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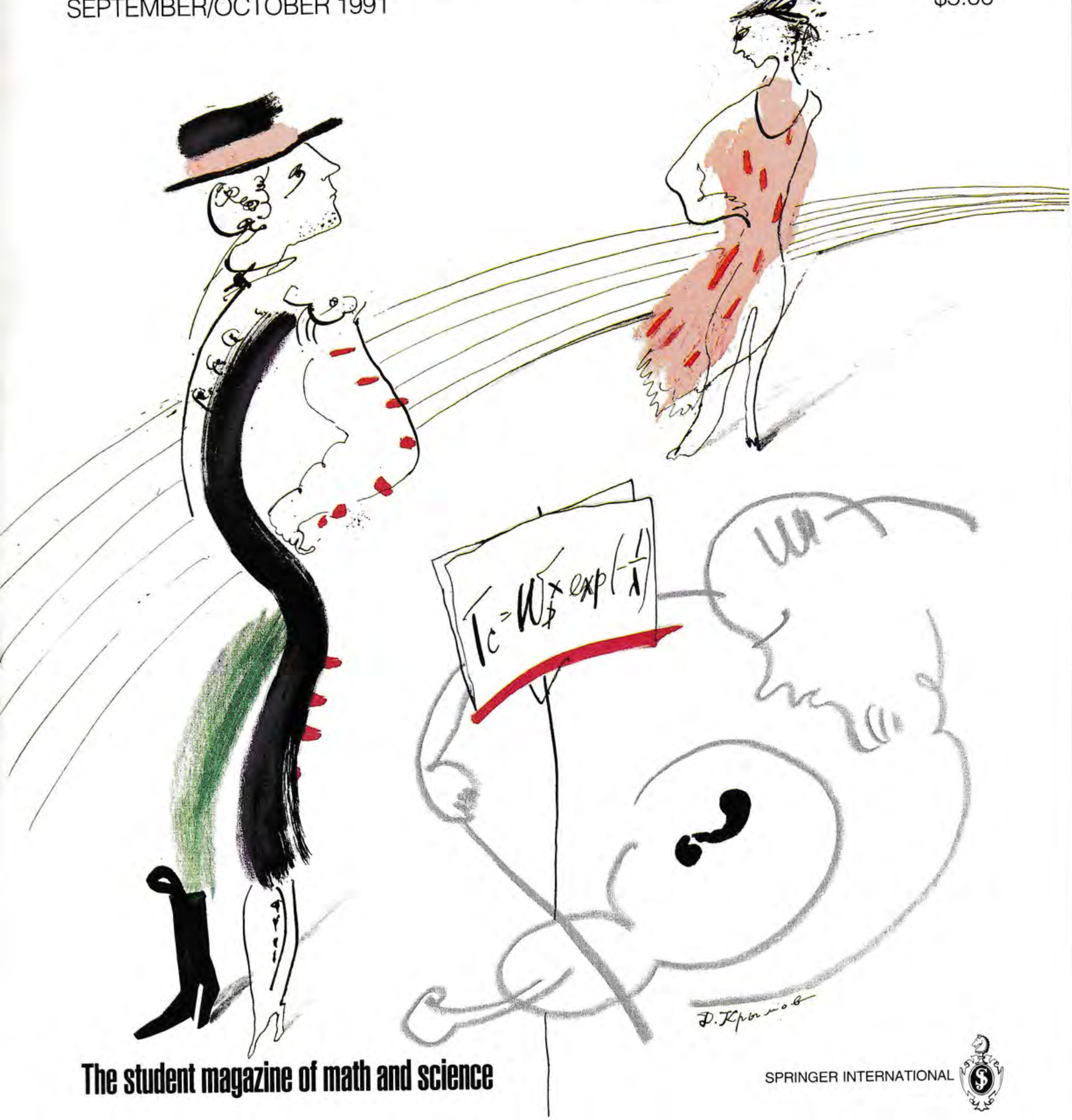
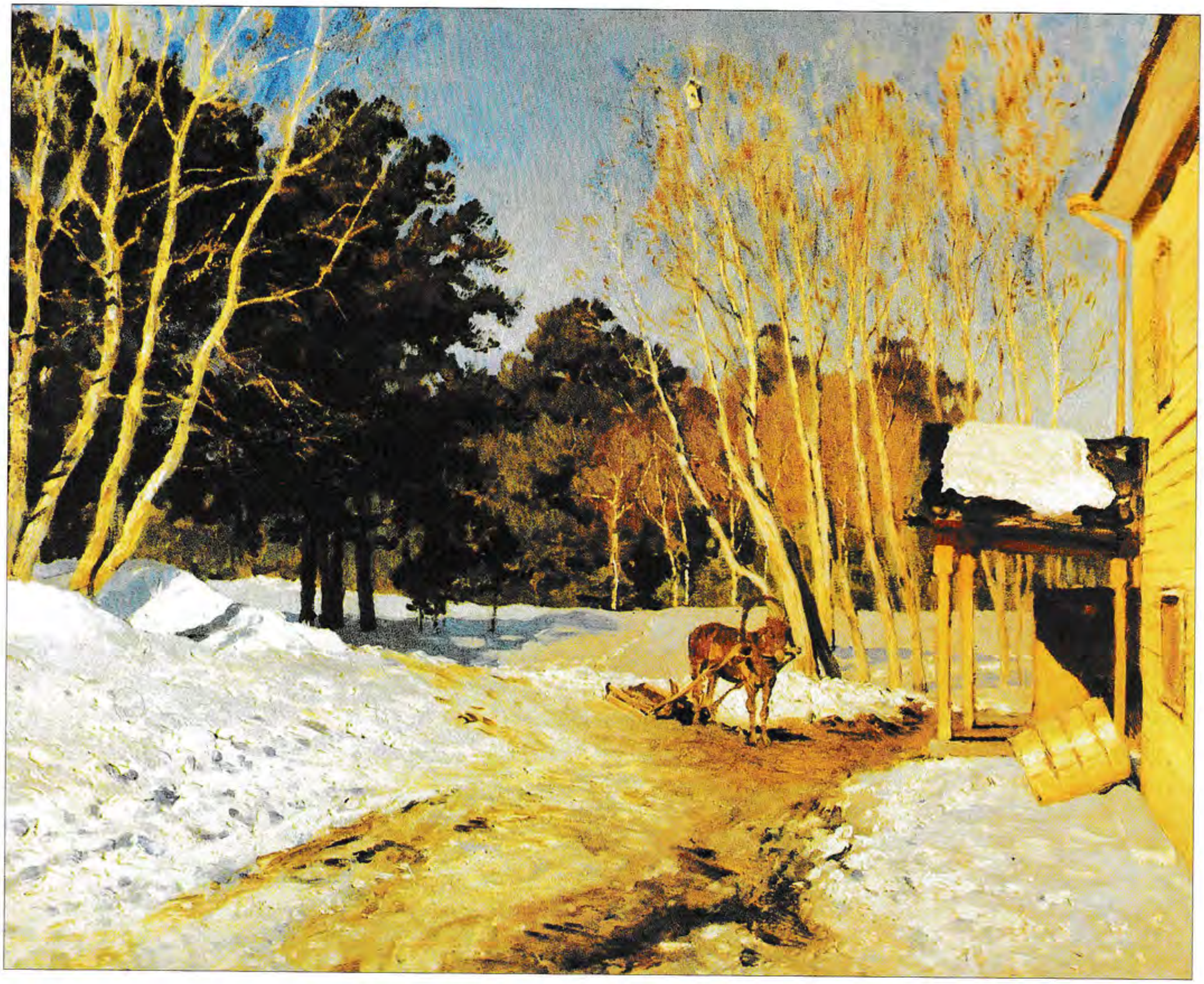


The student magazine of math and science

SPRINGER INTERNATIONAL



March by Isaac Levitan

ONE WAY TO APPROACH THE WORK of Isaac Ilyich Levitan (1861–1900) is by way of his lifelong friend, Anton Chekhov. The stories and plays of Chekhov depict in words the wistful beauty seen in Levitan's landscapes. But while Chekhov's works are populated by people caught between their comfortable old way of life and a frighteningly unknown future, human beings are eerily absent from Levitan's paintings. In the last twenty-five years of his life, he did not place a single human figure in any of his paintings.

In fact, it was Levitan who suggested the symbol of a dead gull as the central image in Chekhov's first great play, *The Sea Gull*. As James Billington writes in *The Icon and the Axe*, "through Chekhov's plays the symbol became equated with the slow and graceful sliding out to sea of old aristocratic Russia." Billington sees in Levitan's work "the afterglow of nature rather than daylight or the promise of springtime."

Perhaps "March" is an exception. At the cusp of winter and spring, March is depicted by Levitan with bright sun and sharply delineated shadows on a persistent blanket of snow. But, psychological interpretation aside, do you notice anything curious about those shadows? See problem P35 on page 16.

QUANTUM

SEPTEMBER/OCTOBER 1991 VOLUME 2, NUMBER 1

Cover art by Dmitry Krymov

High-temperature superconductivity is a hot topic (even though the "high temperatures" hover around the 125K mark—about –148°C). In fact, the June issue of *Physics Today* was devoted to this busy field. What exactly is high-T_c superconductivity, and what does it mean for physics, other sciences, and everyday life? See "Meeting No Resistance" on page 6 for some answers, and maybe more questions. Among other things, you'll read about "Cooper pairs," depicted on our cover as two dancers in a strangely cool dance—attraction and interaction from a distance.

FEATURES

DEPARTMENTS

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QUANTUM

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The first US–Soviet Conference of Science Teachers

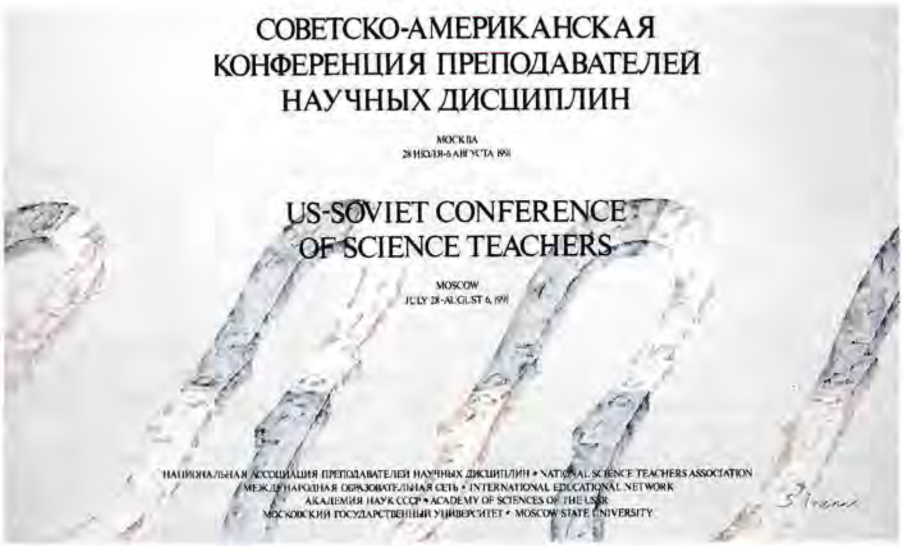
MOSCOW (August 5)—This editorial is being written on the computer in Moscow that our Soviet colleagues use to prepare material for *Quantum*. When I have finished the editorial, I will dial a local telephone number here in Moscow, make connections to a satellite, and the article will be sent to a teleport in San Francisco. Tim Weber, Managing Editor of *Quantum*, will download the article from the SOVAM Teleport, and he will have the article printed in *Quantum* before I return to the United States.

We're able to transmit either English or Russian by this teleport system, which makes it easier to clear up questions and verify translations. While the Soviet Union is opening up, communications are still quite difficult. The teleport is essential to our magazine.

I am in Moscow with 311 American science educators. We have held a convention at Moscow State University with 600 Soviet teachers. Some have traveled all the way from Siberia. It has been very exciting and interesting. Sessions were all provided with simultaneous translation, using electronics and interpreters. Many Americans have been visiting Russian homes, and everyone has found the joint meeting to be one of the finest experiences of their lives. Some 250 of these Americans have just returned from two days in Leningrad (or should we now call it St. Petersburg?).

As you may know, the summit meeting between presidents Gorbachev and Bush occurred while we were here. We were all most regretful that President Bush could not meet with the American teachers.

My most interesting visit here was to Leo Tolstoy's estate and home, some 200 km south of Moscow. It's fascinating to see the very desk at which he wrote *War and Peace* and *Anna Karenina*. And to walk along the lanes he describes in his books is an awesome experience.



Tolstoy's grave is very simple. It's nothing but a small mound of grass in a beautiful wood, without any monument or even marking. He tried to live a simple life and insisted that his place of burial be in the midst of unadulterated nature.

The weather here has been extraordinary. Each day has been clear and warm, but never more than 75 degrees Fahrenheit. Nights have been no colder than 60 or 65. There's very little smog, and there has been no rain. I don't believe I've ever experienced a time with more beautiful weather anywhere in the world.

Our Soviet friends are going through very difficult times. But in spite of these difficulties, they are enormously generous and friendly. As in difficult times before, they're able to survive. And one day they'll prosper, for this is a land of great wealth in natural and human resources.

One goal of our convention was to make new friends and help our colleagues form a science teachers association here in the Soviet Union. We also wanted to help encourage exchanges and other forms of cooperation that would make the US and the Soviet Union economic partners and friends.

As the best of American minds, you readers must help achieve these improved relationships in the future. I hope that you all will take advantage of exchange opportunities. Also, when we need a place for Russian or Soviet students, I hope you'll offer your homes and hospitality.

So this is my message from Moscow: let's keep working together!

—Bill G. Aldridge

Meeting no resistance

"Science is always wrong: it never solves a problem without creating ten more."—George Bernard Shaw

by Alexander Buzdin and Andrey Varlamov

THE GREATEST EVENT IN physics in recent years undoubtedly was the discovery of high-temperature superconductors, whose resistance becomes zero at temperatures above 100K. The practical significance of this discovery can be compared to that of magnetic induction at the beginning of the 19th century. It ranks with the discovery of uranium fission, the invention of the laser, and the discovery of the unusual properties of semiconductors in this century.

The beginning of this exciting new stage in the development of superconductivity was the work by K. A. Müller and T. J. Bednorz at IBM's lab in Switzerland. In the winter of 1985–86 they managed to synthesize a compound of barium, lanthanum, copper, and oxygen—the so-called metal oxide ceramic La–Ba–Cu–O, a compound which had superconducting properties at the record temperature, at that (still recent) time, of 35K. The article, cautiously titled "The Possibility of High-Temperature Superconductivity in the La–Ba–Cu–O System," was turned down by the leading American journal *Physical Review Letters*. The scientific association had gotten tired of receiving sensational reports over the past 20 years about the discovery of high-temperature superconductors that turned out to be false, so it decided to pass on this one. Müller and Bednorz sent the article to the German journal *Zeitschrift für Physik*. After the news about finally superconductivity broke out and research on high-temperature superconductors was being done in hundreds of laboratories, every article devoted to investigating the new phenomenon would begin with a reference to this article. But in the fall of 1986 it went practically unnoticed.[1]

Just one Japanese group checked the result and verified it. Soon the phenomenon of high-temperature superconductivity was corroborated by physicists in the United States, China, and the Soviet Union. At the beginning of 1987 the whole world was in a fever, searching for new superconductors and investigating the properties of those already discovered. The critical temperature T_c increased quickly: for La–Sr–Cu–O, $T_c = 45K$, and for La–Ba–Cu–O (under pressure) it reached 52K. In February 1987 the critical temperature of the compound $YBa_2Cu_3O_7$ broke the fabled "nitrogen barrier," having reached 93K.[2] We now know of compounds with $T_c > 100K$—for example, the critical temperature of $Tl_2Ca_2Ba_2Cu_3O_x$ is 125K.

Loners and low budgets

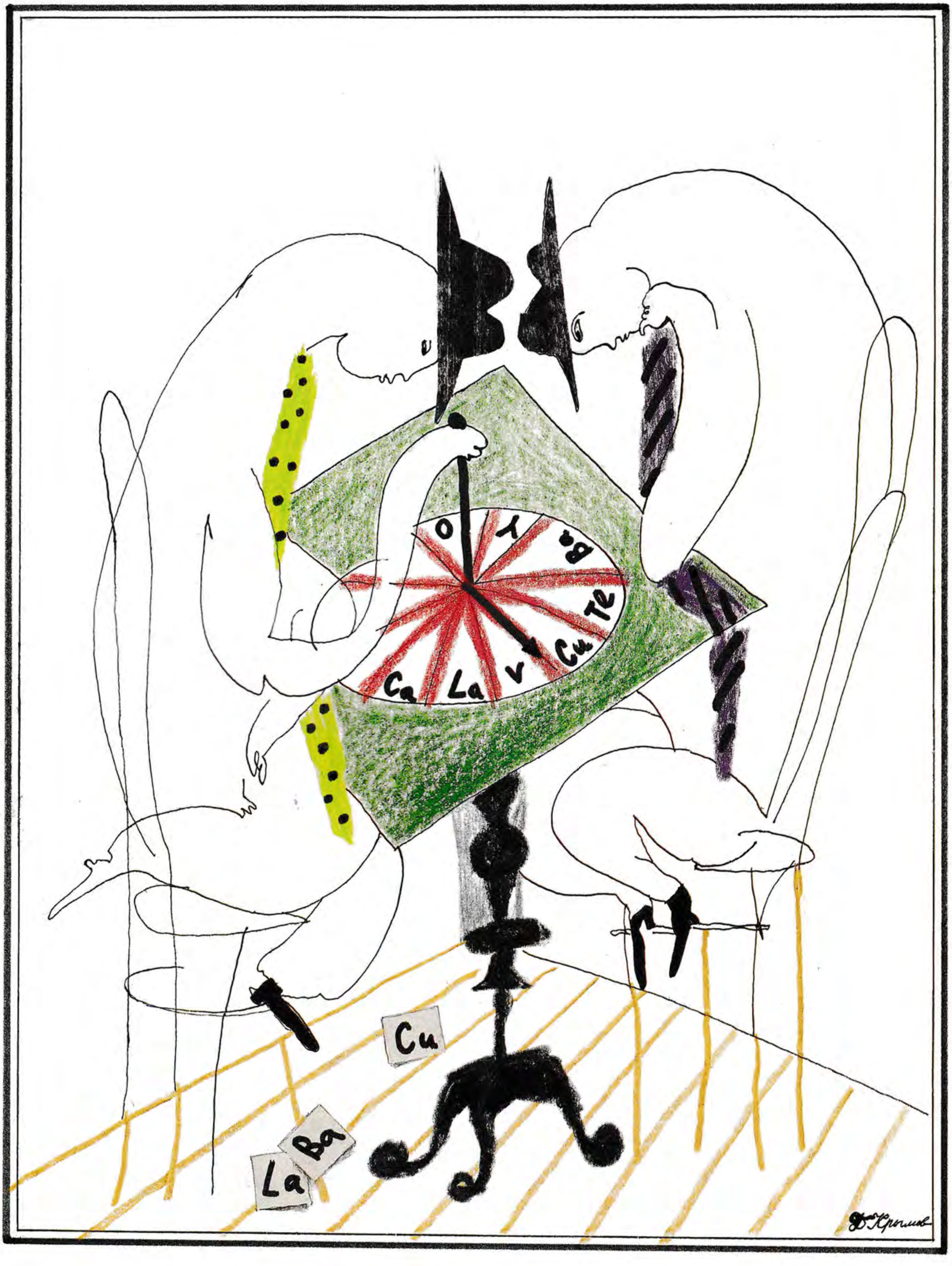
The discovery of high-temperature superconductivity is unique in modern physics. It was discovered by just two scientists with very modest tools. What a contrast with discoveries in other areas of physics—for instance, high-energy particle physics. Here the investigations are conducted by large teams of scientists (the list of authors takes a whole page in a journal article), and the equipment costs millions of dollars. This discovery was cause for optimism: the time of the lone investigator in physics hasn't passed!

This discovery was also striking in that these compounds include elements that are easily obtained—as a matter of fact these superconductors can be made in a high school chemistry lab in a day. And though the discovery had been expected for 75 years, it caught everyone by surprise. Theorists could just shrug their shoulders, and as the critical temperature went up, shrug them even harder.

[1]The Nobel Prize for physics in 1987 was awarded to Müller and Bednorz. The importance of the discovery was evident from the fact that the interval between the publication of the famous paper and the award was just a little over a year.

[2]This temperature could be achieved by using liquid nitrogen, which was much cheaper than the methods used to this point. (See the discussion below about the use of liquid helium and liquid hydrogen in cooling the superconducting compounds.)

Art by Dmitry Krymov



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So was the discovery by Bednorz and Müller a fluke or an inevitability? Could the discovered compound, with its unique properties, have been synthesized earlier? How difficult it is to answer these questions! We have long been accustomed to the fact that everything new is obtained on the edge of the impossible by using unique equipment, superstrong fields, ultralow temperatures, superhigh energies . . . There is nothing of the kind here. It isn't too difficult to "bake" a high-temperature superconductor—a qualified alchemist of the Middle Ages could have managed it.

It's worth recalling that about 10 years ago many laboratories of the world intensively investigated an unusual superconducting compound, Hg_3AsF_6. This substance was called "alchemic gold" because of its yellow luster and high density, which made it resemble the noble metal. It was synthesized by medieval alchemists, passed off as true gold, and advertised as the result of successfully using the "philosopher's stone." Alchemic gold is a complex compound, and who knows, perhaps a high-temperature superconductor could have been baked in the Middle Ages if it had been blessed with a golden luster.

Now that the whole world is intensively investigating the properties of high-temperature superconductors, and instruments based on them are being engineered, many aspects of the history of superconductivity are seen in a different light.

A baffling discovery

Superconductivity, one of the most interesting and unusual phenomena in solid-state physics, first became known on April 28, 1911, at a meeting of the Royal Academy of Sciences in Amsterdam, when the Dutch physicist Heike Kammerlingh Onnes reported a recently discovered effect: the complete disappearance of electrical resistance of mercury cooled by liquid helium to 4.15K. Though no one expected this discovery, and it contradicted the existing classical electron theory of metals, the fact that it was Kammerlingh Onnes who discovered this superconductivity was not accidental. The fact is, he was the first scientist who managed to solve the most complicated scientific and technical problem of the time: obtaining liquid helium (which boils at 4.16K). This allowed scientists to peek into the unknown world of temperatures close to absolute zero.

We'd like to emphasize that the resistance of a sample in the superconducting state is not approximately but exactly equal to zero. That's why electric current in the closed circuit can circulate as long as you like without damping. The longest duration of nondamping superconducting current of about two years was recorded in England. (This current would have circulated in the ring right up until now but for a break in the supply of liquid helium to the laboratory, caused by a transport workers' strike.) Even after two years, no damping of current was detected.

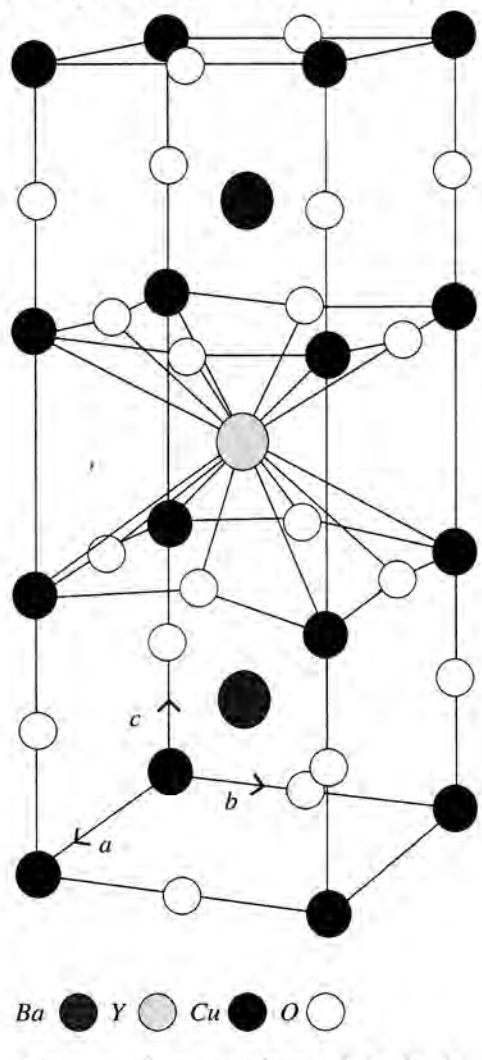
Very soon superconductivity was discovered not only in mercury but in other metals as well. The prospects for practical applications of the discovered phenomenon seemed unlimited: energy transmission over power lines without waste, superpowerful magnets, electric motors, new types of transformers . . . But there were two obstacles. First, the extremely low temperatures at which superconductivity was observed in the materials known to elicit the phenomenon. To cool superconductors to these temperatures, scarce helium is used (its stocks are limited, and helium is very expensive to produce). This makes many projects to apply superconductivity simply unprofitable. The second obstacle, discovered by Kammerlingh Onnes, is connected with the fact that superconductivity turned out to be rather sensitive to magnetic fields (and also to the maximum value of current). In fact, it was destroyed in strong fields.

It took nearly half a century to understand the nature of this wonderful phenomenon and to create a consistent theory. This period can be considered the first stage in the investigation of superconductivity: the stage of gathering information about this complicated effect.

The next fundamental property of the superconducting state discovered, in 1933, was the Meissner–Ochsenfeld effect: the complete expulsion of the magnetic field from the volume of the superconductor. But again experimental investigations were complicated by the need to work with scarce liquid helium—before World War II it was produced in about 10 laboratories throughout the world.

The strange dance of Cooper pairs

The 1950s can be considered the beginning of the second stage of superconductivity research. By this



Structure of superconductive $YBa_2Cu_3O_7$.

time there was qualitative progress in understanding the nature of the phenomenon. Using experimental data and theoretical notions of solid-state physics, based on quantum mechanics and statistical physics, Ginzburg and Landau (USSR) developed the phenomenological theory in 1950, which was followed by a consistent microscopic theory of superconductivity, created by John Bardeen, Leon Cooper, and J. Robert Schrieffer (USA) in 1957. It was found that superconductivity is linked with the appearance of a peculiar attraction of electrons in metals. This phenomenon exhibits strong quantum characteristics.

Here we might find an analogy with two balls lying on a rubber rug. If the balls are far from each other, each of them deforms the rug, making a little "valley." But if we put one ball on the rug and then another one near the first, their holes will come together and the balls will roll down to the bottom of their combined valley. If the temperature is low enough, some electrons form pairs, called "Cooper pairs." The size of these pairs on the atomic scale is really quite large, reaching hundreds and thousands of interatomic distances. According to the strikingly visual comparison suggested by Schrieffer, they should be imagined not as two electrons connected like a double star but like two dancers in a discotheque who come together but may dance in different corners of the hall, separated by dozens of other dancers. Electrons bound in a Cooper pair have a definite binding energy of 2Δ. This is the energy needed to break the pair into isolated particles. Its order at $T = 0$ determines the value of the critical temperature.

According to the laws of the quantum world, the behavior of Cooper pairs completely differs from that of electrons. It was discovered that they are in the common lowest energy—the "ground state." The more particles there are in this state, the easier it is to catch new members and the harder it is for a separate pair to leave this state. This pile is called the "Bose condensate." The concentration of Cooper pairs in the Bose condensate depends on temperature. The fact of their appearance signifies the metal's transition to the superconducting state (which takes place at the critical temperature). As the temperature decreases further, the number of Cooper pairs increases, and at absolute zero there are no free particles left in the system.

So at temperatures below critical, two types of carriers can take part in charge transfer: free electrons and Cooper pairs. But the first may encounter all the usual dangers of life in metal: impurities, lattice scattering, and so on. (These processes determine the resistance in normal metals.) On the other hand, Cooper pairs that stay in ground state can transfer a charge without interacting with impurities or any other defects. Interaction causing a change of energy (which is what causes resistance) means a change of energy in the Cooper pairs, and that is possible only beyond the threshold of 2Δ. Free electrons stay "idle," so to speak—the services of Cooper pairs in charge transfer cost less. So moderately strong current flows in a superconductor because of the condensate of Cooper pairs without producing any heat. Superconductivity turns out to be the consequence of laws from the quantum world on the macroscopic scale. It is a "macroscopic quantum phenomenon."

