# QUANTUM

JULY/AUGUST 1995

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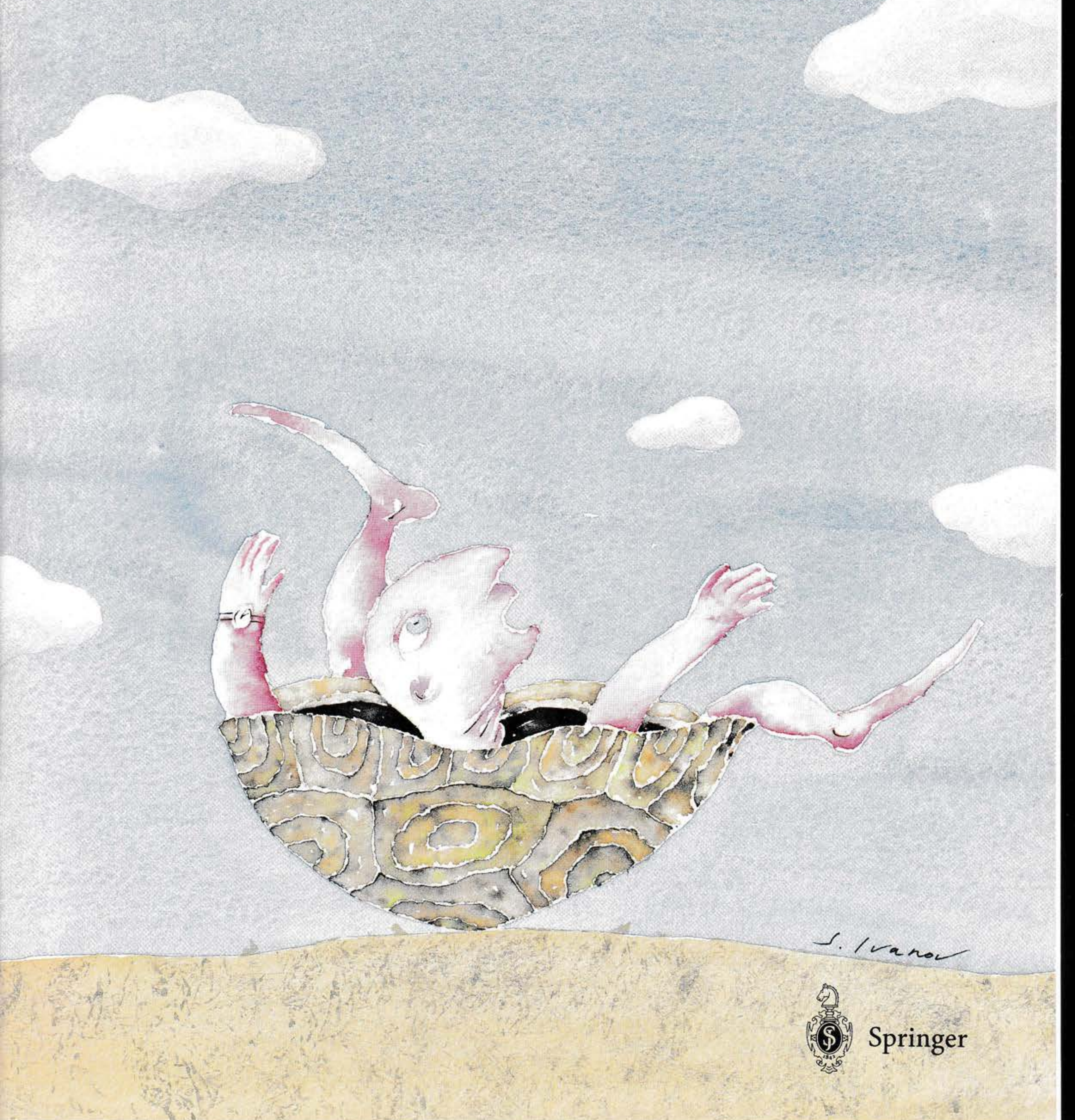


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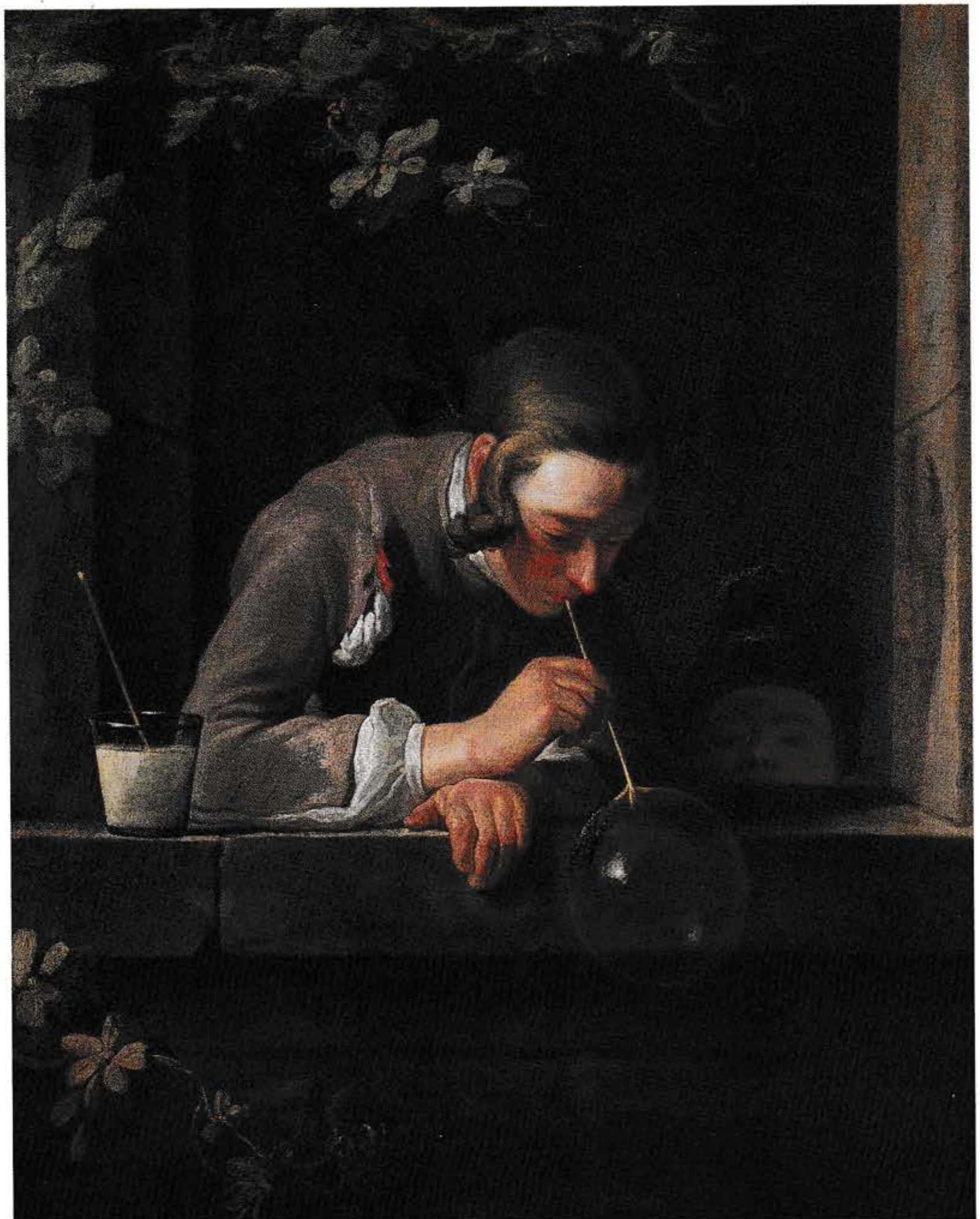



*Soap Bubbles* (1733/1734) by Jean Siméon Chardin

RECOUNTING A FORMATIVE EVENT FROM HIS OWN childhood—glimpsing an undersea "monster" at the age of seven—the naturalist Edward O. Wilson writes:

> A child comes to the edge of a deep water with a mind prepared for wonder. He is like a primitive adult of long ago, an acquisitive early *Homo* arriving at the shore of Lake Malawi, say, or the Mozambique Channel. . . . The child is ready to . . . explore and learn, but he has few words to describe his guiding emotions. Instead he is given a compelling image that will serve in later life as a talisman, transmitting a powerful energy that directs the growth of experience and knowledge.

And he quotes from Rachel Carson's book *The Sense of Wonder*: "If facts are the seeds that later produce knowledge and wisdom, then the emotions and the impressions of the senses are the fertile soil in which the seeds must grow."

A friend of Jean Siméon Chardin's wrote that

> [o]ne day, an artist was making a big show of the method he used to purify and perfect his colors. Monsieur Chardin, impatient with so much idle chatter, said to the artist, "But who told you that one paints with colors?" "With what then?" the astonished artist asked. "One uses colors," replied Chardin, "but one paints with feeling."

One uses electron microscopes, particle accelerators, and supercomputers, but one does science with feeling. As children we watch soap bubbles with fascination, and from out of our hazy, pleasant wonderment, questions begin to form: Where do the rainbow colors come from? How big can a bubble get? Why does it always pop? One can't help but wonder how many scientific careers began with soap bubbles. Alexander Mitrofanov wondered about bubbles in puddles, and he grew up to write an article about them—it begins on page 4.

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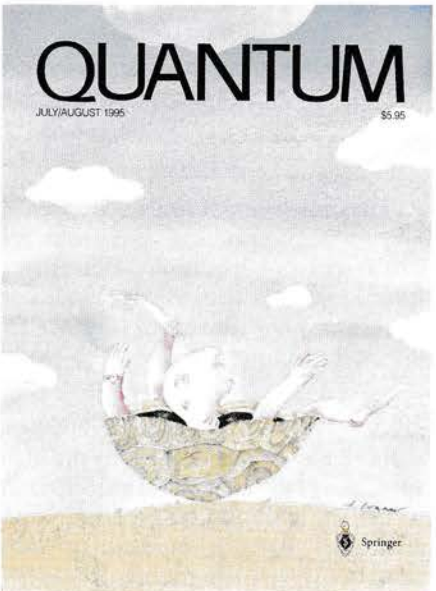


*Cover art by Sergey Ivanov*

The Greek hero Achilles was knocked off his feet by Zeno's paradox, which says that if he gives a tortoise a head start, he can never overtake it in a footrace. Why? Because every time he reaches the point where the tortoise was, the tortoise will have moved on in the meantime. Conceding the field to his plodding opponent, Achilles has become a tortoise himself, flipped on his back, dolefully watching the clouds race by as he flails helplessly on the ground.

Little does he suspect that his problem is all in his head, and that the key to his escape is on his wrist. Turn to page 16 for a clockwork counterproof of the pesky paradox.

Indexed in *Magazine Article Summaries, Academic Abstracts, Academic Search, Vocational Search, MasterFILE*, and *General Science Source*

## FEATURES



## DEPARTMENTS

PUBLISHER'S PAGE

# Curiosity's natural extension

*That's what science is, one reader says, and we're all born curious*

MY LAST PUBLISHER'S PAGE, entitled "Raising the Boats or Lowering the Water," elicited numerous responses from *Quantum* readers. The point I was trying to make was that all students can learn science at substantially greater depths than now occurs or is now expected. Others in the science education community are trying to increase participation in science by broadening it so that it becomes more literacy oriented, and not to expect any significant level of understanding, especially of the quantitative aspects of science or technology.

Here's what some of our readers think.

## Feedback

The letters I received on this topic were mostly supportive. We heard from teachers, college professors, and students at both the high school and college level, and their comments were very interesting.

From Steve Blood, a college student in physics at Williams College:

> I think that anyone and everyone is capable of excelling in science. Some obviously are more inclined than others, but I think this is mainly a function of their upbringing. If only science were taught in an intuitive, exploratory way, in which the student is encouraged to be curious about different phenomena and encouraged to try and figure them out, students of all ages would love it.

From a college student in chemistry, Roderick Ford at Arizona State University:

> I think that students can learn the concepts of all categories of science in a general form early in their grade school years. They cannot [then] understand formal operations but the interest due to science's applicability would aid the development of their mathematical skills.

From a theoretical physicist at Los Alamos National Laboratory, Dr. Leaf Turner:

> I have always been in favor of treating everyone, and especially children, with the highest respect for their potential. . . . [A]s a young student, I always wanted to know what physics and mathematics were about. . . . About the only thing I recall from my high school physics course was Ohm's law and how to convert from Fahrenheit to Centigrade. . . . When I hit physics as a sophomore at Cornell University . . . I chose to take the "macho" course. . . . The year was 1960 and the "teaching assistant" was Professor Hans Bethe. He met with us three times a week. . . . I am currently [in] the Theoretical Division here at Los Alamos. It was because
>
> (a) my potential for learning physics and mathematics was always there, though untapped by my own experience as a high school physics student,
>
> (b) I was unable to realize this potential until my fortuitous taking of a physics course as a prerequisite for medical school, and
>
> (c) the flavor of physics that I received at Cornell was so different from that received in high school
>
> that I took advantage of [an invitation to teach AP physics at Los Alamos High School] in order to attempt to convey the style of physics that I love to the still-idealistic high-school-aged students. . . .
>
> I think that teachers with high standards and expectations who are thoroughly excited about their discipline and love to communicate their feelings can inspire consistently remarkable results in students, results that affect the students far beyond the discipline studied. The students feel great because they are aware how much they themselves have achieved as a result of their own spirit and effort.

And from a high school student from McLean, Virginia, Jeremy Sher:

> The ability to use science words in the right context depends on being able to understand those contexts; otherwise it amounts to nothing more than memorization.

Jeremy expressed well one of the fundamental facts about science and the ability to learn it with the following statement:

> Nobody has an inherent ability to do science, at least not more so than anybody else. We are all equipped from birth with curiosity, and science is curiosity's natural extension.

And from Mark Meyerson, a math professor at the US Naval Academy:

> Teaching children things like the words, effects, and history of science

instead of teaching them the concepts of science not only fails to teach them science, but misleads them into thinking that science consists of boring lists of terms and dates to memorize.

From an e-mail message copied to me, but directed to someone else, from Bettye Anne Case, Florida State University:

Of *course* Aldridge is right!

From a professor of math at SUNY at Stony Brook, Professor Chih-Han Sah:

I totally support your position that young people can learn science at a significant level. Scientific "literacy" has no meaning. One cannot discuss the relevance of science to immediate personal or societal problems without having some deeper understanding of science on a quantitative level. The assumption of educators that scientific methodology would enable a person to solve such problems without quantitative understanding of science is flawed.

From Richard Askey, professor of mathematics at the University of Wisconsin:

There is one group in the United States which is pushing for sufficiently high standards for high school graduates. This is the National Urban League. See their publication on *The State of Black America 1993* for an article on what they propose. . . . [T]hey want all students to graduate from high school with the ability to . . . know mathematics through calculus. . . . There are far too many people who believe that most students cannot learn much of anything, and they are cheating our students. Sometimes I think this is because they don't know much themselves.

I received a very nice, long letter of response from Bernice Kastner, who earned her undergraduate degree in physics and mathematics and taught grades 7–9 in math and science:

[T]here have been fundamental difficulties in this country's mathematics and science curricula in that the intelligence of the learners . . . has consistently been underestimated *and consequently lowered.* . . . [W]e have taught

*CONTINUED ON PAGE 14*


# QUANTUM

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# Bubbles in puddles

*On the shape, size, and longevity of these rainy-day visitors*

by Alexander Mitrofanov

SURELY YOU'VE OBSERVED these lively bubbles. They form on the surface of pools and ponds during a shower, when the water looks like it's boiling from the large drops falling on it. Sometimes you can see these bubbles even after the rain has stopped, when drops are still falling from the boughs of trees. The bubbles are wonderfully stable—not only can you photograph them, you can also watch them swimming on the surface of the water. The bubble's lifetime is anywhere from seconds to tens of seconds. Sooner or later, though, they pop.

In their appearance, all the bubbles are alike. The bubble's shell looks like a hemisphere with a well-defined thin rim near its base. Each bubble is like a transparent round bonnet with a dark, narrow brim. Let's take a look at what this "bonnet" is and how its "brim" was made.

As you probably know, the molecules at the surface of a liquid have a certain potential energy, referred to as the surface energy. The value of this energy is proportional to the surface area—the proportionality factor is called the coefficient of surface tension, and it's different for different liquids. At equilibrium, any system is in a state that corresponds to the minimum energy. For our bubbles, this means that their area should be minimal. A free soap bubble assumes a spherical shape because for a given volume (that is, the volume of air inside the bubble), the spherical surface has the minimum area; the role of gravity is negligible for a bubble because it is very light. If the bubble is deformed slightly (say, by blowing on it), it will regain its spherical shape when the deforming force is removed.

However, the shape of bubbles on the surface of water is different: a large bubble looks like a hemisphere and not a sphere. The shell of a bubble is convex due to the extra pressure $\Delta p$ of the air inside. Since the value of $\Delta p$ is constant inside the bubble (the equilibrium condition), the "bonnet" of the bubble is curved uniformly—in other words, the wall of the bubble that is in contact with the air has the same curvature and forms part of a sphere. But what portion of this sphere is located above the water's surface?

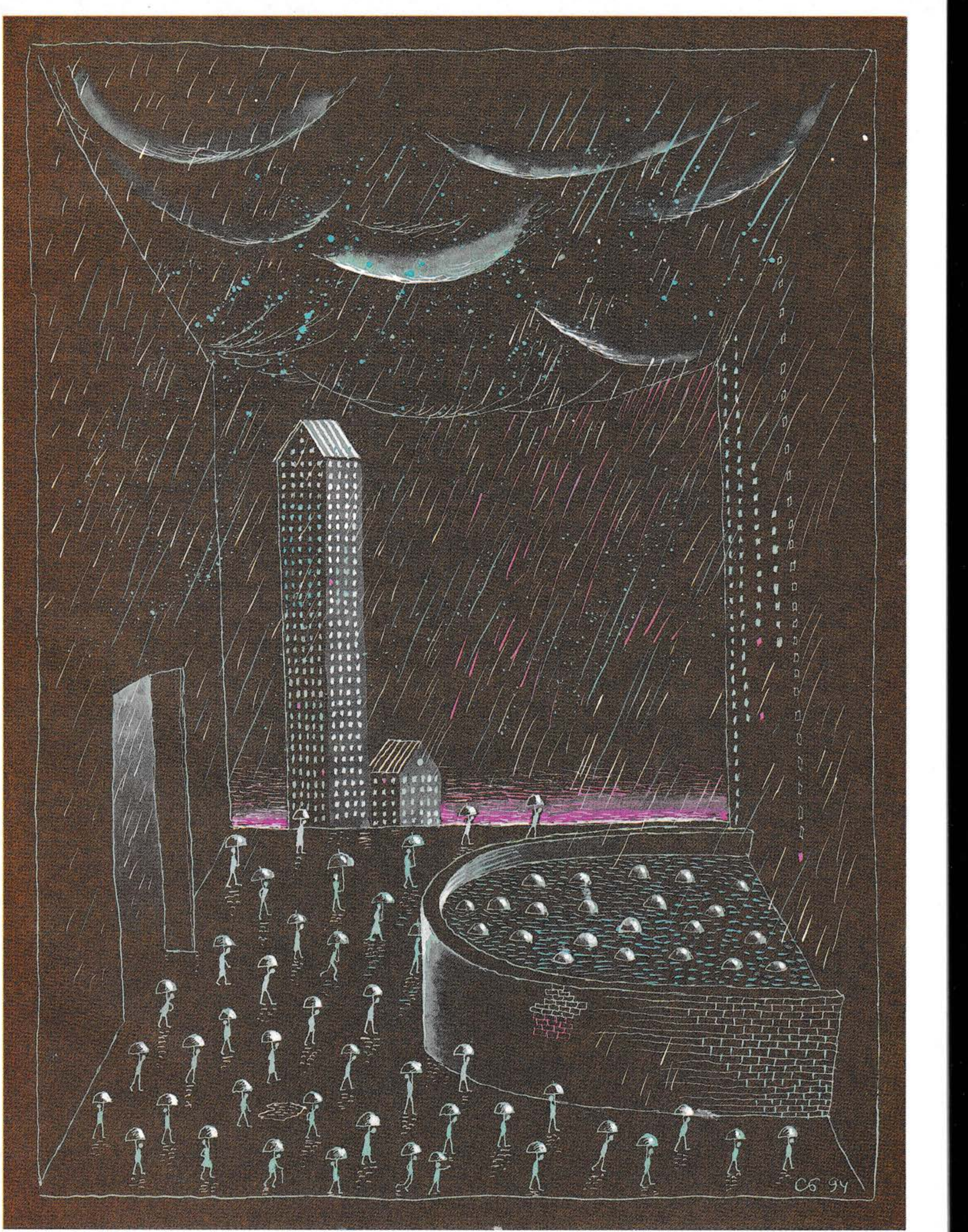
Let's consider the base of a bubble (that is, its "bottom," which is the surface of the pond) to be flat. This assumption, of course, isn't always valid, but the bigger the bubble is, the closer the assumption is to the truth (we'll return to this later). It's not hard to see that for a given volume of air, such a bubble takes the form of a hemisphere (a hemisphere has the least surface area in this case).

Now let's consider the "brim" that encloses the "bonnet" of the water bubble. What is it? It certainly shouldn't be too hard to guess.

You may recall that water rises in a thin capillary tube if its walls are wetted. It's "beneficial" from the energy viewpoint. The smaller the radius of the capillary tube, the higher the level of the liquid in it. Due to the same capillary forces, water rises slightly near the surface of a body of any shape (for example, a flat vertical plate) provided that its surface has been wetted. Lower a small wet plank or clean pane of glass into water such that it sticks out of the water. If the lighting is adequate, you can see the narrow rim created by an increase in the water level due to capillary action near the boundary of the object (see figure 1 on page 6). This is quite similar to the bubble's brim. It is capillary action near the bubble's walls that creates a "brim" on the curved surface.

Those who are interested in calculations and physical estimates are invited to examine in more detail the problem of capillary action near a smooth vertical wall that has been thoroughly wetted.

Art by Sergey Barkhin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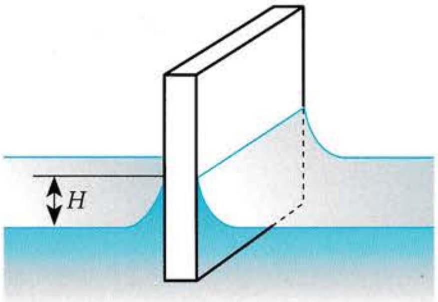



Figure 1

Let's estimate the maximum increase in the water level $H$ near the wall of the plate and find the dependence of $H$ on the coefficient of surface tension $\sigma$. We'll use dimensional analysis to do this.

The equilibrium of a liquid raised by capillary action is determined by surface tension and gravity. So $H$ can depend only on $\sigma$, the density of the liquid $\rho$, and the acceleration due to gravity $g$. We'll look for a function $H = f(\sigma, g, \rho)$ of the form

$$H = \text{const} \cdot \sigma^l \cdot g^m \cdot \rho^n.$$

(Of course, the physical magnitudes determining a phenomenon are not always linked by a power dependence, but very often this is the case.) In order to obtain the mathematical formula for the dependence of $H$ on $\sigma$, $g$, and $\rho$, we need to find the values of $l$, $m$, and $n$.

It's clear that the left- and the right-hand sides of the display equation above have the same dimensions. Let's write down the dimensions of the variables in this equation:

$$[H] = L,\ [\sigma] = MT^{-2},$$
$$[g] = LT^{-2},\ [\rho] = ML^{-3},$$

where $L$, $M$, and $T$ are the units for length, mass, and time.

The corresponding equation is

$$L = (MT^{-2})^l (LT^{-2})^m (ML^{-3})^n.$$

This gives us a system of linear equations for the unknown parameters $l$, $m$, and $n$:

$$l + n = 0,\ l + m = 0,\ m - 3n = 1.$$

The system has a single solution:

$$l = 1/2,\ m = -1/2,\ n = -1/2.$$

Thus,

$$H = \text{const} \cdot \sqrt{\frac{\sigma}{\rho g}},$$

or

$$H \sim \sqrt{\frac{\sigma}{\rho g}}.$$

It's not possible to find the unknown constant in this formula by dimensional analysis. However, in physical formulas obtained by calculation, the numerical coefficients are usually close to unity—in any case, they don't influence the order of magnitude of the result. The exact value of $H$ differs from our estimate by $\sqrt{2}$:

$$H = \sqrt{\frac{2\sigma}{\rho g}}.$$

Now we can estimate how high water rises near a wet vertical wall. Inserting the numerical values $\sigma \cong 73 \cdot 10^{-3}$ N/m, $\rho \cong 10^3$ kg/m$^3$, we have $H \cong 4$ mm. Water near the base of a bubble's spherical surface rises to approximately the same height. Here the curved water surface looks like a dark rim under certain lighting conditions. The rim's width is of the same order of magnitude as $H$ for all large bubbles—that is, several millimeters.

An attentive reader will have observed that the bubbles in the photo are not exactly hemispheres—it looks as if the bubble are slightly sunken—and that the inner and outer rims are different (it's hard to

see such a difference when you watch bubbles form while it's raining). Why is this? Once again, air pressure plays the lead role: the air pressure under the shell is greater than outside. So the water level is lower under a bubble (in the center of its base) than outside it.

Let's estimate how much a bubble is submerged in the water relative to the water level in the pond. The extra air pressure inside a bubble is $\Delta p = 4\sigma/R$. In the surrounding water this extra pressure (in addition to the atmospheric pressure) is created at some depth $\Delta h$ under the surface. Clearly, it is this value $\Delta h$ that equals the "draught" of a bubble (fig. 2a). From the equality $4\sigma/R = \rho g \cdot \Delta h$ we get

$$\Delta h = \frac{4\sigma}{\rho g R}.$$

For the bubbles in the photograph, the value was about 1 to 3 mm. Thus, a large bubble looks like a hemisphere that is slightly submerged in water and is bordered by curved inner and outer edges. By "large" we'll mean a bubble that satisfies the expression $R \gg H \cong \sqrt{\sigma/\rho g}$.

And what would a smaller bubble look like? Try to investigate this question yourself—it's not a simple one. I'll just mention here that if we "prepare" too small a bubble (with a radius of about 1 mm or less), it won't come to the surface at all—it will stay submerged close to the surface as if stuck to it. Small bubbles are spherical. Figure 2 shows how different bubbles look: (a) a large bubble satisfying $R \gg \sqrt{\sigma/\rho g}$; (b) a medium-sized bubble satisfying $R \cong \sqrt{\sigma/\rho g}$; and (c) a small (spherical) bubble satisfying $R \ll \sqrt{\sigma/\rho g}$.

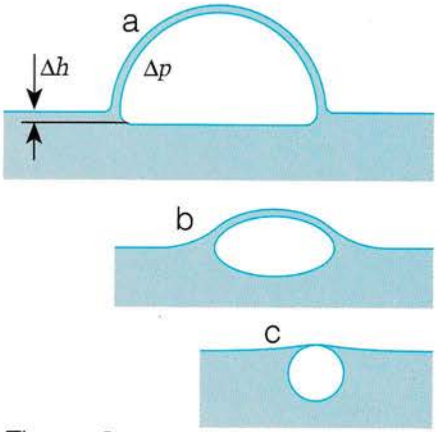


Figure 2

Think about why the shape of a bubble depends on its size—the bigger it is, the more it looks like a hemisphere. And why is it that basically large and mid-sized bubbles appear when it rains?

Let's move on to one final problem. It has to do with the stability of a water bubble. We need to figure out why bubbles enjoy such a long life only if it's still raining or the sky is overcast.

Someone will say that it's because there are so many bubbles when it rains that they seem to be long-lived. There is a grain of truth to that assertion. But if you watch an individual bubble, you'll see that the problem is not far-fetched. Indeed, in rainy or cloudy weather bubbles live far longer than on a dry, sunny day. (They live such a short time in dry weather that no one pays much attention to them.)

There are many different reasons why a bubble is destroyed. Its shell can be torn by a gust of wind, it can be "pierced" by a large raindrop, and so on. The shell can also evaporate or become very thin and nonuniform because of water flowing downward along the bubble's walls. Then again, the presence in water of so-called surface-active substances (for instance, soap) can prolong the lifetime of bubbles by a factor of tens or hundreds. (Think of the foam you've seen in a river, which is remarkably persistent.)

There is another interesting point in this story—the evaporation of liquid from the bubble's walls. There's a saying in Russian: "Bubbles in puddles, the rain will keep raining." This is because the relative humidity is close to 100% when it's raining hard, or if a good shower is imminent or just past. At such times, evaporation from the bubble's walls is negligible. It is precisely this factor that prolongs the life of the large bubbles that are so abundant in a good summer downpour. When the humidity is lower, water evaporates quickly. In dry air the thin water wall of a bubble disappears practically at once. If you have a hygrometer, you can make your own quantitative estimates. Just pour some water in a saucer and note how quickly it evaporates. The result correlates with the reading on the hygrometer, which you've placed some distance away so it won't be influenced by the more humid air near the saucer. ◘

*With the invention of photography, particularly high-speed photography, many details of the events occurring when a raindrop strikes the water's surface became known. A large raindrop falls on the glasslike surface . . . a deep crater forms . . . the crater collapses . . . and now there is a quiet bubble with air "imprisoned" inside it, floating on the surface of the water.*

*In the beginning of our century, one of the first who recorded such wonderful transformations on a photographic plate was A. Worthington, professor of physics at the Royal Naval Engineering College in Devonport. The series of photos Worthington made long ago became classics and have been reprinted in* Beyond Vision: A Collection of Scientific Photographs *by Jon Darius (Oxford University Press, 1984).*

*Smaller drops are destined to another fate. They also make dents in the water's surface, but the indentations aren't very deep and they don't "collapse." Sometimes water "spikes" can be observed—small, thin vertical fountains rising out of the center of the indentation. These water spikes look particularly impressive during a heavy rainfall at night when there is a bright flash of lightning.*


B.B.

# Knots, links, and their polynomials

## *A topological approach to a tangled subject*

by Alexey Sosinsky

KNOTS CAME INTO BEING IN prehistoric times, together with the first threads and ropes. Perhaps they even predated these works of human hands—after all, vines can be knotted. The skill of knotting reaches far back in our history: knots were indispensable to the first navigators, weavers, and builders. But it wasn't until relatively recently that scientists began to study knots—at the end of the last century.[1] It seems that the rigid framework of traditional geometry was too restrictive for these thin and flexible objects. Progress in the theory of knots became possible only with the development of *topology*—the branch of science concerned with the deepest properties of shapes and their position. There it took its modest but respectable place. Modest at first, anyway.

Recently knot theory has stopped being merely a way for a small number of specialists to amuse themselves. It has unexpectedly turned into one of the most fashionable of mathematical activities, and physicists (both "classical" and "quantum") and even geneticists have joined in the fun.

But we'll get to that later. First, let's look at knots from a mathematician's point of view, and let's ask the most natural question . . .

