QUANTUM

MAY/JUNE 1996

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GALLERY Q

Autumn—On the Hudson River (1860) by Jasper Francis Cropsey

WITH THIS PAINTING BY JASPER CROPSEY (1823–1900), the Hudson River school makes a return appearance in Gallery Q. Readers will perhaps recall the painting by Thomas Cole in the November/December 1995 issue, which depicts clouds massing on one side of a mountain peak near Crawford Notch in New Hampshire. (Sharp-eyed readers will have noted that the loosely knit group of American painters was mistakenly referred to there as the "Hudson Valley" school.)

Some consider Cropsey the greatest colorist of the Hudson River school, and this painting lends support to that view. But as with Cole, it is Cropsey's rendering of a meteorological phenomenon that brings him into our gallery. We have all seen the fanlike splay of sunlight through broken clouds at the close of day. Despite many a hackneyed depiction by less talented painters, it remains a majestic sight. We have seen it so often, it seems quite natural. But if we *think* about it . . . if we know some elementary physics . . . we might be momentarily confused by the way the light rays are behaving.

Question 17 in the Kaleidoscope (page 33) brings this confusion into sharper focus.

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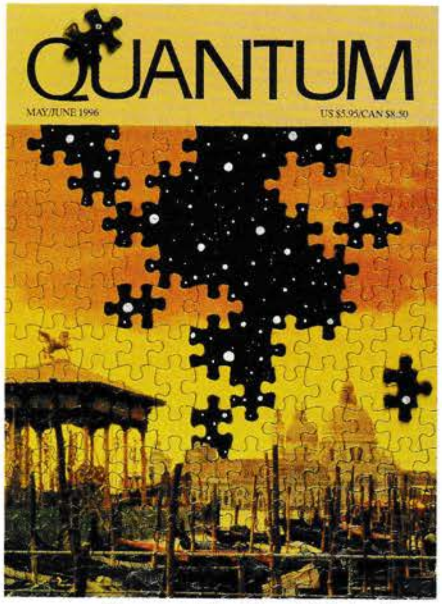

Cover art by Vera Khlebnikova

In a way, the cover for this issue could be a cover for *any* issue of *Quantum*. Aren't there always puzzling things in it? Isn't it sometimes surprising what turns up by the end of an article or problem? Aren't there often holes for you to fill in? And, like a jigsaw puzzle, isn't *Quantum* challenging and satisfying and frustrating and fun?

As usual, though, our cover has a more specific motivation. It has to do with one of our patented silly, yet not so silly, questions: What if the speed of light were everywhere the same? Beginning on page 10, Dmitry Tarasov and Lev Tarasov paint a picture of an imaginary world where the speed of light in any transparent medium is the same as that in a vacuum. It's a bizarre place!

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PUBLISHER'S PAGE

Health and long life

"We had no such thing as printed newspapers in those days to spread rumours and reports of things . . ."
—Daniel Defoe, A Journal of the Plague Year

SINCE OUR ARRIVAL IN THE United Kingdom on April 18, my wife, Alice, and I have been making our way across this great island, from Cornwall in southwestern England to Inverness in Scotland. Even though I have been to the UK many times, I always encounter something new here. I'd like to share a few impressions with the young people who read *Quantum*.

The first concerns science and its implications for social policy. England is in the grips of a crisis caused by bovine spongiform encephalopathy (BSE). No doubt you have read or heard about it under its more colorful name "mad cow disease." BSE is apparently a consequence of using certain animal products in the feed given to beef cattle. Some of these products are contaminated and induce the disease in the cattle. There is also a human disease (Creutzfeldt–Jacob disease) that some scientists believe may be induced by eating beef from cows infected with BSE.

Now, I'm not an expert in this area. I don't really understand the science associated with this problem, and not just because I was trained as a physicist. The science is complex, and there are uncertainties and disagreements even among specialists as to whether England has a problem or not. But, like most of the British populace (and others on the continent), I am an interested onlooker. Who wants to go out feet first because of a bad hamburger?

Beef consumption here is down 30%. The European Union has imposed a ban on British beef. For those who raise cattle in the UK, this is truly a catastrophe. The decisions made so far seem to be strongly motivated by fear, not science. A healthy dose of caution is always acceptable, especially in matters of public health. But how can policy makers keep "concerns" from evolving into "fear" and possibly turning into "panic"? What are the best mechanisms for educating the public on a complex scientific issue that affects them personally? Are we doomed to periodic "crises" in the areas of nutrition and health? Will a consensus ever be reached about what constitutes a proper human diet? Can our governments ever ensure that the food we eat is 100% safe? If not, what level of assurance can we, or should we, demand? And at what cost?

I don't have the answers to these questions. But I'm still eating British beef.

The next impression I'll share with you was, for me, actually more unsettling. I visited the home of the Brontës in Haworth, England. You may recall *Jane Eyre* by Charlotte Brontë, or *Wuthering Heights* by her sister Emily. Well, there was another sister, Anne Brontë, who wrote *Agnes Grey*. There was also a brother, Patrick Branwell, who was an artist.

It seemed truly remarkable that these creative and talented writers should emerge from such depressing and constrained surroundings. During this period Haworth lacked an adequate sewage system, and the average life span was 25 years.

As I looked through the museum and saw the relics and works of these young women, I was struck by the terrible circumstances of their lives, and their passing.

In 1848 their brother, Branwell, died of tuberculosis; three months later Emily died of TB also; and six months later, so too, Anne. Charlotte, the last remaining child, produced two more books, *Shirley* and *Villette*, and married. But in 1854 she too died young, in her first pregnancy. Charlotte was 38 when she died; her brother, 31; her sisters had reached the ages of 30 and 29.

It's hard for me to imagine what it must have been like for Patrick Brontë to lose *three* of his children in less than a year.

It's hard for us to appreciate the benefits of long life and health that science has given us over the past 150 years. This visit certainly raised my consciousness in that regard.

—Bill G. Aldridge

QUANTUM

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S. Ivanov

Cutting facets

Exploring the endless possibilities of a geometric "diamond"

by Vladimir Dubrovsky

ONE DAY I WAS LEAFING through an old issue of *Kvant* (the Russian sister magazine of *Quantum*) and came across the following problem (by V. Shafaryan): Through a point P on the circumcircle of a rectangle, two lines parallel to its sides are drawn. One of the lines meets two sides of the rectangle at A and B; the other line meets the extensions of the other two sides at C and D. Prove that the lines AC and BD (1) are perpendicular and (2) meet on the rectangle's diagonal (fig. 1).

For some forgotten reason it drew my attention. And rather unexpectedly I discovered that this rather ponderous problem has a remarkable variety of solutions based on different useful and instructive ideas. It can also be generalized in several ways leading to important geometric theorems. Like a crude crystal in the process of being cut, this problem kept revealing new facets glittering with the reflected light of other, more attractive and significant properties. Actually, this situation is typical for elementary geometry, a collection of facts so tightly intertwined that when you pull at any of them, it will most probably bring about a long train of other facts from different parts of this field of science (and art, I think).

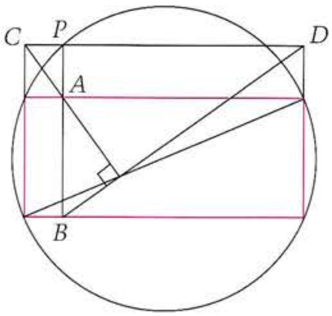

Figure 1

One method that proves to be very efficient in solving this problem is well known in . . . politics! It can be stated in three words: "divide and conquer." In mathematics this means that it's often useful to begin a solution to a problem by "taking it apart," singling out fragments that are really important and ignoring the rest. These important fragments can then be put through certain transformations that preserve properties that are essential for us, but may either reduce the problem to a simple special case or, conversely, lead to interesting extensions. In geometry this is often done by means of transformations of the plane such as translations, rotations, dilations, and so on, which can change the original problem almost beyond recognition. We'll also see how a transformation—in particular a powerful but seldom-used transformation called the "spiral similarity"—can be applied to prove properties that don't seem to have any relation to transformations at all.

But before we step onto the long and winding road of our explorations, try to solve the problem stated above yourself. It's not very difficult, and your solution may well turn out to be different from those you'll find below.

Let's start with the first of the properties stated in the problem.

Proofs of perpendicularity

There are a lot of right angles in our diagram. So it's only natural to try to prove that the angle between AC and BD is equal to one of them.

