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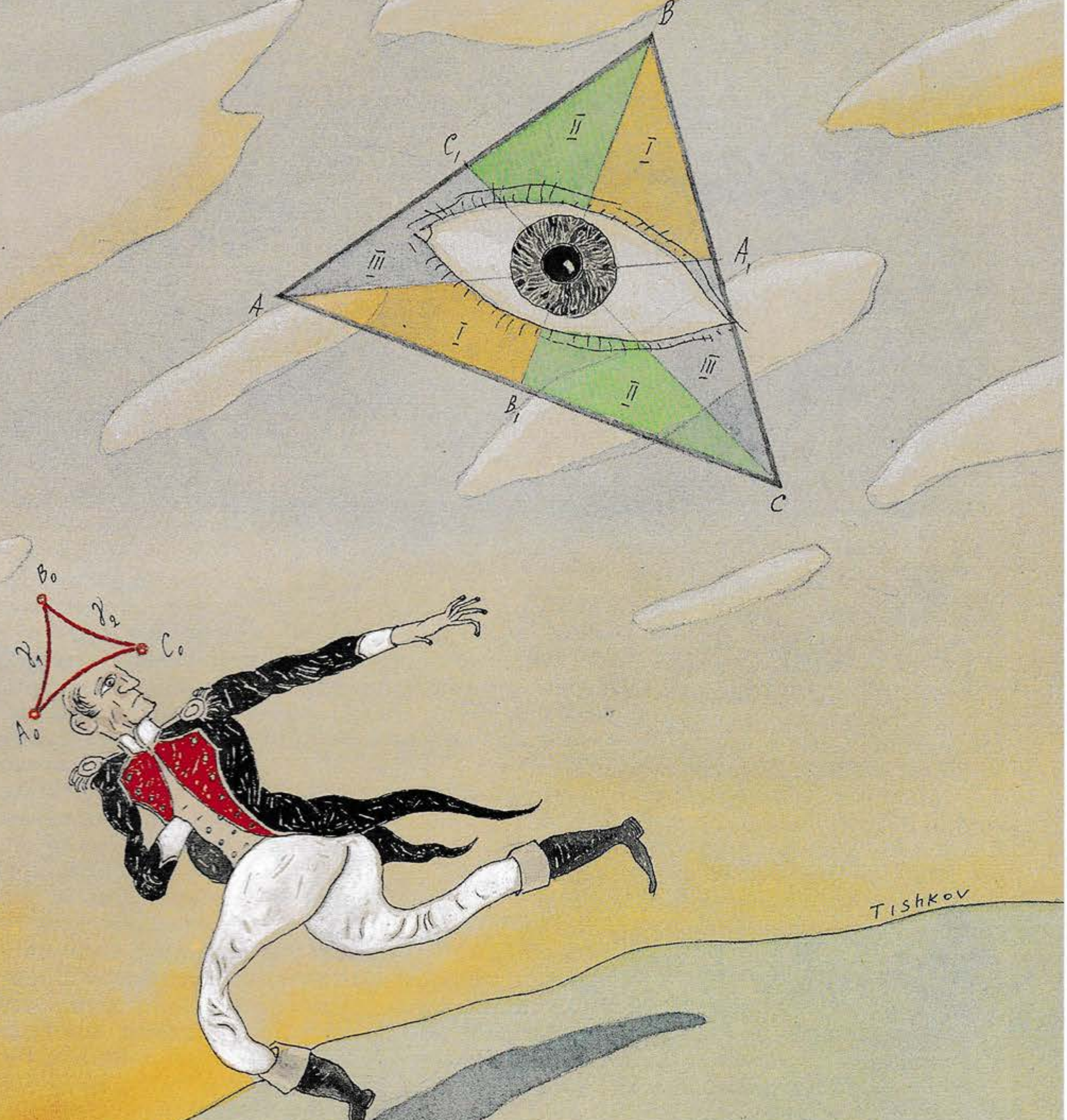

The student magazine of math and science

SPRINGER INTERNATIONAL

National Gallery of Art, Washington (Gift of Marcella Louis Brenner) © NGA

Beta Kappa (1961) by Morris Louis

THERE ARE SOME WORKS OF MODERN ART with such improbable names you think: "Come on! You're pulling my leg!" We can't be certain Morris Louis isn't having a little fun with us when he calls this chromatic collection of lines "Beta Kappa."

Maybe Louis *almost* made it into a certain prestigious honor society. Then again, maybe he's *phi*losophically opposed to such trappings.

Maybe he knows someone whose initials are B.K. (Betty Kaplan?), and this work (α) is dedicated to B.K., (β) was inspired by B.K., (γ) was painted at B.K.'s house while Louis helped himself to whatever was in B.K.'s refrigerator . . .

Maybe Louis was thinking back to his student days, when Greek letters stood for unknown entities or quantities, and used some of them in naming an attempt at depicting an unknown, unknowable, but ordered, rhythmic thing.

When one is struggling to interpret a confusing object, the most farfetched things can pop up, offering to be of use. For instance, in *The Vanished Library* (on the great library at Alexandria in ancient Egypt), Luciano Canfora notes that "Greek *beta* and *kappa* are almost inevitably confused with each other in the small lettering of the ninth and tenth centuries."

Would this painting be any different if it were called "No. 37"? Or "Love is a many splendored thing"? Or "Untitled"? Where does the meaning of such a creation reside? While many artworks of the past seem unambiguously decorative (pretty obvious and obviously pretty), works like "Beta Kappa" make great demands on the viewer. They're like an actor who leans against a prop and says to the audience: "Well? What are *you* looking at?" And nothing more!

The ultimate question in such a case used to be: It it beautiful? But since beauty is in the eye of the beholder (and always has been, really), maybe we should ask several smaller questions: Did it make me think or remember? Was I charmed by its individual elements and overall structure? Did it help me discover or create anything?

You'll have another brush with the Greek alphabet if you turn to page 40. This time, an eminent physicist will be your guide.

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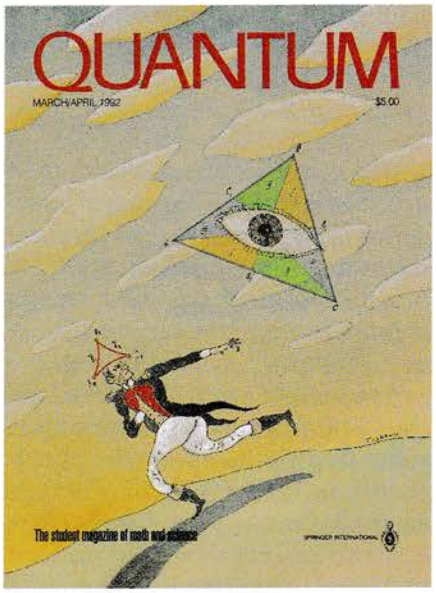

Cover art by Leonid Tishkov

The famed and feared general, the overreaching emperor, Napoleon Bonaparte, is being harassed by a flying triangle! Not just any triangle, but a well-known Masonic symbol—you can see it on the back of a US dollar bill (maybe because George Washington himself was a Mason). Strange—Napoleon and triangles had been on such good terms (see the Kaleidoscope in the September/October 1990 issue of *Quantum*) . . .

The pugnacious polygon seems to have particular designs on Napoleon's hat (and on itself as well—note the smart-looking bisectors and fancy labeling). Apparently there's a price to be paid for abusing this powerful symbol. But don't let this deter you from "Halving It All," which begins on page 6. Perhaps you're immune from the adverse effects of the Trilateral Avenger!

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Global change

And consequences on a human scale

FOR OUR FRIENDS and colleagues in what was the USSR, 1991 was a tumultuous year. The scope and pace of change there was as painful as it was breathtaking. I'm sure *Quantum* readers have been following the course of this "second Russian revolution."

You and I have watched how, on the societal level, old political structures crumbled and the map of the region was redrawn. It was as if the various states of the United States had seceded from the Union and formed separate, independent nations. But the effects in the former Soviet Union were even more devastating. Compared to the US economy, the Soviet economy was much more centralized. In the US many suppliers can be found for most essential products or services. They compete with each other and, ideally, this competition keeps quality up, prices down, and demand satisfied. In the USSR, there was often only one huge supplier for any given item. As various enterprises began to fail, they sent ripples through the rigid, huge, isolated Soviet economy that grew to be a tidal wave.

There is another way in which the former USSR was quite different from the US. There are vast differences in culture and language from one republic to another, and these differences have led to serious and sometimes bloody conflicts.

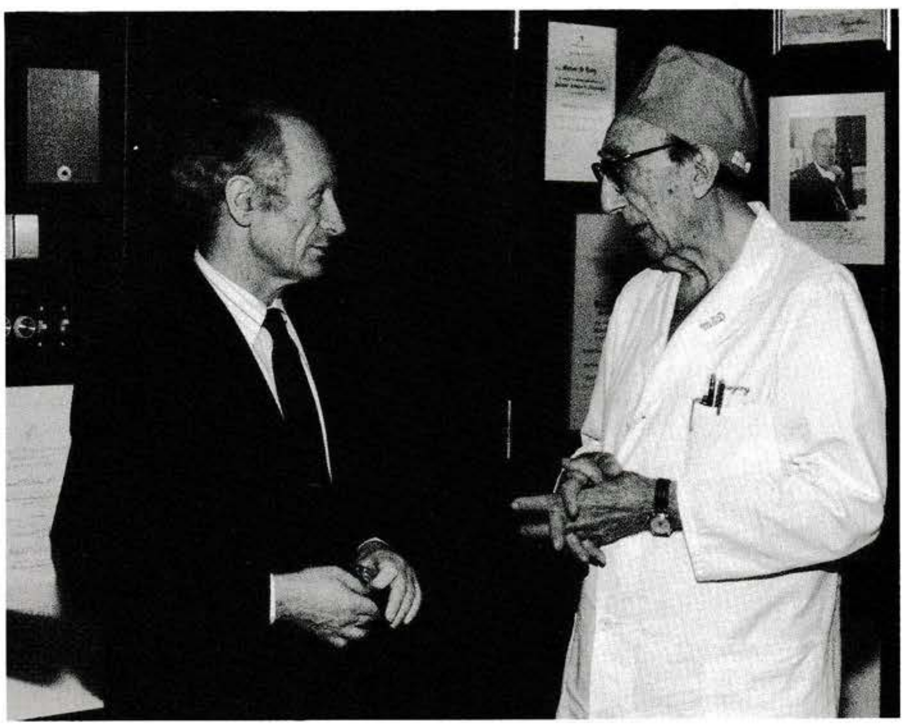

Photo courtesy of Linda Crow

Quantum *founding editor Yuri Ossipyan consults with Dr. Michael Debakey.*

The young Commonwealth of Independent States (CIS) faces a bewildering array of challenges. It needs to create a new financial structure and integrate it into the global economy. It must revamp its economic life so as to provide an incentive for its citizens to work again. Its infrastructure—transportation, communications, and so on—is comparable to that of the US in the 1920s. There are so many problems, and such a long road to travel before the CIS becomes viable in the international market and the standard of living of its people improves. What can we do to help? Why should we help? What are our interests, apart from compassion?

The CIS contains the greatest store of natural resources in the world. And its people are enormously talented. They're well educated, well trained, and capable of surviving under the most adverse conditions. They will, in time, make the CIS one of the two or three leading "nations" in the world (although the CIS isn't actually a "country"). And this time, its member states will be economic powers. It's my view that the US must, for its own economic security, become the leading friend and trading partner of the CIS. Only then are we likely to compete effectively with the Pacific rim nations in the far east and the European Community in the west.

On the personal side, I have some news to share with you. Academician Yuri Ossipyan, formerly vice president of the Academy of Sciences of the USSR, no longer serves in that capacity. First, the academy no longer exists under that name. It has merged with the Russian Academy of Sciences and now carries that name. But also, for reasons of health, and out of a desire to spend more time on his research in physics, Yuri Ossipyan was not a candidate for vice president of the Russian Academy of Sciences during their recent elections. He will continue to work with *Quantum* as one of its founding editors and will serve as president of *Quantum* Bureau, of which Sergey Krotov is executive director.

Yuri Ossipyan has for many years suffered from a serious heart ailment. He was to have been a speaker at the NSTA convention in Houston in March 1991, but because of complications from this chronic ailment he was unable to attend. During a conversation with the renowned heart surgeon Dr. Michael Debakey, who spoke at the convention, I mentioned Yuri's problem. Dr. Debakey generously offered to treat Yuri at his heart center.

On November 23, Yuri entered the Houston Medical Center. While in the hospital, he read the November/December issue of *Quantum* with great interest—especially the article "Heart Waves," which, as fate would have it, dealt with his malady! He was released on December 24, after receiving the finest medical treatment in the world. I am pleased to note that he is recovering well and has returned to his institute in Moscow. We all owe Dr. Debakey, his team of heart specialists, and the Methodist Hospital of the Houston Medical Center a huge debt of gratitude for this act of international good will.

I hope you'll agree that the time to help our friends in Russia and the other states of the CIS is *now.* And we expect *Quantum* to continue forging links between aspiring young mathematicians and scientists throughout this dynamic new world.

—Bill G. Aldridge

QUANTUM

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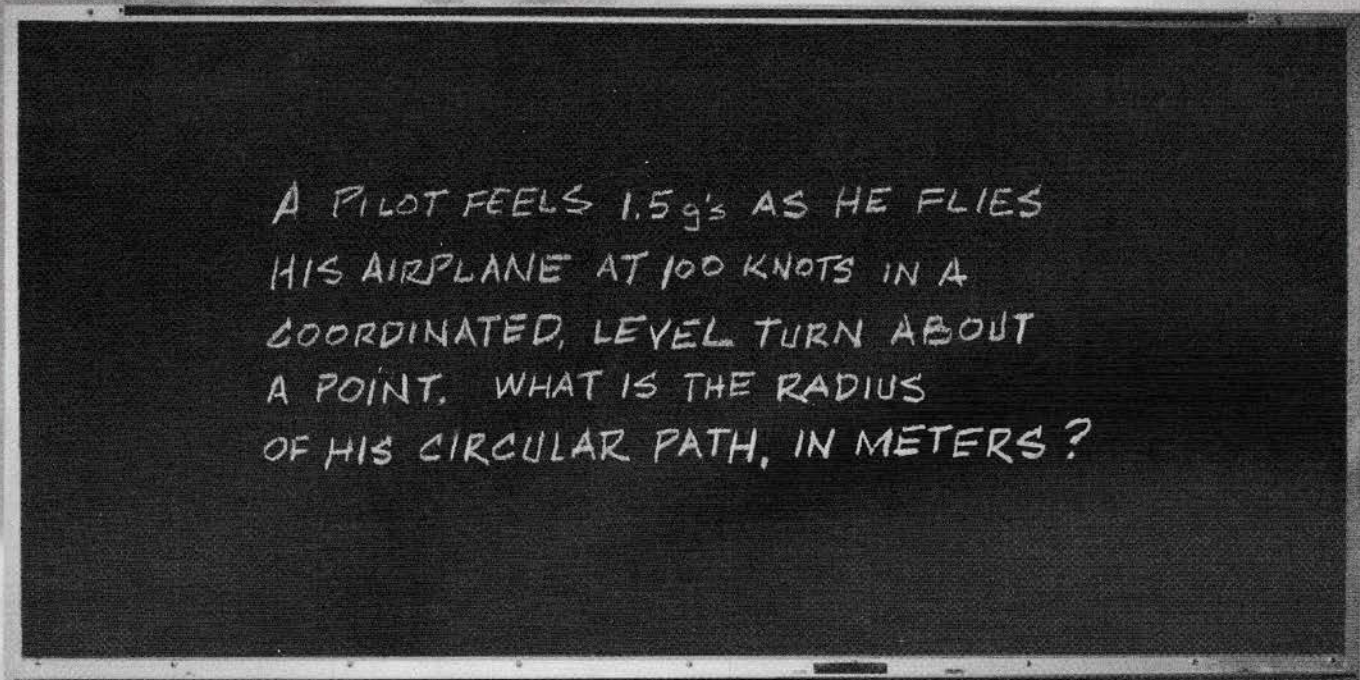

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Art by Sergey Ivanov

Halving it all

Some curious results from planar bisection

by Mark E. Kidwell and Mark D. Meyerson

A PLANAR REGION, WHICH we can think of as a uniform metal plate, has a special point called the *centroid*. To balance the region on a line (think "knife edge" or "tight rope"), just make sure the centroid is on the line, as in figure 1. Warning: these balancing lines usually don't bisect the area of the region. We can cut the area in half with a line in any direction, as in figure 2, but there usually is no "center point of area-bisecting lines." If we look at the family of area-bisecting lines, however, they often "form" (as envelopes) interesting curves that we can describe precisely.

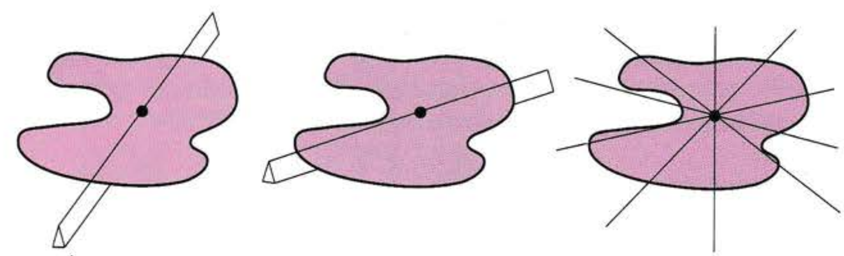


Figure 1
Balancing lines "concur" at the centroid.

In this article we'll take a look at some of these curves. (Many of the ideas that follow can also be found in a 1980 article by Derek Bell, "Halving Envelopes," in the *Mathematical Gazette*.)

Balancing

Take a thin metal plate—say, in the shape of a triangle. Can this plate be balanced on a very thin, taut, horizontal wire? In theory, yes. If the plate falls off to one side of the wire, we can nudge the plate on that side until we go too far and it falls off on the other side (fig. 3). Like the horse that starves to death between two bales of hay, there must be one intermediate position where the plate just balances. Draw a *line* on the plate where it's resting along the wire.

Is there a *point* along this line such that we can balance the plate on a pin, using this point as the point of contact? Again, yes. If the plate doesn't balance at a particular point on this balance line, it won't fall perpendicular to the line, but it will tend to fall "along the line," backward or forward. The points on the line where the plate "falls backward" or "falls forward" must be separated by one (theoretical) point (fig. 4). If the density of the material in the plate doesn't vary from place to place, this balance point is the centroid.

Notice that since the plate will balance on a pin at its centroid, it will surely balance on any wire that passes through the centroid, not just the wire we started with (see figure 5). There will be one such line in any direction.

Recall the *law of the lever*: a lever will balance if the sum of the product of the masses times the distances from the fulcrum on one side of the fulcrum equals the same sum of

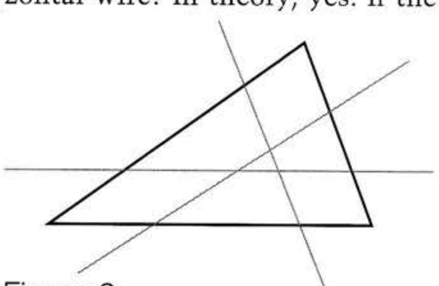


Figure 2
Area-bisecting lines may fail to concur.

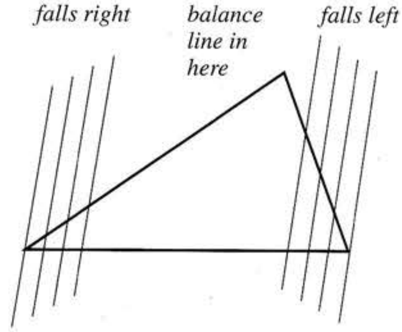


Figure 3

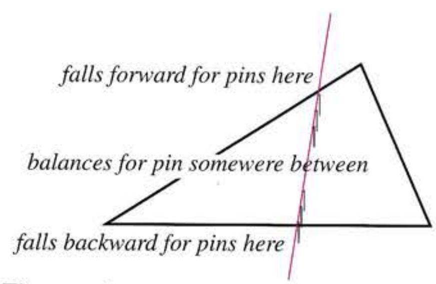


Figure 4
Finding a balance point along a balance line.

Art by Sergey Ivanov

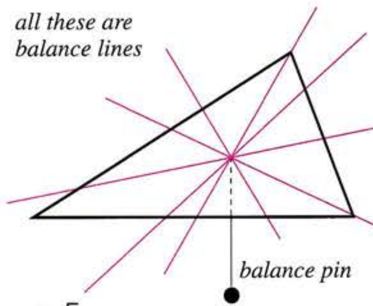


Figure 5

products on the other side of the fulcrum. For plates of uniform density, we can replace mass by area in this computation. In cases where the mass is spread out continuously, as in the plates we've been considering, calculus is generally needed to find the position of fulcrums (balancing lines). Nevertheless, we can give a plausible geometric argument that the *medians* of a triangle are balancing lines.

Consider triangle ABC with a median AM. This means that M is the midpoint of BC. Draw a line on each side of AM, parallel to it and equidistant from it, as in figure 6.

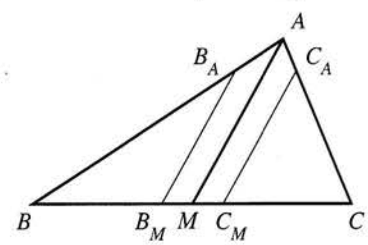


Figure 6

Say these lines meet the triangle as drawn, so we have $B_AB_M \parallel AM \parallel C_AC_M$. Since these lines are parallel and equidistant, any line through them determines two congruent segments. So $MB_M = MC_M$, and so $B_MB = C_MC$. Also, we have two different pairs of similar triangles: triangle CAM ~ triangle CC_AC_M and triangle BAM ~ triangle BB_AB_M. So

$$\frac{C_AC_M}{AM} = \frac{CC_M}{CM} = \frac{BB_M}{BM} = \frac{B_AB_M}{AM}.$$

The middle equality holds because both numerators and denominators are equal and the other two equalities hold by similar triangles. So we conclude that $C_AC_M = B_AB_M$.

Now if we draw lines parallel to AM at distance x and also at distance y, we get figure 7. The two shaded pieces are trapezoids with the same area (their heights are each $y - x$, and we just saw that their corresponding bases are of equal length). They lie about the same distance from AM and so should just balance. Since we can cut triangle ABC into skinny pairs of such trapezoids (see figure 8), the whole plate should balance on the median.

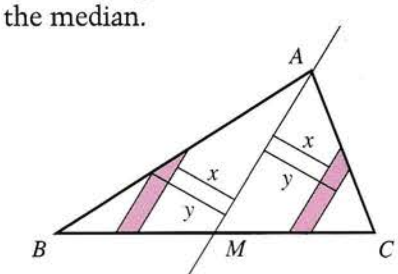


Figure 7

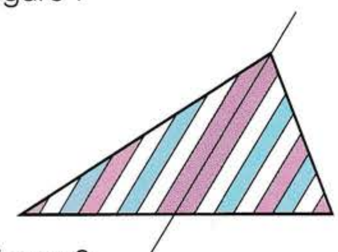


Figure 8
Pairs of trapezoidal regions with the same shading are the same distance from the median and have the same area, so they balance on the median.

On to area

Some of you may be under the (false) impression that any of the balance lines *must* cut the area of the plate in half. Indeed, the medians of a triangle have this property. But consider a line through the centroid that is parallel to one of the sides of the triangle. This creates a small triangle that is similar to the whole plate and also creates a trapezoid (fig. 9). The median AM_1 to the side of the small triangle will also be (when extended) the median AM to the parallel side of the plate. By a theorem of geometry, the little median will have 2/3 of the length of the big median. All the other *linear* dimensions of the little triangle will then be 2/3 of the corresponding dimensions of the

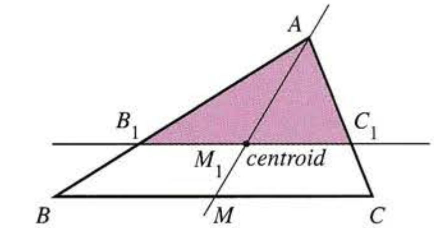


Figure 9
The shaded region has 4/9 the total area.

plate. If r is the ratio between linear parts of similar triangles, then r^2 is the ratio between their *areas*. So the little triangle will have 4/9 of the area of the plate, and the remaining trapezoid will have 5/9 of the area.

Conclusion: not every line through the centroid bisects the area of the plate. More is involved in balancing a plate than just area; distance from the balancing line or point also counts. This is the law of the lever again.

If we slide the line we've just been considering parallel to itself in the direction that increases the area of the little triangle and decreases the area of the trapezoid, we must eventually reach a line that bisects the area of the triangle. The same is true of parallel lines going in any other direction; there must be one line in each direction that bisects the area of the plate. The main question we'll explore is: if these lines don't all pass through one point, then what pattern do they form?

Figure 10 gives a visual answer to this question for a 30°–60°–90° right triangle. (The exact shape of the triangle isn't significant, as it turns out.) The picture presents an optical illusion that curves were drawn near the center; but nothing was drawn

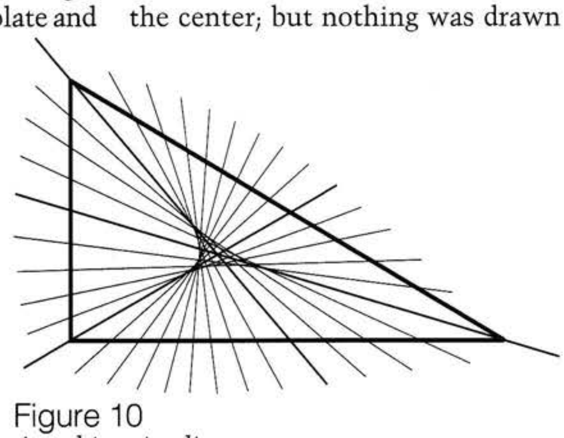


Figure 10
Area-bisecting lines.

but straight lines! These lines can be divided into three classes depending on which pair of sides of the triangle the line exits through. The three medians, which exit through vertices at one end, serve as boundaries between these three classes.

Here's how we can find the area-bisecting line in any direction. Given a triangle with area A and a direction, consider lines in that direction through the three vertices. Exactly one of these lines will meet the triangle in more than one point. Find the larger of the two triangular areas this line determines in the triangle. Move the line toward the larger triangular region, keeping it parallel to the original line, until the area of the triangle cut off is half that of the original. Doing this for a large number of directions produces figure 10.

Finding the curves

What are these illusory curves that appear in the midst of our area bisecting lines? Look at our right triangle as superimposed on an x–y coordinate system, with the right-angled vertex at the origin and the two sides embedded in the positive x- and y-axes. Forget the hypotenuse for now; we'll determine the area-bisecting lines that exit from the triangle through the other two sides. Let A be the area of the original triangle. Consider the family F of all lines that intersect the axes in points with positive coordinates and that cut off area $A/2$ in a triangle as shown in figure 11. F will contain all the area-bisecting lines that meet the legs of the triangle, together with some additional lines that meet the hypotenuse. Now, consider a given point in the first quadrant. There may be no lines from F that pass through this point, or there may be one such line, or there may be two (or more?) such lines. Consider all the lines passing through the given point and having negative slope so that they will cut the positive x- and y-axes. Some of these lines—namely, those that are nearly vertical or horizontal—will cut off triangles that have gigantic area (much bigger than $A/2$). If even the smallest (in area) of the triangles cut off by these lines has area greater than $A/2$, there will be no line in F through the point we're considering.

How do we find the line through the point (x, y) that cuts off the least area? Start with any line through (x, y) with negative slope. The point (x, y) divides the segment in the first quadrant into two segments of length z_1 and z_2, say. Suppose $z_1 > z_2$. Rotate this line through a small angle θ in a direction that tends to make the shorter segment longer and the longer segment shorter. Call the lengths of the corresponding segments on the new line z_1' and z_2' (see figure 12). For small enough θ, we'll still have $z_1' > z_2'$. The areas of the two obtuse triangles in figure 12 are

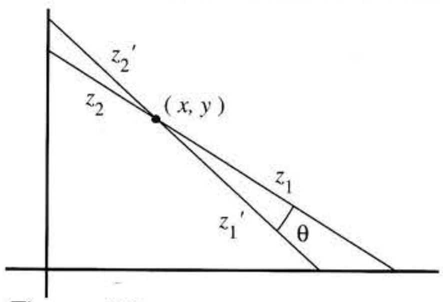


Figure 12

$\frac{1}{2}z_1z_1' \sin\theta$ and $\frac{1}{2}z_2z_2' \sin\theta$; the first area is larger. This means that we lose area in our right triangle as we turn the line until we reach a line where the two segments formed by (x, y) are equal. After that the area gets larger again. By an argument based on congruent triangles, this line with equal segments cuts the two axes at $(2x, 0)$ and $(0, 2y)$, giving a triangle of area $\frac{1}{2}2x2y = 2xy$ (fig. 13).

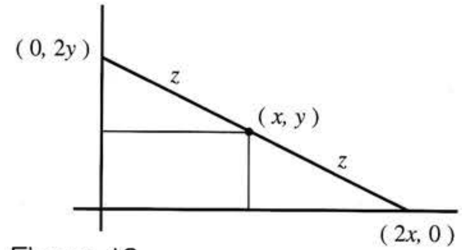


Figure 13

Now, if no line from our original family F passes through the given point, then even this triangle of smallest area has too much area—that is, $2xy > A/2$. The set of all such points in the first quadrant is the region above the hyperbola whose equation is $xy = A/4$. There is one line of F through each point of this hyperbola (F consists of the lines tangent to the hyperbola). It is this curve that we seem to see when we stare at our original family of lines. It's called the *envelope* of the family of lines.

A similar description holds for the rest of the right triangle and, in fact, for any triangle. The envelope of each of the three families of area-bisecting segments "between" the two medians to sides a and b is part of a hyperbola with asymptotes a and b. Of the hyperbolas with these asymptotes, the one to choose is the one that has the medians as tangent lines, and the part of this hyperbola to use is cut off by their points of tangency (at the midpoints of the medians).

Some other shapes

Notice that if we have a uniform plate with symmetry about a central point—as with elliptical regions, or rectangular regions, or other regions as in figure 14—all lines through this central point are both balancing lines and area-bisecting lines. Given any such line, the plate can be cut into pairs of skinny balancing pieces of the same area, as in figure 8.

It is, however, possible for a figure that is *not* centrally symmetric, as in figure 15, to have all the area-bisecting lines concur at one central point.

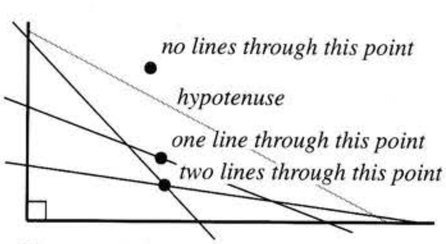


Figure 11
Solid lines form triangles in the first quadrant with area A/2.

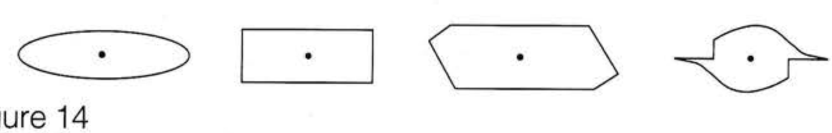


Figure 14
Each of these regions is bisected by all lines through a single central point.

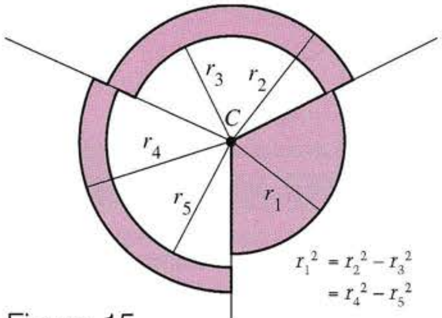


Figure 15
Every line through C bisects the area; C is not the centroid.