Searching for new materials

The creation of a theory of superconductivity was a powerful impulse to investigate it in earnest. Without fear of overstatement, we can say that great progress has been achieved in producing new superconducting materials in subsequent years. The Soviet scientist A. A. Abrikosov's discovery of an unusual superconducting state in a magnetic field played a significant role in this development. Before this the magnetic field was thought to be incapable of penetrating the superconducting phase and so was unable to destroy it (which is actually true for most pure metals). Abrikosov theoretically proved that there was another possibility: under certain conditions the magnetic field could penetrate the superconductor as current vortices whose core turned into the normal state but whose periphery remained superconducting! Depending on their behavior of in a magnetic field, superconductors were divided into two groups: superconductors of the first type (old) and those of the second type (discovered by Abrikosov). It's important that a superconductor of the first type can be changed into one of the second type if we "spoil" it by adding impurities or other defects.

Among superconductors of the second type, scientists managed to find compounds capable of carrying a high-density current and bearing gigantic magnetic fields. And although many problems remained to be solved before they could find practical application (the compounds were brittle, high currents were unstable), the fact remained: one of the two major obstacles to the widespread use of superconductors in technology was overcome.

But increasing the critical temperature remained problematic. If the critical magnetic fields were increased thousands of times in comparison with Kammerlingh Onnes's first experiments, the changes in critical temperature weren't very encouraging: it only managed to reach 20K. So for the normal operation of superconducting instruments, expensive liquid helium was still necessary. This was particularly vexing because a fundamentally new quantum effect, the "Josephson effect," had been discovered. This made it possible to use superconductors widely in microelectronics, medicine, instrumentation, and computers.

The problem of increasing the critical temperature was extremely acute. Theoretical evaluations of its peak value showed that within the boundaries of normal phonon[3] superconductivity (that is, superconductivity determined by electron attraction caused by interaction with the crystal lattice), this temperature

[3]A phonon is a quantum of vibrational energy.

could not exceed 40K. But the discovery of a superconductor with such a critical temperature would be a great achievement, since it could be achieved with relatively cheap and available liquid hydrogen (which boils at 20K). It would open the era of "mid-temperature superconductivity" and provide the impetus for research to modify existing superconductors and create new ones. But the ultimate dream was to create a superconductor with a critical temperature of 100K (or, even better, above room temperature), which could be cooled by cheap and widely used liquid nitrogen.

The best result of the search was the alloy Nb_3Ge with a critical temperature of 23.2K. This record temperature was achieved in 1973 and stood for 13 years. Until 1986 the critical temperature couldn't be raised by even one degree. It seemed that the possibilities of the phonon mechanism of superconductivity had been exhausted.

From *Hoechst High Chem Magazine*

A levitating magnet, which has become a symbol of high-T_c superconductivity. The current in the superconductor is induced by the magnet. As long as the temperature is low enough, the current flows continuously. It builds up an opposing field, causing the magnet to hover.

In view of this, in 1964 an American scientist by the name of Little and the Soviet scientist V. L. Ginzburg proposed the following idea: if the possibility of increasing the critical temperature is limited by the nature of the phonon mechanism of superconductivity, this mechanism of electron attraction should be replaced by some other—that is, electrons should form Cooper pairs by means of some mechanism other than phonon attraction.

During the last 20 years many theories were offered, tens or hundreds of thousands of new substances were investigated in detail. In his work Little found his attention drawn to quasi-one-dimensional compounds—long molecular conducting chains with side branches. According to theoretical evaluations, a noticeable increase in the critical temperature could be expected there. Despite attempts by many laboratories throughout the world, such superconductors were not synthesized. But in the process physicist and chemists have made many wonderful discoveries: they obtained organic metals, and in 1980 crystals of organic superconductors were synthesized (the current record for the critical temperature of an organic superconductor is over 10K). They managed to obtain two-dimensional metal–semiconductor "sandwiches"—layered magnetic superconductors where, at last, superconductivity and magnetism coexist peacefully. But there were no new prospects for high-temperature superconductivity.

By this time superconductors had extended their range of application, but the need to cool them with liquid helium remained their weak spot.

Modern alchemists

Let's come back to the discovery by Müller and Bednorz. In the mid-1970s strange ceramic compounds appeared as candidates for high-temperature superconductivity. In their electrical properties at room temperature they were "poor metals," but they became superconducting not too far from absolute zero. "Not too far" means about 10 degrees below the record value at the time. But the new compound could hardly be called a metal. According to theory, the value obtained for the critical temperature wasn't low but was actually very high for such substances.

We should honestly acknowledge here that there was no serious theoretical support for experimental interest in these compounds. Since 1983 Müller and Bednorz worked like alchemists with hundreds of different oxides, varying their composition, quantity, and conditions of synthesis. In this painstaking way they stealthily approached a compound of barium, lanthanum, copper, and oxygen that showed superconducting characteristics at 35K. This happened at the end of 1985.

It isn't easy to find the logic of discovery in modern material science. Until now the main role has been played by intuition, experience, perseverance, and, yes, sheer luck. We'd like to borrow an example from the Soviet scientist A. S. Borovik-Romanov: "For a long time physicists failed to sinter[4] francium on germanium. Then the Dutch physicist H. Kazimir suggested rhenium as an interfacial layer. The rationale behind his choice was that France and Germany are fastened together by a natural element—the Rhine river. The results exceeded all expectations."

And here we'd like to draw an important conclusion: the search for technologically effective superconductors proved to be very unusual. Scientists started with pure metals, moved on to "dirty" alloys, and ended up with metal oxides, which hardly look like metals at all—they're really a kind of clay. Things generally went from simple to complex. Sometimes theory lagged behind experiment; sometimes it gave a powerful impetus to further investigation. Today theory is again in debt to experiment: scientists have created new high-temperature superconductors by "feeling their way," and a satisfactory theoretical explanation for these discoveries has yet to be found. Many characteristics of these superconductors can't be explained within the framework of traditional approaches.

This certainly doesn't mean that theoreticians have been idle: we can enumerate at least 50 new theories of high-temperature superconductivity that have been suggested in the last few years. But we don't need 50, we need one, and that one true theory hasn't been found yet. Elucidating the nature of high-temperature superconductivity remains the most important problem posed its discovery.

Dazzling prospects

We'd like to say a few words about practical applications based on the superconductors discovered. The prospects are truly fantastic. Many of the global projects suggested earlier have been put on the agenda in the hopes that high-temperature superconductors will make them commercially feasible. For example, at present 20–30% of all electrical energy produced is wasted in power transmission lines. Using high-temperature superconductors for energy transmission could eliminate these losses. Scientists have already managed to sinter high-temperature superconducting films capable of transmitting currents with a density of $106 A/cm^2$ at the temperature of liquid nitrogen.

All projects involving thermonuclear synthesis entail the use of giant superconducting magnets to keep high-temperature plasma away from the walls of the chamber. To maintain the superconducting state streams, if not rivers, of liquid helium are needed. The helium would be replaced by nitrogen at a tremendous cost saving.

Gigantic superconducting coils would serve as accumulators of electrical power, which would be tapped during peak periods.

Supersensitive equipment for making magnetocardiograms and magnetoencephalograms, based on the use of superconducting Josephson elements, would come to be used in every hospital.

A new generation of supercomputers based on superconducting elements and cooled by liquid nitrogen would be created.

Don't think we've lost our heads over superconductivity. Since its discovery, the ardor of many investigators has cooled significantly. The same thing happens when an Olympic record stays out of reach for years. But the record has been set, and now it serves as a benchmark. The possibility of producing materials with the necessary unique characteristics has been confirmed. Certainly many serious problems connected with the production of technologically effective high-temperature superconductors remain to be solved. Economic considerations will affect how the projects mentioned earlier are carried to fruition. But what's important is that today we know *the impossible has become accessible*. And this has irreversibly changed the reference point in our attitude toward superconductivity. ◼

[4]"Sintering" means creating a coherent mass out of components by heating without melting them. Superconducting compounds are synthesized by pressing and sintering finely ground oxides at high temperatures.

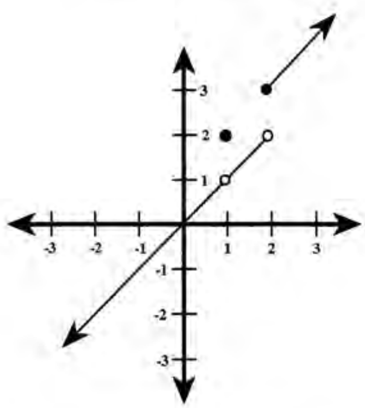
Holes in graphs

Considerations of discontinuity

by Michael H. Brill and Michael Stueben

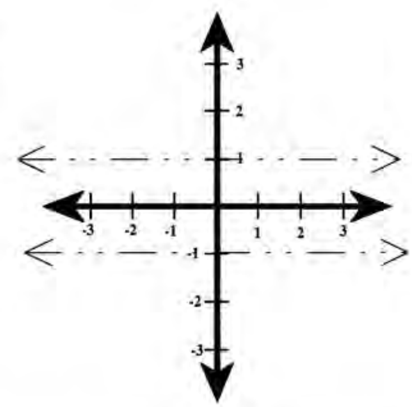
HOLES IN GRAPHS ARE surprisingly counterintuitive and can give rise to functions with astonishing properties. Everybody knows that $g(x) = x$ is a continuous function and that $h(x)$, as given below, is a discontinuous function:

$$h(x) = \begin{cases} x+1, & \text{if } x > 2 \\ 2, & \text{if } x = 1 \\ x, & \text{otherwise.} \end{cases}$$



Technically h is described as an "almost everywhere continuous function" because it is discontinuous only at a finite number of points. We call these points "holes." More interesting is a function that has holes everywhere:

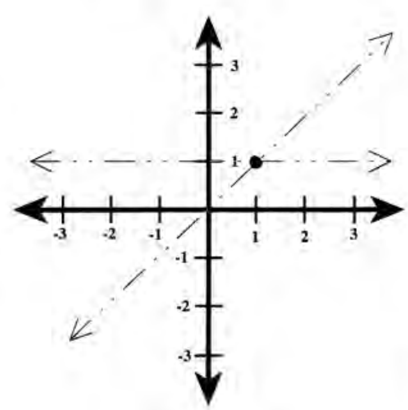
$$j(x) = \begin{cases} +1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational.} \end{cases}$$



Notice that the absolute value of j has no holes.

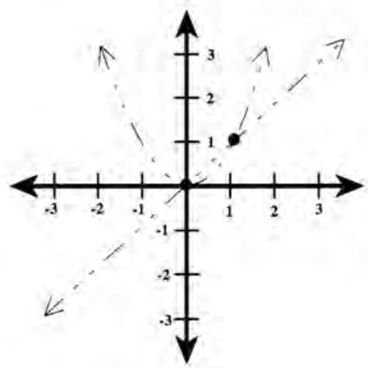
Most people think of a continuous function as being continuous on the points of an interval, but a function can be continuous at an isolated point—for example,

$$p(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$



A slight modification gives us another function that is continuous at exactly two points:

$$q(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ x^2, & \text{if } x \text{ is irrational.} \end{cases}$$



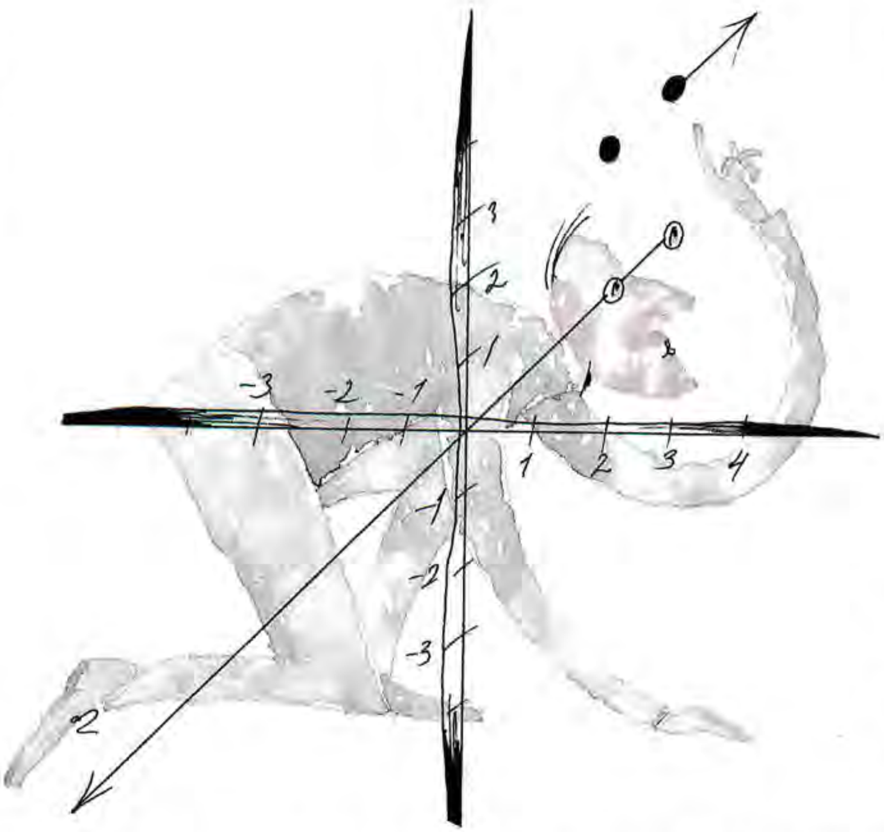
Can a real function be discontinuous everywhere except on the integers? Yes, it can.

Problem 1. Taking some hints from the previous examples, see if you can invent one yourself.

Consider the function $k(x) = x^2/x$ defined on the positive and negative numbers. Is this function continuous? Clearly k is continuous in its domain[1] but discontinuous over the entire set of real numbers, because it

[1]A function is sometimes taken to be a rule applied to the set of real numbers. The largest subset of the real numbers for which this rule gives a result is called the "domain" of the function.—*Ed.*

Art by Sergey Ivanov



is undefined at zero. The phrase "continuous function" is ambiguous without a reference set. In some textbooks the function k is referred to as having a "removable discontinuity"; in other books it's considered continuous. In this article we'll use the second definition of continuity and call a real function continuous if and only if it is continuous over its domain.

DEFINITION. *(1) A function f is continuous at a number c in its domain if and only if to each positive ε there is a positive δ such that $|f(x) - f(c)| < \varepsilon$ whenever x is in the domain of f and $|x - c| < \delta$.*

(2) A real function is a continuous function if and only if it is continuous at all points in its domain.

Now that we have defined continuity, two questions arise:

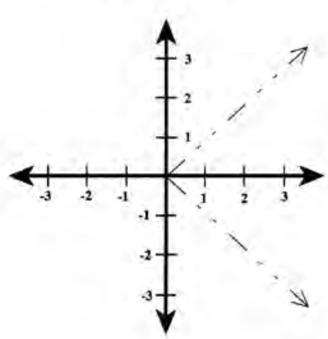
> (1) Can the graph of a continuous function also be the graph of a discontinuous function?
>
> (2) Could a function be discontinuous and its inverse continuous?

Our intuition says "no" and "no." But, incredibly, the answer to both questions is "yes." Take a few minutes to try to construct such a function.

We'll offer a simple algebraic function (that is, something from high school algebra) that has both properties. The function we have in mind maps the positive x-axis onto a subset of the y-axis.

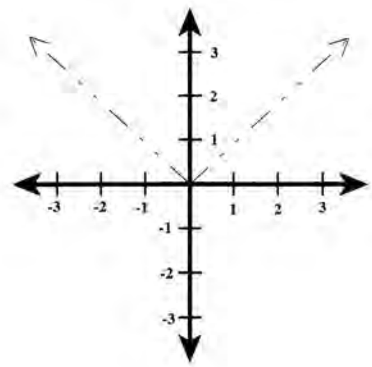
The simplest example we know is mapping the positive rational numbers onto the positive y-axis and the positive irrational numbers onto the negative y-axis. Some people call this a "salt and pepper" function:

$$F(x) = \begin{cases} +x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational.} \end{cases}$$



The inverse of F can also be written as

$$F^{-1}(x) = |x|,$$



where x is constrained to be positive if x is rational, and x is constrained to be negative if x is irrational. Clearly F^{-1} is continuous even though F is discontinuous. But the graph of F^{-1} is the same as the graph of F except that it is reflected about the line $x = y$. The reflection should not disturb continuity. It then appears that, depending on your point of view, F is continuous or discontinuous.

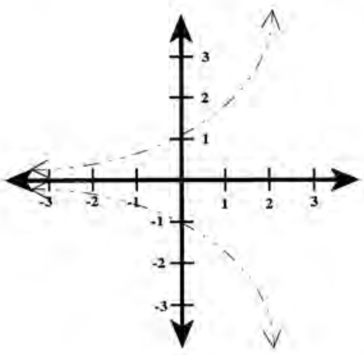
How can the same function be both continuous and discontinuous? The answer is that the domain of F^{-1} (the nonnegative rationals and negative irrationals) is different from the domain of F (the nonnegative reals). So F^{-1} is continuous on an emaciated domain, but F is discontinuous over a robust domain. This is a striking reminder that any definition of a function must include its domain. Change the domain and you'll change the properties of the function.

We'll call discontinuous functions with continuous inverses Thurston functions.[2] Here's a Thurston function that maps all the reals into a subset of the reals:

---

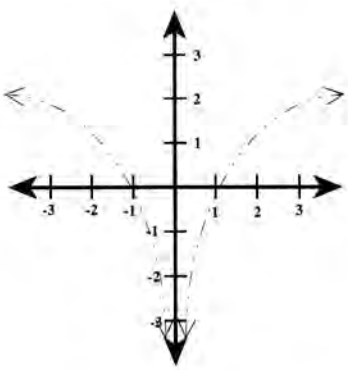
[2]Hugh Thurston writes that "a function can be continuous even though its domain is, so to speak, full of holes. Intuition really comes into play for functions without this property, such as functions with compact domains." (*American Mathematical Monthly*, vol. 96 (1989), p. 814) Thurston is speaking of domain-continuity, not line-continuity.

$$T(x) = \begin{cases} +2^x, & \text{if } x \text{ is rational} \\ -2^x, & \text{if } x \text{ is irrational.} \end{cases}$$



This is a discontinuous function. The continuous inverse function dovetails two distinct sets into the real number line:

$$T^{-1}(x) = \log_2|x|,$$



where x is constrained to be positive if $\log_2|x|$ is rational, and x is constrained to be negative if $\log_2|x|$ is irrational. (Question: Does there exist a Thurston function that maps the entire x-axis onto the entire y-axis? We bet not.)

Regarding such curiosities as Thurston functions, Felix Klein once said, "Every person believes that he knows what a curve is until he has learned so much mathematics that the countless possible mathematical abnormalities confuse him. The fundamental idea is that we think of the curve as the limit of an inscribed polygon."

Similarly, we all believe we know what properties a continuous function can have until we encounter "pathological" counterexamples. The fundamental idea is that we should think of a domain-continuous function in terms of the function, not the graph.

Is it possible for a real function to be defined on the entire x-axis so that it is continuous on all the rational numbers and discontinuous on all the irrational numbers? No, but surprisingly the reverse is possible: a function can be defined on the entire x-axis so that it is continuous on the irrationals but discontinuous on the rationals.

The following ingenious example of this assertion is sometimes called Dirichlet's function:

$$D(x) = \begin{cases} 0, & \text{if } x \text{ is irrational or zero} \\ 1/Q, & \text{where } x = P/Q \text{ and} \\ & P/Q \text{ is reduced (that} \\ & \text{is, } \mathrm{GCF}(P, Q) = 1). \end{cases}$$

It's not hard to show that this function is continuous on the irrationals. For example, let's show that $D(x)$ is discontinuous at the irrational number π. Since $D(\pi) = 0$, we place an interval around 0. Say the interval is $(-0.1, 0.1)$. Can we find an interval around π on the x-axis so that all points in this x-interval are mapped into the y-interval? Yes: just plot all reduced fractions on the x-axis that have a denominator Q, where $|Q| \leq 10$. Then place a small interval around π that doesn't include one of these points. All points in this interval must be mapped into the y-axis so as to be in the interval $(-0.1, 0.1)$. So Dirichlet's function is continuous at π.

Another continuity curiosity involves the use of axes with different properties. The word isotropic[3] as applied to the xy-axes means that the x-axis and the y-axis have the same properties. But suppose we allow the metric (that is, the definition of distance) on the y-axis to be different from the metric on the x-axis. Consider any one-to-one function f and an anisotropic xy-coordinate system in which the distance between points a and b on the x-axis is $|a - b|$, as usual, but the distance between points $f(a)$ and $f(b)$ on the y-axis is not $|f(a) - f(b)|$ but rather $|a - b|$. This means that all one-to-one functions must be continuous regardless of how violently they rip the domain apart. Although rarely mentioned, the property of continuity is dependent on the metric(s) as well as the function involved.

Many of the functions we've presented are easy to understand but psychologically difficult to find. Our final example is a striking instance of this difficulty. Believe it or not, if you remove all the rational numbers from the real number line, it's possible to rearrange the remaining irrationals to fill the holes without creating any new holes. In other words, there exists a one-to-one function from the irrationals onto the real numbers. To understand the trick involved, first consider the function that maps the half-open interval $[0, +1)$ one-to-one onto the closed interval $[0, +1]$. Each

CONTINUED ON PAGE 19

Timeout for terminology

A **function** $y = f(x)$ is a mapping that takes any element in its domain of possible x values to exactly one value y.

A function $y = f(x)$ is said to be **one-to-one** if, when a particular x is mapped to a particular y, there is no other value of x that maps to that value of y. Example: Whereas $f(x) = x^3$ is a one-to-one function, $f(x) = x^2$ is not.

With respect to a prespecified range of y values, a function $y = f(x)$ is said to be **onto** if, for any y in the prespecified range, there is an x such that $f(x) = y$. Examples: The function $f(x) = x^3$ maps the real numbers onto the real numbers. The function $f(x) = x^2$ maps the real numbers *into*, but not *onto*, the entire range of real numbers; however, $f(x) = x^2$ does map the reals onto the *nonnegative* reals.

[3] From the Greek *isos* (equal) and *tropikos* (turn or change).

HOW DO YOU FIGURE?

Challenges in physics and math

Physics

P31

Two sacks. A sack whose mass is m_1 slides along the horizontal surface of a table. It is connected with another sack whose mass is m_2 by a weightless string. This string goes through a little hole in the table (fig. 1). The string's length is L, the table's height is H, and $H < L$. To what height will the second sack be lifted after it touches the floor if at first all the string was on the table and the sacks were at rest? (Omit the influence of friction.) (G. Kotkin)

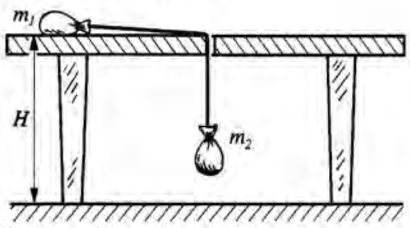


Figure 1

P32

Ice bucket. A bucket contains a mixture of ice and water of mass $m = 10$ kg. The bucket is brought into a room, after which the temperature of the mixture is immediately measured. The dependence $T(\tau)$ is plotted in figure 2. The specific heat of water is $c_w = 4.2$ J/(kg · K), and the latent heat of fusion of ice is $\lambda = 340$ kJ/kg.

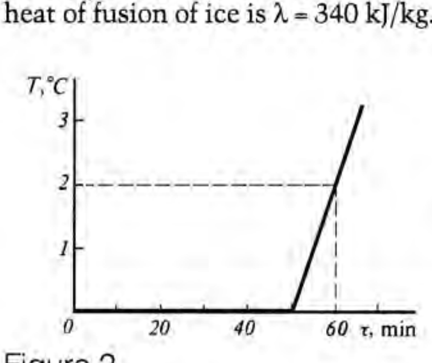


Figure 2

Determine the mass m_1 of ice in the bucket at the moment it is brought into the room, ignoring the heat capacity of the bucket. (A. Buzdin)

P33

Bouncing molecule. A cubic vessel whose volume is $V = 1$ liter contains $m = 0.01$ g of helium at a temperature $T = 300$K. Let's follow the motion of a molecule. How many times will it hit the top of the vessel during $t = 1$ min? (A. Zilberman)

P34

Alpha scattering. A point source of alpha particles emits particles uniformly in all directions. A 20 cm x 20 cm photographic plate is set 10 cm from the source. After 10 seconds of exposure it has 200 traces of alpha particles. How many particles does the source emit in an hour? (V. Volkov)

P35

Snow shadows. Gallery Q in the November/December 1990 issue of *Quantum* contained a reproduction of Franz Marc's "Siberian Dogs in the Snow." Curiously, the dogs' shadows are blue. Now take another look at the inside front cover of this issue: blue shadows again! Wouldn't it be more correct (from a physicist's point of view) to paint the shadows as dark and colorless (that is, black or gray)? (A. Buzdin, S. Krotov)

Math

M31

Maximin and minimax. Some ink was spilled on a sheet of paper. For every point of the blot, the shortest distance and the greatest distance to the blot's boundary were measured. Let r be the greatest of the shortest distances and R the shortest of the greatest distances. What shape is the blot if $r = R$? (A. Blokh)

M32

Inevitable divisibility. The set of all positive integers 1, 2, 3, ... is partitioned into several arithmetic sequences. Prove that the first term of at least one of these sequences is divisible by its difference. (A. Kelarev)

M33

Four intersecting circles. Two congruent circles intersect at points A and B. Two more circles of the same radius are drawn: one through A, the other through B (fig. 3). Prove that the four points of the paired intersection of all four circles (other than A and B) are the vertices of a parallelogram. (V. and I. Kapovich)

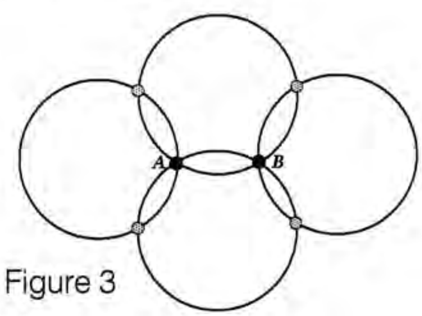


Figure 3

M34

Midpoint sums. Positive integers are written at the points of a segment according to the following rule: at the first step two 1's are written at the ends of the segment; at the second step their sum 2 is written in the middle; at each subsequent step the sum of every pair of neighboring numbers (obtained from the previous steps) is written in the middle of the

CONTINUED ON PAGE 27

AT THE BLACKBOARD

Is this what Fermat did?

