2{,}608$, we obtain the average number of alpha particles emitted per interval (or the average number of alpha particles emitted in 7.5 s), which is equal to $\Sigma k n_k / n = 3.87$. Now we can compare the values of the ratios n_k/n found in the experiment with the numbers $p(k, \lambda) = p(k, 3.87)$ given by Poisson's formula for $\lambda = 3.87$ found above. These values are given in the table at left, which we took from the paper by

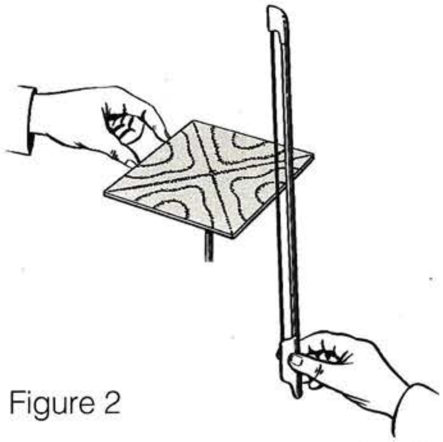


Figure 2

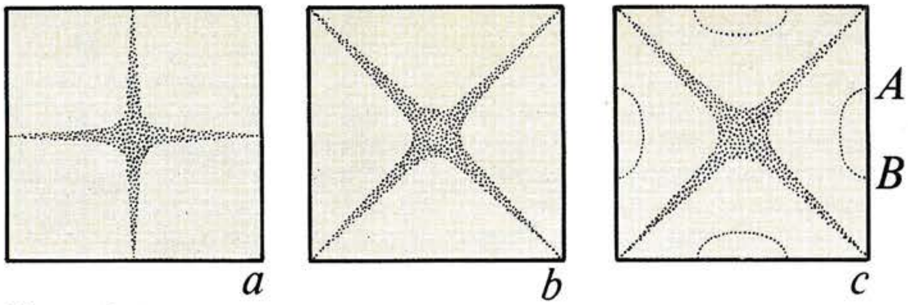


Figure 3

Rutherford and Ellis, and graphically presented in figure 1, in which the black points correspond to the numbers $n \cdot p(k, 3.87)$ and the gray ones to n_k. We see that either set of points fits the same smooth curve that illustrates the Poisson distribution.[2]

The symmetry of vibrations . . .

From his earliest childhood Poisson was taken with the physics of oscillations—quite literally! It seems his nanny wasn't all that conscientious. Rather than be bogged down with little Simeon Denis, she would wrap a wide towel around his waist and fasten it to a horizontal beam. So the little boy spent many an hour swinging like a pendulum, back and forth, back and forth. Years later, Poisson would joke that God Himself ordered him to study the theory of oscillations.

Poisson's results in this field are both numerous and important. Here we'll discuss only one of them, which arose from evaluating the frequencies of vibrations of small glass or metallic plates. The German physicist Chladni (1756–1827) was the first to work out an experimental method for studying the physics of vibrating plates, as early as 1787. In 1809 he presented demonstrations with vibrating plates to a fascinated audience of members of the French Academy of Sciences.

In Chladni's experiments a plate supported at the center and covered with a layer of fine sand is made to vibrate by drawing a violin bow across its edge; at the same time a finger is applied at another point on the edge (fig. 2). The sand collects along lines, called nodal lines, where the plate doesn't vibrate at all (fig. 3). It's worth noticing that the nodal lines pass through the points where the finger touches the plate. The bizarre but always symmetric figures created by the particles of sand are called Chladni figures. You can use square, rectangular, or round plates when experimenting with them. Designs that Chladni himself obtained in experiments with round plates are shown in figure 4.

Poisson's achievement in studying Chladni figures was that he found the dependence of the vibration frequency on the number of nodal lines. For the specific case of a square plate and square Chladni figures (as in figure 3a), the square of the vibration frequency is proportional to $(m + 1)^2 + (n + 1)^2$, where m and n are the numbers of perpendicular nodal lines that partition the plate's surface.

Looking again at the simple nodal lines in figure 3, we conclude that in figure 3a a finger touched the plate at a point in the middle of a side of the square; in figure 3b, at a corner; and in figure 3c two fingers touched the plate at points A and B. The pitch of the sound created in the second case is higher than in the first, and higher in the third case than in the second.

. . . and something about corks

Finally, let's look at the problem, solved by Poisson, of finding the relation between the longitudinal and transverse deformations of a body under stress. The essential point is that, if a force acts on a body, its longitudinal size—that is, in the direction parallel to the force—changes differently from that in the transverse direction. (For an example of this, watch what happens when you stretch a rubber band.) Poisson found the coefficient, named after him, that provides a quantitative description of the phenomenon.

Let's look at a specific example. Consider a cylindrical rod of length l and radius r subjected to a force directed along the rod's axis, giving rise to a tension σ_l and relative deformation $\varepsilon_l = \Delta l / l > 0$. The transverse size of the rod also changes, so that the radius

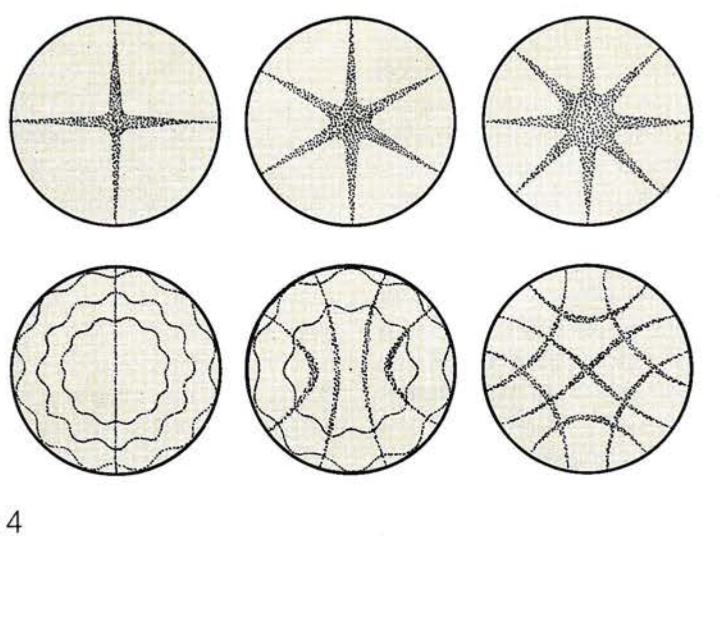


Figure 4

[2]For more on Rutherford and alpha particles, see page 26.—*Ed.*

decreases by Δr. The radial relative deformation $\varepsilon_r = \Delta r/r$ has the sign opposite to that of ε_l. The Poisson coefficient is defined by the equation

$$k = \left| \frac{\varepsilon_r}{\varepsilon_l} \right| .$$

Let's consider the limits within which the Poisson coefficient might change. We assume that the volume of a body under deformation doesn't change. Consequently, we have the relation

$$\pi(r + \Delta r)^2(l + \Delta l) = \pi r^2 l.$$

Opening the parentheses and neglecting the product of the small quantities Δr and Δl, we get

$$r\Delta l + 2l\Delta r = 0,$$

or, put another way,

$$\varepsilon_l = \Delta l/l = -2(\Delta r/r) = -2\varepsilon_r.$$

So the Poisson coefficient in this case is equal to 1/2. But in real life the volume of a body changes under tension, so we have to write the inequality

$$(r + \Delta r)^2(l + \Delta l)^2 > r^2 l,$$

from which we can infer that $k < 1/2$. (The same inequality is also true for noncylindrical bodies.) On the other hand, from its definition it's evident that the Poisson coefficient isn't negative, so we have $0 < k \leq 1/2$.

For different materials the values of the Poisson coefficient can vary quite a bit, within the limits indicated above. Cork, for instance, has a Poisson coefficient very close to zero—that is, the transverse size of a piece of cork changes very little when subjected to stretching (or compression)—as long as the deformation isn't *too* severe! That's why bottles are usually *corked*—a plug made of rubber wouldn't work as well. The Poisson coefficient of rubber is close to 1/2, so that under stress the transverse size of a rubber plug changes considerably—the plug puts up a fight when we try to shove it in the neck of the bottle. To get around this difficulty, rubber plugs are usually made in a conical shape.

We'll leave you with the interesting fact that the Poisson coefficient of the most common building materials—metal, stone, concrete—usually falls between 1/4 and 1/3.

Corrections

Ted Rice, a ninth-grade geometry student in Davenport, Iowa, pointed out an error in the Mathematical Surprises article "Play It Again. . ." (Nov./Dec. 1990). In the section "A Very Mysterious Sequence," the fifth line should read "111221."

Professor Richard Askey of the University of Wisconsin wrote to remind us that several years ago David and Gregory Chudnovsky calculated π to one billion digits (see Kaleidoscope, Jan./Feb. 1991) and that the mathematician mentioned in I. M. Gelfand's talk in the last issue is Hurwitz (not Gourvits).

A strange box and a stubborn Brit

Rutherford's experiments with alpha particles

by M. Digilov

IN THE AUTUMN OF 1903 A thirty-two-year-old professor of physics named Ernest Rutherford sailed from Europe to Canada. He had a small metal box that contained something very precious to him: 30 milligrams of radium salt.

The lead box caused some apprehension among the New York customs officials—there weren't any laws yet concerning the importation of radium. Was it some kind of chemical or was it a precious metal? Was it subject to taxation, and if so, how much should the duty be? Government officials are the same everywhere. The American customs officials decided to send the weird cargo to the higher-ups. But researchers are also the same everywhere. So the report drawn up by the officials on the scene informed those at the top that Dr. Rutherford flat-out refused to part with his treasure. And only the promise made by Dr. Rutherford to transport the box intact through the territory of the United States (that is, not to sell the substance) permitted the Americans to shift the problem onto the shoulders of their Canadian colleagues. It's possible that these few milligrams of radium were responsible for many remarkable scientific discoveries being made.

The name of Rutherford is mentioned in physics textbooks in connection with the planetary model of the atom. But Rutherford made many other discoveries that were as valuable for physics. These include his experiments investigating alpha rays, and that's what we'll be talking about here.

As early as 1899, while working at the Cavendish Laboratory (at Cambridge University), Rutherford found that radiation from radioactive elements is not homogeneous. Here's one of his experiments.

Experiment 1. Two zinc plates were placed horizontally one above the other. The first plate was connected to the pole of a grounded battery, the second to a grounded galvanometer. A thin layer of radium salt was spread on the lower plate. Radiation from the salt formed ions in the air. The air between the plates ceased to act as an insulator, and an electric current appeared that was registered by a special device.

If the layer of radium salt was covered by a thin sheet of metal, part of the radiation was absorbed and the current became weaker by a factor of more than two. If the radiation was screened by two sheets, the current became weaker by a factor of almost six, and if three sheets were used, its value dropped off by a factor of 11. According to the exponential law the current flow should continue to decrease smoothly. But, surprisingly, the experiments didn't corroborate expectations—from the fifth sheet on, there was no appreciable decrease in the value of the current.

It was only natural to assume that ionization of the air is caused by at least two things. Or, to put it differently, that the radiation consists of two types: one that accounts for intense ionization and is well absorbed by the metal, and another whose ionization is weaker but whose penetrability is greater. Rutherford called the first type of radiation alpha particles and the second beta particles. Now the problem for scientists was to study the nature of these particles.

It didn't take much time or effort to find out that beta radiation is a flow of free electrons. At any rate, beta particles behaved exactly like electrons in electric and magnetic fields.

As for alpha particles, their deflection in a magnetic field couldn't be detected for a long time since even a strong field caused only a small deflection. Finally, in 1903 Rutherford achieved positive results and proved that alpha radiation must consist of positively charged particles moving at high speed.

The next task was to determine the value of the alpha particle's charge.

Experiment 2. To determine the charge of a single alpha particle, two things were measured experimentally: (1) the overall quantity of electricity carried by the total radiation of a grain of radium in a unit of time and (2) the quantity of alpha particles emitted by

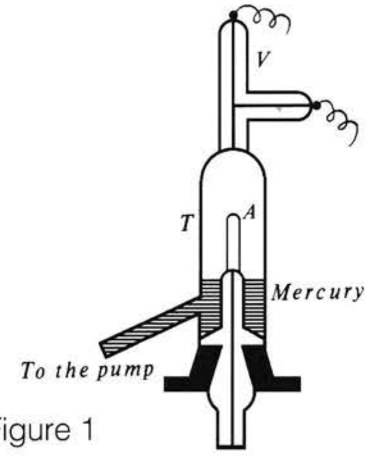


Figure 1

that radium in the same time. Detecting one particle was the most complicated problem. To this end, in 1908 Rutherford and Hans Geiger developed a special method of counting alpha particles, based on their ionizing activity, and also created a special instrument (the "Geiger counter" you've heard about).

A bronze cylinder, 60 cm long, was filled with rarefied air. A thin wire went down the center of the cylinder. The wire was connected to one pole of a battery, while the surface of the cylinder was connected to the other one, and the voltage applied—about 1,000 V—was just about enough to cause an electric discharge. The alpha particle passing through the cylinder ionized the air inside; because of collisions these ions increased the ionization by a factor of about 2,000, which resulted in a sharp increase in the electric current passing through the device.

In order to ensure that only individual particles penetrated the cylinder, the grain of radium was placed in the far end of a narrow glass tube, about 4.5 m long, so that only a small portion of the alpha particles emitted by the radium in various directions would actually reach the cylinder.

Dividing the quantity of electricity that passed through by the number of particles registered by the counter, Rutherford obtained the value of the charge of one alpha particle.

At almost the same time, in 1909, Rutherford proved experimentally that alpha particles are by nature double-ionized helium atoms. He conducted this experiment jointly with his student Thomas D. Royds.

Experiment 3. A sufficiently large quantity of the radioactive gas radon was injected into a glass tube *A* (fig. 1) with walls so thin that most of the alpha particles easily penetrated it. This tube was put inside a wider tube *T*, the top of which was joined to a small vacuum tube *V* with electrodes fused into it. Air was removed from tube *T* and mercury was introduced into the bottom to fill the empty space below tube *A*. The alpha particles accumulating in tube *T* formed gas. Raising the mercury, this gas was compressed, and some of it was transferred to tube *V*. By creating a discharge in the gas there, the researchers could study its spectral composition. It's curious that the first results were obtained in only two days—the yellow line (the brightest in the helium spectrum) became visible first. In six days the whole helium spectrum could be observed.

Finally, it was possible to measure the mass of alpha particles by studying their deviation in a magnetic field.

Experiment 4. A Wilson chamber (a device for observing the tracks of alpha particles by their scintillation) was placed in a very strong magnetic field. As the radius of the circular orbit of an alpha particle was in direct proportion to the particle's mass multiplied by its speed and in inverse proportion to its charge, the mass of the alpha particle could be measured according to known values. It proved to be equal to $6.62 \cdot 10^{-24}$ g.

So thanks to the experiments carried out by Rutherford and his colleagues, the nature, charge, and mass of the alpha particle became known. Not only that, physicists acquired a powerful method of exploring the structure of the atom itself. As Rutherford began sounding the depths of atoms in these experiments, here's how scientists conceptualized the structure of the atom.

The atomic model proposed by Rutherford's teacher Sir Joseph John Thomson in 1882 reminded one of pudding with raisins in it, where the raisins were electrons and the pudding was atomic space itself. The virtue of the model lay in the fact that it permitted one to explain the atom's neutrality and to determine its size quite satisfactorily. At the same time there was a theorem in physics (the Earnshaw theorem) that said the system of charges at rest was unstable. In addition, the nature of a positively charged sphere that spread all over the atomic volume wasn't understood at all.

Bombarding atoms with alpha particles made it possible to determine the structure of the atom.

Experiment 5. Thin plates of the particular substance being examined were subjected to alpha bombardment, and the deviation of the alpha particles was studied. Figure 2 presents a sketch of the alpha-scattering experiment. The bombarding particles, emitted by a radioactive substance, passed through a collimator and fell as a narrow beam onto a target made of very thin gold foil. The alpha scattering was observed by means of a screen coated with a scintillating substance. Scattering angles for most of the particles were small—of the order of 1°; yet a small number of particles scattered at great angles, and some of them even went in the opposite direction.

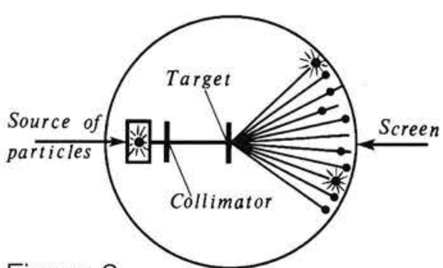


Figure 2

Analyzing the results obtained, Rutherford came to the conclusion that such a strong deviation of alpha particles could only be caused by an extremely strong electric field inside the atom that is caused by a charge linked with a large mass (the nucleus of the atom). Rutherford also worked out the quantitative theory of alpha scattering, which determines the distribution of particles according to the scattering angles. The following fact is of interest in this regard.

To gain a solid understanding of the probabilistic processes occurring when an alpha particle passes through a substance, Rutherford—the world-famous scientist, the Nobel prize winner—wished to become a student for a while. He went to a well-known mathematician named Lamb in Manchester and asked his permission to attend his course of lectures on probability theory. He also wanted to do all the homework involved in the course. As his contemporaries wrote, "It was not a trivial sight: a world

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Some mathematical magic

Identical sums in squares, hexagons, and hypercubes

by John Conway

ONE OF THE OLDEST AND most familiar mathematical recreations is designing magic squares. The problem is to arrange the numbers from 1 to n^2 in an $n \times n$ array so that all the rows and columns, as well as the two diagonals, have the same sum (the magic sum).

First, let's ask: what is the magic sum? The average of all the numbers from 1 to n^2 is the same as the average of the first and last of them—namely, $1/2(n^2 + 1)$. So the average of the row-sums is n times this—namely,

$$\frac{n(n^2+1)}{2}.$$

For $n = 3$, this is 15. There is really only one order 3 magic square, discovered by the Chinese many centuries ago and traditionally called the Lo-Shu. One story is that it was observed written on the shell of an enormous turtle that was found in the Yellow River.

Here it is:

8	1	6
3	5	7
4	9	2

This is in fact one of eight squares:

8	1	6
3	5	7
4	9	2

6	7	2
1	5	9
8	3	4

2	9	4
7	5	3
6	1	8

4	3	8
9	5	1
2	7	6

8	3	4
1	5	9
6	7	2

4	9	2
3	5	7
8	1	6

2	7	6
9	5	1
4	3	8

6	1	8
7	5	3
2	9	4

But since these are obtained by rotating and reflecting the first one, we usually count them as just one square.