### Can any knot be unraveled?

For instance, can we unravel the knots shown in figure 1? A quick glance at the knot $K_1$ will certainly suffice for you to see that it can easily be smoothed out into a circle. But you'll hardly see at once how to disentangle the knots $K_2$, $K_3$, $K_4$, and $K_5$. Our intuition tells us that $K_2$ can't be unraveled without cutting it. As for knot $K_6$, it's simply baffling: you can't tell from the figure how the knot is arranged in space.

In order to move from an everyday discussion of knots to a serious mathematical treatment, we need exact definitions. We start with a definition of a *knot*. A knot is a closed polygonal path in space that does not intersect itself and is connected (it consists of a single whole piece) and oriented (supplied with a certain direction in which it is traced). Knots are depicted on the plane in a special way—by diagrams. A *diagram* starts as a projection of a knot onto a plane. In performing such a projection, self-intersections may occur. These self-intersections are recorded as isolated double points (the intersection of only two segments) with a blank space indicating

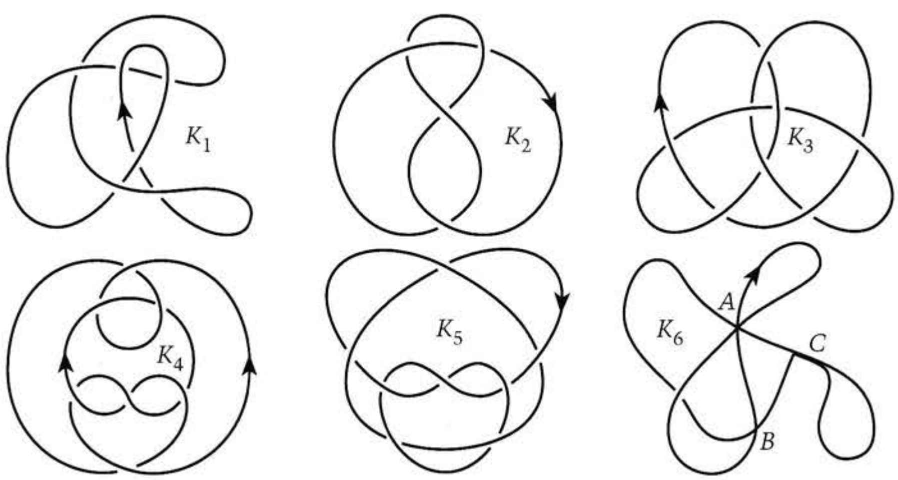



Figure 1

*Examples of knot diagrams: $K_1$—a trivial knot (that is, it can be converted into a simple loop); $K_2$—the "figure 8" knot; $K_3$—the tabulated knot $8_1$; $K_4$—?; $K_5$—the tabulated knot $8_{11}$; $K_6$—an illegal portrait of a knot (not a diagram).*

Art by Vasily Vlasov

[1]You may find it useful to look into the introductory article by the same author in the January/February 1995 issue of *Quantum*. However, the present article can be understood without doing so.—*Ed.*

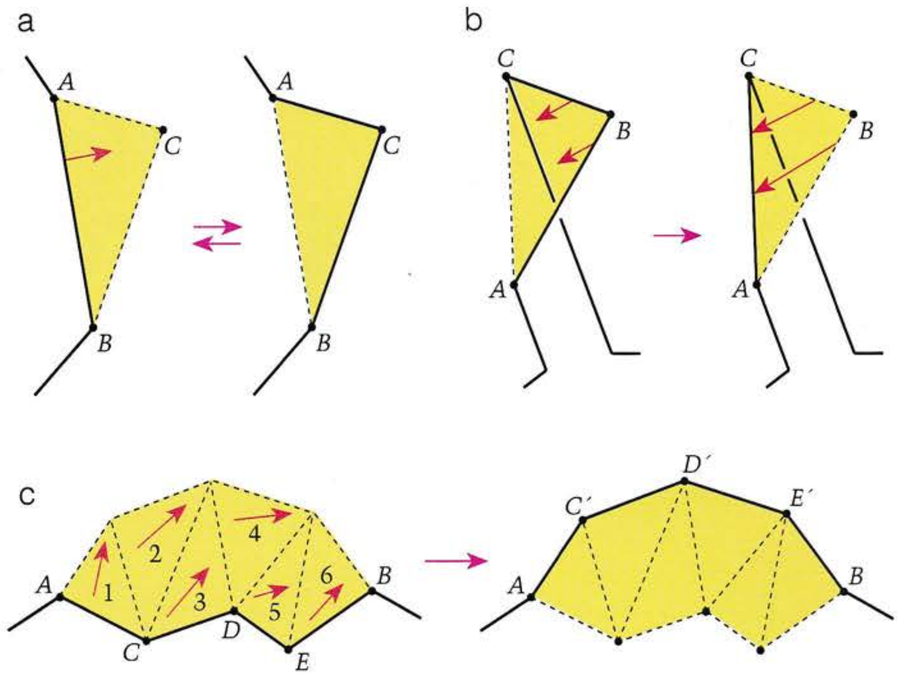



Figure 2

*Elementary moves and equivalence of knots: (a) replacing the segment AB by a two-edge path ACB (and vice versa) under the condition that triangle ABC has no points in common with the knot (except, of course, the points of the segment AB); (b) removing a loop by one elementary move; (c) shifting the portion ACDEB of a knot into a new position AC′D′E′B by means of a sequence of six elementary moves.*

the lower segment of the knot. Also, for the sake of appearance, a polygonal diagram is often drawn as if it consists of smooth curves.

In figure 1 there are only five genuine diagrams. The projection of $K_6$ isn't a knot because it has a triple point ($A$), a double point without a blank ($B$), and an entire segment of (non-isolated) double points ($C$). The direction of circumnavigation is shown on the diagrams by arrows.

Two knots are considered *equivalent* (that is, the same) if one of them can be transformed into the other by moving, shrinking, and stretching it (without cutting and pasting).[2] These transformations allow the knot $K_1$ to be turned into a trivial knot—that is, simply into a circle (fig. 3). There is one more trivial knot in figure 1 (try to guess which), but it's much more difficult to disentangle than the knot $K_1$.

Have you guessed which knot it is? Right! The other trivial knot is $K_4$. But a person with a normal three-dimensional imagination usually can't discover a method of unraveling it immediately: it requires a certain amount of work with a pencil, eraser, and paper (or chalk and blackboard).

**Exercise 1.** Draw the unraveling process for knot $K_4$.

Now the following general questions will sound quite natural.

1. *The problem of unraveling.* Find an algorithm that can determine for any knot whether it is trivial, judging from any of its diagrams.

This question is a particular case of an even more general one.

2. *The problem of comparison.* Find an algorithm that determines whether two given diagrams depict equivalent knots. (The problem of unraveling is the problem of comparing a knot with the trivial knot.)

How can we tackle these problems? There is a tool at hand . . .

## Reidemeister moves

Let's analyze how we solved the problem of unraveling the knot $K_1$. As seen in figure 3, in the process of unraveling, the shape of the knot changes substantially only when there is a change in the number or arrangement of double points. A double point can disappear when a small loop is untwisted (fig. 3e); when a pair of double points disappears after a section of the string slides off another one (fig. 3a); and finally, when a pair of double points crawls across a third double point (fig. 3b). In figure 3, the situations before these operations are shown within yellow

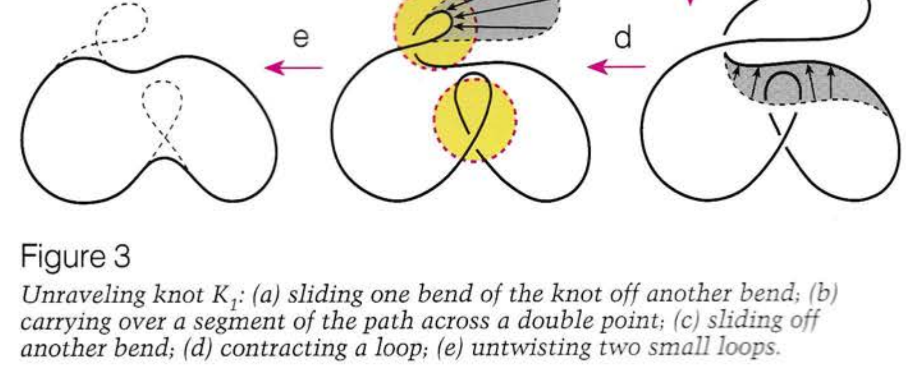



Figure 3

*Unraveling knot $K_1$: (a) sliding one bend of the knot off another bend; (b) carrying over a segment of the path across a double point; (c) sliding off another bend; (d) contracting a loop; (e) untwisting two small loops.*

[2]A stricter definition: two knots are equivalent if it's possible to pass from any of them to the other by a finite chain of *elementary moves* (replacing segments of the first polygonal path, one after another, by two-segment paths and vice versa—see figure 2).

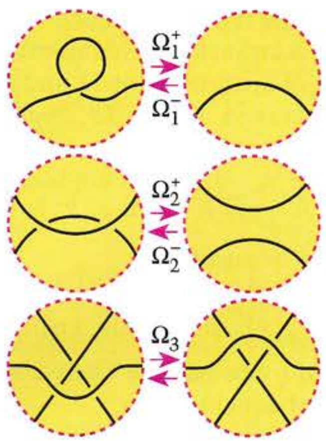



Figure 4

*Reidemeister moves: $\Omega_1^\pm$—untwisting (and twisting) a loop; $\Omega_2^\pm$—sliding one bend of a knot off another and inverting the "slide on" move; $\Omega_3$—carrying the segment of a knot over a double point. The moves marked with a plus sign decrease the number of double points and thus simplify a knot's diagram; the moves marked with a minus sign increase this number; and the move $\Omega_3$ leaves it unchanged.*

circles. All three types of moves are reproduced separately in figure 4. They are called the Reidemeister moves after the German mathematician Kurt Reidemeister, who introduced them in the 1920s and proved the following theorem.

THEOREM. *Two knots are equivalent if and only if their diagrams can be converted into each other by means of a finite number of moves $\Omega_1^\pm$, $\Omega_2^\pm$, $\Omega$.*

Reidemeister's theorem reduces the difficult three-dimensional equivalence problem for knots to a simpler plane problem on converting one knot diagram into another using the three given moves $\Omega_1^\pm$, $\Omega_2^\pm$, $\Omega$ (fig. 4). I'm not going to give a proof of this theorem here—that wouldn't be very interesting.

**Exercises**

2. Unravel the knot $K_1$ using only $\Omega_1^+$ and $\Omega_2^+$ moves.

3. Show that the moves $\Omega_1^+$ and $\Omega_2^+$ cannot be applied to knot $K_2$.

Our success in solving these two exercises inspires the hope that the unraveling problem can be solved. Applying the Reidemeister moves $\Omega_1^+$ and $\Omega_2^+$ that reduce the number of double points, as long as this is possible, we can simplify the diagram step by step until the knot is unraveled. If we arrive at a situation where these moves can't be applied, we conclude that the knot can't be unraveled.

Unfortunately, this idea is fallacious . . .

**Exercise 4.** Show that the moves $\Omega_1^+$ and $\Omega_2^+$ can't be applied to the (trivial!) knot $K_4$.

We've reached a dead end. To unravel a knot, we may have to increase its number of double points, and only after that it becomes possible to simplify the diagram. Notice that the problem is not that we don't know in advance how many new double points must be added—we have nothing that's even close to being an algorithm. Why? Regardless of how many systematic attempts to unravel a knot using the Reidemeister moves have failed, we can't be sure that this is really impossible—that is, that the knot is nontrivial. Our failure could just as well be explained by an insufficient number of additional double points.

And so our naive geometric approach has been completely unsuccessful. At this point we don't have a proof of nontriviality for even a single knot! Perhaps any knot can be unraveled . . . In any case, we need some new ideas.

A reader who has participated in math olympiads will immediately suggest a promising line of inquiry . . .

## Invariants

We need to find an invariant that obstructs the unraveling. Now, what does that mean? Each knot diagram must be coupled with a certain algebraic object—an *invariant*—such that *invariants of diagrams of equivalent knots are the same.*[3] Then no two knots with different invariants will be equivalent. If a knot's invariant isn't equal to the invariant of a trivial knot, then the given knot is nontrivial. We can say that the invariant "obstructs" the unraveling of a knot. At any rate, it can provide a proof of nontriviality for a given knot.

So, if the invariants of two knots are different, the knots themselves are different. If the converse is also true (the equality of invariants implies the equivalence of knots), then the invariant is said to be complete. The problem of comparison for knots is very complicated, and a simple complete invariant for this problem is not known. However, we'd like to have a sufficiently subtle invariant—that is, an invariant that discerns many, if not all, types of knots.

How can we find such an invariant? We'll have to enlarge the class of objects under consideration: we'll study not only knots, but also more general structures . . .

## Links

A *link* is a finite set of closed, disjoint, oriented polygonal paths in space (fig. 5). A knot is a particular

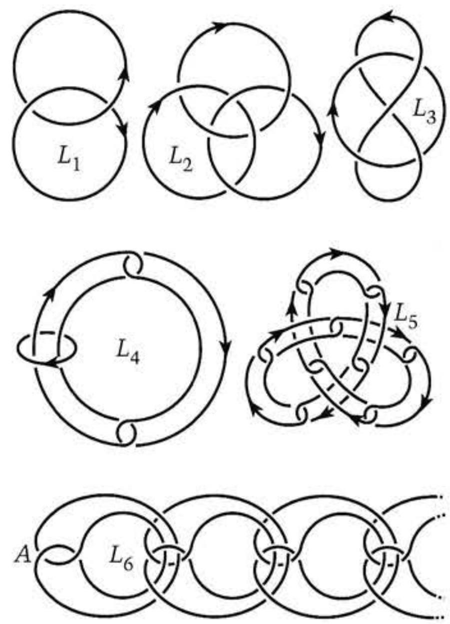



Figure 5

*Examples of links: $L_1$—the simplest right-handed link of two loops; $L_2$—the Borromean rings; $L_3$—the pierced "figure 8"; $L_4$—a link equivalent to the Borromean rings (check it!); $L_5$—a chain knotted into a trefoil; $L_6$—an unstable chain (it falls apart if the front and rear curves at point A are exchanged).*

[3] Applying invariants to olympiad problems was discussed in "Some Things Never Change" (September/October 1993); see also the Toy Store in the first two issues of this year for invariants in puzzle-solving.

case of a link (a set of one path). *Equivalence* of links is defined in the same way as for knots.

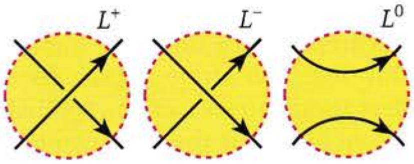
The presence of several disjoint polygonal paths rather than just one creates radically new knotting effects. For instance, two loops (trivial knots) can be combined into a pair that will never come unhooked ($L_1$). The *Borromean rings* ($L_2$) are even more interesting: three circles no two of which are linked, but all three together can't be disentangled. Linking small loops one by one, we can build up long chains and even knot them ($L_5$). A chain of more tricky loops ($L_6$) is also amusing: after the leftmost "lock" is cut (that is, its left double point is replaced with the opposite), the entire chain successively comes loose. This effect resembles a nylon stocking or woolen sweater coming unraveled.

**Exercise 5**. How are the upper and lower threads in a sewing machine linked with each other? Try to draw a picture showing this link.

I'm not going to dwell any further on geometric games with links. Let's pass directly to the construction of the invariant we need. It's named after the British mathematician John Horton Conway . . .

## The Conway polynomial

In the scientific literature this polynomial is more frequently called the *Alexander–Conway polynomial*. James W. Alexander, an outstanding American topologist, invented his polynomial before Conway, in 1933 (the Conway polynomial differs from Alexander's only by a simple change of variable). But Alexander's construction—exquisitely beautiful!—is by no means elementary. And, of course, Conway's contribution doesn't come down to a change of variable—he discovered a completely elementary axiomatic construction of the polynomial. And this is the construction I describe below, leaving the polynomial only under the name of Conway, one of the brightest and most versatile mathematicians of this century, the creator of the Game of Life, surreal numbers, several "sporadic monsters" (certain important examples of groups), and many other clever constructs.[4]




$P_{L^+}(x) - P_{L^-}(x) = x \cdot P_{L^0}(x)$

Figure 6

*The skein relation (axiom (3)) for the Conway polynomial.*

Conway postulated that each knot or link diagram $L$ is associated with a polynomial in $x$ with integer coefficients denoted by $P_L(x)$. This correspondence must satisfy the following three axioms:

1. *The polynomials corresponding to equivalent diagrams L and L′ are equal to each other:*

$$P_L(x) = P_{L'}(x).$$

2. *A trivial knot is associated with the zeroth-degree polynomial equal to 1:*

$$P_0(x) = 1.$$

3. (The skein relation.) *The polynomials corresponding to three links $L^+$, $L^-$, and $L^0$ that coincide with one another everywhere except in a small circle, where they appear as shown in figure 6, are related by the equation*

$$P_{L^+}(x) - P_{L^-}(x) = x \cdot P_{L^0}(x).$$

At this point I'm not going to discuss the question why the polynomial $P_L(x)$ exists for any $L$ and why it is uniquely determined by the axioms. Instead, let's learn how to compute the Conway polynomials based on the axioms above.

Figure 7a shows how to get the Conway polynomial of a pair of unlinked loops. We start with a diagram $L^+$ of a trivial knot with one double point. By axioms (1) and (2), $P_{L^+}(x) = 1$. Replacing the double point with the opposite one and removing it (in accordance with the skein relation (3)) yields another diagram of a trivial knot $L^-$ and a pair of unlinked loops $L^0$. Applying axiom (3), we get

$$P_{L^+}(x) - P_{L^-}(x) = x \cdot P_{L^0}(x)$$

—that is, $1 - 1 = x \cdot P_{L^0}(x)$. It follows that $P_{L^0}(x) = 0$—that is, a pair of unlinked loops has the Conway polynomial equal to zero.

Now let's compute the Conway polynomial for a pair of linked loops. First we note that, because we consider oriented paths, there are two kinds of pairs: *right-handed* pairs and *left-handed* pairs. If I'm going to explain how they differ, we'll have to keep the loops straight in our minds (so to speak), so choose one of the two

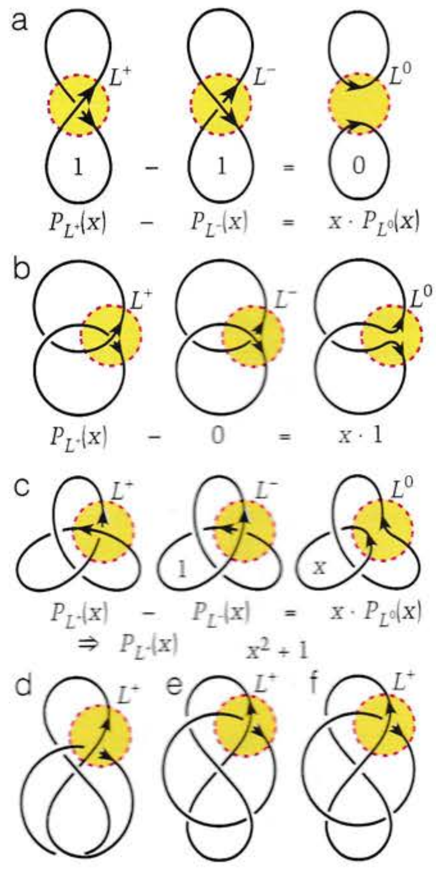



Figure 7

*The Conway polynomials for (a) a pair of unlinked loops $L^0$ ($P_{L^0}(x) = 0$); (b) the left-handed pair (fig. 8) of linked loops $L^+$; (c) the trefoil $L^+$ ($P_{L^+}(x) = x$); (d) the "figure 8" knot $L^+$; (e) the "pierced figure 8" link; (f) the other modification of the "pierced figure 8."*

[4] John Conway has written a number of articles for *Quantum*: see the Mathematical Surprises department in the May 1990 pilot issue and in all four issues of volume 1 (1990–1991).—*Ed.*

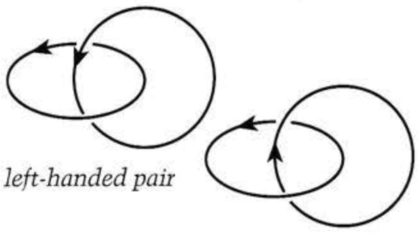



Figure 8

*The simplest left-handed and right-handed links of two loops.*

loops in a pair and call it the "first," and call the other one the "second."

If the second loop pierces the plane of the first loop in the direction defined by the "right-hand rule" (which is the direction of your right thumb when the fingers of your right hand are curled in the direction of the first loop), then this linked pair is *right-handed*; otherwise it's *left-handed* (see figure 8).

You can verify that a right-handed (left-handed) link remains right-handed (left-handed) if we renumber the loops, but that it becomes left-handed (right-handed) if the direction on *one* of the loops is reversed. In figure 7b, $L^+$ is a left-handed pair of linked loops. Applying the skein relation to its right double point, we get, successively, the diagram $L^-$ (equivalent to an unlinked pair) and a trivial knot (with one double point) $L^0$. By axiom (1) and our previous calculation, $P_{L^-}(x) = 0$. Then, by axioms (1) and (2), $P_{L^0}(x) = 1$. Now, substituting into axiom (3), we find $P_{L^+}(x) = x$—that is, the *Conway polynomial of the simplest left-handed link of two loops is equal to* x. For the right-handed link the polynomial equals $-x$. We can see this if we reverse the arrows on the upper loops in figure 7b—then links $L^+$ and $L^-$ exchange roles with respect to the skein relation (and, in addition, the trivial knot $L^0$ changes its pattern).

Figure 7c shows the calculation of the Conway polynomial for the trefoil knot. Notice that the application of the skein relation here creates a *link* ($L^0$)—the knot falls into a left-handed pair of linked loops. This is why we were forced to consider links rather than just knots.

**Exercises**

6. Using figure 7d, compute the Conway polynomial for the "figure 8" knot.

7. Compute the Conway polynomials for the left-handed and right-handed modifications of the "pierced figure 8" link in figures 7e and 7f. (Reminder: don't forget to distinguish between the simplest right-handed and left-handed links.)

To go on with our calculations, we'll need a certain general property of the Conway polynomials . . .

## The polynomial of an unhooked link

We say that a link is *unhooked* if it consists of two pieces that aren't linked with each other.

THEOREM. *The Conway polynomial of an unhooked link is equal to zero.*

To prove this assertion, imagine that we enclosed the two pieces of the given unhooked link $L^0$ in two disjoint boxes and put the boxes not far from each other. Draw a small piece of the string from each box and lay the pieces near each other, as shown in figure 9a. Applying the skein relation, we obtain two new links $L^+$ and $L^-$ (fig. 9b, 9c). Then

$$P_{L^+}(x) = P_{L^-}(x).$$

Indeed, a 360° clockwise rotation of the upper box about its vertical axis turns $L^+$ into $L^-$, so the above equality follows from axiom (1). Then, by the skein relation,

$$P_{L^0}(x) = x^{-1}\left(P_{L^+}(x) - P_{L^-}(x)\right) = 0,$$

and we're done.

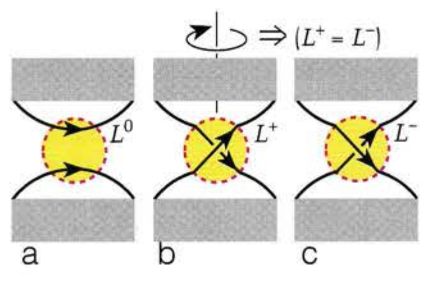



Figure 9

*A proof of the theorem on unhooked links (when the upper box is rotated 360°, the link $L^+$ turns into $L^-$).*

**Exercises**

8. Verify that the Conway polynomial of the Borromean rings is equal to $x^4$.

9. Compute the Conway polynomials for the link and knot in figure 10.

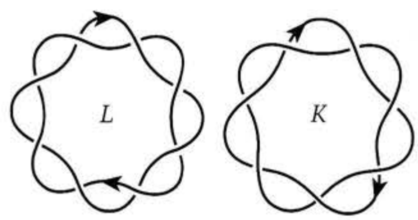



Figure 10

*A toroidal link and toroidal knot (these can be viewed as windings of a torus).*

10. Show that the Conway polynomial isn't a complete invariant—that is, find different knots with the same Conway polynomials. Hint: the simplest example is provided by the trefoil and its mirror image. To prove they aren't equivalent, we can modify the Conway polynomial into the *Jones polynomial* defined in the same way except for the skein relation, which takes the form

$$3.' \quad \frac{1}{\sqrt{x}}P_{L^+} - \sqrt{x}P_{L^-} = \frac{1+\sqrt{x}}{\sqrt{x}}P_{L^0}.$$

## Summing up

Just from the few calculations we've carried out you can see that the Conway polynomial is quite a fine tool, enabling us to identify knots and links and establish their nontriviality. Having calculated, say, the polynomials for the trefoil, "figure 8," and Borromean rings (to see that they are neither 0 nor 1), we gave a strict proof that they can't be unraveled. Or, having counted the corresponding polynomials, we could have proved that the links in figures 7e and 7f aren't equivalent—which, after all, is not that obvious.

Of course, all these proofs are legitimate only if the fact of the *existence* and *uniqueness* of the Conway polynomial for any knot and link has been established. Merely by stating axioms we can't guarantee that an object satisfying these axioms actually exists (what if they contain internal contradictions?). How can such a guarantee be obtained? Only by proving the existence and uniqueness of the Conway polynomial. But the elementary proof I know is very cumbersome, so I won't give it here.

## What's next?

I hope you liked the transparent and elegant constructions described above. It should be said that there aren't many actively developing fields in modern mathematics whose essential ideas could be explained to a high school student. As a rule, contemporary mathematics relies on a vast body of knowledge that goes far beyond the scope of the high school (and even college) curriculum. But the part of knot theory that I've presented here is almost elementary, though relatively new (the Conway polynomial, for instance, was devised in the 1970s). But we mustn't delude ourselves. Despite all its attractiveness, what I've managed to describe wouldn't have sufficed to cause the increased interest in knot theory that I mentioned at the beginning of this article.

The cause lies elsewhere. First, it has to do with the deep connections between polynomials of knots, statistical physics (specifically, the so-called Potts model for ice), and quantum physics (the so-called quantum groups) that were discovered by the British mathematician Vaughan F. R. Jones. Second, knots have appeared in molecular biology (the knotting of the DNA double helix). And the third important recent advance in knot theory was the solution of its fundamental problems (of comparison and unraveling) based on the profound geometric work of the German mathematicians W. Haken and H. Waldhausen.

Today you can find a description of the algorithms for comparing and unraveling knots in the mathematical literature. But these algorithms are still too complicated for computer realization. Also, there is a complete invariant for knots invented by the Russian mathematician Sergey Matveyev. But its practical calculation is also beyond the reach of existing computers. Research in this area continues.

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*PUBLISHER'S PAGE*
*CONTINUED FROM PAGE 3*

> children *not* to use their intelligence in school. Of course our students can learn quantitative science. . . . but we have to teach science and mathematics honestly, and in developmentally appropriate ways.

Not all responses were supportive of my position. One e-mail message comes from hmliu@aol.com, who said, among other things:

> I don't think so. There are some who I truly think cannot learn the stuff (and they are all in 8th period . . .). And some who could spend a lot of time pounding away, and eventually get it, but what is the point?

## Recap

The position I have expressed in these pages does not imply that there are not great differences among young people, and I certainly agree that there are exceptional people who learn very quickly the kinds of material found in the quantitative sciences. There are others who must work much harder, and they can learn such material well, but only with great effort. The problem is that our existing methods for identifying such talent are so faulty that we do not select, and give the necessary resources to, the most talented who should be going on to careers in science and technology. Our present system rewards and provides resources to large numbers of young people who succeed on conventional tests, who memorize well, who conform and follow directions, or who have unusual advantages. As a consequence of our deficient system of discovering talent in young people, we fail to identify from among the very best and brightest large numbers of young women and minorities. We lose the benefit of the best of these groups, while providing many resources and advantages instead to a large number of mediocre young white males, as we necessarily select the best but then move down in that distribution to fill the need.

Providing good science for all students solves the problem of early identification, while simultaneously providing the kinds of solid science people need to live in our complex, technological society. The best and brightest will, if all are kept in science, become recognizable at the right time, irrespective of their ethnicity, gender, or economic status. Also, if our science requires depth of understanding and some measure of creativity and thought instead of memorization of facts, information, or algorithms, we will identify the real talent needed for science and engineering careers.

I want to thank our readers for responding, most of whom did so through the Internet. Please continue to send us your views on these and other matters in math or science education in which you have an interest. I have tried to respond to each message or letter that I have received.

—Bill G. Aldridge

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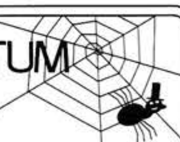

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# Just for the fun of it!

## B146

*A 1995 problem.* How many numbers end in the four digits 1995 and become an integer number of times smaller when these digits are erased? (V. Dubrovsky)


## B147

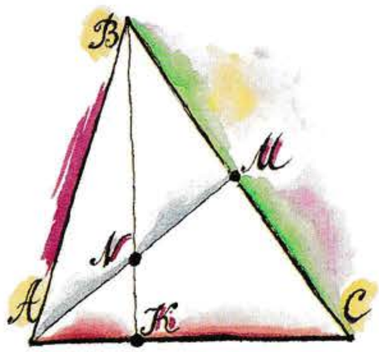
*Number diamonds.* Natural numbers are grouped in diamonds as shown in the figure. The sums of numbers in the diamonds obey a certain law. Guess the law and prove your conjecture. (V. Proizvolov)

## B148

*Rock density.* How can the density of a rock be measured without measuring its volume? (Archimedes)

## B149

*Match figures.* It's not difficult to make two regular pentagons from ten matchsticks. But it's not that easy to arrange the same number of matches so as to form one regular pentagon and five regular triangles. Try it! And what about two regular pentagons and ten isosceles triangles? (A. Abbasov, V. Dubrovsky)




## B150

*Through a median's midpoint.* The line $BK$ bisects the median $AM$ of a triangle $ABC$. In what ratio does it divide the side $AC$? (V. Chichin)

*ANSWERS, HINTS & SOLUTIONS ON PAGE 59*

Art by Pavel Chernusky

# True on the face of it

## *A clockwork proof of the convergence of an infinite series*

by Gordon Moyer

PROBABLY FEW OF US ADvance to second-year calculus without learning the names of Newton, Leibniz, and, strangely, Zeno of Elea. If the first two are the names of the great founders of the subject, the last is surely the name of its great *con*founder. This ancient Greek philosopher, who predates higher mathematics by over two millennia, cunningly insists on the complete impossibility of integration.