An excursion into factorization

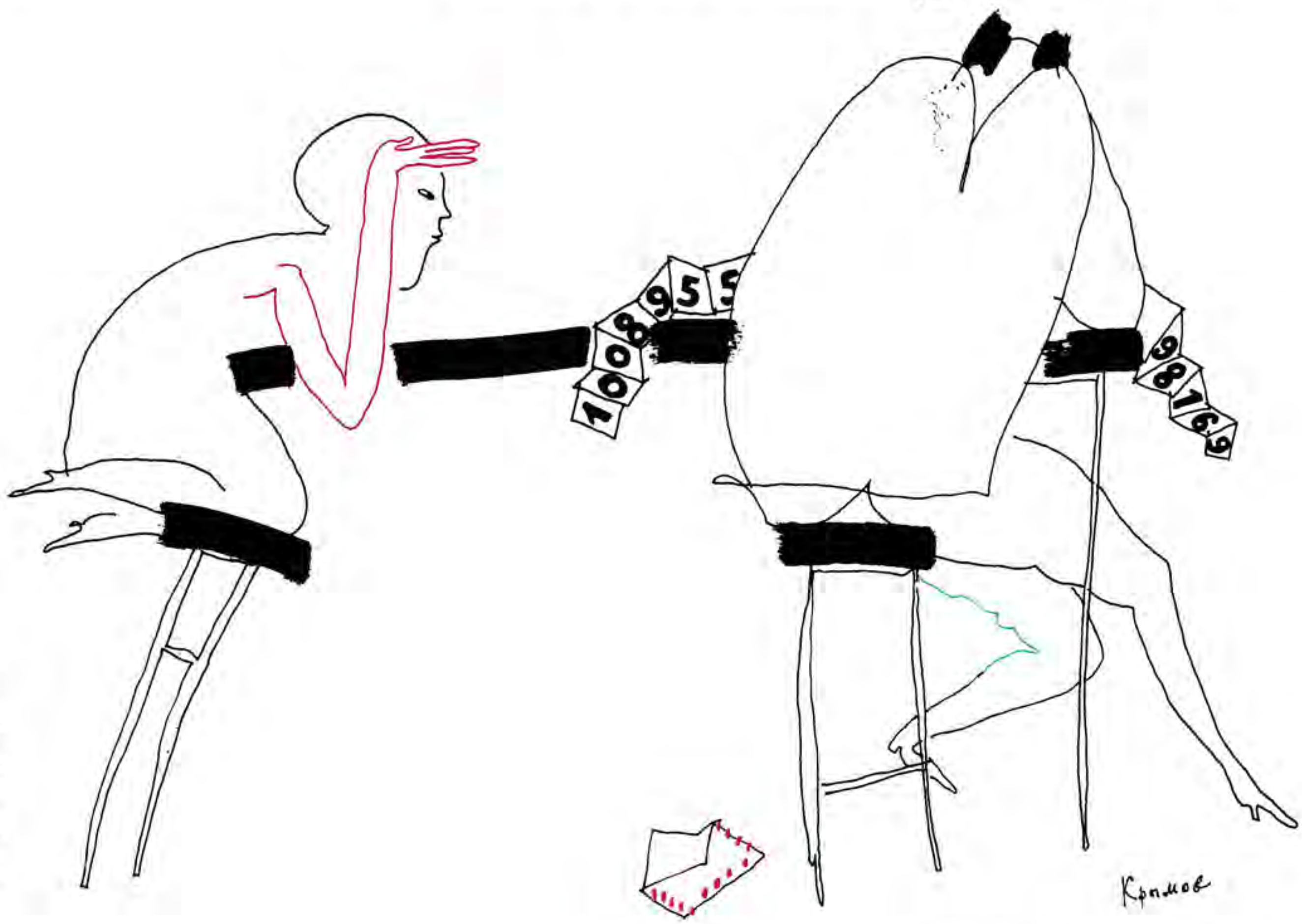
by B. A. Kordemsky

MANY MYSTERIES SURROUND the name of the famous French mathematician Pierre Fermat (1601–1665). Once he received a letter with the question: "Is 100,895,598,169 a prime number?" Fermat answered without delay that the twelve-digit number was the product of two prime numbers: 898,423 and 112,303.

He didn't say how he arrived at that.

Finding prime factors for a natural number is called "factorization." Even with the help of modern computers, the factorization of a large number is an exceptionally tedious task, to say nothing of the "manual approach." Several of the first prime numbers (2, 3, 5, 7, 11, ...) can easily be tested for their viability as possible factors of a number in question—there are well-known indications of divisibility by these numbers. (See exercise 1 at the end of this article.) Knowing these tell-tale signs that subsequent prime numbers are factors greatly simplifies calculations.

It's also clear that in searching for prospective factors of any given number N, it is sufficient to test prime numbers smaller than $\sqrt{N}$. Indeed, if a number N has a factor $m > \sqrt{N}$, then it also has a factor, obtained by dividing N by m, that is less than $\sqrt{N}$.



Art by Dmitry Krymov

One of the ways of finding the prime factors of N is to calculate the greatest common divisor (GCD) of N and another number having a known factorization. This can be done by using the well-known algorithm of Euclid.

The underpinnings of Euclid's algorithm and examples of how to apply it are given in the article "Divisive Devices" on page 36. Here's a brief reminder of how it works. First, we find the remainder r_1 from the division of the larger of two numbers by the smaller; then we find the remainder r_2 from dividing the previous divisor by r_1; then the remainder r_3 of r_1 divided by r_2; and so on. The last nonzero remainder (which must certainly exist, since each successive number r_i is less than the preceding one) is the GCD of the given numbers. (If it's equal to 1, the numbers are coprime).

By way of example, let's apply this algorithm to the numbers 104 and 39:

$$104/39 = 2 \text{ (remainder 26)};$$
$$39/26 = 1 \text{ (remainder 13)};$$
$$26/13 = 2 \text{ (remainder 0)}.$$

Answer: GCD (104, 39) = 13.

How can we use Euclid's algorithm for factorization?

To find the prime factors of a number N let's construct another number P that is the product of all successive prime numbers from the lowest "suspected" factors of N to the largest of all primes that are less than $\sqrt{N}$. N and P are the numbers we'll plug into Euclid's algorithm.

For instance, let $N = 851$. Notice that $\sqrt{N} < 31$. Looking for signs of divisibility, we determine that N is not divisible by 3, 7, 11, or 13. It's also obvious that 851 yields a remainder 1 when divided by 17 (notice that $85 = 5 \cdot 17$). We are left to examine whether N is divisible by 19, 23, or 29. For a number as small as 851, this can be done easily by a direct computation, dividing 851 by each of the prospective factors. But to give you a better grasp of the method, we'll proceed in a way that can later be applied to larger numbers.

Let's construct the number $P = 19 \cdot 23 \cdot 29 = 12{,}673$ and run Euclid's algorithm: $12{,}673/851 = 14$ (remainder 759); $851/759 = 1$ (remainder 92); $759/92 = 8$ (remainder 23); $92/23 = 4$ (remainder 0). So 23 is the GCD of N and P and, consequently, one of the factors of 851. Dividing 851 by 23 we get 37, which is also a prime number.

The factorization of 851 is now complete: $851 = 23 \cdot 37$. For the number that was proposed to Fermat, similar calculations would have taken much longer. (Try it yourself!) Fermat used a different approach, no doubt. But what was it?

On the verge of discovery?

A modern book on mathematics suggests that "certain mathematicians of the 17th century who devoted a great deal of effort to developing number theory had ways, unknown to us now, of recognizing prime numbers." But since these computational wizards didn't disclose their secrets of factorization to their descendants, some methods invented later might actually have repeated their discoveries.

Fermat, one of the creators of number theory, used many properties of numbers in his calculations. In particular, he undoubtedly knew that any odd number N (as well as any even number divisible by 4) can be represented as the difference of the squares of two integers x and y:

$$\begin{aligned} N &= a \cdot b \\ &= \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 \\ &= x^2 - y^2, \end{aligned}$$

where a and b ($a > b$) are any possible odd factors of the odd number N (then $a + b$ and $a - b$ are even numbers, and x and y are integers).

If N is a prime number, then $a = N$, $b = 1$, and the factorization $x^2 - y^2 = (x + y)(x - y)$ is unique and yields no other factors except N and 1. But if N is a composite number, there is a factorization $(x + y)(x - y)$ that gives at least one pair of factors different from N and 1. For instance, the prime number 17 has only one representation as a difference of squares: $17 = 9^2 - 8^2 = 17 \cdot 1$; the composite number 203 has two such representations:

$$203 = 102^2 - 101^2 = 203 \cdot 1$$

and

$$203 = 18^2 - 11^2 = 29 \cdot 7.$$

So the "factorization laboratory" is equipped with yet another tool, which we'll call "factorization by the difference of squares." To select the required squares x^2 and y^2, we can use the following algorithm: (1) find the least square x^2 exceeding the given number N (for instance, using a table of squares or taking the square root of N and rounding up); (2) subtract N from x^2.

If the difference is a perfect square (that is, if $x^2 - N = y^2$), the selection process is over: $N = x^2 - y^2 = (x + y)(x - y)$. If it isn't, we keep trying: we subtract N from the next square, and repeat the procedure until we get a difference that's a perfect square.

Let's see how this algorithm works by looking for the factors of a couple of numbers: say, $N_1 = 153{,}583$ and $N_2 = 689$.

For N_1 we have $\sqrt{153{,}583} \cong 392$; $392^2 = 153{,}664$; $153{,}664 - 153{,}583 = 81 = 9^2$. So $153{,}583 = 392^2 - 9^2 = 401 \cdot 383$, and both factors are prime numbers. Notice that they're quite close to one another and, consequently, to $\sqrt{N}$. That's why we found the answer so quickly.

For $N_2 = 689$, the nearest square in excess of that is $729 = 27^2$. So we get

$$\begin{aligned} 27^2 - N_2 &= 729 - 689 = 40; \\ 28^2 - N_2 &= 784 - 689 = 95; \\ 29^2 - N_2 &= 841 - 689 = 152; \\ &\dots\dots\dots\dots \\ 33^2 - N_2 &= 1{,}089 - 689 = 400 = 20^2. \end{aligned}$$

Consequently, $689 = 33^2 - 20^2 = 53 \cdot 13$.

It took us seven tries before we succeeded! Comparing the factors of 689, we see that the difference between them is quite large. That's what lengthened our computation.

A neat trick

When we start factoring a composite number N, we don't know in advance, of course, whether its factors are close to each other. But if a number of consecutive steps of the algorithm didn't produce the desired perfect square, it's clear that the factors in question are far from $\sqrt{N}$.

Here we can be a little tricky: we start the procedure all over again after multiplying the given number N by, say, 3 (to make sure that it stays odd). This triples the smallest of the two factors of N and makes the factors of $3N$ closer to one another and, consequently, to $\sqrt{3N}$.

If there is reason to assume an even greater difference between the factors of N, we can immediately multiply N by 5, 7, or 8 (in the last case we get an even number, but it's of the sort that can be represented as the difference of squares of integers). Multiplying N by 2 would in any case be useless, and multiplying by 4 would be pointless. (You can prove this yourself.)

Let's get back to $N_2 = 689$ and do our trick, multiplying N_2 by 5. This yields $5 \cdot N_2 = 3{,}445$; $\sqrt{3{,}445} \cong 59$; $59^2 = 3{,}481$; $3{,}481 - 3{,}445 = 36 = 6^2$. So we get $3{,}445 = 59^2 - 6^2 = 65 \cdot 53$; $5N_2 = 65 \cdot 53$; $N_2 = 53 \cdot 13$.

We've succeeded on the first attempt—last time it took seven tries!

Maybe that's what Fermat did

We want to apply the technique of factorization by the difference of squares to $N = 100{,}895{,}598{,}169$, but now we'll be so bold as to introduce an additional factor. Let our intuition lead us to the factor 8 (we'll just say the prospect of trying smaller factors didn't inspire us with confidence).

We get $8N = 807{,}164{,}785{,}352$. Then we find the smallest number whose square is greater than $8N$:

$$\sqrt{807{,}164{,}785{,}352} = 898{,}424$$

(rounded up).

Then: $898{,}424^2 - 8N = 898{,}424$.

Although this difference isn't a square, there's no point in applying the algorithm further: our boldness has been crowned with unexpected success—the common divisor 898,424! Factorization of $8N$ is now achieved by a simple calculation:

$$\begin{aligned} 8 \cdot N &= 898{,}424 \cdot (898{,}424 - 1) \\ &= 8 \cdot 112{,}303 \cdot 898{,}423. \end{aligned}$$

Finally: $N = 112{,}303 \cdot 898{,}423$.

We don't know whether this was what actually happened in Fermat's "laboratory." The facts are lacking. At any rate, I hope you took some pleasure in our speculative excursion into the past.

Exercises

1. Let $\overline{a_n a_{n-1} \cdots a_1 a_0}$ be the decimal notation of a number N. Prove that the remainders of N when divided by 7, 11, or 13 are the same as the respective remainders of the number $\overline{a_2a_1a_0} - \overline{a_5a_4a_3} + \overline{a_8a_7a_6} - \cdots$. In particular, when N has no more than 3 digits, $N = \overline{a_2a_1a_0}$, the three remainders are equal to those of the numbers $2a_2 + 3a_1 + a_0$, $a_2 - a_1 + a_0$, and $-4a_2 - 3a_1 + a_0$, respectively.

2. Find the GCD of 80,887 and 40,091.

3. Prove that $N = 55{,}637$ has only one prime factor smaller than 30 (use the number $P = 17 \cdot 19 \cdot 23 \cdot 29 = 215{,}441$) and find all the other factors of N.

4. Apply factorization by the difference of squares to break 131,289 down into prime factors.

5. Factor 500,207 by the difference of squares. (Apply the "trick" of multiplying by 3.)

6. Applying factorization by the difference of squares directly to the number $N = 20{,}099$, verify that $20{,}099 = 199 \cdot 101$.

How many steps did it take? After you know the result, explain why the best factor for reducing the number of steps is 8. How many steps are necessary to factor $8N$? ◙

"HOLES IN GRAPHS" CONTINUED FROM PAGE 14

point maps to itself except $f(1/2) = 1$, $f(1/4) = 1/2$, $f(1/8) = 1/4$, and in general $f(1/2^N) = 1/2^{N-1}$:

$$f(x) = \begin{cases} 2x, & \text{where } x = \pm\dfrac{1}{2^N} \\ x, & \text{otherwise.} \end{cases}$$

This clever function was shown to us by David Rosen of Carnegie-Mellon University. How clever is this idea? Ask the next algebraist you meet to find it in a day. And here's a challenge for you.

Problem 2. Extend Rosen's idea by finding a one-to-one "onto" function from the irrational numbers to the set of real numbers.

There are many different solutions, and you may be able to improve and extend our answer.

If you work through the problems presented here, we think you'll agree with us that pathological functions have a beauty all their own! ◙

ANSWERS ON PAGE 63

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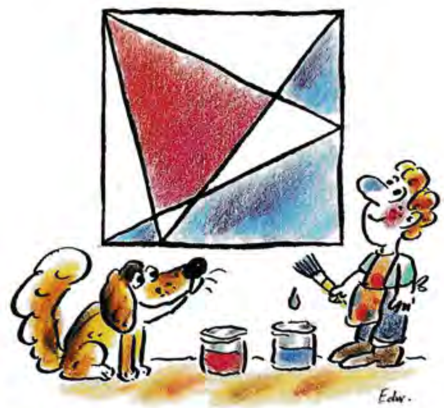
Just for the fun of it!

B31

Prove that at least one of any 18 successive three-digit numbers is divisible by the sum of its digits. (S. Yeliseyev)

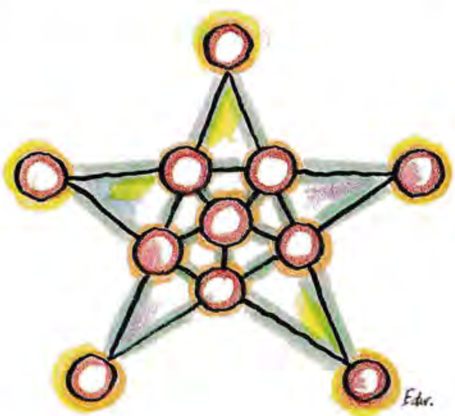


B32

Which part of the square in the figure at left has the greater area: the red part or the blue part? (V. Proizvolov)

B33

In some antique clocks intended for operation in the open air, the pendulum was a long tube with a container of mercury at the bottom. What was the purpose of this design? (A. Buzdin)



B34

Write the numbers 1 to 11 in the circles in the figure at left so that the sum of the four numbers at the vertices of each of the five sectors of the star equals 25. (N. Avilov)

B35

Is it possible to wrap a cube in the stairlike piece of paper in the figure at right so that its entire surface is covered without overlaps? (N. Dolbilin)

ANSWERS ON PAGE 60

Art by Edward Nazarov


TISHKOV

The power of likeness

But analogy can take us only so far

by S. R. Filonovich

"ANALOGY" IS A WELL-used word. We've gotten so used to it that sometimes we don't pay much attention to its meaning. One of the meanings we can find in Webster's Ninth New Collegiate Dictionary is "resemblance in some particulars between things otherwise unlike." Using analogies, we can acquire knowledge about an object by studying another, different object.

Analogies play an important role in physics. Through analogy we can come to significant conclusions without strict and painstaking calculations. For instance, an analogy between sound and light was important in developing the wave theory of light in the beginning of the 20th century. In the 1920s a famous optical–mechanical analogy greatly facilitated the creation of quantum theory.

Analogies can help you when you study physics in school. But every analogy has its weaknesses: if we push the analogy too far we can draw the wrong conclusions. The intuition we develop in solving physics problems can help us use the powerful tool of analogy properly. In this article we'll look at some examples that will help us understand how to use physical analogy.

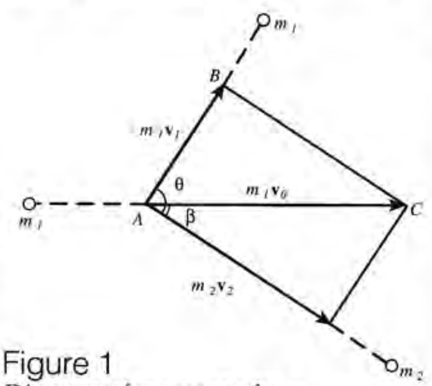


Figure 1
Diagram of momenta for a noncentered collision of two balls.

Art by Leonid Tishkov

A problem for kids

In school textbooks you often come across problems in which balls collide. These problems are popular because they are a relatively simple manifestation of fundamental laws of the conservation of energy and momentum.

To a certain extent the study of collisions is a bow to tradition. The problem was the focus of attention of Descartes and Huygens in the 17th century, and in the 19th century the French physicist Gustave-Gaspard Coriolis published his *Mathematical Theory of the Game of Billiards*, which became a classic. But we mustn't suppose that nowadays the problem of the collision of balls can be used only to illustrate physical laws. It turns out that the model of collision is closely related to modern physics problems. To see how, let's look at a very simple problem.

Problem. A ball of mass m_1 moving with velocity v_0 strikes an immobile ball of mass m_2. After an elastic impact, the balls move apart. Determine the relative change in the kinetic energy of ball m_2 after the impact if we ignore rotation of the balls.

Solution. Let's consider the general case of a noncentered collision, such that after the impact the velocities of the balls form angles θ and β with the original direction of $\mathbf{v}_0$. Figure 1 illustrates the law of conservation of linear momentum for the case of elastic impact we're examining. It follows that

$$(m_2v_2)^2 = (m_1v_1)^2 + (m_1v_0)^2 - 2m_1^2v_1v_0\cos\theta. \quad (1)$$

The law of conservation of energy (since the collision is elastic!) gives

$$\frac{m_1v_0^2}{2} = \frac{m_1v_1^2}{2} + \frac{m_2v_2^2}{2}. \quad (2)$$

From equations (1) and (2) we can find v_1 for any fixed scattering angle θ. We won't write the general expression (you can do that yourselves). Let's write the answer for the particular case of a head-on collision, such that after the impact the balls move along the line AC:

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2}v_0.$$

For the same case we'll write the relative change in the kinetic energy of the approaching ball m_1:

$$\frac{\Delta T}{T} = \frac{v_0^2 - v_1^2}{v_0^2} = \frac{4(m_1/m_2)}{(1 + m_1/m_2)^2}. \quad (3)$$

There's nothing particularly surprising about this expression. (Maybe some of you have already found it on your own.) Nevertheless we'll study it in some detail. (This effort can be taken as a verification of the solution by means of "common sense.") When $m_1 << m_2$, $\Delta T/\Delta T_0 \to 0$; that is, ball m_1 bounces off ball m_2 in the direction opposite the velocity $\mathbf{v}_0$ but without change in the absolute value of its velocity. When $m_1 >> m_2$, $\Delta T/\Delta T_0 \to 0$ as well, since the collision with the very light ball m_2 doesn't disturb the motion of ball m_1. Obviously at the intermediate values of the ratio m_1/m_2 the relative change in the kinetic energy $\Delta T/T_0 \neq 0$. We can easily see that the maximum of $\Delta T/T_0$ corresponds to $m_1/m_2 = 1$ (fig. 2).

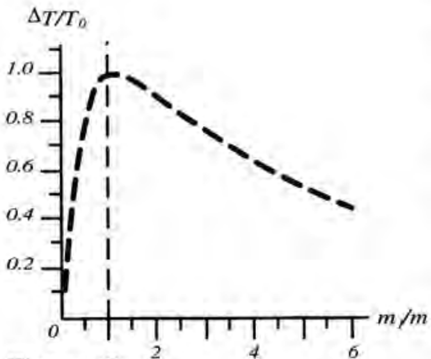


Figure 2
Relative change in the kinetic energy of a moving ball in a head-on elastic collision as a function of the mass ratio.

Using strict scientific terminology we can say that the moving ball scatters its energy most efficiently when it strikes an immobile ball of the same mass. Since we ignored the rotation of the balls and their internal structure in our solution to the problem, we can try to apply this conclusion to phenomena in other domains of physics.

Useful losses

Physicists had reason to recall the problem of colliding balls in the 1940s with regard to the chain reaction of nuclear fission. We can describe the gist of the problem as follows.

When a uranium nucleus absorbs a neutron, it splits into two almost equal parts and an enormous amount of energy is released. New neutrons are released as well. The mean number of these new neutrons is more than one, so "multiplication" of neutrons occurs. The neutrons born in the fission reaction may be absorbed by uranium nuclei, which will cause new fission reactions, and so on. The number of neutrons will increase, and more and more nuclei will split. This process is called a chain reaction, and the energy obtained can be put to some use.

Unfortunately, this simple description doesn't correspond to reality. In the natural state there are two kinds of uranium, called isotopes, with different mass numbers: ${}_{92}U^{235}$ and ${}_{92}U^{238}$. The nuclei of both isotopes can be split, but for a fission reaction with ${}_{92}U^{238}$ the captured neutron must have kinetic energy of more than 1 MeV. If the energy is less than that but not too low, the ${}_{92}U^{238}$ nucleus captures the neutron without fission. On the other hand, ${}_{92}U^{235}$ nuclei capture only slow neutrons with energy much less than 1 MeV—from 70 to 200 eV. But after the uranium nucleus splits, the emitted neutrons have energy less than 1 MeV but much more than 200 eV. As a result, they can't cause fission of ${}_{92}U^{238}$ nuclei. In natural uranium the ratio of ${}_{92}U^{238}$ nuclei to ${}_{92}U^{235}$ nuclei is about 140:1, which means that in the natural mix of uranium isotopes a chain reaction will never happen: after rare instances of nuclei splitting, the neutrons will most probably be absorbed by ${}_{92}U^{238}$ nuclei without any fission.[1]

It's possible to overcome this unpleasant obstacle. First, we can increase the number of ${}_{92}U^{235}$ nuclei in the mix of isotopes. But at the same time we can try to make the neutrons slow down—that is, make their energy so low that ${}_{92}U^{238}$ nuclei won't absorb them and they'll be captured mainly by ${}_{92}U^{235}$ nuclei. And so, while we lose some of the kinetic energy of the neutrons, we can gain a great deal of the energy stored in the uranium nuclei. A fruitful loss indeed!

But how can we slow neutrons down? To find the answer, we'll use an analogy. Let's consider the collision of a neutron with a nucleus (not necessarily a uranium nucleus) as a collision of two balls that have the same ratio of masses as the neutron and nucleus. The analogy will work only if no nuclear reactions take place during the collision. This means that the neutron and nucleus must move apart with changed velocities. Then we can say immediately when the loss of neutron energy is greatest: when the masses of the neutron and nucleus are equal. As the mass of the proton is almost equal to that of the neutron, the most efficient collisions might be collisions of a neutron with the immobile nucleus of a hydrogen atom—in such a collision it will lose all its kinetic energy. But in practice ordinary hydrogen can't be used to slow neutrons down.[2] For this purpose physicists use other substances (called moderators) such as heavy water or graphite (pure carbon)—see figure 3.

Heavy water is a chemical substance like ordinary water but instead of normal hydrogen, whose nucleus consists of one proton, heavy water includes deuterium. Deuterium is a hydrogen isotope whose nucleus consists of a proton and a neutron, so it's twice as heavy as normal hydrogen.

In a collision with an immobile deuterium nucleus a neutron loses 8/9 of its kinetic energy, as we can see from equation (3). So after one collision it keeps only $(1 - 8/9)$ of its initial energy; after two collisions, $(1 - 8/9)^2$; and so on. To reduce the neutron energy from $T_i = 1$ MeV to $T_f = 100$ eV, n collisions between the neutron and deuterium nuclei must occur; the value of n is determined by the obvious relation $T_f = T_i(1 - 8/9)^n$.

[1]To simplify the picture we haven't mentioned other processes that cause a loss of neutrons. In constructing a nuclear reactor these effects must be taken into account.

[2]The hydrogen in ordinary water can't be used for this purpose because the interaction of a molecule of water with a neutron causes specific processes in which the neutron is lost for a chain reaction.

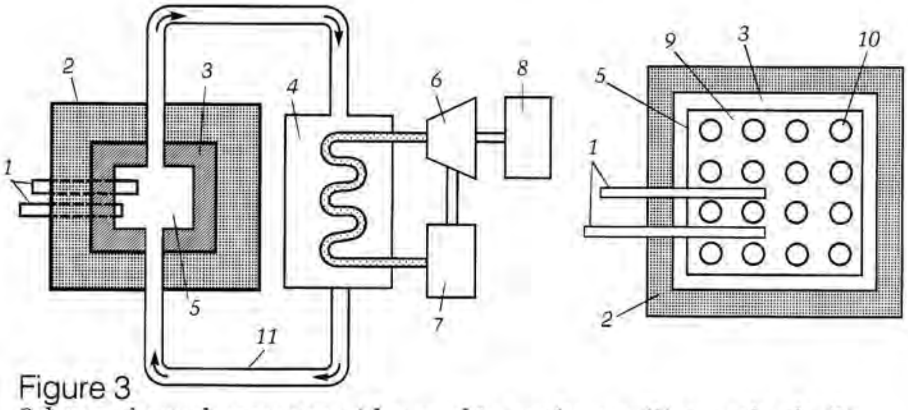


Figure 3
Scheme of a nuclear reactor with a graphite moderator: (1) control rods; (2) shield; (3) reflector; (4) heat exchanger; (5) active zone; (6) turbine; (7) condenser; (8) generator; (9) moderator; (10) fuel; (11) coolant.

Calculations give $n \cong 4$. When a neutron scatters its energy in collisions with an immobile nucleus of carbon $_6C^{12}$, it loses $24/49 \cong 1/2$ of its kinetic energy. So we can find the number of collisions necessary to make a slow neutron: $n \cong 13$. For comparison, the number n typical for collisions of a neutron with heavy nuclei of mass number $A = 90$ is $n \cong 210$.

The reason why physicists try to decrease the number n is that in the interval between "good" collisions (that is, good for slowing down neutrons) some neutrons may be captured by the nucleus of an impurity and be lost for the production of a chain reaction. If the number n of collisions is too great, the neutron may leave the reaction zone and again be lost.

So a simple analogy between the collision of balls and neutron scattering by atomic nuclei helped us understand the basic idea of the moderated reactor and even perform some calculations. But we must remember that this analogy is valid only if no reaction occurs in the collision of a neutron with the moderator. In this case the analogy makes no sense. Now let's look at a different sort of example.

The Compton effect

In the first decades of the 20th century physicists all over the world discussed the problem: what is light—waves or particles? The great Einstein first showed that not only is light emitted in packets but that these packets of radiation (quanta) preserve their individuality during the propagation of light. At the beginning of the century this idea was met by many physicists without enthusiasm, to put it mildly. Skepticism toward the idea was still alive after 1914, when the American scientist Robert A. Millikan confirmed experimentally Einstein's equation for the photoelectric effect. That equation was based on the notion that the energy of a quantum of light (or photon) is determined by its frequency ν and is equal to $E_{ph} = h\nu$ (where h is Planck's constant, equal to $6.63 \cdot 10^{-34}$ J · s). In 1916 Einstein proposed that the photon has not only energy but momentum as well: $P_{ph} = h\nu/c$ (where c is the speed of light). But the value of the photon's momentum in the optical range is very small, and for a long time there was no experimental evidence that every photon has a momentum. It wasn't until 1923 that the young American physicist Arthur Holly Compton found such evidence. This effect is now called the Compton effect.

Compton studied the scattering of X rays with the apparatus shown schematically in figure 4. A beam of X rays coming from the tube T was scattered by an object R and after passing a number of slits was studied by a spectrometer (crystal and ionization chamber).

According to classical theory the process of X-ray scattering is as follows. Electrons in the substance start to oscillate in the alternating electric field of an electromagnetic wave (it had been shown earlier that X rays are a sort of electromagnetic radiation). The frequency of the oscillations is equal to the frequency of the wave. As the oscillating electrons accelerate, they become sources of radiation. So the electrons in the substance become sources of secondary waves, whose direction cannot coincide with that of the primary, excitational wave. This is how scattering arises.

An important feature of the classical theory of scattering is the equality of the frequencies (wavelengths) of the incident and scattered radiation at any angle between the directions of the primary and secondary waves. Only the intensity of the scattered radiation was thought to depend on the angle of scattering. But Compton found that in the scattered radiation there is a component with a wavelength λ' different from the wavelength λ of the primary radiation and that $\lambda' > \lambda$ (see figure 5). This difference between wavelengths $\Delta\lambda = \lambda' - \lambda$ (called the Compton shift) depends on the angle of scattering and increases with the angle.