This region was created from three 120° pieces of equal area. To create this figure, start with any $r_1 > 0$. Then choose r_3 between 0 and r_1 and let $r_2 = \sqrt{r_1^2 + r_3^2}$. Notice that $r_2 > r_1$. Similarly, choose r_5 between r_1 and r_2, and let $r_4 = \sqrt{r_1^2 + r_5^2}$. Then any two small congruent central angles, each in one of the three 120° pieces of the figure, will cut off the same area, since

$$\pi r_1^{\,2} = \pi(r_2^{\,2} - r_3^{\,2}) = \pi(r_4^{\,2} - r_5^{\,2}).$$

But by adding such angles together we see that any two congruent central angles at all will cut off the same area. In particular, all lines through the central point (straight lines) bisect the area.

The region in figure 15 lacks one important geometric property. A region R is *starlike* from a point X if, for any point Y in R, the entire line segment XY lies in R (fig. 16).

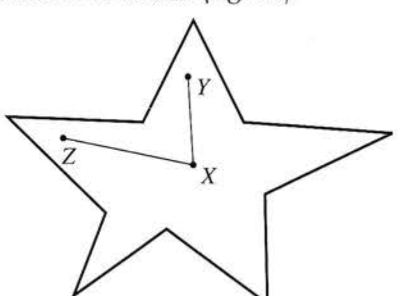


Figure 16
Starlike from X, but not from Y or Z.

Suppose the region is starlike from the point X and has a boundary loop whose distance from X varies continuously as we turn. If the region is *not* centrally symmetric about X, then there is a line l through X cutting off segments of length z_1 and z_2 such that $z_1 > z_2$, as in figure 17. Now rotate the line about X to create two congruent vertical angles θ_1 and θ_2. If this rotation angle is small enough, then the shortest distance from X to a point on the boundary loop inside angle θ_1 will be longer than the longest distance from X to a point on the boundary loop inside angle θ_2, since these distances vary continuously and $z_1 > z_2$. The area of the part of the region inside angle θ_2 is thus smaller than the area of a circular sector with angle θ_2, which in turn is smaller than the area of a circular sector with angle θ_1, which is smaller than the area of the part of the region inside θ_1.

Conclusion: as we turn line l about X, we gain area on one side of l and lose it on the other. So the lines that bisect the area of the region are *not* all concurrent at X. We can turn this statement around through the logical operation called "taking the contrapositive": if a region is starlike from X, has continuous boundary distance from X, and all its area-bisecting lines concur at X, then the region must be symmetric about X.

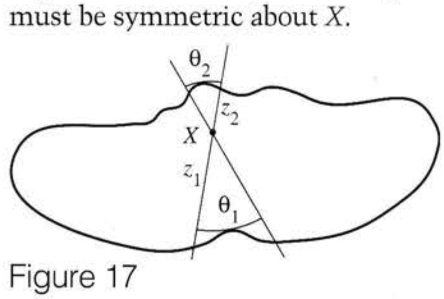


Figure 17

Other questions

There are several similar questions that have related answers. For example, suppose instead of halving the area of a triangle we wish to cut off one third of the area. In each direction, there will be two lines, with one third of the area between them and one third of the area in each outside region. An argument based on constant area, similar to that in an earlier section, shows that we still get pieces of hyperbolas with the triangle's sides for asymptotes. But instead of medians helping to determine the pieces, lines from the vertices that cut off one third of the area are needed. There are six such lines, and we get six families of cutting lines, each forming a piece of a hyperbola (fig. 18).

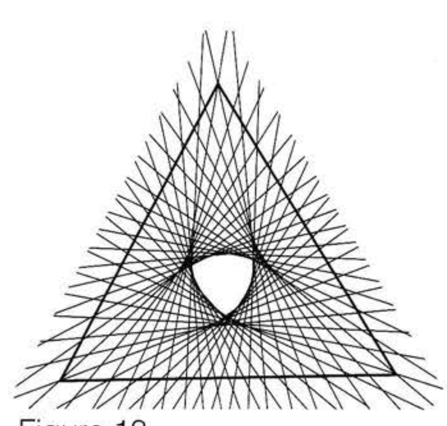


Figure 18
These lines cut off 1/3 of the area. Can you identify the six hyperbolic parts of the envelope and their asymptotes?

We can ask the same question for other polygons. Figure 19 shows that we can still get hyperbolic pieces for the envelope.

What about nonpolygons? By symmetry, we can see that lines that cut off, say, one fourth of a circular region have a circle for an envelope. If we "stretch" these circles in one direction by a factor of k, we multiply all areas by k and get two ellipses as in figure 20. It follows that for ellipses, lines that cut off a fixed proportion (other than half) of the area have el-

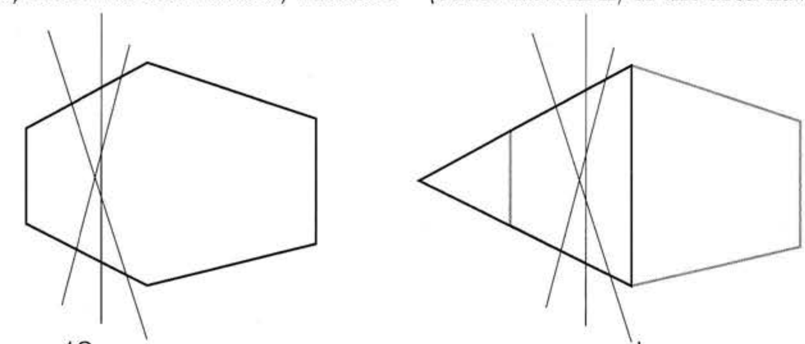


Figure 19 a b
Lines near those in (a) that cut off 1/7 of the polygon correspond to lines in (b) that cut off 2/3 of the triangle formed from the polygon.

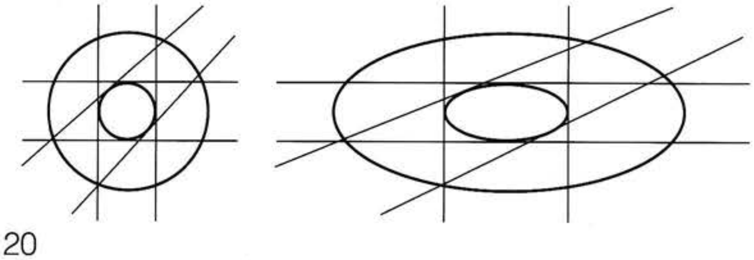


Figure 20

liptical envelopes. (If we exactly halve the area, we get the family of lines through the centroid.)

For regions with infinite area, we can try to cut off pieces of various constant areas. For example, in figure 21 we cut off a constant area from a hyperbola. The envelope is another hyperbola. Similarly, the envelope formed by cutting off a constant area from a parabola is a parabola.

We'll leave you with some problems to work through so that you, too, can "halve it all!"

Problems

1. Make a careful sketch of an isosceles right triangle and the hyperbolic envelopes of the area-bisecting lines.

2. Make a careful sketch of a regular pentagon and the hyperbolic envelopes of the area-bisecting lines.

3. Make a careful sketch of a square and the hyperbolic envelopes of the lines that cut off 1/8 of the area. (Because of the parallel sides, there will be corner points along the envelope through which infinitely many cutting lines pass.)

4. In order to construct the hyperbolic envelope curves in this article, one needs to construct a hyperbola from its asymptotes and a point on the curve. Show how to find a vertex and a focus of a hyperbola with straightedge and compass, given the asymptotes and one point of the hyperbola.

5. Figure 19 shows a situation in which cutting off 1/7 of the area of a hexagon is equivalent to cutting off 2/3 of the area of a triangle. Find the exact fraction of the area of the triangle that must be cut off when the hexagon is regular and we're trying to cut off 1/7 of its area.

6. Figure 20 shows a small circle whose tangents cut off 1/4 of a larger circular region. If the larger circle has radius 1, what is the radius of the smaller circle? (Find an approximate value, using a calculator or computer.)

7. Using the ideas in the discussion of figure 20 and the formula for the area of a circle, prove that an ellipse with semimajor axis a and semiminor axis b has area πab.

The following problems may require calculus.

8. Show that the tangent line to the hyperbola with equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

at (x_1, y_1) is described by the equation

$$\frac{yy_1}{a^2} - \frac{xx_1}{b^2} = 1.$$

Find the coordinates of the intersection of the tangent line with the asymptotes and show that (x_1, y_1) is the midpoint of the segment they determine.

9. Show that the tangent lines to the hyperbola with equation $xy = 2$ cut off a constant area from the hyperbola with equation $xy = 1$ (as illustrated in figure 21a) by finding that area.

10. Show that the tangent lines to the parabola with equation $y = -x^2$ cut off a constant area from the parabola with equation $y = 1 - x^2$ (see figure 21b) by finding that area.

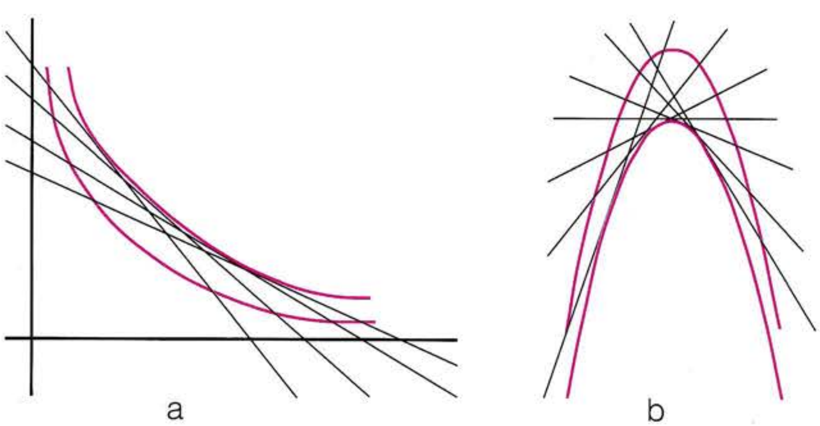


Figure 21

The two Marks who authored this article went to the same high school in suburban Washington, D.C., and are now professors at the US Naval Academy. **Mark E. Kidwell** *got his doctorate from Yale in 1976, and his main research interest is the theory of knots, links, and braids.* **Mark D. Meyerson** *got his doctorate from Stanford in 1975, and his main research interest is geometry and topology.*

ANSWERS ON PAGE 62

Baby, it's cold out there!

Or is it? On "cosmic cold" and thermal radiation

by Albert Stasenko

SCIENCE FICTION WRITERS and space engineers have a lot to say about how cold it is in outer space. An interesting question is how people living on Earth could have come up with the idea of "cosmic cold"—the cold of the universe.

Up to now mankind has worked out a pattern, averaged over regions and seasons, of how the temperature changes with altitude (fig. 1). This pattern is one of the most important aspects of the description of the so-called standard atmosphere. Figure 1 shows that, indeed, up to altitudes of about 10 km the temperature monotonically decreases. Since the highest mountain doesn't reach this high, even the bravest mountain climber would conclude that "the higher you go, the colder it gets." Airline pilots persuade their passengers of the same thing when they inform them that the temperature outside the aircraft is –50°C or even –60°C. So, let's conditionally call the idea of a continuous decrease in temperature with altitude the "extrapolation of mountain climbers and airline pilots."

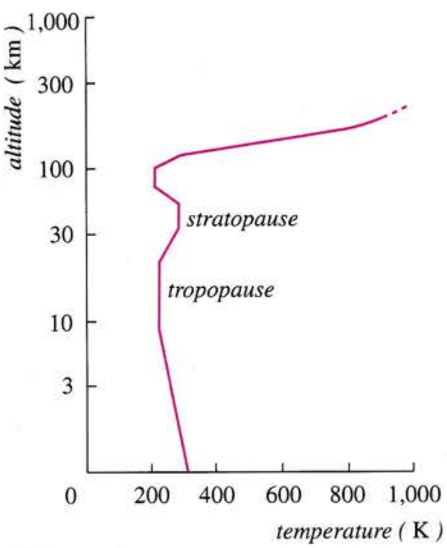


Figure 1

Extrapolation, however, is a pretty unreliable thing. For example, imagine a conscientious student who measures the temperature of water heated in a kettle. Let's assume that the temperature of the water in the kettle (initially at room temperature, or 20°C) increases by 10°C in one minute, 20°C in two minutes, and 30°C in three minutes (which already makes 50°C). Now assume that the student became bored with the experiment and after quick calculations drew the conclusion that in half an hour the water will be heated to 320°C! This is an example of illegitimate extrapolation: by continuing the experiment, the student would have seen (as we already know without any experiments) that the water in the kettle can't be heated to more than 100°C—at this point a new phenomenon—boiling—comes to the foreground.

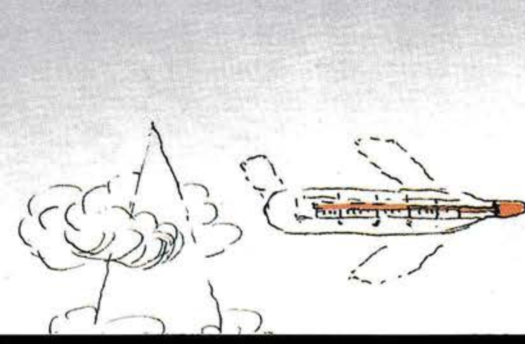
Exactly the same thing happens with the atmosphere: as you ascend higher and higher, new processes come into play. After falling initially, the temperature increases, then falls off again, then increases a final time, reaching (in accordance with figure 1) the value of approximately 1,000 K at an altitude of about 300 km. But this is the altitude at which satellites orbit and astronauts take space walks. So—is it *hot* or is it *cold* in outer space?

Well, what do the words "hot" and "cold" mean? Let's first define these concepts more accurately. We'll take it that when it's "cold" we give out a lot of heat—say, Q^- joules per second. And since some people are short and others are absolute giants, it would be more appropriate to consider the amount of heat released by a unit area per unit time. Let's take the amount of heat released by one square meter of our skin per second as a measure of "cold":

$$q^- = \frac{Q^-}{S}\left[\frac{\mathrm{J}}{\mathrm{s}\cdot\mathrm{m}^2}\right].$$

Here on Earth we've learned not to think about why we're not too hot. Living in a temperate climate, we think more about how not to get too cold (we worry about coats and hats and so on). And if it gets too hot, there's always ice cream, a swimming pool, and the wind in our faces. The water and air ensure the required value of q^- immediately without being very noticeable. The medium surrounding a heated body flows all over its surface and removes the heat. (With some ingenuity you can even take a picture of the column of shim-

Art by Pavel Chernusky


П.ЧЕРН.


mering air above your head that looks like heat waves on a hot, sunny day.)

And what will you do if it gets too hot for you in outer space? In a vacuum there is neither air nor water, and it would be a shame to throw anything overboard. There's only one thing to do: *radiate* the heat. This process is also observed on Earth—think of an electric stove or a fire (you can feel this radiation if you screen your face with your hand). On Earth, however, other processes compete with it. These are thermal conductivity (heat transfer arising from a difference in the temperatures of two adjoining bodies) and convection (removal of heat by a moving medium). In space the only way to cool off is through radiation. It's clear that the higher the body's temperature, the more heat it radiates. This paradox—the fact that in order to cool off faster you have to become overheated—is associated with the well-known Stefan–Boltzmann law:

$$q_r = \sigma T^4, \qquad (1)$$

where σ is a physical constant that you can find in any handbook: $\sigma = 5.7 \cdot 10^{-8}$ W/(m$^2 \cdot$ K^4). The equal sign in formula (1) refers only to so-called black bodies; in practice one always gets the inequality

$$q_r < \sigma T^4. \qquad (2)$$

Formula (2) shows how the rate at which the heat is removed increases as the temperature of the radiating body increases. Given a threefold increase in the body's temperature, the quantity q_r will increase by a factor of $3^4 = 81$ (almost a hundredfold increase!). We just have to remember that the temperature T is measured in kelvins.

In order to move on and arrive at any conclusions, we have to do some calculations.

We should note that cooling can also be caused by the evaporation of water in the human body through the pores of the skin. We won't discuss this effect now—for the time being we'll assume that the body being considered is placed in an absolutely transparent plastic bag (although sitting in a plastic bag isn't exactly pleasant for human beings). We'll return to this phenomenon later, when we look at the example of a "naked" evaporating drop.

Processes occur in a living organism that lead to the release of heat Q^+. It's an established fact that an office worker needs about 3,000 kcal daily; a manual laborer—about 5,000 kcal. Since an astronaut in space will be involved in both mental and physical activities, let's take 4,000 kcal/day to be a rational diet, of which one quarter is transformed into muscular energy and the remaining three quarters are left over as heat Q' to be radiated. Let's express Q' in SI units (see box at upper right).

$$Q' = \frac{3 \cdot 10^3 \text{ kcal/day} \cdot 10^3 \text{ cal/kcal} \cdot 4.2 \text{ J/cal}}{24 \text{ hours/day} \cdot 3{,}600 \text{ seconds/hour}} \cong 150 \text{ W}.$$

Now let's evaluate the area S of our body's surface. How can we do this? Let's imagine ourselves to be cylinders (not so pleasant a thing either) of height h (for a student let's take $h = 1.5$ m) and choose its radius r so as to obtain the body's volume V (fig. 2). This volume, expressed in liters, equals the mass m of the body, expressed in kilograms. (You'll recall that when you go swimming, you float when you take a deep breath but sink when you exhale; so the body's density must be close to that of water.) Then for $m = 50$ kg we get $r = 0.1$ m and $S = 2\pi rh + 2\pi r^2 = 1$ m^2. (Of course, a human being isn't a cylinder—it's a much more complex shape; for physical estimates, though, this model will do.)

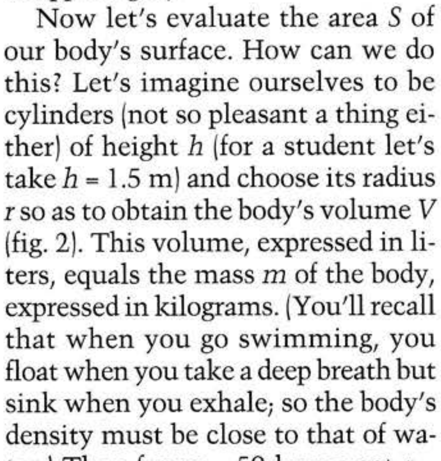


Figure 2

Now let this model of a person—a cylinder of water at a normal body temperature $T = 37°$C with a constant heat release of $Q' = 150$ W—suddenly find itself in outer space, and let's assume that no radiation from stars, planets, or other celestial bodies reaches the cylinder (this is obviously the "coldest" case). How will the cylinder cool off? Its surface radiates the energy $Q_r \le \sigma T^4 = 525$ J/s. The internal heat release of Q' partially compensates for this loss of heat, so the total heat loss is about $Q_r - Q' \le (525 - 150)$J/s $= 375$ J/s. It would be interesting to find the time it takes for its temperature to decrease by, say, $\Delta T = 2$ K. The heat capacity of the water cylinder $mc = 50$ kg $\cdot$ 1 kcal/(kg $\cdot$ K) $\cong 200$ kJ/K. Assuming the heat loss to be constant, we get

$$\tau \ge \frac{mc\Delta T}{Q_r - Q'} = \frac{200 \cdot 10^3 \text{ J/K} \cdot 2\text{ K}}{375 \text{ J/s}} \cong 18 \text{ min}.$$

So this is the time needed to cool the body from 37°C to just 35°C! Even in this "coldest" case, we can hardly speak of turning instantly into a block of ice. (Try and calculate the steady-state temperature as $t \to \infty$.)

By the way, what about the portion of the power—one quarter—that we excluded from our calculation? We assumed it to be spent on some useful work; in fact, this work is directly related to the efficiency coefficient $\eta = 1/4$. But if this work is expended by doing exercises in a plastic bag, it will ultimately be expended on heating the muscles, and this heat has to be removed from the body. How? Again, through radiation. And what if the astronaut hammers on the outside of the spacecraft (repairing something or just knocking at the door, asking to be let in)? In this case, some of this power is spent on imparting kinetic energy to the hammer, on deforming and heating the spacecraft's skin; and ultimately it will also be partially radiated, but this time outside the astronaut's body. All these considerations lead us to the conclusion that we overestimated the rate of heat removal; consequently, the body of an actively engaged astronaut will cool off still more slowly.

Now let's imagine that we're traveling along the Earth's orbit around the Sun. Let's find the steady-state temperature T_s of our body, assuming that it radiates not only the internally released heat but also the heat absorbed each second by the side of our body turned to the Sun. The density q_S of the flow of solar radiation at the Earth's orbit—the so-called solar constant—is approximately 1,400 W/m². We have the equation

$$q_S S_\perp + Q' = \sigma T_s^4 S$$

($S_\perp = 2rh$), which yields the estimate

$$T_s = \sqrt[4]{\frac{1{,}400\ \mathrm{W/m^2} \cdot 0.3\ \mathrm{m^2} + 150\ \mathrm{W}}{5.7 \cdot 10^{-8}\ \mathrm{W/K^4}}} = 317\ \mathrm{K} = 44°\mathrm{C}.$$

This temperature is much higher than what your physician will allow you to maintain. Of course, you can polish the side of your body turned to the Sun (though in this case the polished side won't radiate either) and make the other side of your body (the side that will radiate your internal heat Q') absolutely black (fig. 3). But your motley, harlequin appearance shows that you're more afraid of being *fried* than being *frozen* in the supposed realm of terrible, eternal cold.

It's appropriate to mention here that a "bit of outer space" (but with a rather weak vacuum) is used as the best possible insulation in flasks designed to keep hot drinks hot and cold drinks cold. Its walls are coated with a reflective material that neither absorbs nor radiates heat. So the almost complete absence of matter in a vacuum is in itself the best "fur coat."

Well, what if you want to cool not your own body, with its "little furnace" inside ($Q' = 150$ W), but a huge spacecraft intended for flights all over the solar system and equipped with a nuclear reactor? Let's assume that the spacecraft is equipped with a reactor that releases thermal energy at the rate of about ten million kilowatts ($Q = 10^{10}$ W). A fraction of this energy ($\eta \sim 10\%$) is converted into the kinetic energy propelling the spacecraft and expended on lighting the ship, sending out radio waves, and performing many other useful tasks. Why not all the energy? This is forbidden by an important prohibition known as Carnot's law, which states: if you want to do useful work at the expense of chaotic thermal energy, you should provide not only the "hot end" T_1 of a heating device (the temperature of the power source—say, a nuclear reactor) but also its "cold end" T_2 ($T_2 < T_1$). In the most favorable case the heat engine's efficiency will equal

$$\eta = \frac{T_1 - T_2}{T_1} < 1. \qquad (3)$$

A heating device can work (that is, the condition $\eta > 0$ is satisfied) only if the temperatures of the "heater" and the "cooler" are different. To ensure the maximum efficiency you must either bring T_2 closer to zero or raise T_1 to infinity. In either case the efficiency will be close to unity, which is mankind's eternal dream. However, the reactor temperature T_1 can't go too high—it's limited by the melting temperature of its materials, for one thing. The temperature T_2 of the cooler also can't go too low—to ensure the effective removal of useless heat from the spacecraft through radiation, T_2 should be increased. This compromise gives $\eta \cong 10\%$.

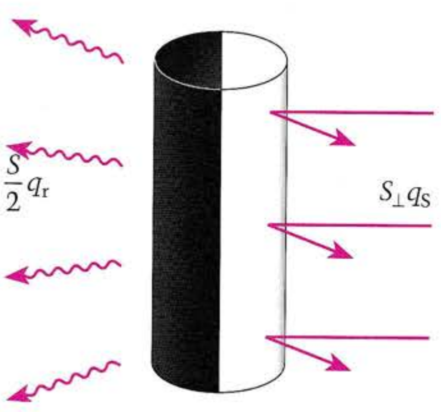


Figure 3

Let's take, for example, T_1 = 2,000 K (metals like tungsten can withstand such temperatures). Then, for $\eta = 10\%$, from formula (3) we get T_2 = 1,800 K. Then, from formula (1), we find the maximum density of the radiation flow $q_r = \sigma T_2^4 = 6 \cdot 10^5$ W/m².

So to effectively radiate the useless reactor energy $Q' = (1 - \eta)Q^+$, we'll need an area of no less than

$$S = \frac{(1-\eta)Q^+}{q_r} = \frac{0.9 \cdot 10^{10}\ \mathrm{W}}{6 \cdot 10^5\ \mathrm{W/m^2}} = 1.5 \cdot 10^4\ \mathrm{m^2}.$$

One and a half hectares of surface heated to 1,800 K and exposed to meteors and molecular and corpuscular streams (fig. 4)! And all this surface (this mass of pipes with molten metal or gas heated to a high temperature used as a heat-transfer medium bathing the reactor) is needed to prevent overheating and only because there is no such thing as "cosmic cold."

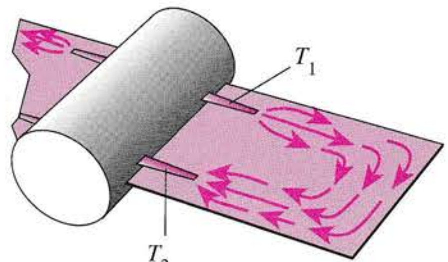


Figure 4

But how can we protect the spacecraft from meteor damage? Keep in mind that it will spend months or even years on its trip to other planets in the solar system.

Here's one interesting solution. Take a long, thin, closed band and wind it around the cylinder you want to cool (we want to keep it at the "low" temperature T_2). When a portion of the band comes in contact

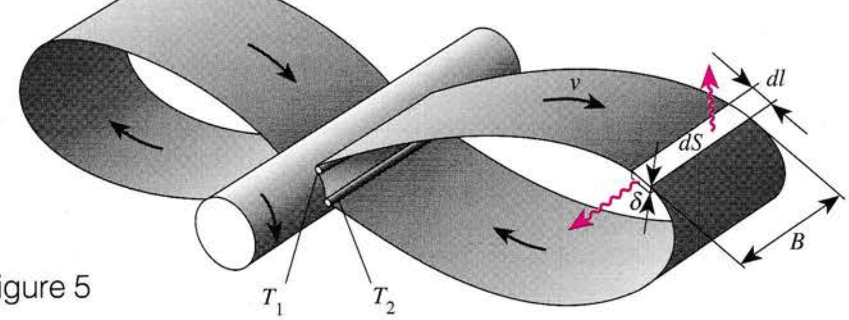


Figure 5

with the cylinder, it heats up and removes heat from the cylinder; later, when it loses contact with the cylinder, it radiates the heat acquired into outer space; when it comes into contact again with the cylinder, it's already cooled to the former temperature; and the whole cycle is repeated (fig. 5).

What are the advantages of this solution? First, meteors aren't as threatening to such a band—a hole in it won't cause any trouble, and the area of the cylinder's surface is much smaller and can easily be protected with small additional shields.

But how can we press the band tightly against the cylinder? One possible solution is to rotate the band as a whole so that all of it would be in the field of centrifugal forces, just like the hula hoop kids used to (and maybe still) twirl around their waists (fig. 6). In outer space the band will meet no resistance!

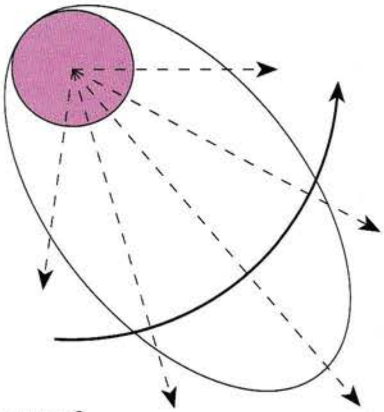


Figure 6

Or how about this: let's not simply glue the band into a cylindrical shape but twist one of its ends when we glue it to obtain a very interesting surface that has only one side—the Moebius band (fig. 7). Why does this suit our purposes better? Portions of this surface "radiate" directly into outer space and not onto each other, as is the case of a cylindrical band, so it radiates and removes heat more effectively.

Of course, this is easier said than done. For a flight to Mars this band would have to be about 10 meters wide, 100 meters long (a football field!), and thinner than a razor blade. In addition, the band may become fused to the surface of the cylinder to be cooled. And there are other dangers. But the main thing is that physics supplies designers and engineers with ideas like this as possible solutions.

Now let's consider how the temperature changes from the highest temperature T_1 to the lowest T_2 over the length of the band.

Let a band of width B and thickness δ move at a constant speed v, radiating heat from both sides. Its temperature will change over the band's length, and so it will be a function of the distance l measured, say, from the line where the band last touches the cylinder to be cooled (fig. 5). Let's write the equation for the change in the band's heat content. We'll take a small section of its length dl. The mass of this section equals $\rho dS\delta$ (where $dS = Bdl$ and ρ is the density). Multiplying this mass by its specific

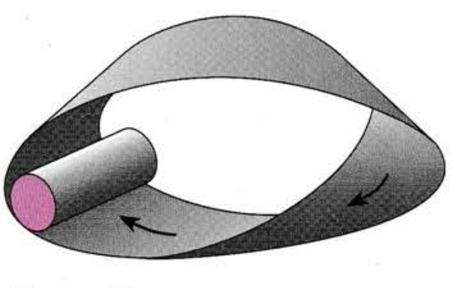


Figure 7

heat capacity c, we get the total heat capacity of the section $C = c\rho dS\delta$. The temperature $T(l)$ of this section is taken to be constant throughout, so the energy it radiates per unit time is

$$Q_r = \sigma T(l)^4 2dS$$

(where $2dS$ is the total area of the two sides of this section of the band). This loss of radiated energy will decrease the temperature of the section by a small value dT, so the total heat content will decrease by $CdT = -Q_r dt$ in the time interval $dt = dl/v$.

Substituting the value of C into this expression and canceling dS on both sides, we get

$$c\rho\delta \frac{dT}{dt} = -2\sigma T^4,$$

or, taking into account that in time dt this section of the band covers the distance $dl = vdt$, we get

$$c\rho\delta v \frac{dT}{dl} = -2\sigma T^4.$$

This differential equation can be rewritten as

$$\frac{dT}{T^4} = -\alpha dl,$$

where $\alpha = 2\sigma/c\rho\delta v$ is a constant. Now there's nothing simpler (if you know calculus) than to integrate both sides of the equation. You may recall that, if $n \neq -1$,

$$\int x^n dx = \frac{x^{n+1}}{n+1}.$$

If you don't believe me, check it by differentiating the right side of the last equation:

$$d\frac{x^{n+1}}{n+1} = \frac{n+1}{n+1}x^n dx = x^n dx.$$

We have $n = -4$, $x \equiv T$, so after integrating we'll get

$$\left.\frac{T^{-3}}{-3}\right|_{T_1}^{T} = -\alpha l.$$

Here we allowed for the so-called boundary condition: at $l = 0$ (the line where the band leaves the cylinder) we have the highest temperature T_1, and then the temperature can only decrease. So, substituting the limits of integration, we get

$$-\frac{1}{T^3}+\frac{1}{T_1^3}=-3\alpha l.$$

or

$$T(l)=\sqrt[3]{\frac{1}{\frac{1}{T_1^3}+3\alpha l}}.$$

So we've found the distribution of temperature over the band's length l. It looks like the distribution shown in figure 8. The figure shows that the temperature reaches its minimum value T_2 at point $l = L - s$ (where s is the length of arc where the band touches the cylinder, L is the total length of the band) and then, after the band touches the cylinder the next time, it's again heated from T_2 to T_1. Each section of the band repeats this cycle over and over with a period $\tau = L/v$.

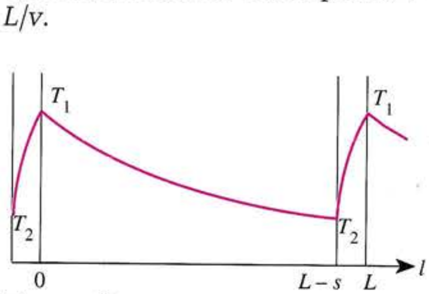


Figure 8

Now we can raise this dependence to the fourth power and integrate it over the band's length to find the power radiated into space. Or, conversely, given the power to be radiated, we can find the length of the band and the temperatures T_1 and T_2 that will ensure this heat removal. And that's exactly what spacecraft designers did.