1. *Proof by shifting and symmetry.* Denote by $KLMN$ the given rectangle (fig. 2). The lines AC and BD are diagonals in the rectangles $PALC$ and $PBND$. The other diagonals of

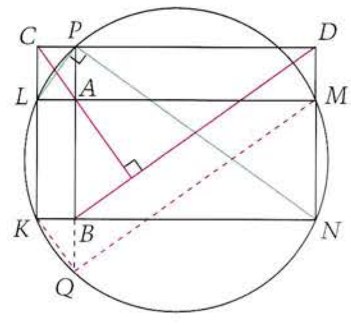

Figure 2

Art by Sergey Ivanov

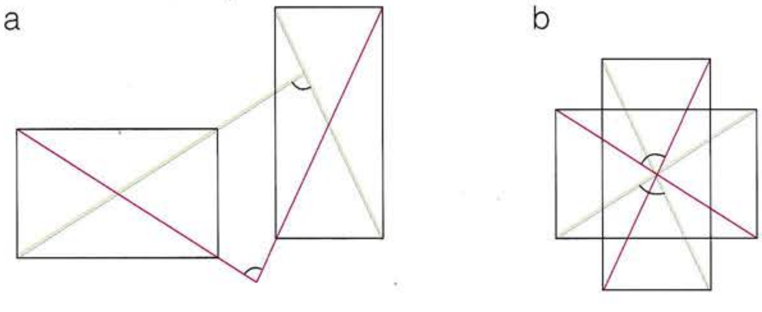

Figure 3

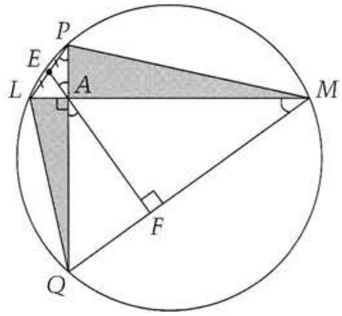

Figure 5

this rectangle are LP and PN. Now $\angle LMN = 90°$, so LN is a diameter of the circle. This means that $\angle LPN = 90°$ also. Thus it suffices to show that *if two rectangles have parallel sides and a diagonal is chosen in each of them, then the angle between these diagonals is equal to the angle between the two other diagonals.*

This becomes obvious if we shift one of the given rectangles (fig. 3a) parallel to itself (which doesn't change the angles between diagonals) so that its center coincides with the center of the other rectangle (fig. 3b). Now the two angles in question are equal simply because they are symmetric about one of the midlines of the rectangles.

2. *Proof by translation.* The same idea of comparing angles is used in the following argument. Let Q be the point where PA meets the given circle for the second time (fig. 2). Then, as before, $\angle KNM = 90°$, so KM is a diameter of the circle, and $\angle KQM = 90°$. The perpendicularity of AC and BD follows from the fact that $AC \parallel KQ$ and $BD \parallel QM$.

Exercise 1. Prove that $AC \parallel KQ$, $BD \parallel QM$.

We can say that in this argument the angle formed by lines AC and BD is moved onto the angle KQM by parallel translation.

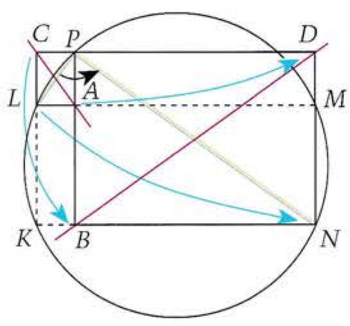

Figure 4

3. *Proof by rotation and dilation.* Notice that a 90° rotation about P takes the lines PC, PL, PA into PB, PN, PD, respectively (in figure 4 the rotation is counterclockwise). This allows us to construct a new proof, based on this rotation followed by a dilation with center P and ratio PN/PL. This composite transformation moves point L onto point N. Where is the image of point A? Call this image X. Since line PA moves onto line PB, X must be on line PB. Now $\angle PAL = 90°$, and neither the roation nor the dilation changes any angle measures. Hence $\angle PXN = 90°$, and $X = D$. Similarly, the image of point C is point B. So the line CA is taken by this transformation to BD. But the rotation turns CA by 90° and the dilation doesn't change the directions of lines. Therefore BD, the image of CA, makes a right angle with CA.

Rotation followed by dilation relative to the same center results in the transformation of the plane that was called "spiral similarity" by H. S. M. Coxeter and S. L. Greitzer in their wonderful book *Geometry Revisited.* (Neither this nor other names for this type of transformation are universally accepted.) Under this transformation all lines turn through the same angle—this property was used in the proof above. Another property of spiral similarity will be used below in a proof of the second statement of our problem.

Brahmagupta's theorem

According to exercise 1, the line MQ in figure 2 is parallel to BD. This observation allows us to give a more elegant wording to the fact that $AC \perp BD$, which is known as Brahmagupta's theorem: *If the diagonals of a quadrilateral inscribed in a circle are perpendicular, then the line through their point of intersection that bisects any of the sides is perpendicular to the opposite side.*

(In figure 5, which preserves our notation, the quadrilateral in question is $PLQM$ and the line is EF—the extension of AC in figure 2.)

You can try to find a direct proof of this theorem (for instance, by establishing the equality of the four angles marked in figure 5), or you can adjust one of the proofs above. I won't dwell on this theorem because it has a beautiful generalization.

Imagine that the triangles AQL and AMP in figure 5 are hinged at their common vertex A. Turn one triangle with respect to the other (fig. 6). Then we will show that the statement of Brahmagupta's theorem (for the sides PL and QM) remains true. In other words, *if a point A is chosen inside a quadrilateral $LPMQ$ such that AQL and AMP are similar right triangles (with right angles at A and $\angle L = \angle P$), then the line bisecting PL and passing through A is perpendicular to QM.*

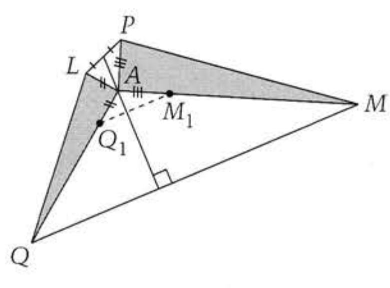

Figure 6

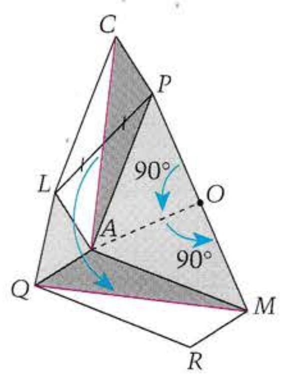

Figure 7

It will suffice to prove this theorem for the more familiar special case where $AQ = AL$ and $AM = AP$ (we can always replace the points Q and M with Q_1 and M_1 on AQ and AM, respectively, such that $AQ_1 = AL$, $AM_1 = AP$, because $AQ_1/AQ = AL/AQ = AP/AM = AM_1/AM$, and so Q_1M_1 is parallel to QM). In this special case we complete the triangles LAP and QAM to form parallelograms $LAPC$ and $QAMR$ (fig. 7). We now see that the 90° rotation about the midpoint O of PM takes $APCL$ into $MAQR$ (AP goes to MA, $\angle LAP = 180° - \angle QAM = \angle RMA$ and $AL = AQ = MR$, so L goes to R, and so on). In particular, the diagonal AC is taken by this rotation into MQ. But this just means that the line bisecting PL (that is, AC) is perpendicular to MQ.[1]

Proofs of concurrency

We have given three proofs of the first part of our original theorem and linked it to the classic result of Brahmagupta. Now we turn to proofs of the concurrency of the three lines described in the second part of the problem.

1. *Proof by using circumcircles.* In a geometric diagram, when a line segment subtends several right angles, it's usually a good idea to draw a circle, with the line segment as diameter, through all the vertices of the right angle. Our diagram is no exception.

Let J be the intersection point of CA and BD (fig. 8). Let us draw segments KJ and KM as shown. We've already shown that $\angle CJB = 90°$. Therefore point J lies on the circumcircle of rectangle $PBKC$, and so $\angle PJK = 90°$, too. Similarly, point J lies on the circumcircle of the rectangle $PAMD$ and $\angle PJM = 90°$. It follows that the segments KJ and JM make one straight line, so the three lines AC, BD, and KM are concurrent.

2. *Proof by spiral similarity of intersecting circles.* We used a 90° spiral similarity in the third proof of the perpendicularity statement above. In a much less obvious way, a useful property of this kind of transformation can be applied to prove the concurrency directly, without proving $AC \perp BD$ in advance. Here is the property.