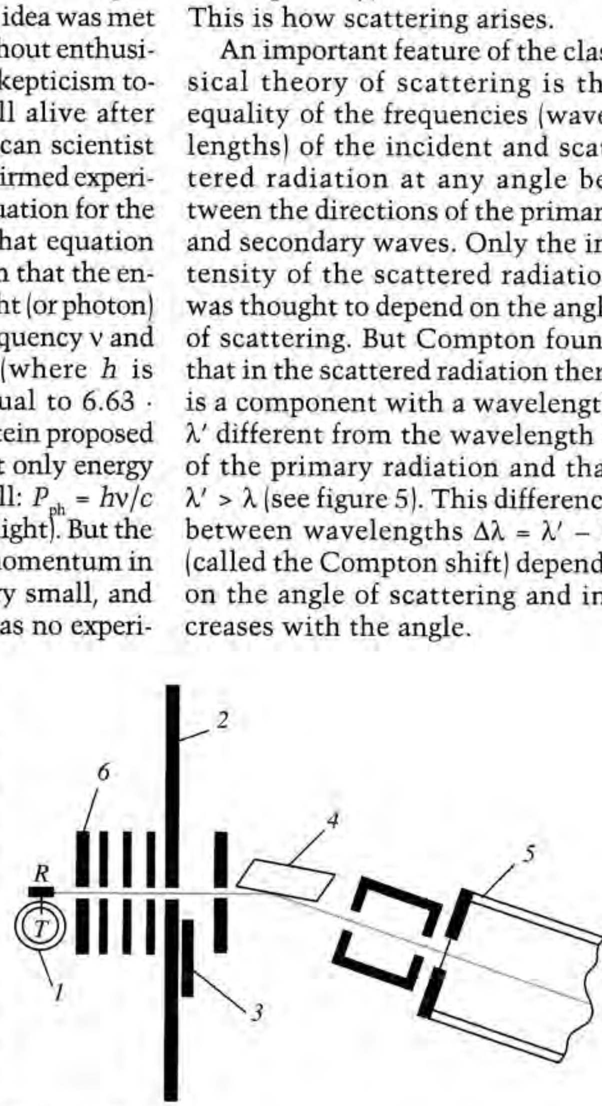


Figure 4
Scheme of Compton's apparatus. The scattering angle is the angle between two segments of the X-ray beam at point R. The scheme corresponds to $\theta = 90°$: (1) X-ray tube; (2) lead box; (3) shutter; (4) crystal; (5) ionization chamber.

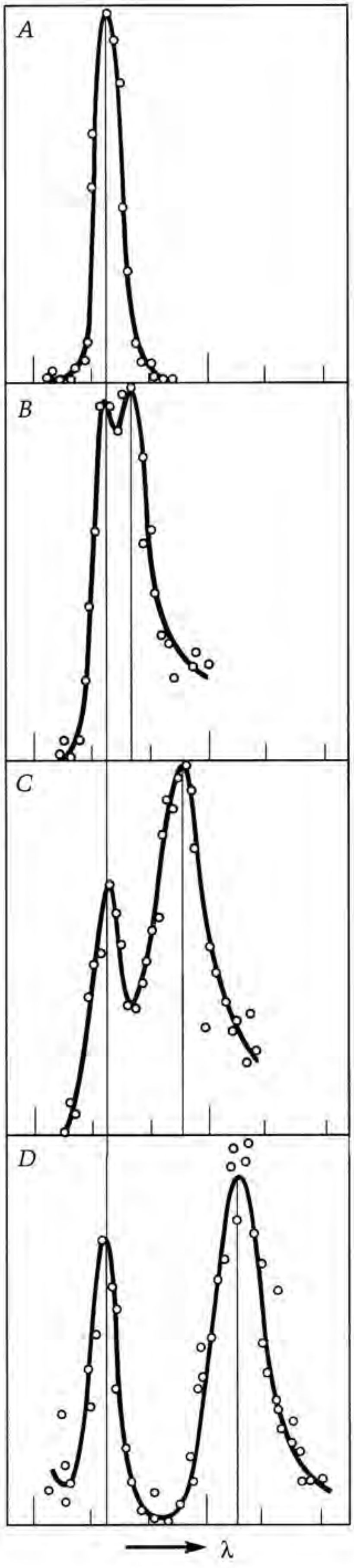


Figure 5
Dependence of the Compton shift on the scattering angle. A *is the spectrum of the incident radiation;* B, C, *and* D *are the spectra of scattered radiation at* $\theta = 45°$, $90°$, *and* $135°$.

It's remarkable that Compton, an experimentalist, managed not only to explain the effect qualitatively but to provide its elementary theory. He supposed that X rays are a beam of quanta with large energy $h\nu$ and relatively large momentum $h\nu/c$. In the scattering substance the quanta interact with free "immobile" electrons. Compton likened this interaction to the noncentered elastic impact of two particles. This case differs from our problem with two balls only in that the photon is a particle moving at the speed of light, and writing the equations for the conservation laws we must use the relativistic formula for the energy and momentum of an electron. In general, the diagram of momenta will be the same: instead of $|m_1\mathbf{v}_0|$ we must write $h\nu/c$; instead of $|m_1\mathbf{v}_1|$, $h\nu'/c$; and instead of $|m_2\mathbf{v}_2|$, p_e (where ν and ν' are the frequencies of incident and scattered radiation, p_e is the momentum of an electron).

I won't write out the system of equations, though it's not very complicated. This is the solution of the system:

$$\Delta\lambda = \frac{h}{mc}(1-\cos\theta). \qquad (4)$$

This formula is interesting in that the Compton shift doesn't depend on λ but only on θ and the fundamental constants m, c, and h.[3] The correctness of this conclusion is confirmed by the fact that the experimental value of $\Delta\lambda$ doesn't depend on the nature of the scattering substance, since electrons are the same everywhere.

At the same time, the results Compton obtained for the dependence of $\Delta\lambda$ on θ are described pretty well by equation (4): when we change θ from 0 to π, $\Delta\lambda$ increases from 0 to $2h/mc$.

So the analogy between photon scattering on free electrons and the collision of balls turned out to be fruitful. But is it absolute? When a photon interacts with an electron, the electron acquires a recoiling momentum and kinetic energy, while the photon loses energy and its wavelength increases. But though there are common features, there is an important difference. To prove it, let's determine the amount of energy a photon loses when it is scattered backwards ($\theta = \pi$) as a function of its initial energy:

$$\begin{aligned}\Delta E &= h\nu - h\nu \\ &= \frac{ch}{\lambda} - \frac{ch}{\lambda + \Delta\lambda} \\ &= \frac{2c^2\nu^2}{mc^2 + 2h\nu},\end{aligned}$$

since at $\theta = \pi$, $\Delta\lambda = 2h/mc$. The relative energy loss is

$$\frac{\Delta E}{h\nu} = \frac{2h\nu}{mc^2 + 2h\nu}.$$

Figure 6 shows this function. We can see that the curve differs drastically from that in figure 2. It has no peak. This means that more and more of the photon's energy is transferred to the electron as the energy of the photon increases. But the curve approaches the limit only when $h\nu \to \infty$. So a photon can't pass all its energy to the electron. And that's

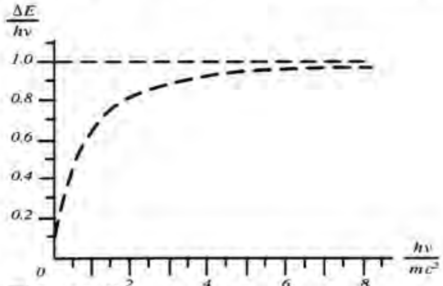


Figure 6
Dependence of the relative loss of photon energy on the energy of the initial photon in the Compton effect.

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[3]Now it's clear why it's almost impossible to detect the Compton effect in the optical range. If for X rays the maximum ratio $(\Delta\lambda/\lambda)_{max}$ is equal to several percent (for $\lambda = 10^{-10}$ m, $(\Delta\lambda/\lambda)_{max} = 2.4 \cdot 10^{-2}$), for visible light ($\lambda \cong 5 \cdot 10^{-7}$ m) $(\Delta\lambda/\lambda)_{max} = 4.8 \cdot 10^{-6}$—that is, it's very small.

how this case differs from that of the collision of balls. What's the reason for this difference? The answer is very simple: the photon's mass is equal to zero. A photon doesn't exist at rest, so it can't stop the way a ball does. This is one of the restrictions that prevents the analogy from being absolute. We examined the ball's collision on the basis of classical mechanics, but the Compton effect is relativistic by nature.

Another problem, another analogy

We've looked at two examples that, on the one hand, are like the case of the collision of balls and, on the other hand, differ from it. Can we consider the analogy exhausted? No, not at all. We considered only elastic collisions. Can we find anything in modern physics that is analogous to the inelastic collisions of balls? There is such an analogy; it has to do with elementary particle physics. You'll have an easier time understanding this analogy after we look at another very simple problem.

Problem. Determine what portion of the kinetic energy of a moving ball of mass m is converted to heat in an absolute inelastic collision[4] with an immobile ball of the same mass.

Solution. Let the first ball move with velocity v_0. Its kinetic energy is $T_0 = mv_0^2/2$. When a collision is inelastic we can't use the law of conservation of mechanical energy. But the law of momentum conservation is still valid:

$$mv_0 = 2mv',$$

where v' is the velocity of the body $2m$ formed after the collision. So the kinetic energy after the collision is

$$T' = \frac{2mv'^2}{2} = \frac{T_0}{2}.$$

This means that half of the kinetic energy of the moving ball was converted into heat.

Let's change the problem a little. Two balls of equal mass moving in the opposite direction at the same velocity strike inelastically. In this case the momentum of the system before the collision is zero. After the collision the balls will stop; so all the kinetic energy will be converted to heat.

These two very simple problems are closely related to elementary particle physics. The most popular method in experimental studies in this field is the collision method. In the collision of a fast particle with a stationary target, a portion of the kinetic particle's energy is used to form new particles. This portion can be considered a quantity analogous to the amount of heat in the collision of balls. All the kinetic energy can't be expended in giving birth to new particles since the momentum of the system before the interaction was finite. Consequently, the products of the reaction must have nonzero momentum and kinetic energy. This means that, when immobile targets are used, a portion of the kinetic energy given to the particle in the accelerator is wasted. If we continue the analogy, we can draw the conclusion that when the same particles collide (when one of them is moving and the other is immobile), one half of the kinetic energy is always lost. Can scientists build increasingly powerful accelerators with stationary targets and just accept an "efficiency" of 50%? The answer is no.

Why? Because the analogy is invalid. The fact is, the analogy can't help us find *what portion* of the energy is expended to form the new particles. The collision of elementary particles is a relativistic process, so in studying it we have to use the formulas provided by the special theory of relativity. Using these formulas, physicists found that the proportion of "useful" energy decreases as the kinetic energy of the moving particles increases. So accelerators with immobile targets become less and less effective. What can be done then? Analogy might provide an answer.

Do you remember how all the kinetic energy in a collision of balls is converted into heat? The same idea is valid for elementary particles. A new type of accelerator, the "collider," is based on this very notion. Particles (protons) moving in opposite directions at the same velocity collide. The hopes of particle physicists are now pinned on this type of accelerator. Analogy with a classical and very simple problem helped us understand why scientists need such huge facilities as accelerators that cost billions of dollars.

By now you'll probably agree that analogies are useful in scientific research as well as in studying physics as an academic subject. Discovering the common features in physical phenomena and clarifying their differences, we gain a better understanding of physical laws. ◘

[4]After an inelastic collision of two balls, they move together as one body.

"CHALLENGES" CONTINUED FROM PAGE 16

segment between them, so that $n-1$ numbers are added at the nth step. How many 1991's will be written after the 1991st step? (G. Galperin)

M35

Angry lion. A lion rushes about a circus ring with a radius of 10 m. It runs 30 km along a broken line. Prove that the sum of the angles of all the turns on its route is greater than 2,998 radians. (I. Bernstein)

ANSWERS, HINTS, AND SOLUTIONS ON PAGE 58

PHYSICS CONTEST

A snail that moves like light

"The cause is hidden but the effect is known."—*Ovid,* Metamorphoses

by Arthur Eisenkraft and Larry D. Kirkpatrick

MOST *Quantum* READERS know that light bouncing off a mirror travels along a path that can be adequately described as "the angle of incidence equals the angle of reflection." Light traveling from a point in air to a point in water is certainly more complicated. In this case, the light bends (refracts) at the boundary between the two surfaces. The amount of bending is a property of the water and the color of the light. Light entering other transparent substances, like quartz or diamond, refract by different amounts. Willebrord Snell in 1621 was able to give a mathematical description of the behavior of light, which is now known as Snell's law:

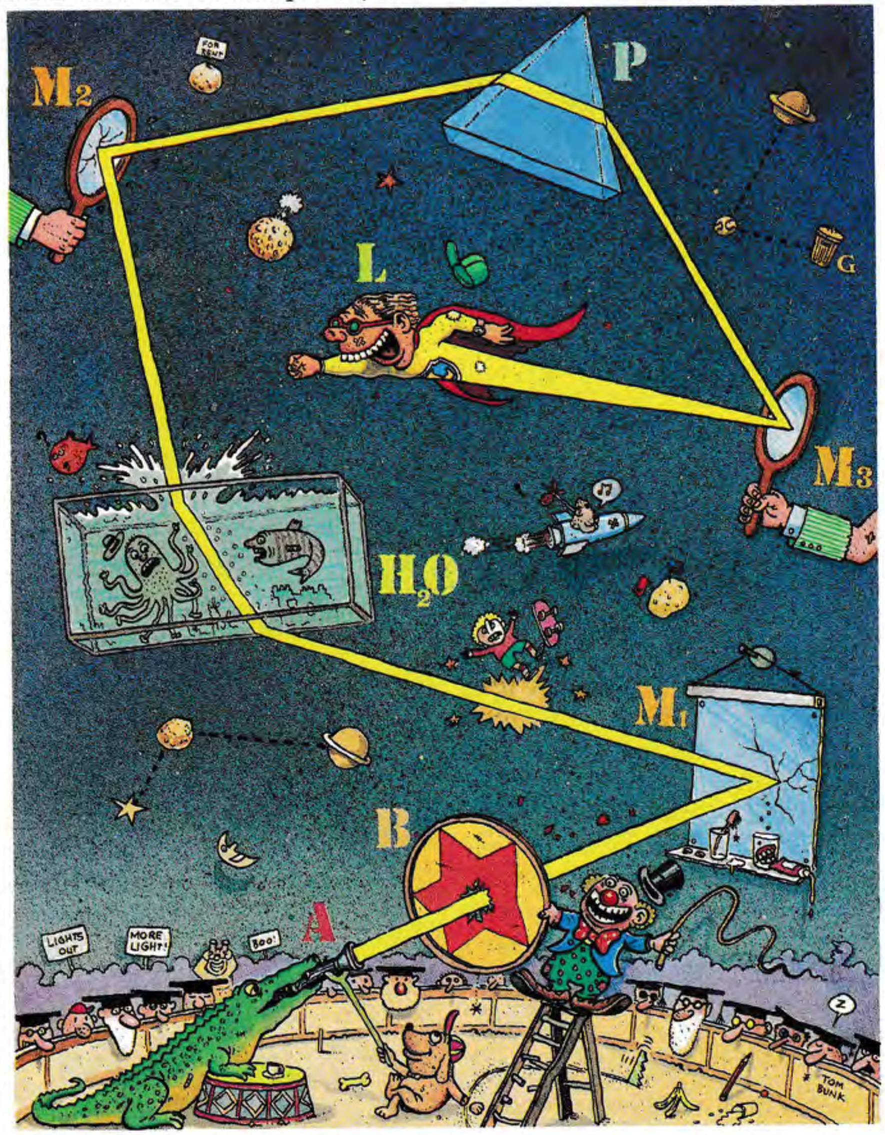
$$n_1 \sin \theta_1 = n_2 \sin \theta_2,$$

where n_1 and n_2 are the indices of refraction. We can see that if the light enters water (n = 1.33) from air (n = 1.00) at an angle of 30°, the angle in water would be 22°:

$$\begin{aligned} n_1 \sin \theta_1 &= n_2 \sin \theta_2, \\ 1.00 \sin 30° &= 1.33 \sin \theta_2, \\ \theta_2 &= 22°. \end{aligned}$$

Measuring the angle of refraction is one way to tell whether that's a diamond or a piece of glass in that ring you bought.

What fascinates many people about the study of physics is the alternative ways of explaining phenomena. The great mathematician Pierre de Fermat recognized (in 1657) that the path of light is the path that requires the least time.[1] If you try all possible paths from the light source *A* to the object *B* after they hit the



Art by Tomas Bunk

Light is bending the rules a bit here. (Can you see where?)

[1]The "extremum path."

mirror, you'll find that the shortest path, and so the quickest, is the path through point D (fig. 1), where the angle of incidence equals the angle of reflection.

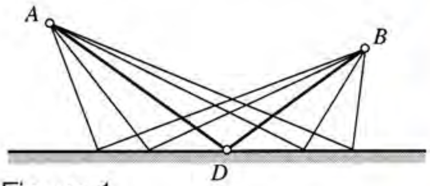


Figure 1

You can demonstrate this for yourself by drawing lots of paths and measuring them. You can also prove it with some simple geometry or by using some calculus.

Fermat's theorem is also valid for refraction: the path light takes when it passes from air to water must be the path requiring the least time. In this case least time is not identical to least distance, since light travels more slowly in water that in air. The speed of light in a substance is equal to the speed of light in a vacuum divided by the substance's index of refraction n.

Proving that the path of the light is the quickest one takes some ingenuity. You can draw lots of paths of light traveling from point A in air to point B in water (fig. 2). You can then measure the lengths of the lines in air and water. But Fermat's theorem states that the path should take the least *time*, not the least distance. We can multiply the lengths in water by 1.33, since the light takes longer to travel in water by a factor of 1.33. Then add this distance to the distance in air. The path that minimizes this sum is the path the light takes. And—guess what? It's the same path described by Snell's law! Those of you who have some calculus background can prove it mathematically. (See the Kaleidoscope for more on bouncing and bending light.)

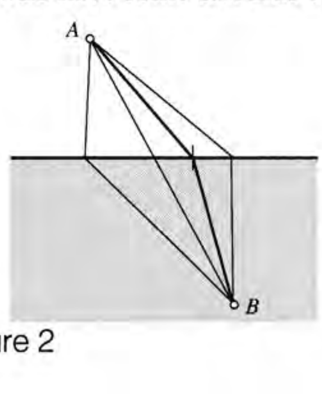
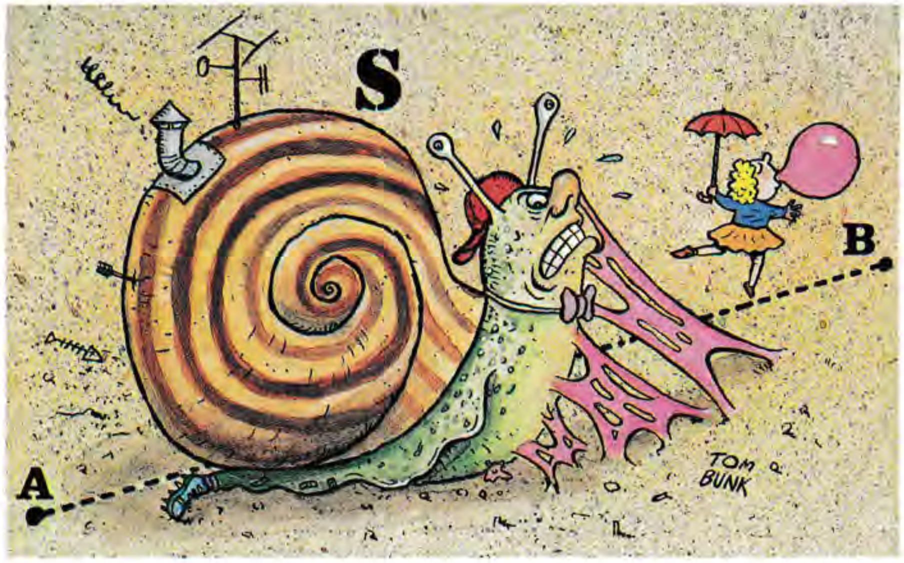


Figure 2



Leaving light behind, we enter the world of slow-moving mollusks to find our contest problem. A snail must get from one corner of a room (dimensions 5 m x 10 m x 15m) to the diagonally opposite corner in the least time. The snail can walk on any of the four walls but may not walk on the floor or ceiling. What is the path that the snail should take? In part B of the contest problem, for our more advanced readers, the snail finds that the 15 meter wall that must be traveled is sticky—that is, the snail can only travel at a fraction of its normal speed. If the snail on the sticky wall travels at 1/3 of its normal speed, what is the path that requires the least time for the snail? Finally, in part C, for our most advanced readers, what happens if the snail finds that the stickiness of the first wall is not constant but increases linearly along one dimension of the wall? Specifically, the speed at one end of the wall is the normal speed and the speed at the far end of the wall is 1/3 the normal speed. What will be the path of least time? You may need to use graphical or computer techniques to solve parts B and C. Our best students are encouraged to see if they can find general proofs for any room (dimensions l x w x h) and a stickiness factor of s. We are not sure ourselves if such general proofs exist.

When you submit a solution, please indicate your age, your school, and your physics background. That way, we can recognize beginning physics students as well as our most advanced readers.

What the seesaw taught

In a previous contest problem (January/February), we asked you to explain why two fingers supporting a horizontal meterstick will wind up under the center of mass if they are brought together slowly. Excellent proofs were supplied by Philip Miloslavsky of Hunter College High School in New York and Daniel Louzonis of Worcester, Massachusetts. They will both receive a subscription to *Quantum*.

The hand closer to the center of mass will support more weight and will necessarily have more frictional force. So the other hand will slide under the stick until its sliding friction is less than the static friction of the stationary hand. At this moment the other hand begins to slide. The hands soon meet at the center of mass. Philip pointed out that if you move your hands too quickly, they will not meet at the center of mass unless they start at equal distances from the center of mass. Philip wonders if it's possible to calculate a "critical velocity" (or critical velocity and distance). This seems like a good challenge for some of our readers.

In the second part of the problem, we asked what would happen if the meter stick were supported by two counter-rotating cylinders. Philip's solution follows (see figure 3).

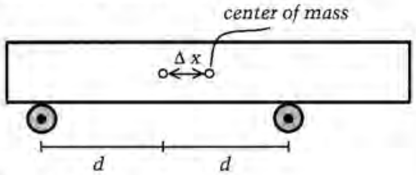


Figure 3

Let x be the displacement of the center of mass, d be 1/2 the distance between the cylinders, N_1 and N_2 be the force on the cylinders, μ be the coefficient of rolling friction. We know that the net vertical force must be zero:

$$N_1 + N_2 = Mg.$$

Since the meterstick has no rotational acceleration, the net torque must also be zero:

$$N_2(d - x) = N_1(d + x).$$

Solving for N_1 and N_2, we get

$$N_1 = \frac{W}{1+\dfrac{d+\Delta x}{d-\Delta x}}, N_2 = \frac{W}{1+\dfrac{d-\Delta x}{d+\Delta x}}.$$

The total frictional force on the meterstick is result of the rolling friction of the two cylinders:

$$F_{\text{total}} = \mu(N_1 - N_2) = \mu\left(\frac{W}{\dfrac{2d}{d-\Delta x}} - \frac{W}{\dfrac{2d}{d+\Delta x}}\right)$$
$$= -\mu\frac{Mgx}{d}.$$

This last equation is the standard equation for simple harmonic motion, with

$$\omega = \sqrt{\frac{\mu W}{md}} = \sqrt{\frac{\mu g}{d}},$$

or

$$T = 2\pi\sqrt{\frac{d}{\mu g}}.$$

KALEIDOSCOPE

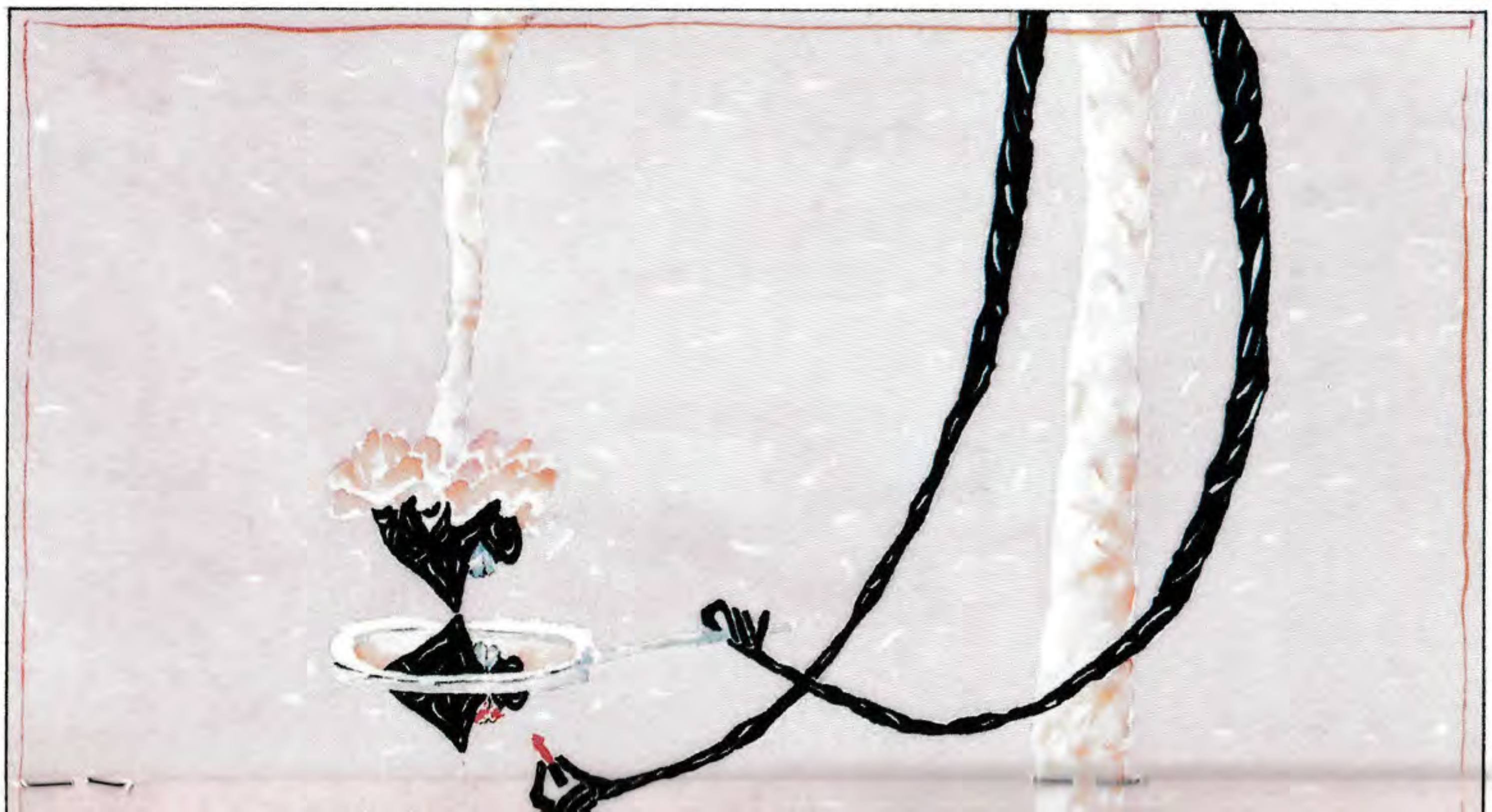
Reflection and refraction

Can you keep them straight?

"I would like to believe that those who love to learn the reasons for certain phenomena . . . will find some pleasure in these ruminations on the subject of light . . ."
—Christian Huygens (1629–1695), Treatise on Light

THESE CONCEPTS, SO simple at first glance, are profound and integral notions in one of the most important branches of physics: optics. There are few areas in physics whose development from the very beginning has achieved such weighty results, which have retained their value even to the present. This is why research in optics has had such a great influence on the establishment of other branches of natural science, forced a reexamination of the foundations of classical physics, and played a significant role in the birth of the theory of relativity and quantum mechanics.