In the most famous of his paradoxes, Zeno asserts that the swiftest Greek warrior could not beat even a tortoise in a footrace, for to do so would involve, in effect, the summing of an endless series of intervals. Unfortunately, the original text of this paradox no longer survives, but Aristotle, in a chapter of his *Physics*,[1] cites the hero Achilles as the favored contestant; a much later commentator on Zeno tells us that Achilles' challenger was a tortoise.[2]

Fair-minded as he is fleet-footed, the brassy young hero in the tale gives his dawdling adversary a head start. With ease, Achilles reaches his opponent's starting-point, but, in the interim, his hard-shelled challenger has moved on. Dashing Achilles continues toward the tortoise's new position, but by the time he reaches it, the animal has once again crept ahead. And so it goes, with our hero constantly gaining ground on his lowly competitor only to find his reptilian rival, meanwhile, has moved on. Exasperated, Achilles concedes the race; he never finds himself so much as neck and neck with the tortoise!

Common sense, of course, dictates a different winner. And, in light of the theory of limits, most calculus teachers are quick to show students the error of Zeno's ways. Nevertheless, as historian G. J. Whitrow has pointed out, calculus professors have not convinced everybody. Philosopher–mathematician Bertrand Russell, for one, argued that there are *physical* implications, core problems involving the structure of space and time, that Zeno aimed at and which limit theory does not resolve. Russell believed that one way to escape the paradox is to assume that, unlike a number line, actual space and time are composed, not of dimensionless points or instants, but of discrete segments that cannot be infinitely subdivided.

Such a theory may resolve the paradox, but it does so by calling into question the legitimacy of using calculus to treat bodies in motion. Others have offered similar "quantum" arguments, and for many logicians the Achilles paradox endures. A conundrum that seemed to philosopher Charles Sanders Peirce as nothing more than a "silly little catch presenting no difficulties to a mind trained in mathematics" remains for many other scholars just as Russell saw it, "all immeasurably subtle and profound."

Discovering that the resolution of Zeno's paradox might not be so cut and dried may shake a student's confidence in limits and infinite series, or at least in the applicability of those concepts to Achilles' chase.[3] Of course, whether space or time, or "space–time," actually constitutes a

[1] See Aristotle, *The Physics*, trans. P. H. Wicksteed and F. M. Cornford, vol. II, Loeb Classical Library no. 255 (Harvard University Press, Cambridge; and William Heinemann Ltd., London, 1980), Book VI, Chap. IX, p. 180.

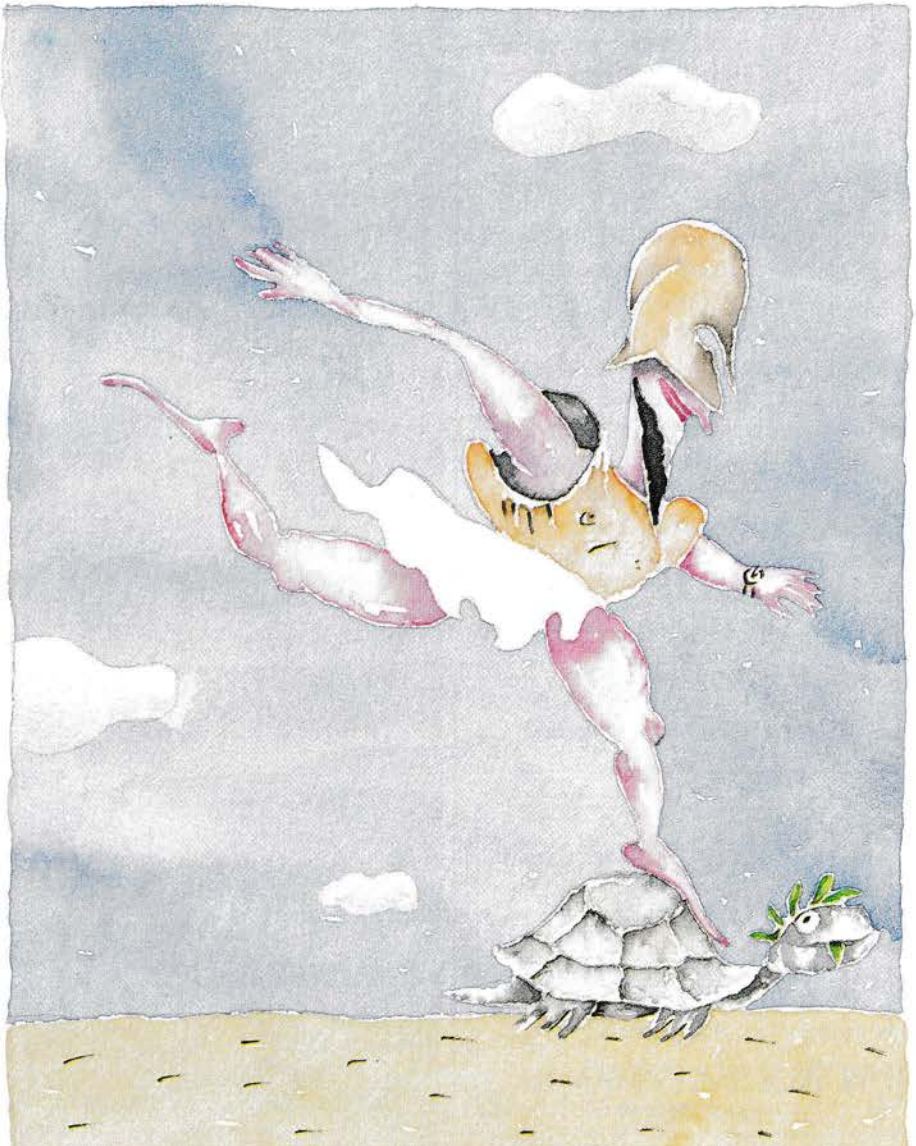
[2] Simplicius of Cilicia, a Neoplatonic scholar, who wrote about Zeno's paradox in the 6th century A.D., fully a thousand years after Zeno.

[3] According to H. G. Zeuthen, a historian of ancient mathematics, Zeno was probably familiar with finding the sum of an infinite geometric series. Zeuthen contends that Zeno may well have been questioning the validity of using infinite series at all to solve a problem involving real, physical distance: if the world is composed of indivisible bits of matter, then a physical distance on a racetrack cannot actually be broken up into an infinite array of points corresponding to the mathematical points of an infinite series.

continuum and is therefore truly integrable is a matter that need not hold up anyone in Calculus I. Yet, Russell's contentions aside, there are other "little catches" in Zeno that may confound and divert a learner.

Could one expect even an epic hero, not to mention a tortoise, to run a perfectly straight course at a precise uniform velocity? The scrupulous among us have been known to pick at such nits. What about initial acceleration—do Achilles and his inhuman challenger achieve their respective speeds instantaneously? Simultaneously? These more mundane physical considerations also complicate attempts to describe Zeno's race using strict geometric series. No doubt such problems are easily sidestepped by granting idealized motion to the contestants, but how much the contest already stretches belief! Since infinite series already seem "unreal" to many of us, the topic could better be introduced in an example which is a good deal more plausible, and decidedly less controversial, than the paradox of Achilles and the tortoise.

An ordinary mechanical clock provides such an example. Here, another Zenonian race is run, but instead of a tortoise and a hero out of Homer, we have two far more convincing figures: clock hands. Unquestionably, the big, leaner hand outstrips the little one, completing a lap around the dial in exactly an hour. In comparison, the "little hand" takes twelve times as long to go around—exactly twelve hours. With the hour hand running only $\frac{1}{12}$ as fast as the minute one, the latter quite visibly overtakes it.



Art by Sergey Ivanov

We can actually see where, *and when*, the faster hand overruns the slower. The point at which the two hands coincide marks the limit of a convergent series. From the outset, we get an intuitive sense of the actual location of a limit in time and space. If the time is one o'clock, for example, it is clear that the minute hand will catch up to its mate just a bit beyond five after. There, as the one hand gradually closes in on the other, the "convergence to a limit" concept becomes palpably real.

The truth of limit theory reveals itself in the time ticking away. At precisely one o'clock, the hour hand has a 30-degree "head start" on the swifter minute hand. The latter, starting at the twelve, makes up this distance in exactly five minutes, but the hour hand, in the interim, will have moved on a bit. How far? Since it runs only $\frac{1}{12}$ as fast as the minute hand, it will have crept on only $\frac{1}{12}$ of its 30-degree lead, or $2\frac{1}{2}$ degrees. The minute hand covers *this* distance in very short order, less than half a minute; of course in that brief interval its slower mate will have once more moved on. Little further thought, though, is required to realize that the pair in this "Achilles-and-tortoise" race will be running in a dead heat right around five-and-a-half minutes passed one.

The face of a clock is a perfect, yet physically real, analogy to Zeno's legendary racecourse, complete with precisely paced runners on a conveniently numbered track. The footrace is now a "hand race," the clock hands providing grounds for believing that a continuously diminishing geometric series indeed converges to a limit. Seeing that this limit really exists—at a point a little past 1:05, for example—may give us confidence in the so-called "sum to infinity" formula for the limit sum of a geometric series:

$$s = \frac{a}{1} - r \qquad (1)$$

(where $r < 1$), which confirms, as

we'll see below, that one place where the hands converge is indeed found a few degrees beyond the five-minute mark on the clock—that is, a bit beyond five minutes after one.

If our great clock-hand chase begins at one o'clock sharp, we can represent each distance that the minute hand closes in on the hour hand by the constantly diminishing series

$$30° + \frac{5°}{2} + \frac{5°}{24} + \frac{5°}{288} + \cdots, \qquad (2)$$

where the first term equals the hour hand's head start. Each of the hour hand's subsequent leads is always $\frac{1}{12}$ of the previous one. This stands to reason, since it moves only $\frac{1}{12}$ as fast as its running mate. All of this is more succinctly given as

$$\sum_{n=0}^{\infty} \frac{30°}{12^n}. \qquad (3)$$

Measured clockwise from the twelve-hour mark, the precise position where our pair of runners will go "hand to hand" is readily found by substituting the appropriate values into equation (1), where *a*, our first term in the series, represents the hour hand's initial head start; and *r*, the common ratio of our series, is the rate at which the hour hand runs compared to its swifter mate:

$$s = \frac{30°}{1 - \frac{1}{12}} = 32\tfrac{8}{11}°. \qquad (4)$$

Starting on the twelve, the minute hand will sweep through $32\frac{8}{11}°$ before it exactly coincides with the hour hand. What time does this arc distance represent? Since the minute hand travels 360 degrees in 60 minutes, it covers 1 degree every $\frac{1}{6}$ of a minute; to cover $32\frac{8}{11}$ degrees, then, it must take exactly $5\frac{5}{11}$ minutes, or 5 minutes $27\frac{3}{11}$ seconds. These, demonstrably, are the limits in space and time upon which the swifter hand converges in order to "catch" the slower.

There are, of course, other places on the clock where the minute hand overruns the hour hand. At first glance it might seem obvious that twelve different places exist, but a little experimentation will show that the number is really *eleven*. This horological fact can actually be used to verify the limit we obtained in equation (4) by infinite series: if the minute hand overtakes the hour hand eleven times during the latter's 360-degree circuit, then each point where it does so must occur at consecutive intervals of 360°/11—that is, $32\frac{8}{11}°$.

We can wind the clock hands manually and locate each of the eleven limit points for ourselves, checking the spacing of these points with a protractor. Although only an expensive vernier protractor can come close to confirming the $32\frac{8}{11}$ degrees found by equation (4), the measurements, if taken carefully with an ordinary protractor, will prove significantly close to what the theory of infinite series "predicts."

In this way, Zeno's mythic footrace becomes a credible laboratory exercise for students of beginning calculus. It gives a real sense of the location of a limit in both space and time. Additionally, the clock-hand analogy shows that there is very good reason for trusting the formula for the sum of a geometric series (equation (1))—it squares precisely with the division of 360 degrees by 11, furnishing an empirical check on our mathematical theory, which would be impossible to do with any pair of runners on a straight course. ◘

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HOW DO YOU FIGURE?

# Challenges in physics and math

## Math

### M146

*In the summertime.* Mary will spend all her 90 days of vacation in the country. She's very well organized and she decided to strictly follow this daily routine: every other day she's going to swim in a lake, every third day she's going to wash and clean, and every fifth day she'll do math problems. (On the first day Mary tried to do the first, the second, and the third, and got very tired.) How many of the 90 days will be "pleasant"—that is, she will only have to swim? And how many "boring" days—with no planned activities—will Mary have? (N. Vasilyev)

### M147

*Zero cyclic sum of products.* Each of the numbers $x_1, x_2, \ldots, x_n$ equals one or minus one. Prove that the equality $x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n + x_nx_1 = 0$ is possible only if $n$ is divisible by four. (A. Leontovich)

### M148

*Factorial numeration.* (a) Prove that any integer $a > 0$ can be uniquely represented in the form

$$a = a_n n! + a_{n-1}(n-1)! + \ldots + a_2 \cdot 2! + a_1 \cdot 1!,$$

where the coefficients $a_k$, $k = 1, \ldots, n$, are integers, $0 \le a_k \le k$, $a_n > 0$.

(b) Prove that any rational $b$, $0 \le b < 1$, can be uniquely written in the form

$$b = \frac{b_2}{2!} + \frac{b_3}{3!} + \cdots + \frac{b_n}{n!},$$

where $0 \le b_k < k$ for $2 \le k \le n$, $b_n > 0$.

### M149

*Triangles around a pentagon.* Each diagonal of a convex pentagon cuts off a triangle from it (fig. 1). Prove

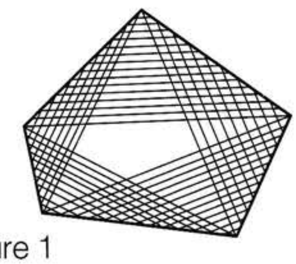


Figure 1

that the sum of the areas of these triangles is greater than the area of the pentagon. (N. Vasilyev)

### M150

*Taking roots in iterations.* Prove that there exists a function $f(x)$ whose $n$th iteration $f(f(\ldots f(x)\ldots))$ (with $f$ applied $n$ times) is equal to (a) $2x + 1$ for all $x$, (b) $1 + x + 2\sqrt{2}$ for all $x \ge 0$, (c) $x/(x + 1)$ for $x \ge 0$. (O. Izhboldin, K. Kohas)

## Physics

### P146

*Fox and hare.* A fox pursues a hare by heading straight for it. The hare happens to be cross-eyed, so it doesn't run along a straight extension of the line between the fox and itself—its velocity at any moment makes an angle of 60° to this line. The initial distance between the fox and the hare is $L$, and their speeds are $v$. How long will it take the fox to catch the hare? How far from the fox's starting position will that occur? How does the answer change if the hare becomes even more crossed-eyed, making an angle of 90°? Or if it wears eyeglasses so that the angle is only 40°? (O. Shpyrko)

### P147

*Oscillation of a parallelogram.* The construction shown in figure 2 consists of four light rigid rods of length $l$ and a light spring of length $2l$. The

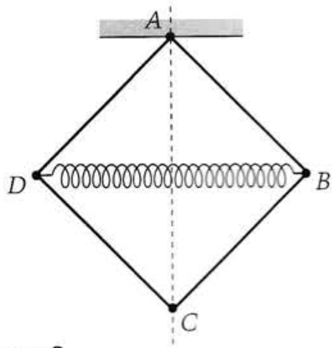



Figure 2

rods are connected by small identical massive balls. The rods are hinged so that their angles are free to change. The system is fixed at point $A$. In the system's equilibrium state the rods form a square. Find the period of small oscillations of this system when point $C$ moves along a vertical line. (S. Krotov)

### P148

*Heat engine.* The cycle carried out by a heat engine using a monatomic rarefied gas consists of two isochores (constant volume) and two isobars (constant pressure). Find the maximum efficiency of such a cycle. (Y. Krementsova)

### P149

*Variable capacitor.* A variable capacitor of initial capacitance $C_0$ is charged to a voltage $V$ and connected to a resistor $R$ (fig. 3). How must the capacitance be varied to keep the electric current constant?

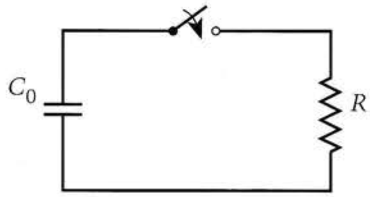


Figure 3

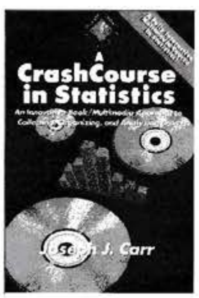
What power must be developed by external forces to change the capacitance? (P. Zubkov)

## P150

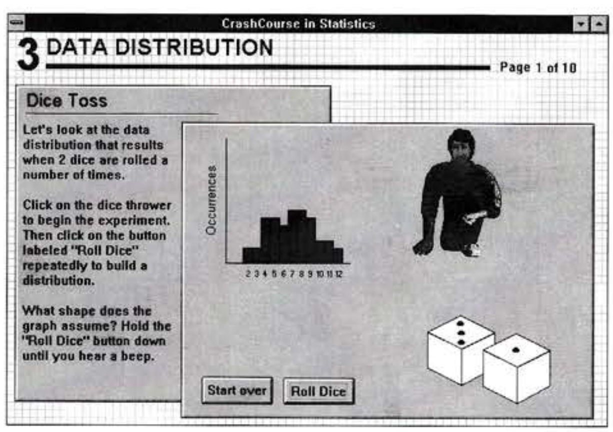
*Microscope and glasses.* While adjusting a microscope, a *Quantum* reader found that she saw the image of an object clearly with both eyes when it was placed at a distance $d = 6.5$ mm from the objective. The length of the microscope tube is $L = 100$ mm. The focal length of the objective is $f_1 = 6$ mm, and that of the eyepiece $f_2 = 26$ mm. What is the prescription for the glasses worn by this reader? (A. Yudin)

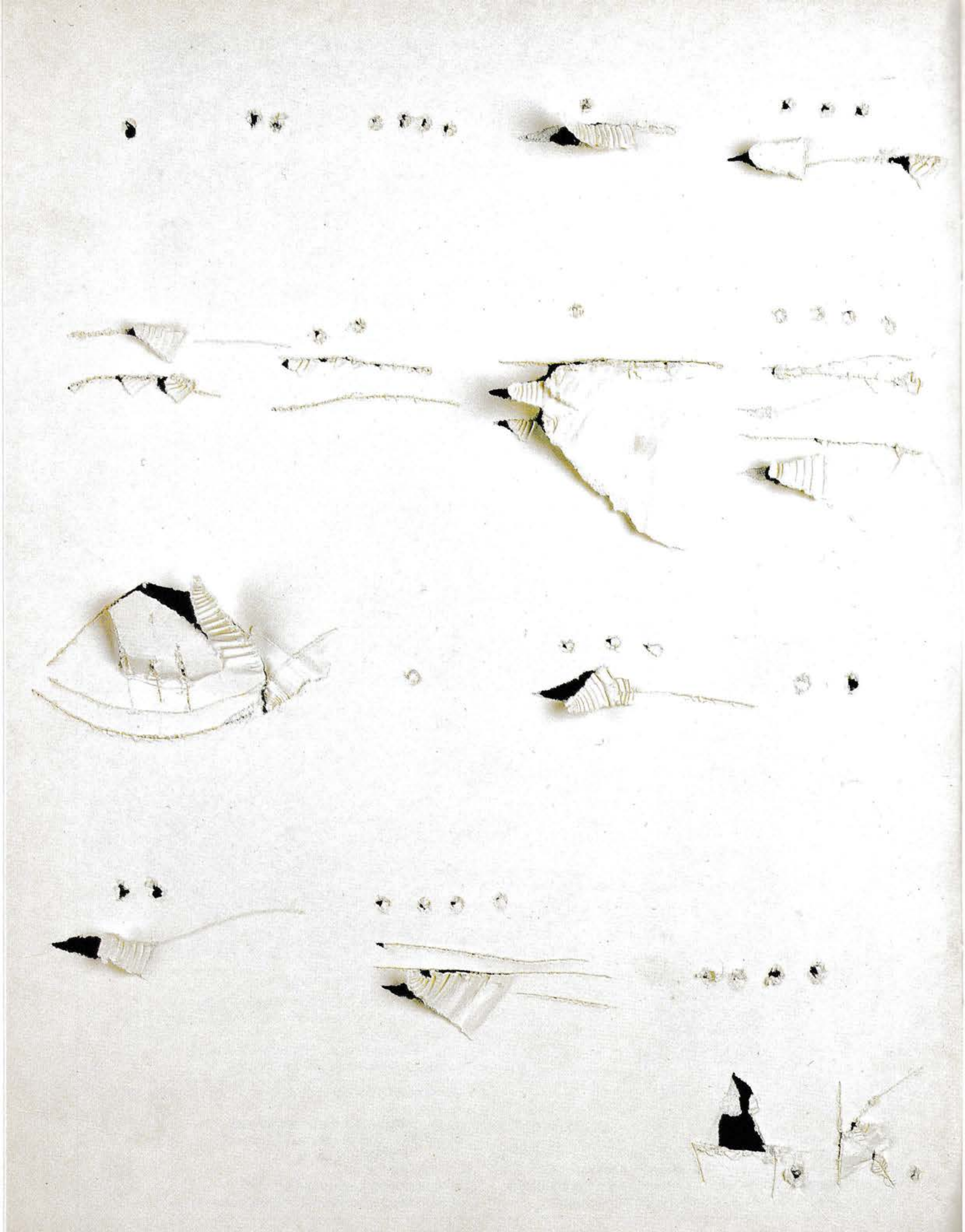
*ANSWERS, HINTS & SOLUTIONS ON PAGE 55*









# Number systems

## *Mayans, Romans, Babylonians—lend us your calculators*

by Isaak Yaglom

WE DESIGNATE THE FIRST nine natural numbers by the special symbols

$$1, 2, 3, 4, 5, 6, 7, 8, 9.$$

But it would be inconvenient in practice to assign a symbol to every number we use. Even if we didn't need to make any calculations beyond the first thousand, we'd have to memorize a thousand special symbols. So it's only natural that long ago people chose one or another set of "key" numbers, and only those were given special signs. Take, for instance, the Roman numeral system—it's based on the key numbers

$$1, 5, 10, 50, 100, 500, 1{,}000,$$

which are denoted by the Latin letters I, V, X, L, C, D, and M, respectively.

Some of these notations arose out of pictures that once represented these numerals (the Roman numeral I looks like an upraised finger, V looks like a hand with the fingers spread apart, X looks like two of these stylized hands in a mirror image), some came from the first letters of the corresponding Latin words (*centum* means one hundred, *demimille* means five hundred, *mille* means one thousand).

Since we need the Roman notation only as an example, we'll consider its older, simplified version, where the number "four" was written as IIII rather than IV, "four hundred" as CCCC rather than CD, and so on. In this old Roman system the number 3,477 is written as

$$\begin{gathered}\text{MMMCCCCLXXVII}\\ = 3 \cdot 1{,}000 + 4 \cdot 100 + 50 + 2 \cdot 10 + 5 + 2.\end{gathered}$$

The same rule of representing numbers is used by a cashier who has bank note of the denominations 100 rubles, 50 rubles, 25 rubles, 10 rubles, 5 rubles, 3 rubles, and 1 ruble. For the cashier, the key numbers are

$$100, 50, 25, 10, 5, 3, 1.$$

In order to pay, say, 499 rubles, she first gives out as many hundred-ruble notes as possible so as not to give more than is required:

$$499 = 4 \cdot 100 + 99.$$

Then she gives out as many fifty-ruble notes as possible without exceeding the amount remaining to be paid (99 rubles):

$$499 = 4 \cdot 100 + 1 \cdot 50 + 49,$$

and so on. Sometimes the remainder might be less than the next key number. In our example, this will be the case after two ten-ruble notes are given:

$$499 = 4 \cdot 100 + 1 \cdot 50 + 1 \cdot 25 + 2 \cdot 10 + 4.$$

The next key number is 5, but since $4 < 5$, five-ruble notes need not be given out. However, for consistency we can assume that there were zero five-ruble notes paid and include the term $0 \cdot 5$ in our sum. Then the entire process of payment will be written as

$$\begin{gathered}499 = 4 \cdot 100 + 1 \cdot 50 + 1 \cdot 25 +\\ 2 \cdot 10 + 0 \cdot 5 + 1 \cdot 3 + 1 \cdot 1.\end{gathered}$$

Now let's try to generalize these considerations, taking as key numbers (the *basis*) an arbitrary increasing sequence of natural numbers

$$q_0 = 1 < q_1 < q_2 < \ldots < q_n < \ldots. \quad (1)$$

Let's see how an arbitrary number $N$ can be written in the *number system with the basis given in expression (1)*.

Find the largest key number $q_n$ not exceeding $N$ and divide $N$ by $q_n$ to obtain the (incomplete) quotient $a_n$ and the remainder $r_{n-1}$:

$$N = a_n q_n + r_{n-1},$$

where $0 \le r_{n-1} < q_n$.

Now divide the first remainder $r_{n-1}$ by the next key number $q_{n-1}$:

$$r_{n-1} = a_{n-1} q_{n-1} + r_{n-2},$$

where $0 \le r_{n-2}$, or

$$N = a_n q_n + a_{n-1} q_{n-1} + r_{n-2}$$

(we don't exclude the case $r_{n-1} = 0$—in this case all the subsequent quotients and remainders will be zero). Now we divide the new remainder $r_{n-2}$ by $q_{n-2}$, which gives us

$$r_{n-2} = a_{n-2} q_{n-2} + r_{n-3}$$

where $0 \le r_{n-3} < q_{n-2}$, or

$$N = a_n q_n + a_{n-1} q_{n-1} + a_{n-2} q_{n-2} + r_{n-3},$$

and so on.

Finally, dividing the penultimate remainder $r_1$ by $q_1$, we obtain

$$N = a_n q_n + a_{n-1} q_{n-1} + \dots + a_2 q_2 + a_1 q_1 + r_0,$$

where $0 \le r_0 < q_1$. (Since $q_0 = 1$, it's redundant to divide the last remainder $r_0$ by $q_0$—clearly, $r_0 = a_0 q_0 = a_0$.)

A cashier might write this repeated division as one long column. I'll demonstrate this with the example of the sum of 499 rubles we encountered above:

$499 \div 10 = 4$
$\underline{400}$
$99 \div 50 = 1$
$\underline{50}$
$49 \div 25 = 1$
$\underline{25}$
$24 \div 10 = 2$
$\underline{20}$
$4 \div 5 = 0$
$\underline{0}$
$4 \div 3 = 1$
$\underline{3}$
$1 \div 1 = 1$
$\underline{1}$
$0$

(Here the quotients $q_n, q_{n-1}, \dots$ are printed in red and the number $N$ together with the remainders $r_{n-1}$, $r_{n-2}, \dots$ are printed in blue.)

## Positional number systems

I guess you must have figured out by now that the basis

$$q_0 = 1,\ q_1 = 10,\ q_2 = 10^2,\ \dots,\ q_n = 10^n,\ \dots$$

generates the usual *decimal* number system. But instead of writing, say, $4 \cdot 10^5 + 0 \cdot 10^4 + 3 \cdot 10^3 + 0 \cdot 10^2 + 1 \cdot 10 + 7$, we simply write 403,017.

An absent-minded cashier, or one who is overly meticulous, could prepare an expense sheet for writing down her payments, where the number of paid bank note of each denomination would be recorded in a separate column. Here's how the payment of 499 rubles would look in such an expense sheet:

| | 100 | 50 | 25 | 10 | 5 | 3 | 1 |
|---|---|---|---|---|---|---|---|
| 499 | 4 | 1 | 1 | 2 | 0 | 1 | 1 |

We can say that in the "cashier's number system" the number 499 is written as 4112011.

Number systems like those we considered in the last two examples (that is, the decimal system and the cashier's system) are called *positional* (I'll explain the meaning of this term later). Instead of the cumbersome expression $N = a_n q_n + a_{n-1} q_{n-1} + \dots a_2 q_2 + a_1 q_1 + a_0$ in the positional number system with the basis (1), it's convenient to use a more compact notation consisting of $n + 1$ "digits": "$a_n a_{n-1} \dots a_2 a_1 a_0$" (note the quotation marks around the string of variables). Of course, we assume that the "digits" $a_k$ are obtained by the method (algorithm) described above.

It follows from this description that every natural number $N$ has a unique notation for a given basis. Further, since the "leading digit" $a_n$ is obtained by dividing $N$ by $q_n$, where $N < q_{n+1}$ (because otherwise we'd begin with dividing $N$ by $q_{n+1}$ rather than $q_n$), we have $a_n < q_{n+1}/q_n$. Therefore, if $A$ is the (unique) integer satisfying $A - 1 < q_{n+1}/q_n \le A$, then the "digit" $a_n$ is no greater than $A - 1$—that is, it can take $A$ values: $a_n = 0, 1, 2, \dots, A - 1$. (For the *leading* digit of a number the value $a_n = 0$ also has to be excluded.) Similarly, the digit $a_{n-1}$ is obtained in the process of dividing the first remainder $r_{n-1}$ by $q_{n-1}$. But since $r_{n-1} < q_n$, the inequality $a_{n-1} < q_n/q_{n-1}$ holds. This reasoning can be extended to all the other digits as well. In particular, since the last digit $a_0$ coincides with the last remainder $r_0$, obtained after division by $q_1$, this digit can take $q_1$ values $a_0 = 0, 1, 2, \dots$, or $q_1 - 1$.

It took some time for people to come up with the idea of a positional number system. One of the difficulties on the road to this discovery was the absence of the number zero—and, of course, a special sign for it. The use of a sign for zero was pioneered by the Babylonians in their sexagesimal (that is, base-60) number system. In Babylonian texts written in their peculiar cuneiform characters (see figure 1), the numbers from 1 to 59 were denoted according to the *decimal* system. But the main

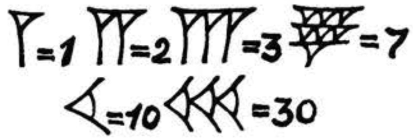



Figure 1

number system of Babylonian mathematics was sexagesimal—with the basis $1, 60, 60^2, \dots, 60^n, \dots$. Mathematicians who used cuneiform arrived at the idea of having a special character for zero rather late (but certainly not later than the third century B.C.). This character is shown in figure 2.

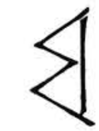


Figure 2

After these explanations, you'll be able to read the notation in figure 3 yourself.

Figure 3

Yes, it denotes the number $12 \cdot 60^3 + 0 \cdot 60^2 + 21 \cdot 60 + 32 =$ 2,593,292. The number system of the ancient Mayan civilization was very close to the Babylonian system. The creation of this system dates back to the first century A.D. While the Babylonian system combined the features of the decimal and sexagesimal systems, the Mayan system combined the bases 5 and 20. The first 19 numbers were written using bars, denoting fives, and dots, denoting ones (fig. 4).