Notice that the obtained dependence $T(l)$ can easily be rewritten in a form in which temperature is given as a function of time $T(t)$ by substituting the simple relation $l = vt$. Then the band's "boundary condition" can be called the "initial conditions" for the section considered. By the way, a similar dependence is typical of the filament of a vacuum tube when it's turned off at $t = 0$. Try to think it through by analogy: imagine this filament to be a cylinder and neglect heat losses due to thermal conductivity through the filament supports. Take only the radiative heat losses into account.

A similar dependence will also be observed for a solid particle that is injected at $t = 0$ into a vacuum and begins to cool only through radiation, starting from temperature T_1. (Try to verify this fact on your own.)

What if the particle also evaporates? Let the particle be a drop of water in a vacuum. It will cool primarily because of evaporation rather than radiation.

Let's try to write down the following idea: the change in the thermal energy of the drop ($mcdT$, where m is mass, c is specific heat, and T is temperature) is due to the latent heat of vaporization of the mass dm that evaporates. But if we merely multiply this mass by the handbook value for the latent heat of vaporization $\mathcal{L}$ (J/kg), we'll be wrong, because the handbook value is obtained from a simple experiment: a certain amount of heat is added to a kettle, the mass of the evaporated water is measured, and then the former is divided by the latter. But the experiment is performed at atmospheric pressure, while our drop is in a vacuum. This means that the vapor leaving its surface doesn't have to do any work to overcome atmospheric pressure. This work, calculated per unit mass, is $p/\rho = R_0T/\mu$, where p is the pressure, R_0 is the gas constant, and μ is the molar mass. (In obtaining this expression I have used the Clausius–Clapeyron–Mendeleyev law.) So this work must be subtracted from the handbook value for the latent heat of vaporization.

On the other hand, in a vacuum the motion of the molecules leaving the water surface through evaporation is perpendicular to the surface, and after leaving the surface the molecules don't meet any resistance (in contrast to evaporation in the atmosphere). Since the flow of molecules is perpendicular to the drop's surface, it's said to possess "one degree of freedom." But each degree of freedom carries the thermal energy $(1/2)(R_0T/\mu)$.[1]

So, to determine the energy carried away with an evaporated unit mass in a vacuum, you have to subtract the work of "counterpressure" $p/\rho = R_0T/\mu$ that isn't done in a vacuum from the handbook evaporation heat value and add the energy corresponding to the selected direction along the line perpendicular to the surface $(1/2)(R_0T/\mu)$. As a result, we see that a kilogram of evaporated mass carries away the energy

$$\mathcal{L}-\frac{R_0T}{\mu}+\frac{1}{2}\frac{R_0T}{\mu}=\mathcal{L}-\frac{1}{2}\frac{R_0T}{\mu}.$$

So let's express the change in energy of the evaporating drop as follows:

$$mcdT=dm\left(\mathcal{L}-\frac{1}{2}\frac{R_0T}{\mu}\right).$$

Here we have two variables: m and T. The first mathematician you encounter will immediately perform a simple operation with this equation—"separating the variables"—to get

$$\frac{dm}{m}=c\frac{dT}{\mathcal{L}-\frac{R_0T}{2\mu}}. \qquad (4)$$

You can see why the operation is named as it is: the left side of the equation now contains only one variable, m; the other variable, T, appears only on the right side. The next thing the mathematician will do is introduce a new notation—say,

$$\mathcal{L}-\frac{R_0T}{2\mu}=y.$$

[1]You'll recall that in an ideal gas each molecule has three degrees of freedom—up and down, forward and back, and left and right—all having equal rights, and it has an average specific kinetic energy equal to $(3/2)(R_0T/\mu)$.

Then, differentiating both sides of the last equation and taking into account that differentiation "devours" solitary constants (in our case, $\mathcal{L}$) and has mercy on constant factors, we write

$$dy = d\left(\mathcal{L} - \frac{R_0}{2\mu}T\right) = -\frac{R_0}{2\mu}\,dT.$$

From this we'll derive

$$dT = -\frac{2\mu}{R_0}\,dy$$

and substitute this entire expression in equation (4):

$$\frac{dm}{m} = -\frac{c2\mu}{R_0}\frac{dy}{y}.$$

Now it's time to integrate both sides. Nothing could be simpler, if you recall the tabulated integral

$$\int \frac{dx}{x} = \ln x.$$

(This integral resembles the one we encountered before with $n = -1$, but this time the solution is different.) Once again let's allow for the initial condition: when the drop had mass m_0, its temperature was T_0. So

$$\ln m\Big|_{m_0}^{m} = -\frac{c2\mu}{R_0}\ln\left(\mathcal{L} - \frac{R_0 T}{2\mu}\right)\Bigg|_{T_0}^{T},$$

or

$$\ln\left(\frac{m}{m_0}\right) = -\frac{c2\mu}{R_0}\ln\left(\frac{\mathcal{L} - \dfrac{R_0 T}{2\mu}}{\mathcal{L} - \dfrac{R_0 T_0}{2\mu}}\right)$$

$$= \ln\left(\frac{\mathcal{L} - \dfrac{R_0 T_0}{2\mu}}{\mathcal{L} - \dfrac{R_0 T}{2\mu}}\right)^{\frac{2c\mu}{R_0}}.$$

Finally, taking antilogs of both sides of the last equation to get rid of the logarithms, we get

$$\frac{m}{m_0} = \left(\frac{\mathcal{L} - \dfrac{R_0 T_0}{2\mu}}{\mathcal{L} - \dfrac{R_0 T}{2\mu}}\right)^{\frac{2c\mu}{R_0}}.$$

This equation shows that even as the temperature tends to zero, the drop's mass still doesn't vanish, since its internal thermal energy is insufficient to tear the drop completely into separate molecules. The dependence represented by the last equation is qualitatively illustrated in figure 9. And we mustn't forget radiative heat losses—although they continually decrease as the temperature tends to zero, they'll also exhaust the energy stored in the drop so that it will lose an even smaller portion of its mass.

Not only that, as the drop moves towards absolute zero, new phenomena will appear: the drop may begin to crystallize, and the evaporation will be replaced by sublimation (fig. 9). This is why we agreed to put the astronaut in a plastic bag—so that evaporation wouldn't mask the only process we wanted to study: cooling by thermal radiation.

Let's solve one more problem. It has to do with the "Stinger" missile, but we'll be using it for peaceful purposes. Suppose a body was headed toward the Earth—an asteroid, say, or a decommissioned orbiting lab. If it struck the Earth, it might cause a great deal of damage. The Stinger might come in handy here. Let the temperature of the target body, which is uniform all over its surface, be T (we'll assume the body is rotating or has a very large thermal conductivity, which leads to equalization of temperature at all points); the body itself is a sphere of radius a (fig. 10). The body will radiate the energy $4\pi a^2\sigma T^4$ per unit time. At a distance r, where the Stinger's self-guidance mechanism is situated at the moment, the energy $q^+ = 4\pi a^2\sigma T^4/4\pi r^2 = a^2\sigma T^4/r^2$ will fall per unit area of its surface per unit time (this is the density of the energy flow).

The Stinger will "notice" this heated body if the density of the energy flow is no less than some mini-

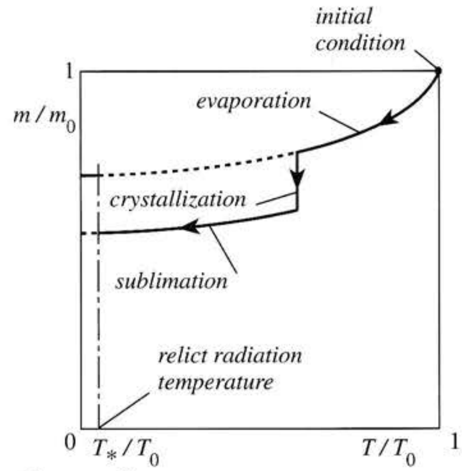


Figure 9

mum value $q_{\min}$ that it can sense: $q \geq q_{\min}$. So

$$r \lesssim aT^2\sqrt{\frac{\sigma}{q_{\min}}}.$$

Let's take advantage of the handbooks, where we can find $q_{\min} \sim 5 \cdot 10^{-7}$ W/m^2, and take $a = 1$ m and $T = 1{,}000$ K. We then get the estimate

$$r \lesssim 300 \text{ km}.$$

The time is now ripe to recall that the thermal radiation of any body contains electromagnetic waves of all wavelengths λ, but the Stinger "sees" only those waves with wavelengths ranging from 3 to 5 μm. Analogously, the human eye can see in the so-called optical range—from 0.4 to 0.8 m (which constitutes the so-called "window of transparency" of the atmosphere). Figure 11 shows Planck's energy distribution as a function of wavelength for the thermal radiation of an absolutely black body at the temperature $T = 1{,}000$ K, and the two regions mentioned above

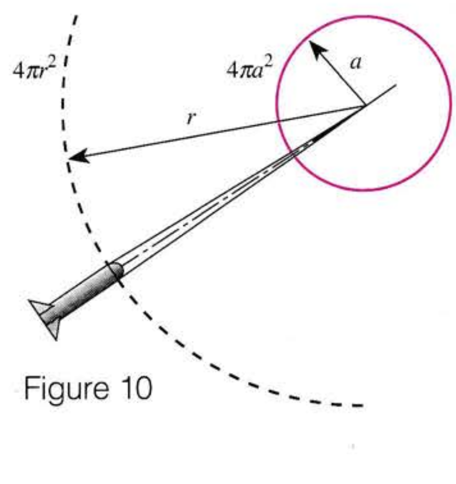


Figure 10

are highlighted in the figure. If you take all the area below this curve as a function of temperature, you'll get the Stefan–Boltzmann law (1) that we started with. So to determine more accurately the range for detecting an alien body, the quantity σT^4 must be multiplied by the fraction of the total area below the curve that is highlighted. You might try to do this on your own.

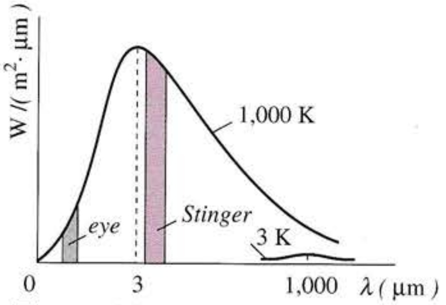


Figure 11

But what's that little hump there under the "tail" of Planck's distribution, near the wavelength 1,000 μm—that is, about 1 mm (fig. 11)? This is the so-called relict (or background) radiation, corresponding to a temperature $T.$ of about 3 K (2.7 K, to be precise). The universe is filled with this radiation left over from the big bang that occurred about 20 billion years ago.[2] So no body, including the evaporating drop, can cool below this temperature (this fact is shown in figure 9 by the vertical line at $T = T.$). So why, when we calculated the radiation of heat from the human-cylinder, didn't we take into account the fact that this cylinder is exposed to relict radiation from all directions, which brings additional energy? Because the ratio of this energy to the radiated energy is of the order

$$\left(\frac{2.7\ \text{K}}{310\ \text{K}}\right)^4 \lesssim \left(10^{-2}\right)^4 = 10^{-8}$$

—that is, one part per hundred billion.

Let's take one last look at figure 11 and make an interesting observation. Multiply the wavelengths corresponding to the maxima of Planck's distribution curves by the temperatures these curves are plotted for—3 μm · 1,000 K and 1,000 μm · 3 K. You'll get the same number. The suspicion arises that some universal law is contained in this coincidence. Wilhelm Wien (1864–1928) was the first to suspect this, so the law he discovered is known by his name.

With this law we can, for example, estimate the temperature T_S on the Sun's surface without flying there with a thermometer on our summer vacation. Since wise Nature has made our eye sensitive to wavelengths of visible light (of the order of half a micrometer), we get T_S = 3,000 μm · K/0.5 μm = 6,000 K. And now we can go back to the beginning of this article and prove that the density of the energy flow from the Sun onto the Earth's orbit q_S has precisely the value we used when calculating the astronaut's equilibrium temperature. ◘

[2]See "A Universe of Questions" in the last issue.—*Ed.*

Just for the fun of it!

B46

Water fractions. A barrel was full of water. All the water was poured in equal portions into three pails. It turned out that the water took up 1/2 of the volume of the first pail, 2/3 of the volume of the second pail, and 3/4 of the volume of the third pail. The tank and all three pails hold integer numbers of liters. What is the smallest possible volume of the tank? (N. Antonovich)

B47

Morning meteors. Explain why we see more meteors from midnight to dawn than from evening to midnight. (V. Surdin)

B48

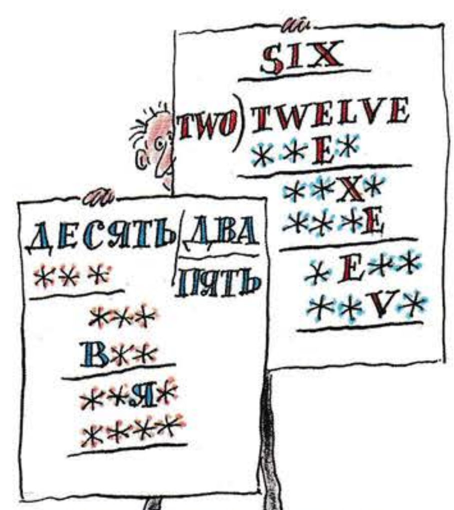
Halving the pentagram. Prove that the area of the red portion of the star is exactly half the area of the whole star. (N. Avilov)

B49

Vacuumed tubes. Air is pumped out of the tubes of some solar telescopes. Why is that? (V. Surdin)

B50

English, Russian, and math. The accompanying number rebuses present two instances of long division (the same letters in each rebus correspond to the same digits, different letters correspond to different digits, and stars stand for any digit). One of the rebuses is in Russian (the words are ДЕСЯТЬ = TEN, ДВА = TWO, ПЯТЬ = FIVE) and written as they are in Russian schools (so we have $10 \div 2 = 5$). But you don't need to know Russian (or even English) to restore all the digits in the rebuses. (E. Rekstins)



ANSWERS, HINTS & SOLUTIONS ON PAGE 62

Art by Edward Nazarov

HOW DO YOU FIGURE?

Challenges in physics and math

Math

M46

Unlucky series. A six-digit number is called "lucky" if the sum of its first three digits equals that of the last three; numbers having less than six digits are also included—zeroes are added on the left until they have six digits (for example, 1001 becomes 001001). (See the article "A Conversation in a Streetcar" for an explanation of the term "lucky.")

(a) What is the longest series of consecutive *unlucky* numbers? (b) How many such longest series are there? (S. Orevkov)

M47

Isosceles triangle on an integer grid. All three vertices of an isosceles triangle have integer coordinates. Prove that the square of the length of its base is an even number. (V. Proizvolov)

M48

$X \leftrightarrow v$, or x-changing variables. Prove that if

$$x^2 + y^2 = u^2 + v^2 = 1, \ \ xu + yv = 0,$$

then, exchanging the variables x and v, we get the true equalities

$$v^2 + y^2 = u^2 + x^2 = 1, \ \ vu + yx = 0.$$

(S. Duzhin)

M49

Halving still more. (a) A straight line l divides the area of a convex polygon in half. Prove that the ratio in which this line divides the projection of the polygon onto the line perpendicular to l (fig. 1) does not exceed $1 + \sqrt{2}$. (b) Each of three lines divides the area of some figure in half. Prove that the part of the figure enclosed in the triangle formed by these lines (fig. 2) has an area not greater than 1/4 that of the whole figure. (V. Prasolov)

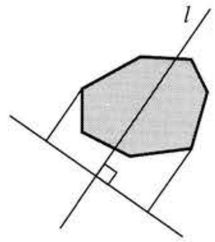


Figure 1

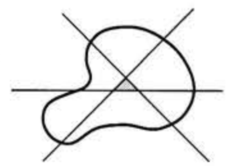


Figure 2

M50

Moving at the movies. (a) A movie theater has $N + k$ seats; their numbers are indicated on the tickets. The first N people (including Howard) who come to a show take N seats without paying attention to the seat numbers. But the remaining k ticket holders are sticklers. If any of them finds her or his assigned seat occupied, the person sitting there is evicted; that person then looks for his or her proper seat and evicts the usurper; and so forth. This "migration" ends with the spectator whose assigned seat is unoccupied. Find the probability that Howard won't have to change his seat (in other words, the ratio of the number of arrangements of spectators favorable for Howard to the total number of arrangements) for (a) $k = 1$, (b) an arbitrary positive integer k. (I. Alexeyev-Astafyev, V. Dubrovsky)

Physics

P46

Uniform motion. A body moving in a straight line covers exactly 1 meter in any given second. Will its motion necessarily be uniform? (A. Zilberman)

P47

Pincer pressure. Pincers consist of two identical pieces fastened with a pivot at point O (fig. 3). What is the force exerted on the pivot if the handles of the pincers are squeezed with a force F? Assume that there is no friction in the pivot. (L. Markovich)

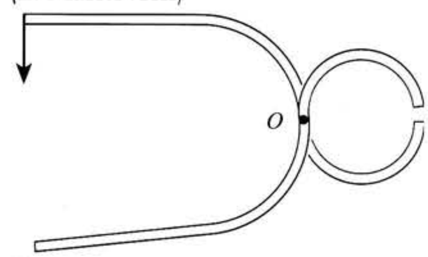


Figure 3

P48

Bigger sphere. An uncharged metal sphere is placed in a uniform electric field. When the field is turned off, an amount of heat Q is released inside the sphere. How much heat would be released inside the sphere if its radius were three time greater? (S. Krotov)

P49

Infinite circuit. The circuit diagram in figure 4 consists of a very large (infinite) number of elements. The

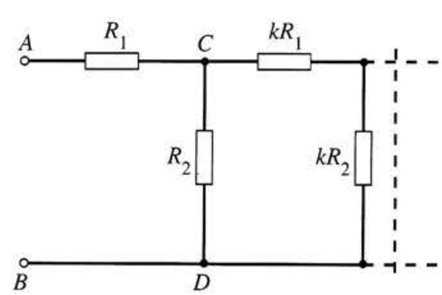


Figure 4

CONTINUED ON PAGE 25

IN YOUR HEAD

A conversation in a streetcar

What are the chances of getting a "lucky ticket"?

by A. Savin and L. Fink

I WAS RIDING A STREETCAR in Leningrad with my nephew Misha. I dropped 6 kopecks into a ticket machine and tore off two tickets.

Misha grabbed one of the tickets. "That one's mine, okay?"

"Good grief! Take either one. They're all the same, aren't they? Good for a whole trip."

"They're the same, but not *completely*. This one's just a regular ticket—number 286 357. But the next one's a 'lucky' ticket: the sum of the first three digits equals the sum of the last three."

At this point I remembered a popular superstition: that a ticket with equal sums brings good luck. Misha got ticket 286 358, so $2 + 8 + 6 = 3 + 5 + 8$.

"Do you get these 'lucky tickets' very often?"

"Oh, no, hardly ever. About once a month. And since I go to school and back every day except on weekends—that makes one lucky ticket for every 50 regular tickets on average."

"Nonsense," one of our fellow passengers interrupted. "I got on at the last stop and drew a lucky ticket from the same machine: 286 349. As a matter of fact, right now someone's tearing off ticket 286 367, also a lucky one, and soon 286 376 will appear, and then 286 385. So there's one lucky ticket in every 10, approximately."

"Excuse me, that's not quite right," another passenger piped up—the one who got 286 367. "Your example doesn't prove anything. There's going to be one more lucky ticket in the next ten: 286 394. But then there'll be none for a long time—not until ticket 286 439. So there we have an interval of 44 unlucky tickets in a row between the two lucky ones. And you can find plenty of examples like that. In this roll of tickets, whose numbers all begin with 286, there isn't a single lucky ticket between tickets 286 097 and 286 169, which is a run of 71 tickets."

"That's just what I was saying!" Misha readily supported him. "On average, one in fifty tickets is lucky."

"That's still a hasty claim," I remarked. "To answer our question correctly we need to study it. But first we have to give it an exact formulation. Something like this: *How many lucky numbers—that is, numbers from 000 000 to 999 999 such that the sum of the first three digits equals the sum of the last three—are there?*"

Art by Edward Nazarov

"Well, gosh," Misha said after thinking a bit, "I can't give the exact answer right away, but I can describe a method for finding it, at least in principle. We can simply write out all the numbers from 000 000 to 999 999, check the equality of the two sums for every one of them, and count up the lucky ones."

"Sure, that's one way to tackle the problem. It's called an *exhaustive search*. It can be applied to problems in which some finite sets of numbers or other objects are to be examined. However, the search method has two drawbacks. First and most important, it's too laborious and time consuming. Just imagine, you have to check one million numbers. Even at the rate of one ticket per second, you'll need . . . almost 278 hours, or about 35 eight-hour working days."

"But you can get a computer to do it."

"Of course you can, but it's like cracking a nut with a sledgehammer. Besides, the exhaustive search has another flaw, one that the computer can't avoid either: it gives a solution to a single particular problem, a solution that usually doesn't allow one to generalize or discover unknown laws. That's why solutions obtained by an exhaustive search are, in a certain sense, uninteresting."

"Let me intervene again," the owner of lucky ticket 286 367 said. "I've taken an interest in your problem, and I've found a solution. Not an exact one, though, but an approximate one. Or rather, what we mathematicians call an 'estimate.' Oh, excuse me, I haven't introduced myself. My name is Anatoly Pavlovich, I'm a professor of mathematics.

"So, young man," he turned to Misha, "I propose that we come up with a new definition of 'lucky ticket.' Better yet, let's come up with a new term—say, 'pretty ticket.' We'll call a ticket 'pretty' if the sums of the first three and the last three digits have the same remainders when divided by 9. Are you with me?"

"Yes, I am," Misha answered, "but why 9?"

"Because in decimal notation every number gives the same remainder when divided by 9 as the sum of its digits does. This property makes it very easy to find the number of pretty tickets.

"Look, there are exactly $999 \div 9 = 111$ numbers from 0 to 999 that have a remainder of 1 when divided by 9; that many again with a remainder of 2; and so forth, except for the remainder of 0—that is, numbers exactly divisible by 9: these include both 0 and 999, so there are 112 of them. So how many pretty ticket numbers can we compose using two three-digit numbers that have a remainder of 1? Clearly, $111 \cdot 111 = 12{,}321$, since any of the three-digit halves of the 'pretty number' can be chosen independently of the other one from the 111 numbers. And there are again as many pretty tickets with remainders of 2, 3, ..., 8. As for the remainder of 0, the number of pretty tickets is $112 \cdot 112 = 12{,}544$. All in all, we get $8 \cdot 12{,}321 + 12{,}544 = 111{,}112$ pretty tickets."

"But what does all this have to do with *lucky* tickets?" Misha asked.

"Well, it's really quite simple: if the sums of the digits are equal, so are their remainders when divided by 9. So every lucky ticket is pretty too! Not every pretty ticket is lucky, though; for instance 100 748 is pretty, but not lucky. So we've proved that *the number of lucky tickets is less than 111,112.*"

"Still, that's not a complete solution," Misha said. "We can see that there are less than 111,112 lucky tickets, but we don't know by how much. Is it possible to show that there are more lucky tickets than some fixed number? I've heard that this is called a 'lower estimate.' "

"I can give a lower estimate too," the professor answered, "but I'm afraid it's rather rough. If one half of the ticket's number is a precise copy of the other, like 287 287, then such a ticket—let's call it 'wonderful'—is impeccably lucky. Since there are exactly 1,000 wonderful tickets—000 000, 001 001, ..., 999 999—we have the following lower estimate: *the number of the lucky tickets is greater than 1,000.*

"The upper bound in my estimates (111,112) is more than 100 times the lower bound (1,000), so this estimate can hardly be regarded as a real solution of the problem."

"Maybe I can improve your estimates a bit," I broke in. "For the lower bound we can count the numbers whose second half not only copies the first half but may consist of the same digits in a different order: they're lucky as well. Three digits can be rearranged in six different ways, so we get six times as many of these new 'special' lucky tickets, or whatever you want to call them, as there were 'wonderful' tickets. So the new, and better, lower bound is 6,000."

"Hold on," the professor replied, "you're slightly mistaken. Three digits can yield less than six different rearrangements if some of the digits are the same! For instance, the wonderful number 222 222 can't be modified by your method. Let me see . . . Yes! With this correction we get 4,600, not 6,000, guaranteed lucky tickets. At any rate, it's better than 1,000."

"I'm lost," Misha complained.

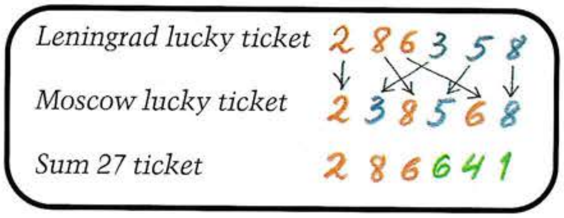
"It's a good problem for you to think over at home," I said, "but now I'd like to finish with the upper bound before we get off. I'll make use of the test for divisibility by 11."

"What's that?" Misha asked. "I've never heard of it—we haven't studied it at school."

"That's okay, I'll teach you—it's easy. You simply add up all the digits of a number that are in the odd decimal places, do the same for the even places, and subtract one sum from the other. If the difference is divisible by 11, then the initial number is divisible by 11 too. Conversely, any number divisible by 11 has this property. You can prove this test yourself—there's another task for you to do at home."

"I'll try, but I still don't see the relationship between your test and lucky tickets."

"Don't worry, you'll see it. There's a direct connection. But tell me, have you ever heard about 'Moscow lucky' tickets?"



"More riddles! Yes, I've heard that Muscovites think a ticket is lucky if the sums of its digits in even and odd places are the same. What a bunch of weirdos!"

"First of all, *you're* a weirdo if you seriously believe that any ticket whatsoever can bring you luck. Second, Muscovites call 'lucky' the very same tickets that Leningraders do, and our 'Moscow lucky' tickets are 'Leningrad lucky' for them. Just as 'American hills' [the Russian name for a roller coaster—*Ed.*] are called 'Russian hills' in America. But that's not the point. [Indeed!—*Ed.*] According to our divisibility test, Moscow lucky numbers are divisible by 11, right?"

"Right."

"So the number of such tickets is not greater than the number of multiples of 11 in the range 0 to 999,999 . . ."

"You mean, not greater than 999,999 ÷ 11 + 1 = 90,910," the professor interrupted. He couldn't pass up the chance to show off his calculating ability.

Misha lost all patience. "Will you *please* tell me what you're driving at?!"

"Just a second. It's easy to prove that there are as many Moscow lucky tickets as regular lucky ones."

"Right, 'easy,' " Misha grumbled. "We don't even know how many tickets of each kind there are."

"But we don't need to know that," the professor interjected. "Put the first three digits of a lucky number into the even decimal places and the last three into the odd places, and you'll get a Moscow lucky number. You can also reverse this transformation and turn a Moscow lucky ticket into a regular lucky ticket. So we've established a one-to-one correspondence between the two kinds of tickets. It follows, then, that they're equal in number. Right?"

"Right!" Misha exclaimed. "Terrific! So we've proved that there are less than 90,910 lucky tickets!"

"Thank you, professor," I said. "I couldn't have explained it better. In the meantime, I've figured out another way of arriving at your initial upper estimate. Let's replace the last three digits of a lucky ticket number with the differences between these numbers and 9. Misha, what's the sum of the digits of the new number?"

"Just a sec . . ." Misha concentrated on his computations. "Hmmm . . . three times nine is 27 . . . minus . . . plus . . . It's 27! And we have another one-to-one correspondence! This means that the number of lucky tickets is equal to the number of tickets whose digits add up to 27."

"Good job. Now, since 27 is divisible by 9, all the numbers whose digits add up to 27 are multiples of 9, so there are not more than 999,999 ÷ 9 + 1 = 111,112 such numbers from 0 to 999,999. And this is precisely the professor's upper bound."

"But, after all this, can you just tell me how many lucky tickets there are?" Misha asked hopefully.

"I'll give you the answer right away: 55,252—that is, on average every eighteenth ticket is lucky. But where this number comes from, I'll tell you some other time. Let's say good-bye to professor—it's time to get off."

Editor's Note. Even if you don't know much about Russia, you would notice that this article was written a while ago. Suffice it to say that the city of Leningrad doesn't exist any more—its old name, St. Petersburg, was restored last year. And that's not all. For 6 kopecks now you'd be able to buy a little more than 1/3 of one ticket, and not from a ticket machine in the streetcar: now tickets are purchased in advance and punched when you get in. (Maybe that's one reason why the game of lucky tickets is no longer popular.) Of course, we could easily have updated the setting of the story. But we decided against it to emphasize that the problem of lucky tickets is still as interesting as it was 15 years ago, when this article was published in *Kvant*. Still, you may want to use your computer to crack this nut, since there are more computers now than sledgehammers.

"CHALLENGES"
CONTINUED FROM PAGE 21

resistances of the resistors in each subsequent element differ by a factor of k from the resistances of the resistors in the previous elements. What is the resistance of the entire circuit? (A. Buzdin)

P50

Two images. A fast cosmic particle moving at close to the speed of light passes through the chamber of an experimental apparatus filled with a liquid with a refractive index $n = 1.6$. The particle emits radiation when passing through the liquid.

A device situated at point C is turned on some time after the particle reaches the chamber and records two bright points A and B. The experimental layout, scaled down proportionally, is shown in figure 5.

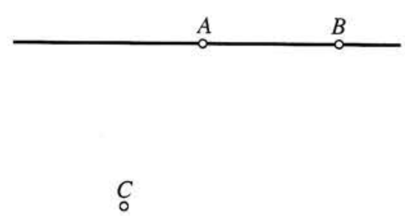


Figure 5

Explain the phenomenon observed and find the particle's speed (using the figure). Neglect the deceleration of the particle in the liquid. (A. Buzdin)

ANSWERS, HINTS & SOLUTIONS ON PAGE 59


B A C
Tishkov

Halving some more

Notes on segments of constant area

by Dmitry Fuchs and Sergey Tabachnikov

THIS ARTICLE was originally written as a kind of addendum to an article about bisecting the areas of polygons that was published recently in our sister magazine *Kvant*. It now takes on a second life as an addendum to the article "Halving It All" (page 6), written by two Americans quite independently on a similar topic especially for *Quantum*. In spite of some differences between the American and Soviet articles, the addendum is equally appropriate for both, demonstrating once again that mathematics (and science in general) knows no borders.[1]

A family of chords and its envelope

Let a smooth convex curve be drawn on the plane (fig. 1). Fix a number a, greater than zero but smaller than the area of the figure enclosed in the curve, and draw all possible chords cutting off segments of area a from this figure (fig. 2).

This family of chords has an *envelope*. If the family is plotted densely enough, the envelope is seen as the curve around which the lines of the family cluster most thickly.

Digression 1: What is an envelope? *Quantum* has already touched on this notion (see the January 1990 issue, p. 19). We'll briefly remind you of the crux of the matter. Sketch a number of successive lines of the family (fig. 3a). Mark the points where the first line crosses the second, the second crosses the third, and so on. The marked points are the nodes of a polygonal curve formed by segments of our lines. Add some more lines of the family (colored red in figure 3b)—the polygonal curve acquires new edges. The curve emerging as the limit of this polygon is the envelope. By the way, from this explanation we get an idea of how to construct points of the envelope. You fix a line l of the family and follow the point of its intersection with another line l'. When l' approaches l, this point approaches the spot where l touches the envelope (fig. 4).