Suppose a spiral similarity S about point A takes a circle ω_1 drawn through A into a circle ω_2. Let B be the second intersection point of ω_1 and ω_2 (other than A). Then the line joining any point X_1 on ω_1 to its image $X_2 = S(X_1)$ on ω_2 always passes through B (fig. 9).

To prove this property, we take any point X_1 on circle ω_1 and its image X_2 on ω_2 (figure 9 shows two possible choices). In triangle AX_1X_2, $\angle X_1AX_2$ is equal to the angle of rotation, and the ratio AX_1/AX_2 is equal to the constant of dilation. So for any two choices of X_1, the corresponding triangles AX_1X_2 are similar. Therefore, $\angle AX_1X_2$ does not depend on the position of point X_1 on the circle ω_1. This means that the intersection point B_1 of the line X_1X_2 and the circle ω_1 is always the same, because the arc AB_1 of ω_1 subtends at X_1 an inscribed angle of constant measure, and so this arc itself has a constant angular measure.

As we see in figure 9, the last argument is good for the case where B_1 lies on the ray X_1X_2. But if B_1 lies on the *opposite* ray, as in figure 10, we should more accurately say that the arc $AB_1 = AX_1B_1$ is complementary to the arc that subtends the angle adjacent to the angle AX_1X_2. However, the angular measure of this arc remains the same in this case as well.[2]

Similarly, the second circle ω_2 meets all lines X_1X_2 at a fixed point B_2. Points B_1 and B_2 both belong to all the lines X_1X_2, so $B_1 = B_2$, and this is the second common point B of ω_1 and ω_2.

Now let's get back to our problem (fig. 8). Just as in the third proof of perpendicularity, we can transform the rectangle $PCKB$ into $PAMD$ by a 90° spiral similarity about P. It remains to apply the property considered above to this similarity, the circumcircles of the rectangles,

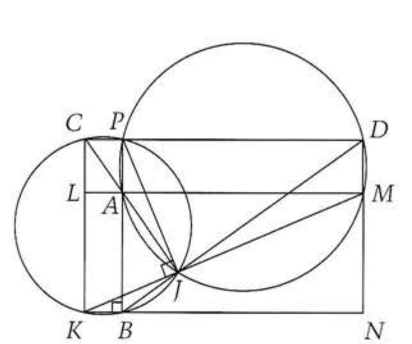

Figure 8

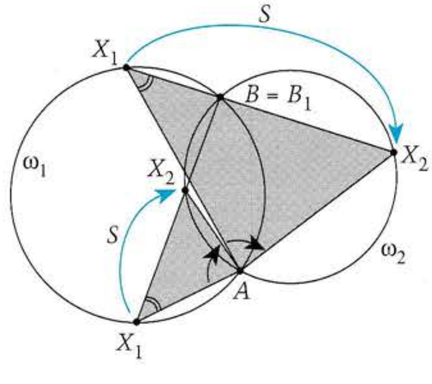

Figure 9

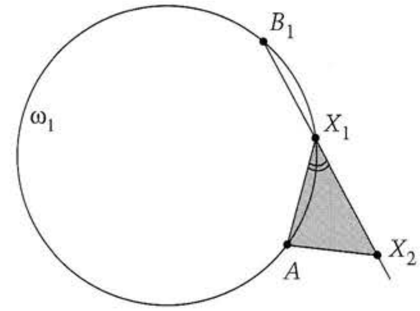

Figure 10

[1]By replacing the 90° rotation here with a suitable spiral similarity, we could have dealt directly with the general situation in figure 6.

[2]To be absolutely accurate, we should have used *oriented* angles and arcs, but I think the figures are convincing enough for us to allow ourselves a certain looseness of expression.

points C, K, B, and their respective images A, M, D: by this property, the lines CA, KM, and BD pass through the common point J ($\neq P$) of the circumcircles, and we're done.

Before we go further, let's look at a couple of other applications of the property of spiral similarity we studied.

Exercise 2. A line drawn through an intersection point P of two given circles meets them for the second time at points A and B. Find the locus of the midpoint of segment AB.

Exercise 3. Let Q be the second intersection point of the circles in the previous exercise. Draw through point A a tangent to the circle on which A lies and through B a tangent to the other circle. Suppose these tangents meet at C. Prove that the points Q, A, C, B are concyclic.

The same property can also be used to prove a classical theorem that will allow us to look at our initial problem from a new vantage point.

Simson lines

The theorem I'm talking about was proved by William Wallace in 1797, but was erroneously attributed to Robert Simson (such historical mistakes are not uncommon in mathematics). It reads as follows: *The feet of the perpendiculars from a point on the circumcircle of a triangle to its sides (or their extensions) lie on a straight line.* This line is called the *Simson line* of the point with respect to the triangle.

To prove this theorem, let ABC be the triangle, P the point, A_1, B_1, C_1 its projections (fig. 11). Consider also the circumcircle ω of $\triangle ABC$, the circle β with diameter PB (which passes through A_1 and C_1), and the circle γ with diameter PC (which passes through A_1 and B_1).

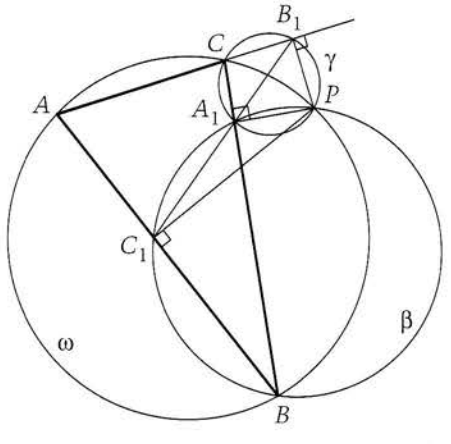

Figure 11

Let's perform the spiral similarity S_1 about P that takes β into ω and then the spiral similarity S_2 about P that takes ω into γ.

It's not hard to see that the resulting transformation S is the spiral similarity about P that takes β into γ. Let's see what happens to point C_1 under these transformations. By the property we proved above, S_1 takes C_1 into the second point of intersection of the line BC_1 with ω—that is, into A. Similarly, $S_2(A) = B_1$. So $S(C_1) = B_1$, and therefore C_1B_1 passes through the intersection point of β and γ that is not P—that is, through A_1. This completes our proof of Simson's Theorem.

Now we apply the theorem to prove the concurrency required by our problem. Look again at figure 8. From the statement of the problem point P lies on the common circumcircle of triangles KLM and MNK. Its Simson line with respect to the first triangle is AC, and with respect to the second it's BD. Therefore, both these lines pass through the projection J of P onto the common side KM of the two triangles, completing the proof of concurrency.

It's interesting that the statement concerning perpendicularity can also be derived using Simson lines, although perhaps less elegantly. The half-turn about the center of the given circle takes the triangle MNK into KLM, point P into the diametrically opposite point P', and line BD into the Simson line l of P' with respect to KLM. So l is parallel to BD. On the other hand, it can be shown (I leave this as an exercise) that the angle between the Simson lines of two different points P and P' with respect to the same triangle is half the angular measure of the arc PP'. In our case this arc measures 180°, so the angle between l and AC (the Simson lines of P' and P relative to KLM) is 90°. Since $l \parallel BD$ and $l \perp AC$, we get $BD \perp AC$.

Pappus's theorem

We saw that the first statement of our initial problem has a far-reaching generalization. The second statement is a particular case of an even more imposing and fundamental fact. It turns out that we don't need any right angles or circles for it to be true. The only essential fragment of the configuration is a triple of parallel lines crossed by another triple of parallel lines. They form a number of parallelograms (how many?). Choose any three of these parallelograms such that any two of the three have exactly one common vertex. Joining these vertices, we'll get a triangle whose sides are diagonals in our parallelograms (the red triangle in figure 12). *Then the other three extended diagonals always meet at the same point.*

I'll prove this statement for the parallelograms $ALCP$, $BNDP$, and $KLMN$ in figure 12, which corresponds to the original problem. The proof will be equally valid, however, for any other choice of parallelograms after a suitable change of labels.