I chose the starting orientation for the Lo-Shu in order to illustrate de la Loubere's rule for writing down magic squares of any odd order at sight. De la Loubere was the first French ambassador to Siam, and this rule was one of many interesting things he learned there.

It will be easier to understand this rule if we regard the square as "wrapped around," as in many video games, so that a step off its top edge takes us to the bottom row, while a step from the rightmost column takes us onto the leftmost one. Then de la Loubere's rule is that we write the numbers in order, starting from the middle of the top row and marching diagonally up and right when this leads to an empty square, otherwise dropping straight down one cell.

Let's see how this leads to the Lo-Shu. The first step upward from 1 takes us off the top edge onto the bottom one by the "wraparound rule":

(2)

	1	
		2

and then the step upward from 2 to 3 takes us off the right edge and onto the left one:

	1		
3			(3)
		2	

But now a step diagonally upward from 3 leads us to 1, so instead we drop straight down from 3 (*not* from 1!) and continue:

	1	6
3	5	
4		2

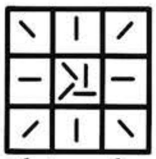
and the square completes itself readily after the second drop, from 6 to 7:

8	1	6
3	5	7
4	9	2

With a little practice it becomes easy to write these squares down. Here is the de la Loubere square of order 5:

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

It's easy to give a general formula for all order 3 magic squares, even when the numbers used may not be just 1 to 9. The first step is to prove that the magic sum S is just 3 times the middle number m. There are 4 lines through the center, so if we add them all up we get $4S$ for the sum of all the numbers shown here:



We put a mark in each cell when we counted the number in it—you see that the middle cell is marked 4 times, and so the sum we get is the sum of all entries—that is, $3S$—plus three times the middle one. So

$$4S = 3S + 3m,$$

from which we get

$$S = 3m.$$

Now the numbers on any line through the center must have the form

$$m + x \qquad m \qquad m - x$$

since they add up to $3m$. So if we suppose the top two corners are $m + a$ and $m + b$, the bottom corners will be $m - b$ and $m - a$:

$m+a$	?	$m+b$
?	m	?
$m-b$	?	$m-a$

and now, since each border line sums to $3m$, the square easily completes itself:

$m+a$	$m-a-b$	$m+b$
$m-a+b$	m	$m+a-b$
$m-b$	$m+a+b$	$m-a$

The general order 3 magic square

The proof that the Lo-Shu is unique is now pretty easy. The magic sum must be one third of $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$, which is 15, and so the middle number m must be 5. We are discounting rotations and reflections, so we can suppose that a and b are positive, since changing the sign of either just reflects the square in a diagonal, and that a is larger than b, since interchanging them reflects the square left-to-right. Now the smallest number that appears is $m - a - b$, so this must be 1, and since $m = 5$, we must have $a + b = 4$, from which we get $a = 3$ and $b = 1$, and everything is known.

In 1693 Frenicle de Bessy wrote out all the order 4 magic squares. He found that, if rotations and reflections are discounted, there are precisely 880 of them. Here's an easy way to get most of them. There are lots of ways to arrange the numbers from 1 to 16 as an addition table:

$A+a$	$A+b$	$A+c$	$A+d$
$B+a$	$B+b$	$B+c$	$B+d$
$C+a$	$C+b$	$C+c$	$C+d$
$D+a$	$D+b$	$D+c$	$D+d$

and whenever we've done that we can rearrange them to make a magic square thus:

$A+a$	$C+d$	$D+b$	$B+c$
$D+c$	$B+b$	$A+d$	$C+a$
$B+d$	$D+a$	$C+c$	$A+b$
$C+b$	$A+c$	$B+a$	$D+d$

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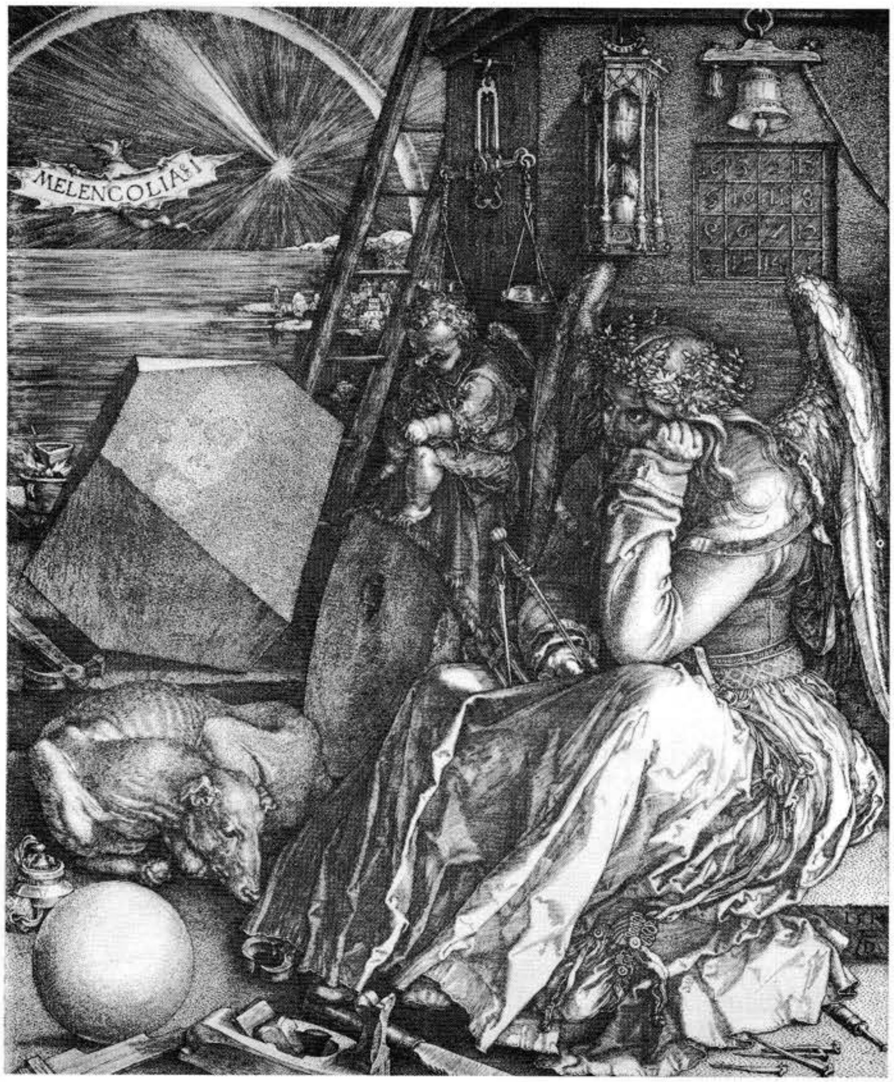


Melencolia I (1514) by Albrecht Dürer, National Gallery of Art, Washington DC (©NGA, Rosenwald Collection)

Challenges in physics and math

Math

M26

Accident on a lake. Boat 1 and boat 2 depart at the same time from docks A and B, respectively, on the bank of a round lake. If they made their way straight to docks C and D, respectively, they'd collide. Prove that if boat 1 goes to D and boat 2 goes to C, they'll arrive simultaneously. (N. Vasilyev)

M27

Prime factors and difference of squares. An odd number is a product of n different primes. Prove that there are exactly 2^{n-1} distinct ways to represent this number as a difference of squares of positive integers. (O. Goncharik, S. Sergey)

M28

Points that paint. A circle of circumference 1 rolls along the outside of a fixed circle of circumference $2^{1/2}$. Initially their point of contact is marked by a dot of sticky red paint. When the circle rolls, new spots are painted on both circles (fig. 1). How many red points will be painted on the fixed circle by the end of the 100th revolution of the rolling circle around the fixed one? (D. Bernshtein)

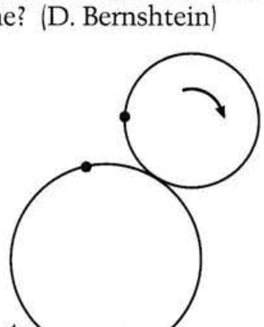


Figure 1

M29

Bulky polyhedron in a cube. A cube contains a convex polyhedron whose projection onto any of the cube's faces covers the entire face. Show that the volume of the polyhedron is not less than 1/3 that of the cube. (V. Prasolov)

M30

Traveling king. A chess king made the rounds of all the squares on the chessboard, visiting every square once. (A king can move to any neighboring square.) The center of each square was joined to the center of the next square on the king's route (the last center was joined to the first one). The closed broken line thus created has no self-intersections. What are the smallest and the greatest lengths of this line if the side of a square is 1 unit long? (A. Klimov)

Physics

P26

Thrown ball. A ball is thrown upward. Which is longer: the time it takes to go up or the time it takes to come down?

P27

Weight on a spring. A weight of mass m is placed on a weight of mass M suspended on a spring (fig. 2). At first, weight M is supported in its original position; then the weights are released. Find the maximum force exerted by weight M on weight m. (P. I. Zubkov)

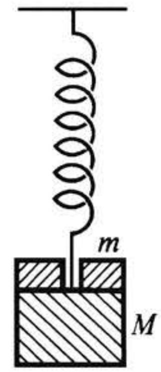


Figure 2

P28

Boiling water. A test tube filled with water is immersed in a retort in which water is boiling. Will the water in the test tube boil? What will we see if toluene is poured on the water? (Toluene is a lighter liquid that doesn't mix with water and has a boiling temperature of 111°C.) (A. Buzdin)

P29

Lamp connections. A lamp designed for voltage 2.5 V and current 0.2 A is connected by long wires to a battery. An ammeter, connected in series with the lamp, gives a reading of $I_1 = 0.2$ A. When the lamp is connected in parallel with the ammeter, it burns exactly as in the first case. What was the reading on the ammeter then? The battery is assumed to be ideal, and the resistance of the wires is 2 ohms. (A. R. Zilberman)

P30

Circular light rays. The refractive index of a certain planet's atmosphere decreases with altitude over the planet's surface according to the formula $n = n_0 - \alpha h$, where h is the altitude above the planet's surface. The radius of the planet is R. Find the altitude h at which light rays can circle the planet, staying at a constant altitude. (N. Sedov)

ANSWERS, HINTS, AND SOLUTIONS ON PAGE 58

KALEIDOSCOPE

Electromagnetic induction

The intertwined lives of electricity and magnetism

MICHAEL FARADAY DEVOTED his entire life to proving that no electric or magnetic process in nature occurs in isolation. He firmly believed that all natural forces are interconnected, and after many years this belief finally led him to a brilliant discovery.

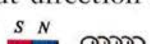
This new effect, as often happens, was then found in many apparently diverse phenomena, which were linked by a single qualitative conclusion: varying magnetic fields cause electric fields. This is the basic operating principle of all existing electrical machines. Faraday's discovery made it possible to transform mechanical energy into electrical energy and to transmit this energy over great distances, laying the foundation of modern technological civilization.

The work of Faraday and his brilliant contemporaries led gradually to the creation of a coherent view of electromagnetism.

Mastering this part of physics will not only enable you to explain facts you already know, it will provide you with a tool for making sense of electromagnetic phenomena on the cosmic and microscopic scales.

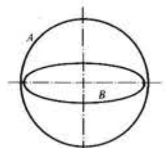
Questions and problems

1. In what direction should you move a mag-



9. The rotor of a working electric motor experiences a braking force. What is the nature of this force?

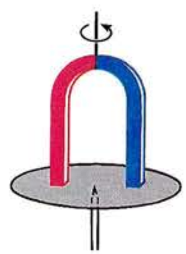
10. Two circular wires are positioned at right angles to each other. Will a current be induced in wire A if we change the current in wire B?



11. A magnetic flux Φ passes through a superconducting ring placed near a constant magnet. No current flows through the ring. Find the magnetic flux through the ring if the magnet is removed.

Microexperiment

Suspend a horseshoe magnet on a string over a disk made of aluminum foil that can rotate freely about its central axis. If you rotate the magnet, the disk will also rotate. In which direction? Why?



It's interesting that . . .

. . . some modern types of electrical machines have no mechanical moving parts. In the so-called MHD (magnetohydrodynamic) generator, plasma—produced by the combustion of oil or natural gas—moves between the poles of the magnet instead of a wire. The carriers of charge in the plasma are deflected by the magnetic field to the electrodes, and current arises in the external circuit.

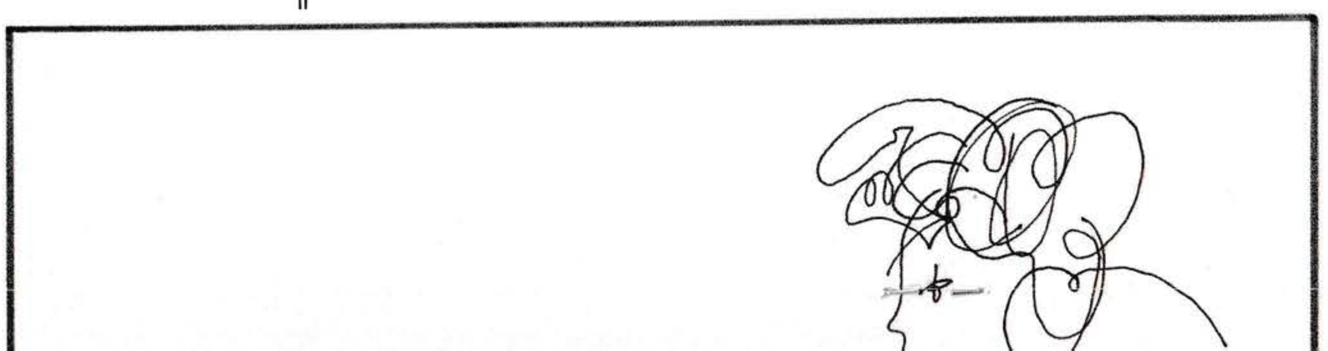
. . . for many years Faraday would carry a small magnet and a wire coil in his vest pocket as a constant reminder of the unsolved problem of how a current is generated by a magnetic field.

. . . eddy currents (or Foucault currents), like friction, can be useful as well as harmful. Here are but three examples: induction furnaces for heating and even melting metals, "magnetic damping" in measuring instruments and circular saws, and the familiar electric meter.

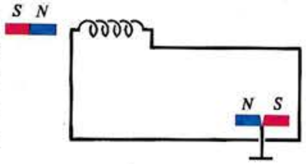
. . . when Faraday arrived at the idea of electromagnetic rotation, he induced continuous rotation of a magnet about a current-carrying wire by means of a mercury contact. This first electric motor started operating in December 1821.

. . . Lenz's law, which determines the direction of an induced current, was formulated almost immediately after Faraday's discovery—in 1833. Nowadays you can see a striking manifestation of this in the school lab by putting a superconducting ceramic disk on a magnet: the disk will hover over it.

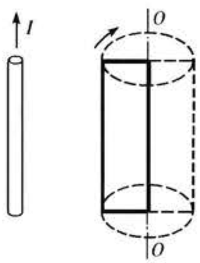
ANSWERS, HINTS, AND SOLUTIONS ON PAGE 61



1. In what direction should you move a magnet to turn the north pole of the magnetic needle in your direction?



2. You are looking down at a round frame placed horizontally in a magnetic field directed upward. The field decreases with time. Find the direction of the induced current.



3. A frame rotates with constant angular velocity about a straight wire through which a current flows. At what position of the frame will the electromotive force in it be the greatest? the least?

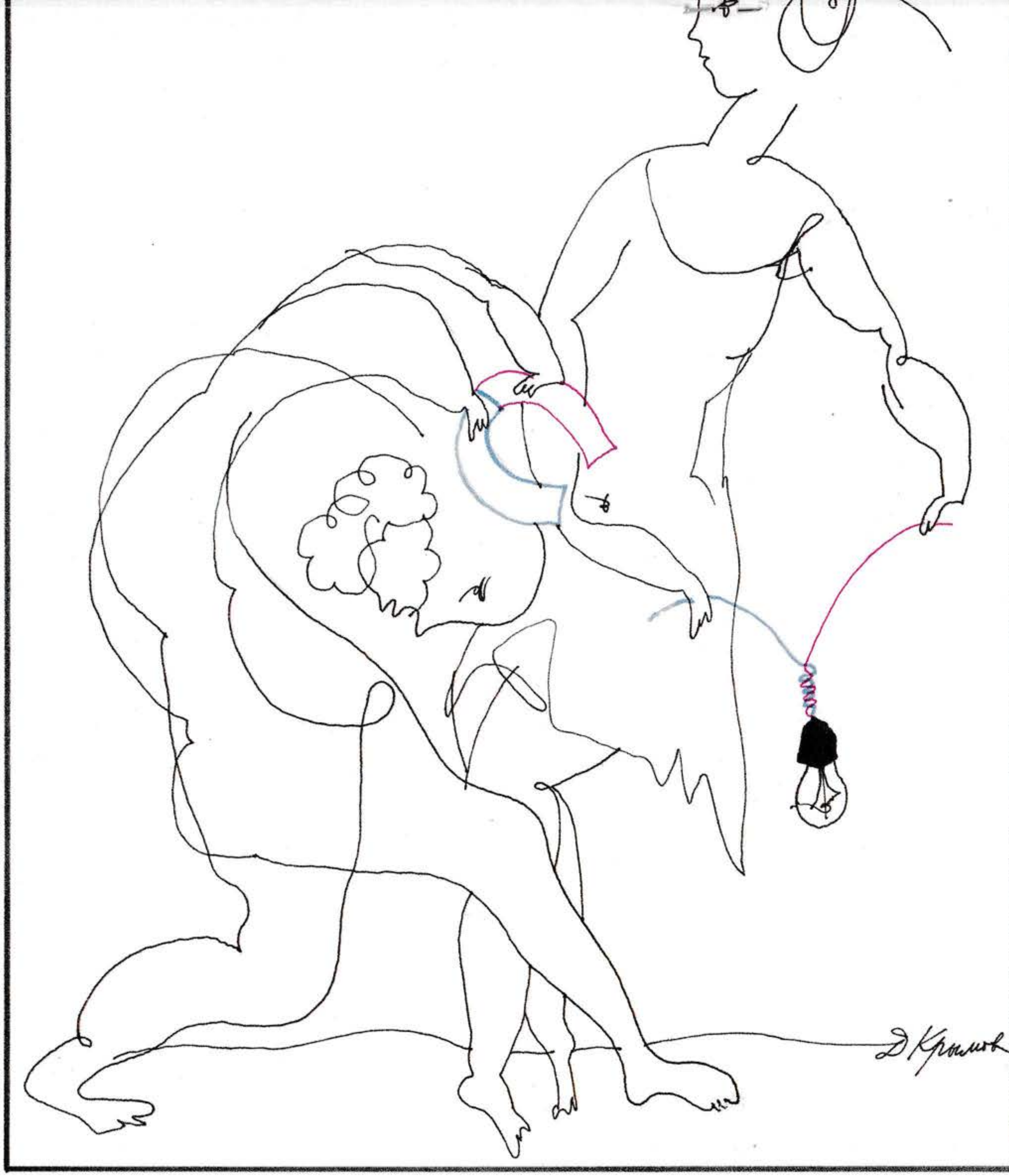
4. What happens when a magnet falls down a long copper tube? (Ignore any air resistance.)