[1] As a matter of fact, the authors of this piece are now teaching at American universities.—*Ed.*

Art by Leonid Tishkov

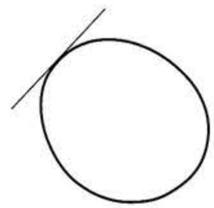


Figure 1

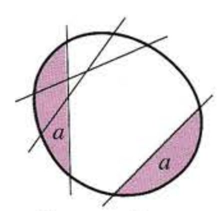


Figure 2

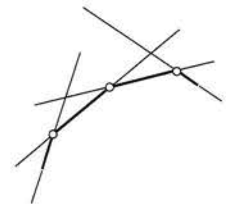


Figure 3a

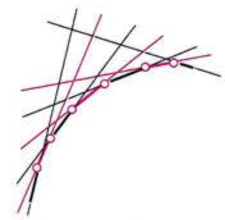


Figure 3b

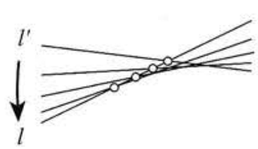


Figure 4

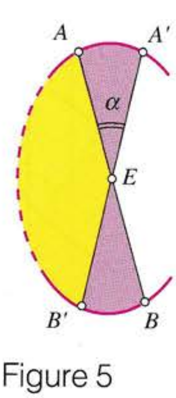


Figure 5

The envelope of our family of chords is the locus of their midpoints

The very title of this section is a statement of a theorem whose proof can be derived from figure 5.

Let AB and $A'B'$ be chords of our family that meet at an angle α. Since both chords cut off segments of equal area, the pink sections, which are obtained by subtracting the yellow section from the segments cut off by the chords, have equal areas too. These areas are approximately equal to $(AE^2 \sin \alpha)/2$ and $(BE^2 \sin \alpha)/2$, and the relative error of the formulas becomes arbitrarily small as α gets smaller. In the limit we get $AE = BE$, so E is the midpoint of AB. (Compare this proof to the similar argument in "Halving It All."—*Ed.*)

Digression 2: The problem of two ovals. Two convex curves, one inside the other, are drawn on the plane (fig. 6). Prove that there's a chord of the bigger curve that touches the smaller one and is bisected by the point of contact. (This problem isn't as innocuous as it might seem. Try to solve it before you read on.)

And now—the solution. Draw all the tangents to the inner curve (red

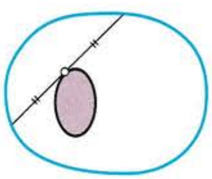


Figure 6

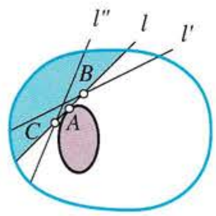


Figure 7

Figure 8a

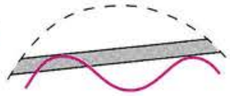


Figure 8b

Figure 9a

Figure 9b

oval). Find the line l that cuts off the segment of minimum area from the oval bounded by the outer (blue) curve. Draw two lines l' and l'' close to l that cut off segments of the same area a. Let l touch the inner curve at point A and meet l' and l'' at B and C (fig. 7). Since a is the smallest value of the segment area, lines l' and l'' do not contain any interior points of the red oval. Therefore, A lies between B and C. When l' and l'' approach l, points B and C tend to each other and so to point A. It follows that A is the point of contact of l and the envelope of chords cutting off segments of area a from the blue oval. Now we can apply the theorem from the last section, according to which point A bisects chord l.

Replacing the minimum area in this reasoning by the maximum, we'll come up with the other chord satisfying the condition of the problem.

It's worth mentioning that the fact just proved has a three-dimensional generalization: If one convex body lies inside another, then there exists a plane touching the boundary of the inner body at a point that is the center of mass of the outer body's section by this plane. The interested reader may want to prove it. However, it's time to return to our envelope.

What does the envelope not have?

The envelope doesn't have two things: bitangents and inflections. A *bitangent* is a line touching a curve at two points (fig. 8a); a *point of inflection* is a point where a curve crosses its tangent (fig. 9a). Arbitrarily close to a bitangent, as well as to the tangent at an inflection point, we can draw mutually parallel tangents to the curve (fig. 8b, 9b). Since these parallel lines can't cut off segments of equal area, the envelope we're investigating can have neither bitangents nor inflections.

What does the envelope have?

This has been made clear enough in the article "Halving It All": the envelope has cuspidal points like the one you see in figure 10 (see also figure 18 in "Halving It All"—there we find 6 cusps on one envelope). Actually, any time we talk about envelopes, cusps are there too. So you shouldn't be surprised. Let's go further, though, and try to answer this question:

Where on the envelope do cusps appear? (We'll answer this question first for curves that have well-defined tangents at each of their points. Although this family of curves excludes triangles, we'll explain later how to apply our theory to triangles and other polygons.)

THEOREM. *If the midpoint of a chord AB is a cuspidal point of our envelope, then the tangents to the curve at the ends A and B of the chord are parallel.*

PROOF. As we've seen, chord $A'B'$ belongs to our family if the pink triangles in figure 5 have equal areas. So the chords of the family close to AB, along with the piece of the envelope close to the midpoint of AB, are determined by the small segments of the initial curve near A and B. If the tangents to the curve at points A and B aren't parallel, the curve near points A and B can be replaced by these tangents without essentially changing the envelope. It's proved in "Halving It All" that the envelope of lines cutting off a fixed area from an angle is a hyperbola—that is, a smooth curve without cusps. Applied to the angle formed by our two tangents, this means that our envelope can't have a cusp at the midpoint of AB if the tangents at A and B aren't parallel. (The actual behavior of the envelope near this point is clarified by figure 11.)

Now, conversely, suppose that the tangents at points A and B are parallel. Then we need additional data: at which of these points is our curve "more curved" ("has a greater *curvature*," to put it in mathematical terms). We're not going to give here a rigorous definition of this notion because we simply can't believe you don't understand it as it is. For instance, when a car goes into a sharper turn (that is, the curvature is greater)

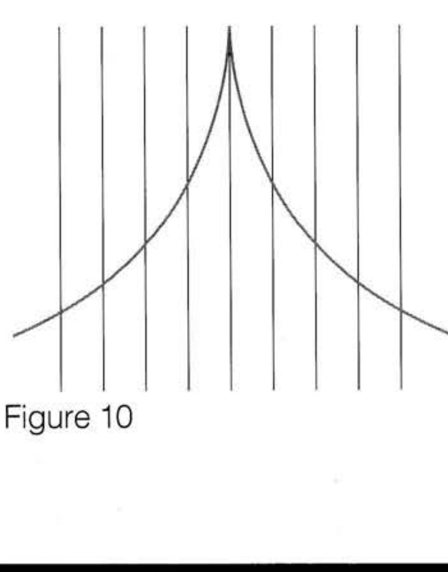


Figure 10

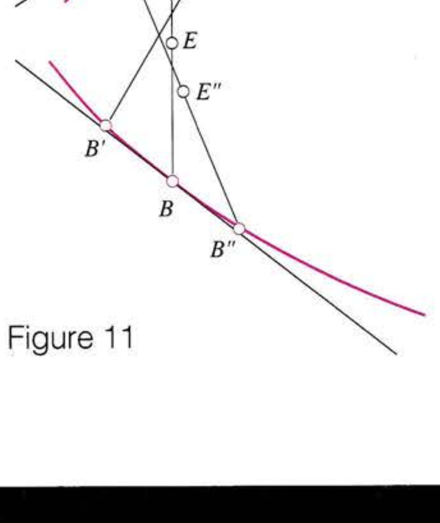


Figure 11

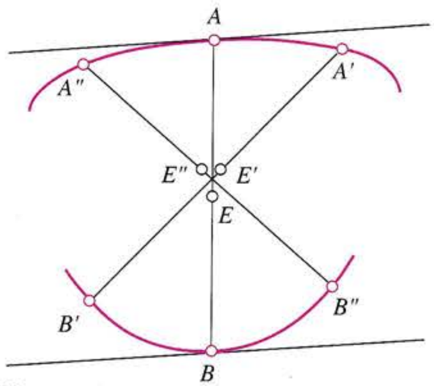


Figure 12

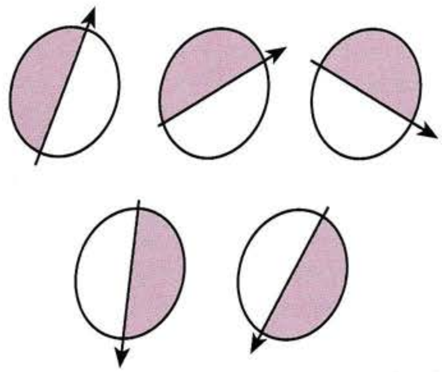


Figure 13

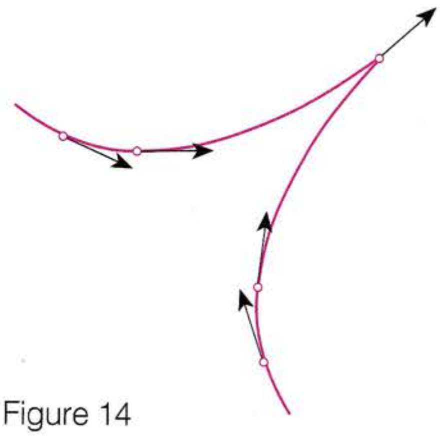


Figure 14

but still travels at the same speed, it skids more sharply.

So, suppose the curvature at point A is less than at B. You can see from figure 12 that in this case the midpoints E' and E'' of the neighboring chords $A'B'$ and $A''B''$ are on different sides of chord AB, displaced from the midpoint E toward A. So there's a cusp at E pointing at B.

To sum up, *cuspidal points on a curve correspond to chords whose endpoints meet at parallel tangents; the cusps point in the direction of the endpoints with less curvature.*

However, the question remains: what happens if not only the tangents at the endpoints of a chord are parallel but the respective curvatures also are equal? This question is so difficult we'll give two different, even opposite, answers, and both will be correct. The first answer: it practically never occurs. There are only a few isolated chords in our family with parallel tangents at their ends, and it's virtually impossible for the curvatures at the ends of these rare chords to be equal.[2] The second answer: this case doesn't affect us right now, but we'll come across it later. So for now, we'll merely introduce a term for it—let's call it the "case of maximum degeneration."

How many cusps?

Quite unexpectedly, the number of cusps depends on whether or not the area a of the segments cut off from the given figure equals half the area of the entire figure.

Case 1: Area a is half the entire area. In this case there's a single chord in each direction cutting off a segment of area a. This means that when a point on the envelope makes a full circuit around it, the tangent to the envelope at this point makes a half-turn and fits on itself, having changed direction (fig. 13). So, however strange it may seem, when we circumnavigate the envelope, the tangent changes its direction to the opposite. How can that happen? Look at figure 14, which shows a cusp. Draw an arrow on the tangent to the envelope in the direction of the contact point's motion. Right away you see that when the contact point passes the cusp, the arrow turns in the direction opposite to its motion. And so it happens at every cuspidal point. But we know that, having traced the circle, the direction has changed. This means . . . Right!—

The number of cuspidal points on the envelope is odd.

How many cusps are there? More than one! Why? Let the tangent at a cusp be vertical, as in figure 10. Each direction of a tangent occurs once, so the envelope has no other vertical tangents. If it also has no other cusps, then on the left side of the cusp the curve runs all the way to the left, and on the right side it runs all the way to the right. So the "ends" of the curve never meet—it's not closed. Therefore—

The number of cuspidal points is not less than 3.

Case 2: The area a is not equal to half the entire area of the figure. In this case, there are two tangents in each direction in our family (fig. 15), and after tracing the whole envelope the tangent makes a full 360° turn (fig. 16). Repeating the above argument, we conclude:

The number of cuspidal points on the envelope is even.

Digression 3: The Moebius theorem. While exploring the first case, we almost proved the statement known as the Moebius Theorem, which states that *a curve without bitangents and inflections, whose*

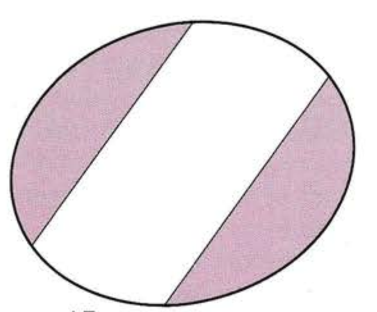


Figure 15

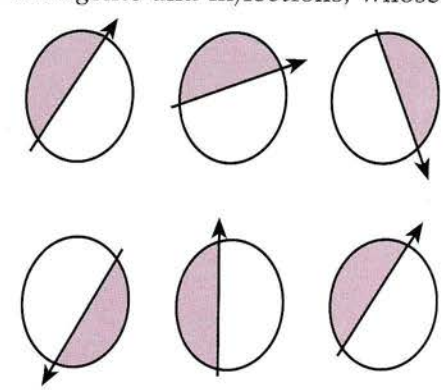


Figure 16

[2]The very simple case of a centrally symmetric figure, in which we seek chords bisecting the area, is excluded here. The envelope is simply the point of symmetry.—*Ed.*

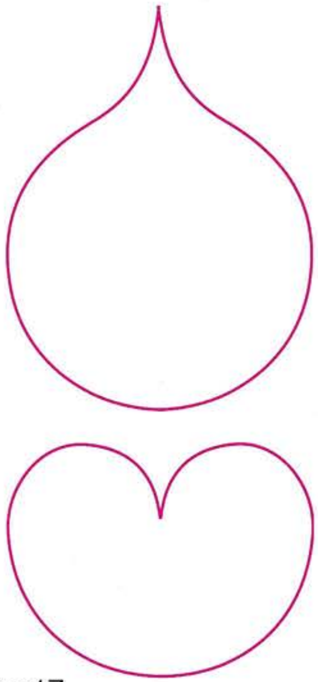


Figure 17

tangent turns 180° after tracing it all way around, has an odd number of cusps not less than 3.

For instance, the curves in figure 17 are excluded by the theorem's condition. As to its proof, you can find it yourself by using figure 10.

In one and the same diagram

Now let's try to plot in one diagram the envelopes of families of chords cutting off segments of constant area for different values *a* of the area ranging from zero to half the entire area. In figure 18a we have chosen a curve with tangents at each point, which approximates a triangle. This curve is the outermost one, shown in blue. The envelopes are black. The innermost black triangle

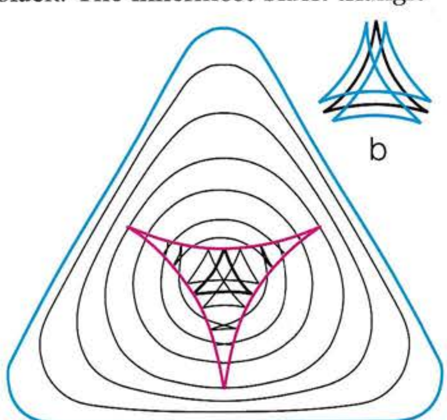


Figure 18

is the envelope of chords bisecting the area. We zoom in on this triangle in figure 18b. The blue curve in this figure, which traces twice the perimeter of the black triangle, is the envelope of chords cutting off segments whose area is a bit less than half that of the entire figure. When the area of the segments decreases, the envelope dilates and approaches the original curve (the outermost curve in figure 18a). For small enough values of the area, the envelope becomes smooth and eventually merges with the original curve, which corresponds to an area of zero. Not only that (a new observation!), during this evolution the cuspidal points of the envelopes glide along some curve (it's red in figure 18a) on their own.

It follows from what was said above that each point of the red curve is the midpoint of a chord connecting points at which the tangents to the original curve are parallel. And this curve has its own cuspidal points! At these points, pairs of cusps of black envelopes meld and disappear (or are born in pairs). So what are these mysterious points? We know them! They're those "points of maximum degeneration"—that is, the midpoints of chords with both parallel tangents and equal curvatures at their ends.

How many points of maximum degeneration are there?

There are at least as many points of maximum degeneration as there are cusps on the envelope of chords bisecting the area (which means not less than three). Why? Because the envelope of chords that cut off a little less than half the entire area (the blue curve in figure 18b) has twice as many cusps as the bisector's envelope; and all of them disappear in pairs at the points of maximum degeneration during this evolution. So the number of points of maximum degeneration is not less than the number of cusps of the bisecting chord's envelope.

In addition, the number of points of maximum degeneration is odd, because the number of cusps of the

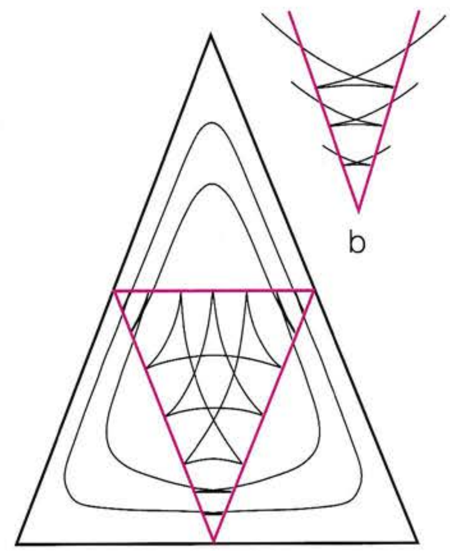


Figure 19 a

blue curve—which is twice an odd number—changes by two at these points and vanishes by the end of the evolution of envelopes. So the number of alterations must be odd.

The case of a polygon

Let's apply our theory to polygons. In order to do so we must approximate a polygon with a smooth curve—almost straight sides and sharp turns at the corners.

Let's start with a triangle. Points with parallel tangents here are a vertex of the triangle and an arbitrary point on the opposite side. It follows that the red curve of figure 18a is in this case a triangle formed by the midlines of the initial triangle. Figure 19a shows the evolution of the envelopes for this particular case. *All* the envelopes here have cusps (since the red triangle reaches the sides of the initial triangle—see figure 19b) and consist of hyperbolic segments.

In the case of an arbitrary polygon, the envelopes likewise consist of seg-

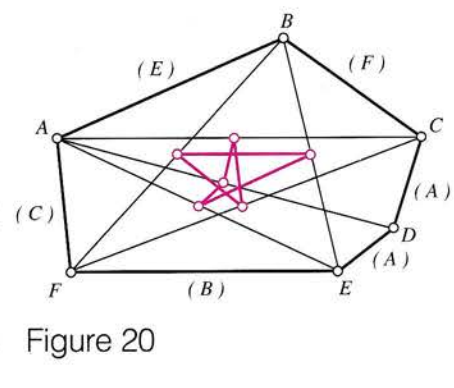


Figure 20

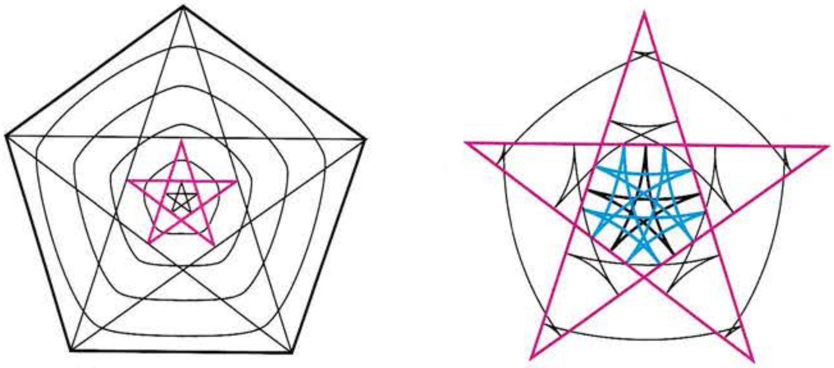


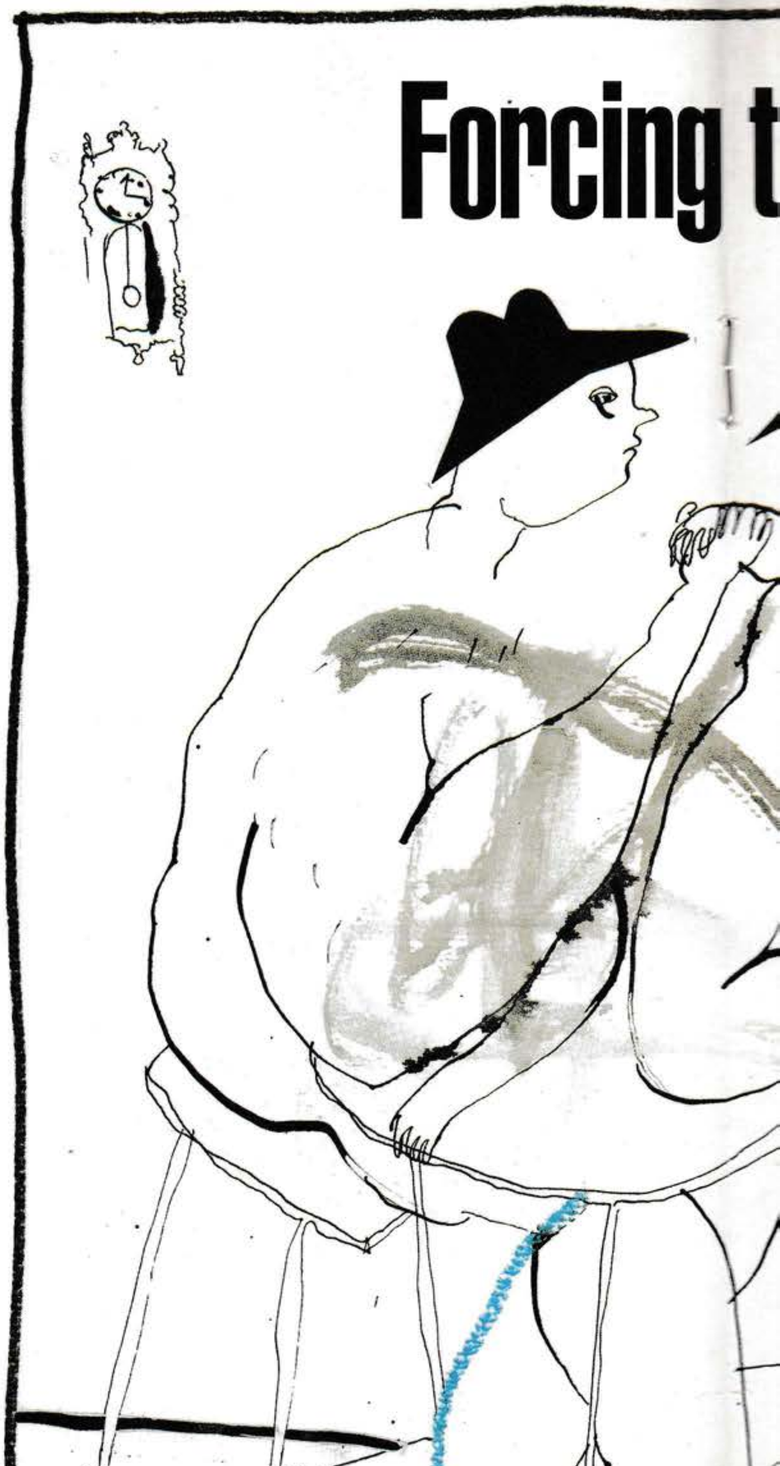
Figure 21

ments of hyperbolas. Let's call a vertex A of the polygon and its side s "opposite" each other if the straight line drawn through A parallel to s lies outside the polygon. Each side is opposite a single vertex (unless the polygon has parallel sides), but a vertex can be opposite several sides or no side at all. In figure 20 each side is labeled with the "name" of the opposite vertex. The red curve here consists of the midlines of triangles whose bases are sides of the polygon and whose vertices are the polygon's vertices opposite the respective sides; so the red curve is a polygonal line whose edges contain all the cusps of all the envelopes, while its nodes are the points of maximum degeneration. In figure 21 we've tried to depict another particular case—that of a regular pentagon (the diagram on the right is the magnified central portion of the diagram on the left).

In conclusion, we'd like to point out a surprising difference between "even-gons" and "odd-gons" (polygons with an even and odd number of sides, respectively). The envelope of the family of chords bisecting the area of an odd-gon that is close to regular (or is in fact a regular polygon) has as many cusps as there are vertices of the polygon and resembles a regular star. The regular even-gon has a center of symmetry, so the envelope of area-bisectors shrinks to a point (the center). But the envelope for a near-regular even-gon has fewer cusps than the polygon has vertices (remember—the number of cusps is odd!). For a quadrilateral, the envelope is a curved triangle; if there are more than four vertices, the envelope's behavior is a bit more complicated. ◘

KALEIDOSCOPE

"It seems that the entire difficulty in physics consists of recognizing natural forces in phenomena of motion and then explaining all other phenomena in terms of theseforces . . ."
—Isaac Newton, Principia Mathematica



ARE YOU ALL THAT FAMILIAR with the notion of *force*? The epigraph summarizes the "research program" that fascinated and inspired Newton and his followers. The mechanics created by Newton was the basis for almost all aspects of physics up until the 20th century, and it has continued to play a role in most areas of technology right to the present.

Newtonian mechanics is based on three famous laws that include the fundamental notion of force as the measure of interaction between material objects. In suggesting that you work on this notion, we should stress that Newton's laws don't reveal the origin and properties of forces but allow us to predict the behavior of bodies when these forces act on them.

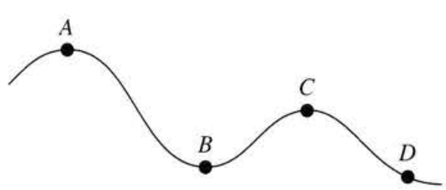
Questions

1. What is the total force of gravity acting on the fragments of a shell after it explodes?

2. In 1864, to demonstrate the effectiveness of the air pump he invented, Otto von Guericke evacuated two brass hemispheres set rim to rim so that the atmospheric pressure held them together. He then hitched two teams of eight horses and had them try to pull the hemispheres apart. Would the traction force have been different if he had attached one hemisphere to a wall and harnessed sixteen horses to the other hemisphere?

3. Is it possible to stretch a rope so that it's absolutely horizontal?

4. A roller coaster car rolls along a hill with the profile shown in the figure below. Where is it necessary to place the sturdiest boards?



g the issue

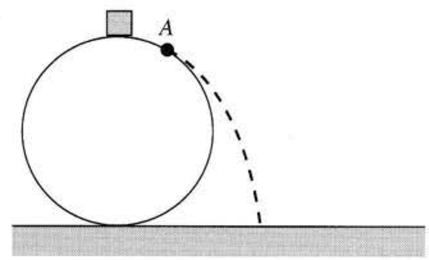
Art by Dmitry Krymov

5. Is it easier to hold a sled on the slope of a hill or to pull it uniformly upward?

6. A truck starts moving. What force acts on the load at the center of the truck's bed? What is its direction?

7. Two identical springs are connected, first in series, then in parallel. Show graphically how the dependence of the elastic force on the stretch of the springs differs in these two cases.

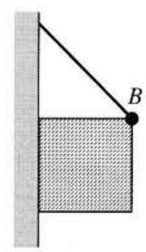
8. In the figure below, what forces act on the body at the separation point A?



9. What is the direction of the net force acting on a pendulum bob (a) at its extremes and (b) at its equilibrium position?

10. We know that all objects are gravitationally attracted to each other. So why don't all the objects in the room start moving toward one another?

11. Will the box in the figure below remain in place against the wall in the absence of friction?



12. Moving horizontally from east to west, an electron enters a magnetic field and is deflected downward. What is the direction of the magnetic field?

It's interesting that . . .

. . . the German physicist Heinrich Hertz managed to devise a mechanics that doesn't make use of the notion of force at all. But the main concepts of the mechanics Hertz formulated became so involved that his system as a whole was never accepted.

. . . the French philosopher and scientist René Descartes (1596–1650) believed that the only forces that exist are those acting in collisions between particles—that is, forces of contact.

Microexperiment

Drive two identical nails, one into a dry board and the other into one that has been soaked. Compare the amount of effort it takes to pull the nails out. Is it different in each case? Why?

ANSWERS, HINTS & SOLUTIONS ON PAGE 63

Make yourself useful, Diana

The Moon as a radio telescope antenna

by P. V. Bliokh

DO YOU KNOW HOW WILD elephants are taught to work? They're so strong that when people catch them, they're still unable to deal with them. But tame elephants come to the rescue and teach their wild relatives their new profession. This story comes to mind when we think of the paths astronomers have taken to unveil the mysteries of distant celestial bodies. Logically the simplest way to scientific understanding is this: we design ever more sensitive and accurate devices and with their help acquire new knowledge. There is, however, another way—in our quest to discover the mysteries of nature, we make allies of nature's creations.

Among the celestial bodies that we have already "taught to work" (they're used as a sort of antenna for radio telescopes) are the Moon, the Sun, and even entire galaxies. Here I'll only tell you about our nearest neighbor, the Moon—how with its help scientists have increased the accuracy of radio telescopes, what role it played in the discovery of quasars, and how new devices (intercontinental radio telescopes) compete with it.

Difficulties in the first radio astronomy observations

The earliest radio astronomy observations were rather modest in their aims. First, it was necessary to detect the sources of cosmic radio emissions and determine their positions on the celestial sphere. In other words, scientists began to map the "radio sky" just as astronomers of old compiled celestial atlases.

Unlike their predecessors, though, radio astronomers didn't fool around with naming the discovered sources after gods and heroes, nor did they group them into "radio constellations," because what they saw wasn't sparkling myriads of stars but columns of figures and graphs on plotters. No wonder the names of radio sources are so prosaic. They only specify the catalog and the number under which the source is to be found in it.

The generally accepted system of naming radio sources isn't just a sign of our businesslike times, it also provides a nice hiding place for the poor quality of initial radio astronomy observations. The coordinates of radio sources were determined so roughly in comparison with the accuracy achieved in optical astronomy that it was very difficult to find any correspondences between optical objects and radio sources. The culprit here is the relatively small sizes of the antennas of radio telescopes compared to optical instruments when expressed in terms of the wavelengths of the radiation each one detects.

Under ideal conditions, when radiation undergoes no scattering in the atmosphere, the uncertainty δ in the coordinates of a source depends only on the wavelength λ of the radiation being detected and the instrument's dimensions a (the long dimension of the radio antenna or the diameter of the telescope's lens).

The formula for calculating the uncertainty is very simple:

$$\delta \cong \frac{\lambda}{a}.$$

In the optical range we have $\lambda \cong 5 \cdot 10^{-5}$ cm, $a \cong 10^2$ cm, and $\delta \cong 5 \cdot 10^{-7}$ rad $\cong 0.1''$. To attain the same accuracy in the radio range, even in its shortwave region ($\lambda \cong 1$ cm), we need antennas whose sizes are $a \cong \lambda/\delta \cong 2 \cdot 10^6$ cm = 20 km.

So, the first maps of the radio sky and the stellar atlases lived their own separate lives. It wasn't until the 1950s that scientists began to identify the optical counterparts of radio sources. How did they do it? They managed to use the Moon as a kind of radio telescope antenna. This method, proposed by G. G. Getmantsev and V. L. Ginzburg in 1950, can't be used for all radio sources—it's suitable only for radio sources that the Moon covers up as it moves along the celestial sphere. Nevertheless, the number of such radio sources is not so small, and the

Art by Dmitry Krymov


Д. Кроллов

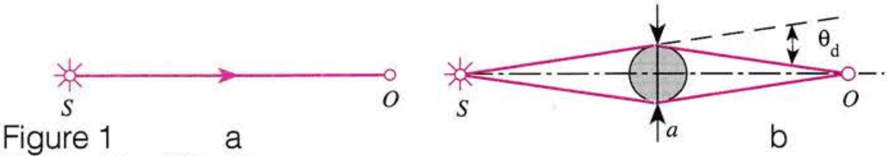


Figure 1 a b
Constructing diffraction rays.

Moon–radio telescope enabled scientists to discover many interesting things. I'll tell you about one such discovery and then explain the gist of the method.

Wave diffraction

Electromagnetic waves are able to skirt obstacles. This phenomenon is called diffraction. It was discovered back in the 17th century when light propagation was being studied (light and radio waves are both electromagnetic waves, differing only in their wavelengths).