Denote the intersection point of BD and LM by U and that of CA and KN by V. By the similarity of triangles ALC and VBA and the equalities of opposite sides of parallelograms, we have

$$\frac{KB}{BV} = \frac{LA}{BV} = \frac{CL}{AB} = \frac{PA}{AB}.$$

From the similarity of triangles AUB and MUD we get

$$\frac{MU}{UA} = \frac{DM}{AB} = \frac{PA}{AB}.$$

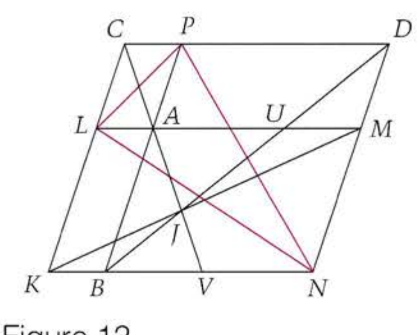

Figure 12

So $KB/BV = MU/UA$, or $KB/MU = BV/UA$. Let J be the intersection point of KM and BD, and let K (not shown in the diagram) be the intersection point of BD and CV. We will show that points J and K are identical (which is why only J appears in the diagram). Since triangles KBJ, MUJ are similar, we have $BJ/JU = KB/MU = BV/AV$. So both points J and K divide segment BU in the same ratio, and so must be the same point.

Exercise 4. Two pairs of opposite sides of a quadrilateral are extended to meet at two points. Prove that the midpoint of the segment joining these points and the midpoints of the quadrilateral's diagonals are collinear. (The line through these midpoints is called the Gauss line of this quadrilateral.)

How could we have come up with the idea of slackening the conditions for the concurrence statement by ridding the problem of all perpendicularities? Via transformations again! We can draw our diagram (or any diagram) on one plane and pass a series of parallel lines through each point of it. Then we consider the intersections of these lines with any other plane. These points form a new diagram, called a *parallel projection* of the old one. Parallel projections preserve parallel lines, whereas such things as right angles and circles are generally destroyed under this mapping. So if we put the diagram to our problem (fig. 1) through parallel projection, we'll get no perpendiculars in the output and no circumcircle on which point P lies—there will simply be no rectangle to circumscribe. However, the straight lines AC, BD, and KM will remain concurrent straight lines, and the resulting diagram will look something like figure 12. Not only that—it can be shown that *any* two intersecting triples of parallel lines can be obtained as a parallel projection of the six lines described in our problem.

Here we used parallel projection to set off the features of the configuration that are responsible for the concurrence we have to prove. We can go even further and subject our configuration to an even more dramatic distortion produced by *central projection*. To perform a central projection, we choose a point outside the plane of our diagram and connect each point in the diagram to the given point. Then we pass a plane through the resulting set of (concurrent) lines. The intersection points of the lines and our new plane form a new diagram, called a central projection of the old one. Like parallel projection, central projection also preserves collinearity of points, but it generally maps parallel lines onto concurrent lines, as illustrated in figure 13. So the diagram in figure 12 under a central projection turns into the configuration in figure 14, while the corresponding statement about concurrent diagonals turns into the following theorem:

Let a, b, c and a', b', c' be two triples of concurrent lines. Then the three lines joining the pairs of intersection points $a \cap b'$ and $a' \cap b$, $b \cap c'$ and $b' \cap c$, $c \cap a'$ and $c' \cap a$ are concurrent.

It's interesting that this theorem is equivalent to the theorem obtained from it by interchanging the words "line" and "point," "concurrence" and "collinearity":

Let A, B, C, and A', B', C' be two triples of collinear points. Then the three intersection points of the pairs of lines AB' and $A'B$, BC' and $B'C$, CA' and $C'A$ are collinear.

This fact is one of the most fundamental theorems in projective geometry. It's called Pappus's theorem.

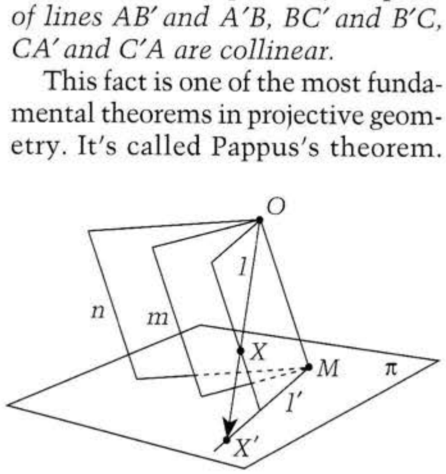

Figure 13

The central projection from the center O on the plane π sends point X into X' and line l through X into $l' = X'M$, where M is the point in π such that OM is parallel to l. The projections of lines n and m parallel to l also pass through M.

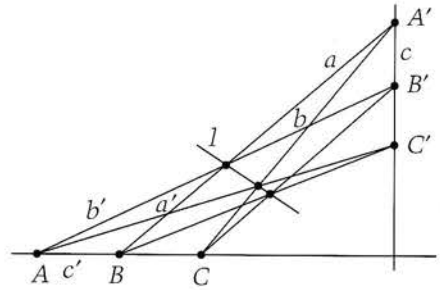

Figure 14

Two statements about lines and points obtained from each other by interchanging these two notions are called *dual*. In projective geometry (which studies the properties preserved under central projections) any statement dual to a true statement is always true.

Exercise 5. Show that Pappus's theorem and its dual theorem are simply rewordings of each other (use figure 14). Explain why the theorem about concurrent diagonals of parallelograms (fig. 12) is equivalent to the dual theorem of Pappus's theorem.

I won't prove Pappus's theorem separately. One way to do this is to go back from figure 14 to figure 12 by a suitable central projection. It can be chosen so as to "send points A and A' to infinity"—that is, make the lines meeting at these points parallel. We can also send other elements of the construction "to infinity" (line $A'B'$ or line l, for example) and thus reduce the theorem to what is called its various "affine versions." You'll certainly enjoy creating and investigating these versions, and I'll leave you with this exciting pastime, although once the words "Pappus's theorem" are pronounced, I have an excuse to spin my tale further, and further, and further . . .

The play of light

What would happen if we changed the rules?

by Dmitry Tarasov and Lev Tarasov

"BUT WHAT IF . . .?" CHILdren are always posing this kind of question. And the scenarios painted after the "if" are extraordinarily diverse. Quite often they touch on physical processes and phenomena. And every time, these questions force us to look anew at some physical situation, and we gain a deeper understanding of the essence of this or that phenomenon, or the role and range of applicability of a certain law of physics.

What if the speed of light in a transparent medium were the same as the speed of light in a vacuum? Maybe you think it's a silly question, given our current understanding of physics. But as we'll see below, to answer it we need to delve into many familiar and seemingly obvious things. Then again, just two hundred years ago this question could be posed in all seriousness.

We should recall that the purely mechanical approach to optical phenomena prevailed in the 17th and 18th centuries. Some scientists favored Newton's particle theory of light, which treated light as a flux of small, fast-moving "corpuscles." Others were persuaded by Huygens's wave theory and considered light a propagation of elastic waves in a particular medium—called the "ether"—that filled the entire universe, including transparent objects. However, to explain known optical phenomena they needed to ascribe strange and sometimes inexplicable properties to the ether. For instance, they were forced to assume that the speed of these "elastic" light waves in the ether depended on what kind of objects are "filling" the ether. This caused a bit of confusion: at first glance it seemed that the speed of propagation of the elastic perturbations in the omnipresent ether must be everywhere the same. And so the question could have arisen: Is the speed of light indeed the same in a vacuum and in different objects?

Let's suppose this were true, and let's look at the "consequences" of this assumption. Of course we'll make use of our current understanding of the true nature of optical phenomena, which corresponds to our everyday experience. In particular, we know that the speed of light in any medium (v) is always less than the speed of light in a vacuum (c). So $c > v$—always. The ratio c/v is the refractive index n of a particular medium.

Suppose the speed of light in all transparent bodies were equal to its speed in a vacuum: $v = c$. This would mean that for every medium $n = 1$. In this new world there would be no refraction at the boundary of different media. For example, a light beam passing from air into water wouldn't change direction. A spoon in a glass of water wouldn't look "broken," and a coin on the bottom of a swimming pool wouldn't seem closer that it actually is. The phenomenon of total internal reflection will also disappear, as you can clearly see in figure 1, which illustrates scenes from two optical worlds—real and imaginary. In our

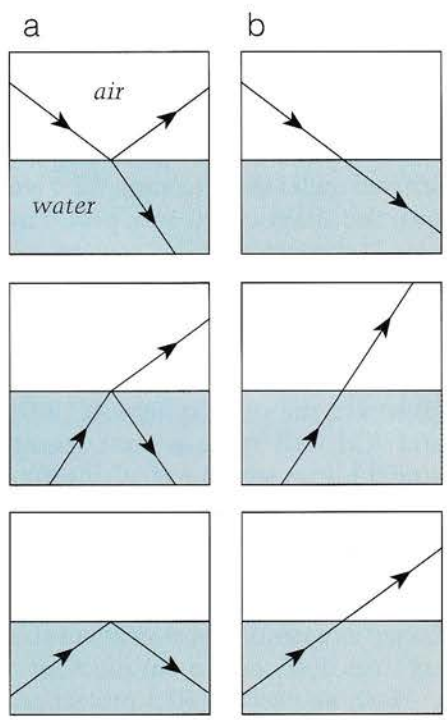

Figure 1

A light beam at the boundary of two media in the (a) real and (b) imaginary worlds.