5. A wire is folded in half and its ends are connected to a galvanometer. Why does the instrument's needle show zero when the wire crosses the induction lines of a magnetic field?

6. A coin is placed atop a vertical coil. Why does the coin heat up when an alternating current flows through the coil, but stays cool when direct current passes through it?

7. A high-frequency current flows through a straight wire. How does the resistance of the wire change if the wire is wound into a solenoid?

8. Two aircraft of the same design are flying in a horizontal direction at the same speed. One of them is flying near the pole and another along the equator. Which of them has the greatest drop in potential between the tips of its wings?



Art by Dmitry Krymov

Two physics tricks

Reluctant water turns eager

by V. Mayer and E. Mamayeva

TAKE A GLASS TUBE THAT tapers at the end (like a pipette) and show it to your audience. With your other hand, carefully pick up a glass of water heated to 80–90°C and show it to the spectators. Put the end of the tube into the glass and wait until some water gets into the tube. Then close the tube's other end with your finger and take it out of the glass (fig. 1).

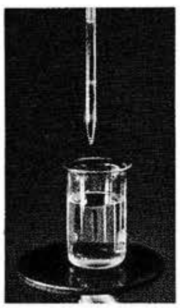


Figure 1

The spectators now see that there are small air bubbles near the lower end of the tube. They expand, separate from the walls, and climb up the tube. But the water doesn't pour out of the tube!

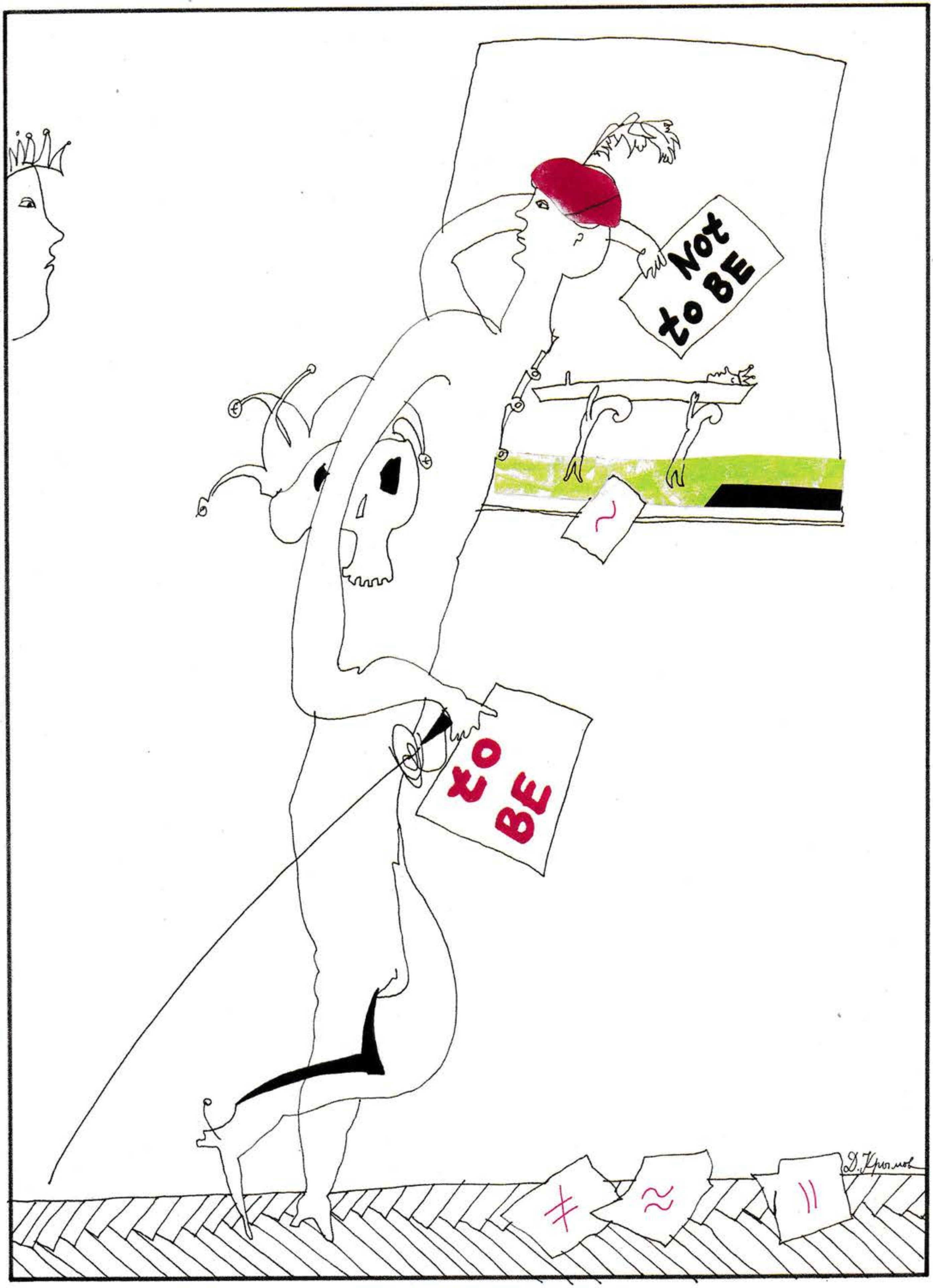
Figure 2

Opening the top end of the tube, pour the water back into the glass, wave the empty tube slowly in front of your audience, and again draw some water from the glass into the tube. Closing the top end of the tube with your finger, quickly take the tube out of the glass and turn it over (fig. 2)—a forceful fountain of water more than a meter high bursts out of the tube.

It's almost certain that no one will be able to discover the secret of your tricks. The explanation is really quite simple, though: the temperature of the water in the glass is 80–90°C, while the temperature of the water in the tube is about 20°C (room temperature). Try to explain the first trick yourself, and we'll help you understand the second.

When the hot water from the glass enters the tube, the air in the top part of the tube stays practically at room temperature (because of the poor thermal conductivity of air). After you close the top end with your finger and turn the tube over, the hot water starts flowing down the walls, heating the air in the tube very quickly. The pressure inside increases and the expanding air expels the water (which had no time to drop down) out of the tube.

We recommend that you use a glass tube about 8–12 mm in diameter and 30–40 cm long, narrowing to about 1 mm at the lower end. During the time between the tricks, be sure to cool the tube well (you might even blow into it), because the height of the fountain depends on the difference between the temperatures of the air and water. The optimal amount of water taken into the tube ranges from 1/4 to 1/3 of its volume—you'll have no trouble finding the best ratio by trial and error.



Art by Dmitry Krylov

Think fast!

The art of estimating

by G. V. Meledin

OUR LIFE IS FULL OF EVALUATIONS and rough estimates of all sorts. Will I get there in time? Do I have enough money with me? Am I strong enough to carry this load?

In science the ability to make correct evaluations is a professional requirement. They're indispensable in planning new projects and programs. A rough estimate—an evaluation of the order of magnitude—is a necessary stage in designing an experiment, building an installation, or working something out theoretically. Sometimes an evaluation suggests a path to an exact solution and gives the range of values for which the solution may be valid. You can also estimate how the problem should be modified if a solution is required that's outside this range.

Along with intuition, the ability to make evaluations is quite important in creative work.

The physical setting of a problem—the choice and development of the simplest physical model—is the most important and most difficult stage. You have to select the parameters that are crucial to the problem and neglect those of minor importance. The correct use of physical laws and definitions is vital. Occasionally the rough version of a definition or the qualitative interpretation of a physical law is sufficient.

Two comments before we look at some problems. First, let's agree on what we mean by "order of magnitude." Two numerical values are said to differ by an order of magnitude if their ratio is approximately equal to 10; if it's approximately equal to 10^2, the values are said to differ by two orders of magnitude; and so on. From this point of view the number 89 is considered to be of the same order of magnitude as 10^2, and the number 15 is of the order of 10. If the ratio of two numbers is, for instance, equal to 1.3, they are said to have the same order of magnitude. The same is true when this ratio is equal to 2.3 or even 5. For rough estimates these errors aren't important.

Second, let's establish what the notation means. The sign "=" means an exact equality, while "≅" denotes an approximate one. We'll also use the symbol "~." Its traditional meaning is that the values on each side of it are proportional. Here it will mean that the values are equal by their order of magnitude, which underlines the fact that the dimensionless proportionality factors in our formulas have an order of magnitude of one. I'd like to emphasize that if a "true" factor is several times the "estimated" one (or vice versa), the difference isn't considered important for our purposes.

Now let's look at a few relatively simple problems. We'll start with ones whose physics is absolutely clear so we merely have to make a reasonable choice of parameters.

Problem 1. Evaluate the pressure of a ballpoint pen on paper when someone writes with it.

To make this evaluation, we make direct use of the definition of pressure: $p = F/S$. What numerical values should be substituted for the force and the area? A line drawn by a ballpoint pen consists of a series of individual points. A point can be considered a ring with diameter d equal to the width of the track made by the pen on the paper: $S = \pi d^2/4$. Let $d \sim 0.2$ mm (which is likely enough). The force F applied to the pen can also be roughly evaluated: it does not exceed the weight of the hand but is greater than the pen's weight. Let $F \sim 1$ N. Then

$$p = \frac{F}{S} \sim \frac{4F}{\pi d^2} \sim 3 \cdot 10^7\,\text{Pa}.$$

To get a sense of whether this is a lot or a little, let's make a comparison with a 1-kg weight sitting on top of a table. Its diameter is about 4 cm, so the pressure on the table is of the order of $8 \cdot 10^3$ Pa. This means the pressure of a ballpoint pen is several thousand times that of a 1-kg weight.

Art by Dmitry Krymov

Problem 2. Evaluate the velocity of steam coming out of the spout of a kettle of boiling water.

Denote the power of the heating element by W and the specific latent heat of vaporization of water by L. Let η be the proportion of power spent on creating the vapor. Then $\eta W/L$ is the mass of steam created per

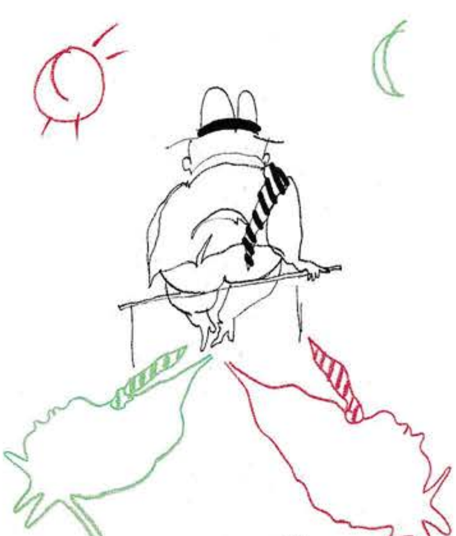
unit of time. Obviously the amount of steam leaving the kettle is the same as that produced by the heater:

$$\frac{\eta W}{L} = \rho v S.$$

Here ρ is vapor density at the boiling point, v is the velocity of outgoing steam, and S is the cross section of the kettle's spout. According to Clapeyron's law the vapor density $\rho = p\mu/RT$, where p is the pressure, m is the molar mass of water, R is the universal gas constant, and T is the temperature of the steam. Finally, we get

$$v = \frac{\eta W}{L\rho S} = \frac{\eta WRT}{Lp\mu S}.$$

If the power of the heater $W \sim 1$ kW, $\eta \sim 0.5$, $S \sim 1\ \text{cm}^2$, $T \sim 373$ K, $p \sim 10^5$ Pa (since the pressure of saturated vapor at the boiling point is equal to the atmospheric pressure), and the constants are equal to $R = 8.3$ J/(kg · K), $L = 4.2$ kJ/kg, and $\mu = 18 \cdot 10^{-3}$ kg/mole,



we obtain

$$v \sim 2\ \text{m/s}.$$

Problem 3. Evaluate how much it is brighter on a sunny day than at night with a full Moon shining.

The Moon doesn't emit any light itself but only reflects that coming to it from the Sun. We'll assume that the illumination of the Earth and the Moon by the Sun is approximately the same and denote it by E_S. The power of the light falling on the Moon's surface is $E_S\pi R^2$ (where R is the radius of the Moon). Some of the light reflected by the Moon falls on the Earth, creating the following amount of illumination:

$$E_M = \frac{E_S \pi R^2 k}{2\pi l^2},$$

where k is the coefficient of reflection of the lunar surface and l is the distance between the Earth and the Moon (we'll assume that the Moon reflects the incident light homogeneously within the solid angle equal to half of the maximum one). Now we can compute the ratio

$$\frac{E_S}{E_M} = \frac{2}{k}\left(\frac{l}{R}\right)^2 = \frac{8}{k}\left(\frac{2R}{l}\right)^{-2},$$

where $2R/l$ is the angular size of the Moon (of the order of 0.01 rad). Finally, taking $k \sim 0.2$, we get

$$\frac{E_S}{E_M} \sim 4 \cdot 10^5.$$

Problem 4. Evaluate how much further a grenade travels if an athlete throws it on the run.

Let's assume that during its flight the grenade reaches the height H. Then its flight time is

$$2\sqrt{\frac{2H}{g}}.$$

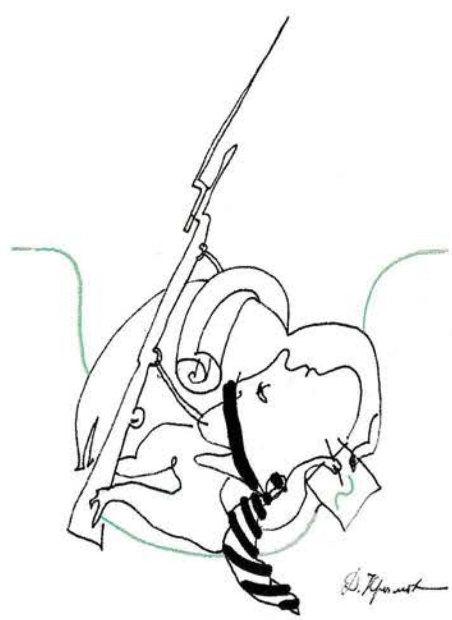
If the horizontal projection of the grenade's velocity at the moment of release is increased by v and the vertical projection remains practically the same, the flight time doesn't change but the distance to the landing point is increased by

$$l = 2v\sqrt{\frac{2H}{g}}.$$

It's reasonable to assume that $H \sim 5$ m and $v \sim 8$ m/s (you'll recall that a good athlete runs the 100-meter dash in about 10 to 12 seconds). And so

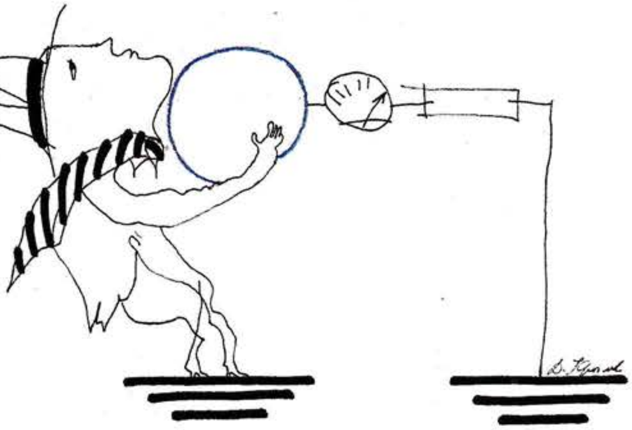
$$l \sim 20\ \text{m}.$$

This value seems reasonable enough.



Problem 5. Estimate the discharge time of a charged metal sphere connected to the ground through a resistor with a known resistance.

Denote the potential of the charged sphere by φ and its charge by $Q = C\varphi$, where $C = 4\pi\varepsilon_0 a$ is the capacity of the sphere (a is the sphere's radius). After the sphere is connected to the ground, a current I starts to pass through the circuit until its potential and charge



become equal to zero. The current depends on time, but we'll ignore this. We then get

$$I \sim Q/t \sim \varphi/R,$$

where R is the resistance and t is the discharge time. From this we get

$$t \sim QR/\varphi = CR = 4\pi\varepsilon_0 aR.$$

For $a \sim 1$ m, $R \sim 1$ megohm, and $\varepsilon_0 = 8.85 \cdot 10^{-12}$ F/m, we have

$$t \sim 10^{-4}\ \text{s}.$$

The book *Amusing Physics* by Y. I. Perelman includes a story entitled "Out of the Water Dry." It begins with the following problem: "Place a coin on a big dinner plate, cover it with a thin layer of water, and ask your friends to extract it from under the water without getting their fingers wet.

"This apparently insoluble problem can easily be solved by means of an empty glass and a burning piece of paper. Set fire to the paper, place it inside the glass, and quickly put the glass upside down on the plate near the coin. The paper stops burning, the glass fills with white smoke, and the water gets sucked inside the glass. The coin stays where it was, and after it dries you can pick it up without getting your fingers wet."

Let's look at a problem related to this story.

Problem 6. Evaluate the minimum temperature to which the glass should be heated so that all the water on the plate is sucked into the glass.

Let's first solve the problem exactly (making certain assumptions, of course, which will be specified later on). Before the glass comes in contact with the water, the pressure inside is equal to the atmospheric pressure p_a. Denote the unknown temperature of the glass by T_x. After the glass cools down and sucks in an amount of water of mass m, the pressure inside it will be p, its temperature will be equal to the room temperature T, and the volume of the air inside the glass will decrease by the volume of the water—that is, it becomes equal to $Sl - m/\rho$, where S and l are the cross section and height of the glass, respectively, and ρ is the density of water. By Clapeyron's law we can write

$$\frac{p_a Sl}{T_x} = \frac{pS\left(\frac{l-m}{\rho S}\right)}{T}.$$

The equilibrium condition for the uplifted column of water yields $pS + mg = p_a S$. So

$$T_x = T\frac{1}{1-\frac{mg}{p_a S}}\cdot\frac{1}{1-\frac{m}{\rho l S}}.$$

Now we can specify the indirect assumptions used in solving the problem. We assumed that the temperature of the air inside the glass was the same as that of the sides of the glass. We also assumed that the glass was being put into the water gradually, so that the initial inner pressure immediately after the contact was equal to the atmospheric pressure. We also assumed we could ignore the pressure of the water vapor inside the glass and that the capillary effects are negligibly small.