There's a simple rule of thumb that allows us to construct rays bent because of diffraction. Imagine an elastic string stretched between the observer and the source. If light meets no obstacles in its path, the string is a straight line—this is a light ray in a uniform medium (fig. 1a). Now imagine that a body is placed between the observer and the source (fig. 1b). Then the elastic string joining the source S and the observation point O will bend to bypass the obstacle. This is like the bending of rays due to diffraction. And what about the fact that diffraction is a distinctive property of waves? What do ray–strings have to do with waves? It turns out they're directly related. Each ray can be assigned a certain wave amplitude and phase. The phase is determined by the length of the ray segment joining the source and the observer; the amplitude depends on how much the ray is bent. Most of the radiant energy is transmitted by straight or slightly bent rays. These rays have the largest amplitude. The maximum angle of bend with an appreciable amplitude—that is, the diffraction angle (let's denote it by θ_d)—depends on the radiation wavelength λ and a typical size of the obstacle a:

$$\theta_d \cong \frac{\lambda}{a}.$$

You can use this formula only if the observer is far away from the obstacle. The rays skirting the obstacle on various sides must be able to reach the observation point and, at the same time, the bending angle must not exceed θ_d. Nevertheless, diffraction phenomena also manifest themselves at smaller distances, when waves from only one edge of the obstacle reach the observation point (fig. 2a). In this case we can also construct diffraction rays using elastic strings that are either "stretched out" by an obstacle (ray 1) or are "hooked" by its edge (ray 2).

The main rule still holds true: the more the ray is bent, the less energy it carries. However, the measure of maximum bending now is not the angle but the linear displacement of the observer from the edge of the shadow. Let's denote it by d. It depends on the wavelength and the distance between the obstacle and the source and between the obstacle and the observation point. If the distance L to the source is very large, the formula for determining d takes its simplest form:

$$d \cong \sqrt{\lambda L}.$$

Figure 2b plots the radiation intensity at the transition across the shadow's edge. The thick line jumping suddenly from zero to I_0 (which is the intensity of the light that isn't blocked) is plotted without allowing for diffraction. The thin line illustrates the effect of diffraction: the intensity gradually falls off as you move deeper into the shadow; oscillations arise in the illuminated area and damp as you move away from the edge. The oscillations are caused by the interference of two rays: a straight ray and a diffracted ray (rays 3 and 2 in figure 2a). If the phases of the waves corresponding to these two rays coincide, you're at the crest of the intensity plot; if the phases differ by 180°, a trough appears.

Now let's turn back to figure 1. Imagine that the obstacle is a sphere whose center is on the line SO. Then the diffracted rays that skirted the obstacle on different sides will converge at the observation point after traveling paths of the same length. This means that the waves all arrive in phase and the intensity at this point will be a maximum.

As you shift the observation point away from the symmetry axis, the lengths of the rays that arrive from different directions are no longer equal. The corresponding waves will no longer be in phase and will cancel each other. The light intensity abruptly falls off. This means that a bright spot should be observed at the center of the shadow cast by an opaque sphere.

Don't be distressed if you find this a strange conclusion that contradicts your everyday experience. You're not the first to fall victim to "common sense." Siméon-Denis Poisson, one of the most famous French scientists of the early 19th century, followed the same line of thought. He was a member of the commission specially created by the French Academy of

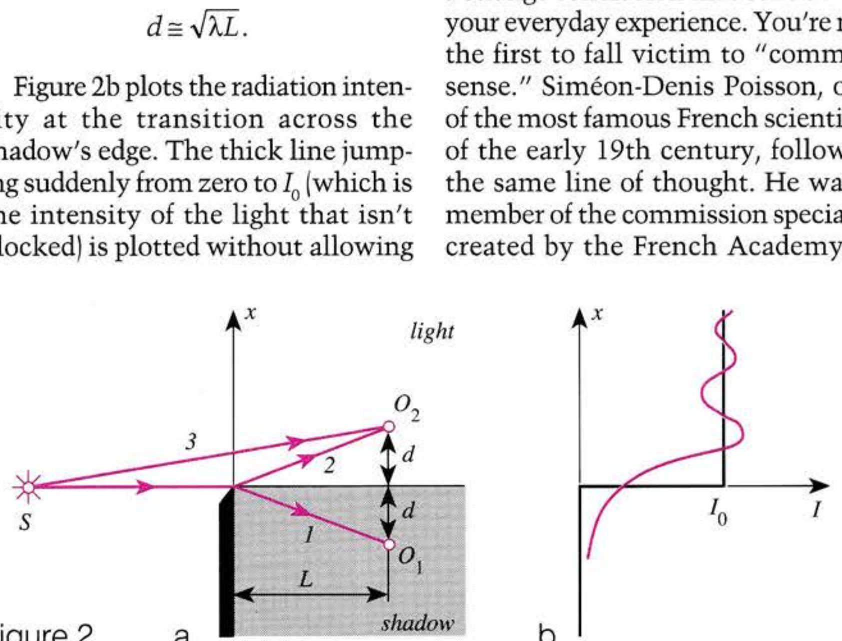


Figure 2 a b
Light diffraction at the edge of the obstacle.

Sciences in 1818 to hold a competition for the best study on the nature of light. He analyzed the theory presented in Augustin Fresnel's entry, and the conclusion about a bright spot at the center of the shadow seemed so absurd to him that he took this conclusion as an argument against the wave theory of light. The commission proposed that this paradoxical statement be tested experimentally. The experiments were carried out, and a bright spot was actually observed at the center of the shadow![1] After this observation the wave theory became generally accepted.

The bright spot at the center of the shadow is just an image of the point source S—that is, the nontransparent sphere acts as a convex lens. Using such a lens, one can create even more complex images. In his book on optics, Arnold Sommerfeld has a portrait of a woman. What's interesting is that it was produced by a camera whose lens was a solid metal disk!

The Moon's role in refining coordinates

Figure 3 shows a point source S, the Moon, and its "radio shadow" cast on the Earth. The circular shadow's radius is r, which is approximately the radius of the Moon ($r \cong 1{,}400$ km). The transition from the illuminated area to the shadow is accompanied by diffraction oscillations with a period $d \cong \sqrt{\lambda L}$, where $L \cong 380{,}000$ km is the distance between the Earth and the Moon. In the meter range of radio waves we have $d \cong 20$–60 km. The Moon's shadow moves along the Earth's surface at a speed of about 1 km/s. So the whole diffraction pattern passes the observer in about a minute, which is quite sufficient for recording it.

The location of the Moon on the celestial sphere is well known. So we can determine the coordinates of source S with an accuracy to 1″. Moreover, examining the fine structure of the diffraction oscillations (that is, small variations of intensity $I(t)$ with time), you can draw certain conclusions about the source itself: what its angular diameter is and even how its radio brightness is distributed (in particular, we can reliably resolve double sources).

Since the intensity changes appreciably as the Moon moves a distance d along its orbit, the effective angular resolution of the Moon–antenna is calculated according to the formula

$$\delta_{\text{Moon}} \cong \frac{\sqrt{\lambda L}}{L} = \sqrt{\frac{\lambda}{L}}.$$

If we wanted to achieve the same resolution using ground-based devices, we'd need an antenna whose size is

$$a \cong \frac{\lambda}{\delta_{\text{Moon}}} \cong \sqrt{\lambda L}.$$

For $\lambda = 10$ m, we'd get $a \cong 60$ km. The incredible difficulties associated with such a project need no elaboration.

If we have a device of sufficient sensitivity, we can detect the change in intensity caused by the Moon's displacement through an angle less than $\sqrt{\lambda L}$. The important thing is that interference be eliminated. In other words, the resolution is ultimately determined by the signal-to-noise ratio.

The method described takes advantage of the diffraction at the edge of the lunar disk. There is, in principle, another possibility. You may recall that a bright spot surrounded by diffraction fringes is formed at the center of the shadow cast by a sphere. The angular dimensions of the central spot are $\theta_d \cong \lambda/r$—that is, the resolution in this case is equivalent to that of a circular antenna with a radius $a \cong r \cong 1{,}400$ km!

However, it's very difficult to make use of such a tempting opportunity. No one has succeeded so far, for two reasons. First, the roughness of the Moon's surface breaks the sphericity of the Moon and leads to irregular disturbances of the diffraction pattern that are practically impossible to take into account. We can overcome this difficulty by making our measurements at wavelengths long enough for the irregular phase shifts of the rays skirting the lunar mountains to be small (not larger than several degrees). An estimate yields a minimum allowable wavelength of the order of 100 m. This range, however, is hardly useful for measurements from the Earth's surface because of the screening effect of the ionosphere. Second, with the suggested method it's possible to study only sources that are covered exactly by the center of the lunar disk. The allowable deviations should be substantially smaller than the angular width of the central spot.

Maybe at some point in the future all these difficulties will be overcome if a radio telescope is mounted on a spacecraft. Choosing the appropriate flight trajectory (the spacecraft should cross the line passing from the source to the Moon's center), one will be able to use the Moon as an antenna at sufficiently long wavelengths without worrying about disturbances induced by the ionosphere.

The first observations in which the Moon was used for a more accurate determination of the coordinates and structure of radio sources were made in the mid-fifties. At first, they offered only limited possibilities since the measurements were made with stationary antennas. In this case, we need a lucky coincidence of the moments when the Moon covers the source and the Moon is within

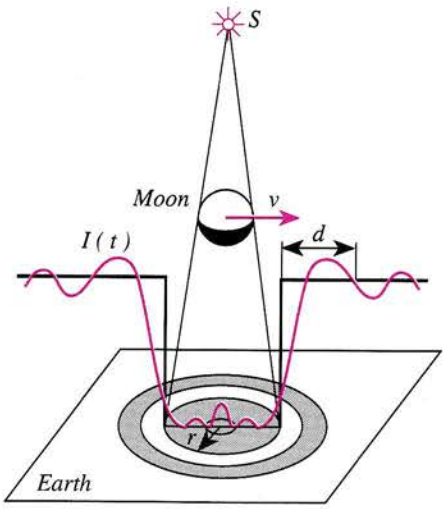


Figure 3
Eclipse of a cosmic radio source by the Moon.

[1] We never notice this spot because the usual light sources are extensive, while here we're speaking of rays emerging from a single point.

the cone of the nonadjustable directional pattern of the radio telescope. The situation changed drastically when first radio telescopes with tracking antennas were placed in operation. With these radio telescopes the antenna can be oriented toward any point in the heavens. A special radio telescope for performing measurements with the Moon was constructed in India. Its antenna is a rotating parabolic cylinder with dimensions 530 m × 30 m situated on a hill so that the rotation axis of the antenna is parallel to the Earth's rotation axis. This makes it possible to keep the Moon within "sight" for considerable periods of time by compensating for the daily rotation of the Earth.

The Moon's role in the discovery of quasars

Attempts to find the optical counterparts of radio sources led to one of the most surprising astronomical discoveries of the 1960s. In 1963 a group of Australian scientists, operating a 64-meter radio telescope, used the Moon to determine more accurately the coordinates of source 3C-273 (3C is the third edition of Cambridge University's catalog of heavenly bodies). It proved to be a double source. Its components have angular dimensions of less than 10″, and the distance between the components is 19.5″. Their radio coordinates were determined with an accuracy better than 1″ and exactly coincided with the coordinates of an optical object that likewise turned out to be a double object.

One of the components of 3C-273 (the B-component) coincided with a star, while the second (the A-component) coincided with a fountainlike nebula that is believed to have been thrown out of the star. The very small angular dimensions of the radio source justify its name: a quasi-stellar source—quasar, for short. One quasi-stellar source, 3C-48, was already known before this discovery. However, the surprising properties of quasars were discovered only after studying the visible spectrum of the 3C-273 source, which is linked with the brightest optical object of any quasars (a star of greater than 13th magnitude).[2]

It was established that all the lines in the quasar's spectrum are shifted toward longer wavelengths—to the red region of the spectrum. The phenomenon of "redshift" of spectral lines was well known before the discovery of quasars, but for quasars this shift turned out to be incredibly large. The shift deformed the spectra so much that for a long time astrophysicists couldn't even recognize them. Our current understanding is that the redshift results from the Doppler effect and indicates that the radiation source is moving away from us. The redshift allows astronomers to determine the distances to extragalactic objects according to Hubble's law, which states that the farther away an object is, the faster it moves and the larger the redshift in its spectral lines. At present it's practically our only way of measuring such huge distances.

Applying Hubble's law to quasars showed that they are among the most distant objects that can be observed. And there are "champions" among quasars that race away from us at speeds reaching 93% of the speed of light.

More than 2,000 quasars have been discovered to date. The distinctive feature of quasars, apart from their redshift, is the enormous power of their radio emissions. Their radiation can be many times more intense than the radiation emitted by entire galaxies (which perhaps is why quasars are also called "superstars"). At the same time the angular dimensions of a quasar are extremely small. In fact, the angular dimensions of quasars haven't yet been measured directly despite the fact that radio astronomers can now take advantage of antennas comparable in size to the Earth. I'm talking about *interferometers with a superlong base*. They deserve to be discussed in greater detail.

Intercontinental radio interferometers

A radio interferometer in its simplest form consists of two antennas situated some distance apart. The signals received by these antennas are transmitted to a station, where by comparing the phases of the signals one can determine the direction the wave came from and the angular dimensions of the source. The larger the distance D between the antennas (called the base of the interferometer), the better the resolution. I won't discuss here the peculiarities of the directional pattern of an interferometer compared to a lens or parabolic antenna. Suffice it to say that its resolution is calculated by the same formula: $\delta_{int} \cong \lambda/D$.

Interferometers with bases several meters long were used in optical astronomy to measure the diameters of star as early as the 1920s. In the radio range the base sizes can be enlarged significantly, since the radio signals received by the interferometer arms are much more "transportable" and can be transmitted easily to a common processing point. If the signals are transmitted by cable, the base may be several kilometers long. If one uses microwave radio relay links, the antennas can be moved even further apart, to distances of about 100 km, which yield a resolution better than 1″. This value is already close to the resolution of optical telescopes, but it's still insufficient for investigating the structure of quasars. To accomplish this task we need a resolution of the order of 0.001″. But such a resolution can be attained only with an interferometer whose base is comparable to the Earth's diameter. It's clear that neither cables nor radio relay links will help here.

The problem was solved by independently recording the signals on magnetic tape; the tape recorders are synchronized with the utmost precision by means of atomic clocks. The recordings are then sent to a computer processing center to be analyzed.

[2]See also "The View Through a Bamboo Screen" in the last issue for more on the search for optical counterparts of radio sources.—*Ed.*

The first measurements with an interferometer whose base ranged over Canada and the US were performed in 1967 at wavelengths of 75, 50, and 18 cm. They confirmed that quasars are small (of the order of 0.01″), but many objects remained unresolved. In 1968 and 1969 the first intercontinental radio interferometers appeared, whose arms were in the US, Sweden, and Australia. At wavelengths of 6 cm they provided a resolution of 0.001″. This is almost three orders of magnitude better than in the visible range, where quasars and even the nuclei of galaxies look like formless spots of light. But even such an unbelievable resolution doesn't allow us to make out the individual components of many radio sources. For this we need to expand the interferometer's base even more. But the longest base, between Australia and California, already measures 10,500 km—more than 80% of the Earth's diameter. A further increase is possible only if receivers are moved into space—to the Moon or even to other planets.

There is, however, another way of improving the resolution: instead of enlarging the base, we can switch to shorter wavelengths. Radio telescopes operating at rather short wavelengths had been built in the US and USSR, and they were included, with great difficulty, in an interferometer system. Additional complications arose because American and Soviet time standards had never been directly correlated. To correlate the time standards, clocks were first synchronized at Pulkovo (outside Leningrad), where "American time" was delivered from Sweden, and then the atomic clock adjusted at Pulkovo was flown from Leningrad to the Crimea (in the south of Russia), where the first Soviet radio telescope is located. This procedure is reminiscent of events that took place more than 100 years ago. In 1843–44 it was necessary to correlate the time at Pulkovo and at Greenwich, England, in order to determine the difference in longitudes of these cities' observatories. The clocks were synchronized in two stages. First, in 1843, time readings in Pulkovo and Altona, Germany, were correlated; then, in 1844, time readings in Altona and Greenwich were correlated. In the first stage 81 chronometers traveled between the two places; in the second stage—44. Several trips were made, and finally, after two years of tedious work, the time in the two places was synchronized with an accuracy better than 0.1 s. Such accuracy was more than sufficient then, but it exceeds the synchronization error allowable in an interferometer system by a factor of about 100,000!

Observations with the first Soviet–American interferometer were begun in 1969. The 42-meter antenna of the National Radio Astronomy Observatory at Green Bank, West Virginia, operated at one end of the interferometer, and the 22-meter antenna of the radio telescope at Simeiz in the Crimea operated at the other. In 1971 an additional 64-meter NASA radio telescope, at Goldstone, California, was included in the program. Using the base Simeiz–Goldstone at a wavelength of 3 cm, a resolution of about 0.0003″ was achieved. To give you an idea of what this figure means, here's an example: using an optical instrument with this kind of resolution, you'd be able to read this text—that is, make out letters about 1 mm tall—from 1,000 km away.

A global network of radio telescopes now exists that includes radio telescopes in Russia, the US, England, Australia, South Africa, and other countries and uses even shorter wavelengths (1.35 cm). And this is how the maximum resolution attainable on Earth—10^{-4} arc-seconds—was achieved.

GETTING TO KNOW...

The Greek alphabet

A physicist's guide to those pesky little shapes

by Sheldon Lee Glashow

α: **Alpha** rays are helium nuclei that are emitted from nuclei in the process of alpha decay, one of three forms of natural radioactivity.

β: **Beta** rays are energetic electrons that are emitted from nuclei in the process of beta decay, another form of natural radioactivity.

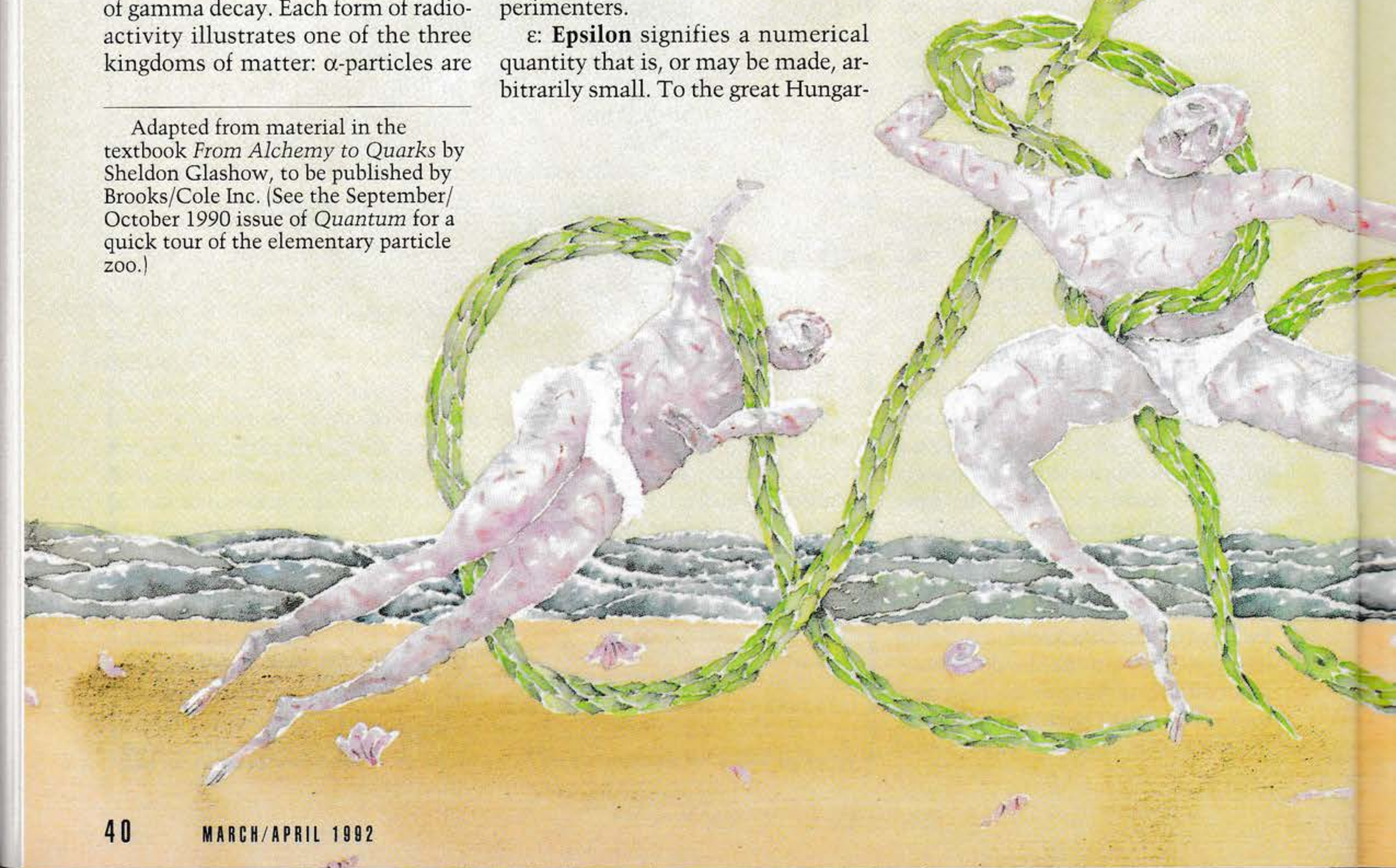
γ: **Gamma** rays are energetic photons emitted by nuclei in the process of gamma decay. Each form of radioactivity illustrates one of the three kingdoms of matter: α-particles are *hadrons*, β-particles are *leptons*, and γ-particles are *gauge bosons*.

δ: **Delta** *something* (whether in upper or lower case) often stands for a small change in that something, as in δx or Δt. Moreover, the delta function $\delta(x)$ (which is zero everywhere except at $x = 0$, where it is infinite) is a must if you are a theoretical physicist, and δ-rays are important to experimenters.

ε: **Epsilon** signifies a numerical quantity that is, or may be made, arbitrarily small. To the great Hungarian bachelor mathematician Paul Erdös, children are epsilons.

ζ: **Zeta** was the name suggested by its "discoverers" for a particle that turned out not to exist at all. It has not been popular since.

Adapted from material in the textbook *From Alchemy to Quarks* by Sheldon Glashow, to be published by Brooks/Cole Inc. (See the September/October 1990 issue of *Quantum* for a quick tour of the elementary particle zoo.)



η: **Eta** denotes the eighth and heaviest member of the octet of spinless mesons. Its existence was predicted by Murray Gell-Mann in 1961. Capitalized, η becomes H to honor the Johns Hopkins University, where the η-particle was discovered.

θ: **Theta** should remind you of trigonometry, since it is a common choice for the measure of an angle.

ι: **Iota** looks exactly like an *i* that someone forgot to dot.

κ: **Kappa**, like its predecessor, is too much like its Latin equivalent to be of great use, although I once invented the κ-particle. It doesn't exist.

λ: **Lambda** is for the length of a wave. Big Λ is the lightest strange baryon.

μ: **Mu** stands for the muon, about which I. I. Rabi, a famous American physicist, asked "Who ordered that?!" half a century ago. We still don't know the answer.

ν: **Nu** can stand for any of the three known species of neutrinos: ν_e, ν_μ, and ν_τ. Moreover, μ's and ν's often appear as numerical subscripts, as in the basic equation of general relativity: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}$.

ξ: **Xi** is easy to say ("zy") but much too hard to pen in lower case. Capital Ξ stands for a baryon with two units of strangeness.

ο: **Omicron** looks, for all the world, like an *o*.

π: **Pi** denotes the ratio of the circumference of a circle to its diameter and it's an important meson to boot. The pion usually decays into a muon and a neutrino: $\pi \rightarrow \mu + \nu$.

ρ: **Rho** could be a density. The ρ-meson is made of the same quarks as the pion but with their spins lined up.

σ: **Sigma** signifies a particle's spin. Big Σ is a mathematical sum or any of the three strange baryons: Σ^+, Σ^0, and Σ^-.

τ: **Tau** is the heaviest of three known charged leptons, except when it's a fixed time interval (like a half-life).

υ: **Upsilon** denotes the particle made of a *b*-quark and its antiquark, discovered by Leon Lederman in 1977 after an embarrassing false start. The original was known as the "Oops, Leon!"

ϕ: **Phi** is an angle when you have more than one and you've already used θ—for example, $\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$, a memorable identity.

χ: **Chi** ("ky") is a conveniently uncommitted symbol often standing for what you just thought of.

ψ: **Psi** is preempted for quantum-mechanical wave functions, although it could be an angle when you've got more than two. The *J*-particle (first seen in New York) is the same as the Ψ-particle (spied at the same time in California). Today it's known as the *J*/Ψ, or "gypsy," particle.

ω: **Omega** is the overworked last letter of the Greek alphabet: ω is both a meson and a favorite for frequencies expressed in radians per second. Capitalized, Ω can be the ratio of the mean mass density of the universe to its critical value, or the triply strange baryon discovered by the Greek–American physicist Nicolas Samios. In fourteenth-century England omega was denoted by *oo*, as in: "I am alpha and oo, the bigynnyng and the endyng . . ." (John Wyclif, 1382).

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Classic writings from the history of science

"Knowledge of the classics of science themselves, and not merely their exposition in textbooks, is of great importance in forming one's scientific worldview."
—L. S. Landsberg, "On the occasion of the second edition of the Russian translation of Newton's Opticks*"*

by Yuli Danilov

IN ORDER TO CARRY OUT their own scientific research, modern scientists must be acquainted with information that has been accumulated by their predecessors. It's a lot to cope with, and scientists are sometimes tempted to ignore it. But in so doing they lose sight of many important things. For example, they may forget that their favorite science is an element of the much broader endeavor of human culture (some people talk about "science *and* culture," as if science isn't an integral part of culture).

In order to help young people to avoid this danger, we've decided to introduce a new department called "Anthology." In it you'll find authentic (albeit translated) excerpts from works by scientists of different centuries, countries, and viewpoints. To begin, we chose a fragment from an almost completely unknown work of the Greek historian Plutarch (ca. 45–ca. 127 A.D.). His best-known work is *Parallel Lives*, in which he recounts the noble deeds and analyzes the characters of Greek and Roman soldiers, legislators, orators, and statesmen.[1] All his other surviving compositions are collected under the title *Moralia*.

The selected work is written in the form of a conversation. Lucius and Lampria (especially Lampria, the narrator) express the author's point of view. Apollonides is a mathematician, Pharnaces represents the Stoic school of thought, and Theon is a grammarian—that is, a literary expert. The Aristotle in the dialogue, however, isn't the historical Aristotle (384–322 B.C.)—he is meant to represent the views of Aristotle and his followers.

You may find Plutarch's "science" strange reading (especially if you've already read the other physics articles in this issue.) You should bear in mind that in Plutarch's time there was no such thing as experimental or observational science. The Greeks believed that knowledge about nature could be gained by speculative thought alone. But we won't judge them too harshly because, after all, ancient Greece was the cradle of our civilization and, in particular, of our science.

The conversation begins in midstream because the opening has been lost. This excerpt constitutes about one sixth of the surviving manuscript.[2]

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[1]Plutarch presented the lives in pairs—one Greek, one Roman—chosen for the subjects' similarity of character or career. Each pair of biographies was followed by a formal comparison.

[2]The text that follows was translated from the Greek by Harold Cherniss (*Plutarch's Moralia*, vol. XII, Cambridge: Harvard University Press; London: William Heinemann Ltd, 1957). (Footnotes by Yuli Danilov and the editor.)

Concerning the face which appears in the orb of the moon

(Excerpt)

1. . . . These were Sulla's words. "For it concerns my story and that is its source; but I think that I should first like to learn whether there is any need to put back for a fresh start to those opinions concerning the face of the moon which are current and on the lips of everyone." "What else would you expect us to have done," I said, "since it was the difficulty in these opinions that drove us from our course upon those others? As people with chronic diseases when they have despaired of ordinary remedies and customary regimens turn to expiations and amulets and dreams, just so in obscure and perplexing speculations, when the ordinary and reputable and customary accounts are not persuasive, it is necessary to try those that are more out of the way and not scorn them but literally to chant over ourselves the charms of the ancients and use every means to bring the truth to test.

2. Well, to begin with, you see that it is absurd to call the figure seen in the moon an affection of vision in its feebleness giving way to brilliance, a condition which we call bedazzlement. Anyone who asserts this does not observe that this phenomenon should rather have occurred in relation to the sun, since the sun lights upon us keen and violent (as Empedocles[3] too somewhere not infelicitously renders the difference of the two:

> The sun keen-shafted and the gentle moon,

referring in this way to her allurement and cheerfulness and harmlessness), and moreover does not explain why dull and weak eyes discern no distinction of shape in the moon but her orb for them has an even and full light, whereas those of keen and robust vision make out more precisely and distinctly the pattern of facial features and more clearly perceive the variations. In fact the contrary, I think, should have been the case if the image resulted from an affection of the eye when it is overpowered: the weaker the subject affected, the clearer should be the appearance of the image. The unevenness also entirely refutes the hypothesis, for the shadow that one sees is not continuous and confused but is not badly depicted by the words of Agesianax:

> She gleams with fire encircled, but within
> Bluer than lapis show a maiden's eye
> And dainty brow, a visage manifest.

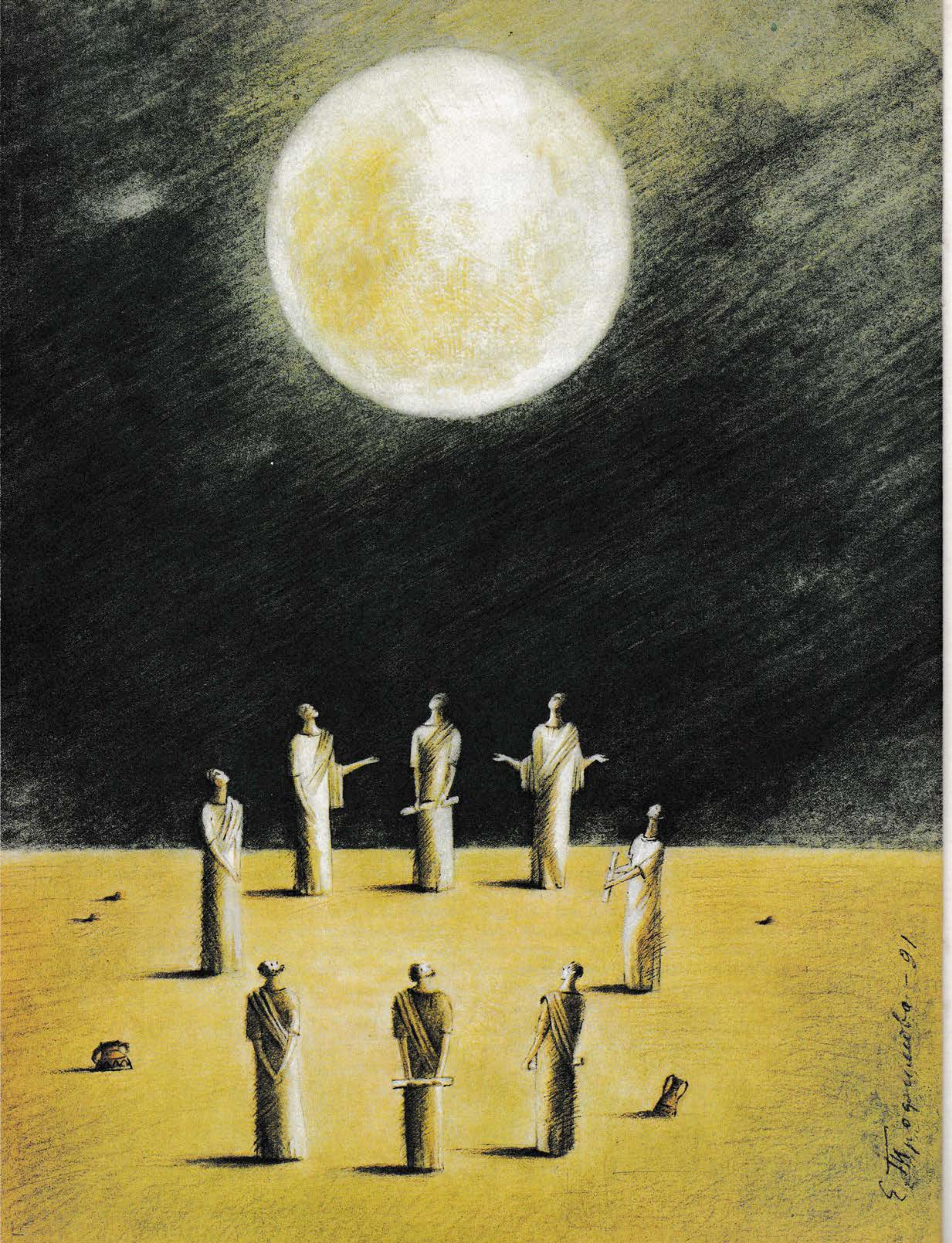
In truth, the dark patches submerge beneath the bright ones which they encompass and confine them, being confined and curtailed by them in turn; and they are thoroughly intertwined with each other so as to make the delineation of the figure resemble a painting. This, Aristotle, seemed to be a point not without cogency against your Clearchus also. For the man is yours, since he was an associate of the ancient Aristotle, although he did pervert many doctrines of the School."[4]

3. Apollonides broke in and inquired what the opinion of Clearchus was. "You are the last person," I said, "who has any right not to know a theory of which geometry is, as it were, the very hearth and home. The man, you see, asserts that what is called the face consists of mirrored likenesses, that is images of the great ocean reflected in the moon, for the visual ray when reflected naturally reaches from many points objects which are not directly visible and the full moon is itself in uniformity and lustre the finest and clearest of all mirrors. Just as you think, then, that the reflection of the visual ray to the sun accounts for the appearance of the rainbow in a cloud where the moisture has become somewhat smooth and condensed, so Clearchus thought that the outer ocean is seen in the moon, not in the place where it is but in the place whence the visual ray has been deflected to the ocean and the reflection of the ocean to us. So Agesianax again has somewhere said:

> Or swell of ocean surging opposite
> Be mirrored in a looking-glass of flame."