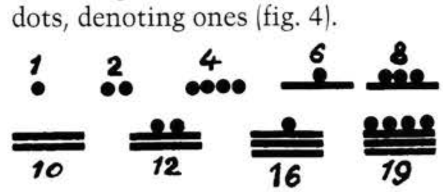



Figure 4

But the main role was played by a "distorted" system in base 20. The "digits" were written one under the other, the most significant one on top and the least significant one on the bottom. Here's what the distortion was: the third key number (after 1 and 20) was $18 \cdot 20 = 360$ rather than $20^2 = 400$, and it was followed by $18 \cdot 20^2$, $18 \cdot 20^3$, $18 \cdot 20^4$. There was also a special character for zero, which resembled a half-closed eye (fig. 5).

Figure 5

Figure 6 shows several examples of Mayan numbers.

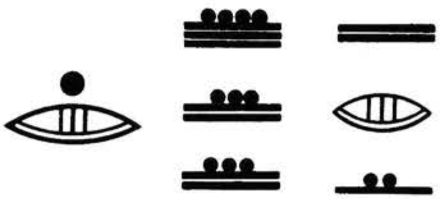


Figure 6

*The numbers depicted are $1 \cdot 20 + 0 = 20$; $19 \cdot 360 + 13 \cdot 20 + 13 = 7{,}113$; $10 \cdot 360 + 7 = 3{,}607$.*

The main difference between the Babylonian and Mayan number systems, on the one hand, and the Roman system, on the other, consists of the positional principle of the first two systems: while Romans always understood the letter I as one and V as five regardless of where these letters stood, for Babylonians and Mayans the value of a digit significantly depended on its position. This is why number notations of this kind, which include our decimal system (created in India in the 8th–9th centuries or a little earlier) are called *positional.*

**Exercises**

1. In the "complete cashier system" (based on the key numbers 10,000, 5,000, 2,500, 1,000, 500, 300, 100, 50, 20, 15, 10, 5, 3, 2, and 1, which express in kopecks the values of all bank note and coins used in the Soviet Union 10 to 15 years ago), write the sum of 233 rubles 87 kopecks.[1] Write down the operation of converting 23,387 kopecks into the cashier system using continuous division (see the end of the first section).

2. Read the number written in the Babylonian system in figure 7.

Figure 7

[1]One ruble equals 100 kopecks.

3. Read the number written in the Mayan system in figure 8 (the largest number found in Mayan texts).

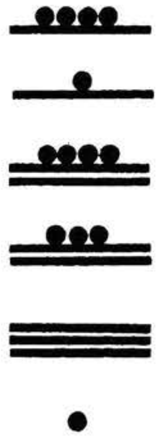


Figure 8

## Number systems to a given base

That's what we call systems of numeration with the basis

$$q_0 = d^0 = 1,$$
$$q_1 = d^1 = d,$$
$$q_2 = d^2,\ q_3 = d^3,\ q_4 = d^4,\ \ldots, \qquad (2)$$

where $d$ is any integer greater than one. The number $d$ is called the *base* of this number system.

Such systems include the familiar decimal system to the base $d = 10$; the Babylonian sexagesimal system ($d = 60$); and the binary system ($d = 2$), widely used in computers. The system to an arbitrary base $d$ is referred to as *d-nary* system.

As before, we use the notation $N = $ "$a_n a_{n-1} \ldots a_1 a_0$" in a $d$-nary system (with quotes around the variables) to mean that $N = a_n d^n + a_{n-1} d^{n-1} + \ldots + a_2 d^2 + a_1 d^1 + a_0$. Clearly, any digit $a_k$ in this notation can take $d$ values $0, 1, 2, \ldots, d-1$. In particular, with $d = 10$—that is, with the basis $q_0 = 1$, $q_1 = 10$, $q_2 = 100$, …—we arrive at the common decimal system (counting by ones, then by tens, hundreds, thousands, and so on); in this system, no digit ever exceeds 9.

The simplest $d$-nary system is the binary system of numeration with the basis $q_0 = 1$, $q_1 = 2$, $q_2 = 4$, $q_3 = 8$, $q_4 = 16$, … . There are only two digits in this system, 0 and 1. Here's a list of the first 15 natural numbers in the binary notation (as you may recall, a subscript indicates the base of a number):

| | | |
|---|---|---|
| $1_{10} = 1_2$ | $6_{10} = 110_2$ | $11_{10} = 1011_2$ |
| $2_{10} = 10_2$ | $7_{10} = 111_2$ | $12_{10} = 1100_2$ |
| $3_{10} = 11_2$ | $8_{10} = 1000_2$ | $13_{10} = 1101_2$ |
| $4_{10} = 100_2$ | $9_{10} = 1001_2$ | $14_{10} = 1110_2$ |
| $5_{10} = 101_2$ | $10_{10} = 1010_2$ | $15_{10} = 1111_2.$ |

In this system, calculations are rather long but extremely simple. If we always used this system, pupils would have to memorize only this "multiplication minitable":

$$0 \cdot 0 = 0,\ 0 \cdot 1 = 1 \cdot 0 = 0,\ 1 \cdot 1 = 1$$

(and an "addition table" that reduces to the equality $1 + 1 = 10$, because $10_2$ is $2_{10}$!).

**Exercises**

4. Rewrite the decimal number $N = 123{,}456$ in the (a) 7-nary number system; (b) duodecimal (12-nary) system, which involves 12 digits: 0, 1, 2, …, 9, X = 10, Y = 11; (c) binary system.

5. What is the decimal notation for the binary numbers $P = 100100$, $Q = 101010101$?

6. Write out the addition and multiplication tables in the ternary (base-3) system of numeration.

## Number systems with other bases

Number systems with bases that are not geometric sequences $1, d, d^2, d^3, \ldots$ do not have many applications. But they can sometimes prove useful in solving certain mathematical problems.

Let's consider some examples of such systems.

I. *Mayan system.* As you already know, the basis of the Mayan number system has the form $q_0 = 1$, $q_1 = 20$, $q_2 = 18q_1 = 18 \cdot 20$, $q_3 = 20q_2 = 18 \cdot 20^2$, $q_4 = 20q_3 = 18 \cdot 20^3$, … .

This system is similar to 20-nary system in all respects but one: in the Mayan notation $N = $ "$a_n a_{n-1} \ldots a_2 a_1 a_0$" of an arbitrary number (which we write horizontally rather than in the genuine Mayan vertical way), the second digit (from the right) takes 18 values—

$$0 \le a_1 < \frac{q_2}{q_1} = 18,$$

whereas all the other digits take 20 values 0, 1, 2, …, 19.

This is how matters stand in any number system with a basis for which $q_{n+1}$ is exactly divisible by $q_n$ for all $n = 0, 1, 2, \ldots$:

$$q_0 = 1,\ q_1 = d_0,\ q_2 = d_1 q_1 = d_1 d_0,$$
$$q_3 = d_2 q_2,\ q_4 = d_3 q_3,\ \ldots, \qquad (3)$$

where $d_0, d_1, d_2, d_3, \ldots$ are *any* integers greater than one—they can be identical or different. In this number system the notation

$$N = "a_n a_{n-1} \ldots a_2 a_1 a_0"$$

has the first digit from the end, $a_0$, taking $d_0$ values 0, 1, ..., $d_0 - 1$; the next digit, $a_1$, taking $d_1$ values from 0 to $d_1 - 1$; the next digit, $a_2$, taking values from 0 to $d_2 - 1$; and so on.

Not only that—just as for a $d$-nary system (which occurs in the case $d_0 = d_1 = d_2 = \ldots = d$), in the number system with basis (3) *any* expression $N = "a_n a_{n-1} \ldots a_2 a_1 a_0"$ with nonnegative integers $a_0, a_1, \ldots, a_n$ less than $d_0, d_1, \ldots, d_n$, respectively, has meaning. Indeed, it's easy to see that converting the number $N = a_n q_n + a_{n-1} q_{n-1} + \ldots + a_1 q_1 + a_0$ into our number system by the "continuous division" method,[2] we'll successively obtain the digits $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$.

However, such a smooth state of things isn't the general rule, as we'll see in an example of one very simple "number system."

II. *Even-number system.* Let's take as the basis the number $q_0 = 1$ and all *even* numbers: $q_0 = 1$, $q_1 = 2$, $q_2 = 4$, $q_3 = 6$, $q_4 = 8$, ... .

Then $q_1/q_0 = q_1 = 2$, and

$$\frac{q_{n+1}}{q_n} = \frac{2(n+1)}{2n} = \frac{n+1}{n} = 1 + \frac{1}{n} < 2$$

for all $n > 1$.

So this number system, like the binary system, knows only two digits—0 and 1. Printing the numbers written in this "even" system of numeration in boldface, we come up with the following representations:

$$2 = \mathbf{10},\ 3 = \mathbf{11},\ 4 = \mathbf{100},$$
$$5 = \mathbf{101},\ 6 = \mathbf{1000},\ 7 = \mathbf{1001},$$
$$8 = \mathbf{10000},\ 9 = \mathbf{10001},$$

and so on. And in general, all numbers are represented either as a one with several zeros (if the number is even) or as two ones separated by a string of zeros (odd numbers).

Thus, in the even numeration any number is written in the form $N = "a_n a_{n-1} \ldots a_2 a_1 a_0,"$ where each of the digits $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ can only take one of the two values 0 or 1. But the overwhelming majority of 0–1 sequences *do not* express any numbers in "even" numeration, because the "meaningful" sequences can have ones only in the first and, sometimes, the last places.

And the last example.

III. *Seller's system.* The standard set of weights for a pan balance usually included a set of 12 weights: 10 g, 20 g (two of these), 50 g, 100 g, 200 g (two of these), 500 g, 1 kg, 2 kg (two of these), and 5 kg.[3] A seller uses these to weigh out any multiple of 10 g much as a cashier uses bank notes. Weighing a certain item—say, a piece of meat—the salesperson puts on the balance the heaviest weight that doesn't outweigh the meat, then adds the heaviest of the remaining weights, and so on. For instance, if the meat weighs 3,460 g, the following weights will be used: one 2-kg weight, one 1-kg weight, two 200-g weights, one 50-g weight, and one 10-g weight.

This system of weights was very familiar to tradespeople and their customers in many parts of the world until recently. To generalize this to a mathematical representation of numbers (weights), we add three smaller weights (1-g, 2-g, and 5-g) to get a numeration system with the basis

$$1, 2, 5, 10, 20, 50, 100, 200,$$
$$500, 1{,}000, 2{,}000, 5{,}000,$$

in which the weight 3,460 g is written as 11020101000.

The digits in the expression $N = "a_n a_{n-1} \ldots a_2 a_1 a_0"$ of an arbitrary number $N$ in this system can take the following values: the last digit $a_0$ can be equal to 0 or 1 (since $q_1 = 2$); the next one, $a_1$, takes the values 0, 1, or 2 (since $2 < q_2/q_1 < 3$); the third digit from the end, $a_2$, again is either 0 or 1 (since $q_3/q_2 = 2$); $a_3$ is also 0 or 1, but $a_4$ can take three values, 0, 1, or 2 (since $2 < q_5/q_4 < 3$). In general, in the "seller's numeration" the digits $a_1, a_4, a_7, a_{10}, \ldots, a_{3k+1}$ can take the three values 0, 1, and 2, while all the rest can only equal 0 or 1. (This is why our seller, using the set of weights originally described, can manage with only one 1-kg and one 5-kg weight, but needs two 2-kg weights.)

**Exercises**

7. Write the numbers $X$, $Y$, $Z$ in decimal notation if (a) $X$ = 1000000001 in the even notation; (b) $Y$ = 121121 in the seller's system; (c) $Z$ = 20120 in the seller's system.

8. Describe all possible sequences of digits "$a_n a_{n-1} \ldots a_2 a_1 a_0$" that can be read as a certain number written in the seller's system.

9. In a number system with basis (1) two numbers are represented as $N$ = 10211004, $M$ = 10210437. Can you tell which of them is greater?

10. The system of weights described above is more convenient for weighing on a pan balance than the decimal system would be, but it isn't the most economical in terms of the number of weights. To see why, solve the following problems. (a) What smallest number of weights is necessary to weigh out any integer number of kilograms from 1 through 30 on a pan balance if the weights are allowed to be placed on only *one* pan? (You can choose any weights you like.) (b) Answer the same question, assuming you're allowed to put weights on both pans. (c) Answer the same questions (with both assumptions (a) and (b)) if you have to weigh out any integer number of grams from 1 to 1,000.

11. Prove that in the ternary system (which has the basis (2) with $d = 3$) any number $N$ can be represented as $A - B$, where all the digits in the ternary representations of $A$, $B$, and $A + B$ are only zeros and ones. Prove that such a representation is always unique. For instance, $2 = \mathbf{2} = \mathbf{10} - \mathbf{1}$, $7 = \mathbf{21} = \mathbf{101} - \mathbf{10}$, $28 = \mathbf{1001} = \mathbf{1001} - \mathbf{0}$ (the numbers in boldface are ternary).[4]

12. Prove that the condition "$q_{n+1}$ is divisible by $q_n$ for all $n$ = 0, 1, 2, ..." is necessary for any expression $N = "a_n a_{n-1} \ldots a_2 a_1 a_0"$ with $a_0 < q_1/q_0$, $a_1 < q_2/q_1$, ..., $a_n < q_{n+1}/q_n$, to make sense in the numeration system with basis (1).

[2]An interesting example of this type of number system is discussed in math challenge M148 in this issue.—*Ed.*

[3]Let's not hear from the physicists, who insist that "grams" designate mass only, and not weight!—*Ed.*

[4]This is the *balanced ternary system,* which was used in some computers.—*Ed.*

13. What conditions must numbers of the basis $q_0, q_1, q_2, \ldots$ satisfy such that the notation of any number consists only of the digits 0 and 1? What additional requirement must be imposed on the basis so as to eliminate any notations with two ones in a row?

14. Recall that the Fibonacci series is defined as $q_0 = 1$, $q_1 = 2$, $q_2 = 3$, $q_3 = 5$, $q_4 = 8$, $q_5 = 13$, $q_{n+1} = q_n + q_{n-1}, \ldots$ .[5] Prove that the "Fibonaccian number system" satisfies all the requirements of the previous problem. Find the following sums in Fibonaccian notation:

(a) $\underbrace{100\ldots00}_{k} + \underbrace{100\ldots00}_{l}$;

(b) $\underbrace{10101\ldots01}_{2m+1} + 1$;

(c) $\underbrace{10101\ldots01}_{2m+1} + \underbrace{100\ldots00}_{2m+1}$.

15. Is it possible to split all natural numbers into two increasing sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ such that $b_k - a_k = k$ for any $k = 1, 2, 3, \ldots$?

After some thought you'll of course understand how to choose these sequences (this can be done uniquely—see figure 9). But it's not very easy to figure out the rule for generating the pairs $(a_k, b_k)$. How can we learn, for example, which of the two sequences—$a_k$ or $b_k$—contains 100, what the corresponding number $k$ is, and what number makes a pair with 100 without writing out all the preceding $a_k$ and $b_k$?

| | |
|---|---|
| $a_1$ | 1 |
| $\boldsymbol{b_1}$ | **2** |
| $a_2$ | 3 |
| $a_3$ | 4 |
| $\boldsymbol{b_2}$ | **5** |
| $a_4$ | 6 |
| $\boldsymbol{b_3}$ | **7** |
| $a_5$ | 8 |
| $a_6$ | 9 |
| $\boldsymbol{b_4}$ | **10** |
| $a_7$ | 11 |
| $a_8$ | 12 |
| $\boldsymbol{b_5}$ | **13** |
| $a_9$ | 14 |
| $\boldsymbol{b_6}$ | **15** |
| ⋮ | ⋮ |

Figure 9

It turns out that this rule can be formulated in terms of the Fibonaccian rather than the decimal representation of the number in question. Try to find and prove this rule.

*ANSWERS, HINTS & SOLUTIONS ON PAGE 60*

[5]*Quantum* ran an article devoted to this famous sequence—see the July/August 1992 issue, p. 15.—*Ed.*

# Call for manuscripts

NINETEEN NINETY-SIX marks the 25th anniversary of The Club of Rome's study *The Limits to Growth.* To provide its young readers with both information and current perspectives on this study, *Quantum* invites the submission of papers for a special issue on *The Limits to Growth* and its 1992 sequel, *Beyond the Limits.*

Several authors have already expressed an interest in writing for such a special issue. Victor Gorshkov of the St. Petersburg Nuclear Physics Institute will prepare a paper that presents ideas from his recent book *Physical and Biological Bases of Life Stability* (Springer, 1995). Kurt Kreith will show how a spreadsheet investigation from "Look, Ma—No Calculus!" (*Quantum,* November/December 1994) illustrates "the four generic ways in which a population can approach its carrying capacity."

We seek additional papers that analyze this study and its implications from a variety of points of view. Such papers might address the changes (and growth!) that have occurred since the publication of *The Limits to Growth.* They might also address the advances in computer technology that make such models ("state of the art" in 1970) accessible via desktop computers available at most American high schools and in many secondary schools around the world. Or they might review both the study and its critics, shedding light on the ways in which science and public opinion interact in the search for solutions to the environmental challenges confronting the current generation of students.

Prospective authors are invited to send a query to

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AT THE BLACKBOARD I

# Circuits and symmetry

*An eye for symmetry can help your hand hack through the algebra behind the diagrams*

by Gary Haardeng-Pedersen

THE CONCEPT OF SYMMETRY is very useful in many physical situations. Many fundamental physical laws express symmetry in their formulation—for example Newton's law of universal gravitation or Coulomb's law. Other physical laws are derived explicitly from the requirement of symmetry. For example, the laws of relativity—whether Galilean relativity or Einstein's special relativity—are based on the symmetry of the observed physics for different observers.

The remainder of this article will deal specifically with electrical circuits where there is a symmetry evident in the circuit diagram. (See, for example, problem P39 in the November/December 1991 issue of *Quantum* and the article by S. N. Lykov and D. A. Parshin in the same issue.)

A classic example of an arrangement of resistors that is difficult to analyze using standard techniques—but which is very simple using the evident symmetry—is a set of identical resistors along the edges of a cube (fig. 1). If each of the 12 resistors has resistance $R$, what is the equivalent resistance between two corners that are diagonally opposite?

By way of review, let's consider a system where two identical resistors, each of resistance $R$, are placed in parallel with a current $I$ entering (and leaving) the combination (fig. 2). The symmetry makes it obvious that half of the current passes through each of the resistors so that the voltage across the system is

$$V = \tfrac{1}{2}IR,$$

and the equivalent resistance is

$$R_{\text{eq}} = \frac{V}{I} = \tfrac{1}{2}R.$$

Also, when two identical resistors are in series (fig. 3), the current through one must be the same as the current through the other. The voltage drop across the pair, from the symmetry of the situation, must be twice the voltage drop across each individual resistor. Hence a voltage drop $V = 2IR$ occurs across the pair of resistors, so that the equivalent resistance of the pair is

$$R_{\text{eq}} = \frac{V}{I} = 2R.$$

A standard textbook problem is to determine the equivalent resistance for the system shown in figure 4a. Five identical resistors, each of resistance $R$, are connected in a simple symmetrical arrangement that cannot be further reduced using series or parallel equivalences. No two of the resistors are in series and no two of the resistors are in parallel.

Nevertheless, the symmetry of the circuit is the key to its solution. Label the four junctions $a$, $b$, $c$, $d$ as in figure 4a and assume that a current $i$ flows from $b$ to $c$ through the resistor that joins these two junctions.

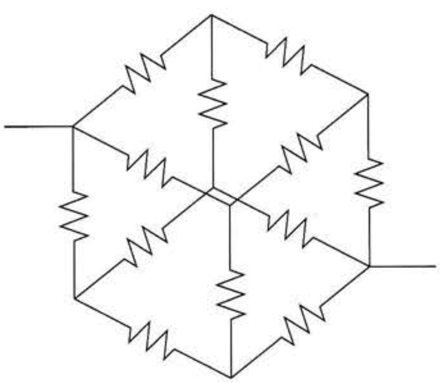


Figure 1

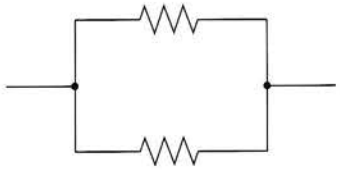


Figure 2

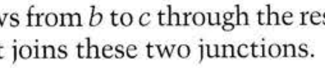


Figure 3

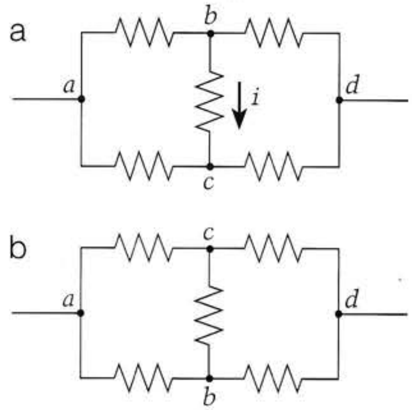



Figure 4

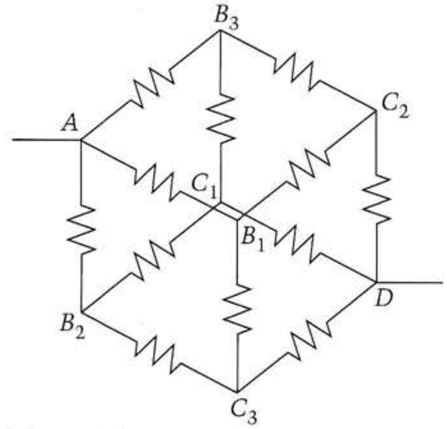



Figure 5

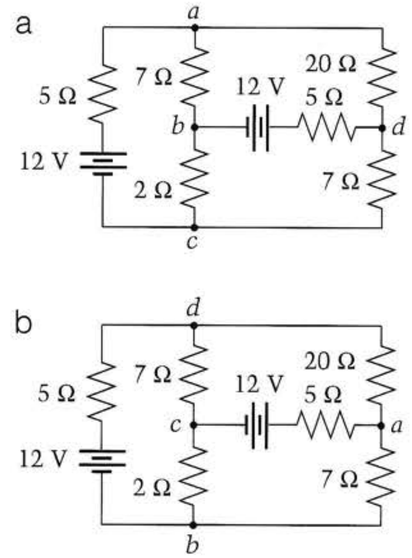



Figure 6

Now redraw the circuit as shown in figure 4b—the only difference is the interchange of the upper and lower branches (and consequently the junctions $b$ and $c$). In this modified circuit, the current $i$ flows from $c$ to $b$ because of the equivalence between the original and the modified circuit, but flows from $b$ to $c$ according to our original assumption. The only solution to this conundrum is that the current $i$ must be zero—only thus can it fulfill both conditions.

Once it has been determined that the current between $b$ and $c$ is zero, we can see that the branch between $b$ and $c$ can be deleted without altering the flow of current in the circuit. After this deletion, the two remaining resistors joined at $b$ are in series, with an equivalent resistance $2R$; those joined at $c$ are in series with an equivalent resistance $2R$. These two equivalent resistors are in parallel between junctions $a$ and $d$, so that the equivalent resistance between $a$ and $d$ is $2R/2 = R$. In fact, it is now apparent that any value of the resistance between $b$ and $c$ will carry a zero current whenever the resistances $R_{ab} = R_{ac}$ and $R_{bd} = R_{cd}$, where $R_{ab}$ is the resistance between points $a$ and $b$, and so on.

Now let's return to the problem illustrated in figure 1: the set of 12 identical resistors along the edges of a cube. Assume that a current $I$ enters one corner (label this corner $A$) and leaves the corner diagonally opposite (label it $D$). Of the remaining six corners, we'll label the three that are one resistor from $A$ as $B_1$, $B_2$, and $B_3$. Label the remaining three corners that are one resistor from $D$ as $C_1$, $C_2$, and $C_3$ (see figure 5).

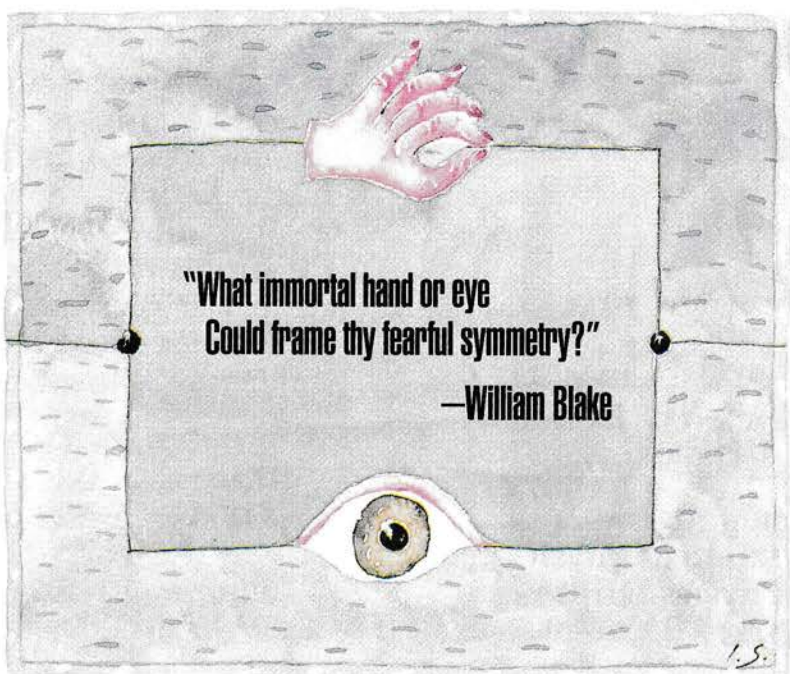
From the symmetry, it's apparent that the current $I$ splits at the junction $A$ into three equal parts. One third of $I$ goes from $A$ to $B_1$, one third from $A$ to $B_2$, and one third from $A$ to $B_3$. Again, from the symmetry, at the junction $B_1$ the current (from $A$) splits equally. One sixth of $I$ goes from $B_1$ to $C_2$, the other one sixth of $I$ from $B_1$ goes to $C_3$. Junction $C_3$ also receives a current of one sixth of $I$ from $B_2$, so the total current from $C_3$ to $D$ is one third of $I$. By any path from $A$ to $D$, the total voltage drop is

$$V = \frac{I}{3}R + \frac{I}{6}R + \frac{I}{3}R = \frac{5}{6}IR,$$

and the single resistance that would have this voltage drop while carrying the same current $I$ is

$$R_{\text{eq}} = \frac{V}{I} = \frac{5}{6}R.$$

Now consider the circuit in figure 6a. It has four junctions, labeled $a$, $b$, $c$, $d$. It has a pair of equal batteries, a pair of 5-Ω resistors and a pair of 7-Ω resistors, so there is a certain amount of symmetry. Once again redraw the circuit, in the hope that it may appear simpler when




Art by Sergey Ivanov

redrawn. Place the battery and the resistor in the branch between $b$ and $d$ on the left of the diagram, with $b$ and the negative terminal of the battery near the bottom. Two other paths connect $b$ to $d$. One path is through $c$, so in the redrawn circuit, put $c$ midway between $b$ and $d$, with the 7-Ω resistor in the branch $dc$ and the 2-Ω resistor in the branch $bc$. Place $a$ to the right of $c$. Junction $a$ is connected to $b$ through a 7-Ω resistor, to $d$ through a 20-Ω resistor, and to $c$ through the 5-Ω resistor and the 12-V battery. In the branch $ca$ the positive terminal of the battery is connected to $a$.

It's obvious that the redrawn circuit (fig. 6b) isn't any simpler than the original circuit; on the other hand, it isn't any more complicated. In fact, it appears at first glance to be exactly the same as the original circuit. But there are two important differences. Assume that a current $i_1$ flows from $b$ to $c$ in the original circuit—downward through the 2-Ω resistor. Then in the redrawn circuit, a contradiction appears. Since the circuit is the same as the original circuit, the current should flow downward through the 2-Ω resistor; but this would be a current not from $b$ to $c$ but from $c$ to $b$! The only way to avoid the contradiction is for the current through the 2-Ω resistor to be zero. In exactly the same fashion, the current through the 20-Ω resistor can also be shown to be zero. All of the current actually flows through the path that has the two batteries, the pair of 5-Ω resistors, and the pair of 7-Ω resistors. In fact, the current strength is 1 A. The symmetry argument used depends on the equality of the battery voltages and the matching of the two pairs of resistances. The actual values taken for the other two resistances are immaterial.

So—look for symmetry in physical situations. When you find it, use it to reduce the amount of algebra you need to solve the problem.

I leave one final problem for you to puzzle out. Consider the arrangement of the 12 identical resistors that make up the edges of a cube. What is the equivalent resistance between two corners that are on the same face, diagonally opposite each other (fig. 7)?

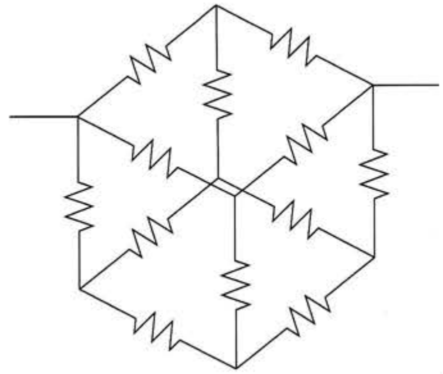
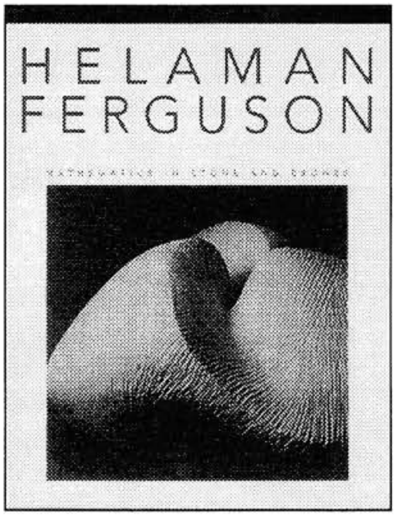


Figure 7

**Gary Haardeng-Pedersen** *is an associate professor of physics at Sir Wilfred Grenfell College (a campus of Memorial University of Newfoundland), located in Corner Brook, Newfoundland, Canada.*


HELAMAN
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# Lazy-day antidotes

*Some things to quicken your mind in the good ol' summertime*

IT HAS BECOME A TRADITION in our summer issues to publish collections of recreational problems that you can use in a math competition at summer camp, or just solve yourself when you don't have anything better to do (like the character in problem M146 on page 20). Here's another set of such problems, along with five games that have winning strategies for one of the players. Have fun solving the problems and finding the winning ways!


**Problems**

1. Ten coins are arranged in a triangle as shown in the drawing at right. What smallest number of coins must be removed so that the centers of no three of the remaining coins are the vertices of an equilateral triangle?

2. Think of four integers whose sum and product are both odd.

3. A clock reads 8:20. What is the angle between the hands?

4. Count the fingers on your left hand in the following order: the thumb will be 1, forefinger 2, middle finger 3, ring finger 4, little finger 5; then reverse yourself: the ring finger is 6, middle finger 7, forefinger 8, thumb 9; now reverse yourself again: forefinger is 10, and so on. Which finger will get the number 1995?