Art by Vera Khlebnikova

imaginary world, the light beam simply "pays no attention to" the boundary between different media. In the optical sense, this boundary just doesn't exist!

The "metamorphosis" will necessarily result in many losses. All optical devices that use lenses, from ordinary eyeglasses to high-tech microscopes, will be useless, because the lenses can no longer focus or unfocus the light beams. Fiber-optic communication won't work either, because light is "held" in the transparent fibers only due to total internal reflection at the fiber's wall.

There will certainly be changes in some natural phenomena, primarily optical ones, that take place in the Earth's atmosphere. As you know, the atmosphere is an optically heterogeneous medium: its refractive index varies with altitude, and it also depends on temperature, humidity, the presence of impurities, and the motion of the different atmospheric layers. In particular, the refractive index of the air gets smaller as the air's density increases. Thus light beams travel in the Earth's atmosphere not along straight lines, but along smoothly curving trajectories. (This phenomenon is referred to as atmospheric light refraction.) It can be shown that a light beam bends in such a way that its trajectory always curves in the direction of the larger refractive index. This property underlies the formation of mirages (fig. 2). The refraction of light in the Earth's atmosphere causes optical illusions at sunset, such as the flattening of the solar disk, or the appearance of a horizontal gap in the Sun (fig. 3). The twinkling of stars is also due to refraction.[1]

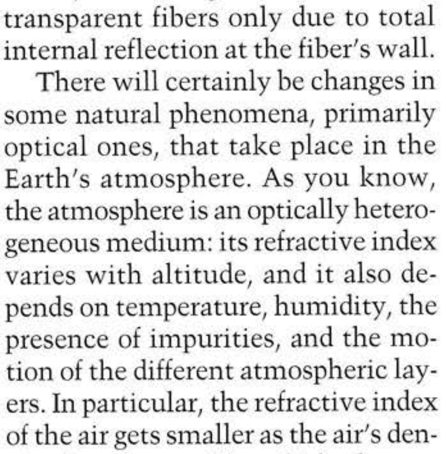

Figure 2

The origin of a mirage. (a) The air is warmer near the ground, so its density (and thus its refractive index) is less than that higher up. This causes the beam to curve upward. (b) If the air is warmer in the upper layers, the refractive index is smaller there, and so the light beam bends downward.

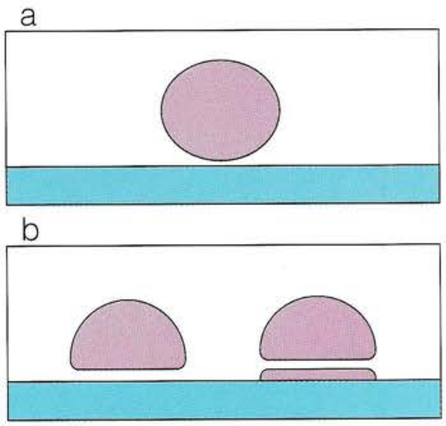

Figure 3

Both (a) the "flattened" Sun at sunset and (b) the gap in the setting Sun are the result of light refraction in the Earth's atmosphere.

In the new optical world that we've created, the atmosphere is a homogeneous optical medium. Regardless of variations in the density of the air, the refractive index is the same everywhere in it—it's equal to 1. So light will no longer be refracted in it, and other phenomena will be gone as well—mirages, twinkling stars, and the kinds of sunsets described above.

We know that a beam of sunlight passing through an ordinary triangular prism is decomposed into all the colors of the rainbow. This is because the refractive index depends not only on the type of medium but also on the wavelength of the light (this property is known as "dispersion"). Violet light is characterized by a larger refractive index, while red light has a smaller index.

Now, when we say that the refractive index is always the same, we're also saying that it depends neither on the medium, on the wavelength of the light, nor on any other factor. And so dispersion of light will also disappear, and a prism will not break light into different colors (fig. 4). Alas, many devices will become useless—almost all spectrometers and spectrographs, for instance. And we'll never see a rainbow in the sky . . .

Thus the world has become so much the poorer after adopting our new optical criteria. There is no refraction at the boundaries of media, no refraction in the atmosphere, no dispersion. And what about reflection? We're talking about the ordinary reflection of light from the surface of a transparent medium. Well, it will disappear as well! (We could have discerned that from looking at figure 1b.) A striking picture emerges: standing on the shore of a lake, we can't see the reflections of trees, bushes, or clouds in the water. We can't see the bright path on the water leading to the moon hanging low in the sky. Not only that, we can't see the water itself—all we see is the bottom of the lake! You have to admit, this would really change the landscape. And we're far from reaching the end of our story—some other remarkable changes will also take place.

Why is the sky blue? Because of light scattering in the atmosphere.

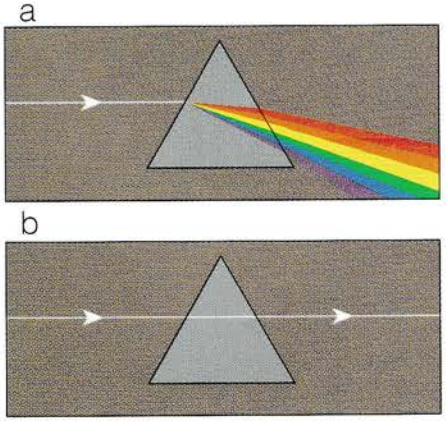

Figure 4

(a) A "rainbow" that appears after light passes through a prism is the result of dispersion. (b) In our imaginary optical world, there is no dispersion.

[1]See the article "What Little Stars Do" in the March/April 1994 issue of *Quantum*.—*Ed.*

The intensity of the scattered light is inversely proportional to λ^4, where λ is the wavelength of the light. So in diffused light, the colors near the violet end of the spectrum are more intense. As a result, the spectrum of the diffused light seems to be shifted toward the short waves: we see blue instead of white. When we look at the sky, we see light scattered by the atmosphere, so the sky is blue. When we look at the Sun at sunset (or simply look in the direction of the Sun), we perceive not the diffused light but the direct rays of the Sun, which have passed through a thick layer of atmosphere without scattering. The spectrum of this light is shifted toward the longer waves. So the setting Sun is red, and the sky nearby takes on red and orange coloration.

Now let's think things through: what will happen to these features in our new optical world? Well, how is light scattered in the real world? By variations in air density, which are in turn caused by chaotic molecular motion. These variations are random: the density varies chaotically from point to point and from moment to moment. The scattering of light results from random variations in the refractive index of air, which is caused by changes in air density. In other words, light is scattered in optically heterogeneous parts of the atmosphere that arise because of the thermal motion of its molecules. In the world of no refraction, variations in density won't produce any optical heterogeneities. So light will not be scattered in the atmosphere—we would see only the black, star-studded expanse of space and the bright, white disk of the Sun in it.

Come to think of it, would we see anything at all in this strange new world? If the speed of light in the lens of our eye is equal to that in a vacuum, the lens would no longer function as "designed." Our eye would be more like a camera obscura. What would it be able to make out? We would be very "far-sighted" indeed! But things near at hand would be wrapped in murk.

Who would have thought such a simple question—"What if the speed of light were everywhere the same as the speed of light in a vacuum?"—would lead to the creation of such a bizarre world! ◘

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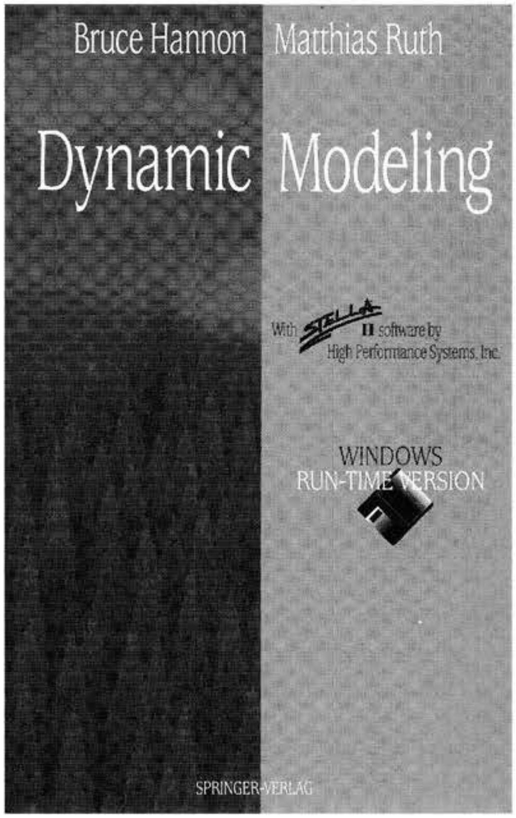

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BRAINTEASERS

Just for the fun of it!