It's interesting that the expression for T_x includes the change in the pressure (because of water flow inside the glass) and the change in the volume of air as independent factors. In that case it's a good idea to analyze their effects separately.

The first term can be transformed in the following way:

$$\frac{1}{1-\frac{mg}{p_a S}} = \frac{1}{1-\frac{p_w}{p_a}} \cong 1 + \frac{p_w}{p_a}.$$

The atmospheric pressure corresponds to the pressure of a column of water 10 meters high. The height of the water inside the glass cannot exceed the size of the glass—that is, about 10 cm. So we can ignore the pressure of the water and take this factor to be approximately equal to 1:

$$\frac{1}{1-\frac{mg}{p_a S}} \cong 1.$$

The second term describes the change in the volume of air inside the glass. The difference between the volume of water and that of the glass isn't as great as the pressure difference, which means that the volume of water mustn't be neglected. Thus,

$$T_x \cong \frac{T}{1-\frac{m}{\rho l S}}.$$

Taking the numerical parameters $T \sim 300$ K, $m \sim 30$ g, $l \sim 10$ cm, and $S \sim 20$ cm^2 (since the volume of the glass is about 200 cm^3 and its height $l \sim 10$ cm), we have

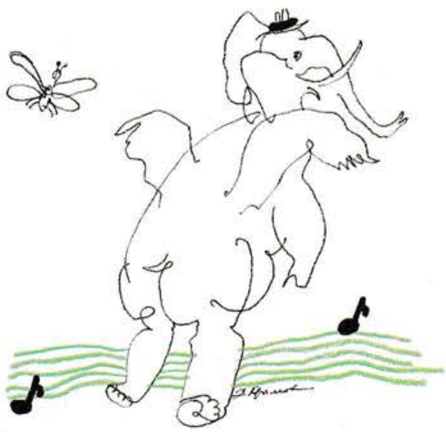
$$T_x \sim 353\ \text{K},$$

or

$$t \sim 80^\circ\text{C}.$$

The correction needed to take the pressure change into account is $\Delta T_x \cong T_x(mg/p_a S) \sim 0.1$ K, which is tiny compared to the value of T_x. It's funny that we've virtually neglected the change in air pressure during cooling—the very phenomenon responsible for the statement of the problem.

Sometimes a more complicated situation calls for a more careful approach to the evaluation. Here's a good example.



Problem 7. Evaluate the frequency of the sound generated by a flying mosquito.

It's natural to assume that the sound is generated by the periodic flapping of the mosquito's wings. In fact, the physics of mosquito flight isn't so simple. We will, however, use an extremely crude model, assuming that flapping generates lift that compensates for the pull of gravity. The lift is provided by the change in momentum transferred to the air under the wings per unit of time:

$$\Delta P/\Delta t = mg.$$

We'll denote the area of the wings by S and their velocity by v. Moving downward, the wings push down a mass of air $\Delta m = \rho_a v \Delta t S$ over the time interval Δt. This is accompanied by the momentum transfer $\Delta P = \Delta m v = \rho_a v^2 \Delta t S$. The resulting upward lift F is

$$F \sim \Delta P/\Delta t \sim \rho_a v^2 S,$$

where ρ_a is the density of air. The length of a mosquito is, say, $l \sim 4$ mm. The area of the pair of wings is $S \sim l^2$ (we assume that the length of a mosquito's body is of the same order of magnitude as its wingspan). It's reasonable to take $(1/10)l^3$ as the mosquito's volume, since a mosquito's width and height are considerably less than its length. We'll take the density of the insect's body to be equal to the density of water ρ_w. Denoting the flapping frequency by ν, we have $v \sim l\nu$. The equilibrium condition gives us

$$F \sim \rho_a v^2 l^2 \sim \rho_a \nu^2 l^4 = mg \sim \rho_w l^3 g/10,$$

which leads to

$$\nu_s \sim \nu \sim \sqrt{\frac{\rho_w}{10\rho_a} \cdot \frac{g}{l}} \sim 400\,\text{Hz}.$$

This result gives a reasonable order of magnitude, although we can't be very certain of the digit "4."

This formula predicts that the frequency changes as the inverse square root of l. In other words, the bigger the insect, the lower the sound it generates. Indeed, comparing the buzz of a bumblebee with the high pitch of a mosquito shows that this prediction is justified.

Quite often evaluations are made by using the dimensionality technique. This method is based on the assumption that the parameters appearing in a problem combine as factors in the final result. Of course, this approach can't give the numerical values of the factors. Occasionally, they can be estimated by considering a special case, but more often they're assumed to be equal to 1. This is reasonable if we're interested only in the order of magnitude of the result. Let's look at an example.

Problem 8. Evaluate the time it takes for the sound of thunder to reach an observer who saw a tree get hit by lightning 3 kilometers away.

The speed of light is approximately $3 \cdot 10^5$ km/s, so the observer will see the flash in $\sim 10^{-5}$ s. The speed of sound in the air is much lower. Let's try to estimate it by means of the dimensionality technique.

The speed of sound v depends on the parameters that characterize the medium in which it propagates. For air let them be pressure p and density ρ. We'll assume that

$$v \sim p^x \rho^y,$$

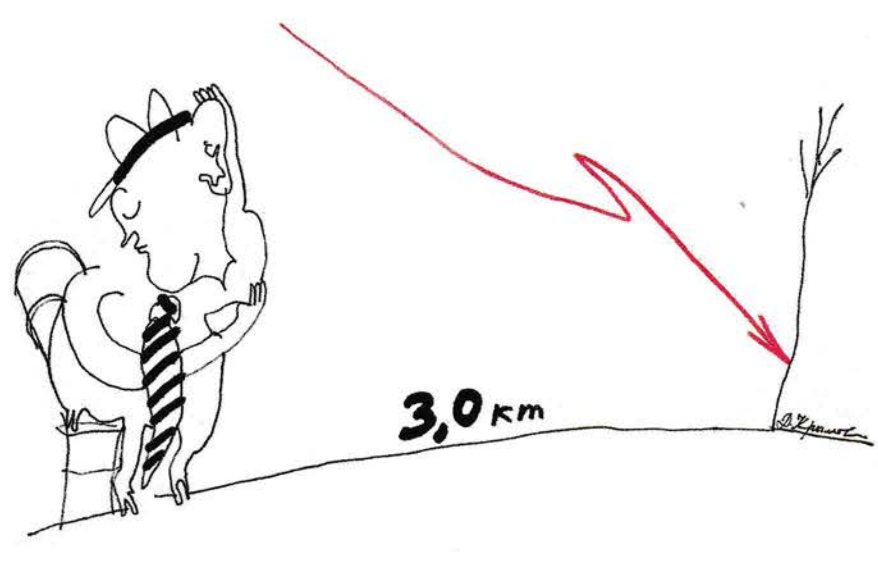
where x and y are as yet unknown. If such a relation does indeed exists, the dimensionalities of its left and right sides must be the same.

Let's agree to denote the dimensionality of A by $[A]$. Then

$$[v] = \text{m} \cdot \text{s}^{-1}$$
$$[p] = \text{Pa} = \text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2},$$
$$[\rho] = \text{kg} \cdot \text{m}^{-3},$$

and we can write

$$\text{m} \cdot \text{s}^{-1} = (\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2})^x (\text{kg} \cdot \text{m}^{-3})^y.$$



This equality holds under the condition

$$x + y = 0,$$
$$-x - 3y = 1,$$
$$-2x = -1.$$

From this we get

$$x = \frac{1}{2},$$

$$y = -\frac{1}{2},$$

which leads to

$$v \sim \sqrt{\frac{p}{\rho}}.$$

The numerical factor in this formula can't be obtained by the dimensionality technique. We'll assume that it's of the order 1 (generally speaking, this would have to be verified somehow).

To estimate the speed of sound we have to substitute numerical values for the atmospheric pressure ($p \sim 1$ bar $\sim 10^5$ Pa) and the density of the air ($\rho \sim 1.3$ kg/m^3 under normal conditions). Then the speed of sound is

$$v \sim \sqrt{\frac{p}{\rho}} \sim 300\,\text{m/s},$$

so that the time it takes for the observer to hear the thunder is

$$t \sim \frac{3 \cdot 10^3\,\text{m}}{300\,\text{m/s}} \sim 10\,\text{s}.$$

This is six orders of magnitude greater than the time it took the light to reach the observer, and our experience tells us that it's quite reasonable.

I'll leave you with some estimation problems to figure out on your own. Try to resist the temptation to peek at the answers!

Problems to ponder

1. Evaluate the change in the atmospheric pressure if all the water in the oceans evaporated.

2. Estimate the rate of descent of a parachutist with an open parachute.

3. Evaluate the mean density of the Sun.

4. Estimate how many revolutions (somersaults) an automobile make as it plummets freely at full speed into a precipice 1 km deep.

5. Evaluate the pushing force of an athlete putting (throwing) the shot.

6. Evaluate the tension of a bicycle chain during uphill peddling.

7. Evaluate the velocity of a drop of water if, upon hitting a stationary wall, the pressure of the drop on the wall at impact is about 10 Pa.

8. Estimate the tension of a car's safety belt if the car crashes into a pillar at a speed of 30 km/h, producing a dent 30 cm deep.

9. Estimate the distance at which a person wearing bright clothes vanishes from sight in a pine woods. (There is no underbrush.)

ANSWERS, HINTS, AND SOLUTIONS ON PAGE 61

LATIN RECTANGLES
FROM PAGE 20

to assume "only" that

$$m < \sqrt[3]{n}.$$

I've put "only" in quotes because mathematicians would much prefer to find, if not exact, then at least asymptotic formulas with no restrictions on the relation between the length and width of the Latin rectangle in order to approach Latin squares, which are now used in the theory of experiment planning. But a solution to this problem remains a challenge for future mathematicians.

The Moscow Correspondence School in Quantum

Directed by Professor I. M. Gelfand
with the assistance of M. Berkinblit, E. Glagoleva, N. Konstantinov, V. Minachin, and V. Retakh

A Word of Introduction

by I. M. Gelfand

TWENTY-FIVE YEARS AGO I organized an unusual mathematics school by correspondence in the Soviet Union, and I still continue to direct it.

I'd like to tell you a little bit about this school. The Soviet Union, as you surely realize, is a large country, and there are simply not enough good teachers throughout the country who can show all the students how wonderful, how simple, and how beautiful the subject of mathematics is. The fact is that everywhere, in every country, and in each part of a country there are students interested in mathematics. Realizing this, we organized the Mathematics Correspondence School so that students from 12 to 17 years of age from any place could study. Since the number of students we could take in had to be restricted to about one thousand, we chose to enroll those who lived outside of such big cities as Moscow, Leningrad, and Kiev, and who resided in small cities and villages in remote areas. The books were written expressly for them. They, in turn, read them, did the problems, and sent us their solutions. We never graded their work—it was forbidden by our rules. If anyone was unable to solve a problem, some personal help was given so that he or she could then complete the work.

Of course, it wasn't our intention that all the students who studied with these books or even completed the school should choose mathematics as their future career. Nevertheless, no matter what they later chose, the results of this training remained with them. For many, this had been their first experience in being able to do something on their own—completely independently.

The project proved so fruitful that we extended it and opened the biology and linguistics branches, since we do not live by mathematics alone.

The Correspondence School is now publishing its books in the United States, and with this article we are opening a new department in *Quantum*, Math by Mail. Its aim is to give you a feel for the things we do in the school and to offer advice to those who would want to study on their own. We'll start by presenting some material used in the school. These columns are not directed only to those students who will later become mathematicians or embark upon a related career but to all who want to study mathematics with the help of our books, our advice, and our school. They are in no way intended to provide special mathematical training for its own sake. Rather, we consider mathematics to be an important part of human culture.

With this goal in mind we have written several books and will continue writing them. Two books are being translated into English and will be published by the Birkhäuser publishing house. No additional knowledge is required, but sometimes familiarity with parts of the high school math curriculum is necessary.

One more remark. How were these books written? The things that are now taught in high school mathematics courses were in their time great achievements of the human mind, and intellectuals of the Stone Age or ancient Greece were undoubtedly very enthusiastic about these discoveries. Because of repetition in school, these things have lost their freshness, but each student learns all this anew, just as ancient mathematicians did. So in writing our books we tried to forget that we already knew all this and to look at mathematics with fresh eyes. Maybe this approach accounts for the great popularity these books enjoy in the Soviet Union.

Sample problems

The first homework given to a student who wants to be enrolled in the school is the "entrance examination." It is composed in such a fashion that no prerequisite knowledge is expected, but the results can show whether a would-be student wants (and that generally means is able) to study mathematics.

Here's a set of problems you might encounter in the entrance examination.

1. You have a glass of wine and a glass of water. You take a spoonful of the wine and pour it into the glass of water, stir the mixture thoroughly, then take a spoonful of that and pour it into the glass of wine. Is there now more wine in the water or more water in the wine?

2. Is it possible to wrap a cube with sides of length 1 in a square sheet of paper with sides of length 3?

3. Into how many parts can four distinct straight lines divide a plane? Draw an example for each case.

4. Find all three-digit numbers such that writing three digits before each of them turns it into its square.

5. What is the maximum number of Saturdays there can be in a year?

6. You have a chess knight on an infinite chessboard. What is the number of squares it can reach in no more than 10 moves? (A knight must make L-shaped moves: two squares horizontally or vertically and then a right-angle turn for one more square—see figure 1.)

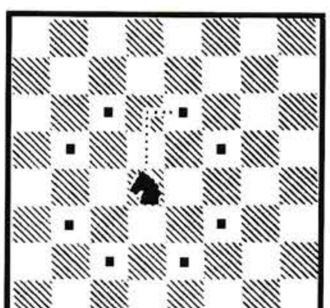


Figure 1

7. There are 10 bags with 10 coins in each. In one bag all the coins are counterfeit and each weighs 11 grams, while in all the remaining bags all the coins are genuine and weigh 10 grams each. How can you decide, by a single act of weighing, which is the bag with the counterfeit coins? (Your scales are accurate to the gram.)

8. Find all positive integers that satisfy

$$xy = x + y + 1990.$$

9. Decide which of these two numbers is larger:

$$\frac{13^{15}+1}{13^{16}+1} \text{ or } \frac{13^{16}+1}{13^{17}+1}.$$

How do you know?

10. Can there be a triangle whose area is greater than 100 square meters and whose three heights are all shorter than 1 cm?

11. Is it possible to draw a curve on a Rubik's cube (fig. 2) that begins in one square, ends in another square, and enters each square, excluding these two, exactly once?

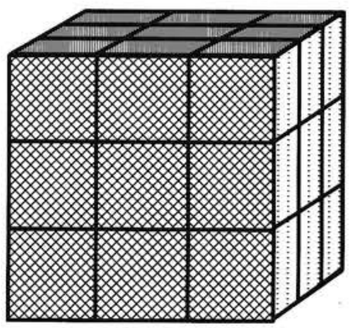


Figure 2

If you want to send us your solutions to these problems, we'll certainly read them and send you our comments. We'll print the solutions in the next issue of *Quantum* (September/October).

This year I'm staying at Rutgers University, so please send your letters to this address:

Professor I. M. Gelfand
Center for Mathematics, Science, and Computer Education
Rutgers University
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Piscataway, NJ 08855

We welcome our new correspondents in the US, and we'll try to give you useful advice. We plan to open a similar school in America, so any contact with you, our first correspondents, will be most valuable to us.

The Correspondence School Library

Books written for the Correspondence School are very popular in the Soviet Union. Hundreds of thousands of copies of each book have been sold. Their success is apparently due to the fact that they were intended for students in the most far-away places, where they often can't find a mathematics teacher who can give them good advice. So they turned out to be very suitable for independent study.

Now that we've begun publishing a new series of books for students, we've decided to reprint the best of the Correspondence School books, those that have stood the test of time. I think even today we'd write them as they are.

It seems like a very good idea to combine these books with a computer, so the second edition will include a software package so you'll be able to work with a book and a PC simultaneously. But please don't think that now, in the computer age, the effort of studying mathematics can be transferred to a computer. The computer can help you solve a problem, but it can neither think nor understand for you.

I'd like to make a comment here. Some of my American colleagues have explained to me that American students aren't really accustomed to thinking and working hard, and that for this reason we must make our materials as attractive as possible. Permit me to not completely agree with this opinion. From my long experience with young students all over the world, I know that they are curious and inquisitive, and I believe that if they have some clear materials presented in a simple form, they'll prefer these to all the artificial means of attracting their attention—much as one buys books for their content and not for their dazzling jacket designs that engage your mind only for the moment.

The most important thing a student can get from studying mathematics is the attainment of a higher intellectual level. In this light I'd like to point out as an example the famous American physicist and teacher Richard Feynman, who succeeded in writing both his popular books and his scientific works in a simple and attractive manner.

Unfortunately, most of our Correspondence School publications are in Russian. However, English translations of the first two books in the series, *Method of Coordinates* and *Functions and Graphs*, are about to be published by the Birkhäuser publishing house. Algebra, geometry, calculus, and combinatorics will be treated in subsequent books in the series. Of course, each of the books can be used for independent study.

Would you like a sample? Here's a problem from *Geometry* by T. V. Alexeyevskaya and I. M. Gelfand.

Straight lines in a plane are said to be in general position if no two of them are parallel and no three of them have a common point. We can consider these lines infinitely long fences dividing the plane into a number of regions. Some of the regions are bounded—we'll call them "cells." In each cell a wolf can be placed (since it can't run away from a bounded region). If there are 4 lines in general position, what is the maximum number of wolves we can keep? (This case, along with the case of lines that are not in general position, is considered in problem 3 above.) What is the maximum number of wolves you can keep if you have 5 lines in general position? What is the minimum number?

SURFACE AREA
FROM PAGE 9

its length and the area of its cross section (a circle of radius h—see exercises 4 and 8). This version of the definition also holds for curves that do not lie in a plane.

Now you know three definitions based on a common idea, and you probably understand that this string of definitions can be continued by adapting them to the dimensionality of the object we're going to measure and the surrounding space. You'll find two more examples in exercise 9, but further generalizations lie outside the scope of this article.

Exercise 8. The body created by rotating a circle around an axis that does not intersect it is called a torus. Its volume is computed by the formula $2\pi^2Rr^2$, where r is the radius of the circle and R is the distance from the center of the circle to the rotation axis. Show that the spatial h-neighborhood of a circle is a torus (fig. 11) and derive once again the formula for the length of the circle.