We hope the *Quantum* Kaleidoscope will help you look with fresh eyes at certain everyday phenomena and de-



eryday phenomena and devices—for instance, a lens or camera—and maybe prompt you to think seriously about the mechanism of human sight. At any rate, we hope it will help you discover something new and interesting in optics.

Art by Sergey Ivanov

Questions and problems

1. Why do air bubbles in water glitter?

2. Can you light a fire by using ice?

3. Why does a person see so poorly underwater, even when the water is crystal clear?

4. A scuba diver can see a person fishing on the shore, but that person can see the scuba diver only at rare moments. Why is that?

5. What one characteristic of an unknown substance do we need to know in order to determine the speed of light in it?

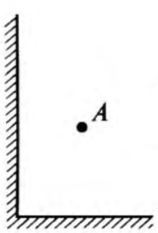
6. A man approaches a mirror at a speed of 2 m/s. At what speed does he approach his own reflection?

7. The beam of a searchlight is easy to see when it's foggy, harder to see when the weather is clear. Why?

8. How do you explain the way precious stones glitter?

9. Why don't we see the stars during the day?

10. How many images of point A (see the figure) can we get by using a system of two perpendicular mirrors?



11. Nearsighted people squint to see better. Why do they do that?

It's interesting that . . .

. . . extremely accurate measurements of light refraction in various media were made by Claudius Ptolemaeus (2nd century A.D.). But he wasn't able to establish the correlation between the angle of incidence and the angle of refraction.

. . . the total internal reflection in natural monocrystals is closer to the ideal than the reflection from specially made metal mirrors, which always absorb some of the energy of the incident beam.

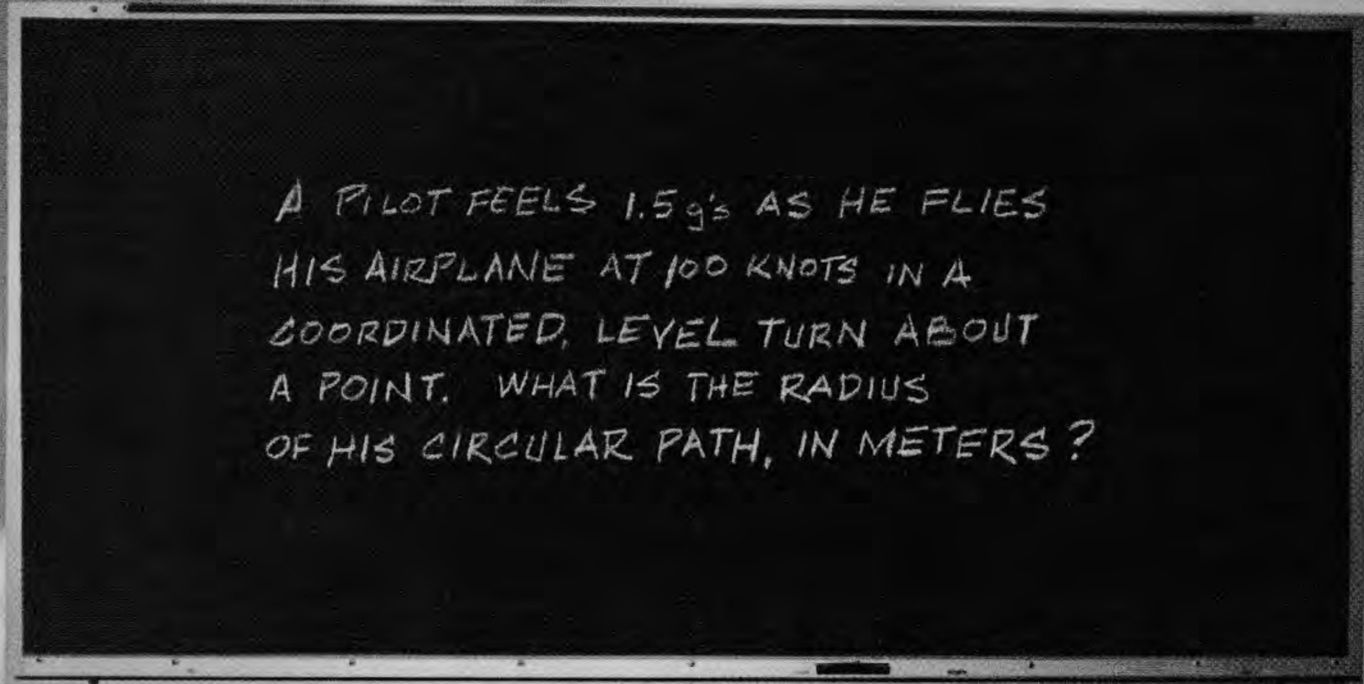
. . . during his lifetime Newton was most famous for his optics research, and not just in scientific circles. His *Optics* was his most popular work, which remained for many decades a veritable encyclopedia of knowledge about light.

Microexperiment

Design an experiment to distinguish between eyeglasses for myopia (nearsightedness) and those for hyperopia (farsightedness). Collect some eyeglasses and conduct the experiment. ◘

ANSWERS ON PAGE 62

If you like solving this problem, then Embry-Riddle Aeronautical University has a program for you.



If you enjoy solving problems like this, why not consider a career in aviation and aerospace? You'll be challenged with complex problems which require creative solutions.

Aviation offers many diverse fields which require a strong scientific education.

An Embry-Riddle education.

Embry-Riddle programs include aerospace engineering, electrical engineering, engineering physics, avionics engineering, aviation computer science and aeronautical science.

You'll learn to design air and space vehicles by applying scientific and mathematical principles. You'll conduct experiments in laboratories devoted to materials, composites, structures, instrumentation and aircraft performance. Or, delve into the worlds of software engineering, simulations and artificial intelligence.

It won't be easy. You'll work hard.

The reward?

You'll receive a superior education and the opportunity to join more than 25,000 Embry-Riddle alumni involved in the most exciting association on earth — the fraternity of aviation.

Our Campuses

The Daytona Beach, Florida campus has 55 buildings and is home to 5,000 students. More than 1,800 students attend the Prescott, Arizona campus with its 75 buildings on 510 acres.

With a student/faculty ratio of 25:1, you'll find yourself working alongside faculty who have outstanding academic credentials and impressive aviation experience. In this total aviation environment, you'll tackle challenging questions and recognize solutions to problems — as in this case, 242 meters.

So, when you're looking for the right tools, look to Embry-Riddle Aeronautical University.

Aviation does.

Please send me more information about Embry-Riddle Aeronautical University. Or call toll-free 1-800-222-ERAU. MZ

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EMBRY-RIDDLE AERONAUTICAL UNIVERSITY

Circle No. 4 on Reader Service Card

Going around in circles

Wire structures, feeler gages, and circular racetracks

by George Berzsenyi

IN PROBLEM 5 OF ROUND 2/Year 2 of the USAMTS (USA Mathematical Talent Search, conducted via COMAP's *Consortium*), the participants were presented with the challenge of determining n for which one can construct a wire structure by connecting n wire triangles at their vertices so that at each vertex exactly three wire triangles meet. Many of the over 200 contestants succeeded in showing that such structures exist for all $n \geq 7$. The necessity of $n \geq 7$ is easily verified, while its sufficiency follows from observing that if $A_1, A_2, ..., A_n$ are n points (in space) in general position, then the triangles can be chosen as

$$\begin{aligned} T_1 &= \{A_1, A_2, A_4\}, \\ T_2 &= \{A_2, A_3, A_5\}, \\ &\dots \\ T_{n-3} &= \{A_{n-3}, A_{n-2}, A_n\}, \\ T_{n-2} &= \{A_{n-2}, A_{n-1}, A_1\}, \\ T_{n-1} &= \{A_{n-1}, A_n, A_2\}, \\ T_n &= \{A_n, A_1, A_3\}. \end{aligned}$$

Noting the cyclic arrangement above, in the "official" solutions I commented on the fact that the problem is roughly equivalent to determining the location of three markers on a circular path of integer length so that the markers are of integer distance from one another and that the clockwise distance between any two pairs of them is different. The path of minimal length is shown in figure 1a, with the points $A_1, A_2, ..., A_7$ unit distance apart. The placement of the markers at A_1, A_2, and A_4 corresponds to the construction of T_1; the shifting of the markers to A_k, A_{k+1}, A_{k+3} (with the subscripts reduced modulo 7) corresponds to the construction of triangle T_k; while the fact that the clockwise distances (A_1, A_2), (A_1, A_4), (A_2, A_4), (A_2, A_1), (A_4, A_1), and (A_4, A_2) are all different assures that each pair of triangles intersects in at most one point.

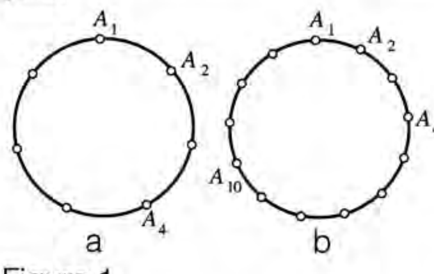


Figure 1

I also noted that in a similar fashion one can construct wire structures from n quadrilaterals with four of them meeting at each vertex, whenever $n \geq 13$. The circular path of length 13 with 4 markers is shown in figure 1b. Moreover, I challenged the students to find wire structures from pentagons with five of them meeting at each vertex.

The purpose of this column is to further generalize these challenges by asking for the minimum value of $n(k, m)$ that will ensure the construction of a wire structure from wire polygons of k vertices with m of them meeting at each vertex.

The special case of $k = m$ was treated earlier by T. H. O'Bierne in his "Puzzles and Paradoxes" column in *New Scientist* exactly 30 years ago, where he connected the problem (for $k = m = 6$) to an interesting finite geometric structure. He also noted that the gaps between the A_i's allow for the construction of gages arranged on a ring so as to allow for the measurement of different thicknesses.[1] For example, if gages of thickness 1, 5, 2, 10, and 3 are placed on a ring in that order (fig. 2), then one can measure with them thicknesses of 1, 2, ..., 21 units.

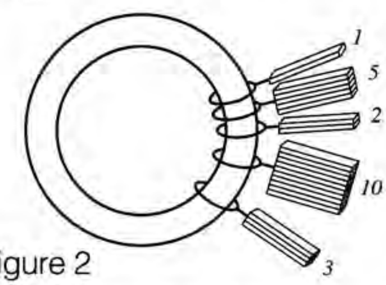


Figure 2

Some related problems are also discussed by L. R. King and Harold B. Reiter in an article entitled "Graceful Graphs and Sparsely Marked Rulers" in the May 1991 issue of *The College Mathematics Journal.*

CONTINUED ON PAGE 40

The purpose of this column is to direct the attention of *Quantum*'s readers to interesting problems in the literature that deserve to be generalized and could lead to independent research and/or science projects in mathematics. Students who succeed in unraveling the phenomena presented are encouraged to communicate their results to the author either directly or through *Quantum*, which will distribute among them valuable book prizes and/or free subscriptions.

[1]See *New Scientist*, nos. 261–64, or chapter 6 and the accompanying postcript in *Puzzles and Paradoxes*, published by Dover.

Divisive devices

From Euclid's algorithm to the fundamental theorem of arithmetic

by V. N. Vaguten

EVERYBODY KNOWS THAT any natural number—that is, any positive integer—can be broken down into the product of prime factors. For example,

$$400 = 2^4 \cdot 5^2,$$
$$1001 = 7 \cdot 11 \cdot 13,$$
$$290{,}981 = 43 \cdot 67 \cdot 101.$$

Why is such a "factorization" unique? Or take a simpler fact: if the product mn is divisible by 43, then at least one of the numbers m or n is divisible by 43. How can we prove this?

These facts seem pretty obvious, but it's not so easy to prove them. The proofs will come at the end of this article. Let's start with the simplest statements about the divisibility of integers and see how to find the greatest common divisor (GCD) of two numbers without breaking them down into prime factors. I hope you enjoy solving the problems that we'll encounter along the way.

Throughout the text the letters a, b, c, ... will denote integers.

Division with a remainder

There's a well-known procedure that divides a number a by a number b. Let's take, for instance, $a = 1991$ and $b = 31$:

$$\begin{array}{r} \underline{\quad 64} \\ 31\,)\,1991 \\ -\,\underline{186} \\ 131 \\ -\,\underline{124} \\ 7 \end{array}$$

The process of division is continued until the remainder becomes smaller than the divisor. In this example, the remainder is 7. This means that $1991 = 31 \cdot 64 + 7$. This may prompt us to formulate the following statement (see figure 1):

> *If a and b are integers and b is greater than zero, then there is a number q such that $a = bq + r$, where the "remainder" r is an integer satisfying the inequality $0 \le r < b$.*

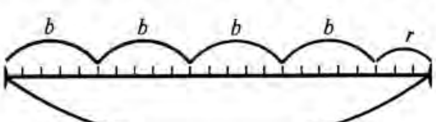


Figure 1

Problem 1. Find the remainder when 1991 is divided by (a) 100; (b) 3; (c) 7; (d) 11.

Problem 2. There is an eight-story apartment building with a number of separate, numbered stairways. On one of the floors off one of the staircases, the apartments are numbered from 97 to 102. On what floor, and off which staircase, is apartment 211? (There are the same number of apartments on each floor, and all the staircases are of the same design.)

Problem 3. Imagine 5 sheets of paper. Some of the sheets are cut into 5 pieces. Then some of the smaller pieces are cut into 5 pieces again. This is done several times. Is it possible to end up with 1991 pieces?

Problem 4. Find the smallest six-digit number divisible by 3, 7, and 13.

Problem 5. What is the remainder when 98,765,432,123,456,789 is divided by (a) 4; (b) 8; (c) 9?

The greatest common divisor

Let a and b be nonzero integers. Take all the common divisors of a and b and choose the largest of them. We'll denote this "greatest common divisor" by GCD (a, b). For instance, GCD $(4, 12) = 4$; GCD $(21, 91) = 7$; GCD $(15, 28) = 1$.

If GCD $(a, b) = 1$, the numbers a and b are said to be coprime.

Problem 6. Prove that if d = GCD (a, b), $a = kd$, $b = ld$, then GCD $(k, l) = 1$.

Problem 7. The product of two numbers is equal to 600. What maximum value can their GCD have?

Problem 8. What is the greatest number of identical bouquets that can be made out of 264 white and 192 red tulips? (No flowers should be left out.)

Problem 9. (a) A 10×15 rectangle is drawn on a sheet of graph paper (fig. 2). It has 6 nodes of the grid on its diagonal. Let there be an $m \times n$ rectangle whose sides run along the lines of the grid. How many nodes lie on this rectangle's diagonal? (b) Determine the number of solutions in natural numbers x, y of the

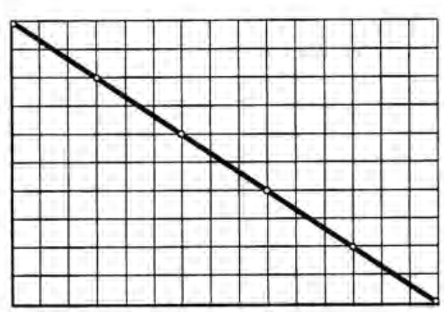
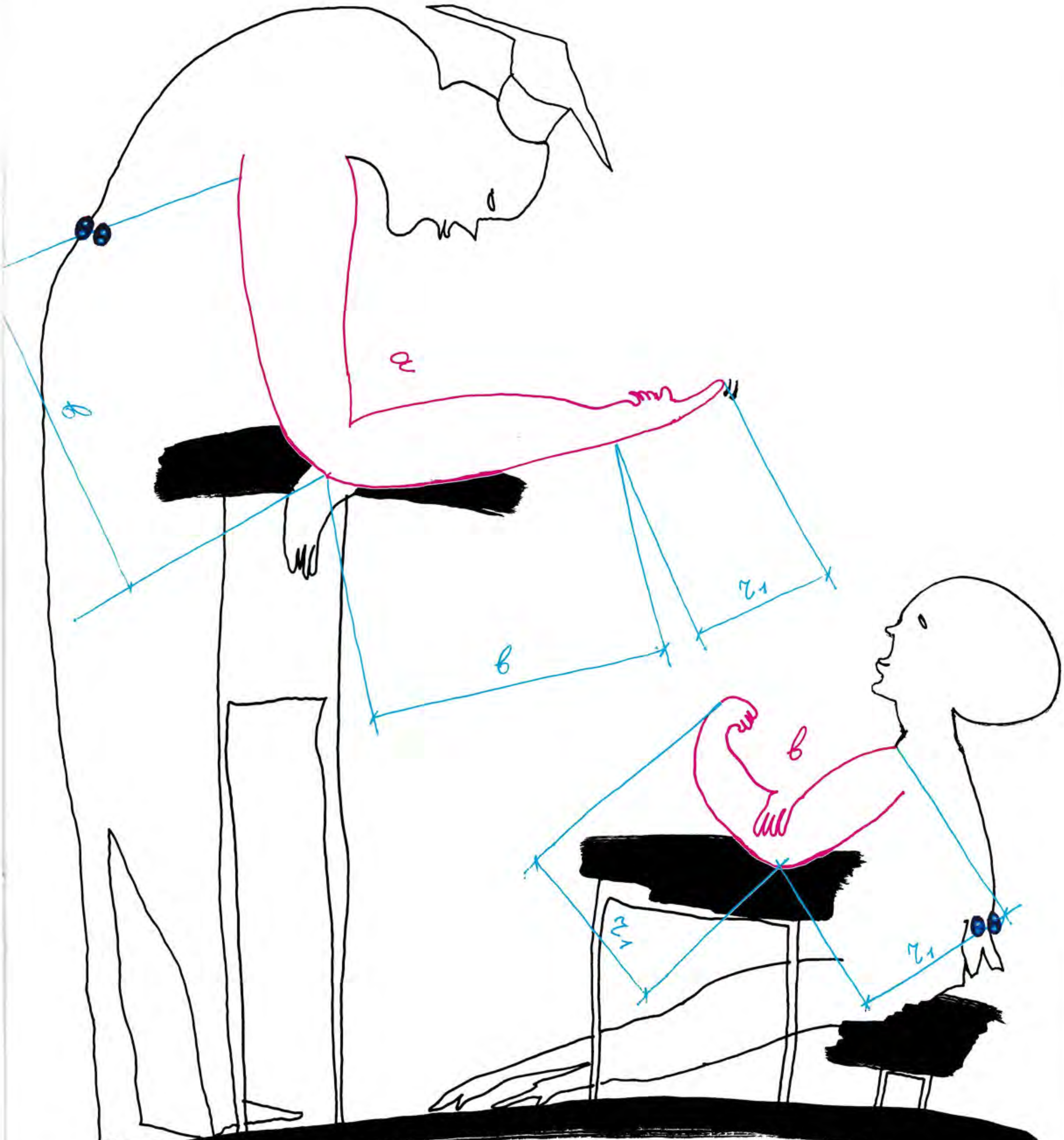


Figure 2

Art by Dmitry Krymov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Крымов

equation $mx + ny = mn$, where m and n are given natural numbers? (Just a reminder: positive integers are called natural numbers.)

Euclid's algorithm

To find the GCD of two numbers you can, of course, write down all the divisors of each of the numbers, choose all the common divisors, and then take the largest of them. This is easy when you can factor both numbers into primes—for example, $600 = 2^3 \cdot 3 \cdot 5^2$, $288 = 2^5 \cdot 3^2$, so GCD $(600, 288) = 2^3 \cdot 3 = 24$. (Some methods of factorization are explained in the article "Is This What Fermat Did?" on page 17.) There is, however, a different approach to this problem, one that doesn't require that you search for divisors of each of the numbers separately.

Let's prove the following important lemma.

LEMMA 1. *Let $a = bq + r$. Then GCD (a, b) = GCD (b, r).*

It's enough to show that a pair of numbers (a, b) has the same set of common divisors as the pair of numbers (b, r): this will mean that the GCDs of these pairs are also the same. So we have to prove that each common divisor of a and b is also a divisor of r and, conversely, that each common divisor of b and r is a divisor of a.

We'll start by proving the first statement. Let a and b be divisible by k. Then bq is divisible by k, and $r = a - bq$ is divisible by k.

Now for the second statement. If b and r are divisible by m, then bq is divisible by m, and $a = bq + r$ is divisible by m. The lemma provides a quick and easy way of finding the GCD of two numbers. Let's have a look at how this is done.

Example. Find GCD (943, 437).

Solution. We divide 943 by 437 and get a remainder of 69, which we'll rewrite

$$943 = 437 \cdot 2 + 69.$$

According to the lemma,

$$\text{GCD}(943, 437) = \text{GCD}(437, 69).$$

Now we have to find GCD (437, 69). We divide 437 by 69:

$$437 = 69 \cdot 6 + 23.$$

Using the lemma again, we see that GCD (437, 69) = GCD (69, 23). But 69 is divisible by 23 without a remainder:

$$69 = 23 \cdot 3,$$

so GCD (69,23) = 23, and consequently

$$\begin{aligned} 23 &= \text{GCD}(69, 23) \\ &= \text{GCD}(437, 69) \\ &= \text{GCD}(943, 437). \end{aligned}$$

Answer. GCD (943, 437) = 23.

The method of finding the greatest common divisor by the consecutive application of lemma 1 is called Euclid's algorithm. Try it yourself! Take other pairs of numbers as large as you want and find their GCD using this algorithm. By the way, the probability that you'll choose a pair of coprimes has quite an unexpected value: it's equal to $6/\pi^2$, where π is the circumference of a circle with a diameter of 1, which seems to have nothing in common with the GCD!

Problem 10. Find the greatest common divisor of the following numbers: (a) 987,654,321 and 123,456,789; (b) 7,777,777,777 and 777,777.

Problem 11. A number of squares whose sides are 141 cm long are cut from a 324 cm x 141 cm rectangle until a rectangle is left with a side shorter than 141 cm. Squares with sides equal to this second rectangle's smaller side are cut as long as it's possible (fig. 3); and so on. Into what kinds of squares will the original rectangle be cut? (Give their sizes and quantities.)

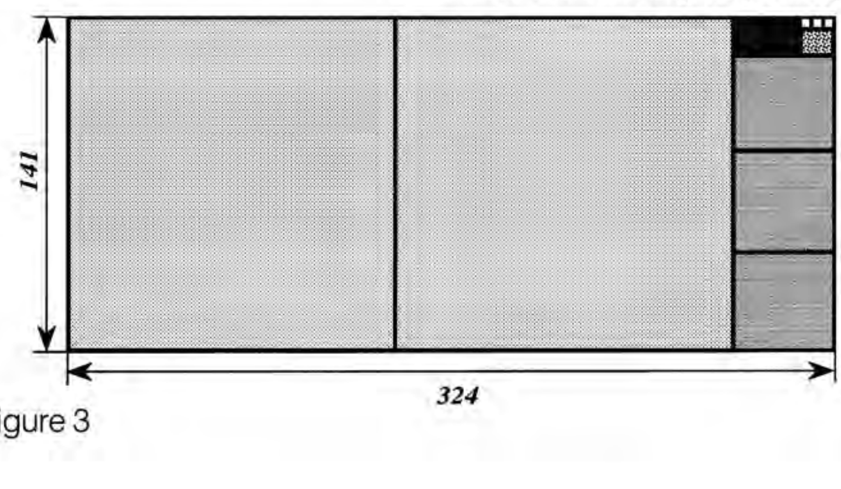


Figure 3

Euclid's algorithm is a simple method of finding the greatest common divisor of two numbers. Given two numbers a and b such that $a > b > 0$, we first divide a by b and get a remainder r_1, which is smaller than b. Then we divide b by r_1 and get a remainder r_2, which is smaller than r_1. Then we divide r_1 by r_2 and get a remainder r_3 smaller than r_2, and so on, until some remainder r_{n-1} is divisible by the remainder r_n without a remainder (that is, until $r_{n+1} = 0$).

It's clear that the process has to end sooner or later, since each successive remainder is smaller than the preceding one and all the remainders are nonnegative numbers. The last remainder r_n is in fact the GCD of a and b:

$$\begin{aligned} r_n &= \text{GCD}(r_n, r_{n-1}) \\ &= \text{GCD}(r_{n-1}, r_{n-2}) \\ &= \dots \\ &= \text{GCD}(r_2, r_1) \\ &= \text{GCD}(r_1, b) \\ &= \text{GCD}(a, b). \end{aligned}$$

Problem 12. Prove that for an arbitrary pair of integers (a, b), $0 < a < b < 1{,}000$, the number of steps in Euclid's algorithm for finding GCD (a, b) is not greater than that for the pair (610, 987). What is this number?

A geometrical illustration of Euclid's algorithm has been given in problem 11. A better known and more important geometrical formulation of Euclid's algorithm is the algorithm of finding the greatest common unit of measurement for two line segments (fig. 4).

Problem 13. Find the largest number α such that $15/28\alpha$ and $6/35\alpha$ are integers. In other words, find the length of the interval α that is the greatest common unit of measurement for intervals of lengths 15/28 and 6/35.

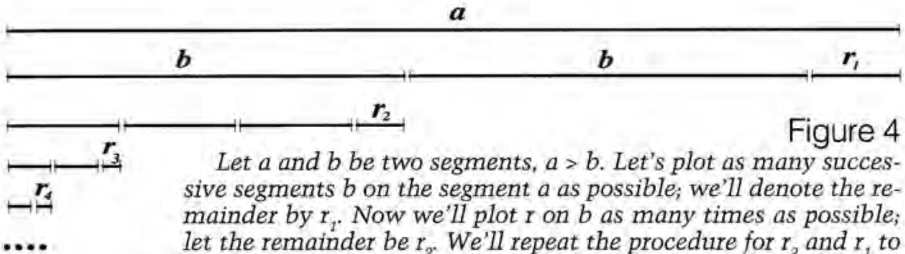


Figure 4

Let a and b be two segments, a > b. Let's plot as many successive segments b on the segment a as possible; we'll denote the remainder by r_1. Now we'll plot r on b as many times as possible; let the remainder be r_2. We'll repeat the procedure for r_2 and r_1 to obtain the remainder r_3, and so on.

If at some step, when trying to fit a segment r_n inside r_{n-1}, we get no remainder (that is, if $r_{n+1} = 0$), then segment r_n is the greatest common unit of measurement for segments a and b. If the lengths of a and b are integers, then all the remainders r_1, r_2,... are also integers. The process ends, and the last nonzero r_n is the GCD of a and b. If the process doesn't end, the intervals are said to be incommensurable (and the ratio a/b is an irrational number).

Linear equations

Finding the GCD is one of many problems that can be solved by Euclid's algorithm. The proof of the following important property of the greatest common divisor involves the numbers that arise in each step of the algorithm.

> LEMMA 2. *If GCD (a,b) = d, then there exist integers x and y such that d = ax + by.*

Indeed, the remainder r_1 obtained after the first division of a by b can be written as $ax_1 + by_1$, since $r_1 = a - bq_1$ (that is, $x_1 = 1$, $y_1 = -q_1$). The next remainder r_2 obtained when b is divided by r_1 can also be written as $ax_2 + by_2$, since

$$\begin{aligned} r_2 &= b - r_1q_2 \\ &= b - (ax_1 + by_1)q_2 \\ &= a(-x_1q_2) + b(1 - y_1q_2) \\ &= ax_2 + by_2. \end{aligned}$$

Obviously the same reasoning is applicable to all the subsequent remainders until we finally arrive at the equality $r_n = ax + by$. But r_n = GCD (a, b). Lemma 2 is proved.

Let's go back to the earlier example in which we calculated GCD (943, 437) and try to find the numbers x and y such that

$$23 = 943x + 437y. \qquad (1)$$

The greatest common divisor was found from the chain of equalities

$$\begin{aligned} 943 &= 437 \cdot 2 + 69, \\ 437 &= 69 \cdot 6 + 23, \\ 69 &= 23 \cdot 3. \end{aligned}$$

The first equality yields

$$69 = 943 - 437 \cdot 2.$$

From the second equality we get

$$\begin{aligned} 23 &= 437 - 69 \cdot 6 \\ &= 437 - (943 - 437 \cdot 2)6 \\ &= -943 \cdot 6 + 437 \cdot 13. \end{aligned}$$

So we've found the numbers $x = -6$ and $y = 13$ satisfying equation (1).

The following statement is an important particular case of lemma 2.

> *If numbers a and b are coprime, then there exist integers x and y such that ax + by = 1.*

We should note that lemma 2 follows from this statement. For instance, instead of solving equation (1), we can immediately cancel 23 out and get an equivalent equation:

$$41x + 19y = 1. \qquad (2)$$

The numbers 41 and 19 are coprime.