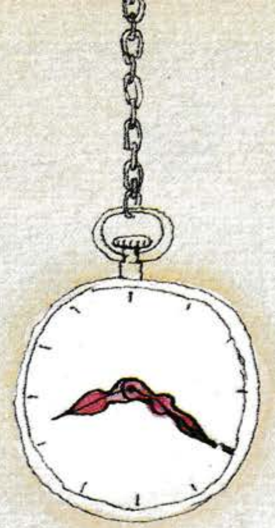
5. Do triangles exist such that the midpoints of their altitudes are on the same straight line?

6. Seven candles were burning, and three of them were blown out. How many were left?

7. The side lengths of a triangle are 17, 35, and 18. What is its area?

8. Find the last digit of the product of all odd numbers from 1 to 99.

9. When Sally walked out of the forest with her dog Rex and headed home, her brother Harry left the house to take a walk in the forest. Rex rushed home, but when he encountered Harry, he immediately turned around and ran to Sally,



only to turn around again and run back to Harry. He kept running to and fro until Harry met Sally. What distance did Rex cover if Sally walked at a speed of 3 km/h, Harry walked a little faster at 5 km/h, and Rex sped along at 8 km/h? The distance from the forest to the house is 4 km.

10. Two spaceships are flying toward each other. Their speeds are 15,000 km/h and 21,000 km/h, and their launching sites are 1995 km apart. How far apart will the spaceships be one minute before they meet?

11. What number is exactly divisible by all numbers?

12. The titles of three famous adventure novels begin with the numbers 2, 20, and 20,000. Do you know what they are?

13. The base of an isosceles triangle is twice as long as its altitude. Find its angles.

14. An angle of 1° is viewed through a lens with 4x magnification. How big will the angle appear to be?

**Games**

1. Two players take turns removing pebbles from two piles—one pile has 13 pebbles, the other has 10. Each player is allowed to do one of two things: (a) take any number of pebbles from either pile or (b) take the same number of pebbles from both piles. The player who takes the last pebble wins.

2. Two players take turns removing one, two, or three pebbles from a pile of 25 pebbles. The player who ends up with an even number of pebbles wins.

3. Two players have two piles of candies, 9 candies in each pile. A move consists of moving a candy from one pile to the other and eating two candies from either pile. The players take turns moving and eating candies. The player who can't make a move loses.

4. One of two players puts a white checker on any square of a chessboard, the other puts a black checker on any other square. Then they move their checkers in turn, each time onto an adjacent square (horizontally or vertically). The player who manages to put his or her checker on the opponent's checker wins.

5. A chess knight is set on a corner square of a chessboard. Two players take turns marking squares (say, with a piece of chalk)—one square at a time—in such a way that the knight can reach any unmarked square without hitting a marked square. The player who can't make a move loses. ◘

—Compiled by A. Savin

*ANSWERS IN THE NEXT ISSUE*

Art by Pavel Chernusky

PHYSICS CONTEST

# Pins and spin

*"Nay, sometimes,*
*Like to a bowl upon a subtle ground,*
*I have tumbled past the throw . . ."*
—The Tragedy of Coriolanus, *Act V, Scene 2*

by Arthur Eisenkraft and Larry D. Kirkpatrick

ALTHOUGH ISAAC NEWTON probably never bowled a perfect 300 game, his physics can be used to analyze the sport of bowling. The collision of the ball with the bowling pins (and the collisions of the pins themselves) must obey the conservation laws. If we ignore the friction of the ball and pins with the floor during the impact, linear momentum in the horizontal directions must be conserved during these collisions. As the pins fly through the air, they conserve linear momentum, angular momentum, and mechanical energy. The path of each center of mass between collisions is the same as those of the projectiles we've all studied in class.

In this contest problem, let's concentrate on the interaction of the ball with the surface of the lane. Those of us who throw a "curve ball" know that this interaction is very important. Let's imagine that we throw a ball parallel to the fourth board in from the right-hand gutter. This ball would obviously not produce a strike as it will miss the head pin and probably only take out the three pins (6-9-10) on the far right-hand side.

At the time of release, we right-handers can lift our fingers to put spin on the ball so that it curves to the left and hits the "pocket" between the 1 and 3 pins. The curve changes the ball's angle of approach toward the pins and yields a higher percentage of strikes than a "straight ball." This pocket is also more forgiving in that it yields fewer splits.

Analyzing the curve ball involves looking at two rotations: the spin and the rotation down the lane. As we usually do when faced with a problem in physics, let's begin with the simplest case—the straight ball. We assume that the ball of radius $r$ and mass $m$ is thrown horizontally with an initial speed $v_0$, with no initial rotation, and at a negligible height above the floor. The ball starts with no rotation, loses speed, and picks up rotation due to the friction with the floor, and at some point it rolls without slipping. We are interested in how long this process takes.

Because there is no acceleration in the vertical direction, we know that the force of gravity on the ball is canceled by the normal force of the floor on the ball. Therefore, the net force is due to the frictional force $f$ with the floor, which we assume to have the standard form $f = \mu mg$, where $\mu$ is the coefficient of kinetic friction. Therefore, Newton's second law tells us that

$$f = ma = -\mu mg,$$

or the linear acceleration $a$ of the ball is

$$a = -\mu g.$$

The frictional force also exerts a torque $\tau = fr$ on the ball about its center. According to Newton's second law for rotation we have

$$\tau = I\alpha = fr = \mu mgr,$$

where $I = 2/5\ mr^2$ is the moment of inertia of the ball about its center. The resulting angular acceleration is given by

$$a = \frac{5\mu g}{2r}.$$

From the kinematics equations for translational and rotational motion, we have

$$v = v_0 + at = v_0 - \mu gt$$

and

$$\omega = \omega_0 + \alpha t = \frac{5\mu gt}{2r}.$$

Using the condition for rolling without slipping, $v = \omega r$, we can solve for the time when this first occurs:

Art by Tomas Bunk

f=ma=
BOWLING
Tom BUNK

$$t = \frac{2v_0}{7\mu g}.$$

At the instant the ball starts rolling without slipping, the speed of the ball is $5v_0/7$ and the ball has traveled a distance

$$d = \frac{12v_0^2}{49\mu g}$$

down the alley. Furthermore, we can use the final speed to calculate that the ball loses 2/7 of its initial kinetic energy. However, it is very interesting to note that the loss in kinetic energy is *not* equal to $fd$. How can you reconcile this?

Let's now return to the curve ball. The ball is initially spinning sideways without any sideways translational motion. This is the inverse problem and leads us to this month's contest problem. Let's use a cylinder instead of a ball to simplify the numbers as we did when this problem was given on the semifinal exam to select the 1995 US Physics Team. Besides, if the cylinder is long enough, bowling would be a lot simpler!

Assume that a uniform cylinder of mass $m$ and radius $r$ has an initial rotational velocity $\omega_0$ about its axis, which is horizontal. Assume further that we drop the spinning cylinder onto the floor from a negligible height.

A. How long is it before the cylinder rolls without slipping?

B. What is the translational speed of the cylinder at this time?

C. How far down the alley does this occur?

D. Use the speed of the cylinder to calculate the fraction of the initial rotational kinetic energy that is lost.

E. Show that this loss of energy can be explained using the work–energy theorem.

We leave it to you to apply the ideas in these two problems to the motion of the curving bowling ball. And we hope that your analysis improves your score!

Please send your solutions to *Quantum*, 1840 Wilson Boulevard, Arlington VA 22201-3000 within a month of receipt of this issue. The best solutions will be noted in this space and their authors will receive special certificates from *Quantum*.

## Cloud formulations

The clouds certainly cleared for Alex Lee, a student at Choate Rosemary Hall, as he solved the contest problem in the January/February issue. We will follow along with Alex as the problem unfolds.

A. The first part of the problem asked readers to determine the temperature $T_1$ at $M_1$, where the cloud forms.

Since the air is streaming adiabatically, we have two equations that hold:

$$PV^\gamma = \text{constant},$$
$$\frac{PV}{T} = \text{constant}.$$

We can combine these two equations to get

$$\frac{T_1}{T_0} = \left(\frac{P_1}{P_0}\right)^{1-\frac{1}{\gamma}}.$$

Therefore,

$$T_1 = 279.4 \text{ K} = 6.4°\text{C}.$$

B. Consider the pressure difference between $M_0$ and $M_1$. This must be caused by the extra chunk of air below $M_1$ and above $M_0$. Consider an imaginary cylinder with base area $A$ and height $h_1$ from $M_0$. Then we have

$$-P_1A + P_0A = mg,$$

where $m$ is the mass of air within the cylinder. Since the density of air varies linearly, the mass is

$$m = Ah_1\frac{\rho_0 + \rho_1}{2},$$

where $\rho$ is the density of air. The density is calculated from the equation of state:

$$\frac{P_0}{\rho_0 T_0} = \frac{P_1}{\rho_1 T_1},$$

$$\rho_1 = 1.054 \text{ kg/m}^3.$$

Solving for $h_1$ we get

$$h_1 = \frac{P_0 - P_1}{g\left(\frac{\rho_0 + \rho_1}{2}\right)},$$

$$h_1 = 14.08 \text{ m}.$$

C. At $M_2$ the air again streams adiabatically up the slope. In this movement, we must also take into consideration that there is an additional temperature change due to the condensation of water:

$$T_2 = T_i + \Delta T,$$

where $T_i$ is the temperature from the adiabatic process. From this we get

$$\frac{T_i}{T_2} = \left(\frac{P_2}{P_1}\right)^{1-\frac{1}{\gamma}},$$

$$T_i = 264.8 \text{ K}.$$

As for $\Delta T$, we know that the latent heat is

$$Q = mL_V = C_P\Delta T,$$
$$(2.45 \text{ g})(2{,}500 \text{ kJ/kg}) = (1{,}000 \text{ J/kg}\cdot\text{K})\Delta T,$$
$$\Delta T = 6.1 \text{ K},$$
$$T_2 = 264.8 \text{ K} + 6.1 \text{ K} = 270.9 \text{ K}.$$

D. The precipitation separated from the ascending column of air per square meter per second is

$$(2{,}000 \text{ kg/m}^2)(2.45 \text{ g/kg})(10^{-3} \text{ kg/g})(1/1{,}500 \text{ s})$$
$$= 3.3 \cdot 10^{-3} \text{ m}^{-2}\text{s}^{-1},$$
$$(3.3 \cdot 10^{-3} \text{ m}^{-2}\text{s}^{-1})(3 \text{ hr})(3{,}600 \text{ s/hr})$$
$$= 35.3 \text{ kg/m}^2.$$

Since 1 kg/m² results in a precipitation level of 1 mm, the depth of the precipitated water is 35.3 mm.

E. Whatever air that has passed over the mountain will probably descend adiabatically:

$$T_3 = T_2\left(\frac{P_3}{P_2}\right)^{1-\frac{1}{\gamma}},$$

$$T_3 = 300.0 \text{ K} = 26.9°\text{C}.$$

If there were no condensation and rain from the air, $T_3$ should be equal to $T_0$. Because of the rainfall, the air at $M_3$ is colder and less moist than at $M_0$.

## MATH INVESTIGATIONS

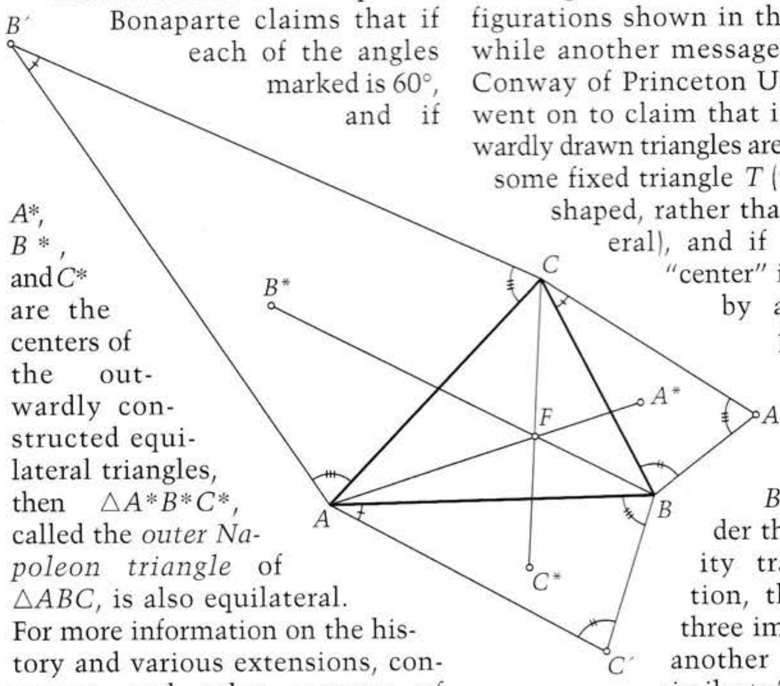
# Revisiting Napoleon's theorem

*"The last thing we want from you, General, is a lesson in geometry!"—Pierre-Simon Laplace*

by George Berzsenyi

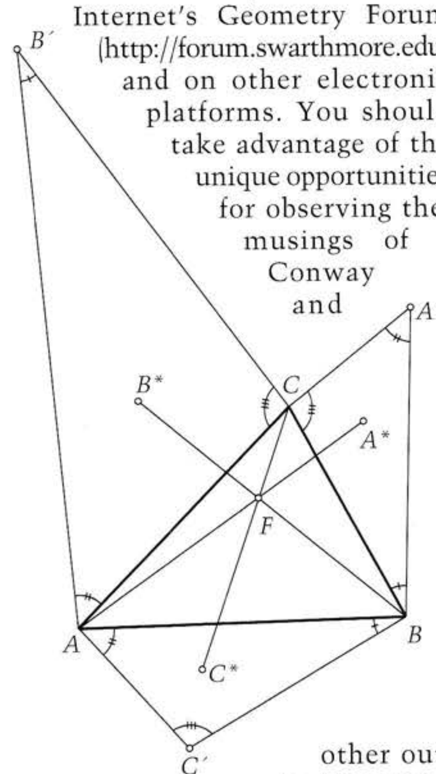
IN THE FIRST FIGURE BELOW, $\triangle ABC$ is arbitrary and points $A'$, $B'$, and $C'$ are chosen so that the angles marked the same way are equal to one another.

The same is true in the second figure; however, it should be noted that in the two figures we require the equality of different pairs of angles. In its simplest form, the theorem attributed to Napoleon Bonaparte claims that if each of the angles marked is 60°, and if $A^*$, $B^*$, and $C^*$ are the centers of the outwardly constructed equilateral triangles, then $\triangle A^*B^*C^*$, called the *outer Napoleon triangle* of $\triangle ABC$, is also equilateral. For more information on the history and various extensions, converses, and other aspects of Napoleon's theorem the reader is referred to John E. Wetzel's recent article in *The American Mathematical Monthly* (volume 99, number 4, pp. 339–351) and the references cited there.




My own interest in Napoleon's theorem was rekindled by a couple of messages that appeared in one of the geometry newsgroups on the Internet (geometry.pre-college or geometry.puzzles) and were called to my attention by my friend and former colleague, Bradley Brock. One of these messages (by Michael de Villiers of South Africa) discussed generalizations to the configurations shown in the figures, while another message (by John Conway of Princeton University) went on to claim that if the outwardly drawn triangles are similar to some fixed triangle $T$ (that is, $T$-shaped, rather than equilateral), and if the word "center" is replaced by any fixed point $P$ of $T$ and its images—say, $A^*$, $B^*$, $C^*$, under the similarity transformation, then these three images form another triangle similar to $T$. Conway conjectures that in the general cases depicted in the two figures, the lines $AA^*$, $BB^*$, $CC^*$ also concur in a point $F$, and that at $F$, if it is interior to $\triangle ABC$, the three sides of $\triangle ABC$ all subtend the same angles as the angles subtended by the sides of $T$ at the point $P$.

My challenge to my readers is to verify the above claims, extend them by constructing the triangles congruent to $T$ inwardly, and join the fun of similar explorations in the Internet's Geometry Forum (http://forum.swarthmore.edu) and on other electronic platforms. You should take advantage of the unique opportunities for observing the musings of Conway and other outstanding practitioners of mathematics, and learn how to follow in their footsteps.




In closing, I want to thank my readers for their responses to the problems posed in the last two columns. Most of my next column will be devoted to their findings.

LOOKING BACK

# The giants

*"A giant is like a hill rising up in the middle of a plain."*
*—Kozma Prutkov*

by Vladimir Belonuchkin

EVERYBODY KNOWS THAT Isaac Newton was one of the greatest geniuses in the history of science. And it was he who flatly disagreed with the "esteemed" Kozma Prutkov.[1] "If I have seen farther than others," wrote Newton, "it is because I have stood on the shoulders of giants." Who were these giants? Kepler, certainly, and Galileo, and Copernicus. But what of the others—the giants who lived before them?

The first scientist we know by name, Thales of Miletus (6th century B.C.), made a contribution to astronomy. Legend has it that he predicted the solar eclipse of May 28, 584 B.C. However, there are reasons to suppose that he used the method developed in ancient Babylon. The Babylonian method was purely empirical: the observations of many years made it possible to discern a regularity in the repetition of heavenly phenomena.

The first "theoretical" postulate, which marked the beginning of a scientific description of the universe, was probably the notion that the Earth is spherical. There were two dogmas underlying the first astronomical systems describing the universe: the apparent immobility of the Earth and the uniform circular motion of the Sun, Moon, and planets around the Earth. The fact that the second dogma was incorrect was known even to the ancient astronomers, but nevertheless it was this dogma that survived for two millennia—until Kepler's time.

Even if it seems that a heavenly body moves nonuniformly along a noncircular path, it nevertheless moves uniformly about a point, which in turn revolves uniformly about another center, and so on, until finally there is a point revolving uniformly about the Earth. The ancient Greek astronomer Eudoxus was the first to construct a complete model based on this reasoning, in the 4th century B.C. His scheme consisted of 27 circles (spheres), and it accounted for the motion of the Sun, Moon, and the five known planets with an accuracy quite impressive for that period.

In the course of time the accuracy of the observations improved, and this led to the need for more and more spheres. Callippus, a pupil of Eudoxus, needed 33 spheres. Aristotle brought the number to 56. This system of circles and spheres—improved by the hard work of many astronomers (Hipparchus first among them), perfected by Ptolemy (and named after him), and sanctioned by the authority of Aristotle—was the only accepted scheme of the universe even for such a great scientific revolutionary as Copernicus, who "merely" shifted the center of the universe from the Earth to the Sun once and for all. Only Kepler managed to finally discard the idea of the circular uniform motion of the planets.

As you probably know, Kepler was prompted to reject the old postulate because of an eight-minute deviation in the position of Mars from its predicted location. Tycho Brahe, whose observations Kepler used, had achieved such accuracy in his measurements that these eight minutes could not be attributed to observational error. If the truth be told, one and a half centuries before Brahe, the great Muslim astronomer Ulugh Beg achieved the same—and even higher—accuracy. But Samarkand is far from Europe, and Ulugh Beg also fell victim to his own religion: the upholders of Islam attempted to erase the memory of this heretic, even though he was the ruler of a great central-Asian state.

The accuracy achieved by Eudoxus was about 0.5°; Tycho Brahe managed an accuracy of about 2′, while Ulugh Beg's accuracy was of the order of 1′. With the invention of the telescope, the accuracy of measurements was greatly improved. But does this mean that the results of ancient and medieval astronomers no longer have any significance for us? Of course not.

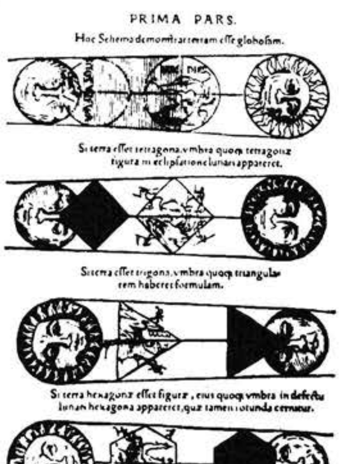
Art by Yury Vashchenko

[1]A 19th-century Russian writer (actually, the pseudonym of three writers) whose aphorisms are more quotable than deep.—*Ed.*

"Despite the railway, keep your horse-drawn cart," advises the inimitable Kozma Prutkov. The records of ancient scientists—Greek, Egyptian, Babylonian, Chinese, Inca—help establish the long-term regularities in the motion of the Earth, planets, and "stationary" stars. Here is an example.

**Problem 1.** The world largest reflecting telescope (in the former Soviet Union) has an objective mirror with a diameter $D = 6$ m. How long must a star be observed with this telescope in order to find its velocity more accurately than can be found using the data of Eudoxus? What about the data of Tycho Brahe and Ulugh Beg?

To calculate the velocity of a star we must determine its position at least two times. The accuracy of telescopic measurement of the coordinates of heavenly bodies depends on the resolving power of the telescope, which is fundamentally limited by diffraction phenomena. The minimum angle that can be measured with an objective of diameter $D$ is approximately $\lambda/D$, where $\lambda$ is the wavelength of the radiation used in the observations. For example, the wavelength of yellow light is $\lambda \cong 6 \cdot 10^{-7}$ m, which means that the maximum measurement accuracy for the Soviet telescope is about $10^{-7}$ radians. I leave the remaining calculations to the reader (you can check your answers in the back of this issue).




*A proof that the Earth is spherical (from a handbook for making sundials, 1531). The text reads: "This drawing shows that the Earth is round." And further: "If the Earth were square, triangular, or hexagonal, we would see a shadow with the corresponding shape during a lunar eclipse."*

*"The Skeptic, or Pilgrim on the Edge of the Earth"—a 19th-century woodcut by Camille Flammarion.*

The struggle against the dogma postulating the immobility of the Earth began in the 3rd century B.C. The first astronomer to do battle on this front was Aristarchus of Samos. Although even Pythagoras and Heracleitus considered the Sun the center of the universe, Aristarchus was the first to try to substantiate this notion. Comparing the relative sizes of the Sun and the Earth (which he had calculated himself), he came to the conclusion that since the Sun was much larger than the Earth, it should be the center around which the planets, including the Earth, revolved. And this was eighteen hundred years before Copernicus!

Let's try to follow the calculations of Aristarchus in the next example.

**Problem 2.** The Sun is located much farther from the Earth than the Moon is. The angular sizes of the Sun and the Moon are practically the same, which means that the Moon's shadow on the Earth is a point. The Earth's shadow at the Moon's orbit is approximately twice the Moon's diameter (the precise value is 2.67). How much larger than the Moon is the Earth? Calculate this ratio using modern data.

To determine the size of the Sun, Aristarchus measured the angle between the Sun and Moon at the Moon's first and last quarters—that is, when exactly half of the moon is illuminated. But he made a serious blunder at this point: according to his observations, this angle differed from a right angle by 3° while the correct figure is 8.6′. It is difficult to measure this value in general, in part because of the "ashen" character of the light coming from the Moon: the Moon re-reflects solar light that the Earth scatters in its direction. This is why Aristarchus arrived at a result so different from the solution to the next problem.

**Problem 3.** Determine the ratio of the diameters of the Sun and Moon using data from Aristarchus. What will the result be when modern data are used? (Hint: Aristarchus considered the angle between the lines connecting the Earth and Moon and the Moon and Sun to be an exact right angle.)

The relative sizes of the Earth and the Sun obtained by Aristarchus was not doubted for more than two thousand years. Not until the middle of the 17th century did the founder of the Paris Observatory, Giovanni Domenico Cassini, make direct measurements of the distance between the Earth and the Sun and "increased" the size of the Sun.

As for the Earth, its circumference had been measured with good accuracy by Eratosthenes of Cyrene long before—in the 3rd century B.C. (do you remember the "sieve" of Eratosthenes, used to obtain prime numbers?). He knew that once a year in Syene (modern Aswan) objects do not

# The Astronomical Giants

## A partial roster

**Thales of Miletus** (fl. c. 580 B.C.)—Greek philosopher, founder of ancient Greek philosophy.

**Pythagoras** (c. 580–c. 500 B.C.)—ancient Greek mathematician and philosopher. Created an astronomical system positing a central fire around which the celestial bodies revolve, including the Sun, the Earth, and a "counter-Earth." He is said to have invented the "harmony of the spheres" and to have recognized that the morning star and the evening star are in fact one and the same object (Venus).

**Heracleitus** (c. 540–c. 480 B.C.)—Greek natural philosopher who exerted a great influence on the scientific thought of his time. He believed change to be the only certainty of the universe and fire its main principle. He held that all men have a universal soul. He was known as the "weeping philosopher" for his pessimistic view of life.

**Eudoxus of Cnidus** (c. 400–c. 350 B.C.)—Greek astronomer and mathematician. He considered the motion of planets a combination of 27 concentric spheres uniformly circling around the Earth.

**Callippus** (fl. 4th cent. B.C.)—Greek astronomer, student of Eudoxus. He corrected and improved his mentor's theory of concentric spheres to account for the movement of the Sun, Moon, and planets.

**Aristotle** (384–322 B.C.)—Greek philosopher and scientist. His works cover all branches of ancient knowledge. In astronomy Aristotle's interests were concentrated mainly on the structure of the universe. He also made some observations of celestial phenomena, comets, and shooting stars.

**Aristarchus of Samos** (fl. c. 270 B.C.)—ancient Greek astronomer. He was the first to maintain that the Earth revolves around the Sun, which he stated was fixed and located at the center of the sphere of the fixed stars.

**Eratosthenes of Cyrene** (c. 276–c. 194 B.C.)—ancient Greek scientist who worked in many fields of knowledge. In mathematics, for example, he invented a way of finding prime numbers. He was the first person known to have measured the Earth's circumference. He worked out a calendar that included leap years and tried to fix the dates of literary and political events since the fall of Troy.

**Hipparchus** (fl. 146–127 B.C.)—ancient Greek astronomer, one of the founders of astronomy. He determined the distance to the Moon, calculated the length of a year, and compiled a catalogue of 850 stars, which he subdivided into six classes according to their brightness.

**Poseidonius of Apamea** (c. 135–c. 51 B.C.)—ancient Greek philosopher. His scientific works covered all the fields of knowledge and gave ancient natural philosophy its final form. He ascribed tides to the combined action of the Sun and Moon and recognized the connection between tides and the phases of the Moon.

**Ptolemy** (fl. 2nd cent. A.D.)—ancient Greek astronomer, creator of the geocentric system of the universe. He developed a mathematical theory of the motion of planets about a stationary Earth, which made it possible to calculate the future positions of the planets in the sky.

**Roger Bacon** (c. 1220–1292)—Franciscan philosopher and naturalist. Wrote about optics, astronomy, geography, mathematics, and chemistry.

**Ulugh Beg** (1394–1449)—central-Asian statesman, scientist, and enlightener. Built one of the most significant observatories of the Middle Ages. Compiled an original catalogue of the positions of 1,018 fixed stars determined with unusual precision.

**Nicholas of Cusa** (1401–1464)—German clergyman, natural philosopher. One of the precursors of the Copernican cosmology and experimental natural science. He suggested that the Earth might rotate on its axis, and that observers on the Sun would see the Earth revolving about them.

**Nicolaus Copernicus** (1473–1543)—Polish astronomer, creator of the heliocentric system of the universe. He made a revolution in science by rejecting the dogma of the central position of the Earth, which was accepted for centuries. He explained the apparent motion of celestial bodies by the rotation of the Earth about its axis and the revolution of the planets, the Earth included, around the Sun.

**Tycho Brahe** (1546–1601)—Danish astronomer, reformer of practical astronomy. For more than twenty years he made astronomical observations in Uraniborg, in the large observatory he built. The observations were made with the highest precision possible for his time. Using his data, Kepler formulated the laws of planetary motion.

**Francis Bacon** (1561–1626)—English philosopher and statesman, forerunner of the English materialist school of philosophy. Formulated general principles of experimental investigation.

**Galileo Galilei** (1564–1642)—Italian scientist, founder of modern physics and telescopic astronomy. His scientific activity was of great importance for the victory of the heliocentric system of the universe.

**Johannes Kepler** (1571–1630)—German scientist, one of the founders of modern astronomy. He discovered the laws of planetary motion (known as Kepler's laws) and on this theoretical basis calculated the planetary tables.

**Giovanni Domenico Cassini** (1625–1712)—Italian astronomer. He became the first director of the Paris Observatory in 1671. Cassini discovered the rotation of Jupiter and Mars, two of Saturn's moons, and a dark division in its ring that now bears his name.

**Ole Rømer** (1644–1710)—Danish astronomer. Determined the velocity of light by observing the eclipses of Jupiter's moons.

cast shadows. Eratosthenes traveled to Syene and confirmed this. Thus he realized that during the summer solstice the Sun in Syene was at the zenith. In Alexandria, where Eratosthenes lived, the Sun at the summer solstice fell short of the zenith by 1/50 of a circumference. The distance between Alexandria and Syene is about 5,000 stadia (a stadium was a Greek unit of length), and the towns are located on virtually the same meridian. It was a simple problem for Eratosthenes, and it is now being offered to you.

**Problem 4.** Find the Earth's circumference using the data given above.

A solution in stadia can be obtained at once: 250,000. But how does it look in kilometers? Those who want to flatter Eratosthenes choose a value of 157 m for one stadium. This gives "astronomical" accuracy—less than 2%—for the ancient astronomer's data. However, two points remain unclear. First, Syene is not located exactly at the tropic of Cancer; second, it's strange that a century and a half later, Poseidonius of Apamea measured the Earth's diameter with an even larger error: he obtained a value of 240,000 stadia—that is, the error worsened threefold in a century and a half. The most probable value for a stadium seems to be about 190 m.

Let's return to Aristarchus. He was the first to suggest the idea of heliocentrism, but he failed to subvert the evidence in favor of the existing dogma. So for another two thousand years, the Earth was stuck in place. As time went by, however, the accuracy of the measurements increased, and it was more and more difficult to reconcile planetary motion with Ptolemy's scheme.

In this long story of how the geocentric system of the universe was overthrown, one particular cardinal of the Catholic Church, Nicolaus of Cusa, is worthy of mention. His arguments were completely theological: as the almighty Lord is omnipresent, any point in the universe must be equidistant from Him and can claim the role of the center of the universe. But one mustn't think that the merits of this scholar can be boiled down to that one sentence.

Roger Bacon in the 13th century, Nicolaus of Cusa in the 15th, and Francis Bacon at the beginning of the 17th century laid the groundwork of the modern scientific method, whose main idea can be expressed in the dictum: "Experiment is the criterion of truth." Some may ask: Isn't that a bit meager—one thesis for four centuries? But we should recall that for more than a thousand years the criterion of truth was that it be in accord with the Bible and with Aristotle. Indeed, as early as the 17th century one could readily fall into the clutches of the Inquisition just for counting the number of legs on a spider. Aristotle wrote that a spider has six legs, but if you tried to count them yourself, you'd certainly obtain eight (provided the spider isn't an invalid)—and this would be a great sin. But it was Nicolaus Copernicus who managed to strike the death blow to the geocentric system.