Exercise 9. For what sets F is the limit of the following ratios positive and finite: (a) $V(F_h)/1$; (b) $V(F_h)/(4\pi h^3/3)$? What is the geometric meaning of these limits?

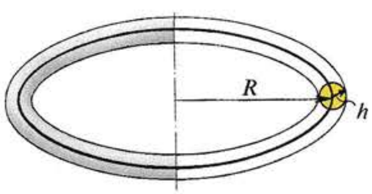


Figure 11

Seeing is believing?

Now the time has come for me to apologize for the little bit of cheating I engaged in at the beginning of the article. Have you discovered it yet? The secret is very simple: after the unrolling is done, the petals of the bud are curvilinear triangles, not the usual ones. It's easy to see that the sum of the angles at the vertices of the triangles in figure 1b is less than 2π, and so if we rolled them back into a bud, we'd get a surface with holes in it. Naturally, this leads to a value that is smaller than the true area of the sphere. If this trick has enticed you into reading this article, it has played its role well.

MATH MAGIC
FROM PAGE 29

Just how many squares does this give? Here are six additional tables:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

1	2	5	6
3	4	7	8
9	10	13	14
11	12	15	16

1	3	5	7
2	4	6	8
9	11	13	15
10	12	14	16

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

1	3	9	11
2	4	10	12
5	7	13	15
6	8	14	16

1	2	9	10
3	4	11	12
5	6	13	14
7	8	15	16

each of which yields 24 x 24 by permuting their rows and columns. So the method gives 6 x 24 x 24 = 3,456 magic squares in all, or 3,456/8 = 432 when we discount rotations and reflections.

One of these squares—namely,

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

obtained from

16	12	15	11
14	10	13	9
8	4	7	3
6	2	5	1

appears in Albrecht Dürer's famous self-portrait, Melencolia I, and the middle two numbers in the bottom row indicate the date of the work (refer back to page 29).

We can use the addition table idea to give a simple formula for the general order 4 square like the one we gave for order 3 earlier. We just adjoin two new parameters, x and y:

A+a+x	C+d+x	D+b+y	B+c+y
D+c+x	B+b+y	A+d+x	C+a+y
B+d+y	D+a+x	C+c+y	A+b+x
C+b+y	A+c+y	B+a+x	D+d+x

The general order 4 magic square

The 10 parameters here are not all independent, since we can decrease either A, B, C, D or a, b, c, d by some amount without changing the square, provided we simultaneously increase x and y by the same amount. So you can take $D = d = 0$, if you like, and then the remaining 8 parameters A, B, C, a, b, c, x, y will be independent.

This raises the general question of how many parameters are needed for the general order n magic square. The answer is given on page 61. And just how many magic squares can be made using the numbers from 1 to n^2? You won't find the answer on page 61, since for n larger than 4 nobody knows!

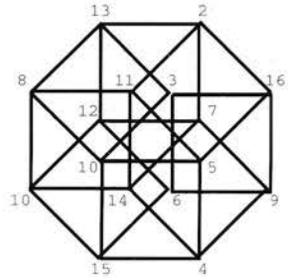
A particularly interesting kind of square is the one called pandiagonal, in which all the "wraparound diagonals" such as

#	·	·	·
·	·	·	#
·	·	#	·
·	#	·	·

also have the magic sum. Of course, an order 3 square can be pandiagonal only if all the numbers are equal, since they must all be one third of the magic sum! Our general order 4 square is pandiagonal only if

$$\begin{aligned} A + C &= B + D, \\ a + c &= b + d, \\ x &= y. \end{aligned}$$

There is a beautiful relationship between the order 4 pandiagonal squares and the four-dimensional hypercube, or tesseract. In the figure, the vertices of the tesseract are numbered from 1 to 16. Each two-dimensional face adds up to 34, so this is a magic tesseract! If you start at any vertex and read around a face, you get the first row of a magic square whose other rows are found by reading around the parallel faces in the same order. We'll leave it to you to work out the precise rules.



The magic tesseract

One square obtained like this is

8	13	2	11
10	3	16	5
6	9	4	15
1	12	7	14

Can you see its rows and columns as faces in the magic tesseract?

There are 384 orderings of the numbers that work, corresponding to the 384 ways of repositioning the tesseract so that it occupies the same portion of four-dimensional space. They give 384 pandiagonal order 4 squares in all, or just 48 when we discount rotations and reflections.

There are many other shapes to make magic with. One of the most remarkable is

15
14 13
9 8 10
6 4
11 5 12
1 2
18 7 16
17 19
3

Adams's amazing magic hexagon

in which all the lines parallel to the sides have the same sum, 38. It is so called because Clifford W. Adams became so obsessed with the problem of rediscovering it that he made 19 numbered ceramic tiles and spent 47 years shuffling them around at odd moments until at last he did find the solution, which Charles W. Trigg later proved unique. In fact, the Adams hexagon had been discovered several times before, in particular by William Radcliffe, who registered it in 1896 at Stationers Hall, London.

Just for the fun of it!

B26

Mix and match. There are three red and five blue sticks of different lengths lying on the table. The total length of the red sticks is the same as that of the blue ones. Is it possible to cut up the sticks and pair the pieces such that the pieces in each pair will be alike in length but different in color? (V. Proizvolov)

B27

Comparing ages. Now I'm four times older than my sister was when she was half as young as I was. In 15 years our combined age will be 100. How old are we now?

B28

Go with the flow. Once I got lost in a forest. I was going to make a fire and spend the night, but fortunately I found a water pipe. Obviously I should go along the pipe, but in what direction? In the direction of the water flow, since the water goes to people. But how could I determine in which direction the water was flowing? (M. Lobak)

B29

The algebra of cooperation. Solve the number rebus USA + USSR = PEACE. (The same letters stand for the same digits, different letters denote different digits.) (B. Kruglikov)

Art by Edward Nazarov

B30

The wisdom of old. King Arthur ordered a pattern for his quarter-circle shield. He wanted it to be painted in three colors: yellow, the color of kindness; red, the color of courage: and blue, the color of wisdom. When the artist brought in his work, the king's armor-bearer said there was more courage than wisdom on the shield. But the artist managed to prove that the proportions of both virtues were equal. Can you tell how? (A. Savin)

ANSWERS, HINTS, AND SOLUTIONS ON PAGE 60

The fast game for math minds

Taking the Twenty-Four Challenge™

MINNEAPOLIS PUBLIC HIGH School students are manipulating numbers to solve math problems that would short-circuit the world's most elaborate pocket calculator. They are practicing this lightning-quick skill in a citywide tournament in which they play Twenty-Four,® a new game that takes an innovative approach to the world of numbers.

Colorful Twenty-Four game cards containing four numbers from one through nine were designed by Robert Sun, an inventor and engineer who has had a lifelong fascination with the world of numbers. Using each number only once, players must add, subtract, multiply, and/or divide to arrive at the solution of 24.

For example, a card has the numbers **3**, **4**, **5**, **5**. Two possible solutions are:

$$\begin{array}{ll} 3 \cdot \mathbf{5} = 15 & \mathbf{5} \cdot \mathbf{5} = 25 \\ 15 + \mathbf{5} = 20 & \mathbf{4} - \mathbf{3} = 1 \\ 20 + \mathbf{4} = 24 & 25 - 1 = 24 \end{array}$$

And that's just an easy card. The infamous **7**, **3**, **7**, **3** combination has stumped some of the best math minds in the country.

According to Sun, the game goes right to the essence of what math is all about. "Students are faced with four seemingly random numbers. They must quickly try to sense the interrelationship between the four numbers and the number 24." The Twenty-Four game challenges students to build a solid foundation of basic math skills and to feel confident in manipulating numbers. Says Sun, "Without this foundation, students cannot experience the excitement of moving on in the mathematical world."

To further encourage students' fascination with numbers, Sun introduced the Twenty-Four Challenge Tournament to Minneapolis last fall. With the support of the Minneapolis Public Schools and a generous grant from the TCF Bank, over 10,000 students competed for the coveted title of Minneapolis Numbers Whiz Kid. The winner was Mike Appelhans of WOC High School.

Similar Twenty-Four Challenge tournaments are being held in chosen cities nationwide, including Philadelphia, Boston, St. Louis, San Francisco, Chicago, Portland, and Tampa. This widespread participation can be attributed, in part, to the fact that the tournaments are extremely simple to administer, and there is no cost to the school systems.

For information about starting a Twenty-Four Challenge tournament in your community, contact Robert Sun or Nan Ronis of Suntex Int'l, Inc., 118 North Third Street, Easton, PA 18042, or call 215 253-5255.

Quantum's Twenty-Four Challenge

Would your class like to try this number game? Take the *Quantum* Twenty-Four challenge, sponsored in part by the Eastman Kodak Company-21st Century Learning Challenge.

How to play. On this page, we've printed four cards from the Twenty-Four game. Use the four numbers on each card to compute 24 as many ways as you can. Do the math step by step. Use only the numbers on the card and the answers from each step. (Note: the 9's on the cards are filled in with red; the 6's aren't.)

How to enter.

1. Send us a list of all the ways you got 24 for each card. Show *each stage* of your work—just like we did.

2. Write on a sheet of paper:

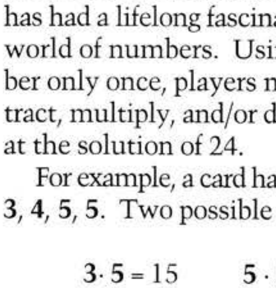
a) your grade;

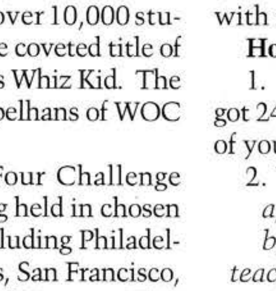
b) the names of your school and teacher;

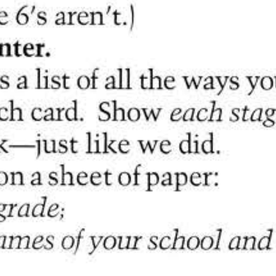
c) your school's address and phone number; and

d) the statement, "We pledge that these answers were derived without the help of any adults except the teacher." Make sure all participants sign it.

3. Send us your entry no later than **April 24, 1991**. Mail it to *Quantum*'s Twenty-Four Challenge, 1742 Connecticut Avenue NW, Washington, DC 20009.







The prizes. All grades K–12 are eligible, and will be judged in two categories: 8th grade and below, and 9th grade and above. Prizes have been generously donated by Kodak.

The class in each category that finds the highest total of correct and different ways to make 24 will win the Grand Prize. Each student will get a Twenty-Four T-shirt and a standard edition of the Twenty-Four game.

The class in each category with the next highest total is first runner-up. Each student in those classes will receive a Twenty-Four T-shirt and a pocket edition of the Twenty-Four game.

The students in the second, third, and fourth runner-up classes in each category will each receive a pocket edition of the Twenty-Four game.

Do's and don't's. Make sure you follow these rules when you enter:

Do use all four numbers on each card.

Don't use a number more than once, unless it appears on the card more than once.

Don't use the commutative property of addition or multiplication to make 24 in more than one way. (If you do, we'll only count one of the answers.) Here's an example using **1**, **6**, **8**, and **9**:

$$\begin{array}{ll} \mathbf{1} \cdot \mathbf{8} = 8 & \mathbf{8} \cdot \mathbf{1} = 8 \\ \mathbf{9} - \mathbf{6} = 3 & \mathbf{9} - \mathbf{6} = 3 \\ 3 \cdot 8 = 24 & 3 \cdot 8 = 24 \end{array}$$

Only one of these combinations would count.

Don't put two digits together to make a larger one. You can't make 23 from 2 and 3.

Don't use exponents. You can't use 2 and 3 to make 2^3, or 8.

—*Elisabeth Tobia*

Bulletin board

The computer revolution

There can be little doubt that computers have changed the face of the world. Now "The Computer Revolution," a fascinating six-part documentary series, offers a look back at the history and development of this technological achievement and a look forward to the computers of tomorrow. From the earliest prototypes and room-size machines of the past to today's compact desktop workstations, this new video series presents a historical and analytical perspective on the computer and its impact on modern society.

Through interviews with top professionals in the field and state-of-the-art computer graphics, "The Computer Revolution" not only explains the leaps and bounds in computer technology but often demonstrates them as well. The series explores applications in agriculture, medicine, communications, business, space exploration, and national defense. It depicts the story the birth and growth of an entire industry, from Silicon Valley to the personal computer. Voice recognition software, machine vision for the blind, natural language comprehension, and artificial intelligence are only some of the new uses this documentary examines.

"The Computer Revolution" is available on VHS and Beta videocassette, for purchase or rental, from Films for the Humanities & Sciences. For more information, write to Dan Maurer, FHS Inc., PO Box 2053, Princeton, NJ 08543, or call 800 257-5126.

—*E. T.*

Soviet and American space art

In the movie "2001: A Space Odyssey" one of the astronauts sketches his hibernating colleagues aboard the Jupiter probe and later shows his work to HAL the computer, who dutifully admires them. This episode says something about the human urge to record reality through the eye and hand as well as through lens and chemical-coated paper, spectrometer and printout. *In the Stream of Stars: The Soviet/American Space Art Book* is a collection of paintings by those who have had the privilege of traveling in space and those who study it and dream about it.

Some of the most interesting work in the book was done by an astronaut and a cosmonaut. Alexei Leonov (after whom the US spacecraft in the

"Galactiscope" by Beth Avery, from *In the Stream of Stars*. Copyright © 1991 by William K. Hartmann, Andrei Sokolov, Ron Miller, and Vitaly Myagkov. Workman Publishing Co., New York. Reprinted with permission of the publisher.

sequel "2010" is named) took colored pencils to space to sketch his ideas, which he developed more fully on his return to Earth. Alan Bean's portraits of fellow Apollo astronauts on the Moon show traces of Monet, a painter he greatly admires, and he offers us a virtual dissertation on the perception of color, texture, and depth on the lunar surface.

As one of the book's editors, William K. Hartmann, points out, "space art" lies somewhere between "fine art" and "illustration." The examples presented in this handsomely printed book run the gamut between the two extremes. Some are highly imaginitive and thought-provoking; others are competent renderings of known facts or speculations without much emotional content. Together, though, they represent a satisfactory mix of the science and poetry that intermingle in humankind's space venture.

In the Stream of Stars presents the work of more than 70 artists, most of them from the US or USSR, the nations at the forefront of large-scale space exploration. It was edited by a Soviet–American team of artists and is graced by a rhapsodic introduction by Ray Bradbury, dean of American science-fiction writers. In addition to the more than 200 full-color illustrations, the book contains illuminating essays on the history and role of space art as well as first-person accounts of space travelers. Cosmonaut Leonov writes, "Probably my strongest impression of the Earth was that it was not so big! It was finite. . . . What astonished me most of all was the thinness of our atmosphere, which seems so thick and full of power to the observer on Earth. If a model were made, the air would be no thicker than a film of tracing paper, covering the Earth."

The timeworn phrase "Spaceship Earth" takes on a new freshness and immediacy in these vibrant paintings, making this a book not only for the armchair star voyager but for the most entrenched homebody on this remarkable planet. (183 pages, $19.95 paperback/$29.95 hardcover, Workman Publishing, New York)

—*Tim Weber*

Index of Advertisers

Problem racing

That's what all your training is for!

by Gary Sherman

I LOVE BIKE RACING AND I follow a rigorous training regimen that includes weight lifting, plyometrics (jumping exercises), and stretching exercises to prepare for races. This training regimen is time consuming and laborious, but it is not an end in itself. In fact, the only measure of its value is how well I do in races. And to take that measure I've got to race.

You're reading this magazine, so I'm betting that you're pretty fond of mathematics and that you're in training. Did you calculate, graph, factor, differentiate, or integrate your way through 1–27 (odd) on pages 254 and 255 last night? Why? To prepare to race. So race! Try a real problem!

What does "real" mean? The problem comes out of your worldly experience, somebody (if only you) cares what the answer is, and nobody (as far as you know) knows what the answer is. Let me illustrate with four examples.

Endless "war"

Did you play the card game "war" when you were growing up? Here's how it goes. After the deck is shuffled and dealt to the two players, they begin matching cards, and the higher card captures the lower card. Stalemates (suits aren't ranked) are broken according to local rules, and the game is over when one player has all the cards.

Back up for a minute—must a game eventually be "over"? Or could it go on forever? This (very natural) question was posed by a frustrated seven-year-old—"Dad, does this game ever have to end?"—not by me. A 52-card deck, the vagaries of breaking stalemates (at our house, usually a wrestling match), and the haphazard way in which we returned cards to the deck prompted the insightful reply, "Uhhhhh, gee, Mike, I'm clueless."

What about war with the integers $1, 2, \ldots, n$? For example, if we shuffle and deal $1, 2, \ldots, 8$, your initial hand might be Y_0: 1, 6, 2, 7 and my initial hand might be M_0: 8, 5, 3, 4. It's clear that I'm not going to lose (it's my article), but must I win? Let's play.

$$\begin{array}{ll} Y_0: 1, 6, 2, 7 & M_0: 8, 5, 3, 4 \\ Y_1: 6, 2, 7 & M_1: 5, 3, 4, 8, 1 \\ Y_2: 2, 7, 6, 5 & M_2: 3, 4, 8, 1 \\ \vdots & \vdots \\ Y_{15}: 1 & M_{15}: 7, 5, 3, 6, 4, 8, 2 \end{array}$$

I win on the 16th match:

$$\begin{array}{ll} Y_{16}: & M_{16}: 5, 3, 6, 4, 8, 2, 7, 1 \end{array}$$

The questions are endless. Must any game with $n = 8$ end? If not, what proportion of games must end? If there is a game that doesn't end, it must cycle, since there are only a finite number of possibilities for your hand and my hand. What can you say about the number of matches that occur before a cycle must begin? How many matches are there per cycle? And the endless part: what if n is an arbitrary positive integer?

Shuffled seating

The atmosphere in my calculus class was getting stale, I was frustrated, and before I realized I was

Art by Nishan Akgulian

saying it, I said it. "Okay, I want everybody out of their seats . . . now stretch, wander around the room for a couple of minutes, and sit back down. *But*—have new neighbors when you sit down!"

About halfway through the stretch-and-wander phase, it occurred to me that

1. Nobody, including me, knew what "neighbors" meant;
2. I didn't know if any such seating rearrangement existed;
3. I had stumbled on a nice problem.