B171

The last digit. Delete all even factors and multiples of five from the product $1 \cdot 2 \cdot 3 \cdot \dots \cdot 1995 \cdot 1996$. What is the last digit of the product of the remaining numbers? (N. Antonovich)

B172

Counting liars. The population of the island of Pianosa is 100. Some of the inhabitants always lie; the others always tell the truth. Each islander worships one of three gods: the Sun god, the Moon god, or the Earth god. One day a visiting anthropologist asked each inhabitant the following questions:

1. Do you worship the Sun god?
2. Do you worship the Moon god?
3. Do you worship the Earth god?

There were 60 "yes" answers to the first question, 40 "yes" answers to the second question, and 30 "yes" answers to the third. How many liars live on the island? (F. Nazarov)

B173

Bubbling tablets. No doubt you've noticed how an effervescent tablet (for instance, Alka-Seltzer™) dropped into water first sinks to the bottom, giving off lots of bubbles, but soon floats to the surface, continuing to release bubbles of gas. Why does the tablet rise? (A. Savin)

B174

Clever arrangement. Arrange four 1's, three 2's, and three 3's around a circle such that no sum of three consecutive numbers is divisible by 3. (S. Berlov)

B175

Equilateral partition. It's easy to cut an equilateral triangle into four equilateral triangles—just draw the midlines. But is it possible to divide it into 8, or 10, or 11 equilateral triangles? And, in general, how many equilateral triangles can it be cut into?

ANSWERS, HINTS & SOLUTIONS ON PAGE 60

Art by Pavel Chernusky


$x=\sqrt[3]{-\frac{q}{2}}$
$-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}$

Surprises of the cubic formula

An equation of much significance and little use

by Dmitry Fuchs and Irene Klumova

Many *Quantum* readers will have heard that, in addition to the formula for solving quadratic equations, a formula exists for cubic equations. All you know about it, though, probably amounts to "you really don't need to know it" (because it's so complicated and unwieldy). At least, you won't find the formula in your high school textbook.

But from time to time you may have encountered a problem that led to a cubic equation. It's then that you begin to wonder about the "utter uselessness" of such a formula, no matter how complex it might be. The aura of mystery surrounding it piques your curiosity—and you start to look for it in mathematical encyclopedias and reference books.

And this is roughly what you'd find.

What does it look like?

A cubic equation is an equation of the form

$$x^3 + ax^2 + bx + c = 0. \tag{1}$$

To solve it, we first of all notice that a simple substitution eliminates the term with x^2. Just set $x = y - a/3$.[1] Then

$$\begin{aligned} x^3 + ax^2 + bx + c &= \left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c \\ &= y^3 - ay^2 + \frac{a^2}{3}y - \frac{a^3}{27} + ay^2 - \frac{2}{3}a^2y + \frac{a^3}{9} + by - \frac{ab}{3} + c \\ &= y^3 + \left(b - \frac{a^2}{3}\right)y + \left(c - \frac{ab}{3} + \frac{2a^3}{27}\right) = 0. \end{aligned}$$

So any cubic equation is reduced to an equation without the square term—that is, to the form

$$x^3 + px + q = 0. \tag{2}$$

This equation is solved by the formula

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \tag{3}$$

which is then followed by the "derivation of the formula"—half a page of calculations.[2]

Something's wrong here

For the time being, let's leave the derivation aside and try to apply the formula to particular equations.

Example 1.

$$x^3 + 6x - 2 = 0.$$

Here $p = 6$ and $q = -2$. Our formula yields $x = \sqrt[3]{4} - \sqrt[3]{2}$. Now, you may have had it drummed into you at school that roots of equations must be expressed in a very simple form, so this answer may strike you as inelegant in the extreme. But you have to admit you never would have guessed that this difference of two cube roots is the solution to such a simple equation. So this result goes on the "credit" side of our ledger.

Example 2.

$$x^3 + 3x - 4 = 0.$$

Formula (3) gives

$$x = \sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}.$$

[1]The sum of the roots of the original equation is $-a$, so if we increase each root by $a/3$, the sum will be zero. This observation motivates the substitution shown.

[2]You can find the derivation in *Quantum*, too—see "The Great Art" in the May/June 1995 issue.—*Ed.*

Art by Dmitry Krymov

This answer is more cumbersome. It can be calculated approximately, and the more exact the calculation, the closer the result is to 1. The reason is very simple: the answer *is* 1![3] But you can't see this just looking at the formula, and this is perhaps its disadvantage. The quadratic formula, however, always shows us if the solution it gives for an equation with integer coefficients is rational or not.

Example 3.

$$(x+1)(x+2)(x-3)=0.$$

We see at once that this equation has three solutions: -1, -2, and 3. But let's try to solve it using our formula. We'll get rid of the parentheses—

$$x^3-7x-6=0$$

—and apply formula (3):

$$x=\sqrt[3]{3+\sqrt{-\frac{100}{27}}}+\sqrt[3]{3-\sqrt{-\frac{100}{27}}}.$$

What's this! A negative number under the square root sign! If this were a quadratic equation, we would conclude that the equation has no solution ("no real solutions," our more educated readers will say, but education is of little help here!). However, we know that the equation *has* a (real!) solution—three, as a matter of fact. So in this case[4] our formula fails completely.

These and other examples suggest that the reason formula (3) is unpopular is not that it's cumbersome (actually, it's not at all cumbersome!). The real reason is that this formula is unreliable: sometimes it works well, sometimes it doesn't give any solutions, and sometimes the form of the solution it gives is unsatisfactory.

This conclusion is, in principle, correct.

However, let's try to explore our formula in order to see when it does and when it doesn't work well. We begin with the simplest.

THEOREM. *If the expression $q^2/4+p^3/27$ is nonnegative, then the right side of formula (3) is a solution to equation (2).*

Proof. We start the proof by setting

$$\alpha=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}},$$

$$\beta=\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}.$$

The right side of equation (3) is simply the sum $\alpha+\beta$. And the product $\alpha\beta$ equals $-p/3$:

$$\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\cdot\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$
$$=\sqrt[3]{\left(-\frac{q}{2}\right)^2-\left(\frac{q^2}{4}+\frac{p^3}{27}\right)}$$
$$=\sqrt[3]{-\frac{p^3}{27}}=-\frac{p}{3}.$$

Now substitute $\alpha+\beta$ into the left side of the equation we are solving (equation (2)):

$$(\alpha+\beta)^3+p(\alpha+\beta)+q=\alpha^3+\beta^3+3\alpha\beta(\alpha+\beta)+p(\alpha+\beta)+q$$
$$=\alpha^3+\beta^3-p(\alpha+\beta)+p(\alpha+\beta)+q$$
$$=-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}+q=0.$$

The proof is done.

How many solutions?

This is a natural question, because to solve an equation means to find all its solutions. So it's useful to know how many solutions a cubic equation has.

As we know, we can confine ourselves to equations like equation (2) in form. Let's see where the function on its left side ($y=x^3+px+q$) increases and where (for what values of x) it decreases. Readers familiar with calculus can undertake this simple investigation on their own. For the rest, we'll give a more elementary argument, one that is, in essence, equivalent to taking the derivative.

Compare the values of our function at two closely situated points x and $x_1=x+\delta$, where δ is a small positive number (fig. 1). Which is greater: x^3+px+q or $x_1^3+px_1+q$? Consider the difference of these values:

$$(x+\delta)^3+p(x+\delta)+q-(x^3+px+q)=\delta(3x^3+p+3\delta x+\delta^2).$$

The sign of the difference is the same as the sign of the factor on the right in parentheses (since $\delta>0$). As to this factor, it's clear that

if $3x^2+p>0$, then for sufficiently small δ it's positive;
if $3x^2+p<0$, then for sufficiently small δ it's negative.