[3]Like Heraclitus before him, Empedocles (ca. 490–ca. 430 B.C.) thought that all matter is composed of four elements: fire, water, air, and earth. They are uncreated and eternal; they cannot change into one another but can only mix mechanically. According to Empedocles, all phenomena are governed by two forces: Love (which causes elements to mix) and Strife (which causes them to separate).

[4]That is, the "peripatetic school" of Aristotle. The name derives from the covered walk (*peripatos*) of the Lyceum in Athens, where Aristotle would stroll as he instructed his students. His adherents are often referred to collectively as the Peripatetics.



Art by Elena Trofimova

4. Apollonides was delighted. "What an original and absolutely novel contrivance the hypothesis is," he said, "the work of a man of daring and culture; but how did you proceed to bring your counter-argument against it?" "In the first place," I said, "in that, although the outer ocean is a single thing, a confluent and continuous sea, the dark spots in the moon do not appear as one but as having something like isthmuses between them, the brilliance dividing and delimiting the shadow. Hence, since each part is separated and has its own boundary, the layers of light upon shadow, assuming the semblance of height and depth, have produced a very close likeness of eyes and lips. Therefore, one must assume the existence of several outer oceans separated by isthmuses and mainlands, which is absurd and false; or, if the ocean is single, it is not plausible that its reflected image be thus discontinuous. Tell me whether—for in your presence it is safer to put this as a question than as an assertion—whether it is possible, though the inhabited world has length and breadth, that every visual ray when reflected from the moon should in like manner reach the ocean, even the visual rays of those who are sailing in the great ocean itself, yes and who dwell in it as the Britons do, and that too even though the earth, as you say, does not have the relation of centre to the orbit of the moon. Well, this," I said, "it is your business to consider; but the reflection of vision either in respect to the moon or in general is beyond your province and that of Hipparchus[5] too. Although Hipparchus was industrious, still many find him unsatisfactory in his explanation of the nature of vision itself, which is more likely to involve a sympathetic compound and fusion than any impacts and rebounds such as those of the atoms that Epicurus[6] invented. Moreover, Clearchus, I think, would refuse to assume with us that the moon is a body of weight and solidity instead of an ethereal and luminiferous star as you say; and such a moon ought to shatter and divert the visual ray so that reflection would be out of the question. But if anyone dismisses our objections, we shall ask how it is that the reflection of the ocean exists as a face only in the moon and is seen in none of all the many other stars, although reason requires that all or none of them should affect the visual ray in this fashion. But let us have done with this; and do you," I said with a glance at Lucius, "recall to me what part of our position was stated first."

They blacken the Moon's eye defiling her with blemishes and bruises, at one and the same time addressing her as Artemis and Athena.

5. Whereat Lucius said: "Nay, lest we give the impression of flatly insulting Pharnaces by thus passing over the Stoic[7] opinion unnoticed, do now by all means address some remark to the gentleman who, supposing the moon to be a mixture of air and gentle fire, then says that what appears to be a figure is the result of the blackening of the air as when in a calm water there runs a ripple under the surface." "You are very nice, Lucius," I said, "to dress up the absurdity in respectable language. Not so our comrade; but he said what is true, that they blacken the Moon's eye defiling her with blemishes and bruises, at one and the same time addressing her as Artemis and Athena and making her a mass compounded of murky air and smouldering fire neither kindling nor shining of herself, an indiscriminate kind of body, forever charred and smoking like the thunderbolts that are darkling and by the poets called lurid. Yet a smouldering fire, such as they suppose that of the moon to be, cannot persist or subsist at all unless it get solid fuel that shelters and at the same time nourishes it; this some philosophers, I believe, see less clearly than do those who say in jest that Hephaestus[8] is said to be lame because fire without wood, like the lame without a stick, makes no progress. If the moon really is fire, whence came so much air in it? For the region that we see revolving above us is the place not of air but of a superior substance, the nature of which is to rarefy all things and set them afire; and, if air did come to be there, why has it not been etherealized by the fire and in this transformation disappeared but instead has been preserved as a housemate of fire this long time, as if nails had fixed it forever to the same spots and riveted it together? Air is tenuous and without configuration, and so it naturally slips and does not stay in

[5]Hipparchus (2nd century B.C.) was the most influential astronomer of antiquity and an able mathematician. He is best known for his discovery of the precessional movement of the equinox—that is, the alterations of the measured positions of the stars resulting from the movement of the points of intersection of the ecliptic and of the celestial equator. This notable achievement was based on painstaking observations (in contrast to the speculative approach noted in the introduction).

[6]The "atomism" of Epicurus (341–270 B.C.) formed the basis of a philosophical system that was primarily concerned with ethical questions. He distinguished three forms of motion in atoms: a natural one of falling straight down, owing to their weight; a forced one due to impacts; and a free motion of declination, or swerving from a straight line.

[7]Stoicism is a system of philosophy that has had a profound effect on thinkers from antiquity to the present. The name derives from the place where its founder, Zeno of Citium (ca. 335–ca. 263 B.C.), customarily lectured—the Stoa Poikile (Painted Colonnade). Its physics had aspects in common with the theories of Empedocles (the same four elements, for example). Among other things, the Stoics thought that the world (cosmos) is a spherical animate body situated in an endless void, and that there are only two principles—one passive, the other active—underlying all phenomena.

[8]The Greek god of fire and metalworking ("Vulcan" in the Roman pantheon).

Art by Elena Trofimova

place; and it cannot have become solidified if it is commingled with fire and partakes neither of moisture nor of earth by which alone air can be solidified. Moreover, velocity ignites the air in stones and in cold lead, not to speak of the air enclosed in fire that is whirling about with such great speed. Why, they are vexed by Empedocles because he represents the moon to be a hail-like congelation of air encompassed by the sphere of fire; but they themselves say that the moon is a sphere of fire containing air dispersed about it here and there, and a sphere moreover that has neither clefts nor depths and hollows, such as are allowed by those who make it an earthy body, but has the air evidently resting upon its convex surface. That it should so remain is both contrary to reason and impossible to square with what is observed when the moon is full. On that assumption there should have been no distinction of dark and shadowy air; but all the air should become dark when occulted, or when the moon is caught by the sun it should all shine out with an even light. For with us too, while the air in the depths and hollows of the earth, wherever the sun's rays do not penetrate, remains shadowy and unlit, that which suffuses the earth outside takes on brilliance and a luminous colour. The reason is that air, because of its subtility, is delicately attuned to every quality and influence; and, especially if it touches light or, to use your phrase, merely is tangent to it, it is altered through and through and entirely illuminated. So this same point seems right handsomely to re-enforce those who pack the air on the moon into depths of some kind and chasms, even as it utterly refutes you who make her globe an unintelligible mixture or compound of air and fire—for it is not possible that a shadow remain upon the surface when the sun casts his light upon all of the moon that is within the compass of our vision."

The moon is saved from falling by its very motion and the rapidity of its revolution, just as missiles placed in slings are kept from falling by being whirled around in a circle.

6. Even while I was still speaking Pharnaces spoke: "Here we are faced again with that stock maneuvre of the Academy:[9] on each occasion that they engage in discourse with others they will not offer any accounting of their own assertions but must keep their interlocutors on the defensive lest they become the prosecutors. Well, me you will not to-day entice into defending the Stoics against your charges until I have called you people to account for turning the universe upside down." Thereupon Lucius laughed and said: "Oh, sir, just don't bring suit against us for impiety as Cleanthes thought that the Greeks ought to lay an action for impiety against Aristarchus the Samian on the ground that he was disturbing the hearth of the universe because he sought to save the phenomena by assuming that the heaven is at rest while the earth is revolving along the ecliptic and at the same time is rotating about its own axis. We express no opinion of our own now; but those who suppose that the moon is earth, why do they, my dear sir, turn things upside down any more than you do who station the earth here suspended in the air? Yet the earth is a great deal larger than the moon according to the mathematicians who during the occurrences of eclipses and the transits of the moon through the shadow calculate her magnitude by the length of time that she is obscured. For the shadow of the earth grows smaller the further it extends, because the body that casts the light is larger than the earth; and that the upper part of the shadow itself is taper and narrow was recognized, as they say, even by Homer, who called night 'nimble' because of the 'sharpness' of the shadow. Yet captured by this part in eclipses the moon barely escapes from it in a space thrice her own magnitude. Consider then how many times as large as the moon the earth is, if the earth casts a shadow which at its narrowest is thrice as broad as the moon. All the same, you fear for the moon lest it fall; whereas concerning the earth perhaps Aeschylus has persuaded you that Atlas

> Stands, staying on his back the prop of earth
> And sky, no tender burden to embrace.

Or, while under the moon there stretches air unsubstantial and incapable of supporting a solid mass, the earth, as Pindar says, is encompassed by 'steel-shod pillars'; and therefore Pharnaces is himself without any fear that the earth may fall but is sorry for the Ethiopians or Taprobanians, who are situated under the circuit of the moon, lest such a great weight fall upon them. Yet the moon is saved from falling by its very motion and the rapidity of its revolution, just as missiles placed in slings are kept from falling by being whirled around in a circle. For each thing is governed by its natural motion unless it be diverted by something else. That is why the moon is not governed by its weight: the weight has its influence frustrated by the rotatory motion. Nay, there would be more reason perhaps to wonder if she were absolutely unmoved and stationary like the earth. As it is, while the moon has good cause for not moving in this direction, the influence of weight alone might reasonably move the earth, since it has no part in any other motion; and the earth is heavier than the moon not merely in proportion to its greater size but still more, inasmuch as the moon

[9] A reference to followers of Plato (ca. 428–348/347 B.C.), who founded the Academy in Athens. In many of Plato's dialogues, his teacher Socrates asks most of the questions and so controls course of the argument, a trait apparently retained by his followers.

has, of course, become light through the action of heat and fire. In short, your own statements seem to make the moon, if it is fire, stand in greater need of earth, that is of matter to serve it as a foundation, as something to which to adhere, as something to lend it coherence, and as something that can be ignited by it, for it is impossible to imagine fire being maintained without fuel, but you people say that earth does abide without root or foundation." "Certainly it does," said Pharnaces, "in occupying the proper and natural place that belongs to it, the middle, for this is the place about which all weights in their natural inclination press against one another and towards which they move and converge from every direction, whereas all the upper space, even if it receive something earthy which has been forcibly hurled up into it, straightway extrudes it into our region or rather lets it go where its proper inclination causes it naturally to descend."

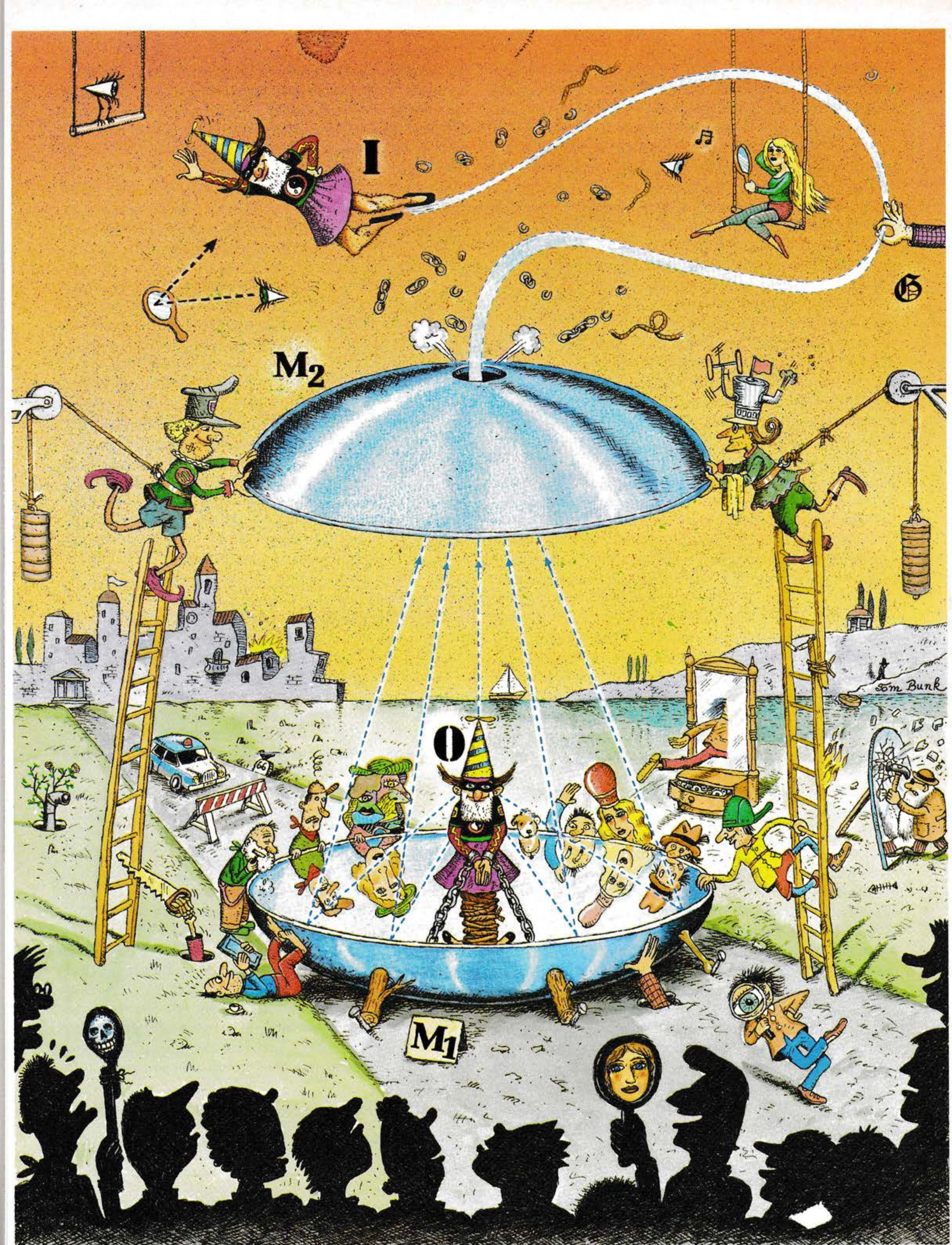
What paradox is not involved in this doctrine? Not that people live on the opposite hemisphere clinging to the earth like wood-worms or geckos turned bottomside up?

7. At this—for I wished Lucius to have time to collect his thoughts—I called to Theon. "Which of the tragic poets was it, Theon," I asked, "who said that physicians

> With bitter drugs the bitter bile purge?"

Theon replied that it was Sophocles. "Yes," I said, "and we have of necessity to allow them this procedure; but to philosophers one should not listen if they desire to repulse paradoxes with paradoxes and in struggling against opinions that are amazing fabricate others that are more amazing and outlandish, as these people do in introducing their 'motion to the centre.' What paradox is not involved in this doctrine? Not the one that the earth is a sphere although it contains such great depths and heights and irregularities? Not that people live on the opposite hemisphere clinging to the earth like wood-worms or geckos turned bottomside up?—and that we ourselves in standing remain not at right angles to the earth but at an oblique angle, leaning from the perpendicular like drunken men? Not that incandescent masses of forty tons falling through the depth of the earth stop when they arrive at the centre, though nothing encounter or support them; and, if in their downward motion the impetus should carry them past the centre, they swing back again and return of themselves? Not that pieces of meteors burnt out on either side of the earth do not move downwards continually but falling upon the surface of the earth force their way into it from the outside and conceal themselves about the centre? Not that a turbulent stream of water, if in flowing downwards it should reach the middle point, which they themselves call incorporeal, stops suspended or moves round about it, oscillating in an incessant and perpetual see-saw? Some of these a man could not even mistakenly force himself to conceive as possible. For this amounts to 'upside down' and 'all things topsy-turvy,' everything as far as the centre being 'down' and everything under the centre in turn being 'up.' The result is that, if a man should so coalesce with the earth that its centre is at his navel, the same person at the same time has his head up and his feet up too. Moreover, if he dig through the further side, his bottom in emerging is up, and the man digging himself 'up' is pulling himself 'down' from 'above'; and, if someone should then be imagined to have gone in the opposite direction to this man, the feet of both of them at the same time turn out to be 'up' and are so called.

8. Nevertheless, though of tall tales of such a kind and number they have shouldered and lugged in—not a wallet-full, by heaven, but some juggler's pack and hotchpotch, still they say that others are playing the buffoon by placing the moon, though it is earth, on high and not where the centre is. . . ." ◘



Art by Tomas Bunk

PHYSICS CONTEST

The clamshell mirrors

"Mirror facing facing mirror
Doubles—exquisite effect;
Between them in the shadow stands a
Crystal cube which will reflect."
—Johann Wolfgang von Goethe

By Larry D. Kirkpatrick and Arthur Eisenkraft

WHAT IS IT THAT CAUSES our fascination of worlds within worlds? A picture in a picture in a picture absorbs us as we wonder what the different worlds are all about. Readers are encouraged to spend a delightful afternoon looking at the drawings of M. C. Escher or reading Lewis Carroll's *Alice's Adventures in Wonderland* to see how two people were also intrigued by these ideas.

A simpler set of worlds occurs with two plane mirrors. Can you remember your last visit to the beauty salon or barber shop, where you observed the infinite number of images of yourself formed by mirrors facing each other? Each mirror produced images of everything in front of it—including the images formed by the other mirror. In this case the images were all of the same size. They just appeared to get smaller and smaller because they were farther and farther away from you. What happened when you moved between the mirrors? How did the images move? How does the number of images vary with the angle between the mirrors? What does the speed of light have to do with how the images move when you raise your hand?

Although you may have less experience with curved mirrors, you probably expect that similar things would happen. A very popular physics toy consists of two concave spherical mirrors facing each other like the shells of a clam, as shown in the photograph. If a coin or button is placed on the surface of the lower mirror, its image appears in the hole at the center of the top mirror. The image looks so real that people will try to push the button or pick up the coin. It's fun to watch their faces as they discover that there is actually nothing there!

The separation of the mirrors in this toy has been carefully chosen so that the real image appears in the hole as if it were sitting on a clear portion of the upper mirror. The image is the same size as the object.

A. Begin by finding the separation of the mirrors that corresponds to the photograph. Draw a ray diagram to show how the light reflects from each surface to form the image.

B. If you haven't already discovered a second separation that also produces a real image in the hole with unit magnification, find it and draw the corresponding ray diagram. Are there any differences in the images produced in the two cases?

C. In fact, there are an infinite number of solutions. Our more advanced readers may want to find a few of them to discover the general method for obtaining additional solu-

Photo courtesy of Larry Kirkpatrick

Art by Tomas Bunk

tions. As this can be done in at least two different ways, this problem should give you a challenge and, we hope, much enjoyment.

Please send your solutions to *Quantum*, 3140 North Washington Boulevard, Arlington, VA 22201 within a month of receipt of this issue. The best solutions will be acknowledged in *Quantum* and their creators will receive free subscriptions for one year.

A snail that moves like light

In the September/October issue of *Quantum*, readers were asked to help a snail find the quickest path from one corner of a room to a diagonally opposite corner.

In the first case, in which all walls were identical and the dimensions of the room were $5 \times 10 \times 15$, there are at least three ways to solve the problem. The first is to choose different crossover points at the edge between the two walls and calculate the total distance that the snail travels. This numerical method may appear to be tedious, but it will actually converge on the correct solution quickly. A second method is to call the height of the crossover point x, write the total distance traveled in terms of x, and differentiate. By setting the derivative equal to zero, the minimum distance will be revealed as the solution to the equation. The third method is the elegant solution. In this case, the wall is opened up. The room is now a large rectangle of dimensions 25×5. The shortest distance will be the diagonal connecting the two corners of the rectangle. If the snail starts at the lower corner of the 15-meter wall, the crossover point can be found by using similar triangles. The crossover point is

$$\frac{x}{15} = \frac{5-x}{10},$$
$$x = 3.$$

In the second case, one of the walls was declared "sticky," meaning that the snail could travel at only 1/3 of its speed on this wall. Unlike the first case, the shortest distance is no longer the shortest time! Since the snail travels at different speeds on the two walls, the quickest path will be the one where the snail travels a greater distance on the faster wall. Once again, the straightforward but tedious solution would be to assign the variable x to the crossover point, write an equation that describes all paths in terms of x, and the minimum time will be revealed.

The more elegant solution in this case is to realize that light always takes the least time to travel, and that this snail traveling on a sticky wall is like light traveling in a slower medium. We then recognize that the solution will be Snell's law (or, if you'll forgive us, "Snail's law"). Even with this knowledge, we are faced with a fourth-order equation, which we choose to solve by numerical techniques:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2.$$

Since the stickiness factor is 3, then $n_1 = 3$ and $n_2 = 1$, and it follows that

$$3\frac{x}{\sqrt{15^2 + x^2}} = \frac{5-x}{\sqrt{(5-x)^2 + 10^2}}.$$

We'll try different values of x and see if the value of the left side of the equation is equal to the value of the right side.

x	left side	right side
1	0.1996	0.3714
2	0.3965	0.2873
1.5	0.2985	0.3304
1.7	0.3378	0.3134
1.6	0.3182	0.3219
1.63	0.3241	0.3194
1.62	0.3221	0.3202

This method can give us any accuracy we desire. It would certainly be easier to plug the equations into a spreadsheet program and have all values given "instantly."

The third part of the problem, to solve for a wall whose stickiness varies along one dimension, was solved by Jason Jacobs of Harvard University. We will leave this problem as a tease for others for the time being.

Short take

In *A Brief History of Time*, the eminent physicist Stephen Hawking tells how the irrepressible George Gamow managed to add a twist to an important paper on the origin of the universe. (You may remember Gamow from the article "Physics for Fools" in the November/December 1990 issue of *Quantum*.)

It seems Gamow persuaded the great Hans Bethe (pronounced "BAY-tuh") to sign on to an article written by Gamow and a graduate student by the name of Ralph Alpher. The result was a treatise authored by Alpher, Bethe, and Gamow.

These physicists sure do love to play with the Greek alphabet!

East and west of Pythagoras by 30°

Where some neglected relatives of Pythagorean triples reside

by George Berzsenyi

THE PURPOSE of this column is to recommend to my readers the investigation of triangles with positive integer sides a, b, c and with one angle measuring 60° or 120°. The resulting triples (a, b, c) share many of the interesting properties of the well-known Pythagorean triples,[1] but have been greatly neglected throughout the history of mathematics. We'll refer to them as Pre- and Post-Pythagorean triples, respectively. For definiteness, we'll assume that c is the length of the side opposite the 60- or 120-degree angle of the triangle and call it the hypotenuse. We'll also refer to a and b as the legs of the triangle. And we'll say that (a, b, c) is primitive if the greatest common divisor of a, b, and c is 1.

My first challenge to you is to show that these triples must satisfy the equations

$$a^2 + b^2 - ab = c^2$$

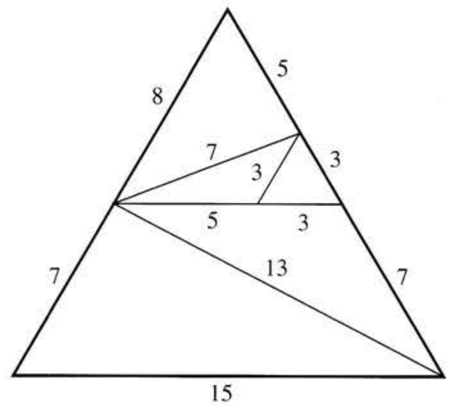
and

$$a^2 + b^2 + ab = c^2,$$

respectively. Our second challenge is to solve these Diophantine equations and to generate your own extensive tables of primitive Pre- and Post-Pythagorean triples. And then you are invited to widen the search for interrelationships, curiosities, and other properties analogous to those of the Pythagorean triples. For ideas, you might want to consult Waclaw Sierpinski's beautiful *Pythagorean Triangles*, published by Yeshiva University (New York) in 1962. Also, most books on elementary number theory devote at least a brief section to the interesting properties of such triples.

To whet your appetite, this article is illustrated with a couple of suggestive drawings; for one of these, the author is indebted to Dr. Paul Mielke, professor emeritus at neighboring Wabash College. Listed below are some facts and curiosities to further challenge interested readers. Valuable books and/or free subscriptions to *Quantum* will be distributed among those who respond to these queries. Write to me at the address below or c/o *Quantum*, 3140 Washington Boulevard, Arlington, VA 22201.

1. Show that if (a, b, c) is Post-Pythagorean, then $(a, a + b, c)$ is Pre-Pythagorean.



2. Prove that there are infinitely many Post-Pythagorean triples whose legs differ by 1.

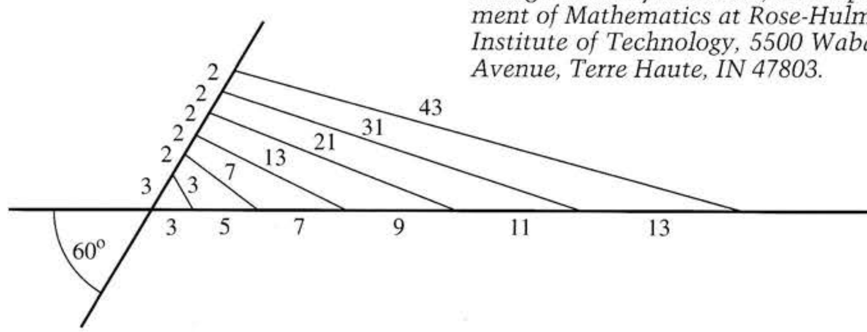
3. Prove that if one of the legs of a Pre-Pythagorean triple is even, then it is a multiple of 8.

4. Prove that if (a, b, c) is a primitive Pre-Pythagorean triple, then c is the product of primes of the form $6k + 1$.

5. Show that if (a, b, c) is Post-Pythagorean, then $abc(a + b)$ is a multiple of 840.

There are many more properties of these poorer relatives of the Pythagorean triples; let's declare open season on them.

George Berzsenyi *is head of the Department of Mathematics at Rose-Hulman Institute of Technology, 5500 Wabash Avenue, Terre Haute, IN 47803.*



[1] See "Genealogical Threes" in the November/December 1990 issue of *Quantum*, as well as Dr. Berzsenyi's article "Shapes and Sizes" in the same issue.—*Ed.*

AT THE
BLACKBOARD

Direct current events

An overview of the laws governing DC machines

by I. Slobodetsky

IN THIS ARTICLE WE WON'T probe the design of different electric machines. We'll be looking at the electric motor and the dynamo (the old name for an induction generator of electric current) using two fundamental laws: Ohm's law and the energy conservation law.

The article basically consists of questions and answers. While you're working on the article, try to answer each question on your own before looking at the answer. Even if you don't succeed, thinking about the question is certainly beneficial, and you'll remember the correct answer better as a result.

Electric motors

All electric motors have the same basic components. They all have a stationary magnet, called the stator, and a rotating electromagnet, called the armature. The stator is the source of a magnetic field. When electric current is running through the winding of an armature that has been placed in the magnetic field, the armature starts to rotate. In this way electrical energy is converted to mechanical energy. If, for example, the shaft of the motor is connected to a lathe, the lathe can be set in motion.

Question. Consider a direct-current motor with independent induction (its stator is either a permanent magnet or an electromagnet whose winding is fed independently of the armature's winding). How is Ohm's law written for the circuit of the armature of such a motor, connected to a DC source?

Answer. There are two electromotive forces (emf) in the armature's electrical circuit: the voltage of the current source V and the induced emf ε_i, which appears in the winding of the armature as it's rotating in the stator's magnetic field. The induced emf, according to Lenz's rule, is opposite in sign from the voltage of the current source. So Ohm's law is written

$$V - \varepsilon_i = IR, \qquad \text{(I)}$$

where I is the current in the circuit and R is the total resistance of the armature winding and the wires supplying the current.

Question. How is the conservation law written for this circuit?

Answer. The energy $W_s = VIt$ supplied by the current source is expended on heat emitted by the resistance R and on performing useful work A. So according to the energy conservation law,

$$VIt = I^2Rt + A,$$

or for power,

$$VI = I^2R + P. \qquad \text{(II)}$$

Many useful consequences can be inferred from equations (I) and (II).

Question. What is the useful power P equal to?

Answer. Multiplying both sides of equation (I) by the current I, we get

$$VI - \varepsilon_i I = I^2R,$$

Figure 1

Figure 2

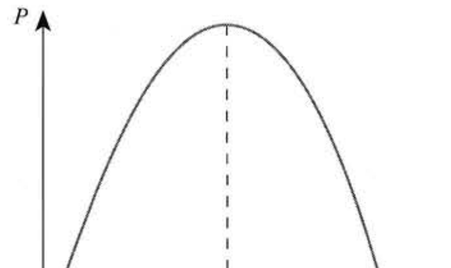


Figure 3

or

$$VI = I^2R + \varepsilon_i I.$$

Comparing this equation with equation (II), you notice immediately that

$$P = \varepsilon_i I$$

—that is, the motor's useful power equals the product of the induced emf and the current.

When V and R are held constant, the magnitude of the current depends on the induced emf:

$$I = \frac{V - \varepsilon_i}{R}.$$

It follows from the law of electromagnetic induction that the induced emf arising in the armature winding is proportional to the rate of change of the magnetic flux through this winding:

$$\varepsilon_i \sim \Delta\Phi/\Delta t.$$

The rate of change of the magnetic flux is in turn proportional to the armature's angular velocity of rotation ω. Thus

$$\varepsilon_i \sim \omega,$$
$$\varepsilon_i = k\omega,$$

where k is a proportionality coefficient. The graph of the dependence of ε_i on ω is shown in figure 1.