Specifically, consider the following 4 x 5 array:

A	B	C?	D?	E?
F	**G**	H	I	J
K	L	**M**	N	O
P	**Q**	R	S	T

The neighbors of F are A, G, K; the neighbors of P are K, Q; and the neighbors of J are G, M, Q, K. Can you rearrange the entries in this array so that each entry gets all new neighbors? What's the probability that a random rearrangement provides all new neighbors for each entry? How many rearrangements preserve only J and M as neighbors? How many rearrangements preserve a total of exactly 1 (2, 3, ...) neighbor(s)? On the average, how many neighbors does a random rearrangement preserve? What happens in an $m \times n$ array? An $m \times 1$ array? An $m \times n \times k$ array? In d (≥ 4) dimensions? Maybe you prefer another definition of "neighbors"?

Scrambled evaluations

During the time I was chair of the mathematics department at Rose-Hulman Institute of Technology, one of my jobs was to read student evaluations of teaching. (Each instructor's teaching is evaluated each quarter in each course by each student.) A week or so after the end of the quarter a stack of $m \times n$ envelopes (m instructors and n courses per instructor), in random order, would appear on my desk. My preference was to read the evaluations by instructor, so I had to sort the envelopes accordingly. Now my desk may have enough clear space for two stacks, but certainly never m stacks. Let me take $m = 3$ (say, instructors A, B, and C) and $n = 3$ to illustrate what I did.

Stack 1		**Stack 2**
?		?
	1-B	
	2-A	
	3-A	
	4-C	
	5-B	
	6-C	
	7-A	
	8-C	
	9-B	

The first three envelopes are easy to sort.

Stack 1		**Stack 2**
1-B		2-A
		3-A
	4-C	
	5-B	
	6-C	
	7-A	
	8-C	
	9-B	

What do I do with 4-C? I'm going to put it on the bottom of the original stack and think of it as the tenth envelope.

Stack 1		**Stack 2**
1-B		2-A
		3-A
	5-B	
	6-C	
	7-A	
	8-C	
	9-B	
	10-C	

Now 5-B goes to the top of Stack 1, 6-C goes to the bottom of the original stack as 11-C, and 7-A goes to the top of Stack 2.

Stack 1		**Stack 2**
5-B		7-A
1-B		2-A
		3-A
	8-C	
	9-B	
	10-C	
	11-C	

Stack 2 is now available to accept envelopes for instructor C, so I can complete the sort.

Stack 1	**Stack 2**
9-B	11-C
5-B	10-C
1-B	8-C
	7-A
	2-A
	3-A

Notice that I had to handle $k = 11$ envelopes to complete the sort using this algorithm. And what is the algorithm? Suppose envelope i-X is at the top of the unsorted stack.

1. If each sorted stack consists of complete sets of envelopes, then place i-X at the top of any sorted stack.

2. If j-X is at the top of a sorted stack, then place i-X on top of that stack.

3. Otherwise, place i-X at the bottom of the unsorted stack and relabel it $(s+1)$-X, where s-Y is the envelope that was moved from the last to the next-to-last place in the unsorted stack.

Now for the questions. Suppose there are m instructors, each teaching n courses, and you have room for s stacks. What's the minimum value of k? What's the maximum value of k? What's the average value of k? And the I'm-a-glutton-for-punishment question: what if the number of courses per instructor isn't constant?

Shoot first . . .

A few years ago the Continental Basketball Association (CBA) held a shooting contest—the $1,000,000 CBA Easy Street Shootout—in an attempt to increase fan support of the league. Each of the fourteen franchises held a local version of the contest to determine its representative for the league contest at the all-star game. Both the local and league contests proceeded according to the following rules:

1. A lottery is held to determine the order in which the contestants will shoot. (At the franchise level ten spectators were selected, and ordered, by lottery using numbers printed on their programs.)

2. In turn, each contestant shoots one shot from whatever location on the court he or she chooses.

3. At the completion of one round of shots, the contestant who made the longest shot is declared the winner.

4. If all the contestants miss their first-round shot, a sudden-death round begins: the shooting order of the first round is repeated, and the first contestant to make a shot wins.

This contest was described in the Scorecard section of *Sports Illustrated* (November 18, 1985), and the editor wondered just how important the shooting order is in such a contest. I doubt that the editor thought of this as a mathematical question. But you should. It's a natural: how important is shooting order, and what is the best shooting strategy for each of the contestants? Here are the assumptions I would make to get started:

1. All shooters are of equal ability.

2. The probability of missing a shot increases monotonically from 0 (a layup) to 1 (a 90-foot bomb).

3. If two shooters hit shots from the same distance, the second shooter wins.

4. There are n shooters—and $n = 2$ for openers.

Et cetera

You can generate problems like these. It's just a matter of developing the mindset to question everything and anything you experience. What's the optimum? How many? What's the minimum? What's the maximum? What's the average? Before long you'll acquire one of the distinguishing characteristics of a good mathematician—more problems than answers. View each of your problems as a race. If you do well, look for stiffer competition. If you don't do so well, adjust your training regimen accordingly and race again!

If you would like to know the status of the four problems I've described, send me a note that includes your effort to solve at least one of them. My address is Math Department, Rose-Hulman Institute of Technology, 5500 Wabash Avenue, Terre Haute, IN 47803.

Dr. Gary Sherman *was the math department chair and is now professor of mathematics at Rose-Hulman, where he is director of a research program in group theory. Dr. Sherman was drafted by an NFL team but went to graduate school instead. He has won several state championships in bicycle racing in Indiana.*

To calculate or guess—you decide!

A baker's dozen curious problems

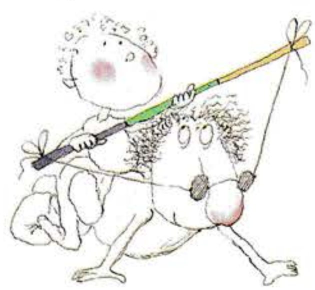
by I. Akulich

LET'S GET RIGHT TO THE problems.

Problem 1. A father is 26 years old and his son is 6 years old. In how many years will the father be three times older than his son?

This problem and others like it are usually solved by one of two methods.

Arithmetic method: If the father becomes three times older than his son, the difference of their ages is two times the son's age. But the difference is constant and is equal to $26 - 6 = 20$ years, so the son's age at the moment we're looking for will equal $20/2 = 10$ years, which will happen in $10 - 6 = 4$ years.



Art by Pavel Chernusky

Algebraic method: Let the father be three times older than his son in x years. Then $26 + x = 3(6 + x)$, so $x = 4$.

Let's add one more method to these two—the *guess-and-choose method*. Let's try to guess the answer or, more exactly, to choose it. Let's use our experience in solving such problems and remember that, as a rule, only integers are used in them. The search area then becomes sharply delimited. In addition, when the father becomes three times older than his son, naturally his age must be divisible by 3. At first this situation will happen in a year—that is, the father will be 27 and his son will be 7 (which doesn't work); then in 4 years—that is, the father will be 30 and his son will be 10 (now it *does* work). That's it!

If any of you is thinking of indignantly rejecting the third method in favor of the first two, I strongly urge you to hold off. It's true, a mathematician (even an amateur) ought not guess. But imagine the following situation. A math competition is taking place. One participant from each team is called upon to solve the above problem. The first to give the correct answer is considered the winner. And so the clock starts ticking! You can't help but agree that the third method seems preferable to the first two. Of course, there's a chance of not finding the answer at all, but the greater speed and more limited number of calculations (and, consequently, the greatly reduced possibility of making arithmetical errors, which are quite likely in the heat of competition) are sure to outweigh that consideration.

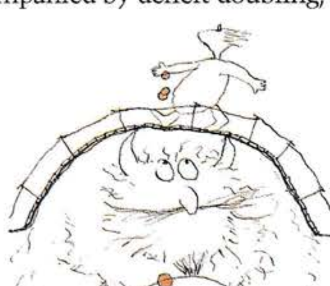
Problem 2. According to legend there was a tombstone with this inscription: "Ye Traveller, lying under this tombstone are the remains of Diophantos, who died in extreme old age. He was a child for a sixth of his long life, a youth for a twelfth, and unmarried for a seventh. Five years after he was married his wife had a baby boy, who lived half as long as his father. Four years after the son's death Diophantos himself went to his eternal rest, and his death was mourned by his relatives. Tell me, if you can count, how many years Diophantos

lived."

If you use the algebraic method, you have to solve a cumbersome equation, and the arithmetic method isn't any simpler. And what about the guessing method? Let's think about it this way: since all the numbers used must very likely be integers, the number of years Diophantos lived must divide evenly by 6, by 12, and by 7—that is, by their least common multiple, which equals 84. This means that the number of years lived by Diophantos is a multiple of 84—in fact, it *is* 84. (Larger numbers are unrealistic.) That's it! In this particular case the advantages of the guessing method are really obvious.

Problem 3. One day the Devil proposed to a certain goof-off that he earn some money. "As soon as you cross the bridge," he said, "your money will double. You may cross the bridge as many times as you like, but every time you do you must pay me 24 cents." The goof-off agreed . . . and after the third crossing he was penniless. How much money did he start off with?

Let's try to guess the answer, using general reasoning. It's clear that the goof-off had less than 24 cents; otherwise he wouldn't have gone bankrupt. In other words, from the very beginning there was a "budget deficit." Money doubling was apparently accompanied by deficit doubling, and



after the third crossing the doubled deficit amounted to exactly 24 cents. This means that at first the deficit was equal to $24/8 = 3$ cents—that is, the goof-off had $24 - 3 = 21$ cents. Of course, we can't guarantee that the answer is right, but it's certainly plausible.

Problem 4 (proposed by Sam Loyd). "Here are two turkeys," the butcher says, "one tom and one hen. Together they weigh 20 pounds. But the price per pound of the hen is 2 cents higher than the price per pound of the tom." Mrs. Smith bought a turkey hen and paid 82 cents for it, while Mrs. Brown paid \$2.96 for a tom. What did each turkey weigh?

Let's take some risks: suppose each turkey weighed an integral number of pounds and the price per pound is an integral number of cents. The total sum paid was 378 cents. If the extra charge for the hen's meat is subtracted

from the total, the remainder must divide by 20—the total weight of the turkeys. The nearest least number divisible by 20 is 360—that is, the extra charge is equal to 18 cents, and so the tom weighed 11 pounds.

We have the answer, but the hen turned out to be a little hefty, didn't it? The difference in weights is quite small, but the price of the hen is one fourth that of the tom, even though the price per pound of the hen is higher. Something's wrong!

Sure enough, this is a case where the guessing method misfires. We made a faulty assumption and so we're doomed to failure. We had no way of knowing this is a "historical" problem, though we might have suspected from the total price—\$3.78 for 20 pounds of turkey—that this problem is set in the not-so-recent past. As late as the 1950s, many grocery items were priced to the *half cent.* (Now, armed with this additional information, try to guess the answer.)

Problem 5 (also proposed by Sam Loyd). A certain lady offered one dollar to a post office worker and said, "Please give me some two-cent stamps, ten times that many one-cent stamps, and five-cent stamps with whatever money is left." How did the post office worker fulfill this rather mind-wringing request?

He could have proceeded this way: let x denote the number of two-cent stamps and y the number of five-cent stamps; from the statement of the problem we have $2x + 10x + 5y = 100$, or $12x + 5y = 100$.

Diophantos, the Greek mathematician mentioned above, worked hard at finding integral solutions of such equations. In fact, they're now named after him—"Diophantine" equations. So all that's left is to solve the Diophantine equation $12x + 5y = 100$. But we can get along just fine without it. Notice that the total price of the two-cent stamps and, consequently, their number must divide by 5 (why?). The number of one-cent stamps is ten times that of the two-cent stamps, so their number divides by 50, which comes to exactly 50 stamps for 50 cents. In this case the number of two-cent stamps is 5, which comes to 10 cents; the rest are five-cent stamps, and their price is 40 cents for the eight of them. Here our answer turned out to be absolutely correct, even though we simply tried to guess it without any guarantee it would be right, just as in the previous problem. (By the way, the problem has one more solution—a "degenerate" one: the number of one-cent stamps is 0, the number of two-cent stamps is also 0, and the number of five-cent stamps is 20.)

Here's a more complex example.

Problem 6. Three brothers received 24 apples. The youngest brother received the least of all, the eldest brother received the most. Seeing this, the youngest brother proposed the following exchange: "I'll keep one half of my apples and we'll divide the rest equally between you. Then let the middle brother keep half of his apples and we'll divide the rest equally between me and the eldest brother. Finally, let the eldest brother keep half of his apples and we'll divide the rest equally between me and the middle brother." The brothers didn't suspect the youngest brother of deceit and agreed. As a result, everybody ended up with the same number of apples. How many apples did each brother have at first?

Such problems are usually solved backwards. But we'll go straight ahead and determine successively the number of apples the youngest brother had, the number of apples the middle brother had, and the number of apples the eldest brother had. It's clear that the youngest brother was given less than 8 apples. On the other hand, the number of his apples divides by 4 (since he managed to divide half of his apples equally). This means that the youngest brother was given 4 apples. Notice that the middle brother was given less than half of the remaining apples—that is, less than 10 apples, so he got from 5 to 9 apples. After adding the apple he received from the youngest brother, we again have to get a number that divides by 4. This number must be within the range of 6 to 10—it must be 8, then. So the middle brother was given $8 - 1 = 7$ apples and the eldest brother was given all the rest—that is, 13 apples. (It turns out that only the eldest brother lost by the redistribution. Maybe the middle brother suspected the youngest brother of trickery, but he kept his mouth shut in his own interests.)

As you can see, guessing can be pretty useful at times, and in any case it shouldn't be dismissed out of hand. In fact, the effectiveness of this method increases as the statement of the problem becomes more complicated and involved, since we don't have to go into details when we're guessing.

And now it's your turn. To get a better grip on this technique, try to guess the answers to the following problems.

Problem 7. A merry hiker set off on a cross-country trek. On the first day he covered 1/3 of the distance to his destination, on the second day he walked 1/3 of the rest of the distance, and on the third day he traveled 1/3 of the distance that remained. As a result the hiker had 32 kilometers left to walk. How far is it from his house to his destination?

Problem 8. Upon being asked how old he is, someone answered, "When I live another one half plus one third plus one quarter of my years, I will be 100 years old." How old is this person?

Problem 9. "Will you tell me, renowned Pythagoras, how many students attend your school?"—"Count them yourself," Pythagoras answered. "One half are doing mathematics, one quarter are learning music, one seventh are keeping silent, and there are three women besides." How many followers did Pythagoras have?

Problem 10. One fifth of the bee swarm is on the cherry blossoms, one third is on the apple blossoms; the tripled difference of the two last numbers flew to the pear blossoms; and one bee is flying back and forth, attracted by the fragrance of jasmine and rose. How many bees are there in the swarm?

Problem 11. Four people donated money to a cause. The second donor gave two times the coins given by the first one, the third donor gave three times the coins given by the second one, the fourth donor gave four times the coins given by the third one, and altogether they gave 132 coins. How many coins did the first donor give?

Problem 12. On being asked how many people there are in his crew, the captain answered, "Two fifths of my crew are on guard duty, two sevenths are at work, one fourth are in the sick bay, and 27 people are right here." How many people are there in the crew?

Problem 13 (and the most complex, I daresay). At present you and I together are 86 years old. The number of my years is 15/16 of the age you'll be when my age is 9/16 of the age you'd be if you lived to the age two times the number of my years when I'm twice your age. How old am I?

I certainly advise you to solve these problems by using the "normal" method, too. Besides, in this way you can check the correctness and uniqueness of your answers. The problems weren't created specially for this article but were taken from highly respected books, including *Mathematical Puzzles* by Sam Loyd (Dover Publications, New York, 1959) and *More Mathematical Puzzles* by the same author (Dover Publications, New York, 1960).

RUTHERFORD
FROM PAGE 27

celebrity sitting down among youngsters and bending over notebooks full of assignments."

In 1913 scientists from Rutherford's laboratory tested his formula for alpha scattering by counting scintillations observed at various angles over identical time intervals and found it to be correct. This certainly showed the truth of the nuclear model of the atom. Insofar as the system of charges at rest couldn't be in a stable equilibrium, Rutherford gave up the static model of the atom and suggested that electrons in an atom move around the nucleus along curved paths. But in that case the electrons had to move with acceleration and, according to classical electrodynamics, emit electromagnetic waves; and this, in turn, must be accompanied by energy loss. In the final analysis, the electron must fall into the nucleus.

It was Nils Bohr who managed to eliminate the contradiction. But that, as they say, is another story. And what about Rutherford's experiments? Are they important now only as an episode in the history of physics? No, their effect is more far-ranging. Almost 60 years after his experiments, in the 1970s, Rutherford's method of probing substances with alpha particles became widely used in laboratories to study crystalline structure, find the positions of various impurities in crystals, and determine their compositions. This is known nowadays as the method of Rutherford backscattering. But, instead of grains of radium, powerful accelerators are used as the source of large flows of high-energy alpha particles. And to think that their prototype was a small lead box with grains of radium that Dr. Rutherford, as you recall, categorically refused to relinquish on his way through US customs.

CONTEST

How the ball bounces

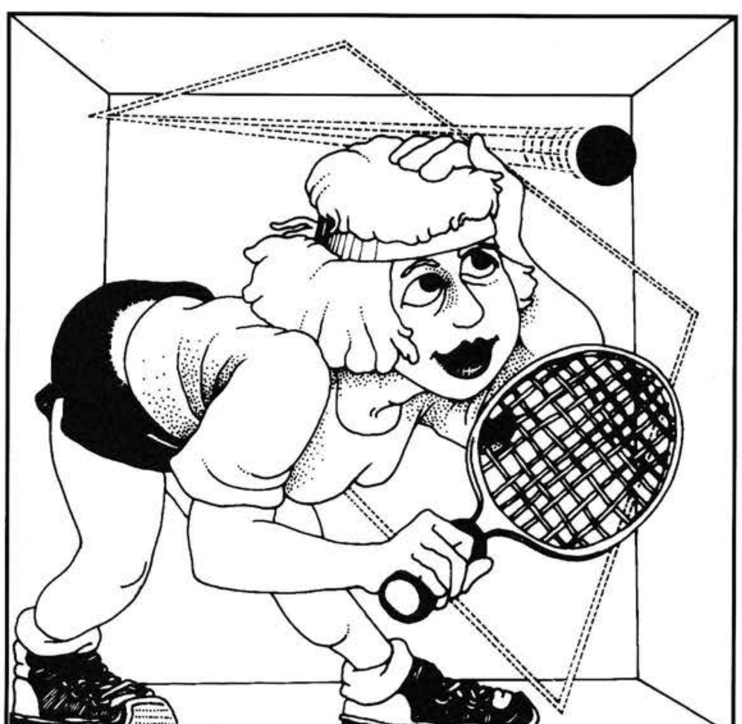
"We should be careful as to the play, but indifferent to the ball."
—*Epictetus*, Discourses

by Arthur Eisenkraft and Larry D. Kirkpatrick

ONE OF THE JOYS OF PHYSICS is being able to apply our textbook knowledge to everyday situations, such as sports. Even though we often need to make simplifying assumptions to obtain answers, we can often gain insight into the situation and learn what variables are important. As we gain more and more sophistication in physics, we can apply these new tools to obtain better answers.