At the beginning of the 17th century, the Dutchman Hans Lippershey built a telescope, fulfilling a prediction made by Roger Bacon long before, and the news spread like wildfire. Two years later, on January 7, 1610, Galileo pointed his own improved model at Jupiter and found that this huge planet had four moons. Now we know that Jupiter has more than a dozen moons, but the four largest are still called Jupiter's Galilean moons. By 1670 Cassini had obtained a realistic value for the radius of the Earth's orbit. Five years later Ole Rømer realized Galileo's dream and measured the velocity of light, using Jupiter's moons and Cassini's data. Let's solve the same problem, though in a somewhat different way.

**Problem 5.** The period of time between two successive eclipses of Jupiter's satellite Io changes during a year from the minimum value of 42 h 28 min 21 s to a maximum of 42 h 28 min 51 s. Find the velocity of light.

Since Jupiter's orbit is much larger than the Earth's, and Jupiter's velocity is far less than the Earth's, we can assume that in the course of a year the relative positions of these planets do not change significantly. The apparent variations in the period of rotation of Io are connected only with the change in the direction of the Earth's velocity. The value for this velocity is constant and equal to 29.8 km/s.

As time passed, more data accumulated, accuracy increased, and empirical laws were obtained. The time has come to explain them. However, this exceeds the limits of the present article. The reader is invited to explore previous articles in *Quantum*, such as "The Fruits of Kepler's Struggle" (January/February 1992) and "The Universe Discovered" (May/June 1992). ◙

*ANSWERS, HINTS & SOLUTIONS ON PAGE 61*

# MATHEMATICAL SURPRISES

# Dr. Matrix on the wonders of 8

## *MS found in a file cabinet*

by Martin Gardner

*EDITOR'S NOTE*: THE NARRATIVE THAT FOLlows is based on a manuscript discovered by the managing editor while cleaning out his files. It bears the date July 28, 1988. As *Quantum* was but a gleam in a Soviet émigré's eyes in the summer of eighty-eight, it is not at all clear how the manuscript came to be lodged in this particular file cabinet. It is offered to our readers as a serendipitous bit of July/August reading matter.

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My good friend Dr. Irving Joshua Matrix, the world's greatest numerologist, was not killed by a Russian KGB agent in 1980 as I once reported. He is alive and well in Casablanca, as I shall reveal in my book *From Penrose Tiles to Trapdoor Ciphers*.

I had occasion to see Dr. Matrix at a recent conference on superstring theory. The centennial celebration of the American Mathematical Society (AMS) will begin on August 8 of this year. The date can be written 8/8/88, a pattern of digits that I was sure would interest my friend. When I asked him for comment about the numbers 8, 88, and 888, he smiled slightly, put his fingertips together, and spoke for almost an hour. Here are some of his remarks that I managed to jot down while he talked.

Eight, he began, is one of the most interesting of digits. From the standpoint of pure arithmetic, it is the second cube, the sixth Fibonacci number, the sum of the first three digits of pi, and so on, but these he considered dull and trivial facts. Applied to the outside world, 8 takes on more glamour as the notes of an octave, the arms of an octopus, the 8-cylinder motor, stock market eighths, the figure-8 knot, the skater's figure 8, the 8-hour workday, the eight Beatitudes, the predicament of being behind the 8-ball, old Spanish pieces of 8, and the eighth wonder of the world.


Art by Sergey Ivanov

Turning to his specialty, bizarre numerology, Dr. Matrix called 8 both the holiest and the largest of the digits. It is the holiest, he said, because it has two holes. The other hole digits—0, 6, 9, and sometimes 4—have only one hole. It is the largest because, given a quarter turn, it becomes infinity.

Half of 8 is 0 or 3, depending on whether you slice it horizontally or vertically. Dr. Matrix credited his friend Gerald J. Cox with noticing that three fourths of 8 is 9 or 6, depending on which of its four arcs you remove, but since 6 actually *is* three fourths of 8, the fact is of no numerological interest. Written as a Roman numeral, VIII is half of XIII, at least the top half, and we should remember that 8 leads the list when all the English names for numbers are alphabetized. Of course, he added, 8 is the past tense of "eat." At this point Dr. Matrix leaned back in his chair, his emerald eyes twinkling, and recited the following line from *Finnegans Wake*, which he said, so help me, I could find on page 142, line 2, of the Viking edition: "And who eight the last of the goosebellies . . ."

When I asked Dr. Matrix for some number problems involving 8, he at once proposed a set of seven tasks, all involving the insertion of plus or minus signs into the ascending sequence 123456789 or the descending sequence 987654321 to produce sums of 8, 88, and 888. He began with 88, the last two digits of the year 1988. There are, he assured me, six ways to insert plus or minus signs into the ascending sequence to obtain a sum of 88. Here is one:

$$1 - 2 - 3 + 4 - 5 + 6 + 78 + 9 = 88.$$

Can the sum be achieved with as few as four signs? Yes, this the minimum number, and the solution is unique. Can you find it without writing a computer program?

The descending sequence has sixteen solutions, but only one with as few as five signs. Can you work it out?

If a minus sign is permitted in front of a sequence, Dr. Matrix continued, other solutions are possible. With a minus sign in front of the ascending sequence, there are eleven ways to make a sum of 88, but none is unique for a specified number of signs. However, if a minus sign is put in front of the descending sequence, there is just one solution. Hint: it uses eight signs, counting the minus sign in front.

If the desired sum is 888, there is no solution for the ascending sequence, either with or without a minus sign in front. For the descending sequence there are three ways to obtain 888 with signs inside the sequence, and one way to do it with a minus sign in front. Hint: it uses five signs altogether.

Now for the digit 8 as the sum. There are, said Dr. Matrix, eight solutions for the ascending sequence without the minus sign in front, but none is unique for a given number of signs. With the minus sign in front, there are ten solutions, but only one with as few as five signs. Can you find it?

There are three ways to obtain 8 for the descending sequence with no minus sign in front. One is unique for seven signs—the maximum. The descending sequence with the minus sign in front has six solutions. The smallest number of possible signs is five. The solution is unique, but not easy to find.

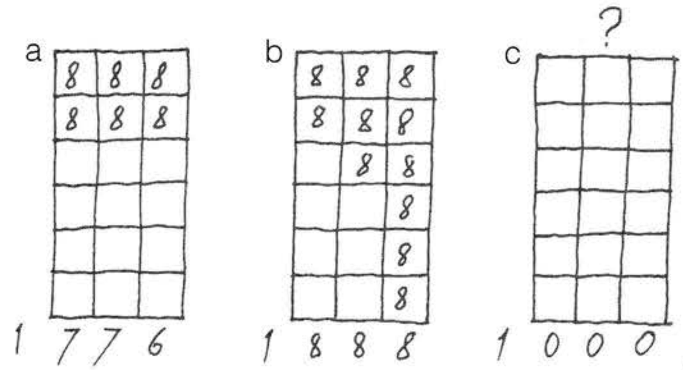


Figure 1

Dr. Matrix drew a $3 \times 6$ matrix on my notepad. First he penciled six 8's in the cells (fig. 1a) to produce a sum of 1776, the most famous date in US history. He added five more 8's (fig. 1b) to raise the sum to 1888 (the year the AMS was founded). Then he erased all the 8's and asked: "Can you place eight 8's in the cells (fig. 1c) to make the sum exactly 1,000?

"Here's an elegant little problem," Dr. Matrix continued. "It involves the location of 8 in a $3 \times 3$ magic square." He sketched on my notepad (fig. 2a) the ancient *lo shu*, or Chinese magic square, and called my attention to the 8 in the lower right corner. Alongside it he drew an empty matrix of nine cells, then penciled 8 into the top middle cell (fig. 2b). The problem: add eight whole numbers, no two of the nine numbers alike, to the vacant cells to form a magic square with the *lo shu*'s constant of 15—that is, the sum of each row, each column, and the two main diagonals is 15. The solution is unique.

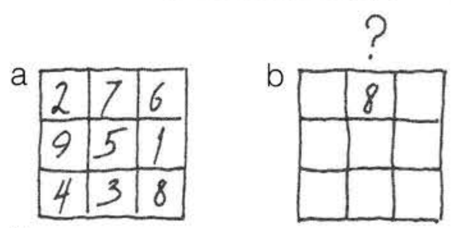


Figure 2

"Would you say that the centennial of the AMS is a special occasion?" Dr. Matrix asked.

"Of course," I replied.

"If you own a New Testament, King James version," he went on, "you might check the eighth book, the eighth chapter, the eighth verse, and the eighth word. Note also that the word has eight letters."

Before we parted, Dr. Matrix asked me to divide 987654321 by 123456789. I took out my pocket calculator and punched in the numbers. The quotient blew my mind!

**Martin Gardner** *wrote the Mathematical Games column in* Scientific American *for 25 years. Among his many books,* The Magic Numbers of Dr. Matrix *(Prometheus Books) contains all of Gardner's interviews with the notorious numerologist before his apparent death in a pistol duel near Istanbul.*

# Canopies and bottom-flowing streams

*Physics in a spoonful of water*

by Ivan Vorobyov

IF YOU'RE EVER ASKED, "WHAT kind of a stream fills a spoon more quickly: a weak stream or a strong one?"—watch out! It's a trick question. In a strong stream of water a spoon stays virtually empty, while a weak stream fills the spoon completely—even "more than completely"! But—first things first.

## Streams weak and strong

You can do the following experiment and see for yourself that the weak stream results in an almost horizontal water surface that curves at the brim, where the water then flows along the outer surface to the middle of the spoon's bottom and again forms a stream (fig. 1). This lower stream isn't particularly stable—it's sensitive to where the initial stream meets the water surface in the spoon, as well as to the angle and cleanliness of the spoon. Sometimes several streams form, which quickly break down into droplets.

A strong stream, however, spreads as a thin layer from the point where it hits the spoon, and the water flows off the edge of the spoon as a broad canopy fringed from below with thin streams and drops (fig. 2). It's pretty clear why the water forms a thin film as it splashes out. The falling water has enough energy to run up the brim with a nonzero velocity. Then in the course of its free fall the small streams come together to form a thin curved surface.

Consider the trajectories of different parts of the water as independent, each of them parabolic—that is, the water particles move under the action of gravitation only. Then it's easy to evaluate the horizontal velocity $v$ at the top of the canopy of water (fig. 3). During the time $t$ that the particles travel between the spoon's brim and the top of the canopy, the horizontal displacement is $x = vt$ and the vertical displacement is $y = gt^2/2$. Having measured $x$ and $y$, we can now find the velocity:

$$v = x\sqrt{\frac{g}{2y}}.$$

In a set of typical measurements, the values obtained were $x = 10$ cm, $y = 4.5$ cm, and $v \cong 1$ m/s.

In order to clarify whether the energy lost during impact and due to the friction between the stream and the spoon is essential for our reasoning, it's appropriate to compare the velocity $v$ with the velocity $u$ of the original stream at the same level as the parabola's apex. Measuring the time it takes to fill a glass (whose volume is 200 ml), we obtain the flow per second $q = \pi r^2 u$, where $r$ is the stream's radius. Thus,

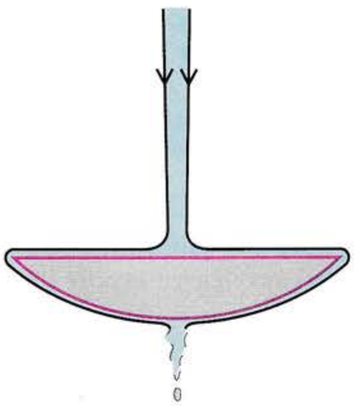

Figure 1

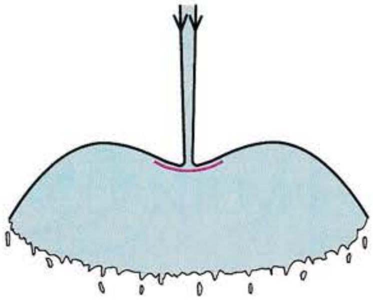

Figure 2

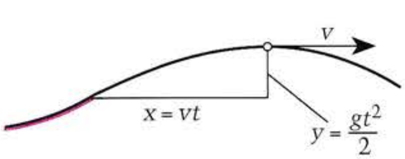


Figure 3

$$u = \frac{q}{\pi r^2}.$$

In the case considered above, we get $u \cong 1.4$ m/s. The velocities aren't identical, but they're close. So for the purposes of our rough estimates, we can neglect the energy losses due to the initial impact and the friction of the water against the spoon.

Now let's return to the weak stream. Paradoxically, in this case the energy of the falling water is also more than enough for the water to fly off the spoon, but this doesn't happen. What is it that slows the water almost completely, if the losses from the impact and friction aren't responsible? There is a good reason to consider carefully the transition from the quiet stream flowing over the brim to the arching film that sprays out from the spoon.

## Braking at the wall

Experience has shown that, even if one is careful in opening a faucet, the splashing is unpredictable. So it's advisable not to touch the faucet once you open it up to produce a moderate stream, but instead slowly lower the spoon to the bottom of the sink. It's interesting that the results depend strongly on preconditions. If you achieve the splashing mode of flow and then raise the spoon a few centimeters, the water will continue to spray over the spoon's brim. But as soon as water fills the spoon, you won't be able to restore the canopy simply by lowering the spoon to the initial position—you need to lower it still more.

If the spoon is rather flat and if you are attentive, you can observe an interesting phenomenon: the falling stream makes a depression in the water down to the surface of the spoon. This depression is bounded by a steep wall of water. The water is calm behind the wall, and the surface of the water behind the wall is a little bit higher than the spoon's brim (fig. 4). The stream is strong enough to push away the water near the point where the stream meets the spoon, but it isn't strong enough to empty the spoon entirely.

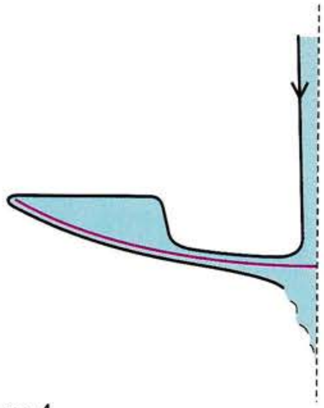

Figure 4

The steplike transition from rapid flow in the thin water film to almost stationary water behind the steep wall can be observed more easily when the surface involved is simpler than that of a spoon. You could use a hand mirror with a low rim created by the frame. In this case the radius of the water depression—that is, the region of rapid flow—varies smoothly with the strength of the stream. The height $H$ of the water wall almost coincides with that of the rim and is much greater than the thickness $h$ of the incoming thin layer of rapidly flowing water in front of the wall (fig. 5). The velocity $v$ of the rapid water decreases in the narrow region of the steep ascent (you can see the turbulence in the water there).

Let's consider the portion of the water delimited by the vertical faces $Hl$ and $hl$ and apply Newton's second law. Water of mass $\rho vhl$ (where $\rho$ is density) enters this fragment per unit time, and at some distance beyond the wall the velocity of the water drops almost to zero. Thus, some force is decreasing the momentum by $\rho v^2 hl$ every second. What is this force?

It turns out that it's the pressure from the almost stationary water.

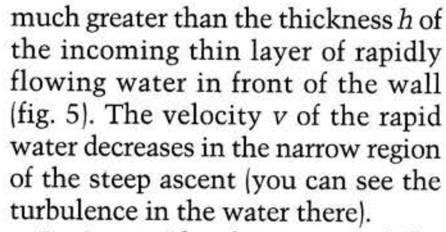


Figure 5

(Friction against the bottom is neglected due to the small horizontal distance in the region where the water rises. The forces that arise in the turbulent water at this step are internal and don't affect the total momentum. The forces of surface tension are also insubstantial.)

The pressure at depth $H$ is greater than the atmospheric pressure by $\rho gH$, but to calculate the force we must take the mean excess pressure. Thus,

$$F = \frac{\rho gH}{2} Hl = \frac{\rho gH^2 l}{2}.$$

It is this braking force that is equal to the rate of decrease in the momentum. From this we obtain the important relationship

$$v^2 h = \frac{gH^2}{2},$$

which can be viewed as the condition whereby the boundary of the water wall stays put. So what happens if the faucet is opened wider—or if the spoon or mirror is lowered? Then $v^2h$ increases and becomes greater than $gH^2/2$, the water wall yields to the pressure of the rapidly flowing water, and the step starts to move in the direction of the flow. We can calculate the velocity of the step using Newton's second law again and the law of conservation of mass. If, however, the pressure of the water wall is greater than the pressure of the incoming water ($gH^2/2 > v^2h$), the step will move toward the flow and the region of almost standing water will increase. (In the same way, for example, the tidal rise of sea water "locks" a river's estuary and produces an abrupt tidal wave that runs counter to the river's flow.)

## Radius of the canopy

Under our experimental conditions, it's not easy to verify the relationship

$$v^2 h = \frac{gH^2}{2}$$

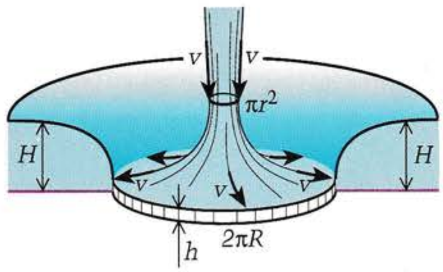


Figure 6

—the thickness $h$ is too small, and the velocity $v$ is hard to measure. So let's choose another way. Let's try to measure as accurately as possible the stream's diameter $2r$ at its narrowest part (its "neck")—that is, at a height of about $H$ from the horizontal platform, and also the radius $R$ of the circular wall (fig. 6). We'll assume that the velocity of the water is unaffected by friction and the small difference in the heights where the smooth overflow occurs.

The same amount of water flows through the neck as through the cylindrical boundary of the wall, which has a height $h$ and diameter $2\pi R$. Thus,

$$q = \pi r^2 v = 2\pi R h v.$$

From this we get the velocity

$$v = \frac{q}{\pi r^2}$$

and the thickness of the layer of spreading water

$$h = \frac{r^2}{2R}.$$

Substituting these values into the relationship $v^2 h = gH^2/2$ gives us the radius of the spreading water:

$$R = \frac{1}{g}\left(\frac{q}{\pi r H}\right)^2.$$

Using measured values of $q$ = 52 ml/s, $r$ = 3.5 mm, and $H$ = 6 mm, my calculated value of $R$ was 9 cm, while the directly measured $R$ was 6 cm. My calculation was off by 50%. I suspect that this is due primarily to our allowing the velocity of the water to be considered constant as it travels the rather long distance in the narrow layer. Assuming a decrease in velocity of 40% results in a passable numerical agreement. I hope your experiments confirm this conclusion.

But no matter how interesting the phenomena in the spoon, and no matter how instructive it is to delve into the fine points of the shape of the surface and its effect on the flow, certain "kindred" phenomena that lie well beyond the spoon's brim are even more interesting. I'll give you just one example. In rapidly flowing mountain streams, the ruggedness of the riverbed can cause the water to stop almost completely and dam up. Such a wall of water can prove hazardous to a person negotiating that stretch of river in a kayak or river raft, due to the steepness of the water and the abrupt change in the velocity of the current.

## AT THE BLACKBOARD II

# Irregular regular polygons

### *A problem owed to Noah Webster's successors*

by Eric D. Carlson and Sheldon L. Glashow

HAVE YOU EVER FOUND A math problem in a dictionary? We have. A regular polygon, according to the *American Heritage Dictionary*, is a polygon "having equal sides and equal angles." Really? Consider the two polygons in figure 1. For each, the sides have equal lengths and the vertices form right angles. According to the dictionary, both are regular polygons.

Figure 1

Polygons like the square, for which all *interior* angles are the same, are truly regular, while those like the Swiss cross on the right, for which some interior angles are supplementary, we shall call irregular regular polygons (IRPs). Regular polygons are a cinch. Aside from size and orientation, there is one such polygon for every $N > 2$. The IRPs to which our dictionary led us are much more interesting.

First, we should clarify what is and is not an IRP. It's reasonable to

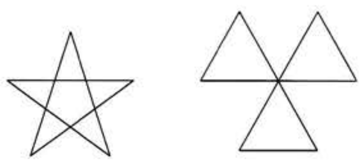

Figure 2

demand that none of its sides touch except at vertices and that all its vertices be distinct. Thus, the polygons in figure 2 are disqualified. The 5-sided star has sides that intersect, and the 9-sided symbol suggesting radioactivity has several coincident vertices at its center. Indeed, there are no IRPs with fewer than 10 sides.

The least IRP is the unique 10-sided polygon shown in figure 3. It

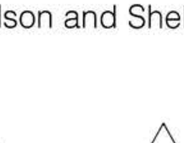

Figure 3

is made from a regular hexagon by inserting an extra wiggle at both the top and the bottom. Since we have added two wiggles, and each wiggle has two sides, the total number of sides $6 + 4 = 10$. We can add wiggle pairs to build larger IRPs with $6 + 4n$ sides, where $n$ is any positive integer. In figure 4, you can find an IRP with $46 = 6 + 4 \cdot 10$ sides built this way. The trick works because 2 divides 6. Any number of wiggle pairs can be added in a symmetrical pattern without spoiling the closure of the polygon. Of course, 3 also divides 6. Thus, we can add wiggle triplets to the hexagon, rather than pairs, to make new IRPs. Figure 4 also shows $(6 + 6n)$-sided IRPs for $n = 1, 2$, and 3.

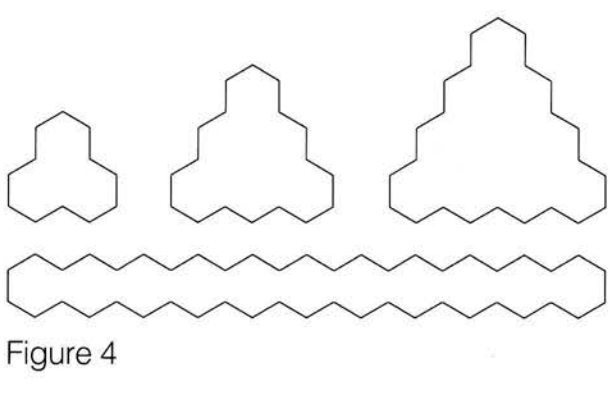

Figure 4

These tricks can be combined to build IRPs with $6 + 4n + 6m$ sides. Figure 5a is a 30-sided IRP made this way with $n = 3$ and $m = 2$. Of course, 1 and 6 also divide 6. One is not a helpful divisor: you can't add single wiggles to a hexagon symmetrically. But you can add 6 wiggles to build IRPs with $6 + 12n$ sides. Figure 5b is the 30-sided IRP made this way with $n = 2$.

Thus far, all our IRPs were built by adding sides to a regular hexagon. We have shown how to build IRPs with any even number of sides not less than 10. What

*Authors' note*: Our research on irregular regular polygons was supported in part by the National Science Foundation under grant number PHY-922-18167.

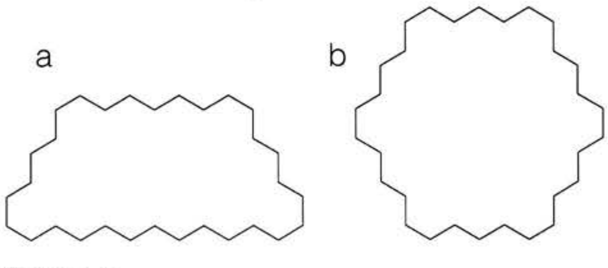


Figure 5

about IRPs with an odd number of sides? The technique used for the hexagon can be applied to a larger regular polygon—say, one with 15 sides. Because 3 divides 15, we can add triplets of wiggles to get IRPs with $15 + 6n$ sides. Because 5 divides

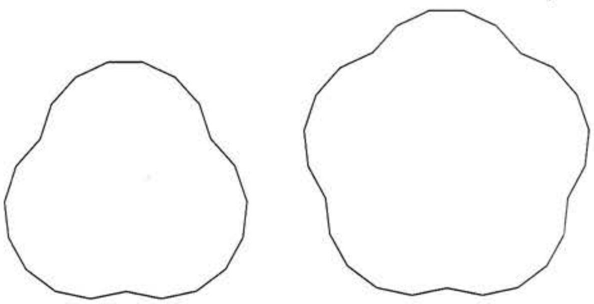

Figure 6

15, we can add quintuplets of wiggles to get IRPs with $15 + 10n$ sides. Figure 6 shows 21- and 25-sided IRPs made with $15 + 6$ and $15 + 10$ sides, respectively.

As for the initial hexagon, the two methods can be combined to build IRPs with $15 + 6n + 10m$ sides. Figure 7 is a 31-sided IRP made in this way with $n = m = 1$. It is the least IRP with a prime number of sides. Two variations on the design can be obtained by replacing one or two portions with the lines shown in color.

Of course, one need not start with a 15-sided regular polygon. Beginning with a regular pentagon, we can build IRPs with $9 + 6n$ sides. These starting points, with $n = 1$, yield two different IRPs with 15 sides. They are shown in figure 8, along with a third 15-sided IRP.

Figure 7

For what values of $N$ are there IRPs? Even values of $N$ have already been dealt with. Beginning with a regular 15-sided regular polygon, and extending it as we have described, we can obtain all integers of the form $15 + 6n + 10m$. These include 21, 25, and 27, as well as all odd integers greater than 30. Having already constructed three different 15-sided IRPs, we are left IRPless for only 15 values of $N$: for primes less than 30 and for any integer less than 10. In fact, none of these IRPs exists.

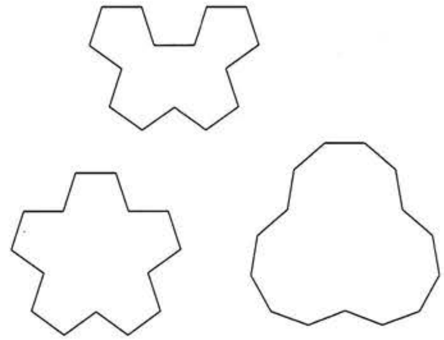

Figure 8

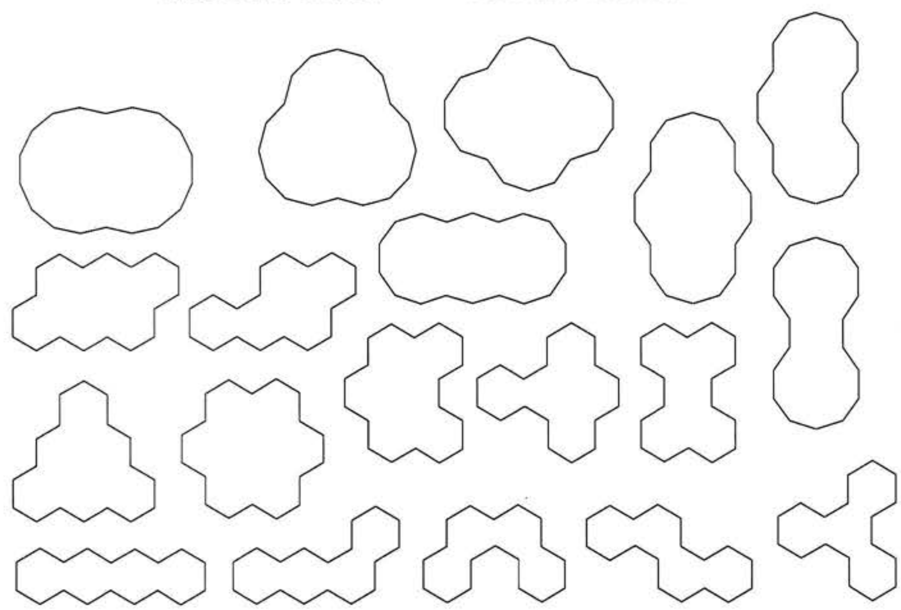

Figure 9

For any $N$, how many different IRPs are there? This is a tough question. The least IRP with 10 sides is unique. For $N = 12$, there are three distinct IRPs. Two of them appear previously in this article—can you find the third? For larger $N$ the number of IRPs grows rapidly. We believe there to be 19 different IRPs with $N = 18$. These are depicted in figure 9. (Four of these IRPs are distinct from their reflected versions. If both handednesses are counted, there are 23 eighteen-sided IRPs.) This is the first instance for which the number of IRPs exceeds the number of sides.

Smaller IRPs possess some level of rotational symmetry. Larger ones, like two of the 18-sided and one of the 31-sided IRPs, can be completely asymmetric. The least completely asymmetric IRP is shown in figure 10. It has 16 sides.

In all the examples given so far, it is implicitly assumed that IRPs are planar figures. Suppose we re-

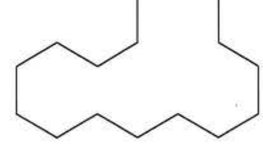

Figure 10

move this restriction and allow the figures to be drawn in three dimensions? This additional dimension permits IRPs to be constructed with any number of sides except 1, 2, 3, and 5. Basically, the freedom allowed by the additional space is so great that for large $N$ it's very easy to make an IRP. It's obvious that there cannot be any IRPs with fewer than 4 sides, but the absence of a 5-sided IRP is intriguing.

For a less tractable and more interesting puzzle, let's demand that three-dimensional IRPs satisfy a more stringent criterion. Suppose that not only the angles between adjacent sides but also those between next-to-adjacent sides are equal. What do we mean by the angle between next-to-adjacent sides? The angle $\theta$ between two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by the relation

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

If the cosines of the angles between next-to-adjacent sides are all equal, then we've shown that IRPs exist for all even $N$ except 2. We have no idea whether they exist for any odd value of $N$. If we further demand that next-to-next-to-adjacent sides make equal angles, then all the even IRPs persist and there are no odd IRPs.

Plenty of unsolved IRPish questions remain for you. For planar IRPs, how does the number of distinct species grow for large $N$? Can you prove that 31 is the largest value of $N$ for which the number of different IRPs is less than $N$? Is there a systematic way to find all IRPs? In three dimensions, with next-to-adjacent angles equal, are there any odd IRPs? Irregular regular polygons in more than three dimensions are also worth studying, but all we have done so far is find a 5-sided IRP in four dimensions.

**Eric D. Carlson** *is an associate professor of physics at Harvard University, specializing in particle theory and particle astrophysics. In August he will join the physics department at Wake Forest University.* **Sheldon L. Glashow** *is Higgins Professor of Physics at Harvard University and a founding editor of* Quantum.

# Bulletin Board

## Making sense of our senses

How do we recognize a friend, distinguish colors, respond to smells? A full-color report from the Howard Hughes Medical Institute takes a fresh look at recent scientific findings in the area of sensory perception.

The 59-page publication, *Seeing, Hearing, and Smelling the World*, reports on new research about the sensory receptors and the intricate pathways in the brain that let us perceive what's around us. It shows that, even within the visual system, the brain has separate pathways for color, shape, and motion. After a stroke, for instance, one woman could no longer see motion.

The beautifully illustrated report also shows how some of the brain's assumptions shape what we see and hear. For example, we assume that light comes from above, like sunlight, and are fooled when light actually comes from below. The publication covers new findings about the genes that determine what colors we can recognize and about pathways that are activated as we see these colors. It explains why very young children need to use both eyes correctly, or lose their sight in one eye.