So, at a sufficiently small distance from point x, our

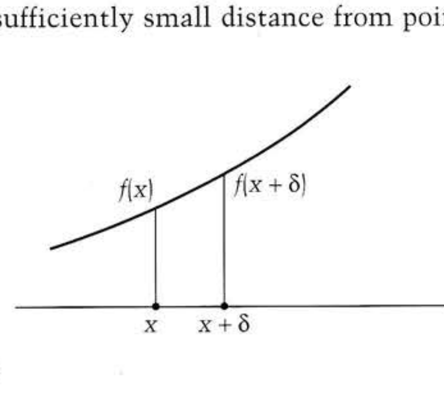


Figure 1

[3]Proof: obviously 1 is a root of our equation, but it has only one real root (as we'll see below).

[4]It's called the *irreducible case* and was mentioned in the article "The Great Art" referred to above as the case that caused Cardano so much trouble.—*Ed.*

function $y = x^3 + px + q$ *increases* if $3x^2 + p > 0$, and *decreases* if $3x^2 + p < 0$. But we know perfectly well that (1) $3x^2 + p > 0$ for all x if $p > 0$, and (2) if $p < 0$, then $3x^2 + p > 0$ for $x > \sqrt{-p/3}$ or $x < \sqrt{-p/3}$, and $3x^3 + p < 0$ for $-\sqrt{-p/3} < x < \sqrt{-p/3}$. To summarize:

1. If $p > 0$, then the function $y = x^3 + px + q$ increases for all x.
2. If $p < 0$, then the function $y = x^3 + px + q$ increases for $x < -\sqrt{-p/3}$, decreases for $-\sqrt{-p/3} < x < \sqrt{-p/3}$, and increases again for $x > \sqrt{-p/3}$.

Notice that our function is certainly positive for any sufficiently large positive x, and negative for any negative x sufficiently large in absolute value. Now we can plot the function $y = x^3 + px + q$ schematically (fig. 2a–2c).

These schematic graphs display only the intervals where the function $y = x^3 + px + q$ increases or decreases and, in addition, the fact that it's negative for remote negative values of x and positive for remote positive values of x. But these graphs suffice for us to say how many solutions our equation has. Our findings:

1. If $p > 0$ or if $p < 0$ and the values of the function at points $-\sqrt{-p/3}$ and $\sqrt{-p/3}$ are of the same sign, then the equation has a unique solution (fig. 2a and 2c).
2. If $p < 0$ and the values of the function at $-\sqrt{-p/3}$ and $\sqrt{-p/3}$ have opposite signs, then the equation has three solutions (fig. 2b).

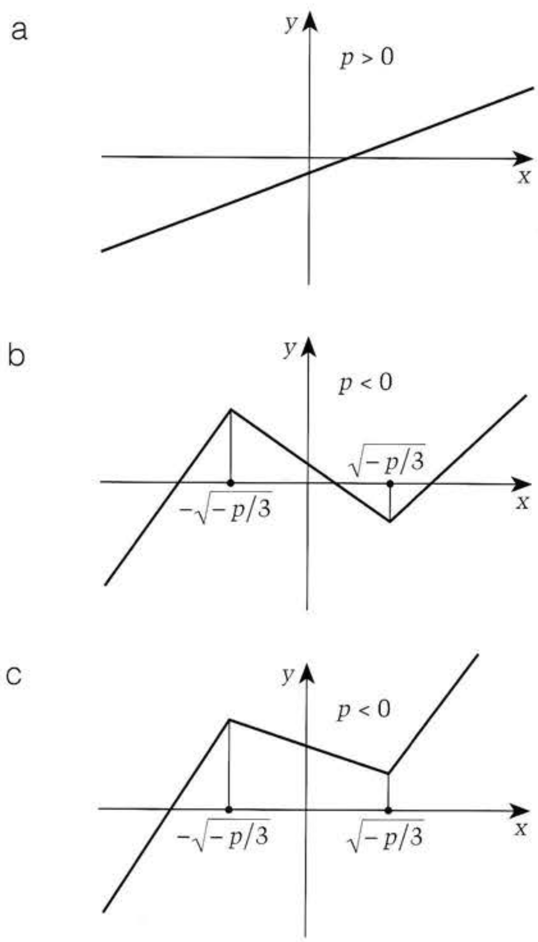


Figure 2

We can come up with a more convenient formulation of this result. Notice that the values at points $-\sqrt{-p/3}$ and $\sqrt{-p/3}$ considered above have the same sign if their product is positive and opposite signs if their product is negative. Let's calculate the product:

$$\left[\left(-\sqrt{-\frac{p}{3}}\right)^3 + p\left(-\sqrt{-\frac{p}{3}}\right) + q\right]\left[\left(\sqrt{-\frac{p}{3}}\right)^3 + p\sqrt{-\frac{p}{3}} + q\right]$$
$$= \left[-\frac{2}{3}p\sqrt{-\frac{p}{3}} + q\right]\left[\frac{2}{3}p\sqrt{-\frac{p}{3}} + q\right]$$
$$= q^2 + \frac{4}{27}p^3.$$

So in the case $p < 0$, the number of solutions is either one or three, depending on whether $q^2 + 4p^3/27$ is greater than or less than zero. Since $q^2 + 4p^3/27 > 0$ for any $p > 0$, we can reformulate as follows:

if $q^2 + 4p^3/27 > 0$, then equation (2) has a single solution;
if $q^2 + 4p^3/27 < 0$, then equation (2) has three solutions.

(We ignored the cases when something vanishes. It's not difficult to see that if $q^2 + 4p^3/27 = 0$, then the equation has two roots except for the case $p = q = 0$, when there is only one root.)

An unexpected consequence

Have you noticed anything surprising in what we've obtained? The expression $q^2 + 4p^3/27$ differs from the treacherous expression under the radical in formula (3) only by an inconsequential factor (as far as the sign is concerned):

$$q^2 + \frac{4}{27}p^3 = 4\left(\frac{q^2}{4} + \frac{p^3}{27}\right).$$

This means that, when the equation has three solutions, the expression under the radical is negative and the formula produces *nothing*. (We saw this in the third example above, but now it's clear that it wasn't an accident.) Otherwise the equation has one solution—the one given by the formula.

So the phenomenon we observed in the second section has now received an explanation. It only remains to see if we can squeeze anything reasonable out of our formula in the case of three solutions and a negative number under the radical.

Help from complex numbers

This section is intended for the readers familiar with complex numbers. In fact, "familiarity" with complex numbers is nothing more than the habit of using certain words in certain situations. Complex numbers appeared in mathematics in much the same way as—somewhat earlier—negative and fractional numbers did. Since we can't extract the square root of every real number, it's convenient to expand our "number stock" with a new number i whose square is -1. (Similarly, negative numbers were introduced to make the operation of subtraction universal.) Along with the number i, we have to introduce all numbers of the form $a + bi$ with real a and b. Our usual algebraic operations are then defined for them by means of the most natural rules:

$$(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i,$$
$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

(in the last equation we took into account the defining relation $i^2 = -1$). The numbers we have constructed are called *complex numbers*. The numbers a and b are called the *real* and *imaginary parts* of the complex number $a + bi$. Two complex numbers are equal if and only if they have equal real and imaginary parts. It's also time to introduce *de Moivre's formula* (we'll be needing it below):

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha + \beta) + i\sin(\alpha + \beta),$$

which is an immediate consequence of the formula for complex multiplication and certain well-known trigonometric formulas.

Now let's get back to formula (3). In the case $q^2/4 + p^3/27 < 0$, we can try to apply it to equation (2) by extracting the cube roots of complex numbers.

Taking the cube root of a complex number $a + bi$ means solving the equation

$$(x + iy)^3 = a + bi$$

—that is, the equation

$$x^3 - 3xy^2 + i(3x^2y - y^3) = a + bi,$$

which is equivalent to the system of real-number equations

$$\begin{cases} x^3 - 3xy^2 = a, \\ 3x^2y - y^3 = b. \end{cases}$$

This system can again be reduced to a cubic, in different ways. For instance, as follows:

$$\underset{\text{(from the first equation)}}{y^2 = \frac{x^3 - a}{3x}} \Rightarrow y\left(3x^2 - \frac{x^3 - a}{3x}\right) = b \Rightarrow y = \frac{3bx}{8x^3 + a};$$

$$x^3 - 3x\frac{9b^2x^2}{\left(8x^3 + a\right)^2} = a \Rightarrow \left(x^3 - a\right)\left(8x^3 + a\right)^2 - 27b^2x^3 = 0;$$

or, putting $x^3 = z$,

$$(z - a)(8z + a)^2 - 27b^2z = 0. \qquad (4)$$

Of course, we can try formula (3) again. But it turns out that applying it to equation (4) always leads to a negative expression under the radical. And this should come as no surprise: there are always three (complex) cubic roots of a complex number $a + bi$, and the cubes $z = x^3$ of their real parts are the three (real) roots of equation (4).