The equations for kinematics that we learn at the beginning of most physics courses can be used to analyze many games played with balls. We usually begin by assuming that the effects of air resistance and friction can be neglected. A great simplification occurs because we can separate two-dimensional motion into two one-dimensional motions. We usually analyze projectile motion in terms of the vertical motion and the horizontal motion, where the two motions are connected by the time.

As an example, consider the following problem. Elisabeth hits a racquetball toward the front wall with a speed v at an angle θ above the horizontal. She hits the ball at a height H above the floor and a distance D from the front wall. We simplify the real situation by assuming that the collision is completely elastic (no kinetic energy is lost) and frictionless (no forces parallel to the wall). **When and where will the ball land? What are the numerical values when v = 6.00 m/s, θ = 36.9°, H = 1.80 m, D = 2.25 m, and the acceleration due to gravity g = 9.80 m/s²?**


Art by Max-Karl Winkler

Please send your solutions to *Quantum*, 1742 Connecticut Avenue NW, Washington, DC 20009. The best solutions will be acknowledged in *Quantum* and their creators will receive free subscriptions for one year.

Neutrinos and supernovas

Interesting correct solutions to the neutrino problem posed in the November/December issue were submitted by W. Richard O'Connell Jr. of Rockville, MD, and Walter Stockwell of Berkeley, CA, with the assistance of Mandeep Gill. We follow Walter's solution and begin by remembering that the total energy of a relativistic particle is given by

$$E = \gamma mc^2, \tag{1}$$

where $\gamma = (1-\beta)^{-1/2}$ and $\beta = v/c$. We'll label the neutrinos that arrive first (15 MeV) with a subscript 1 and the later ones by a subscript 2.

Since the neutrinos in the first burst have twice the energy, we have

$$E_1 = \gamma_1 mc^2 = 2E_2 = 2\gamma_2 mc^2. \tag{2}$$

So

$$\gamma_1 = 2\gamma_2. \tag{3}$$

Let the arrival time of the 15-MeV neutrinos be t. Then the 7.5-MeV neutrinos arrive at $t + \Delta t$, where Δt = 15 s. If the distance to supernova 1987A is L, we have

$$\beta_1 = L/ct,$$
$$\beta_2 = L/c(t+\Delta t).$$

We can use the binomial expansion

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \cdots$$

for $|x| << 1$ to find the value of β_2 to first order in $\Delta t/t$, which is a very small number:

$$\begin{aligned}\beta_2 &= (L/ct)(1+\Delta t/t)^{-1}\\ &\cong (L/ct)(1-\Delta t/t)\\ &= \beta_1 - L\Delta t/ct^2.\end{aligned}$$

Since $\Delta t << t$, we now assume that we can set $t = L/c$ to obtain

$$\beta_2 \cong \beta_1 - c\Delta t/L. \qquad (4)$$

We can now square equation (3) and substitute for the γ's from (1) to get

$$(1-\beta_2^2) = 4(1-\beta_1^2).$$

Substituting β_2 from (4) and rearranging terms, we get a quadratic equation in β_1:

$$\beta_1^2 + (2c\Delta t/3L)\beta_1 - 1 = 0,$$

where we have ignored the term $(c\Delta t/L)^2$. Taking the positive root so that $0 < \beta < 1$ and keeping terms to lowest order in $c\Delta t/L$, we have

$$\beta_1 \cong 1 - c\Delta t/3L.$$

Substituting this into (1), we get

$$\begin{aligned}mc^2 &= E_1/\gamma_1\\ &= (1-\beta_1^2)^{1/2}E_1\\ &\cong (2c\Delta t/3L)^{1/2}E_1,\end{aligned}$$

also to lowest order in $c\Delta t/L$. Using the numerical values stated in the problem, we obtain a rest energy for the electron neutrinos of 20 eV.

Another approach to this problem starts by approximating β:

$$\beta = v/c = t/(t+\Delta t) \cong 1 - \Delta t/t.$$

Denoting $\Delta t/t$ by δ, we get

$$\beta^2 \cong 1 - 2\delta$$

and

$$\gamma \cong \frac{1}{2\delta}.$$

Putting this into (3) we obtain the relationship

$$\delta_1 = 4\delta_2.$$

Since we know that

$$\delta_1 - \delta_2 = \frac{15\text{ s}}{t},$$

we can solve for the value of δ_1 and use it to find γ_1 and E_1, as above.

. . . CONTEST

Adventures among P_t-sets

Maybe you should take your PC along

by George Berzsenyi

FERMAT OBSERVED THAT FOR the set {1, 3, 8, 120}, each of the six numbers $1\cdot3+1$, $1\cdot8+1$, $1\cdot120+1$, $3\cdot8+1$, $3\cdot120+1$, and $8\cdot120+1$ is the square of an integer. Are there other such remarkable sets of integers?

The answer to this question is yes. In an article published in *The Fibonacci Quarterly*, it was shown by the founder and present editor of the *Quarterly* that if F_n denotes the nth Fibonacci number (that is, $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$), then the set of numbers

$$\{F_{2n}, F_{2n+2}, F_{2n+4}, 4F_{2n+1}F_{2n+2}F_{2n+3}\}$$

behaves similarly. Other such sets were found earlier by Euler, who showed that if m, n, and k are integers such that $mn + 1 = k^2$, and if $S = \{m, n, m + n + 2k, 4k(k+m)(k+n)\}$, then the pairwise products of the elements of S increased by 1 always yield perfect squares. Our first challenge to you is to verify the above claims.

More generally, a finite set S of three or more nonzero integers will be called a P_t-set if, for each pair of distinct members x and y of S, $xy + t$ is a perfect square. To date, nobody has managed to construct a P_1-set of five or more elements, and it was only recently that Vamsi Mootha, a sophomore at Stanford University, found a P_t-set of size 5. It is {14, 22, 30, 42, 90} with $t = -299$. Surely, with the clever use of computers, some of you will challenge these records.

There are plenty of challenges even if we restrict the size of P_t-sets to 3. For instance, it's not difficult to verify that {3, 7, 17} is not a P_t-set for any t, that {1, 3, 8} is a P_t-set only if $t = 1$, and that {−5, 10, 23} is a P_t-set for both $t = 131$ and $t = 1139$. Is there a set $\{a, b, c\}$ that is a P_t-set for three or more different t's?

Vamsi and I wrote our first joint article on the subject while he was still a high school student in Beaumont, Texas. He also organized some of his findings into winning projects in the International Science and Engineering Fair and in the Westinghouse Science Talent Search. We'd be happy to share with interested readers our findings, the results of our literature search (which resulted in about 30 articles on the subject), and many more of the unanswered questions still challenging us.

Correspondence concerning the subject of P_t-sets should be sent to *Quantum*, 1742 Connecticut Avenue NW, Washington, DC 20009. In addition to solutions, you are invited to share your own inquiries concerning the subject. The best results will be acknowledged and their authors will receive free subscriptions to *Quantum* and/or book prizes.

Math

M26

Let M be the point where boat 1 and boat 2 collide (fig. 1). By the statement of the problem it would take the same time t for boat 1 to cover the distance AM and for boat 2 to cover BM. So the time for boat 1 to travel from A to D is $t_1 = (AD/AM)t$, and the time for boat 2 to move from B to C is $t_2 = (BC/BM)t$. But triangles AMD and BMC are similar by the equality of corresponding angles (angles AMD and BMC are vertical, inscribed angles $ADB = ADM$ and $BCA = BCM$ are subtended by the same arc). Therefore, $AD/AM = BC/BM$, so $t_1 = t_2$.

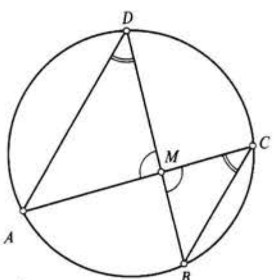


Figure 1

M27

The equalities $q = x + y$, $r = x - y$, and $x = (q + r)/2$, $y = (q - r)/2$ establish a one-to-one correspondence between the representation $a = x^2 - y^2 = (x + y)(x - y)$ of a number a as a difference of squares and the factorizations $a = qr$, where $q > r$. (Since a is odd, the above formulas for x and y yield integer numbers for any two factors q and r.) To find the number of such factorizations, we notice that each of the n primes, whose product is equal to given number a, can and must be a factor of either q or r. This makes 2^n possibilities, but the condition $q > r$ holds for only half of them. Thus, we get the needed number $2^n/2 = 2^{n-1}$.

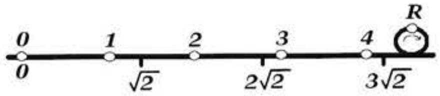


Figure 2

M28

The answer is 142. First, let's find the number of tracks left on the fixed circle f by the very first red point R of the rolling circle r.

Roll circle r along the number axis, starting from the origin 0. Then point R will color all the nonnegative integer points of the axis (fig. 2). Now wind the axis around the circle f, beginning from 0. Each line segment of length $2^{1/2}$ will make exactly one winding, and all the red points of the axis will show up at different points of the circle (the relation $k - l = m \cdot 2^{1/2} \neq 0$ is impossible for integers k, l, m, since $2^{1/2}$ is an irrational number). Thus, the number of red tracks on circle f left by point R in n revolutions equals the number of red points on the segment $[0, n \cdot 2^{1/2}]$ of the axis—that is, $[n \cdot 2^{1/2}] + 1$ ($[x]$ is the greatest integer not exceeding x).

It may seem that, taking into account all the multiple tracks of sticky points on both circles, we'd discover that their number grows exponentially, depending on the number of revolutions: each revolution should approximately double this number, since all the painted points leave new tracks. But this assumption is absolutely wrong: *all the red points emerging on circle f coincide with the tracks of the first point R!* To prove it, suppose there are red points on f that are not the tracks of R, and choose the point A that was colored first among them. Point A was colored by some point R' of circle r, and point R' itself is a track of some point A' of circle f. Since point A' was colored earlier than A, it must be a track of R. So all the tracks of R' after it hit point A' must follow the tracks of R after it hit the same point A'. In particular, point A must coincide with one of the tracks of R, which is a contradiction. (In the same way one can prove that all the red points of circle r are the tracks of the first red point on circle f, so the nth revolution adds the $(n + 1)$th red point on circle r.) Thus, the rate of growth of the number of red points on f is not exponential but linear: $[n \cdot 2^{1/2}] + 1$. For $n = 100$ it yields 142 points.

M29

Each edge of the cube must contain at least one point of the given polyhedron P; otherwise, the projection of P onto a face perpendicular to this edge won't cover the common vertex of this face and the edge. So we can take one point of P on each edge and consider the *convex hull C* of all these points—that is, the smallest convex polyhedron containing all of them. Since polyhedron P is also convex and contains the chosen points, it contains C. It's easy to see that polyhedron C is obtained by truncating every vertex of the cube along the plane drawn through the points of P that were chosen on the edges meeting at this vertex (fig. 3).

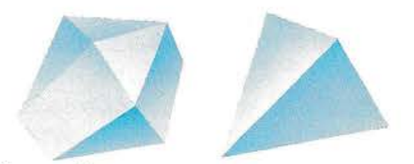


Figure 3

Let's estimate the total volume of the 8 triangular pyramids cut off the cube (some of them may actually degenerate into a point, a segment, or a triangle). Consider two pyramids cut off at the ends of a vertical edge of the cube. Let the edge length be 1 unit, the areas of the horizontal bases of the pyramids be B_1 and B_2, and their respective heights be h and $1 - h$ (fig. 3). Then the sum of their volumes equals

$$\frac{1}{3}\left[B_1 h + B_2(1 - h)\right] \le \frac{1}{3} \cdot \frac{1}{2}[h + (1 - h)] = \frac{1}{6},$$

since, clearly, $B_1 \le 1/2$, $B_2 \le 1/2$. Repeating the same evaluation for the other three vertical edges and summing the results, we derive a total volume of the pyramids not greater than $4/6 = 2/3$, so

$$\text{volume }(P) \geq \text{volume }(C) \geq 1 - \frac{2}{3} = \frac{1}{3}.$$

The tetrahedron in figure 4 satisfies the conditions of the problem and has exactly the volume 1/3.

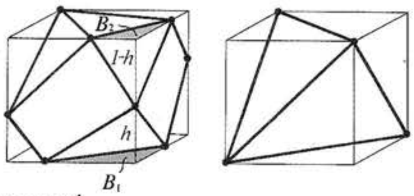


Figure 4

M30

The minimum route is 64, and the maximum one is $28 + 36 \cdot 2^{1/2}$ units long.

Examples of routes of these lengths are shown in figures 5 and 6. Now we must show that any closed and non-self-intersecting chess king's route has a length not less than 64 and not greater than $28 + 36 \cdot 2^{1/2}$.

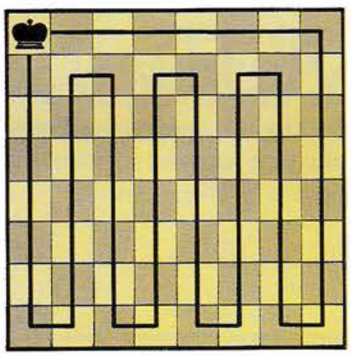


Figure 5

The first (lower) estimate is self-evident: the total number of moves is 64, and every move is either "straight" (parallel to the chessboard's sides) and has the length 1, or "diagonal" having the length $2^{1/2}$.

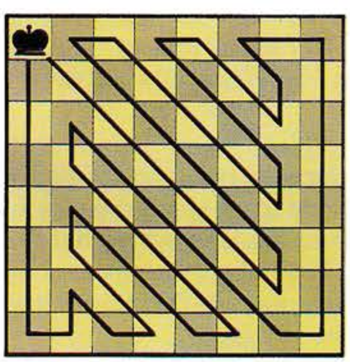


Figure 6

To prove the upper estimate, we must show that any route in question contains at least 28 "straight" moves. Consider two successive border squares A and B on the king's route. They must be adjacent. Otherwise, the part AB of the route would divide the chessboard into two non-empty sets

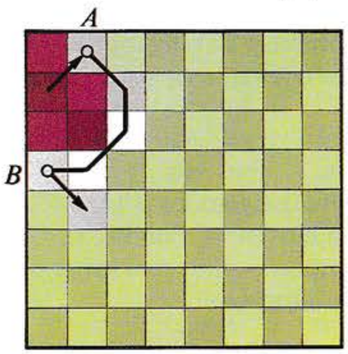


Figure 7

of squares (fig. 7), so the king would have to cross the broken line AB to get from one set to the other; but self-intersections are forbidden. The squares A and B are of different color, and diagonal moves do not change the colors of squares, so there should be a "straight" move between point A and B on the king's route. All the routes can be split into 28 segments connecting neighboring border squares (28 is the number of border squares); each of the segments contains at least one "straight" move. So the number of "straight" moves is not less than 28, and we're done.

Physics

P26

The kinetic energy of the ball at any altitude is greater when it is going up than when it is falling down. Indeed, if there were no air resistance, both energies would be equal. The difference between the two kinetic energies (on the way up and on the way down) is equal to the work done to overcome the resistance of the air. So at any given altitude the velocity of the ball is greater when it's going up than when it's coming down. It's obvious that the average velocity of the ball on the way up is also greater than on the way down. So the ball takes less time to go up than it does to come down.

P27

To find the force N exerted by weight M on weight m, we use Newton's second law for weight m:

$$ma = mg - N.$$

After the weights are released, they enter into harmonic oscillation (as a single weight of mass $M + m$). This means that the acceleration a changes its direction periodically. The force

$$N = m(g - a)$$

reaches its maximum at the moment when the acceleration has the maximum absolute value and is directed upward—that is, at the moment of the maximum stretching of the spring. To find the value of the acceleration, we use Newton's second law for the weight $M + m$:

$$(M + m)a = (M + m)g - kx. \qquad (1)$$

We can find the maximum value $x_{\max}$ by means of the law of energy conservation (taking into account that at the moment of maximum stretching of the spring, the velocity of the weight is zero):

$$\begin{aligned} &-(M+m)\,gx_0 + \frac{k}{2}x_0^2 \\ &= -(M+m)\,gx_{\max} + \frac{k}{2}x_{\max}^2, \end{aligned} \qquad (2)$$

where x_0 is the stretching of the spring when there is no weight m. Taking into account that $kx_0 = Mg$, from (2) we get $kx_{\max} = (M + 2m)g$. Substituting this value in (1), we find the maximum acceleration

$$|a_{\max}| = \left| -\frac{m}{M+m}g \right| = \frac{m}{M+m}g.$$

Knowing $a_{\max}$, we can get $N_{\max}$:

$$N_{\max} = mg\frac{M+2m}{M+m}.$$

P28

It's well known that the process of boiling requires an uninterrupted heat supply. When the water in the test tube heats to 100°C, the heat transfer from the retort stops. That's why the water in the test tube won't boil.

A more interesting situation occurs if toluene is poured over the water in the test tube. Then you'll see the curious phenomenon of "interface boiling." The boiling begins on the interface between the two liquids when the sum of their partial pressures of saturated vapor equals the external atmospheric pressure. It's clear that the pressure of saturated water vapor is less than the atmospheric pressure, and so the temperature is less than 100°C. Thus, when toluene is added to the test tube filled with water, the boiling on the toluene–water interface begins before the water itself starts to boil. The interface boiling temperature is less than the boiling temperature of either liquid.

P29

An ammeter connected in series with the lamp shows the current flowing through the lamp. According to the statement of the problem, the current I_1 is equal to the nominal current $I_n = 0.2$ A. So in the first case, the voltage of the lamp is equal to the nominal $U_n = 2.5$ V. The fact that the lamp, when connected in parallel with the ammeter, glows exactly as much as in the first case means that the current flowing through the lamp is equal to I_n and that the voltage on the lamp and on the ammeter is equal to U_n.

We'll write Ohm's law for both cases. For the serial connection (fig. 8) we have

$$E = I_n (R + r) + U_n , \qquad (1)$$

where R is the resistance of the wires and r is the resistance of the ammeter.