A central section features new brain-imaging techniques that allow scientists to observe the brain in action as volunteers look at faces or objects or hear different kinds of sounds. Another section reports on the astonishingly sensitive bundles of cilia at the top of "hair cells" in the inner ear, whose vibrating motion allows us to hear sounds. It shows experiments with barn owls that are providing clues to how we locate objects by their sound.

Some of the most exciting discoveries involve the sense of smell, which was almost a total mystery until a few years ago. The report describes how scientists identified the genes and proteins that enable us to distinguish some 10,000 odors. And it raises the question of whether there is a secret sense in the human nose that brings social or sexual messages from people around us.

This report is the fifth in a series that began with *Finding the Critical Shapes* in 1990 and continued with *Blazing a Genetic Trail* (1991), *From Egg to Adult* (1992), and *Blood: Bearer of Life and Death* (1993).

For a free copy of *Seeing, Hearing, and Smelling the World*, write to the Howard Hughes Medical Institute, 4000 Jones Bridge Road, Chevy Chase MD 20815-6789. Teachers who subsequently wish to order additional free copies for classroom use can do so by returning the reply card enclosed with the publication.

## Eisenhower National Clearinghouse

Although it is intended primarily as a resource for teachers, the Eisenhower National Clearinghouse (ENC) for Mathematics and Science Education is open to the public, which means that interested parents and students are free to explore its vast holdings. One quick way into ENC is via the Internet. By telnet or gopher, the address is enc.org; ENC's World Wide Web address is http://www.enc.org. You can also dial in directly at 800 362-4448 or 614 292-9040. ENC also offers print materials to users who lack the computer tools for electronic access.

ENC is located at Ohio State University and is funded by the US Department of Education. Its purpose is to improve access to mathematics and science resources available to teachers, students, parents, and others. The Clearinghouse will collect and create the most up-to-date and comprehensive listing of mathematics and science curriculum materials in the nation. The list or catalog of materials, the text of some of the materials, and evaluations of them will be made available in a data base in a variety of formats, including print, CD-ROM, and electronically online.

For more information on ENC, e-mail info@enc.org; write to Eisenhower National Clearinghouse for Mathematics and Science Education, The Ohio State University, 1929 Kenny Road, Columbus OH 43210-1079; or call 800 621-5785 (in Ohio, 614 292-7784)

## Cyberteaser winners

The following visitors to the *Quantum* home page on the World Wide Web submitted correct solutions to brainteaser B146 posted there in advance of publication in this issue:

Matt Nehring (Boulder, Colorado)
Denis E. Baker and friends Cheng-Chih and Don (Columbus, Ohio)
Carl Bosley (Topeka, Kansas)
Cyrus Hsia (Scarborough, Ontario)
Robert Au (Stanford, California)
Arnd Roth (Heidelberg, Germany)
Paul Grayson (Urbana, Illinois)
Francis Trudeau (Montreal, Quebec)
Sean McLennan (Calgary, Alberta)
Louis Smadbeck (Edgartown, Massachusetts)
Hal and Matt Harris—father and son (St. Louis, Missouri)
Yakov Kronrod (Marietta, Georgia)

Congratulations to the winners, and our thanks to all who submitted entries.

To take a crack at the next *Quantum* cyberteaser, aim your browser at http://www.nsta.org/quantum and follow the hyperlink.

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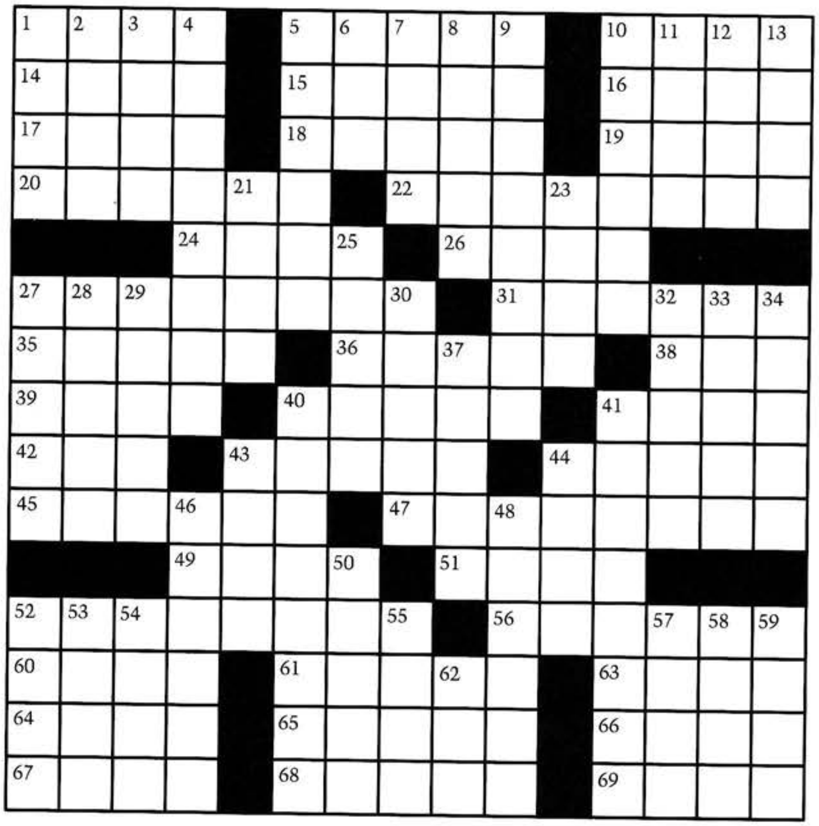
# Volume 5 (1994–95)

# Criss × cross science

by David R. Martin



## Across

1 Liability
5 764,876 (in base 16)
10 52,140 (in base 16)
14 Agent: suff.
15 Admire
16 Vagrant
17 Talipot palm fiber
18 Mountain ash tree
19 Actress ___ Perlman
20 Like some computer files
22 Like some ocean currents
24 Element 10
26 Unit of mass
27 Like a Bishop's move
31 Pacific and Atlantic
35 Seed coverings
36 Number part
38 Abraham's nephew
39 British archaeologist ___ Kathleen Mary Kenyon (1906–1978)
40 Extracted ore
41 Motion picture
42 Devoured
43 More like 24A
44 Not drunk
45 British efficiency apartment
47 $AgAsS_2$
49 Underground stem
51 Type of gin
52 Equation's answer
56 ___ radius (gyromagnetic radius)
60 Rice: comb. form
61 Journalist Herbert ___ (1869–1930)
63 60,138 (in base 16)
64 ___'s law (of opposition)
65 962,030 (in base 16)
66 Dark: comb. form
67 Religious views
68 Winter vehicles
69 New Mexico resort

## Down

1 55,995 (in base 16)
2 Decorative case
3 German physicist Max ___ (1882–1970)
4 Isosceles or right
5 Heavy particle
6 Bustle
7 Dish
8 British actor Edward Gordon ___ (1872–1966)
9 Middle point
10 Element 24 alloy
11 Danish physicist Niels ___ (1885–1962)
12 44,010 (in base 16)
13 Decomposed vegetation
21 Old cars
23 Diplomacy
25 Point opposite zenith
27 896,427 (in base 16)
28 Angry
29 Pointed
30 Shortest paths between points
32 Pretext
33 Musical group of nine
34 Unit of volume
37 Growth sources
40 Arrays of numbers
41 Like laser light
43 Uproar
44 Greek portico
46 SCSI
48 Hungarian poet Gyula ___ (1902–1983)
50 Good
52 Vended
53 A cookie
54 Wild cat
55 Point of minimum disturbance
57 Yucatan Indian
58 Environment: comb. form
59 Rodents
62 Display light: abbr.

*SOLUTION IN THE NEXT ISSUE*

*SOLUTION TO THE MAY/JUNE PUZZLE*

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
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| E | D | G | E | | L | A | R | G | E | | M | O | V | E |
| A | D | E | R | | D | R | I | E | R | | O | G | E | E |
| H | A | L | O | F | O | R | M | | O | C | E | A | N | S |
| | | | S | I | S | | | S | L | A | B | | | |
| M | A | R | I | N | E | | T | H | O | R | A | C | I | C |
| A | M | I | N | E | | S | I | E | G | E | | A | C | E |
| S | I | N | E | | E | A | R | L | Y | | A | M | I | N |
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| | | | N | E | X | T | | | R | I | M | | | |
| P | I | R | A | N | I | | A | D | A | P | I | C | A | L |
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| C | A | D | E | | E | L | E | M | I | | U | B | A | C |
| A | M | E | S | | S | L | E | E | T | | M | U | S | H |

# Math

## M146

The answer to both questions is 24.

Let $S$, $C$, $M$ be the sets of "swimming," "cleaning," and "math" days and $s$, $c$, $m$ the complements of these sets ($s$ is the set of the days when Mary isn't going to swim, and so on). Denoting the intersection of sets $A$ and $B$ by $AB$ (omitting, for the sake of brevity, the sign $\cap$), we can write the sets we have to count as $P = Scm$ (pleasant days) and $B = scm$ (boring days). In figure 1, the set $P$ is shaded blue and $B$ is shaded green; the blue, red, and black circles depict the sets $S$, $C$, $M$, respectively, and the entire big circle is the set of all 90 days. The figure suggests that the number of elements corresponding to any of the eight areas bounded by these circles can be expressed in terms of the numbers of elements in the sets $S$, $C$, $M$ and their double and triple intersections. For instance, to find $|P| = |Scm|$ (here $|X|$ is the number of elements in $X$), we must subtract from $|S|$ the number of days when Mary plans to swim and do something else. So we subtract $|SC|$ and $|SM|$. But then $|SCM|$ is subtracted twice, and we must add one $|SCM|$ back:

$$|P| = |S| - |SC| - |SM| + |SCM|. \quad (1)$$

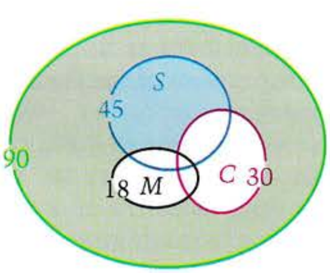



Figure 1

(This is actually an application of the inclusion–exclusion theorem, discussed in "The School Bus and the Mud Puddles" in the January/February 1995 issue.)

Number all the days 1, 2, ..., 90. Then the days in $S$ are $1, 3, \ldots, 1 + 2k, \ldots, 89$; $C = \{1, 4, 7, \ldots, 1 + 3k, \ldots, 88\}$; $M = \{1, 6, \ldots, 1 + 5k, \ldots, 86\}$. These sets can be described as the sets of numbers that give the remainder 1 when divided by 2, 3, 5, respectively. Since 90 is divisible by all these numbers, we have

$$|S| = \frac{90}{2} = 45,$$

$$|C| = \frac{90}{3} = 30,$$

$$|M| = \frac{90}{5} = 18.$$

All the intersections are described similarly: each consists of the numbers $1 + Nk$, $k = 0, 1, \ldots, [89/N]$, where $N$ is either $2 \cdot 3 = 6$, $3 \cdot 5 = 15$, $2 \cdot 5 = 10$, or $2 \cdot 3 \cdot 5 = 30$ for the intersections $SC$, $CM$, $MS$, $SCM$, respectively (since 2, 3, and 5 are coprime). As before, we get

$$|SC| = \frac{90}{6} = 15,$$

$$|CM| = 6,$$

$$|MS| = 9,$$

$$|SCM| = 3.$$

So, by formula (1),

$$|P| = 45 - 15 - 9 + 3 = 24.$$

$|B|$ can be counted by direct application of the inclusion–exclusion theorem for the triple union $S \cup C \cup M$ (using $|scm| = 90 - |S \cup C \cup M|$). But we can save some work by noting that

$$\begin{aligned} |S \cup C \cup M| &= |S \setminus (C \cup M)| + |C \cup M| \\ &= |Scm| + |C| + |M| - |CM| \\ &= 24 + 30 + 18 - 6 = 66, \end{aligned}$$

and so $|B| = 90 - 66 = 24$. (L. Limanov, V. Dubrovsky)

## M147

Since the sum $x_1x_2 + x_2x_3 + \ldots + x_nx_1 = 0$ consists of equally many positive ones and negative ones, the number $n$ of terms in this sum is even. The product of these terms equals

$$(x_1x_2)(x_2x_3)\ldots(x_nx_1) = (x_1x_2\ldots x_n)^2 > 0.$$

Therefore, the number of negative terms (equal to $n/2$) is also even. So $n$ is divisible by 4.

## M148

(a) We'll prove by induction over $n$ a slightly modified statement, which clearly is equivalent to the one in the problem: any nonnegative integer $a < (n + 1)!$ is uniquely representable as

$$a = a_n \cdot n! + a_{n-1} \cdot (n-1)! + \ldots + a_1 \cdot 1!,$$

with $0 \le a_k \le k$ for all $k = 1, \ldots, n$ (here we allow $a_n$ to be zero but fix $n$).

The case $n = 1$ is obvious. Let's prove the statement for $a < (n + 1)!$ assuming it's true for all numbers less than $n!$. Dividing $a$ by $n!$ we get $a = a_n \cdot n! + r_n$, where $0 \le a_n \le n$ (because $(n + 1)n! = (n + 1)! > a$) and $0 \le r_n < n!$. By the induction hypothesis, $r_n = a_{n-1} \cdot (n-1)! + \ldots + a_1 \cdot 1!$ with $0 \le a_k \le k$, so the number $a$ can be represented as required. To prove uniqueness, notice that the number of sets $(a_n, a_{n-1}, \ldots, a_1)$ of coefficients in our representations is $(n+1) \cdot n \cdot \ldots \cdot 1 = (n+1)!$. Each of these sets has been made to correspond to one of the $(n+1)!$ nonnegative integers less than $(n+1)!$. This correspondence goes both ways, since formula (1) restores the number $a$ from any given set $(a_n, a_{n-1}, \ldots, a_1)$. Thus each set corresponds to an integer, and the correspondence is one-to-one.

Using the terminology of the article "Number Systems" in this issue, we can say that formula (1) gives the representation of a number $a$ in the "factorial" number system (with the bases $1, 2!, 3!, 4!, \ldots$). This can be generalized to any base of the form $q_1 = 1$, $q_2 = d_2q_1$, $q_3 = d_3q_2$, ..., $q_n = d_nq_{n-1}$, where $d_k > 1$, $k = 2, \ldots, n$. That is, any nonnegative integer $a$, $a < q_n$, is uniquely representable in the form

$$a = a_{n-1}q_{n-1} + a_{n-2}q_{n-2} + \cdots + a_2q_2 + a_1, \qquad (2)$$

with $0 \le a_k < d_{k+1}$, $k = 1, \ldots, n-1$. The proof above remains valid with just a slight change—substituting $q_k$ for $k!$ and $d_k$ for $k$. (See also exercise 12 in "Number Systems" and its solution.)

(b) As in part (a), it suffices to prove that any fraction $b = p/q$, $0 \le b < 1$, whose denominator $q$ is a divisor of $n!$, is uniquely representable in the form

$$b = \frac{p}{q} = \frac{b_2}{2!} + \frac{b_3}{3!} + \cdots + \frac{b_n}{n!},$$

with $0 \le b_k < k$, $2 \le k \le n$ (we allow for $b_0 = 0$).

Multiply this equality by $n!$. It takes the form

$$a = \frac{pn!}{q} = b_2q_{n-1} + b_3q_{n-2} + \cdots + b_{n-1}q_2 + b_nq_1,$$

where $a$ is an integer less than $n!$ and $q_1 = 1$, $q_2 = d_2q_1 = n$, $q_3 = d_3q_2 = (n-1)n$, ..., $q_{n-1} = d_{n-1}q_{n-2} = 3 \cdot 4 \cdot \ldots \cdot (n-1)n$ (that is, $d_k = n - k + 2$ for $k = 2, \ldots, n-1$). Replacing the coefficients $b_2, b_3, \ldots, b_n$ in this formula with $a_{n-1}, a_{n-2}, \ldots, a_1$, respectively, we arrive at representation (2) discussed at the end of the previous solution, which exists and is unique for $0 \le a_k < d_{k+1} = n - k + 1$, which are equivalent to $0 \le b_{n-k+1} < n - k + 1$ $(k = 1, 2, \ldots, n-1)$. This completes the proof. (V. Dubrovsky)

## M149

We'll show that the problem can be reduced to the case of a degenerate pentagon two of whose vertices coincide. In this case the statement is obvious—see figure 2.

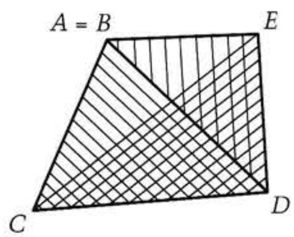



Figure 2

The reduction is based on the simple but very useful fact that can briefly (if not quite correctly) be formulated as follows: the area of a triangle changes linearly as one of its vertices moves along a straight line while the other two stay put. To give a stricter wording, we'll create coordinates for the given line $l$ by arbitrarily choosing the origin and the positive direction on it. Then *the area of a triangle ABC with vertex A on l is a linear function of its coordinate $x_A$ as long as A stays on one side of the fixed base BC.*

Indeed, if $l$ is parallel to $BC$, the area is constant. If $l$ meets the extension of $BC$ at $P$ (fig. 3), then the area of $\triangle ABC$ is equal to $BC \cdot (PA \sin \angle BPA)/2$, which is a constant multiple of $PA$, and $PA = x_A - x_P$ on one side of $BC$ and $x_P - x_A$ on the other side. So the area

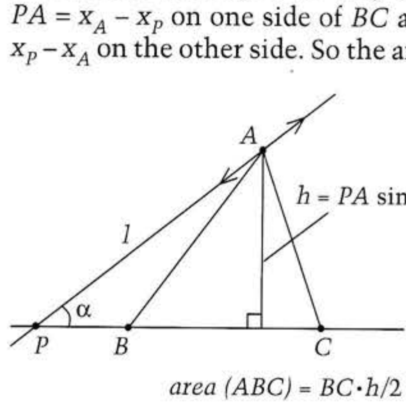



Figure 3

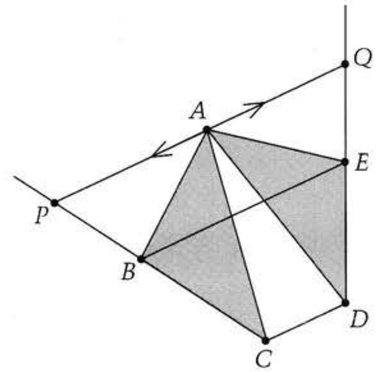
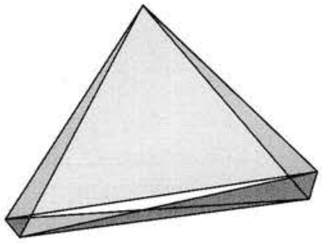



Figure 4

depends linearly on $x_A$ (on either side of $BC$).

Now let's draw a line $l$ parallel to $BE$ through the vertex $A$ of the given pentagon $ABCDE$ (fig. 4). Let $P$ and $Q$ be the points where it meets the extensions of $CB$ and $DE$. When $A$ moves along the segment $PQ$ the area of $\triangle ABE$, and so of the pentagon, remains constant; the areas of triangles $ABC$ and $ADE$ vary linearly; and triangles $BCD$ and $CDE$ are left intact. So the total area of the five triangles in question changes linearly (as the sum of linear functions). Therefore, the minimum value of this area on segment $PQ$ is attained at one of its endpoints—say, at $P$. So if we replace $A$ with $P$, we'll get a convex pentagon $PBCDE$ of the same area as that of $ABCDE$, but with a smaller sum of the areas of the five triangles cut off by its diagonals. (The triangle $ABC$ in this "pentagon" degenerates into segment $PC$.) Then we apply the same argument to the vertex $B$ of the new pentagon. This yields a pentagon that actually coincides with the quadrilateral $PCDE$, with one of the vertices $P$ or $C$ counted twice. As we mentioned at the beginning of the solution, the statement of the problem is true for this pentagon. It follows that it's true for the initial pentagon, because it has the same area and a greater sum of the areas of the five triangles.

It's evident that the total area $s$ of our five triangles is never greater than twice the area $a$ of the pentagon (see the figure accompanying the problem statement). Figure 5 shows that both bounds in



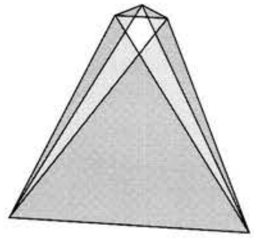


Figure 5

the estimate $a < s < 2a$ are exact. (N. Vasilyev, V. Dubrovsky)

## M150

(a) The answer is $f(x) = 2^{1/n}x + 2^{1/n} - 1$. It's natural to seek a solution in the form of a linear function $f(x) = ax + b$. The second iterate of this function is

$$f^{(2)}(x) = f(f(x)) = a(ax + b) + b = a^2x + b + ab.$$

We can find (and prove by induction) the expression for the $n$th iterate $f^{(n)}$:

$$f^{(n)}(x) = a^n x + b\left(1 + a + \cdots + a^{n-1}\right)$$

$$= a^n x + \frac{b\left(a^n - 1\right)}{a - 1}.$$

Setting this equal to the given function $2x + 1$, we find the coefficients $a$ and $b$ given in the answer above.

This "method of undetermined coefficients," which includes a lucky guess at the general form of the answer and finding particular coefficients from the corresponding equation, can be applied to parts (b) and (c) as well. But there is another, more direct approach that allows us to write out the unknown $n$th "iteration root" straightaway. We'll first illustrate it by means of the simpler problem in part (a).

Our given function $y = g(x) = 2x + 1$ can be viewed as a mapping of the number axis onto itself that takes any $x$ into $2x + 1$. Since this mapping doubles all distances ($|g(x') - g(x)| = 2|x' - x|$), it must be a dilation by 2 relative to a certain center, which can be found as its fixed point from the equation $2x + 1 = x$ (yielding $x = -1$). Indeed, let's shift the origin to the center $x = -1$ (fig. 6). Then the new coordinate (with respect to the new origin) $z_x$ of any point $x$ is equal to $x + 1$, and for the image $y = 2x + 1$ of point $x$ under our mapping we have

$$z_y = y + 1 = 2(x + 1) = 2z_x.$$

So in the new coordinates, our mapping is actually the dilation by 2 relative to the origin: $z \to 2z$. The "$n$th root" of a dilation by a factor of $k > 0$ is the dilation by $k^{1/n}$ relative to the same center. In our case, we get the mapping $z \to 2^{1/n}z$ (in the new coordinates). Returning to the initial coordinates, this yields $x + 1 \to 2^{1/n}(x + 1) = y + 1$, or $y = f(x) = 2^{1/n}x + 2^{1/n} - 1$.

This second solution may seem too sophisticated for such a simple problem, but, stripped of all the hints and explanations, it's even shorter than the first, displaying the core of the problem (taking the "$n$th root" of a dilation) and allowing us to solve parts (b) and (c) in a few lines.

(b) The given function can be rewritten as $y = (1 + \sqrt{x})^2$ or $\sqrt{y} = \sqrt{x} + 1$. So, introducing a new "coordinate"—or, better put, "parameter"—on the positive half-axis by the formula $z_x = \sqrt{x}$, we can represent the mapping of the half-axis defined by our function in a simple form: $z \to z + 1$ (in terms of the new parametrization)—the point with parameter $z$ goes to the point with parameter $z + 1$. We have to perform this mapping in $n$ identical steps, so in each step we can simply increase the parameter by $1/n$: $z \to z + 1/n$. In the initial coordinates the required function $y = f(x)$ is defined by $\sqrt{y} = \sqrt{f(x)} = \sqrt{x} + 1/n$, or $f(x) = (\sqrt{x} + 1/n)^2$.

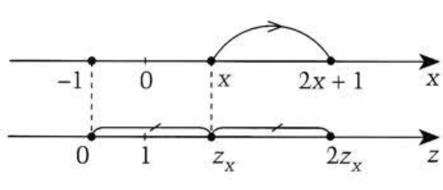


Figure 6

(c) Here we have $y = x/(x + 1) = [(x + 1)/x]^{-1} = (1 + x^{-1})^{-1}$, or $y^{-1} = 1 + x^{-1}$ We can repeat the previous solution replacing $\sqrt{x}$ with $x^{-1}$. So we don't have to do any calculations to write the answer:

$$f(x) = \left(\frac{1}{x} + \frac{1}{n}\right)^{-1} = \frac{nx}{x + n}.$$

(V. Dubrovsky)

# Physics

## P146

The simplest part of the problem is determining the time $\tau$ it takes the fox to catch the hare. We need only consider the projections of their velocities on the straight line connecting them. The relative velocity of the fox and hare along this line is

$$v_{\text{rel}} = v(1 - \cos\alpha),$$

from which we obtain

$$\tau = \frac{L}{v_{\text{rel}}} = \frac{L}{v(1 - \cos\alpha)}.$$

This formula is correct for all three cases.

To find the distance to the point where the fox catches the hare, let's construct a regular hexagon with sides of length $L$ (fig. 7). Let point $A_1$ move directly to point $A_2$ (as the fox pursues the hare), let point $A_2$ move to point $A_3$, and so on. Clearly, this hexagon will rotate and shrink but remain regular, so that the point where the animals

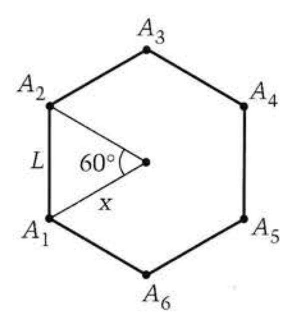



Figure 7

meet coincides with its center, which is located at a distance $x$ from the fox's initial position. If the hare becomes cross-eyed by 90°, we will have a square instead of a hexagon, but if it corrects its eyesight to 40°, a regular nonagon will result. In all these cases, the distance we seek is

$$x = \frac{L}{2\sin\frac{\alpha}{2}}.$$

## P147

Superimpose the origin of the coordinate system with point $A$ and direct the $x$- and $y$-axes as shown in figure 8. Now write the equations (in projections on the $x$- and $y$-axes) for the motion of masses $C$ and $B$:

$$ma_{Cx} = mg - 2T_2 \cos\alpha, \qquad (1)$$
$$ma_{Bx} = mg - T_1 \cos\alpha + T_2 \cos\alpha, \qquad (2)$$
$$ma_{By} = F_{\text{elast}} - T_1 \sin\alpha - T_2 \sin\alpha, \qquad (3)$$

where $F_{\text{elast}} = k(2l - 2l \sin\alpha) = k \cdot 2l(1 - \sin\alpha)$ is the elastic force. Combining equations (1) through (3) yields

$$ma_{Cx} + ma_{Bx} - ma_{By}\frac{\cos\alpha}{\sin\alpha} = 2mg - F_{\text{elast}}\frac{\cos\alpha}{\sin\alpha}. \qquad (4).$$

(This equation is true for any $\alpha$.) Let's convert the left-hand side of equation (4). Keep in mind that the changes $\Delta x_B$ and $\Delta x_C$ of the $B$ and $C$ coordinates are linked by the relationship

$$\Delta x_B = \frac{1}{2}\Delta x_C$$

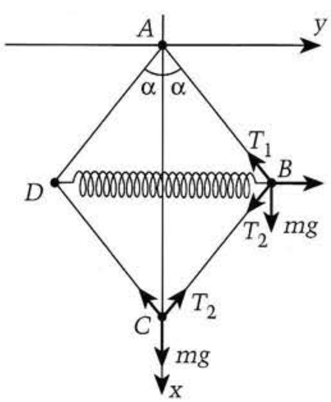



Figure 8

(in other words, this relationship describes the shifts of masses $B$ and $C$ from the equilibrium position). The velocities and accelerations of masses $B$ and $C$ obey the same relationship. Thus,

$$a_{Bx} = \frac{1}{2}a_{Cx}.$$

When the deviations from equilibrium are small (that is, when $\Delta\alpha \ll \alpha_0 = 45°$), the changes in the coordinates of the masses can be found using calculus or some identities from trigonometry. For instance,

$$y_B = l \sin\alpha$$

and

$$\begin{aligned}\Delta y_B &= l(\sin(\alpha + \Delta\alpha) - \sin\alpha)\\ &= l(\sin\alpha\cos\Delta\alpha + \cos\alpha\sin\Delta\alpha - \sin\alpha)\\ &\cong l(\sin\alpha + (\cos\alpha)\Delta\alpha - \sin\alpha)\\ &= l(\cos\alpha)\,\Delta\alpha \cong l(\cos\alpha_0)\Delta\alpha.\end{aligned}$$

Likewise,

$$x_C = 2l\cos\alpha$$

and

$$\Delta x_C = -2l(\sin\alpha)\Delta\alpha \cong -2l(\sin\alpha_0)\Delta\alpha.$$

Therefore,

$$\Delta y_B \cong -\frac{\Delta x_C}{2\tan\alpha_0} = -\frac{\Delta x_C}{2}.$$

Thus,

$$a_{By} \cong -\frac{a_{Cx}}{2\tan\alpha_0} = -\frac{a_{Cx}}{2}.$$

Thus, when the oscillations are small, the left-hand side of equation (4) can be written as

$$ma_{Cx} + \frac{1}{2}ma_{Cx} + \frac{1}{2}ma_{Cx} = 2ma_C.$$

Now let's convert the right-hand side of equation (4). In the equilibrium position with $a_{C0} = 0$, this equation becomes

$$2mg = F_{\text{elast}\,0}\frac{\cos\alpha_0}{\sin\alpha_0} = 2kl(1 - \sin\alpha_0),$$

from which we get

$$2mg = 2kl\left(1 - \frac{\sqrt{2}}{2}\right). \qquad (5)$$

We write down the right-hand side of equation (4) in the following form:

$$\begin{aligned}&F_{\text{elast}\,0}\frac{\cos\alpha_0}{\sin\alpha_0} - F_{\text{elast}}\frac{\cos\alpha}{\sin\alpha}\\ &= 2kl(1 - \sin\alpha_0)\frac{\cos\alpha_0}{\sin\alpha_0}\\ &\quad - 2kl(1 - \sin\alpha)\frac{\cos\alpha}{\sin\alpha}\\ &= 2kl\left(\frac{\cos\alpha_0}{\sin\alpha_0} - \frac{\cos\alpha}{\sin\alpha}\right)\\ &\quad + 2kl(\cos\alpha - \cos\alpha_0).\end{aligned}$$

Because $\Delta x_C = 2l(\cos\alpha - \cos\alpha_0)$, the last term is just $k\Delta x_C$. When the deviations from equilibrium are small,

$$\Delta\left(\frac{\cos\alpha}{\sin\alpha_0}\right) = \frac{\cos\alpha}{\sin\alpha} - \frac{\cos\alpha_0}{\sin\alpha_0} \cong \frac{-\Delta\alpha}{\sin^2\alpha_0}$$

and

$$\Delta x_C \cong -2l(\sin\alpha_0)\Delta\alpha.$$

Therefore,

$$\frac{\cos\alpha_0}{\sin\alpha_0} - \frac{\cos\alpha}{\sin\alpha} \cong \frac{\Delta\alpha}{\sin^2\alpha_0} \cong \frac{-\Delta x_C}{2l\sin^3\alpha_0}.$$

Thus, when the oscillations are small, the right-hand side of equation (4) can be written as

$$\begin{aligned}&2mg - F_{\text{elast}}\frac{\cos\alpha}{\sin\alpha}\\ &\cong -k\cdot\Delta x_C\left(\frac{1}{\sin^3\alpha_0} - 1\right)\\ &= -k\cdot\Delta x_C\left(2\sqrt{2} - 1\right).\end{aligned}$$

Taking into account that

$$\frac{m}{k} = \frac{l}{g}(1 - \sin\alpha_0) = \frac{l}{g}\left(\frac{2-\sqrt{2}}{2}\right)$$

(see equation (5)), we have the final form of the equation for small oscillations:

$$a_{Cx} = -\frac{g(2\sqrt{2}-1)}{l(2-\sqrt{2})}\Delta x.$$

Thus, the period is

$$T = 2\pi\sqrt{\frac{l(2-\sqrt{2})}{g(2\sqrt{2}-1)}}.$$

## P148

The gas ($n$ moles) acquires heat during phases 1–2 and 2–3 of the cycle (fig. 9):

$$Q = \frac{3}{2}nR(T_3 - T_1) + \alpha P(\beta - 1)V$$

$$= \left[\frac{3}{2}(\alpha\beta - 1) + \alpha(\beta - 1)\right]PV.$$

The work performed in the cycle 1–2–3–4–1 is

$$W = (\alpha - 1)(\beta - 1)PV.$$

Thus, the engine's efficiency is

$$\eta = \frac{W}{Q} = \frac{(\alpha-1)(\beta-1)}{\frac{3}{2}(\alpha\beta-1)+\alpha(\beta-1)}$$

$$= \frac{1 - \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\alpha\beta}}{\frac{5}{2} - \frac{1}{\beta} - \frac{3}{2\alpha\beta}}.$$

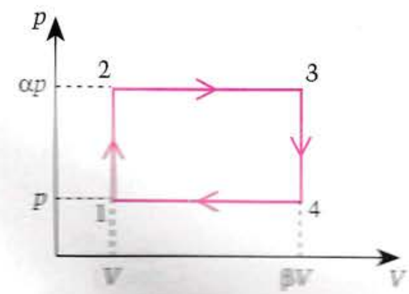



Figure 9

One can see that the efficiency increases with $\alpha$ and $\beta$, and at very large values of $\alpha$ and $\beta$ it approximates to $\eta = 2/5 = 0.4 = 40\%$.