The solution $x = -6$, $y = 13$ satisfies both equations (1) and (2).

One more note. We've shown a way to find only one solution of the equation. In fact, if there is at least one solution, there are infinitely many of them. For instance, the numbers

$$\begin{aligned} x &= -6 + 19t, \\ y &= 13 - 41t \end{aligned} \qquad (3)$$

(t is an arbitrary integer) are also solutions of equation (2):

$$41(-6 + 19t) + 19(13 - 41t) = 1.$$

Actually, every integer solution of equation (2) takes the form of that in (3). To prove it, let's take a solution (x, y) of (2):

$$41x + 19y = 1.$$

Subtracting from this equation the equality

$$41 \cdot (-6) + 19 \cdot 13 = 1,$$

we get

$$41(x + 6) + 19(y - 13) = 0$$

or

$$41(x + 6) = 19(13 - y).$$

Since the left side of the last equality is divisible by 19 and the numbers 41 and 19 are coprime, the number $x + 6$ must be divisible by 19: $x + 6 = 19t$, where t is an integer. Then $y = 13 - 41t$. So we have discovered how to find integer solutions of any linear equation of the form $ax + by = c$. In the general case the result reads as follows.

> *The necessary and sufficient condition for equation ax + by = c to have integer solutions (x, y) is that c is divisible by GCD (a, b) = d. If this condition is satisfied and (x_0, y_0) is one of the solutions of this equation, then all the solutions of the equation are given by the formulas*
>
> $$\begin{aligned} x &= x_0 - b_1t, \\ y &= y_0 - a_1t, \end{aligned}$$
>
> *where*
>
> $$a_1 = \frac{a}{d}, \quad b_1 = \frac{b}{d}.$$

The formulas have a simple geometric interpretation: a straight line meets integer points periodically (as in figure 2).

Problem 14. Find integers x and y such that $85x + 204y = 17$.

Problem 15. Do the following equations have integer solutions: (a) $105x + 56y = 42$; (b) $104x + 65y = 43$?

Problem 16. (a) Is it possible to set up a battery having a voltage of 220 V by serially connecting cells of two types, 6 V and 16 V? If so, how many batteries of each type must be used? (b) The same question, but the voltages of the cells are 6 V and 15 V.

The fundamental theorem of arithmetic

Before we prove the fundamental theorem, let's take one more step forward and prove the following lemma.

> LEMMA 3. *If the product ab is divisible by c and the numbers b and c are coprime, then a is divisible by c.*

Indeed, since GCD $(b, c) = 1$, then, according to lemma 2, there exist integer numbers x and y such that $1 = bx + cy$. Multiplying both sides of the equation by a we get $a = abx + acy$. According the condition, ab is divisible by c, so both abx and, of course, acy are divisible by c, which means that their sum a is also divisible by c.

Lemma 3 is used often in solving various problems, although sometimes this use is pretty inconspicuous. For example, we've already used it in the preceding section while deriving formulas yielding all the solutions of equation $41x + 19y = 1$ (there the corresponding phrase is in italics).

Problem 17. Prove that if a number a is divisible by both the coprimes b and c, then a is divisible by bc.

Problem 18. Which of the following statements is true: (a) if ab is divisible by 15, then at least one of the factors is divisible by 15; (b) if ab is divisible by 17, then at least one of the factors is divisible by 17; (c) if a is divisible by 6 and b is divisible by 10, then ab is divisible by 15; (d) if ab is divisible by 60 and b is coprime with 10, then a is divisible by 20.

Here I'll remind you that *a natural number p is called prime if it has exactly two divisors: p and 1.*

If p is a prime number, then for any integer a one of the following two statements is valid: either a is divisible by p, or a and p are coprime (since GCD (a, p) can be equal only to p or 1). A particular case of lemma 3 can now be formulated as follows.

> *If a product ab is divisible by a prime number p, then at least one of the numbers a and b is divisible by p.*

The fundamental theorem of arithmetic is an immediate consequence of this statement.

> *Every natural number can be uniquely factored into a product of prime factors.*

Indeed, let a number be factored into several factors so that at least one of them is not a prime number. Then this factor itself can be factored; if any of its factors isn't prime, we can factor it again; and so on. Since each factor of any number is smaller than the number itself, this process can't go on forever—at some point we'll arrive at a factorization of the number into prime factors.

Now let's prove that there can't be two different factorizations of a number into primes. Assume that there are two factorizations of some number a: $a = p_1p_2 \ldots p_r = q_1q_2 \ldots q_k$ $(r \le k)$, where p_i and q_i are prime numbers. Since the left side of the equation is divisible by p_1, the right side should also be divisible by p_1, so that one of the numbers q_i has to be divisible by p_1. But q_i is a prime number, so we have $q_i = p_1$. Canceling out the common factor $p_1 = q_i$, we turn to the factor p_2, and so on. Finally, all the factors will be canceled out and the left side becomes equal to 1. Since q_i are positive integers, nothing but 1 will be left on the right side as well. So the factors in both factorizations are the same (though their order may be different), which means that the factorizations are identical.

Problem 19. Factor the numbers 1990, 1991, and 1992.

Problem 20. (a) Prove that if m and n are coprime and $am = bn$, then there exists an integer k such that $a = kn$, $b = km$. (b) Prove that if m and n are coprime and $x^m = y^n$, then there is an integer z such that $x = z^n$, while $y = z^m$.

ANSWERS ON PAGE 61

"GOING AROUND IN CIRCLES" CONTINUED FROM PAGE 35

A pigeonhole for every pigeon

Many thanks to James Quinn (CA) and Brian Platt (UT) for their excellent responses to the problems posed in the January/February issue. Both of them answered the first question by noting that for $n = 85$, the Greedy Algorithm yields the set {85, 84, 83, 81, 78, 72, 61} with a sum of 544, while the sum of the elements of {84, 83, 82, 80, 77, 71, 60, 40} (which is the Greedy Algorithm solution for $n = 84$) is 577. They also observed a similar phenomenon for $n = 162$, and Platt went on to conjecture a rule for obtaining all other integers for which a direct application of the Greedy Algorithm does not yield optimal answers. He also constructed a Greedy Sequence that is conjectured to yield maximal S_n for all other values of n.

Readers interested in Platt's insightful constructions should write to me for a one-page summary.

IN THE LAB

The riddle of the Etruscans

The beautiful legacy of a vanished race

by A. S. Alexandrov

THE ETRUSCANS, A mysterious people about whom archaeologists and historians still patiently gather scraps of information, used to live in north central Italy. By the middle of the first millennium B.C., their civilization had blossomed culturally and economically. Their cities wielded formidable military power, and the Tarquin dynasty ruled in Rome. But after the Etruscan kings were expelled from the city in 510 B.C., the Etruscans encountered dangerous rivals in Italy—the Romans. Protracted wars over the course of several centuries resulted in the total subjugation of the Etruscans. By the beginning of the Christian era they were completely absorbed in the sundry mass of peoples of the Roman Empire. All that is left of them are a few inscriptions in a language that hasn't been adequately deciphered, individual examples of arts and crafts of outstanding quality, and scanty references to them by Roman authors.

While excavating Etruscan cities, archaeologists managed to find a variety of objects indicating the high level of development in this extinct civilization. Etruscan jewelry is universally admired, especially the so-called granulated ornamentation. These masterpieces created by unknown craftsmen are copper plates with elaborate tracery made up of thousands of tiny gold spheres (about 0.2 mm in diameter). No other people has attained such perfection in granulated ornamentation.

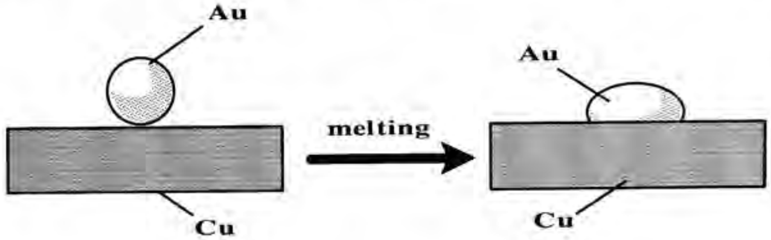
By the end of the first millennium A.D. the art of manufacturing such ornamentation was completely forgotten. Only in the 19th century did scientists try to rediscover the technical secrets involved, but to no avail: for a long time nobody could explain how it's possible to attach a gold granule to a copper substrate without melting the granule. If the granule melts, the drop of liquid gold spreads over the copper. Upon cooling, the flattened drop would certainly be fused to the substrate, but the elegant look of the ornamentation would be lost.

The secret was finally discovered in 1933. The technology turned out to be rather subtle—to understand it, we need refresh our memory on the subject of *diffusion.*

Diffusion is the process by which atoms or molecules of one substance penetrate another. This process is easy to see in liquids. If we drip a drop of ink into water, it keeps a sharply defined outline for a while, but as time goes on the outline will get blurred, the two liquids will mix together, and the drop as such will disappear. Why does that happen?

Before answering, let's recall a certain very famous experiment. In 1827 the British botanist Robert Brown was studying the chaotic movement of pollen grains in water. The particles were quite small (generally about 0.005 mm in diameter), so Brown used a microscope to observe them. He saw that their "routes" had no system to them and were so strange that, at first, he took this movement for some special form of life.

An adequate theory explaining this phenomenon, which is now called Brownian motion, was worked out half a century later. The point is that a pollen grain is huge in comparison with a molecule of water but small enough to feel the impact of a single molecule. Molecules of water are constantly in chaotic motion, "bombarding" the pollen grain unequally from various directions and forcing it to move randomly. So Brownian motion can serve as visible proof of molecular motion that can't be observed even under a microscope.[1]



[1]Recent research suggests that Robert Brown may not have seen the motion that now bears his name. In a paper presented at a meeting of the American Physical Society, chemist Daniel H. Deutsch says that jiggling of Brown's microscope and evaporation

Art by Sergey Ivanov

Now let's turn back to the drop of ink in the glass of water. Molecules of both water and ink move chaotically, some of the water molecules penetrating the ink, and some of the ink molecules penetrating the water. There is diffusion of both liquids, which causes the drop of ink to blur.

Diffusion in gases is a common phenomenon that we regularly observe when we smell substances at a great distance from us. Processes of diffusion may also take place in solids, but they are usually too slow to be observed at room temperature. At higher temperatures, though, the motion of molecules or atoms becomes more intense. For example, a drop of ink is blurred in hot water sooner than in cold water. By keeping solid bodies at high temperatures long enough, we can verify that there is diffusion in them as well.

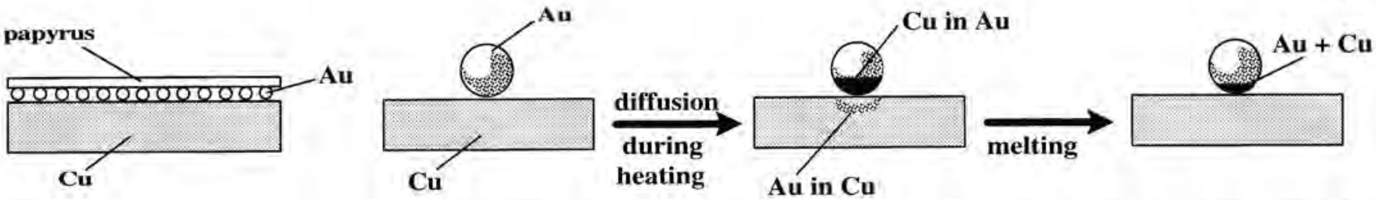
Diffusion in solids was first observed by the British metallurgist William Chandler Roberts-Austen in 1896. He pressed a gold disk and a lead cylinder together and put them in an oven for 10 days, keeping the temperature at 200°C. When the oven was opened, it was impossible to separate the disk and the cylinder: the gold and lead had literally "grown into one another" as a result of diffusion. By now this technology of joining components is widely used and is called diffusion welding. Could it be that the Etruscans attached gold granules in just this way?

We have to reject this supposition outright. First, the process of diffusion welding requires a vacuum—otherwise the oxygen in the air will oxidize the copper and the product will be covered with a black scaly layer; second, diffusion welding requires a rather long exposure to high temperatures.[2] The Etruscans couldn't have accommodated these conditions.

A more likely version of the Etruscan technology is this. First, gold granules were glued in a design on a sheet of papyrus, which was

of the water sample, among other factors, led him to see what was, "in effect, flotsam on the ocean, where you have waves knocking things about." (See *Science News*, May 4, 1991, p. 287.)—*Ed.*

[2]Roberts-Austen was lucky that the objects in his experiments were lead and gold. This pair of metals is one of the "record holders" in the area of diffusion times. At such a low temperature as 200°C, diffusion welding of other metallic pairs would have taken longer than a month.



placed on a copper plate, granules downward. The precious "sandwich" was then gradually heated. During the heating a small amount of the gold diffused into the copper, and vice versa. The result was a copper–gold alloy in the extremely thin area of contact between the granule and the plate.

The melting temperature of gold is 1,063°C; that of pure copper, 1,083°C. But copper–gold alloys melt at lower temperatures. For example, the alloy consisting of equal quantities of gold and copper melts at 910°C. This is the key to unraveling the secret of the Etruscan jewelers. They would increase the temperature to the point at which the gold–copper alloy created by diffusion melts, while the gold and copper themselves stay solid. During subsequent cooling the alloy would solidify, and a granule that was still virtually round would be fused to the copper substrate. This process occurred in all the granules simultaneously, so that the entire design of gold spheres glued to the papyrus would be transferred to the copper. At such high temperatures the papyrus would burn away, and the item was finished. The copper had no chance to oxidize because the process was rapid enough and the combustion of the papyrus used up a considerable amount of oxygen.

But we still haven't figured all the secrets of these ancient jewelers. For instance, it's not clear how the Etruscans managed to make such tiny, perfectly round gold granules. But the most amazing thing of all is how craftsmen in the distant past could have developed such an elaborate technology. What wonderful combination of chance, experience, and insight led the Etruscans to their discovery? Maybe we'll find out some day, but for now we can only pay tribute to this vanquished and vanished people, repeating the words of the Roman historian Sallust (1st century B.C.): "What people can achieve, in tilling the land, melting metals, and erecting buildings, depends on their spiritual strength."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Criminal geometry, or A matter of principle

A methodological handbook in one act

by D. V. Fomin

THE STAGE IS DARK. A QUIET melody is heard (see drawing). The lights are turned up. The drawing room at 221B, Baker Street. Sherlock Holmes is seated, looking through the evening newspaper. Watson enters.

Holmes. Good evening, my dear fellow. I see that you have decided to deal with geometry instead of medicine for a while.

Watson. How could you . . . ?

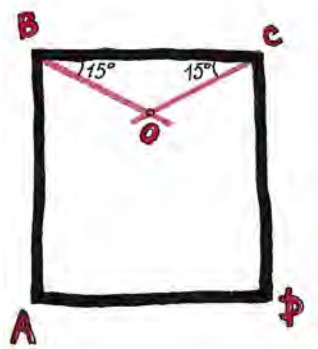
Holmes. Yesterday's *Daily Joke*, with the geometry contest, is sticking out of your pocket. I can see right off that you have wasted quite a lot of ink trying to solve at least one of the problems.

Watson. But how did you know that I have not solved any of them? To tell the truth, you are absolutely right . . . (*Sitting down*).

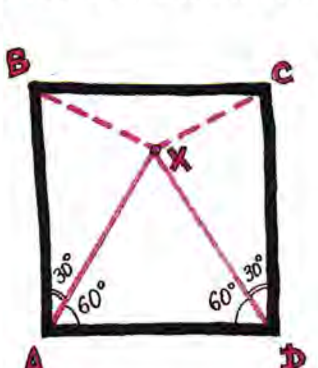
Holmes. Please do not take offense, my dear Watson. By the way, all these problems can be solved practically the same way—if, of course, we find the right approach to them. Frankly, I have not seen these contest problems yet, but . . . Well, let us have a look at them.

Problem 1. Point O is given inside a square $ABCD$. Angles OCB and OBC are both 15 degrees. Prove that triangle OAD is equilateral.

Watson (*spreading his arms in dismay*). The mysteriousness of this problem reminds me of the case of the abducted shah. Do you remember it, Holmes?



Holmes. My dear fellow, whatever are you talking about? I will give you the answer straight away. We will use the "begin-at-the-end principle" here. I hope you will be able to ascertain from the solution what the principle is. Consider a point X that is the third vertex of an equilateral triangle whose other two vertices are A and D.



Watson. But there are two such points.

Holmes. Of course. We choose the one inside the square. Now we find the angles XBC and XCB. Well, Watson, as an adherent of the exact sciences, you know the ropes here. Have you done it?

Watson. Just a moment . . . We must make use of the fact that BAX and XCD are isosceles triangles. Oh! They are 15 degrees each. Hum! What then?

Holmes. This means that points O and X coincide. But this is elementary, my dear fellow.

Watson. Excellent! But . . . this principle of yours will not help us with the second problem.

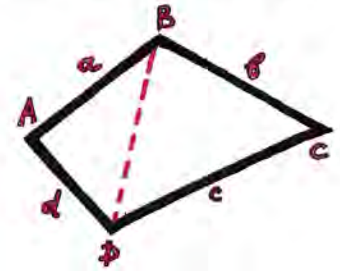
Holmes. Well then, we will use another one. And so,

Problem 2. The lengths of the sides of a convex quadrilateral $ABCD$ are equal to (clockwise) a, b, c, d. Prove that the area of $ABCD$ is not greater than

$$\frac{1}{4}(a+b)(c+d).$$

Yes, this problem is of an entirely different kind . . . inequality. Watson, it suddenly reminds me of professor Moriarty's cipher.

Watson (*dreamily*). Yes, that was a tricky business. He was, to give him his due, a brilliant mathematician . . . But you have digressed, Holmes.



Art by Dmitry Krymov

Holmes. It was *you* who was dreaming, Watson. In the meantime I have solved your problem. First, we open the parentheses:

$$\frac{1}{4}(a+b)(c+d)$$
$$=\frac{1}{4}(ad+bc)+\frac{1}{4}(ac+bd).$$

Keep the "simplification principle" in mind, Watson: first try the simplest and most natural ways of solving the problem.

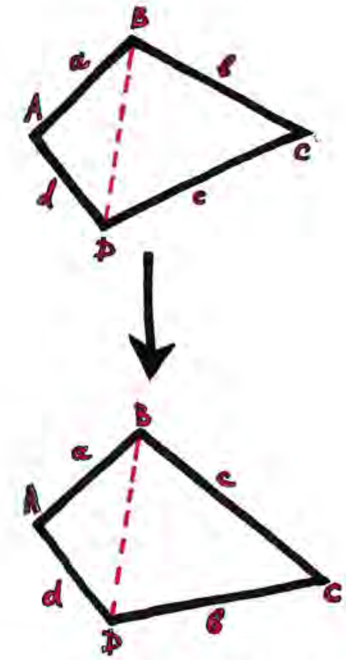
Watson. Well, as for proving that the sum $ad + bc$ is not less than twice the area of the quadrilateral, I can do that myself: ad is not less than twice the area of triangle ABD, bc is not less than twice the area of triangle BCD—we are done. But what can we do with the expression $ac + bd$?

Holmes. The "analogy principle" will help us here. All we need to do, my dear fellow, is think consistently and logically—this is essential in mathematics as well as in criminology. To what did you owe your success in the previous evaluation? You were helped by the fact that sides a and d are situated next to each other, as are sides b and c, correct? So you should do something to bring a next to c.

Watson. What about b and d?

Holmes. Think it through, Watson: if a is next to c, then of course b will be next to d. Always check for unnecessary conditions! But that is by the bye. And so: what will we do with our quadrilteral so that its area remains the same while side a ends up next to c? . . . My dear fellow, what is the matter? . . . Do you not have your scalpel with you?

Watson (*not understanding*). No, I do not. Why do you . . . ? (*Looking at the diagram and suddenly understanding.*) Brilliant! Why, we merely cut $ABCD$ along the diagonal BD and . . . and turn over one of the pieces. Then, of course, reasoning the same way as before, we determine that $ac + bd$ is not less than twice the area of the quadrilateral. Combining that with the previous inequality, we establish what we had set out to prove. Marvelous!



Holmes. Notice that we also used the "dynamic principle" here when we changed the data while solving the problem. It really is quite a remarkable principle. It reads: change anything in the problem you feel like changing—its formulation, the data, the things you have to prove—as long as the solution to the new problem gives you a solution to the old one. In particular, it says: do not take the data of a problem as something chiseled in stone. Well, for example, if you are to catch a criminal, do not forget that he is a living being and can move freely about your "theater of operations."

Watson. There is something I have not quite caught here . . .

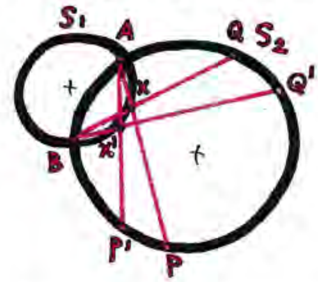
Holmes. Let us take a look at the third contest problem, Watson.

Watson. I beg of you, Holmes, tell me at once every step in your reasoning, not just the solution. I am already weary of wondering. (*Gets up from his chair and walks up to Holmes.*)

Holmes. I will try, my dear fellow. And so,

Problem 3. Two circles S_1 and S_2 intersect at right angles at points A and B. Point X lies on the first circle but inside the second one. Rays AX and BX meet S_2 at points P and Q. Prove that segment PQ is a diameter of circle S_2.

Watson. I simply have not understood the statement of this problem. What does that mean—the "circles intersect at right angles"? Nonsense!



Holmes. Not at all, Watson, it merely means that the tangent lines at the intersection points are perpendicular. Now, look how the dynamic principle works here. Let us move point X along arc AB of circle S_1—it becomes point X', and rays AX' and BX' will intersect circle S_2 at points P' and Q'—look, I will draw it on a sheet of paper.

Obviously, angles $X'AX$ and $X'BX$ are equal. That is why the angle measures of arcs PP' and QQ' are equal. But this means that the angle measure of arc $P'Q'$ is equal to that of arc PQ.

Watson (*interrupting*). But Holmes, where did you get the idea of proving that?

Holmes. My dear fellow, think it through yourself: if segment PQ is a diameter for any position of point X on arc AB, then however we move point X, the angle measure of arc PQ will not change. If what the problem asks us to prove is true, then obviously this must be true as well. The begin-at-the-end principle again.

And now, Watson, let us move our point X all the way to point B. What will we get? In this case, the angle measure of arc PQ will be exactly 180 degrees.

Watson. Why on Earth is that? Ah, yes . . . we use the fact that these circles intersect at right angles. By the way, Holmes, here is one more principle: use all the problem's data and keep in mind that they must be used somehow! What should we call it?

Holmes (*coolly*). I call it the "solution completeness principle."

Watson. Well, I say! One would think you have got a principle in every pocket!

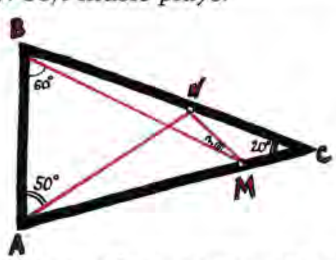
Holmes. No, my dear fellow, I have them all in my *head*. But I must point out to you that the principle that just occurred to you is sometimes inapplicable in real life. Some facts that seem at first glance to be suspicious or directly incriminating turn out to be mere coincidences or diversionary tactics by the real culprit. Remember the case of the beryl coronet? . . . Well, at any rate, here is the last problem.

Problem 4. ABC is an isosceles triangle, and the angle at vertex C is 20 degrees. Points M and N are taken on sides AC and BC such that angle NAB is 50 degrees and angle MBA is 60 degress. Prove that angle NMB is 30 degrees.

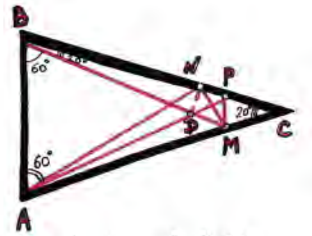
Watson. I was trying to compute that angle by using trigonometry . . .

Holmes. My dear fellow, pray save your strength! Perhaps you could read your newspaper while I spend a few minutes on this problem.

During the next five or six minutes Watson reads the paper while Holmes studies the diagram carefully. Soft music plays.



Holmes. There. Listen, Watson. This problem is indeed rather a tough nut. Let us look at a point P on side BC such that angle PAB is 60 degrees. Clearly, line PM is parallel to AB, and triangle PDM is equilateral (D being the point, Watson, where segments PA and BM intersect). Since triangle BNA is isosceles, the lengths of segments BN, BA, and BD are equal and angles BND and BDN are both 80 degrees. We can easily deduce from this that angle NDP is 40 degrees. So triangle NDP is isosceles and, as a result, MN is the bisector of angle BMP. Therefore, angle NMB is half of angle DMP and equal to 30 degrees. We are finished.



Watson (*astounded*). But . . . but . . . how? How did you think up such a clever solution?

Holmes. Well, my dear Watson, I suppose I could tell you an exciting story of how I found the solution by means of a dozen skilfully selected principles . . . You are laughing, Watson! Certain principles, of course, came in handy. For example, the remarkable "goal principle": always keep in mind what remains to be done to achieve your goal. And, certainly, a few little things here and there . . . At any rate, my friend, to solve a problem you need a bit more than just a set of standard rules of thinking. You need such things as experience and intuition. Do you really think that everything is so simple—that all we need do is memorize a lot of "principles" and learn how to use them in some sort of sequence? Fortunately, human reason is something immeasurably greater . . . though, of course, these principles, which are in essence nothing more than thought-clichés, can still be of some use. One must not ignore anything that is rational, Watson!

Watson (*sits down wearily in his chair, picks up his newspaper*). Oho! Listen to this, Holmes: "Yesterday night unknown malefactors, after breaking into the offices of the *Daily Joke,* cracked the editor's safe and stole the prize for the annual geometry contest: a life-size gold Moebius strip valued at . . ." and so forth, nothing that interesting . . . Oh, wait! "Inspector Robinson declared that the police have no leads in the case. The editor of the *Daily Joke* told reporters that, in order to increase the number of subscribers and improve its financial affairs, the paper will announce a new, special contest of problem solving in a future issue . . ."

My dear fellow, after all I have heard today, you simply must catch these villains—it's a matter of principle!

As the final words are spoken, the stage darkens and the final chords of the accompanying melody are heard in the darkness.

Here are the problems from the special contest announced in the newspaper. Keep in mind that each of them has its own little twist, which you can find more easily if you use the principles expounded by the great detective.

1. Prove that a five-point star cannot be drawn so that the lengths of segments AB, BC, CD, DE, EF, FG, GH, HI, IJ, JA (see the figure) are equal to 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, respectively.

2. Point B lies inside a right angle with vertex O, points A and C lie on its two sides. Prove that the perimeter of triangle ABC is not less than twice the length of segment OB.

3. A quadrilateral $ABCD$ is inscribed in a circle and the length of segment AD is equal to the sum of the lengths of segments AB and CD. Prove that the bisectors of angles B and C intersect on side AD.

4. An armless thief wants to steal a coin from a moneychanger's table, pushing the coin with his nose off the table without clinking it against any other coins. Will he succeed? The coins are round, they are of different sizes (possibly), and they are not touching one another.

5. A square is cut into several rectangles. For each of these rectangles, the ratio of the smaller side to the larger can be computed. Prove that the sum of these ratios is not less than 1.

ANSWERS ON PAGE 61

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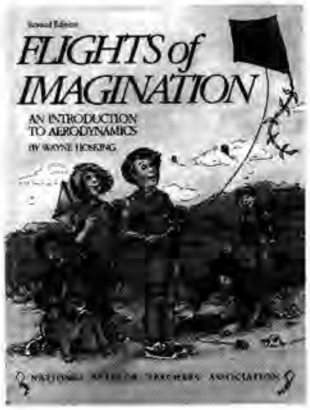

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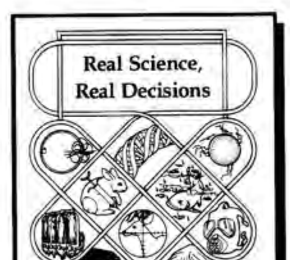

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LOOKING BACK

The duke and his chicken incubator

The ups and downs of Florentine thermal research and its applications

by Alexander Buzdin

HOT, WARM, CHILLY, frosty . . . These are the words we use every day to describe the degree of heat in various bodies. The measure of heat is, of course, temperature—the higher the temperature, the more a body is heated.