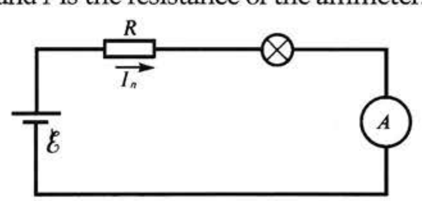


Figure 8

For the parallel connection (fig. 9) we have

$$E = (I_n + I_x) R + U_n , \qquad (2)$$

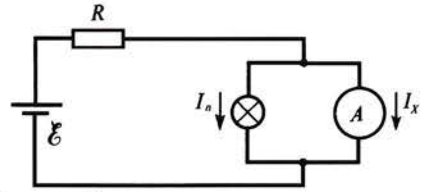


Figure 9

where I_x is the current through the ammeter. Taking into account that $rI_x = U_n$, we can rewrite (1) like this:

$$E = I_n\left(R + \frac{U_n}{I_x}\right) + U_n . \qquad (1')$$

Solving equations (1′) and (2), we find the current I_x flowing through the ammeter connected in parallel with the lamp:

$$I_x = \sqrt{\frac{I_n U_n}{R}} = 0.5 A .$$

P30

In an atmosphere whose refractive index n decreases with altitude, light rays don't travel along straight lines. Each light wave front changes its direction and gets deflected because the speed of light $v = c/n$ decreases when the refractive index increases.

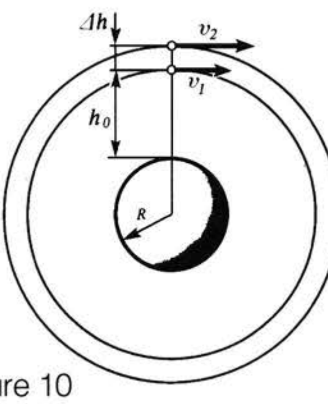


Figure 10

Denote by Δh the width of the optical channel through which the light rays circle the planet at a constant altitude (see figure 10). Consider two extreme rays. The ray at the constant altitude h_0 takes the time

$$t = \frac{2\pi\left(R + h_0\right)}{v_1} = 2\pi\left(R + h_0\right)\frac{n_0 - \alpha h_0}{c}$$

to circle the planet.

The other ray, traveling at a distance $\Delta h << h_0$ from the first, must circle the planet at the altitude $h_0 + \Delta h$ in the same amount of time (only then will all the light wave fronts in the channel be perpendicular to the circle of radius $R + h_0$):

$$t = \frac{2\pi\left(R + h_0 + \Delta h\right)}{v_2}$$

$$= 2\pi\left(R + h_0 + \Delta h\right)\frac{n_0 - \alpha\left(h_0 + \Delta h\right)}{c} .$$

Taking into account that $\Delta h << h_0$, we find that

$$h_0 = \frac{1}{2}\left(\frac{n_0}{\alpha} - R\right).$$

This phenomenon is called circular refraction. Observations show that this situation is actually possible—for example, in the atmosphere of Venus.

Brainteasers

B26

Yes, it's possible cut and pair the sticks in the manner proposed. Place the sticks so as to make two parallel rows, red and blue, one below the other (fig. 11), and cut each row right at the gaps between sticks in the other row.

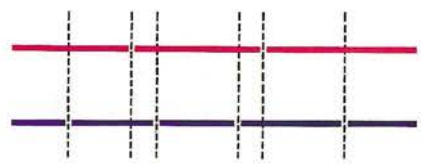


Figure 11

B27

I'm 40 and my sister is 30. If my sister was n years old when she was half as old as I was, I was then $2n$ and am now $4n$ years old. So now she is $n + (4n - 2n) = 3n$, and in 15 years we'll be $4n + 15$ and $3n + 15$. The equation $(4n + 15) + (3n + 15) = 100$ yields $n = 10$.

B28

I made a fire under the pipe, walked a little ways along the pipe in both directions, and put my hand on it to find out where the pipe was warmer. The water was flowing in this direction.

B29

$932 + 9{,}338 = 10{,}270$. Evidently the sum is less than 11,000 but greater than 10,000, so $P = 1$, $E = 0$, $U = 9$. Then $A + R = 10$, $1 + S + S = C$ (if $1 + 2S = C + 10$, then in the next decimal place we'll have $S = A$), and $9 + S = A + 10$. Finally, we get the system of equations $A + R = 10$, $2S + 1 = C$, $S = A + 1$. The letter C denotes an odd number $(2S + 1)$, which is less than 9 ($9 = U$). On the other hand, $C = 2S + 1 = 2A + 3 \geq 2 \cdot 2 + 3 = 7$, since $A \geq 2$ ($1 = P$). Thus, $C = 7$. All that remains is to calculate the other digits and check the answer.

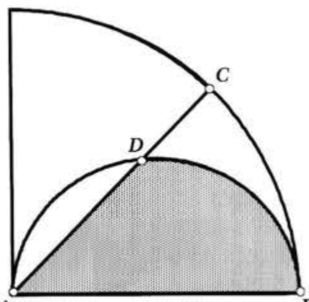


Figure 12

B30

Sector ABC in figure 12 is 1/8 of the circle with radius AB; semicircle AB is 1/2 of the circle with radius $AB/2$. So these two figures are equal in area. Subtracting the shaded area ABD from both of them, we get the required equality.

Math Surprises

The answer is 1 if $n = 1$ or 2; otherwise $n(n-2)$.

Think fast!

1. $\Delta\rho \sim (2/3)\rho_{water} gH \sim 3 \cdot 10^7$ Pa (since two thirds of the Earth's surface is covered by water and the average depth H of the ocean is about 4 km).

2. $\rho_{air} v^2 S \sim mg$, from which we get $v \sim (mg/\pi R^2 \rho_{air})^{1/2} \sim 5$ m/s (for $m \sim 100$ kg and $R \sim 3$ m).

3. $\rho_{Sun} = M_{Sun}/V_{Sun} = (24\pi/GT^2)(D_S/T)^{-3} \sim 10^3$ kg/m^3 ~ 1 g/cm^3. (Here $D_S/r \sim 0.01$ is the angular size of the Sun and $T \sim 3 \cdot 10^7$ s is the period of the Earth's revolution around the Sun.)

4. $n \sim (2gh)^{1/2}/\pi v \sim 1.5$ (for $v \sim 30$ m/s). Hint: when the automobile's center of mass covers the distance of the automobile's length, its vertical velocity attains the value $v_S \sim gl/v$ and its angular velocity the value $\omega \sim v_S/l/2 \sim 2gv$. The number of revolutions then equals $n \sim t\omega/2\pi \sim (2gh)^{1/2}/\pi v$.

5. $Fl \sim mv^2/2 \sim mgL/2$, from which we get $F \sim mgL/(2l) \sim 800$ N (for $m \sim 8$ kg, $L \sim 20$ m, $l \sim 1$ m).

6. $F \cdot R \sim mg \cdot 2R$, where R is the radius of the bicycle's cogwheel, $2R$ is the distance from its axis to the pedal. So $F \sim 2mg \sim 1400$ N (for $m \sim 70$ kg).

7. $mv \sim Ft$ or $\rho \cdot (4/3)\pi r^3 \cdot v \sim p \cdot \pi r^3 \cdot r/v$; then $v \sim (p\rho)^{1/2} \sim 30$ m/s.

8. $F = ma \sim mv^2/2S \sim 7 \cdot 10^3$ N (for $m \sim 60$ kg).

9. Let d be the average diameter of a tree trunk. Suppose the trees are shifted and placed along the circumference of a circle that forms a continuous fence. If you are at the center of the circle you can see nothing behind the "fence." Then the radius of the circle is the desired distance x. There are approximately $2\pi x/d$ trees in a fence of length $2\pi x$. This number of trees is "collected" from an area πx^2. If the average density of the forest is one tree per $\sim l^2$ square meters, then we have $\pi x^2/l^2$ trees in the area πx^2.

Thus, $2\pi x/d \sim \pi x^2/l^2$. For $l \sim 3$ m and $d \sim 0.2$ m, we get $x \sim 2l^2/d \sim 100$ m.

Kaleidoscope

1. Move the magnet inside the coil.

2. The direction of the induced current is counterclockwise.

3. The induced electromotive force is least when the frame lies in the plane passing through the wire and the rotation axis and is greatest when the frame is perpendicular to it.

4. The motion of the magnet in the tube causes the electromotive force of induction. This generates a magnetic field, which hinders the free fall of the magnet.

5. The two halves of the wire produce equal but opposite electromotive forces of induction, which counteract each other.

6. An alternating current in the coin causes eddy currents, while a direct current doesn't.

7. The resistance will increase.

8. The drop in potential between wing tips is greatest for the aircraft near the pole.

9. Along with the usual friction, the rotor experiences a braking electric force generated by the stator's magnetic field.

10. No, since the magnetic flux through wire B doesn't flow through wire A.

11. Since the ring's resistance is equal to zero, the total electromotive force in it must also equal zero. This can happen only if the change in the total magnetic flux through the ring equals zero. So if you remove the magnet, the magnetic flux generated by the induced current will still equal Φ.

Microexperiment. The variable magnetic field of the rotating magnet generates eddy currents in the disk, so that the magnetic field induced by them slows the magnet's motion. By Newton's third law, an equal and opposite force is applied to the disk, causing it to rotate together with the magnet.

Toy Store

Answer: $V = Bh(2 + 1/\cos(\pi/n))/3$. Hint: the volume of one layer of the boot, a polyhedron called an antiprism (fig. 13), as well as the volume of any polyhedron with two parallel faces (bases) containing all its vertices, can be calculated by Simpson's formula: $V = (B_0 + 4B_{1/2} + B_1)h/6$, where B_0, B_1, and $B_{1/2}$ are the areas of the bases and of the section parallel to the bases and equidistant from them, and h is the height. You can make use of Simpson's formula, but to be honest, you ought to try to prove it.

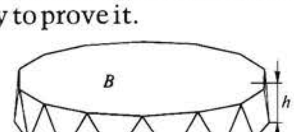


Figure 13

Pilot issues and volume 1 (1990–91)

Latin triangles

And fashionable footwear

by D. Bernshtein

IN THIS ISSUE YOU'VE ALREADY read about "Latin rectangles"—tables of letters in which every line and every column consists of different letters. Here's a similar problem for a triangle table: **put 15 chips of 5 colors onto the nodes of the triangular grid shown in figure 1 such that the colors on every line parallel to a side of the playing board are all different.**

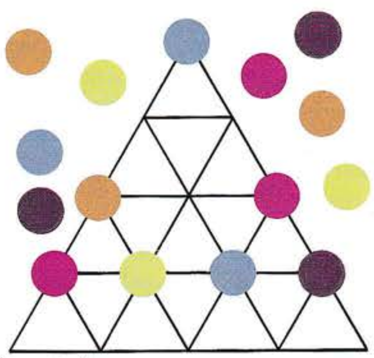


Figure 1

I'm sure this puzzle won't be a hard nut for you to crack. So try to solve these additional problems: (1) find all the solutions to the puzzle; (2) solve the puzzle for a triangular grid with n nodes to a side and n colors of chips ($n = 2, 3, \ldots$).

It's not too difficult to find a general solution to the second problem for odd numbers n. Also, you'll immediately see that the case $n = 2$ is unsolvable. But investigation of the other even values of n will perhaps be a more challenging problem. In particular, you'll have to prove that one more of these values is also unsolvable. (In the case $n = 2k$, the number of chips is $n(n + 1)/2 = k(2k + 1)$, and they can't be divided into n equal groups of uniform color: k groups must have k chips each, and the other k groups must have $k - 1$ chips each.) You'll find the solutions in the next issue of *Quantum*.

Schwarz's boot

The second toy presented here isn't a puzzle, it's a model of the beautiful polyhedral surface described in the article that begins on page 6, "In Search of a Definition of Surface Area." Figure 5 on page 8 gives you an idea of how it looks. This bootlike polyhedron was invented by the German mathematician H. A. Schwarz to demonstrate difficulties that arise when we try to define the area of a curved surface as the limit of the areas of polyhedrons converging to this surface. To make a model of "Schwarz's boot," take a rectangular sheet of sturdy paper, thin cardboard, or any similar material, rule it with a grid of diagonal lines as in figure 2 on the front side and horizontal lines on the back, and score it slightly with a knife or scissors along the lines on both sides. Then roll the sheet up into a tube, pressing on the centers of the grid's rhombi to help the sheet fold along the creases. Glue or tape the opposite edges together at the flap, and the "boot" is done!

This model was designed by the Moscow architect and designer V. Gamayunov, who is also the author of many other beautiful, and much more complicated, polyhedral constructions. He has even developed a general method of inventing them.

Just to keep your head busy while you're admiring your work, try to find the boot's volume V, knowing the area of the base B, height h, and number n of the base's sides.

ANSWER ON PAGE 61

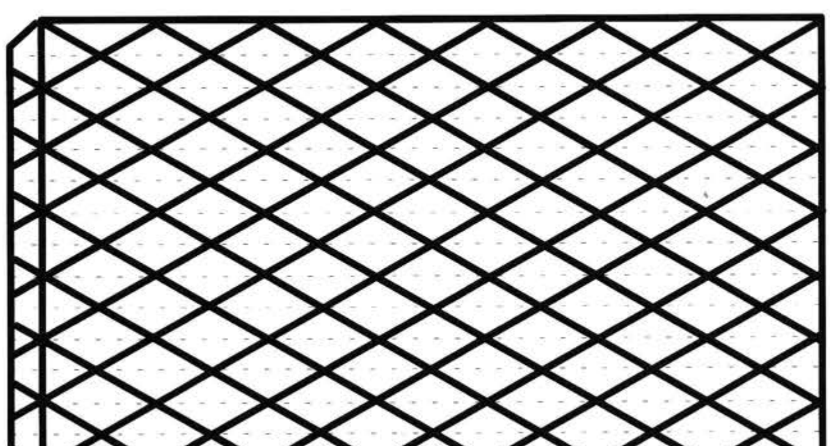
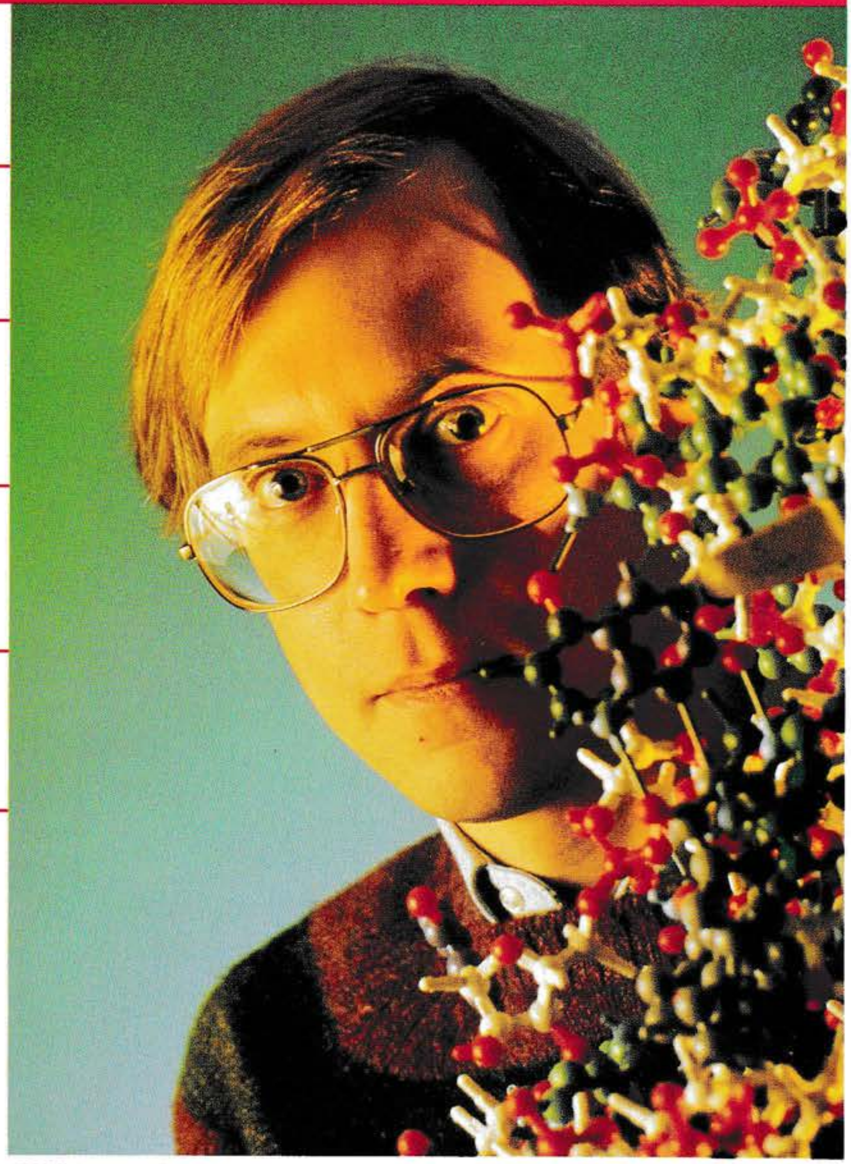


Figure 2

PIONEERING SCIENCE BEGINS AT GRINNELL COLLEGE



1989 Nobel Laureate in chemistry Thomas R. Cech, recognized for his RNA research which may provide a new tool for gene technology, with potential to create a new defense against viral infections.

You may be surprised to learn that Thomas R. Cech, the biochemist who shared the 1989 Nobel Prize in chemistry, is an honors graduate of Grinnell College.

Robert Noyce, the co-inventor of the integrated circuit and the father of the Information Age, also graduated with honors from Grinnell College.

In fact, Grinnell College is one of 48 small liberal-arts colleges that historically have produced the greatest number of scientists in America. Grinnell and these other small colleges compare favorably with major research universities, showing a higher per-capita production of graduates with science degrees. The small colleges comprise five of the top 10 and 13 of the top 20 baccalaureate institutions in the proportion of graduates earning Ph.D.s.

Election to the National Academy of Sciences is an honor second only to receiving the Nobel Prize. Six of the top 10 member-producing institutions, 11 of the top 20, and 15 of the top 25 come from that group of 48 small liberal-arts colleges.

The sciences do not exist in a vacuum in the larger world. Nor do they at Grinnell. The college's open curriculum encourages science students to take courses in other areas.

Students who wish to focus their study may engage in scientific research, usually in a one-to-one relationship, under the direction of a Grinnell College faculty member. Undergraduate student researchers often become the authors of scientific papers with their professors at Grinnell College.

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