Now let's look at how the current in the armature circuit and motor's useful power depend on ω. From Ohm's law,

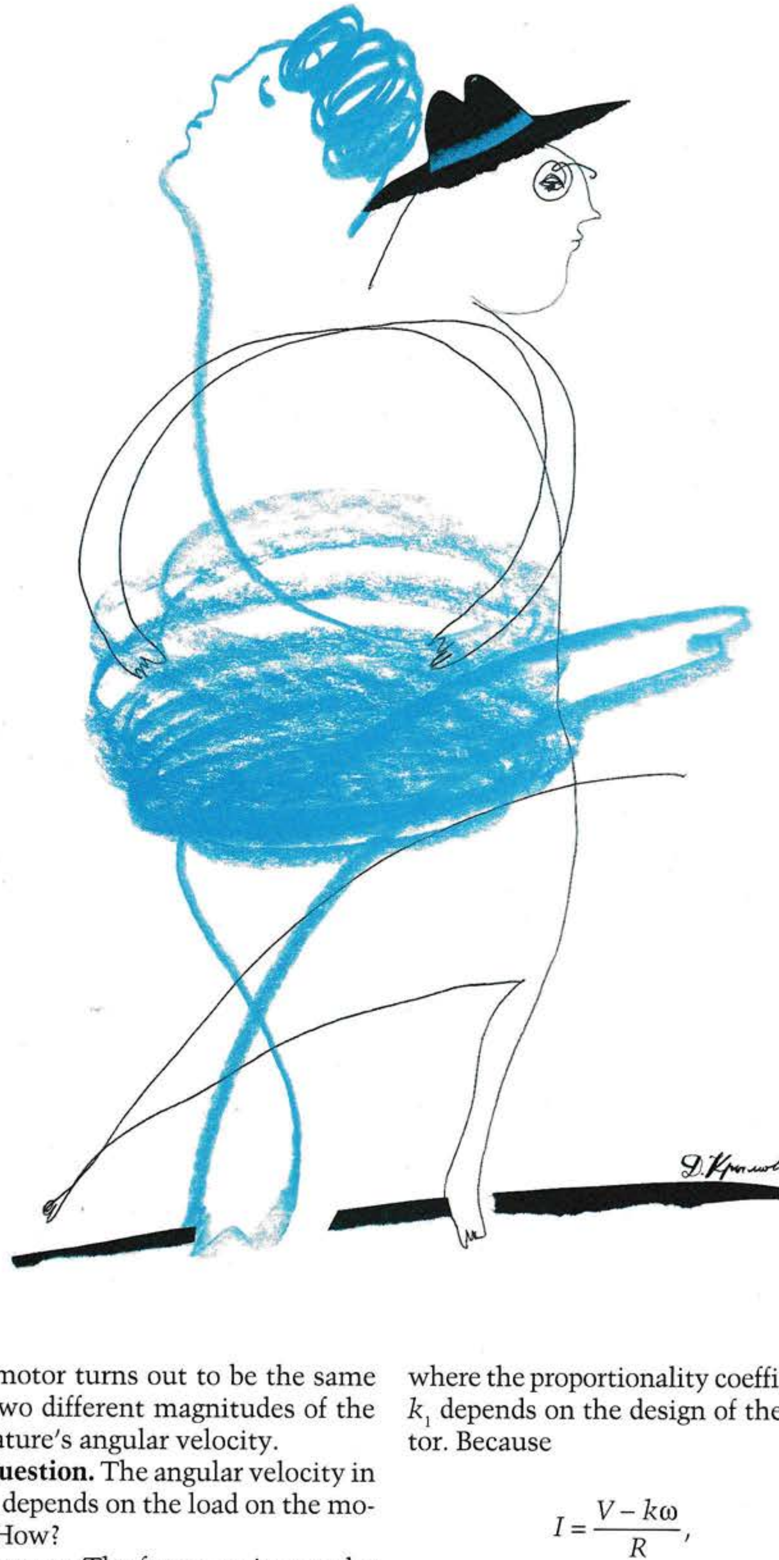
$$I = \frac{V - \varepsilon_i}{R} = \frac{V - k\omega}{R},$$

and from the energy conservation law,

$$\begin{aligned} P &= VI - I^2R \\ &= \varepsilon_i I \\ &= \frac{kV\omega - (k\omega)^2}{R}. \end{aligned}$$

The graphs of the dependence of I and P on ω are given in figures 2 and 3.

So there are certain values of induced emf ε_i and current I that correspond to every value of angular velocity ω. If ω increases, ε_i increases and I decreases. Also, the useful power of



Art by Dmitry Krymov

the motor turns out to be the same for two different magnitudes of the armature's angular velocity.

Question. The angular velocity in turn depends on the load on the motor. How?

Answer. The forces acting on the armature in the stator's magnetic field are proportional to the current through the winding. The torques of these forces are also proportional to the current—that is,

$$M_e = k_1 I$$

where the proportionality coefficient k_1 depends on the design of the motor. Because

$$I = \frac{V - k\omega}{R},$$

we have

$$M_e = k_1 \frac{V - k\omega}{R}$$

—that is, the total torque of the electrical forces acting on the armature depends linearly on the armature's angular velocity of rotation. The torque is high when the angular velocity is low; as the velocity increases, the torque decreases and reaches zero when

$$\omega = \frac{V}{k}.$$

On the other hand, the torque of mechanical forces (the "load torque" M) acts on the motor's shaft. For example, if the motor is smoothly lifting a load on a rope, then the load torque M is equal to the product of the rope's tension (which is equal to the weight of the load) and the radius of the shaft. In the running mode the armature obviously rotates at a velocity ω_0 at which the torque of electric forces equals the load torque. Indeed, let the armature's velocity of rotation be less than ω_0. In this case, the torque of electrical forces would be greater than the load torque, so that the armature's angular velocity of rotation would increase. If the angular velocity is greater than ω_0, the torque of electrical forces would be less than the load torque. This would lead to a reduction in the armature's velocity of rotation until it reaches ω_0.
In this way a velocity of rotation is reached at which

$$M_e = M,$$

or

$$k_1 \frac{V - k\omega_0}{R} = M,$$

from which we get

$$\omega_0 = \frac{k_1 V - MR}{k_1 k}$$

—that is, the armature's angular velocity of rotation depends linearly on the load torque.

So the load determines the current in the armature winding and the angular velocity of its rotation. Let's look at the following problem.

Problem 1. An electric motor is connected to a DC source. At a rate of rotation n_1 = 1,000 rpm the current in the armature circuit equals I_1 = 10 A, while at a rate of rotation n_2 = 900 rpm it equals I_2 = 15 A. Find the rate of rotation when the motor is idling (with no load).

Because $I = (V - k\omega)/r$ and $\omega = 2\pi n$, we have

$$I_1 = \frac{V - 2\pi k n_1}{R}$$

and

$$I_2 = \frac{V - 2\pi k n_2}{R}.$$

In the idling mode $I_3 = 0$, so

$$V - 2\pi k n_3 = 0.$$

Solving these equations together, we get

$$\begin{aligned} n_3 &= \frac{I_2 n_1 - I_1 n_2}{I_2 - I_1} \\ &= \frac{15 \cdot 1{,}000 - 10 \cdot 900}{5} \text{ rpm} \\ &= 1{,}200 \text{ rpm}. \end{aligned}$$

Reversibility of electric machines

All DC machines are reversible. If we supply the machine's leads with voltage the armature will rotate, performing mechanical work—that is, we'll get an electric motor. On the other hand, if we rotate the armature with the help of another machine, then the electric machine will work as a generator (dynamo), producing electric current.

Question. Suppose one and the same machine rotates at the same velocity, working first as an electric motor and then as a dynamo. What can be said about the induced emf in the armature in both cases?

Answer. Because the induced emf depends only on the design of the armature and its angular velocity of rotation, the induced emf is the same in both cases.

Question. And what if the angular velocities are different?

Answer. Then the ratio of the induced electromotive forces equals the ratio of the armature's angular velocities of rotation.

Question. We've talked about Ohm's law and the energy conservation law already, and we've written the corresponding equations (I) and (II). How will the analogous equations look for a generator?

Answer. Let the resistance of the circuit be R. Then according to Ohm's law

$$\varepsilon_g = I_g R,$$

where ε_g is the electromotive force of the generator and I_g is the current. If we ignore energy losses (from friction, for example) we can write the energy conservation law as follows:

$$P_m = \varepsilon_g I_g,$$

where P_m is the mechanical power expended to turn the armature and $\varepsilon_g I_g$ is the electrical power produced by the generator.

Now let's solve a few problems.

Problem 2. A DC motor with independent induction connected to a battery with a voltage V = 24 V rotates at a rate of n_1 = 600 rpm. The current in the circuit I = 0.2 A and the total resistance of the circuit is R = 20 ohms. What emf will the motor produce, operating as a generator, at n_2 = 1,400 rpm?

The emf produced by the generator equals

$$\varepsilon_g = \frac{n_2}{n_1}\varepsilon_i,$$

where ε_i is the induced emf that arises in the armature winding when the rate of rotation is n_1 = 600 rpm. This can be found from Ohm's law:

$$\varepsilon_i = V - IR.$$

So

$$\begin{aligned} \varepsilon_g &= \frac{n_2}{n_1}(V - IR) \\ &= \frac{1{,}400}{600}(24 - 0.2 \cdot 20) \text{ volts} \\ &= 46.7 \text{ volts}. \end{aligned}$$

Problem 3. A load of mass m is suspended from a string wound on the axle of a generator with independent excitation (for example, having a permanent magnet as the stator). The string is unwound from the axle so that the load moves down at a constant speed v_1. The generator is shorted at a resistance R. At what speed v_2 will the same load move up if the generator were used as a motor connected to a DC source with a voltage V and the resistance R is the same?

The power of a motor lifting a load of mass m at a speed v_2 equals mgv_2. It follows from Ohm's law and the energy conservation law that

$$V - \varepsilon_i = IR \tag{1}$$

and

$$VI = I^2R + mgv_2, \tag{2}$$

where I is the current and ε_i is the induced emf in the armature winding. This emf is linked to the emf ε_g of the generator through the relationship

$$\frac{\varepsilon_i}{\varepsilon_g} = \frac{v_2}{v_1}. \tag{3}$$

Now let's write Ohm's law and the energy conservation law for the generator:

$$\varepsilon_g = I_g R$$

and

$$\varepsilon_g I_g = mgv_1.$$

From this, we get

$$\varepsilon_g = \sqrt{mgv_1R},$$

and from correlation (3) we get

$$\varepsilon_i = \frac{v_2}{v_1}\varepsilon_g = \frac{v_2}{v_1}\sqrt{mgv_1R}.$$

Using the expression for ε_i from equation (1) and solving equations (1) and (2) together, we get

$$v_2 = \frac{\sqrt{mgv_1R}V - mgv_1R}{mgR} = V\sqrt{\frac{v_1}{mgR}} - v_1.$$

Exercises

1. An electric motor with no load rotates at n_1 = 1,000 rpm; with a load, at n_2 = 700 rpm. What would be rate of rotation be if the load torque is increased by 20%?

2. An electric motor is connected to a current source with a voltage V = 500 V. When the current is I_1 = 10 A, the motor has a power P_1 = 4 kW. Find its power when the current is I_2 = 20 A (the current changes because of a change in the load).

3. The angular velocity of rotation of the armature of a generator with a permanent magnet as its stator has increased by 10%. How much has the useful power of the generator increased?

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HAPPENINGS

Tartu in the summer of '91

American and Soviet students share math (and more) as the world evolves rapidly around them

by Mark Saul

AMERICANS AND RUSSIANS, former opponents in the Cold War, are taking rapid steps toward each other these days. In the area of mathematics, one giant step was taken by a group of 25 American high school students last summer, in the ARML–Soviet Exchange Program. Accompanied by five teachers, these students traveled to Tartu, Estonia (then in the Soviet Union), for two weeks of study in their favorite subject with Soviet students. The American students came from all over the nation and represented diverse segments of the US population. The Soviet students came mostly from Moscow and St. Petersburg (Leningrad), with some participants from Kiev and Odessa in Ukraine and from Nizhni Tagil in the Ural mountains.

The summer program was sponsored by the National Science Foundation in Washington and the Institute for New Technologies in Moscow. It was run by the American Regions Mathematics League (ARML) and by Symposium Conferences of Moscow. Texas Instrument Corporation donated 80 TI-81 graphing calculators for Soviet students to try out.

The students quickly made friends, both among other American students and with Soviet participants. They enjoyed comparing tastes in clothes, in food, in mathematical styles. The American students found it strange, for instance, that if they went shopping they had to bring their own bag for purchases. On the other hand, the Soviet students found it curious that Americans would buy stationery, mousetraps, and pickles—items familiar to Soviet students, but which the Americans found exotic.

The teachers studied the differences in traditions of working with students of high mathematical ability. The American penchant for short-answer questions, for instance, contrasted sharply with the Soviet tradition of essay-type questions demanding subtle proof or deeper analysis.

On a larger scale, some of the realities of Soviet life just before last August's coup were immediately clear. The free peasant markets in Estonia supplied food readily, while Moscow was plagued with long lines. The Americans found prices in rubles extremely reasonable—when goods were available. They found the Estonians very friendly but very anxious not to be identified as Russian. The tensions in Baltic and Soviet life were obvious as students explored beaches and historical monuments throughout southern Estonia. The American participants got a long and close glimpse of a world that is vanishing

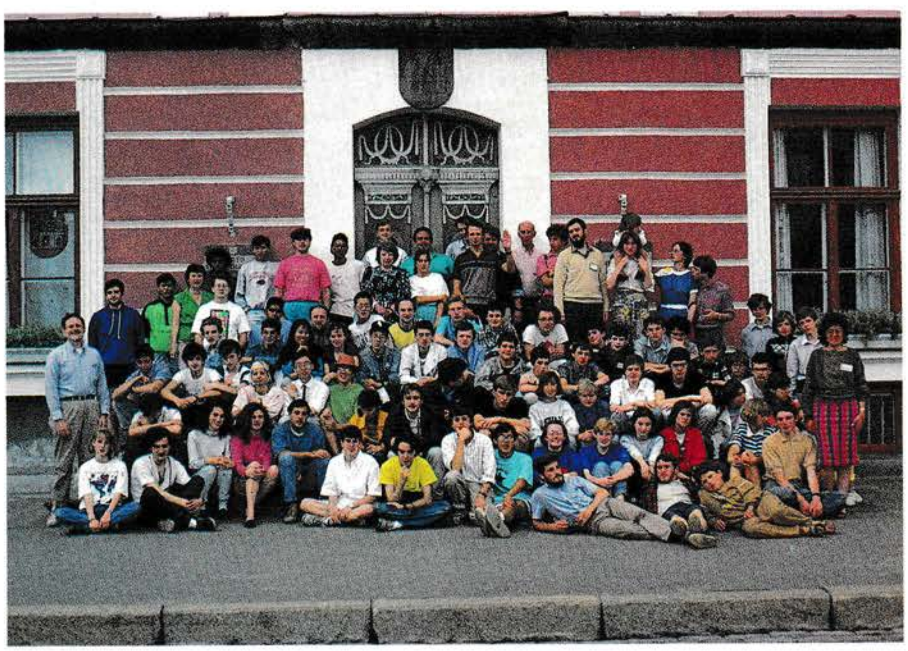

Photo courtesy of Anoop Sinha

ARML–Soviet Exchange Program participants in front of the Tartu City Hall.

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J015

within a matter of months after the program ended.

Most important, students found that they could read through the differences in language, in taste, in economic styles, even in mathematical styles, to get to the heart of the matter: enjoyment of the beauties of mathematics as we explore them by ourselves.

Below are some of the most beautiful and interesting problems explored in the ARML–Soviet exchange program of July 1991.

Problems

1. Points K and L are taken on the sides AB and CD of trapezoid $ABCD$ (with bases AD and BC). Prove that if $\angle BAL = \angle CDK$, then $\angle BLA = \angle CKD$. [Alexander Merkuryev (1988)]

2. Prove that the numbers $(B - C)(BC - A^2)$, $(C - A)(CA - B^2)$, and $(A - B)(AB - C^2)$ cannot all be positive for the same values of A, B, and C. [Dmitry Fomin (1989)]

3. The function $F{:}R \to R$ is continuous and $F(x)^x F(F(x)) = 1$ for all real numbers x. If $F(1{,}000) = 999$, find $F(500)$. [Sergey Genkin (1988)]

4. The number 2 is written on a blackboard. Two persons play a game. Each player chooses a number n that appears on the blackboard and changes n to the number $n + d$, where d is one of the factors of n and $d < n$. The player who must write a number greater then 1,000,000 loses the game. If both players play perfectly, who will win? [Andrey Burago (1989)]

5. If A, B, C, D are positive real numbers, prove that

$$\frac{1}{A}+\frac{1}{B}+\frac{4}{C}+\frac{16}{D} \geq \frac{64}{A+B+C+D}.$$

[Dmitry Fomin (1988)]

6. There is a great heap of coins with values 1¢, 2¢, 5¢, 10¢, 20¢, 50¢, and one dollar. Some set of B of these coins has a value of A cents. Prove that some set of A of these coins has a value of B dollars. [Fyodor Nazarov (1987)]

7. All possible sequences of seven digits are written down one after another in a row, in any order at all, to form a 70,000,000-digit number. Prove that this number is divisible by 239. [Alexander Merkuryev (1989)]

8. N cities are connected by $2N - 1$ one-way roads in such a way that one can reach any city from any other by using these roads. Prove that there exists a road that can be closed while preserving the above condition. [Sergey Genkin and Ilya Itenberg (1988)]

9. The sets $A_1, A_2, \ldots, A_n$ are all subsets of N. Prove that there exist two natural numbers X and Y such that, for every $k \leq n$, A_k contains either both or none of them. [Author unknown (1977)]

10. A pile of 1,001 stones is lying on a table. One "move" consists of choosing any pile containing more than one stone, removing one of its stones, then splitting any existing pile into two non-empty (not necessarily equal) piles. Is it possible that, after several such moves, all the remaining piles contain exactly 3 stones? [Sergey Genkin (1988)]

11. In Ripley's "Believe It or Not" we read that a pear-and-apple orchard was arranged so that on a circle of radius ten meters, centered at the base of any apple tree, there exist exactly ten pear trees. At the same time Ripley claims that there are more apple trees than the pear trees in the orchard. Do you trust him? [Fyodor Nazarov (1988)]

12. Given a 10×10 chessboard covered completely with N 2×2 (possibly overlapping) squares whose vertices lie on the lattice points of the chessboard, prove that it is possible to delete one of these squares so that the board remains covered if (a) $N = 55$, (b) $N = 45$. (c) Try to find the maximal value of N such that the deletion of any one square will result in the chessboard not being completely covered. [Ilya Itenberg (1990)]

ANSWERS IN THE MAY/JUNE ISSUE

This material is based partly on work supported by the National Science Foundation under grant number DMS-9110020. The government has certain rights in this material. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Bulletin board

"Sharing Freedom" from Ohio to Russia

On November 26, 1991, hundreds of high school students in Centerville, Ohio, were linked via satellite with students at Moscow School 23. This "international high school assembly" was undertaken by the Whittle Educational Network (Channel One) to introduce American students to their Russian peers. The event was anchored in Centerville by Tom Brokaw and in Moscow by Sergey Suponev, who has a television program for children in Moscow.

Students asked each other questions about life and culture in the two countries. American students were particularly interested in the rapidly changing political events unfolding in the former Soviet Union.

Because of its cooperative international nature, copies of *Quantum* were distributed to all participating students in Centerville.

"In the Company of Whales"

It has been estimated that only four percent of the oceans contain 95 percent of marine life. The areas where the abundance of sea life resides are precisely those most assaulted by human activity. Is the plight of the whales similar to the warning given by the "canaries in the coal mines," signaling a dark future ahead for all life on Earth?

To explore the magnificence and

beauty of these gentle giants and raise awareness of the state of marine life and our seas, The Discovery Channel presents *In the Company of Whales* as part of a conservation and education campaign. This television special, which makes its world premiere on Sunday, April 5, from 9:00–11:00 P.M., offers an extensive natural history of cetaceans—whales and dolphins—and their lives beneath the waves as it probes the dangers facing them and our oceans today.

The Discovery Channel special will be shown again on April 14, 16, and 18. Check local listings for broadcast times.

Video series explores modern physics

From the indefinably small to the unbelievably large, from chaos to order, from void to living matter, a new video series—*The Subject of Matter*—provides a clear insight and understanding into the concepts of modern physics.

Available from Films for the Humanities & Sciences, *The Subject of Matter* is a compendium of 20th century physics as presented by today's top scientists working at the cutting edge of their fields. Stephen Hawking, Murray Gell-Mann, Roger Penrose, and Benoit Mandelbrot are only a few of the incredible minds that contribute to this collection. The series also includes archival footage of earlier physicists from Mach to Bohr to Feynman. With simplicity engendered by truly staggering intelligence, these great minds—aided by state-of-the-art computer graphics and chalkboard illustrations—make the most complex theories of physics understandable.

The series includes discussions on the states of matter (crystals, superconductors, superfluids, and plasma); the second Copernican revolution; the empirical bases for theories of the big bang and the origin of matter; the formation of galaxies; the methods of particle physics; quantum mechanics and the unification theory; the origins of life; the latest theories of autocatalytic RNA; the physiological, neurological, and philosophical perspectives on human thought; artificial intelligence; the artificial creation of consciousness; and much more.

For rental or purchase information on this eight-part video series, write to Films for the Humanities & Sciences, PO Box 2053, Princeton, NJ 08543-2053, or call 609 452-1128.

1992 ARML competition

The American Regions Math League (ARML) will hold its annual competition on May 30 at two sites—Pennsylvania State University and the University of Iowa.

The ARML competition is the largest on-site event of its kind in the country, drawing 15-member teams of high school students from every region. Teams are organized on a local basis. For information on organizing an ARML team or joining an existing team, write to

Joseph Wolfson
Phillips Exeter Academy
Box 1172
Exeter, NH 03833

or

Barbara Rockow
Bronx High School of Science
75 West 205th Street
Bronx, NY 10468

Students recycle and learn about the environment

Consumers are now spending more time and personal energy recycling—separating trash, turning in aluminum cans or plastic containers. But how do they know that returned items are really being recycled into new products?

They do if they're participating in the program launched last fall by food markets in New York state and the St. Louis area, as well as parts of Illinois, Indiana, and Kansas. They know because they are being paid for their used grocery sacks and can buy the new end product—trash bags—under the supermarkets' labels.

This new environmental "closed loop" system is the brainchild of Phoenix Recycling Inc. of Hartsville, South Carolina, which purchases the used grocery sacks from schools, scout troops, or other local civic organizations and directly recycles them into new trash bags retailed through the stores.

The system is designed to support itself, with program costs covered by sales of the trash bags. The supermarkets provide a fund-raising opportunity for local schools and youth groups, who fan out into the community collecting and boxing used grocery sacks. Phoenix ships the boxed sacks to its plant and grades them, paying $1 per pound of Grade A (clean and neat) sacks to the supermarkets, which in turn remit the funds to the groups for academic or recreational needs. The sacks and bag containers are then recycled into trash bags stocked on store shelves.

—Compiled by Elisabeth Tobia

What's happening?

Summer study ... competitions ... new books ... ongoing activities ... clubs and associations ... free samples ... contests ... whatever it is, if you think it's of interest to *Quantum* readers, let us know about it! Help us fill Happenings and the Bulletin Board with short news items, first-hand reports, and announcements of upcoming events.

What's on your mind?

Write to us! We want to know what you think of *Quantum*. What do you like the most? What would you like to see more of? And, yes—what don't you like about *Quantum*? We want to make it even better, but we need your help.

What's our address?

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ANSWERS, HINTS & SOLUTIONS

Math

M46

(a) The longest "unlucky" series consists of 1,000 numbers. Six-digit multiples of 1,001 are lucky (they have the form $\overline{abcabc}$, where the bar means the decimal notation of a single number as opposed to the product of its six digits), and they divide the remaining positive integers into intervals 1,000 numbers long: [000001, 001000], [001002, 002001], ..., [998999, 999998]. Since any unlucky series must be included in one of these intervals, it can't contain more than 1,000 numbers. At the same time, the first and last of the intervals obviously don't include lucky numbers, and so they're both instances of the longest possible unlucky series of 1,000 numbers.

(b) There are exactly two longest unlucky series—those given above. Since any pretender to this role can only be an interval $[\overline{abca_1b_1c_1}, \overline{a_1b_1c_1abc}]$, where $\overline{a_1b_1c_1} = \overline{abc} + 1$, it suffices to show that each of these intervals, except for the first and last (for $\overline{abc} = 000$ and $\overline{abc} = 998$), contains at least one lucky number. If $c > 0$, $\overline{ab} \neq 99$, then such a number is either $\overline{abc(a+1)b(c-1)}$ or $\overline{abca(b+1)(c-1)}$ (or both, if both a and b are less than 9); if $\overline{ab} = 99$, then $c \leq 7$, and the desired number is $\overline{ab}$; finally, if $c = 0$, then $\overline{ab} \neq 00$, so $\overline{ab1(a-1)b2}$ (for $a > 1$) or $\overline{ab1a(b-1)2}$ (for $b > 1$) will be in the respective interval.

M47

Label the given triangle ABC such that $AB = BC$ and move it by a parallel translation so as to bring vertex A to the origin (0, 0). The new coordinates (b_1, b_2) of B and (c_1, c_2) of C remain integers. Now the equality $AB^2 = BC^2$ in coordinate form

$$b_1^2 + b_2^2 = (c_1 - b_1)^2 + (c_2 - b_2)^2$$

yields

$$AC^2 = c_1^2 + c_2^2 = 2(b_1c_1 + b_2c_2),$$

which is an even number.

M48

A geometric argument makes this problem relatively simple. If we plot points $A(x, y)$ and $B(u, v)$ on a coordinate plane with origin at point O, the first two equations given show that A and B both lie on a unit circle centered at O. We can write the third equation as $x/y = -v/u$, which says that the slope of line OA is the negative reciprocal of the slope of line OB, so that the two lines are perpendicular.

Figure 1 shows one of the two possible positions of point B. Since angles AOP and BOQ are both complementary to angle QOA, these angles are equal, and it's not difficult to show that triangles AOP and BOQ are congruent. This means that $x = \pm v$ and $y = \pm u$. The conclusion follows readily.

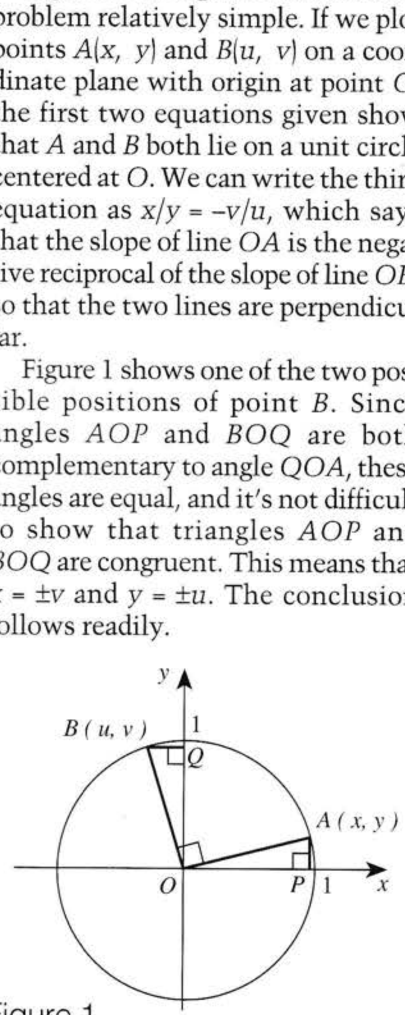


Figure 1

The following identity yields a purely algebraic solution:

$$\begin{aligned}(x^2 + y^2 - 1)^2 &+ (u^2 + v^2 - 1)^2 + 2(xu + yv)^2 \\ &= x^4 + y^4 + u^4 + v^4 + 2 - 2(x^2 + y^2 + u^2 + v^2) + 2(x^2y^2 + u^2v^2 + x^2u^2 + y^2v^2) + 4xuyv \\ &= (x^2 + u^2 - 1)^2 + (y^2 + v^2 - 1)^2 + 2(xy + uv)^2.\end{aligned}$$

We can also prove the assertion by using vectors. Plot points $A(x, y)$ and $B(u, v)$ in the coordinate plane Oxy. The dot product of vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is equal to $xu + yv = 0$, so they're perpendicular. The first two equalities in the statement of the problem mean that $OA^2 = OB^2 = 1$ (fig. 2). If α is the angle between the x-axis and

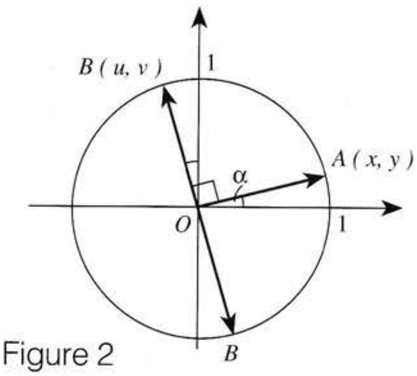


Figure 2

vector $\overrightarrow{OA}$, then $x = \cos\alpha$, $y = \sin\alpha$, and the angle between the x-axis and $\overrightarrow{OB}$ is $\alpha \pm 90°$, so $u = \cos(\alpha \pm 90°) = \pm\sin\alpha = \pm y$, $v = \sin(\alpha \pm 90°) = \pm\cos\alpha = \pm x$. Substituting these values in the equalities that are to be proved, we immediately get

$$v^2 + y^2 = u^2 + x^2 = x^2 + y^2 = 1,$$
$$vu + yx = xy - yx = 0.$$

Try to prove the three-dimensional generalization of the problem: if three unit vectors (x_1, x_2, x_3), (y_1, y_2, y_3), and (z_1, z_2, z_3) are perpendicular to each other, then vectors (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) are also perpendicular and of unit length.

M49

(a) For convenience let's take the given line l to be vertical so we can talk about the left and right sides of it. Label the rightmost point of the figure A and the leftmost B (if there is more than one of either, any one will do). Denote their respective distances from l by a and b (fig. 3). We can assume $a \ge b$, so then we have to prove $a/b < 1 + \sqrt{2}$. Since the polygon is convex, it can intercept an arbitrary straight line in either a single segment, a point, or nothing at all. Let CD be the intersection of the polygon with l. Extend lines AC and AD to meet the line through M parallel to l in points E and F. Points A, C, D, and all of triangle ACD are contained in the polygon; to be more exact, in its right half. So the area of triangle ACD is not greater than half the area of the polygon.

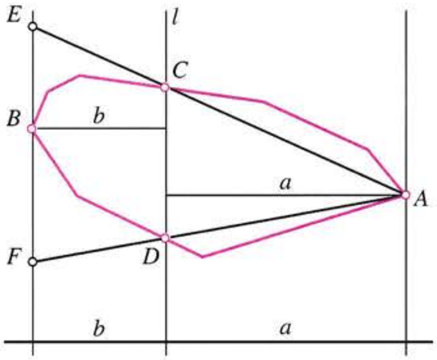


Figure 3

On the other hand, sides CE, EF, and FD of the trapezoid $CEFD$ have no common points with the interior of the polygon, so the trapezoid contains all of the left half of the polygon, and its area is not less then half that of the polygon. It follows that

$$\text{area}(ACD) \le \text{area}(CEFD),$$

or

$$\frac{\text{area}(ACD)}{\text{area}(AEF)} \le \frac{1}{2}.$$

Now, triangle ACD is similar to triangle AEF, the ratio of similarity being $a/(a+b)$ (see figure 3), so the ratio of their areas is $[a/(a+b)]^2 \le 1/2$, which give us

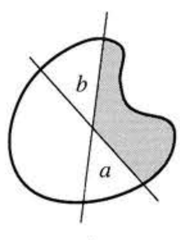
$$\frac{a+b}{a} \ge \sqrt{2},$$

and, finally,

$$\frac{a}{b} \le \frac{1}{\sqrt{2}-1} = \sqrt{2}+1.$$

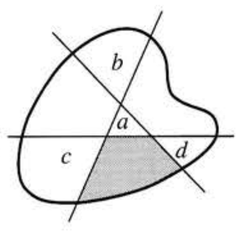
Equality holds, for example, if the original polygon coincides with triangle AEF.

(b) First consider any two of the area bisectors (fig. 4). The areas a and b contained in the vertical angles formed by these lines are equal because each of them, after the same area is added (the shaded portion of figure 4), yields the same result: half the entire area.