We can distinguish a hot object from a cold one just by touching it, but a science begins only when there are measured values. Today every child knows that to measure temperature you need a thermometer. It was with the invention of the thermometer that the study of heat phenomena really began.

Most thermometers make use of the way bodies expand when heated (and contract when cooled). The first thermometer, invented by Galileo at the beginning of the 17th century, used water. Naturally, it was useless for measuring temperatures below the freezing point or above the boiling point of water. Then the mercury thermometer became very popular. But when it's bitterly cold this thermometer also fails to yield accurate readings since mercury freezes at –38.8°C. Below that point the alcohol thermometer can be used—its operational range goes as low as –97°C (the freezing point of alcohol).

The credit for inventing the alcohol thermometer must be given to Duke Ferdinand II, who ruled Florence in the middle of the 17th century. In addition to his royal duties the duke devoted much of his time to the natural sciences. A pupil of Galileo, he made a notable contribution to the development of temperature-measuring devices. Before thermometers came into general use, temperature was monitored by thermoscopes. A thermoscope is a device that indicates whether the temperature is above or below a certain value, but it's incapable of actually measuring it. Ferdinand was very skilled at manufacturing thermoscopes of various designs. He sent one of his thermoscopes

Art by Sergey Ivanov

to Athanasius Kircher, a German Jesuit who was also known for his interest in science. The device consisted of an open glass tube almost completely filled with water. It also contained a number of small pear-shaped vessels that opened downward. Each of the vessels contained an air bubble whose volume was chosen in such a way that at a certain temperature the vessels floated in the water. When the temperature increased, the air inside the vessels expanded, pushing the water out. The resulting increase in buoyancy made the vessels float upward. Ferdinand's description suggests that the temperature at which the vessels were suspended inside the tube was about 15°C. When the temperature dropped below 15°C the volume of the air bubbles decreased, water entered the vessels, and they sank to the bottom.

Together with this device and detailed operational instructions, Ferdinand also sent Kircher another thermoscope. It differed from the first only in that it was completely filled with water and sealed. This thermoscope, however, operated in the opposite way—the vessels went down when the temperature rose and up when it fell. Ferdinand left Kircher to figure out why. We don't know whether Kircher succeeded, but an attentive reader will undoubtedly solve the problem without great difficulty. (You can check your answer by looking on page 63.)

Ferdinand had also devised many kinds of thermometers that actively applied his results in practice. One of his projects dealt with large-scale poultry breeding. He built one of the first chicken incubators, using a thermometer of his own design to monitor the temperature. But even then scientific results were not easy to implement. Only three chicks hatched out of 150 eggs. Why was the yield so low? Maybe because a worker didn't take the thermometer readings seriously enough and instead relied on his own sense of warm and cold. Or maybe Ferdinand himself simply didn't know enough about biology. We are left to guess at the reasons for the experiment's failure.

Ferdinand was more successful in monitoring weather by means of a thermometer. He carried out important meteorological observations and studied temperature in deep wells and underground cavities in different seasons of the year. He found that the change of seasons takes place later underground than on the surface (since time is needed for the ground to warm up or cool down).

For a more detailed treatment of the question of temperature and its measurement, see "Temperature, Heat, and Thermometers" in the May 1990 issue of *Quantum*. ◘

Index of Advertisers

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Bulletin board

Quantum's Twenty-Four Challenge . . .

In our March/April issue, we gave you the opportunity to show off your math skill and agility by playing Twenty-Four®, a game where teams of players must add, subtract, multiply, and/or divide to arrive at the solution of 24. We received more than 70 entries, from classes of all levels, and we judged them in two categories: 8th grade and below, and 9th grade and above. Here are the results.

8th grade and below. *Grand Prize (tie):* Evelyn Maxwell's 8th grade at Greenwood Laboratory School in Springfield, Missouri, and Bernadette Vachetto's 8th grade at Churchville-Chili Middle School in Churchville, New York. *First Runner-Up:* David Defendis's 6th grade at Francis Parker School in Rochester, New York. *Second Runner-Up:* Kathi Lohrman's 8th grade at Jefferson Middle School in Rochester, New York. *Third Runner-Up:* Kathi Lohrman's 7th grade, same school. *Fourth Runner-Up:* Mr. Strothmann's 6th grade at J. L. Buford Grade School in Mt. Vernon, Illinois.

9th grade and above. *Grand Prize:* Jean Kahn's 9th-12th grade at Shoreham-Wading River High School in Shoreham, New York. *First Runner-Up:* Mrs. Schilstra's 9th grade/period 1 at Penfield High School in Penfield, New York. *Second Runner-Up:* Lois Beyer's 10th–12th grade at Valley Christian High School in Campbell, California. *Third Runner-Up:* Mrs. Schilstra's 9th grade/period 2 at Penfield High School. *Fourth Runner-Up:* Mike Appelhans, a 12th-grader at the Work Opportunity Center in Minneapolis, Minnesota.

Prizes were generously donated by the Eastman Kodak Company–21st Century Learning Challenge. Congratulations and thanks to all who participated.

. . . back again

Here's another chance to team up and win prizes for your mathematical ingenuity. Take the second *Quantum* Twenty-Four Challenge, sponsored again by Kodak.

How to play. On this page, we've printed four cards from the Twenty-Four®game. Use the four numbers on each card to compute 24 as many ways as you can. Do the math step by step. Use only the numbers on the card and the answers from each step. (Note: the 9's on the cards are filled in with red; the 6's aren't.)

How to enter.

1. Send us a list of all the ways you got 24 for each card. Show *each stage* of your work.
2. Write on a sheet of paper
   - *a) Your grade*
   - *b) The names of your school and teacher*
   - *c) Your school's address and phone number*
   - *d) The statement: "We pledge that these answers were derived without the help of any computers or adults, except the teacher." Make sure all participants sign it.*
3. Send us your entry no later than **November 15, 1991**. Mail it to *Quantum*'s Twenty-Four Challenge, 3140 North Washington Boulevard, 2nd Floor, Arlington, VA 22201.

The prizes. The class in each category (8th grade and below, 9th grade and above) that finds the highest total of correct and different ways to make 24 will win the Grand Prize. Each student will get a Twenty-Four T-shirt and a standard edition of the Twenty-Four game.

The class in each category with the next highest total is first runner-up. Each student in those classes will receive a Twenty-Four T-shirt and a pocket edition of the Twenty-Four game.

Each student in the second, third, and fourth runner-up classes will receive a pocket edition of the Twenty-Four game.

Do's and don't's. Make sure you follow these rules when you enter:

Do use all four numbers on each card.

Don't use a number more than once, unless it appears on the card more than once.

Don't use the commutative property of addition or multiplication to

make 24 in more than one way. (If you do, we'll only count one of the answers.)

Don't use the number's opposite; 3 is not the same as –3.

Don't put two digits together to make a larger one. You can't make 23 from 2 and 3.

Don't use exponents. You can't use 2 and 3 to make 2^3, or 8.

—*Elisabeth Tobia*

Trailblazing genetics

A special full-color report from the Howard Hughes Medical Institute tells what it means to find the genetic flaw that causes a disease—for example, cystic fibrosis—and how such a discovery fits into the larger picture of medical research and the struggle to conquer genetic diseases. *Blazing a Genetic Trail* takes you down a path that begins with the search for the genetic roots of a disease and then winds its way toward the goal of a treatment or cure. An illustrated fold-out guide shows a map of the genetic trail, highlighting key landmarks.

The second in a series that began in 1990 with *Finding the Critical Shapes*, this 56-page report describes family studies of inheritance; scientific approaches to identifying disease genes; strategies for developing treatments; the need for animal models of disease; efforts to map the human genome; genetic testing; and what the future may hold.

For a free copy of *Blazing a Genetic Trail*, write to the Howard Hughes Medical Institute, Communications Office, 6701 Rockledge Drive, Bethesda, MD 20817.

Soviet space exhibit

Between June 29, 1991, and January 1, 1992, a spectacular array of Soviet space equipment will be on display at the Fort Worth Museum of Science and History. Sponsored by the museum, in affiliation with the Boston Museum of Science and the Soviet Civil Space Agency (Glavkosmos), "Soviet Space" is the first comprehensive collection of Soviet space materials ever to visit the United States. The exhibit features a full-scale model of Sputnik 1, a four-ton space telescope, rockets, interplanetary probes, a space motorcycle, and a lunar rover. For more information, call 817 347-4062.

Publish your research!

Every February and September, Michael Farmer publishes an assortment of papers written by high school students on the topics of botany, chemistry, computers, earth science, mathematics, physics, and zoology, among other sciences. The *Journal of High School Science Research* was founded two years ago by Farmer, a former chemist and teacher who wants to provide young scientists with a model outlet for future research. If you would like to receive the journal or are interested in the submission requirements, write to Michael Farmer, Editor, Applied Educational Technology, PO Box 193, Tigerville, SC 29688.

—*E.T.*

Student supercomputing

The Cornell Theory Center is again offering a summer program called "SuperQuest." Open to all of the 23,000 high schools in the United States, SuperQuest offers advanced supercomputing specifically for high schools. Teams consisting of 3–4 students and one teacher-coach will be selected to come to Cornell for three weeks in the summer of 1992 to learn about supercomputing research and its applications. The students will take classes in supercomputing techniques, meet with supercomputer researchers, and work with Cornell's technical staff to develop their own programs.

Sponsored by IBM and the National Science Foundation, SuperQuest's goal is to foster creativity in devising computational solutions to scientific problems, and no area of scientific endeavor is out of bounds. For an application booklet and more information, write to SuperQuest, Cornell Theory Center, 424 Engineering & Theory Center Building, Ithaca, NY 14853-3801, or call 607 255-4859.

Mathematics and Informatics

The purpose of this Bulletin Board item is to extend a warm welcome to a new journal, which is soon to become a quarterly publication with the new title *Mathematics and Informatics Quarterly*. You can obtain a copy of the pilot issue by writing to me; subscription information will be soon be distributed not only nationally but in all English-speaking countries as well as among students and practitioners of mathematics who recognize the importance of English as the major language in mathematical communications.

In addition to articles and notes written for students and teachers at the high school level, *Mathematics and Informatics* features several problem sections, including the problems and solutions of the International Mathematical Talent Search, which is a take-off on the USA Mathematical Talent Search, presently featured in *Consortium*. Moreover, there is a delightful section of "Forgotten Theorems," which should be of special interest to lovers of mathematical gems.

Mathematics and Informatics is edited by an international team of mathematicians, including Petar Kenderov and Jordan Tabov of Bulgaria, Willie Yong of Singapore, and Mark Saul (*Quantum*'s field editor for mathematics) and myself in the United States. It's supported by the Institute of Mathematics of the Bulgarian Academy of Sciences and the Union of Bulgarian Mathematicians, and is published by Science, Culture & Technology of Singapore. The yearly subscription price is US$12 for individuals and US$20 for libraries. Checks should be made out to *Mathematics and Informatics* and sent to

Dr. George Berzsenyi
Department of Mathematics
Rose-Hulman Institute
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Terre Haute, IN 47803-4999
USA

—*G.B.*


A NOTHER
A WESOME
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HUG


What did the acorn say when he finally grew up?

ANSWERS, HINTS & SOLUTIONS

Physics

P31

The velocities v of the sacks at the moment the second sack touches the floor can be determined by the law of energy conservation:

$$\frac{1}{2}(m_1 + m_2)v^2 = m_2 gH.$$

After that the first sack will move along the table until the string is stretched tight. At this moment the second sack accelerates with a jerk, so the velocities of the sacks become equal.

The changes in the sacks' velocities have the same value, as if the second sack were on the table (and not on the floor). This is because the sacks interact by means of the string, and the force acting on the second sack is the same (though not the direction) as if this sack were lying on the table. So we can find the sacks' velocities u after the jerk by the law of conservation of momentum:

$$m_1 v = (m_1 + m_2)u.$$

After that the sacks' movement is determined by the law of energy conservation, and the highest point of the second sack can be determined by the relation

$$m_2 gH = \frac{1}{2}(m_1 + m_2)u^2.$$

So taking all these equations into account, we have

$$h = \left(\frac{m_1}{m_1 + m_2}\right)^2 H.$$

P32

It follows from the graph (fig. 1) that during the first 50 minutes the temperature of the mixture doesn't

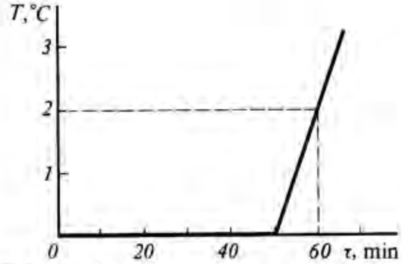


Figure 1

change and is equal to 0°C. The amount of heat the mixture acquires from the room during this time is spent on melting the ice. In 50 minutes, the all the ice melts and the temperature of the water begins to rise. In 10 minutes (from τ_1 = 50 min to τ_2 = 60 min), the temperature increases by ΔT = 2°C. The heat supplied to the water from the room during this time is $q = c_w m_w \cdot \Delta T$ = 84 kJ. Therefore, the amount of heat the mixture acquires from the room during the first 50 minutes is $Q = 5q$ = 420 kJ. This amount of heat was spent on melting ice of mass m_i: $Q = \lambda \cdot m_i$. So the mass of the ice in the bucket brought into the room is

$$m_i = \frac{Q}{\lambda} \cong 1.2 \text{ kg}.$$

P33

The gas is rarefied enough that a molecule would pass through all parts of the vessel during this minute. (To prove this you can estimate the average length of the molecule's mean free path under the given conditions, and you can see that it's substantially greater than the size of the vessel.) The pressure on a side is determined by the blows of all the molecules; if m is the mass of a molecule (for helium, $m \cong 7 \cdot 10^{-24}$ g) and v_x is the average velocity of a molecule's blow, then

$$P = \frac{2mv_x N_b}{S\Delta t}, \quad N_b = \frac{PS\Delta t}{2mv_x},$$

where N_b is the number of blows of all molecules on a side of area S during the time Δt; so one molecule hits the top of the vessel ~ N_b/N times, where N is the total number of molecules. We can estimate the velocity

$$v_x \cong \sqrt{kT/m},$$

so that, finally,

$$N_0 = \frac{N_b}{N} = \frac{PS\Delta t}{2mNv_x} = \frac{\sqrt{kT/m}}{2a} \cong 2.5 \cdot 10^5,$$

where the gas law $PV = NkT$ has been used ($V = a^3$, a = 10 cm).

P34

If we consider the photographic plate as one side of a cube inside which the source is located, then the source will be at the cube's center. So 1/6 of all emitted particles reach the plate, and the number of all particles emitted during an hour can be determined as

$$N = 6 \cdot 200 \cdot 3{,}600/10 = 4 \cdot 10^5.$$

There's no need for more accurate calculations because the emission is a random process and the counted number of traces (200) isn't too large.

P35

The illumination of every part of the Earth's surface is determined by straight solar rays and by sunlight dispersed in the atmosphere. The shadow delimits border of the region into which the straight rays can't reach. If there were no light at all at this spot, the shadow would be black (colorless). But on a sunny day (when there are no clouds in the sky), the dispersed sunlight is blue (the color of the sky). Falling on the shadow (that is, the spot where the straight rays can't reach), this light is reflected from the snow without substantial absorption (this is a property of the white snow), and so these rays give the shadow a blue tint. The cleaner the snow, the more clearly this effect is seen.

So the painters were correct in rendering the shadows blue.

Math

M31

The blot is a circle of radius $r = R$.

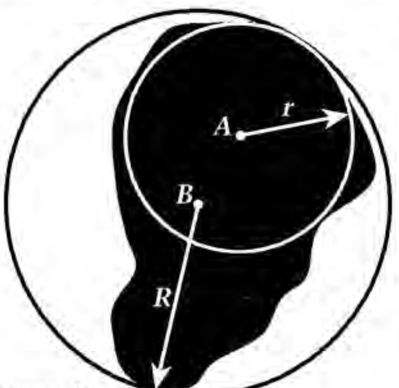


Figure 2

Let A be the point of the blot whose shortest distance from the blot's boundary is r. Then the blot contains the circle with center A and radius r, since all the points of the boundary lie outside this circle (fig. 2). Similarly, if B is the point of the blot whose greatest distance from the boundary is R, then the circle with center B and radius R contains the boundary, and so the blot itself. Since circle B contains the blot, and the blot in turn contains circle A of the same radius, the circles must coincide. So the blot coincides with both circles.

M32

Let $a_1, a_2, \ldots, a_n$ be the first terms of the sequences in question, and $d_1, d_2, \ldots, d_n$ their respective differences. Since the product of all the differences belongs to one of the sequences, the equality

$$d_1d_2\ldots d_n = a_i + kd_i$$

holds for some i, $1 \le i \le n$, and some integer k. It follows that $a_i = d_i(d_1\ldots d_{i-1}d_{i+1}\ldots d_{n-k})$ is divisible by d_i.

M33

Here is one of many possible solutions to this problem.

Denote the centers of the given circles by O_1, O_2, O_3, O_4 and their pairwise intersections by K, L, M, N (fig. 3). Segment O_1O_2 cuts the common chord AK of circles O_1 and O_2 at its midpoint P, and since the circles are congruent, P is also the midpoint of O_1O_2. Similarly, segment O_2O_3 and the common chord AL of circles O_2 and O_3 have the same midpoint Q. It follows that PQ is a midline of both triangles $O_1O_2O_3$ and AKL, so $\overrightarrow{O_1O_3} = 2\overrightarrow{PQ} = \overrightarrow{KL}$. Replacing O_2 in this reasoning with O_4, we get the equality $\overrightarrow{O_1O_3} = \overrightarrow{NM}$. Thus $KL = NM$, meaning that $KLMN$ is a parallelogram.

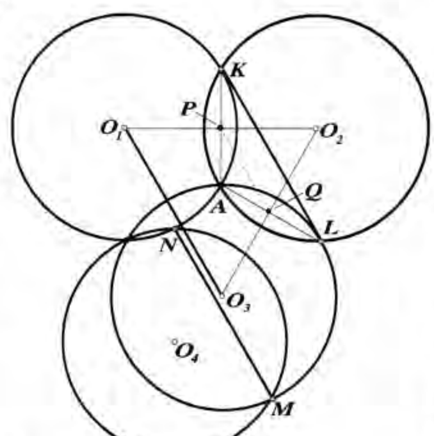


Figure 3

This problem is closely related to problem M6 (May 1990), in which three congruent circles meeting at one point were considered. In the solution to M6 you can find some beautiful properties of such circles and the triangles associated with them; these properties may prompt you to find other solutions to M33.

M34

The answer is 1,800. We'll solve the problem for an arbitrary n and then substitute $n = 1991$.

Writing out the lines of numbers resulting from successive steps of our process, we obtain the table shown in figure 4. Clearly all new numbers that appear in the kth line of the table—that is, at the kth step of the process—are greater than the numbers that appeared in the $(k - 1)$th step, which in turn are greater than the new numbers from the $(k - 2)$th step, and so on. So the numbers written down at the kth step are all not less than k. This means that the nth line of our table contains all numbers n that ever appear on the segment in the course of our process. Let's try to find out how many of these n's there are.

Every time we write down a number n according to our rule, between the neighboring numbers a and $n - a$ we create a pair of consecutive numbers a and n in that line of the table, where $a < n$. Conversely, if a pair (a, n), $a < n$, occurs in the table—that is, n stands to the immediate right of a in some line—then the number n of this pair (and not a) appears for the first time in this very line. So the number of n's in the nth line is equal to the number of pairs (a, n), $a < n$, that occur in our table. We'll prove a little later that every pair (a, b) of coprime numbers a and b occurs in the table exactly once, while every other pair doesn't occur at all.

So the number of pairs (a, n), $a < n$, is equal to the number $\varphi(n)$ of numbers, $a < n$, that are coprime with n. For $n = 1991 = 11 \cdot 181$, all the numbers less than n that have common divisors (other than 1) with n are of the form $181k$, $1 \le k \le 10$, or $11m$, $1 \le m \le 180$. There are $10 + 180 = 190$ such numbers, so $\varphi(1991) = 1{,}990 - 190 = 1{,}800$, and this is the answer to the problem.

Now it remains to prove the above statement. We'll do it by induction over $s = a + b$. For $s = 2$ the statement is obviously true—the only pair with $s = 2$ is $(1, 1)$. Let it be true for all pairs (a, b), $a + b < s$, and let's consider a pair (a, b), $a + b = s$. We can assume $a \le b$, since the table is symmetrical with respect to its vertical midline. Pair (a, b) occurs in the table if and only if pair $(a, b - a)$ occurs in the preceding line (fig. 5). But $a + (b - a)$

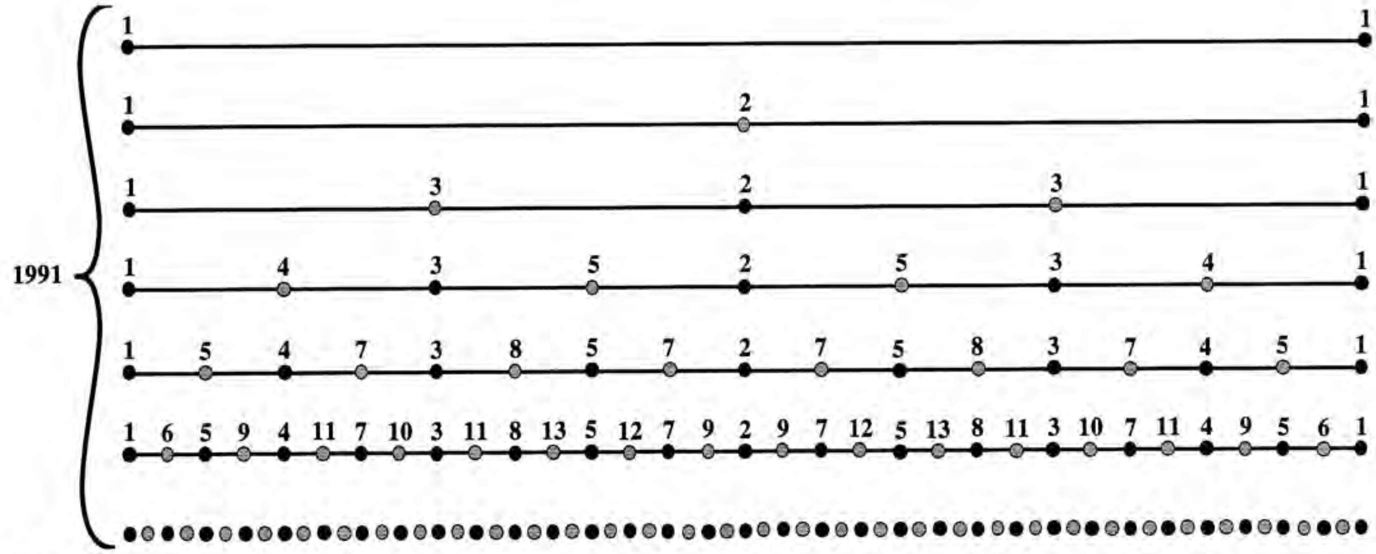


Figure 4

$= b < s$, so by the inductive hypothesis our statement is true for $(a, b - a)$, and common divisors of numbers a and $b - a$ are the same as those of a and b. Therefore, the statement is also true for (a, b), and we're done.

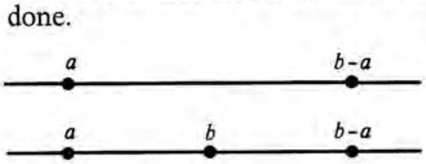


Figure 5

By the way, our table has already appeared in *Quantum* in another form (see "Genealogical Threes" by A. Panov, Nov./Dec. 1990), and the above statement is equivalent to problems 3 and 4 from that article.

M35

Denote the segments of the lion's route by $x_1, \ldots, x_k$ and the angles of its turns by $\alpha_1, \ldots, \alpha_{k-1}$ (the number of turns is 1 less than the number of segments). Let's straighten the route by turning it successively around the ends of segments x_i through angles α_i, to make each x_{i+1} the extension of x_i, and rotating the ring together with the segments (fig. 6). Instead of a broken line we get a straight segment LL' 30,000 m long consisting of the smaller segments $LL_1 = x_1$, $L_1L_2 = x_2$, ..., $L_{k-1}L' = x_k$. Now let's follow the path of the ring's center O during our successive rotations. The first rotation around L_1 takes O into the point O_1 such that $L_1O_1 = L_1O \le 10$ (the lion always stays inside the ring!) and angle $OL_1O_1 = \alpha_1$. Therefore, $OO_1 < 10\alpha_1$. Similarly, the distance from O_1 to the next position O_2 of the center is less than $10\alpha_2$, and so on. It follows that the distance between the first and the last positions of point O can be estimated as

$$OO' < 10(\alpha_1 + \ldots + \alpha_{k-1}).$$

On the other hand, the inequality $LL' \le LO + OO' + O'L'$ (fig. 7) yields

$$\begin{aligned} OO' &\ge LL' - LO - O'L' \\ &\ge 30{,}000 - 10 - 10 \\ &= 29{,}980. \end{aligned}$$

So $\alpha_1 + \ldots + \alpha_{k-1} > 2{,}998$ rad.

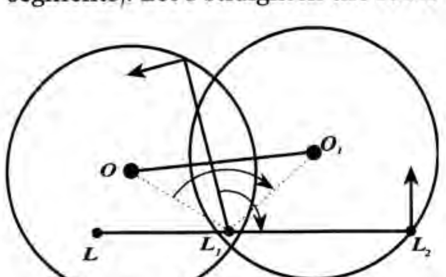


Figure 6

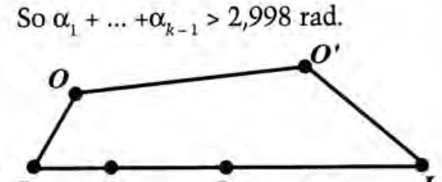


Figure 7

Brainteasers

B31

One of any 18 successive numbers is divisible by 18; so the sum S of its digits is divisible by 9, and the last digit is even. Since the number has 3 digits, S is not greater than $9 + 9 + 8 = 26$; so $S = 9$ or $S = 18$. In both cases S divides into the chosen number.

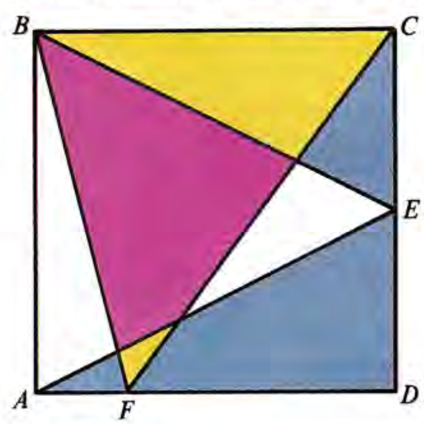


Figure 8

B32

Adding the shaded area to the area of the red part of the square (fig. 8), we get the area of triangle AED, which is half that of the whole square. Similarly, the sum of the same shaded area and the blue area is equal to the area of the square minus the area of triangle ABF, which is also half the area of the square. So the areas in question are equal.

B33

As the temperature increases, the pendulum's length also increases, but the mercury, because its volume increases, goes up the tube. A proper selection of the volume of mercury and the diameter of the tube makes it possible to keep the distance from the pendulum's point of suspension to its center of gravity constant. As a result of this design the pendulum's period of oscillation doesn't depend on temperature and the clock's accuracy is increased.

B34

See figure 9.

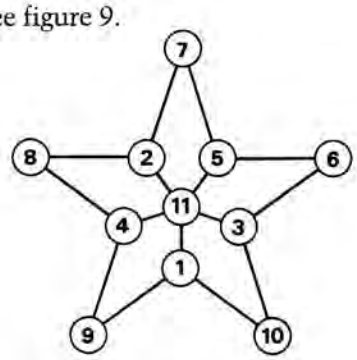


Figure 9

B35

See figure 10 (two opposite faces of the cube are covered with four small triangles each; a third face is covered with the two bigger triangles; the three other faces are covered with the square portions of the given sheet).

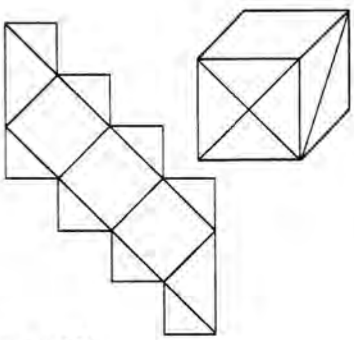


Figure 10

Criminal geometry

1. Since AB must be shorter than BC, angle BAC must be smaller than angle BCA = angle DCE, or, in the notation of figure 11, $\alpha < \beta$. Similarly, $\beta < \gamma < \delta < \varepsilon < \alpha$, leading to a contradiction: $\alpha < \alpha$.