## P149

The electric current in the circuit is determined by the value of the voltage across the resistor, which is just the voltage of the capacitor: $I = V/R$. As the current must be steady, the voltage across the capacitor is also constant—that is,

$$\frac{q_0}{C_0} = \frac{q_0 - \Delta q}{C},$$

where $q_0$ is the initial charge on the capacitor, $\Delta q = It = Vt/R$ is the charge flowing through the circuit in time $t$, and $C$ is the capacitance at moment $t$. It follows that

$$C = C_0\left(1 - \frac{\Delta q}{q_0}\right) = C_0\left(1 - \frac{t}{RC_0}\right).$$

To find the power $P$ of the external forces, we use the law of conservation of energy and consider the change in the energy of the system in time $t$. The change in the system's energy is equal to the work performed by the external forces:

$$\left(\frac{CV^2}{2} - \frac{C_0V^2}{2}\right) + I^2Rt = Pt.$$

Then, taking into account the dependence of the capacitance on time, we get the power of the external forces:

$$P = \frac{V^2}{2R}.$$

## P150

If an object is placed at a distance $d$ ($d > f_1$) from the microscope's objective, its real image will be formed at a distance $d'$ from the objective. The lens formula yields

$$\frac{1}{d} + \frac{1}{d'} = \frac{1}{f_1},$$

which gives us

$$d' = \frac{df_1}{d - f_1} = 78 \text{ mm}.$$

The object's image in the objective should be considered a real source for the eyepiece, since the rays coming from the objective to the eyepiece diverge. This source is placed at a distance $d_1 = L - d' = 22$ mm ($< f_2$) from the eyepiece, so the virtual image in the eyepiece will be formed at the distance

$$d_1' = \frac{d_1 f_2}{f_2 - d_1} = 143 \text{ mm}.$$

This is the image that will be observed, so $d_1'$ must be the best viewing distance for the human eye (we assume that the eye is placed close to the eyepiece).

Wearing eyeglasses, a person should be able to clearly discern objects at a distance $d_0 = 25$ cm from the eyes—that is, at the optimal viewing distance for the normal eye. (We assume here that the lenses are located close to the eyes.) The glasses must therefore produce virtual images of objects at a distance $d_1'$ from the eyes. Thus, we can write

$$\frac{1}{d_1'} - \frac{1}{d_0} = \frac{1}{f_0} = D,$$

where $f_0$ is the focal length of the glasses and $D$ is the optical power of the glasses when the focal length is expressed in meters. So we obtain the result

$$D \cong 3 \text{ diopters}.$$

Thus, our reader is far-sighted and should wear glasses with 3-diopter lenses. Sometimes a physicist can help herself without the services of an optometrist!

# Brainteasers

## B146

Any number in question can be written as $A \cdot 10^4 + 1995$. Clearly, it is exactly divisible by $A$ if and only if $A$ is a divisor of 1995. Since $1995 = 3 \cdot 5 \cdot 7 \cdot 19$, any divisor of 1995 is a product of some of its four prime factors taken no more than once each. So the number of divisors is equal to the number of subsets of a four-element set (including an empty subset, which corresponds to the divisor 1). So the answer is $2^4 = 16$. To put it slightly differently, all the divisors are formed by choosing or not choosing the factor 3, choosing or not choosing the factor 5, and so on, and then multiplying all the chosen factors. The number of choices is $2 \cdot 2 \cdot 2 \cdot 2$.

## B147

The sum in the $n$th diamond is $n^3$. To prove this, we note that for any $1 \le k \le n$ the $k$th row (from the top) in the $n$th diamond consists of $k$ numbers $k$ and the $k$th row from the bottom consists of $k$ numbers $2n - k$, so the sum of the numbers in these two rows is $k(k + 2n - k) = 2nk$. Summing these numbers for $1 \le k \le n - 1$ and adding the sum in the middle ($n$th) row, which is $n^2$, we get

$$2n\left[1+2+\cdots+(n-1)\right]+n^2$$

$$=2n\cdot\frac{n(n-1)}{2}+n^2$$

$$=n^3-n^2+n^2$$

$$=n^3.$$

(V. Dubrovsky)

## B148

Here's one of many possible solutions. Take a container of water filled to the brim and weigh it. Then put the rock in the container (some of the water spills out) and weigh it again. Then take out the rock and weigh the container for the third time. Let $W_1$, $W_2$, and $W_3$ be the three readings of the scales, let $\rho_r$ and $\rho_w$ be the densities of the rock and the water, and let $V$ be the volume of the rock. Then $W_1 = W_3 + \rho_w Vg$, $W_2 = W_3 + \rho_r Vg$. Therefore,

$$\frac{\rho_r}{\rho_w}=\frac{W_2-W_3}{W_1-W_3}.$$

(V. Dubrovsky)

## B149

The first required arrangement can be achieved by "going off into space" and constructing a regular pentagonal pyramid (fig. 10); the second is shown in figure 11.

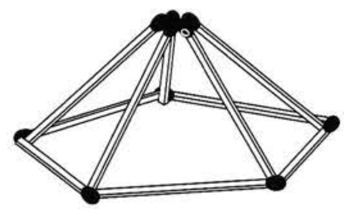


Figure 10

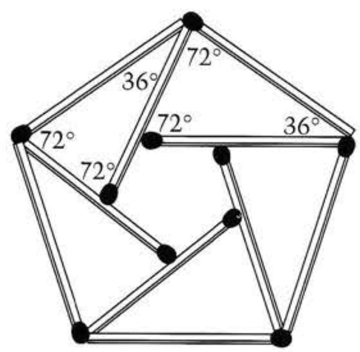



Figure 11

## B150

Let $P$ be the midpoint of $AB$ (fig. 12). Then $MP$ and $BN$ are medians in $\triangle ABM$, so their intersection point $Q$ divides $PM$ in the ratio $PQ/QM = 1/2$. But $PM$ is a midline in $\triangle ABC$, so it's parallel to $AC$. Therefore, triangles $BPQ$ and $BMQ$ are similar to triangles $BAK$ and $BCK$, respectively, with the same coefficients of similarity $BQ/BK = 1/2$. Now we can conclude that $AK/KC = PQ/QM = 1/2$, which is the answer to the problem.

We can avoid the reference to the property of medians by noticing that triangles $AKN$ and $MNQ$ are congruent (by the ASA property applied to sides $AN$ and $NM$), so $QM = AK = 2PQ$ ($PQ$ is a midline in $\triangle ABK$). Then we proceed as above. By the way, this argument can easily be developed into a proof of the property of medians (in the triangle $ABM$), which adds another proof to the collection presented in "The Medians," an article in the November/December 1994 issue of *Quantum*.

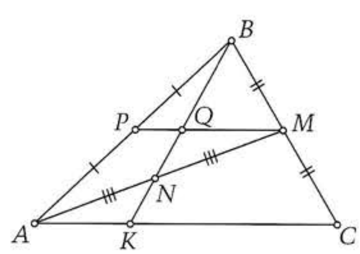



Figure 12

# Number systems

1. 201011011100010.

2. $[(24 \cdot 60 + 2)60 + 32] \cdot 60 + 42 = 5{,}193{,}162$.

3. $[([(9 \cdot 20 + 6) \cdot 20 + 14] \cdot 20 + 13) \cdot 18 + 15] \cdot 20 + 1 = 26{,}889{,}781$.

4. (a) 1022634; (b) 5Y540; (c) 11110001001000000.

5. $P = 2^5 + 2^2 = 32 + 4 = 36$; $Q = 2^8 + 2^6 + 2^4 + 2^2 + 1 = 256 + 64 + 16 + 4 + 1 = 341$—or $Q = ([(4 + 1)4 + 1]4 + 1)4 + 1 = 341$.

6. See the tables below:

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 10 |
| 2 | 2 | 10 | 11 |

| × | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 11 |

7. (a) $X = 18 + 1 = 19$; (b) the string of digits $Y$ can't be obtained according to the seller's system (see the answer to exercise 8); (c) $Z = 2 \cdot 20 + 5 + 2 \cdot 2 = 49$.

8. The sequences $a_n a_{n-1} \ldots a_1 a_0$ in question can be described as those in which the digits $a_{3k+1}$, $k = 0, 1, 2, \ldots$, take any of the values 0, 1, 2 and all the other digits take either of the two values 0 and 1, and in addition the digit 2 is never followed by 1. Check yourself that these conditions are necessary and sufficient for the inequalities

$q_{k+1} > r_k = a_k q_k + a_{k-1} q_{k-1} + \ldots + a_0$ to be true for all $k > 0$.

9. Clearly, $N > M$ (this follows directly from the definitions).

10. Consider the general situation. Suppose we have $n$ weights and they can be put on only one pan. Since any set of $n$ elements has $2^n$ subsets (including the empty subset), these weights can be combined in $2^n$ ways to yield at most $2^n - 1$ different nonzero total weights. On the other hand, taking the powers of two 1, 2, 4, ..., $2^{n-1}$ to be the values of $n$ weights, we can form any weight $W$ between 1 and $2^n - 1$ by representing the number $W$ in the binary system and choosing the weights whose values enter this representation.

If the weights can be put on both pans, there are three possibilities for each weight: to be put on the left pan, to be put on the right pan, or not to be used at all. This amounts to $3^n$ possibilities for $n$ weights, and at most $(3^n - 1)/2$ (positive) weights that could be measured this way (every weight can be obtained on the left or right pan). The set of $n$ powers of three—1, 3, 9, ..., $3^{n-1}$—can be used to obtain any weight from 1 to $(3^n - 1)/2$ by means of the *balanced ternary* representation of the desired weight (see the solution to exercise 11). Now we can find the answers: (a) 5 (since $2^4 - 1 < 30 < 2^5 - 1$); (b) 4 (since $(3^3 - 1)/2 < 30 < (3^4 - 1)/2$); (c) 10 and 7.

11. This problem can be done by induction. It is not difficult to write out the required representations for, say, the numbers 1, 2, and 3. Assume, for $N > 3$, that all the numbers less than $N$ can be written as required. Then let $A = $ "$a_n a_{n-1} \ldots a_1 a_0$" and let $A_1 = $ "$a_n a_{n-1} \ldots a_1$." We can then write $A$ as "$A_1 a_0$," and we can define $B$ and $B_0$ similarly. If $N$ is divisible by 3, then $N/3 < N$, so $N/3$ can be written as $A_1 - B_1$ for some strings $A_1$, $B_1$ of digits 0 and 1. Then it is not difficult to see that $N = $ "$A_1 0$" $-$ "$B_1 0$." If $N$ leaves a remainder 1 when divided by 3, then we represent $(N - 1)/3$ as $A_1 - B_1$. Then $N$ can be represented by "$A_1 1$" $-$ "$B_1 0$." Similarly, if $N$ leaves a remainder of 2, we can write $(N + 1)/3$ as $A_1 - B_1$. Then $N$ can be represented by "$A_1 0$" $-$ "$B_1 1$." This completes the induction.

12. Let $b_k$ be the greatest integer strictly less than $q_{k+1}/q_k$, $0 \le k \le n$. Consider the number

$$B = \text{"}b_n b_{n-1} \ldots b_1 b_0.\text{"}$$

Since it has $n + 1$ digits, $B \le q_{n+1} - 1$. On the other hand,

$$\begin{aligned} B &= b_0 + b_1 q_1 + b_2 q_2 + \cdots + b_n q_n \\ &\ge q_1 - 1 + \left(\frac{q_2}{q_1} - 1\right) q_1 + \left(\frac{q_3}{q_2} - 1\right) q_2 \\ &\quad + \cdots + \left(\frac{q_{n+1}}{q_n} - 1\right) q_n \\ &= q_1 - 1 + q_2 - q_1 + q_3 - q_2 \\ &\quad + \cdots + q_{n+1} - q_n \\ &= q_{n+1} - 1. \end{aligned}$$

This means that $B = q_{n+1} - 1$, which is possible only if $b_k = q_{k+1}/q_k - 1$—that is, if $q_{k+1}$ is divisible by $q_k$.

13. $2q_n \ge q_{n+1}$; $q_{n-1} + q_n \le q_{n+1}$.

14. The proof that the Fibonacci numbers satisfy the conditions of problem 13 can be done, for example, by induction.

(a) $\underbrace{10\ldots0}_{k-l}\underbrace{10\ldots0}_{l}$, if $k - l \ge 2$;

$\underbrace{10\ldots0}_{k+1}$, if $k - l = 1$;

$100\underbrace{10\ldots0}_{k-2}$, if $k = l \ge 3$;

$10 + 10 = 101$;

$1 + 1 = 10$.

(b) $\underbrace{100\ldots0}_{2m+2}$.

(c) $\underbrace{1010\ldots10}_{2m+2}$.

15. Writing the sequence of natural numbers in the Fibonaccian number system, we notice that $(a_k)$ is the sequence of all natural numbers that end (in this numeration) in an even number of zeros or one (no zeros), and $(b_k)$ are the numbers ending in an odd number of zeros; in addition, $b_k$ is obtained from $a_k$ by adding one zero at the end. We leave it to the reader to rigorously prove this observation.

Since the number 100 = 1000010100 in the Fibonaccian system, it belongs to the sequence $a_k$; the corresponding $b_k = 10000101000 = 144 + 13 + 15 = 162$, so $k = 62$—that is, $100 = a_{62}$.

# Toy Store

For the solution to the Packing puzzle, see figure 13. For the solution to the Checkerboard puzzle, see figure 14.

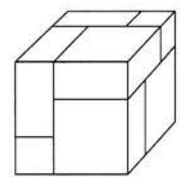


Figure 13

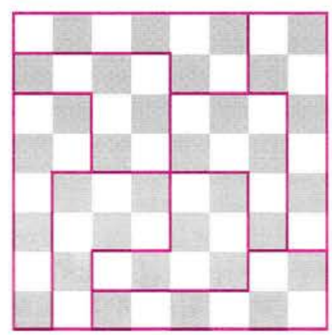


Figure 14

# The giants

1. In order to do better than Eudoxus, one needs 5 days; to do better than Tycho, 25 days; and to best Ulugh Beg's measurements, one would have to collect data for almost 70 days (more than two months). Even these values should be considered low, because the resolving power of diffraction, which was taken into account, is an unattainable ideal. Optical devices are considered perfect if they can resolve an angle that is

twice the resolution.

2. Aristarchus calculated that the Earth is three times the size of the Moon. The modern value is 3.67.

3. Calculation based on data collected by Aristarchus results in a ratio of the sizes of the Sun and Moon of 19:1. According to modern data, this value is 400. The data of Aristarchus himself indicate that the Sun's diameter is at least six times that of the Earth, which means the Sun's volume is almost 250 times that of the Earth. It would strain credulity that such a giant "dances" around the tiny Earth.

4. According to data recorded by Eratosthenes, the Earth's circumference is 47,500 km; data from Poseidonios gives 45,600 km; and the modern value is 40,000 km. Taking a stadium to be 157 m means that Eratosthenes obtained a value of 39,250, while Poseidonius came up with only 37,680. This decrease in accuracy can't be understood until one realizes that Eratosthenes measured the height of the Sun, which has an angular size of about 0.5°, while Poseidonius determined the height of a star (Canopus).

5. As it moves toward Jupiter, the Earth receives the information about the imminent eclipse of Io earlier. This time difference is $v_E t_0/c$, where $v_E$ is the Earth's velocity, $t_0$ is the period of rotation of Io around Jupiter, and $c$ is the speed of light. When the Earth is moving away from Jupiter, the signal is delayed by the same amount. Therefore,

$$t_0 = \frac{t_1 + t_2}{2}$$

and

$$t_2 - t_1 = v_E \frac{t_2 - t_1}{c}.$$

This gives us

$$c = v_E \frac{t_2 + t_1}{t_2 - t_1} = 30.4 \cdot 10^5 \text{ km/s}.$$

Error of the order of 1% is natural, because we have "measured" the time difference between the eclipses with just this accuracy.



# Index of Advertisers

# A party of wise guys

## *A report from the 14th International Puzzle Party*

by Anatoly Kalinin

IN THE SUMMER OF 1994, FOR the second time in my life I found myself in the company of the "cleverest people in the world"—at least, that's what their families and friends say about them. For the organizers and participants it was the 14th International Puzzle Party (IPP).[1] This club of mechanical puzzle lovers and collectors was created in 1978 by the American engineer Jerry Slocum, whose books about puzzles have made him popular all over the world. After the first seven meetings in the United States, the club, which had become international by that time, began to alternate the sites of its annual parties among the US, Europe, and Japan.

On August 19, 1994, puzzle lovers gathered at the Hilton Hotel in Seattle, Washington. The meeting was organized and hosted by Dr. Gary Foshee. Surprises and puzzles began right at the entrance to the hotel's banquet hall. Charming Chelle Foshee gave each participant a card with an anagram of their name on it. My own name was reworked into "a lion ain't lanky." Since I am, in fact, rather lanky, I translated this phrase into Russian for myself as "Anatoly Kalinin ain't a lion." But there was another reason why I couldn't consider myself a lion in the puzzle gathering. A year earlier, at the 13th IPP in Amsterdam, David Singmaster of London presented me with the "Very Nasty" puzzle ball. About two inches in diameter, the ball was neatly cut into seven pieces, one of which was taken out and handed over to me separately. I had to take the ball apart, insert the center block, and put everything back together again. I've been struggling with this puzzle for a year now, but without success.

To take revenge on David I had to invent a puzzle of my own, which I brought to Seattle with me. After all, I've been active in the puzzle world for 25 years now. I've written a lot of articles about them, and my collection remains the biggest within the borders of the former Soviet Union. I called my toy "Decapus's Puzzle." It's a metal cube that rather easily disassembles into ten pieces. The tricky thing is to put it back together—it can't be done piece by piece. You have to arrange the pieces in the proper position in space and then move them together. But this seems to require more than two hands! So Prof. Singmaster will have something to rack his brains over, I think.

The tradition of giving puzzle presents to one another is perhaps the most interesting part of the Puzzle Parties. Each participant brings $N-1$ identical puzzles, where $N$ is the number of participants, and gives them to colleagues, receiving equally many puzzles from all over the world.

The toy is older than the book. This simple fact, which requires no proof, in my opinion, revealed a new aspect to me in Seattle. Jerry Slocum suprised all of us with his gifts—exact copies of a puzzle that is more than 100 years old (because a book was written about it in 1893). This puzzle has the shape of an antique cannon loaded with a cannonball. You have to extract the cannonball from the gun's muzzle without using any tools. Watching others struggle unsuccessfully with this problem, you ask yourself if the average person has really become cleverer over the last 100 years.

At the other pole of the vast universe of intellectual toys—a universe that is hardly known to most of us (and to our children as well, unfortunately)—is the Levitron toy, which I also saw for the first time in Seattle. Levitron is a wooden box without any buttons, lamps, wires, or any other traces of a complex interior arrangement. You take a simple children's top (but heavier than usual) and spin it on the upper face of the box. After twirling for a couple of seconds, the top suddenly takes off into the air and hovers over the box, spinning on and on for more than a minute. A fantastic and mysterious show that specialists declared impossible in their articles a few years ago!

[1]You can read about the previous IPP in the July/August 1994 issue of *Quantum*.

In Seattle I was given several puzzles that at first sight seem impossible like Levitron. Gary Foshee, the hospitable host of IPP '94, invented one such—the "Pickled Nickel." In this puzzle, a lock made of a bolt and a nut prevents you from removing a coin. If you draw its design and show it to a teacher of technical drawing, you'll most probably get the lowest grade: the teacher will immediately say that because of serious errors it's impossible to either assemble or disassemble it. However, Gary Foshee's mechanism stands on my desk, and I can take it apart and put it together.

Since the trip to Seattle, my collection has grown by more than a hundred new puzzles. It wasn't easy to make a selection for this article. I tried to choose items that could give an idea, if only in the slightest degree, of the variety of the mechanical puzzles. Also, the puzzles you see in the figures are not very difficult to make with your own hands. I hope they'll help *Quantum* readers join the world of such clever and enigmatic things—you won't regret it! ◘

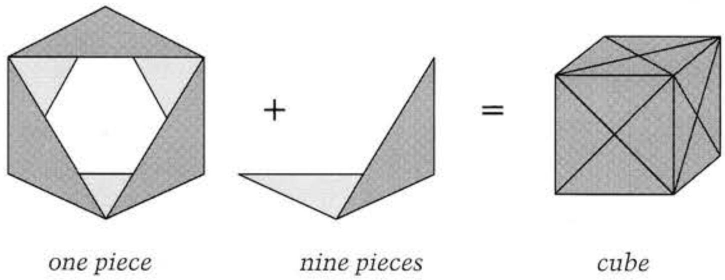



Figure 1

*Decapus's Puzzle by Anatoly Kalinin (Russia). The pieces of this puzzle are made from empty soda cans. The closed piece on the left is a flattened and appropriately bent "belt" cut out from a can.*

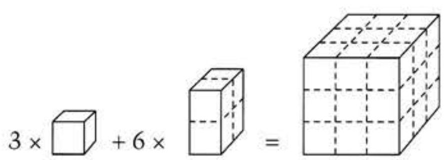



Figure 2

*The Packing puzzle by O. C. Judd (US). This was the easiest puzzle of IPP '94. You have to make a 3 × 3 × 3 cube out of three unit cubes and six 2 × 2 × 1 blocks. Try to solve it in your head—then check the answer on page 61.*

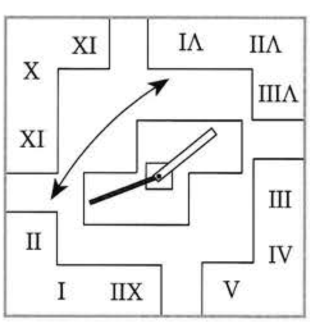



Figure 3

*A sliding-block puzzle by Jean-Claude Constantin (Germany). Shifting and rotating all five movable blocks (including the one with the clock hands), restore the correct arrangement of the figures on the dial. You're not allowed to lift the blocks.*

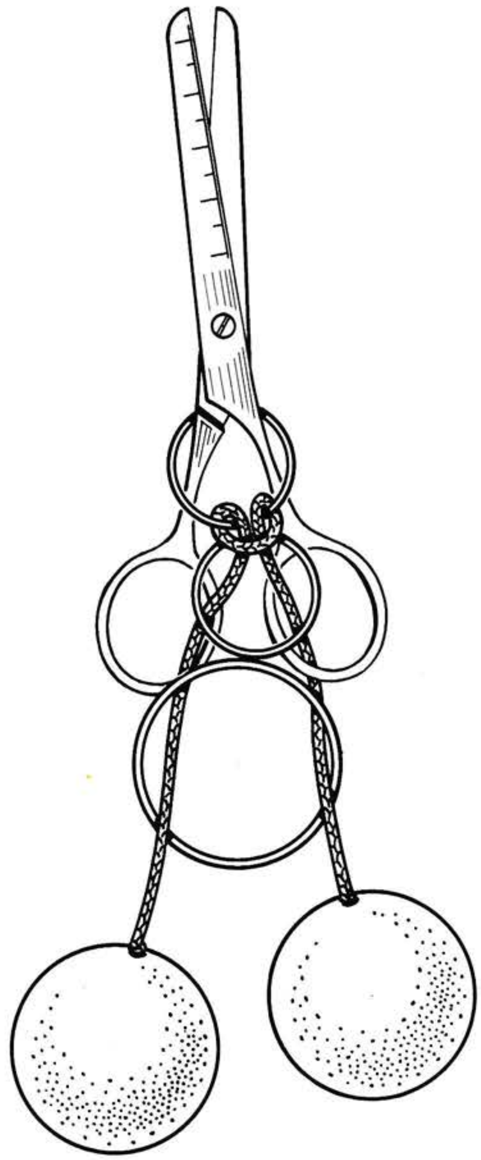


Figure 4

*The Scissors puzzle by W. G. H. Strijbos (The Netherlands). Remove rings 1, 2, and 3.*

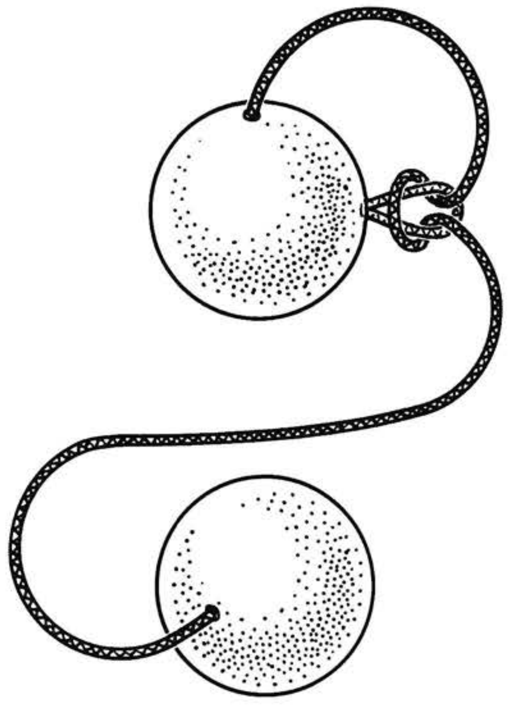


Figure 5

*No-Knot by Howard R. Swift (US). Unravel the knot.*

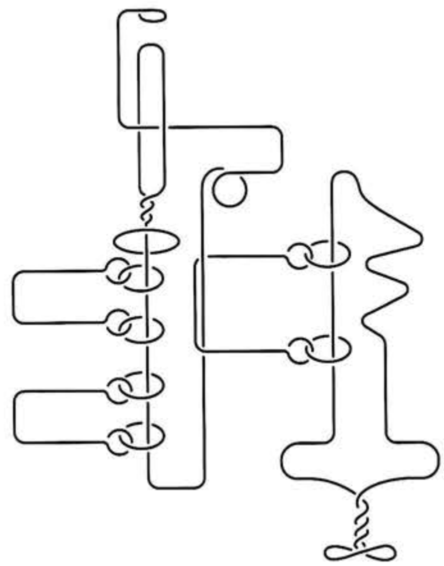


Figure 6

*Key Ring by Richard I. Hess (US). Remove the ring and the key.*

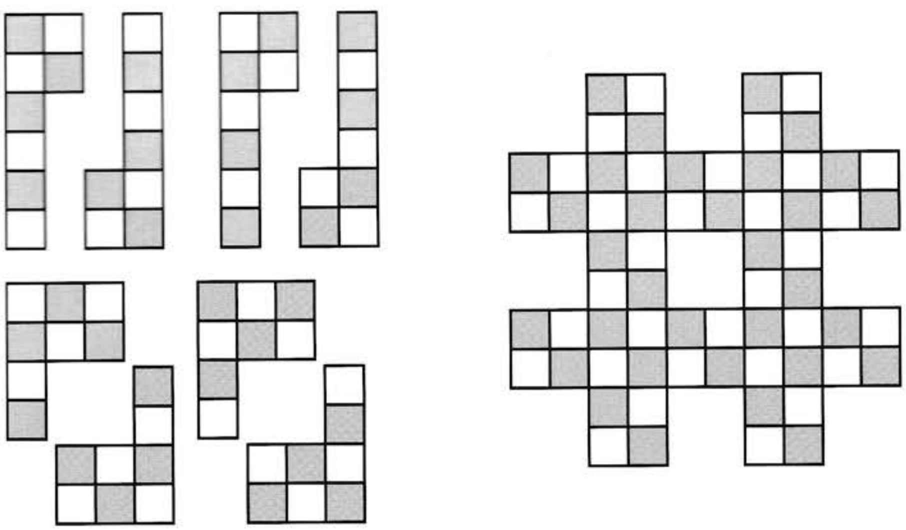


Figure 7

*Checkerboard puzzle by Les Barton (US). Les Barton is the author of the most complete investigation of checkerboard puzzles. He says that they often may seem easy at first, but after a few pieces are in place, it becomes clear that they are very difficult. So before you set about to make a normal checkerboard from the eight pieces on the left, try to solve the easier problem of making the figure on the right.* *(Answer on page 61.)*

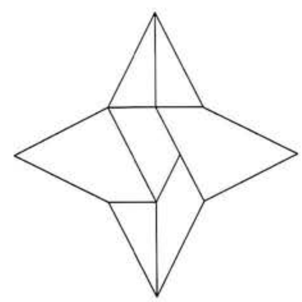


Figure 8

*Swap by Nob Yoshigahara (Japan). This is one of the most difficult puzzles of today. To convince yourself of this, just try to make a rectangle out of its seven pieces. And if you manage to make 14 different convex polygons from them, you can count yourself among the puzzle grandmasters.*