$a = b$

Figure 4



$a \leqslant b$

Figure 5

Now restore the third bisector and label the pieces of the given figure anew (fig. 5). According to the argument above, the sum of the central area a and the shaded area equals area b, so $a \le b$. In the same way, $a \le c$, $a \le d$. It follows that the area A of the entire figure is not less than $a + b + c + d \ge a + a + a + a = 4a$, so $a \le A/4$.

Figure 6 is an example of a figure and three lines for which the inequality turns into an equality. (V. Prasolov, V. Dubrovsky)

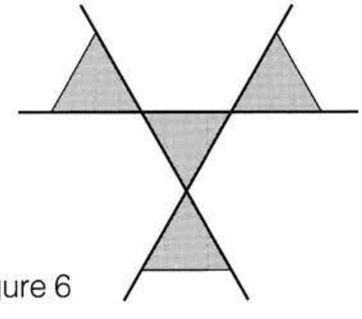


Figure 6

M50

The answer for $k = 1$ is unexpectedly simple: $1/2$. Likewise, in the general case the unknown probability doesn't depend on N and equals $1/(k + 1)$. We proceed directly to the proof for arbitrary k.

We may assume that Howard is the last of the first group of N spectators entering the theater—he goes in right before the sticklers show up. This assumption doesn't affect the situation because the arrangements of the N spectators remain equally likely, which is all we ask of them. When Howard arrives, he randomly chooses one of the $k + 1$ seats that are still unoccupied; therefore, it suffices to show that exactly one of these seats is favorable for him: the probability that he can take it is $1/(k + 1)$.

Consider an arbitrary placement of the first $N - 1$ spectators in the theater. Each of the k sticklers determines a chain of seats: the first seat in the chain is the seat assigned to a certain stickler, each subsequent seat is the seat belonging to the person who has actually taken the preceding seat in the chain, and so forth, until we get to some free seat—the last in the chain. So we have k chains ending in k free seats (the chains are disjoint, since different tickets have different seat numbers). If Howard takes any of these k free seats, he'll be evicted when the stickler holding the corresponding ticket arrives. Otherwise he takes the only remaining free seat that doesn't belong to any of the chains and will avoid the nuisance of moving.

Physics

P46

The motion will not necessarily be uniform. Consider the motion shown in figure 7. We see that imposing a periodic function (with a period of exactly 1 s and a speed equal to zero on

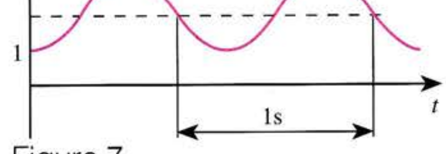


Figure 7

average) onto a graph of uniform motion $v = 1$ m/s also gives a distance traveled of exactly 1 m in any second, though we can hardly call such motion uniform.

P47

Let's determine the force exerted on the pivot by one half of the pincers.

Figure 8 shows the forces exerted on the red piece (the direction of force F_3 exerted by the pivot on this piece is unambiguously determined by the direction of force $F_1 = F$ and of force F_2 exerted by the blue piece on the red

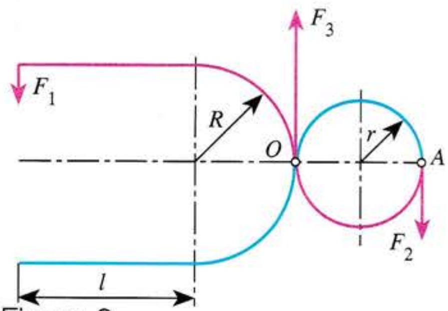


Figure 8

one). The magnitude of force F_3 can be found by balancing the torques. Let's write the equation for the torques about point A (see the figure):

$$2rF_3 = (l+R+2r)F_1$$
$$\Rightarrow F_3 = F_1\left(1+\frac{l+R}{2r}\right)$$
$$= F\left(1+\frac{l+R}{2r}\right).$$

According to Newton's third law the pivot experiences the force $\mathbf{F}_3' = -\mathbf{F}_3$—that is,

$$F_3' = F\left(1+\frac{l+R}{2r}\right),$$

where F_3' is the absolute value of force $\mathbf{F}_3'$. It's understood that a force equal in magnitude is exerted on the pivot by the blue piece. So the force squeezing the pivot point equals

$$F_{\text{sq}} = F\left(1+\frac{l+R}{2r}\right).$$

P48

Placing an uncharged metal sphere in a uniform electric field causes induced electric charges to appear on its surface. They would be distributed over the surface so as to ensure a field intensity of zero inside the sphere. It's clear that the surface charge density σ is determined by the strength of the electric field and by the geometry of the conductor. For a given geometry the stronger the external electric field, the larger the value of σ at each point of the conductor's surface.

When the external field is turned off, the potential energy of the field created by the charges induced on the sphere is transformed into heat. This energy can be calculated as the sum of the potential energies of all possible pairs of charged points on the sphere's surface. Let's divide the surfaces of the spheres into similar small sections. The interaction energy of a pair of sections of the smaller sphere situated at a distance R_{12} from each other equals

$$\Delta W = k\sigma_1\sigma_2\frac{\Delta S_1 \cdot \Delta S_2}{R_{12}},$$

where σ_1 and σ_2 are the corresponding charge densities, ΔS_1 and ΔS_2 are the areas of the corresponding sections. The surface charge densities on the similar sections of the smaller sphere are also equal to σ_1 and σ_2. (Prove this on your own by taking advantage of the superposition principle and the condition of a null electric field inside a conductor.) If the radius of this sphere is n times the radius of the smaller sphere, the energy of the pair of similar sections of the larger sphere equals

$$\Delta W' = k\sigma_1\sigma_2\frac{\Delta S_1' \cdot \Delta S_2'}{R_{12}'}$$
$$= k\sigma_1\sigma_2\frac{n^2 \cdot \Delta S_1 \cdot n^2 \cdot \Delta S_2}{nR_{12}}$$
$$= n^3 \cdot \Delta W,$$

where $\Delta S_1'$ and $\Delta S_2'$ are the areas of the corresponding enlarged sections, R_{12}' is the distance between these sections.

Enlarging the sphere's dimensions by a factor of three ($n = 3$) increases the potential energy contained in the field of induced charges by a factor of $3^3 = 27$. So the amount of heat released when the external field is turned off will also increase by the same factor.

P49

It follows from symmetry considerations that if we remove the first element from the circuit, the resistance of the remaining circuit between points C and D will be $R_{CD} = kR_{AB}$. So the equivalent circuit of the infinite chain will have the form shown in figure 9. Applying to this circuit the formulas for the resistance of resistors connected in series and in parallel, we get

$$R_{AB} = R_1 + \frac{R_2 kR_{AB}}{R_2 + kR_{AB}}.$$

Assuming $k = 1/2$ and solving the quadratic equation for R_{AB}, we get

$$R_{AB} = \frac{R_1 - R_2 + \sqrt{R_1^2 + R_2^2 + 6R_1R}}{2}.$$

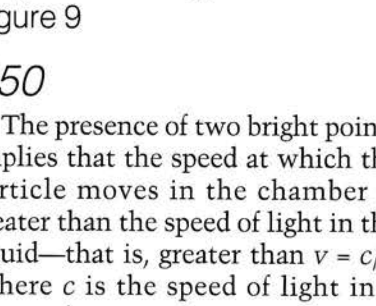


Figure 9

P50

The presence of two bright points implies that the speed at which the particle moves in the chamber is greater than the speed of light in the liquid—that is, greater than $v = c/n$ (where c is the speed of light in a vacuum).

The moment the device is turned on, it records the light previously emitted by the particle from points A

and B. Let the light cover the distance BC in the time t_1, which is greater than the time t_2 taken in traveling the distance AC. So the particle was at point B before point A—that is, the particle moves from right to left (from B to A). In the time $t_2 - t_1$ the light covers the distance $l_2 - l_1$ (fig. 10), and the particle travels along the path l_3. From the figure we see that $l_3 > l_2 - l_1$. This means that, indeed, $u > v$.

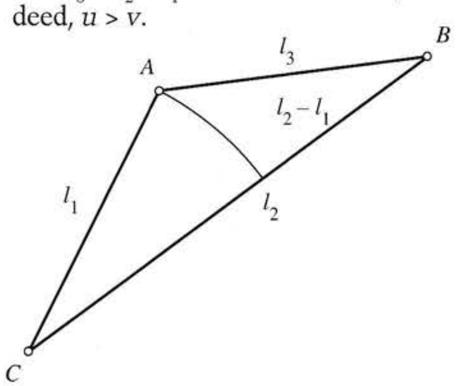


Figure 10

The value of u can be found from the equation

$$\frac{l_2 - l_1}{v} = \frac{l_3}{u},$$

And so

$$u = v\frac{l_3}{l_2 - l_1} = \frac{c}{n}\left(\frac{l_3}{l_2 - l_1}\right).$$

Having measured the distances in the drawing, we can determine the value of the ratio $l_3/(l_2 - l_1)$, and so we get

$$u = 0.9c.$$

Brainteasers

B46

If V is the volume of the barrel, then the volumes of the pails equal $2V/3$, $V/2$, and $4V/9$. The minimum positive integer V for which all these numbers are integers is 18.

B47

Meteorites hit the "morning" hemisphere of the Earth head-on, while those hitting the "evening" hemisphere are chasing after the Earth. That's why after midnight meteorites enter our atmosphere at a higher velocity; as they burn, they sparkle more brightly than before midnight. After midnight our eye will notice meteors of small cosmic particles that before midnight burn imperceptibly in the atmosphere.

B48

Figure 11 shows that the painted and unpainted portions of the star can be cut into the same set of triangles.

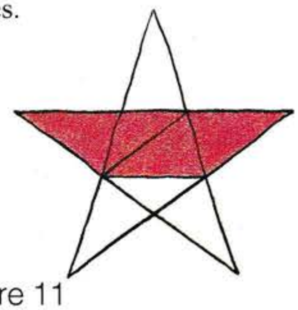


Figure 11

B49

Air is pumped out to lessen turbulent thermal flows of air in the tube, which distort the image of the Sun.

B50

Division "in English":

$$\begin{array}{r}
986 \\
345\overline{)340170} \\
\underline{3105} \\
2967 \\
\underline{2760} \\
2070 \\
\underline{\underline{2070}}
\end{array}$$

Division "in Russian":

$$\begin{array}{l|l}
385024 & 376 \\ \cline{2-2}
\underline{376} & 1024 \\
\;\;902 & \\
\;\;\underline{752} & \\
\;\;1504 & \\
\;\;\underline{1504} &
\end{array}$$

Halving it all

1. See figure 12.
2. See figure 13.
3. See figure 14.

Note: In the first three problems, the hyperbolic pieces start and end at

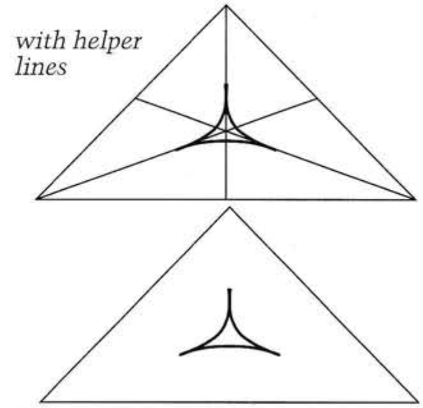


Figure 12

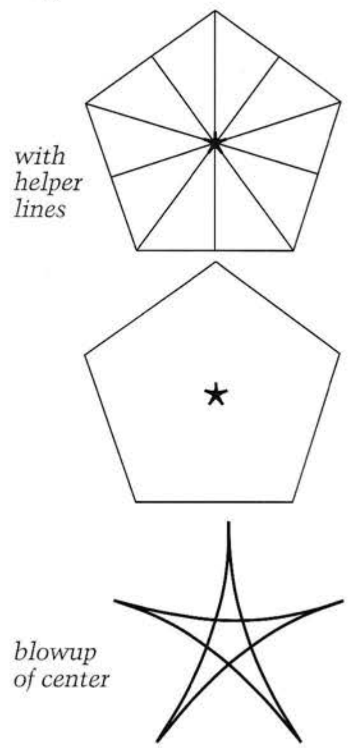


Figure 13

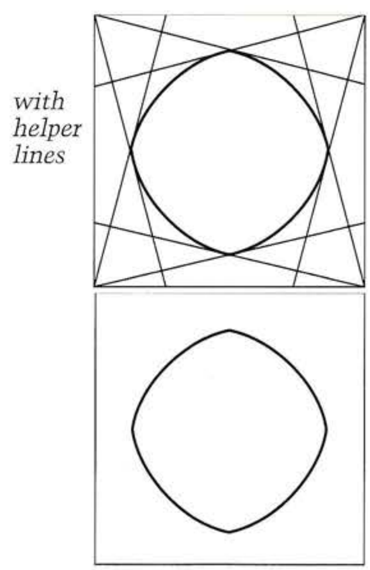


Figure 14

the midpoints of segments drawn and are tangent to them. A good approximation to the hyperbolic piece can be made by using this small amount of information.

4. Given asymptotes l and m and point P, construct l' parallel to l through P and l'' parallel to l' so that the distance from l'' to l' equals the distance from l' to l (fig. 15a). Then for $R = l'' \cap m$, let S be the point of intersection of the line RP with l. The segment RS is bisected by P and so will be a tangent to the desired hyperbola.

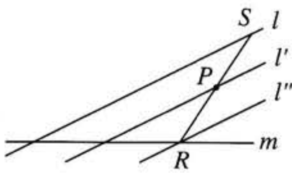


Figure 15a

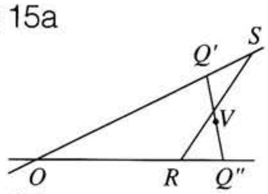


Figure 15b

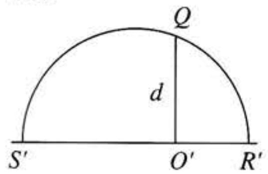


Figure 15c

Let $O = l \cap m$ (fig. 15b). Construct distance $d = \sqrt{OS \cdot OR}$ as indicated (fig. 15c). Then let $d = OQ' = OQ''$ with Q' and Q'' on lines OS and OR as shown. Since ORS and $OQ'Q''$ have the same area, the segment $Q'Q''$ will also be tangent to the desired hyperbola. By symmetry, the midpoint V of segment $Q'Q''$ will be a vertex. And a segment from O through V of length OQ' will end at a focus.

5. If the area of a small triangle in figure 16 is a, then we want to cut off

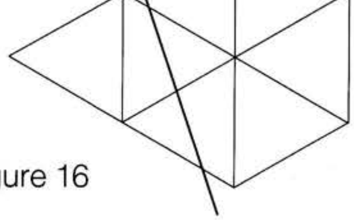


Figure 16

an area of $(6/7)a$. So we need to cut off $(13/7)a$ from the big triangle. That's 13/28 of the area $4a$ of the big triangle.

6. The radius of the smaller circle is 0.40397 to five significant figures.

7. A circle with radius b contains area πb^2. Stretching in one direction by a factor of a/b gives the desired ellipse and multiplies the area by a/b. So the area of the ellipse is πab.

8. Implicit differentiation gives

$$\frac{2yy'}{a^2} - \frac{2x}{b^2} = 0,$$

so

$$y' = \frac{a^2 x}{b^2 y}.$$

Since the line with the given equation has this slope at (x_1, y_1) and passes through (x_1, y_1), it is the tangent line.

The intersection of the tangent line with the asymptotes (given by $y = \pm(a/b)x$) is at $x = x_1 \pm (b/a)y_1$. It follows that (x_1, y_1) is the midpoint of the segment.

9. The tangent line at (x_0, y_0) has the equation $2x + x_0^2 y = 4x_0$. It intersects $xy = 1$ where $x = \left(1 \pm \sqrt{2}/2\right)x_0$. The area cut off is $2\sqrt{2} - \log\left(3 + 2\sqrt{2}\right)$.

10. The tangent line at (x_0, y_0) has the equation $2xx_0 + y = x_0^2$. It intersects $y = 1 - x^2$ where $x = x_0 \pm 1$. The area cut off is 4/3.

Kaleidoscope

1. The total gravitational force will be the same as before the shell exploded.

2. The traction force would double.

3. A real rope (one that has mass) can't be stretched so as to be perfectly horizontal.

4. The boards should be sturdiest at point B, where the track's surface is concave and the force due to the boards must counteract gravity and provide the centripetal force.

5. It's easier to hold a body on a slope than to pull it upward because of the force of friction.

6. The load is subjected to a force of friction in the direction in which the car is moving.

7. See figure 17. Line a shows the dependence of the elastic force on the stretch when the springs are connected in series; line b—when they're connected in parallel.

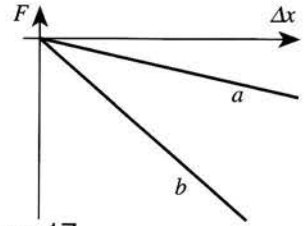


Figure 17

8. The force of gravity acts on the body.

9. At the bob's extremes the net force is tangent to its path; at its lower point the net force is directed toward the pendulum's suspension point.

10. Forces of friction counteract the forces of mutual attraction.

11. No, the box will start rotating about edge B.

12. The lines of the magnetic field are directed from north to south.

Microexperiment. The force of friction depends on the force pressing the objects together; this force is significantly greater for a soaked board than for a dry one.

Direct current

1. The frequency will be $n_3 = n_1 - 1.2(n_1 - n_2) = 640$ rpm.

2. The motor's power will be

$$P_2 = P_1 \frac{I_2^2}{I_1^2} + VI_2 \frac{I_1 - I_2}{I_1} = 6 \text{ kW}.$$

3. The useful power will increase by 21%.

To Flexland with Mr. Flexman

Where you'll find a hexagon that turns itself inside out

by Alexey Panov

ROLL UP! ROLL UP! The magical Flexland tour is waiting to take you away! Flexland is a puzzling country inhibited by flexagons, flexors, and other creatures with a striking ability to change their shape and color. In fact, we've already visited the Topology province of Flexland and explored it thoroughly (see "Flexible in the Face of Adversity" by A. Vesyolov in the September/October 1990 issue). Now we're going to make several trips to the Paper Models county of Flexland. This time we'll have no serious scientific goals, and won't follow any strictly prescribed route. Whenever anything interesting shows up on our way, we won't fail to stop and inquire. But first let me introduce our fellow-traveler, Mr. Flexman, one of Flexland's residents.

Mr. Flexman

To get to know him better, let's simply make him. Cut out a paper square whose sides are 6–8 inches long. Bend it along the diagonals and a midline (see figure 1, in which "crease ridges" are shown as solid lines and "crease valleys" as broken lines, while dotted lines mark future creases). Then fold the sheet along the creases to make a triangle (fig. 2).

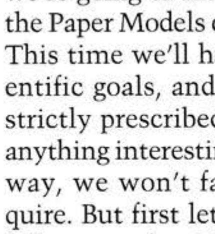


Figure 1

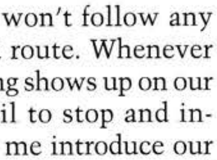


Figure 2

Now repeat this operation four times—bend each of the four "flaps" of the triangle as in figure 3 (this is where the creases predicted in figure 1 appear). The result is seen in

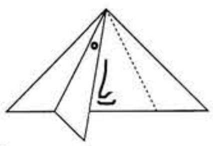


Figure 3

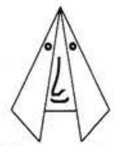


Figure 4

figure 4. All that remains is to provide our Flexman with four pointy feet by bending the corners (fig. 5). Now Mr. Flexman is standing in front of you, ready to walk. As a matter of fact, it's this ability to walk that's his most remarkable feature. Put him on an incline, adjust the slope, and he'll start to waddle down. (If your Flexman skates or shimmies instead of walking, his feet may be too far apart—flatten him out a bit and try again.) Every Flexman (or Flexwoman) has his (or her) own character, or in any case, a unique gait. I think an acquaintance with the Flex family will be to your and their mutual delight.

I wasn't able to trace the origin of Mr. Flexman very far back. But I can

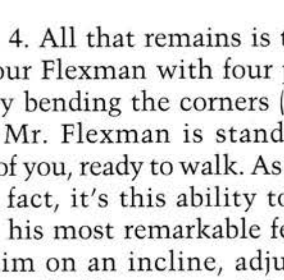


Figure 5

introduce to you one of his relatives. Not long ago I acquired an excellent toy—a walking donkey (fig. 6). This creature walks down an inclined surface, but it can also walk along horizontal surfaces. The donkey has a weight attached to it by a string. When you hang it over the edge of a table, the donkey starts to amble. A weight can be attached to Mr. Flexman too, allowing him to walk on a table. Mr. Flexman and the donkey seem to be creatures of a totally different nature. All the more striking, then, is the fact that the mechanics of their behavior is absolutely identical. I have a feeling that when we finally meet extraterrestrial aliens, in spite of all external differences, our internal likeness will be unquestionable. (I wonder, though, which of us will be more like an ass . . .)

With this brief introduction we can set off on our promised journey together.

Hexaflexagon

The first object—or perhaps, subject—we're going to visit is Hexaflexagon. This is a hexagon that can turn itself inside out and change colors. Its date and place of birth, unlike Mr. Flexman's, can be given with the utmost precision: this remarkable toy was discovered by Arthur H. Stone at Princeton in 1939. (Martin Gardner relates the history of its creation in his book *Mathematical Puzzles and Diversions*.) Mr. Flexman insists that everyone make a Hexaflexagon at least once in their life, and anyone who makes it will certainly want to make another. So, let's get started right away.

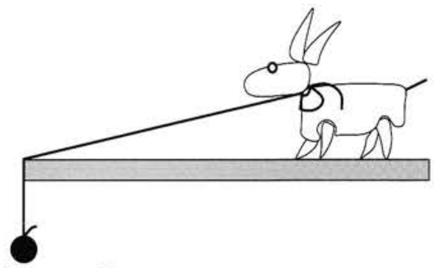


Figure 6

Cut out a paper strip an inch wide consisting of 10 equilateral triangles. Color it on both sides as in figure 7. Bend it back and forth several times along the sides of the triangles and fold it as in figure 8. By the end you must get a red hexagon with a blue triangle sticking out. Fold this trian-

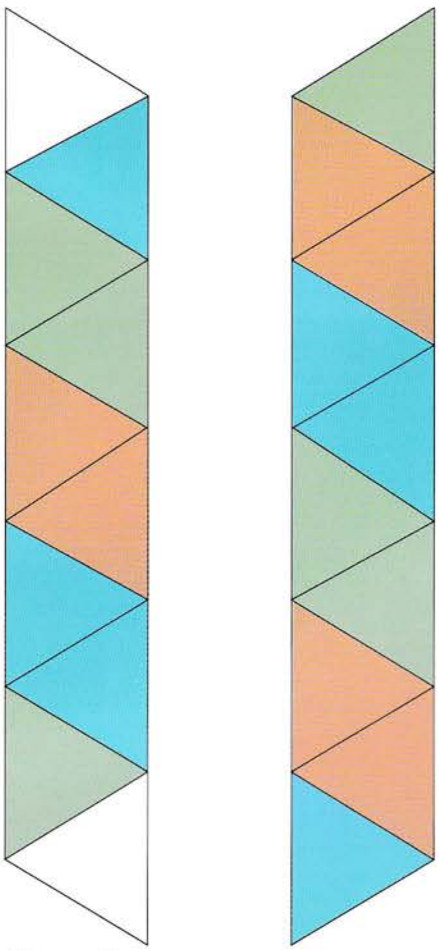


Figure 7

gular flap down and glue the two uncolored triangles together. That's it—the flexagon's ready!

One of its sides is red, the other blue. To extract the third (green) surface we can proceed as follows. Put the flexagon on a table so that it rests on its three lowest points. Then press these points together: when they meet, the flexagon will turn inside out all by itself, revealing its green side (fig. 9). All that's left is to pull apart its upper points, and it's ready for a new transformation. Once you get used to your new toy, you'll find the most comfortable way of performing this operation. In figure 9 you see how a flexagon, shifting from one state to another, displays all three of its surfaces.

The model you've built is the simplest of the large family of flexagons described in Gardner's book, which also tells how to construct more complicated models with a greater number of surfaces. Mr. Flexman recommends that you read this book, paying special attention to the frightening story of a man who got caught in a hexaflexagon, and to be as cautious as possible while manipulating your hexaflexagons.

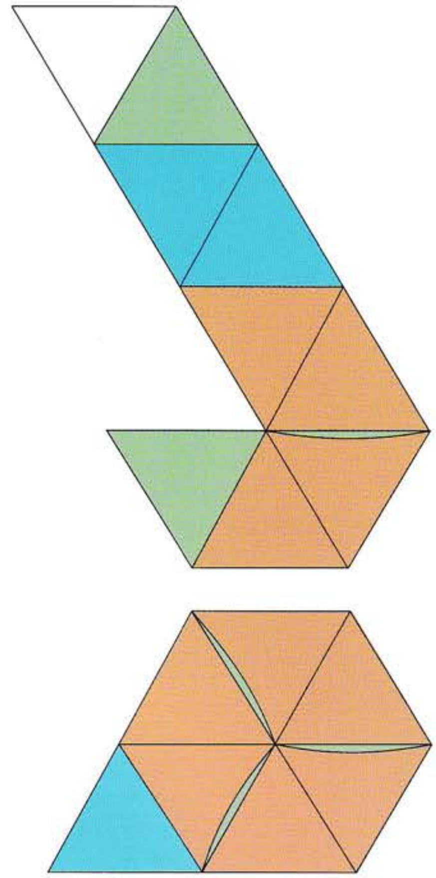


Figure 8

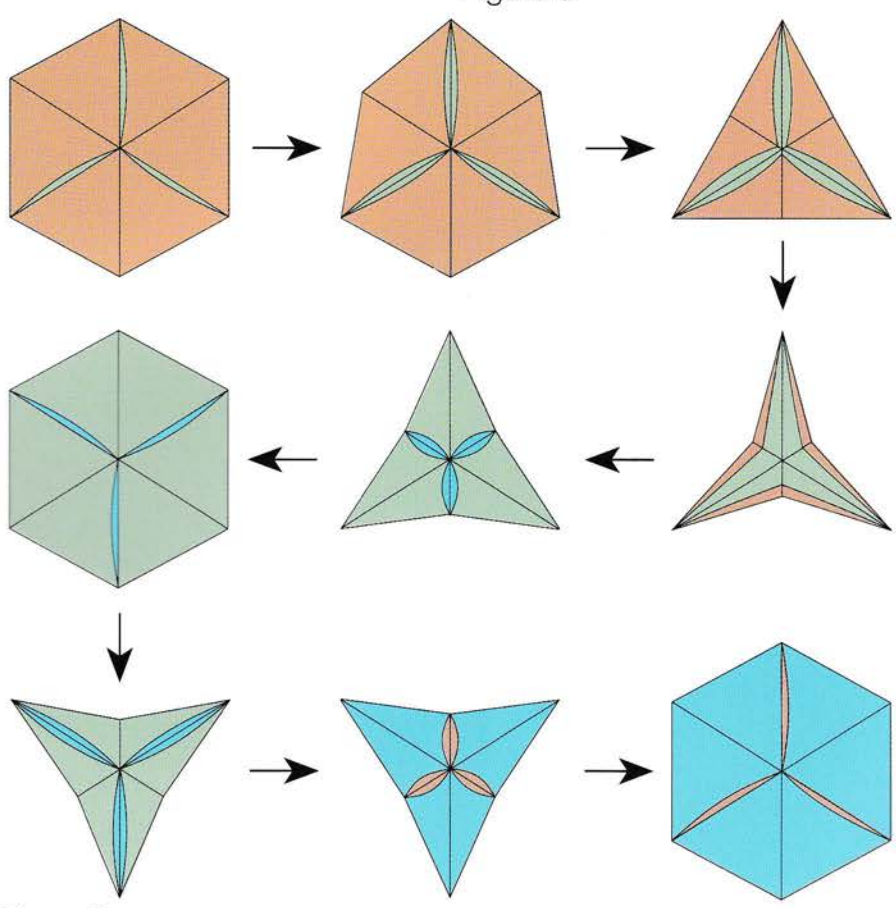
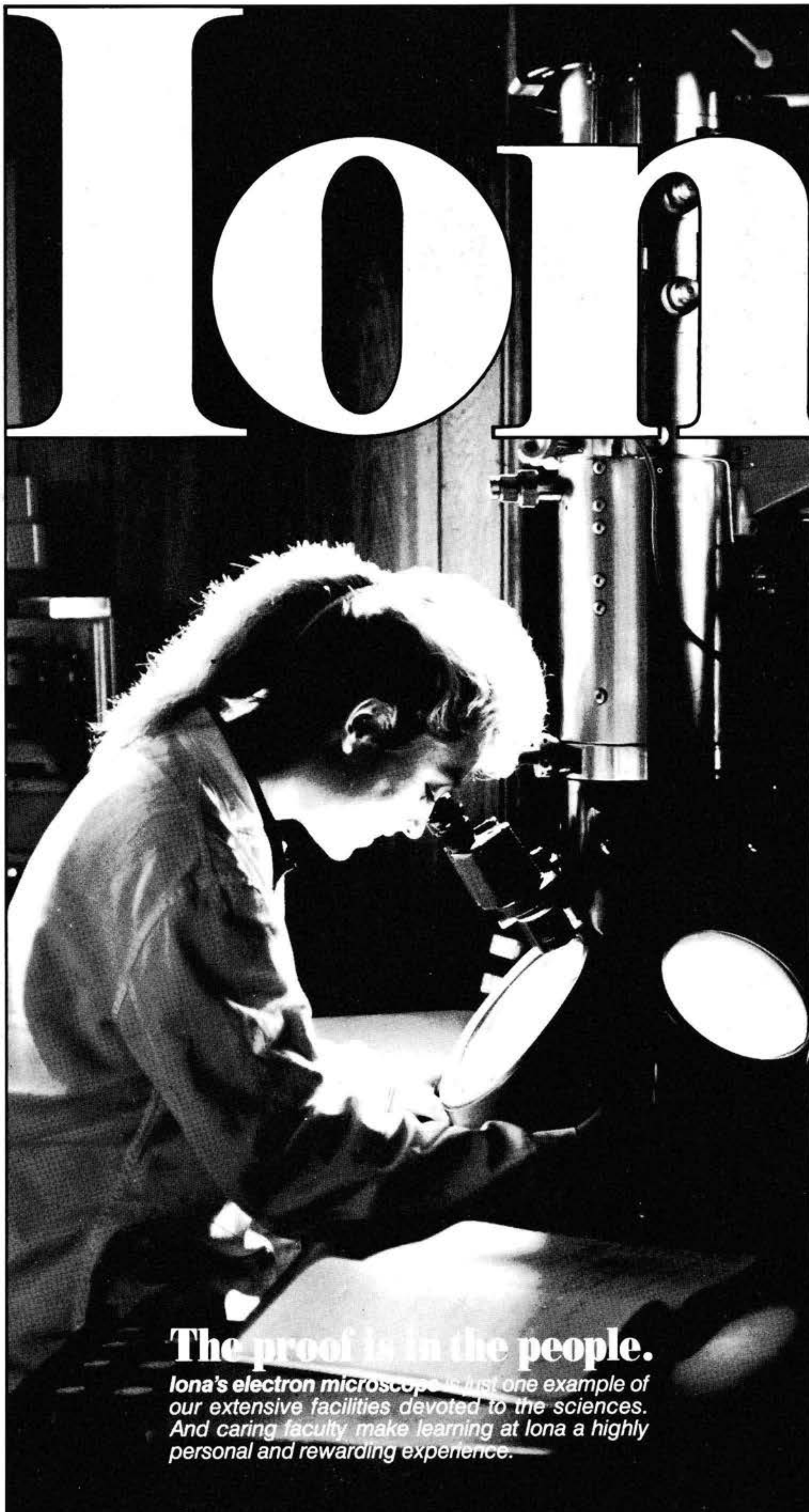


Figure 9



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