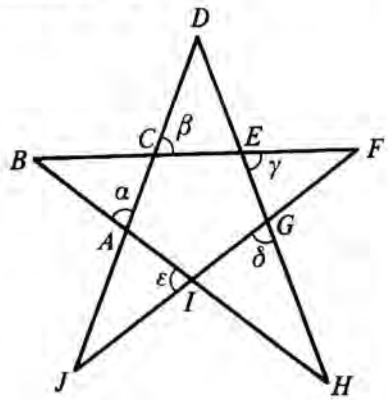


Figure 11

2. The reflections B_1 and B_2 of point B in the sides of the given angle (fig. 12) are symmetrical to each other with respect to the vertex O of the angle. It follows that $BA + AC + AB = B_1A + AC + AB_2 > B_1B_2 = 2B_1O = 2OB$ (since the length of a broken line is greater than the distance between its ends).

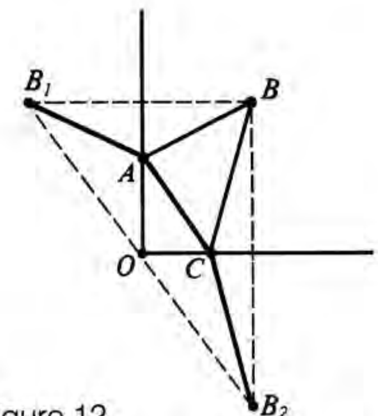


Figure 12

3. Take point M of side AD such that $AM = AB$ (and, therefore, $MD = CD$—fig. 13) and consider the point N at which the circumcircle of triangle BCM meets AD. It suffices to prove that BN and CN are the bisectors of angles ABC and BCD. If angle $AMB = \alpha$, then angle $ABM = \alpha$ (triangle ABM is isosceles), angle $BAD = 180° - 2\alpha$, angle $BCD = 180°$ – angle $BAD = 2\alpha$. On the other hand, angle BCN is subtended by the same arc of circle $BCMN$ as angle BMN, so it is equal to $\alpha = ½$ angle BCD. Therefore, CN is the bisector of BCD. Similarly, if angle $CMD = \beta$, then angle $MDC = 180° - 2\beta$, angle $ABC = 2\beta$, angle $NBC = 180°$ – angle $NMC = \beta$, so BN is also the bisector of the corresponding angle.

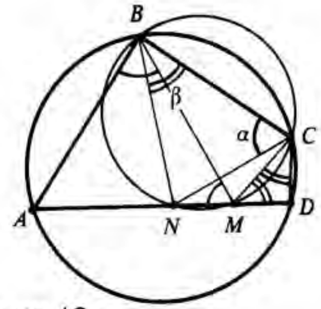


Figure 13

4. Yes, he'll succeed. He must choose the coin whose center is nearest of all the coins' centers to the edge of the table (we'll assume the table is a rectangle) and move it to the edge along the shortest path (fig. 14). If the coin meets some other coin on its way, then the center of that one is closer to the edge than the center of the chosen coin, which contradicts the choice of the original coin.

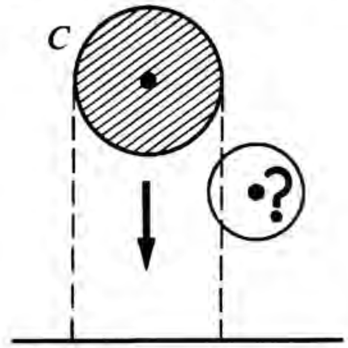


Figure 14

5. We'll assume that the side of the square is of unit length. Then each ratio a/b in question is greater than the product ab, since $b < 1$. Therefore, the sum of all the ratios is greater than the sum of the respective products—that is, the sum of the areas of all the rectangles. But the latter sum is the area of the square and so equals 1.

Divisive devices

1. (a) 100; (b) 2; (c) 3; (d) 0.

2. There are 6 apartments (from 97 to 102) on one of the floors and, therefore, on every floor off each staircase.

So each staircase has $8 \cdot 6 = 48$ apartments and, since $211 = 48 \cdot 4 + 19 = 48 \cdot 4 + 6 \cdot 3 + 1$, apartment 211 is off the 5th staircase ($5 = 4 + 1$) on the 4th floor ($4 = 3 + 1$).

3. When one piece is cut into 5 parts, the number of pieces is increased by 4. So the number of pieces is always of the form $4k + 1$—that is, it yields a remainder of 1 when divided by 4. Since $1991 = 4 \cdot 497 + 3$, the answer is negative.

4. The six-digit number must be divisible by $3 \cdot 7 \cdot 13 = 273$, and $100{,}000 = 366 \cdot 273 + 82$. Adding 191 we get $100{,}191 = 367 \cdot 273$.

5. (a) 1; (b) 5; (c) 8.

6. If GCD $(k, l) = n$, then a and b have the common divisor $nd \geq d$. But d is greater than any other common divisor of a and b. Therefore, $n = 1$.

7. Let a and b be the given numbers, d = GCD (a, b). Then $a = kd$, $b = md$, where k and m have no common divisors greater than 1 (that is, k and m are coprime) and $kmd^2 = ab = 600$. But the greatest square of an integer that divides 600 is 100, so the greatest value of d is 10. Example: $a = 60$, $b = 10$.

8. GCD (264, 192) = 24, so the answer is 24 bouquets.

9. (a) GCD $(m, n) + 1$; (b) GCD $(m, n) - 1$;

10. (a) $987{,}654{,}321 = 8 \cdot 123{,}456{,}789 + 9$; 123,456,789 is divisible by 9, and GCD (987,654,321, 123,456,789) = 9; (b) 77.

11. Two 141 cm x 141 cm squares, three 42 cm x 42 cm squares, two 15 cm x 15 cm squares, one 12 cm x 12 cm square, and four 3 cm x 3 cm squares. GCD (324, 141) = 3, so there are no smaller squares.

12. Successive remainders given by Euclid's algorithm for (a, b) when taken from the last one (equal to the GCD of a and b) will be not less than the so-called Fibonacci numbers 1, 2, 3, $5 = 2 + 3$, $8 = 5 + 3$, ..., 610, 987, 1,597.

13. $\alpha = 3/140$.

14. After division by GCD (85, 204) = 17 we get $5x + 12y = 1$. But $12 = 2 \cdot 5 + 2$, $5 = 2 \cdot 2 + 1$; so $1 = 5 - 2 \cdot 2 = 5 - 2(12 - 2 \cdot 5) = 5 \cdot 5 - 2 \cdot 12$. One of the solutions is $x = 5$, $y = -2$. The general solution is $x = 5 + 12t$, $y = -2 - 5t$, where t is any integer.

15. (a) yes; (b) no.

16. (a) We have to find integers x and y that satisfy the following equality:

$$6x + 16y = 220$$

or

$$3x + 8y = 110.$$

One of the possible solutions is $x_1 = 330$, $y_1 = -110$. The general solution can now be written in the form

$$\begin{aligned} x &= 330 - 8t, \\ y &= -110 + 3t, \end{aligned} \qquad (1)$$

where t is any integer. It's now natural to choose t in such a way that both x and y are nonnegative. (Of course, we can connect the batteries in the opposite way—plus to plus—but we'll try to avoid such a situation.) Taking into account this condition we have

$$330 - 8t \geq 0,$$

that is,

$$t \leq \frac{330}{8} = 41\tfrac{1}{4},$$

and

$$-110 + 3t \leq 0,$$

that is,

$$t \geq \frac{110}{3} = 36\tfrac{2}{3}.$$

Substituting t = 37, 38, 39, 40, 41 into equations (1) we get five solutions:

6-V batteries	34	26	18	10	2
16-V batteries	1	4	7	10	13

(b) In this case we have to solve the equation $6x + 15y = 220$. But GCD (6, 15) = 3, and 220 isn't divisible by 3. So the equation has no integer solution.

17. Since $a = bd$ is divisible by c, and GCD $(b, c) = 1$, then, by lemma 3, d is divisible by c.

18. Statements (b), (c), and (d) are true; (a) is false.

19. $1990 = 2 \cdot 5 \cdot 199$; $1991 = 11 \cdot 181$; $1992 = 2^3 \cdot 3 \cdot 83$.

20. (a) If am is divisible by n and GCD $(m, n) = 1$, then $a = kn$, and so $b = km$. (b) Suppose that the factorization of x includes a prime number p raised to the ath power, and the factorization of y includes p raised to the bth power. Then by the uniqueness property of the factorization, $am = bn$, while problem (a) gave us $a = kn$, $b = km$. Include p^k in the factorization of z and repeat the procedure for all the prime factors of x and y. The resulting value of z is the answer.

Kaleidoscope

1. The bubbles glitter because of light reflection at the air–water boundary.

2. Yes, you can light a fire by making a convex lens out of the ice.

3. The light rays refract slightly at the eye–water boundary and produce a blurred image on the retina.

4. The light reflected at large angles of incidence from the scuba diver undergoes complete internal reflection from the water–air boundary. The light reflected at any angle of incidence from the person fishing onshore passes into the water.

5. All you need to know is the substance's refractive index.

6. He approaches his reflection at 4 m/s.

7. The beam is easier to see in fog because the light is scattered by water droplets in the air.

8. Jewels glitter because of repeated internal reflection of the light beams falling on them.

9. Sunlight scattered by the atmosphere is much brighter than the light from the stars.

10. We can get three images of A (see figure 15).

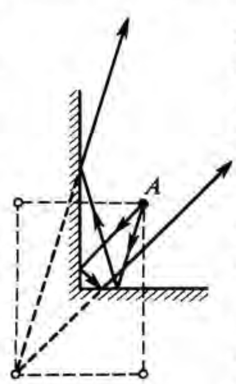


Figure 15

11. By squinting, people decrease the "aperture" of the eye; as with a cam-

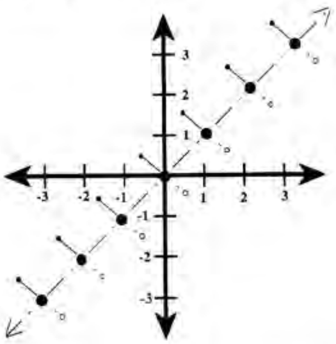
era, the image becomes sharper because of the increased "depth of field."

Microexperiment. For example, you can focus sunlight with glasses for hyperopia but you can't with glasses for myopia.

Holes

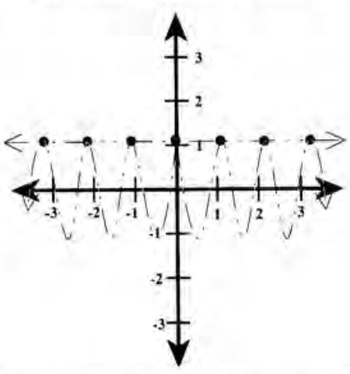
1. First consider the rounding function we all learned in school: $R(2.1) = 2$, $R(2.5) = 3$, $R(2.7) = 3$, and so on. Then the following function is continuous only on the integers:

$$(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 2R(x) - x, & \text{if } x \text{ is irrational.} \end{cases}$$



Another possibility is a periodic function:

$$(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ \cos(2\pi x), & \text{if } x \text{ is irrational.} \end{cases}$$



2. Let $r \neq 0$ be a rational number. We note that $\pi^N r$ is irrational and that there are a countably infinite number of numbers of the form $\pi^N r$, where N is an integer. Simply shift them all, by one jump, to fill the rational holes. In other words, for each rational number $r \neq 0$, map πr to r; map $\pi^2 r$ to πr; map π^3 to $\pi^2 r$; and so on. This takes care of every rational number except zero. We map e to zero, e^2 to e, e^3 to e^2, and so on. The function we desire is

$$f(x) = \begin{cases} x/\pi, & \text{where } x = \pi^N r, \\ & \quad \text{with rational,} \\ & \quad \text{nonnegative } r \\ & \quad \text{and integral } N \geq 1 \\ 0, & \text{where } x = e \\ x/e, & \text{where } x = e^N \\ x, & \text{otherwise.} \end{cases}$$

Further questions that we couldn't answer: (1) Can our solution be simplified? (2) Can the R^3 space be filled with the irrational points on a line segment? Can this be done elegantly?

Latin triangles

Here is the solution to the problem posed in the last Toy Store (March/April).

Figures 16 and 17 illustrate one of the ways to color the nodes of a triangular grid so that the colors are all different on every line parallel to a side of the triangle. Let the number n of nodes on a side of the grid, which is equal to the number of colors, be odd (see the part of figure 16 enclosed in the frame for $n = 5$). Then we draw $2n - 1$ vertical lines through the nodes of the grid and paint the nodes on each line the same color so that moving from left to right we encounter the colors in the order 1, 2, ..., n, 1, 2, ..., $n - 1$ (different numbers here denote different colors). When n is even, this method doesn't work (try it!), but a minor correction will fix that. At first we ignore the bottom line of the grid and paint the remainder (framed in figure 16) as above. Then we extend the vertical lines and the coloring onto the bottom line. This leaves only the two bottom vertices of the big triangle uncolored. We color one of them (say, the left one) the nth color, which hasn't been used so far (in our figures, it's blue). The other one is painted the color that hasn't been used in the bottom line—that is, the top vertex color. Now we have only one problem left: the colors of the top and right vertices are the same. To get rid of this coincidence, swap the colors of the right vertex and its left neighbor and then replace the two new coincident colors (marked with stars in figure 16) with the nth (blue) one. The result is shown in figure 17 (for $n = 6$). After the swap we'll always have only two coincidences, so the method works for all even n except 4. When $n = 4$ the two starred nodes show up on the same line of the grid; but, searching through all the possibilities, we can assure ourselves that in this case the problem can't be solved at all.

Duke

In the second thermoscope the crucial role is played by thermal expansion of water. Since this thermoscope is completely filled with water and sealed, the water can only expand by filling a portion of the space occupied by air in the vessels. The air can't resist the water since air's resistance to compression is many times less than that of water. This results in a decrease in the buoyancy of the vessels, which then sink to the bottom.

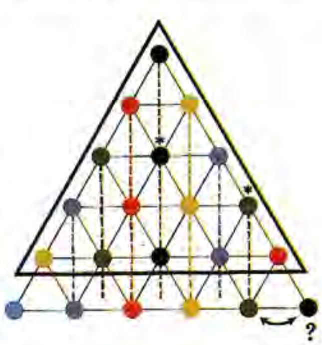


Figure 16

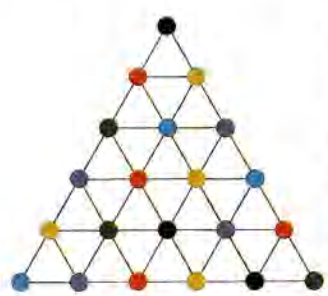


Figure 17

The a-maze-ing Rubik's cube

Watch out for "tunnels" and "dead ends" in this variation of the classic block

by Vladimir Dubrovsky

GO GET YOUR RUBIK'S CUBE (you must have one somewhere among your old toys!) and a roll of cellophane tape: in two minutes you can make a totally new, exciting puzzle. This puzzle was originally created, in a slightly different form, by the French inventor Raoul Raba. Figure 1 shows Raba's "taquinoscope." It looks like three intersecting circles cut into 10 curvilinear triangles so that it's possible to rotate the circles. After several rotations the triangles get scrambled and (just as with the cube) you must try to restore the pristine, regular state. But what has the flat, round taquinoscope to do with the three-dimensional, rectangular cube? To see what, look at figure 2, where a modified Rubik's cube called a "bicube" is shown. The bicube is obtained from an ordinary cube by fastening in pairs 18 little "cubelets" of three adjacent faces of Rubik's cube. (This is why you need the tape.) To make it easier to tell the double blocks from the unit ones, you can tape colored paper over the adjacent cubelets that form a double block. The cubelets on the other side of the cube, which are not visible in figure 2, are left as they are. Comparing figures 1 and 2, we immediately see the correspondence between the "visible" elements of the bicube and the pieces of the taquinoscope. And it's not just a superficial likeness: each transformation of the taquinoscope caused by a series of rotations of its circles corresponds to the transformation of the bicube resulting from respective rotations of the three "visible" faces. So the puzzles turn out to be absolutely equivalent. It's this three-faced version of the bicube that we're going to explore.

Figure 1

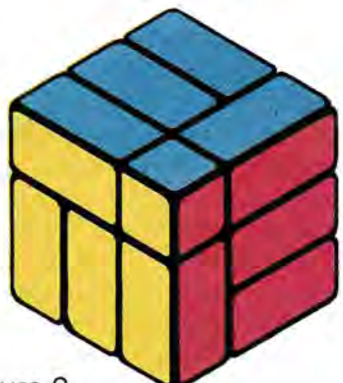


Figure 2

This recasting of Rubik's cube dramatically alters its properties as well as its solution. You notice at once that every double block locks one of the faces, so this face can't be turned. What faces are rotatable depends on the *configuration* of the bicube—that is, on the arrangement of the blocks irrespective of their colors. As opposed to this, an ordinary Rubik's cube has only one configuration, and we can always rotate any of its six faces. So the usual algorithms for solving the cube don't work with the bicube, since standard sequences of face turns used in them are impracticable for most configurations of the bicube. So let's first examine all possible configurations and their interconnections.

Most of the configurations have only one unlocked face. We'll call them *tunnels*, since when we hit any of them after some rotation we can't "turn off the road": all we can do is either return (rotating the same face back) or move ahead (repeating the previous turn of the same face). All other configurations are represented as nodes of the graph shown in figure 3. A colored arrow joining two nodes of the graph means a clockwise quarter turn of the face of the same color in figure 2 (that is, blue arrows correspond to top face turns, and so on). Moving against an arrow means, of course, a counterclockwise quarter turn of the respective face. There are only four *junctions* (configurations with three free faces), and their diagrams are inserted directly into our graph. One of them, in the center, is the *origin* (the configuration of the more move. The next move is determined uniquely every time. But the cube you're playing with is a magic cube, so no wonder the maze you're wandering in is magic, too: after wandering and returning to the origin several times, you may find that the coloration of your bicube has changed, even though the configuration has been restored. This means the blocks have been rearranged.

Here I'll stop and leave you something to solve: try to find all the possible rearrangements of the blocks for a given configuration—say, the origin. I'll just give you a couple of hints. If you start from the origin and move along the black arrow on the graph, turning the corresponding faces, when you come back you'll have five double blocks rearranged in the cyclic order shown in figure 4. You'll get two other five-cycle rearrangements moving along two other similar closed routes on the graph. All possible rearrangements can be obtained by repeating these three in a different order and a different number of times. And I'll reveal one last secret: there are only 60 rearrangements, though one would expect 12 times as many, since the total number of permutations of six blocks is $6! = 720$.

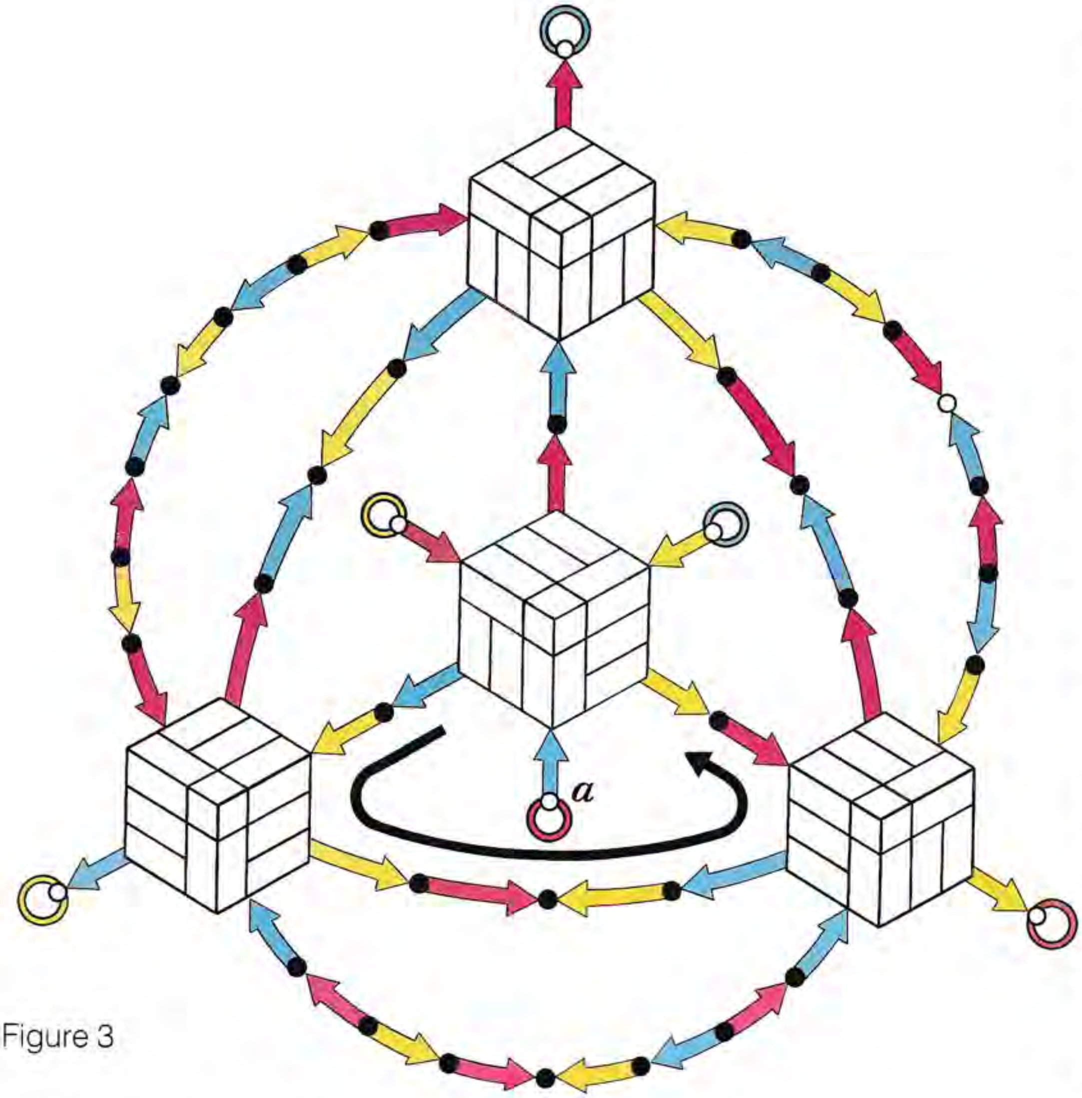


Figure 3

initial state of the bicube). Other configurations represented in our graph have two unlocked faces each, but these are of two kinds. Those denoted by white circles with loops might be called *dead ends*: for example, the red loop at node *a* below the origin means that rotation of the red face in the corresponding dead-end configuration leads us through a series of tunnels back to *a*. Black circles denote *switches* (configurations that allow us to get to another nontunnel configuration by switching the rotated face). The graph shows that, starting from a switch, we have only one way to turn each of the two unlocked faces to get to the next switch or junction. Turning the faces through other angles, we'll get to tunnels that aren't shown so as not to overload the figure.

Now, playing with the bicube is like wandering in a sort of maze. And we can see from figure 3 that this maze isn't too intricate. The rule that will always lead to the origin is quite simple: keep moving forward, avoiding tunnels, until you get to a junction; then repeat the last move (turn the same face) and make one

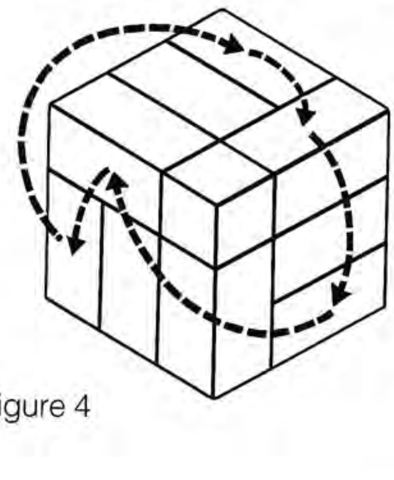
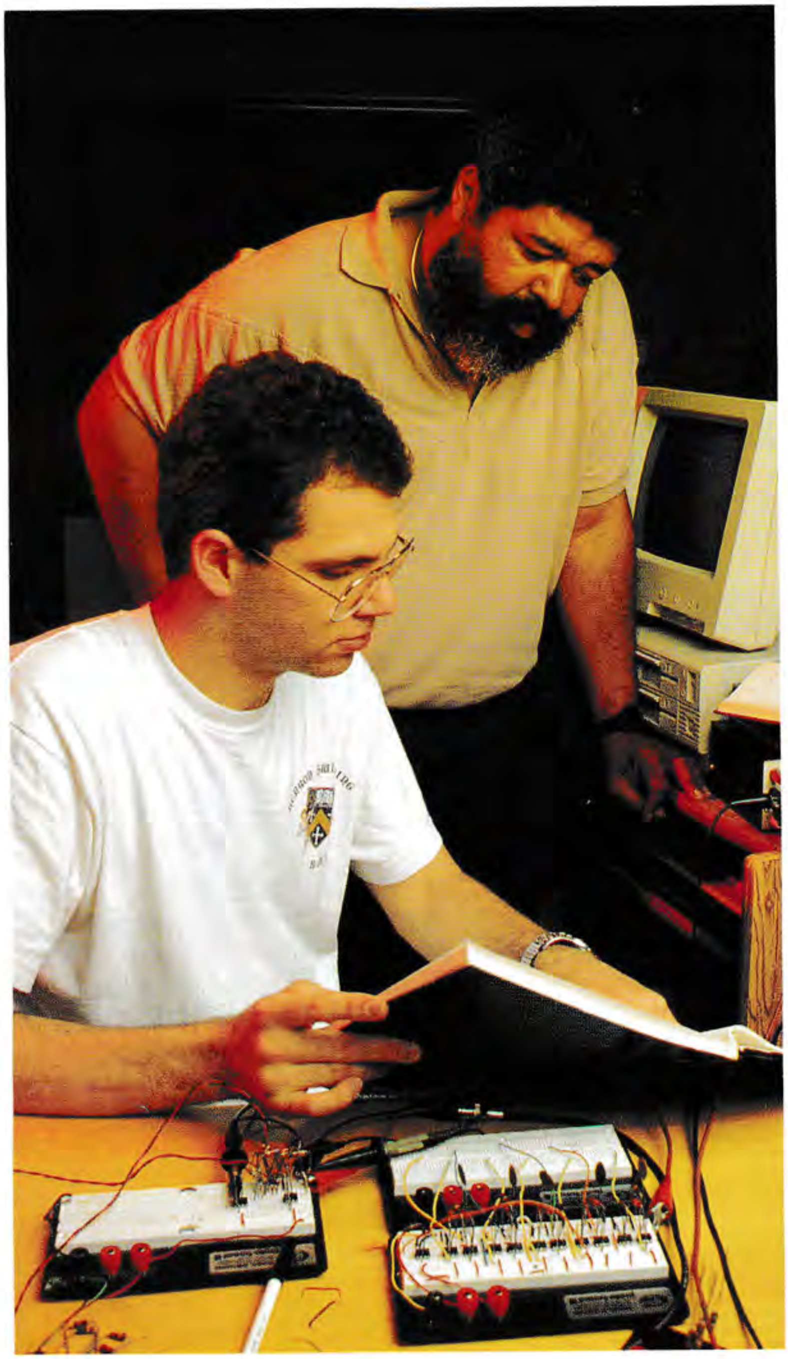


Figure 4

It's rather difficult to prove these statements if you're not going to use a computer (which makes it rather dull). But even more difficult is an investigation of the "superbicube," which is the same puzzle except that all of its six faces can be rotated. In this case, I don't even know the number of possible rearrangements.

"There are often days when I go back to the basics I learned at Kenyon."

—Stephen Carmichael, Kenyon Class of 1967, professor of anatomy, Mayo Medical School



For many science students, the small college's emphasis on strong teacher-student relationships and opportunities to participate in — and be recognized for — solid research with faculty members are powerfully appealing. There is also the promise of access to sophisticated equipment and instrumentation that the small college provides.

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Kenyon physics major Aaron Glatzer (left) consults with Associate Professor of Mathematics James White on his research, which involves building electronic circuits to imitate neurons and neural networks.

Kenyon College