So we can't make use of formula (3) on this path. But it can be used for an approximate computation of the roots of equation (2). This is done by means of de Moivre's formula, which implies that

$$\cos\alpha + i\sin\alpha = \left(\cos\frac{\alpha}{3} + i\sin\frac{\alpha}{3}\right)^3.$$

So to take the cube root of $a + bi$, we first rewrite this complex number as

$$\sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}} + \frac{b}{\sqrt{a^2 + b^2}}i\right).$$

Then we choose α such that $\cos\alpha = a/\sqrt{a^2 + b^2}$, $\sin\alpha = b/\sqrt{a^2 + b^2}$ (we can do this because the sum of the squares of these numbers is 1). Then the cube root of $a + bi$ can be written as

$$\sqrt[6]{a^2 + b^2}\left(\cos\frac{\alpha}{3} + i\sin\frac{\alpha}{3}\right).$$

(Notice that α is defined up to $2k\pi$, which is an even multiple of π. So $\alpha/3$ is defined up to the addition of $2k\pi/3$, which gives three values for $\cos(\alpha/3) + i\sin(\alpha/3)$, as it should.)

In this way we can compute the cube roots in formula (3) approximately. Actually, we only need the *real part of one of them*: if one of the two roots equals $c + di$, then the other is $c - di$ (prove this!) and their sum is equal to $2c$.

But this method of approximate solution (using, say, a hand calculator) is even more cumbersome and inexact than the method of trial-and-error and successive iterations that is usually applied.

This is why we don't memorize formula (3)—it's not the most handy tool for solving cubic equations in practice.

What good is it?

The significance of the cubic formula (or "Cardano's formula"—formula (3)) lies in the answer it gives to the classical question about the "solvability of third-degree equations in radicals." Now, what does *that* mean?

The first irrational numbers we encounter are roots (the very first one is usually $\sqrt{2}$, the diagonal length of a unit square). Extracting roots, together with arithmetic operations, expands our supply of numbers by adding to the rational numbers such numbers as $\sqrt[3]{2} + \sqrt{3}$, $\sqrt{\sqrt{5/\left(\sqrt{2} + \sqrt{3}\right)} + 1}$, and so on. Is this expanded supply enough to solve algebraic equations with integer

CONTINUED ON PAGE 29

Against the current

A look at the forces that seek to slow us down

by Alexander Mitrofanov

HOW IS A MOVING OBJECT affected by the surrounding medium? When you walk down the street, for instance, you generally don't think much about any resistance the air might put up as you move through it. It's unlikely that a student late for class would offer the explanation: "I was held up by aerodynamic resistance." And yet, if you put your hand out the window of a moving car, or try to walk in a strong wind, the question of air resistance stops being just a lame excuse. The air, which is usually so intangible and ethereal, becomes a completely different thing when it's moving quickly—it's more like an elastic wall or an insurmountable obstacle. That's exactly what a pilot feels when forced to eject in flight. The resistance a medium exerts on a moving object is very familiar not only to aviators and astronauts, but also to many others in more "mundane" professions.

One of the first scientific studies of fluid resistance was performed by Isaac Newton.[1] Because data were scarce at that time concerning the interaction of moving objects with a medium, Newton was forced to undertake the necessary experiments himself, tossing various objects and making measurements. In 1710 and 1719 he conducted tests in St. Paul's Cathedral in London with spheres moving in water. From these experiments he obtained the coefficient for his theoretical equation for the resistance acting on an object moving in a fluid.

In our age of high-speed transport, supersonic flight, and space launches, many old problems and experimental methods in aero- and hydrodynamics still have life left in them. So it might be instructive to call up some of these ideas from the past and do some experiments that will introduce you to certain phenomena arising when an object moves in a fluid.

What is the resistance of a medium? An object moving in a fluid (gas or liquid) affects the medium's particles and changes their speed. According to Newton's third law, the object is acted on by an equal and opposite "resistive force." Let's consider a component of the resistive force that is directed along the line of motion (say, the x-axis)—the force pilots call "frontal resistance." The Galilean principle of relativity says that in the case of uniform motion it doesn't matter how one calculates this force—either as an object moving in a stationary medium or a medium flowing with the same speed around a stationary object. We'll begin with a simple example.

Case 1. Let a disk, sphere, or cylinder of radius R move with a speed v along its axis in a medium composed of many stationary particles that do not interact with one another (see figure 1—m is the mass of a particle, n is the concentration of particles). What is this medium similar to? We can imagine a rarefied cold gas where the molecular motion can be neglected. Another useful model would be snowflakes in the air, where the problem is to

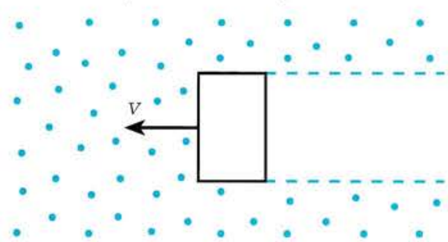


Figure 1
Movement of an object with speed v in a rarefied medium.

Art by Pavel Chernusky

[1]Newton wasn't the only one to study the problem. He was joined by Leonard Euler, Jean D'Alembert, Jean le Rond, Daniel Bernoulli, Charles Coulomb, and many others.

describe the flight of a snowball colliding with the snowflakes. We could come up with others.

An object collides with $N = nvS$ particles per unit time, where $S = \pi R^2$ is the cross-sectional area of the object. If the collisions are nonelastic, the force of resistance is given by

$$F_x = \frac{\Delta p}{\Delta t} = mvN = 2S\frac{\rho v^2}{2}, \quad (1)$$

where $\rho = mn$ is the density of the medium.

Similarly we can calculate the resistive force F_x acting on an object of any shape. To do this we need to know the speed, density, and the cross-sectional area of the column of particles encountered by the object moving in the rarefied medium.

Things are a little more complicated when the collisions are elastic—that is, when the particles bounce away after a collision with an arbitrary surface. However, in this case F_x is also proportional to v^2, although in general the proportionality factor in equation (1) depends on the shape of the object.

Case 2. Up to now we've looked at motion in a rarefied medium. What happens if an object moves uniformly in a liquid (say, water) or in a dense gas (air)—that is, in a continuous medium? How does a continuous medium differ from a system of noninteracting particles? Both media consist of atoms and molecules. However, while particles move independently of one another and collide very rarely in a rarefied medium, motion in a continuous medium looks completely different—here the particles behave like a team or an ensemble.

Indeed, the intermolecular distance (strictly speaking, the *mean free path*) is much smaller than the object's characteristic size. Due to mutual collisions and the interaction of the particles, the medium's perturbations near the object's boundary resulting from its motion are transferred to the adjacent elements of the medium. So the object doesn't only interact with the particles of gas or fluid situated directly in its path—it interacts with the particles of the medium that it pushes aside or drags along with it (fig. 2).

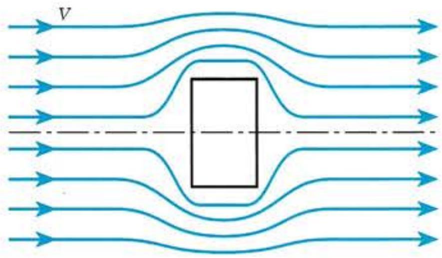


Figure 2
Motion in a continuous medium as seen in the object's reference frame.

How does F_x depend on v and on the medium's characteristics? It's clear that if the speed of the medium varies, the mass of fluid M encountering the object per unit time changes in proportion to v—that is, $M \sim \rho v S$, where S is the object's cross-sectional area and ρ is the medium's density. Each element of this mass transfers momentum to the object that is also proportional to v. So the dependence of the resistive force on the object's speed, which Newton was the first to investigate, looks like this (compare it with equation (1)):

$$F_x \sim \rho v^2 S,$$

or, in different notation,

$$F_x = C_x \frac{\rho v^2}{2} S, \quad (2)$$

where $\rho v^2/2$ is known as the *dynamic pressure*. (Indeed, the unit of measurement for this magnitude is the same as that for pressure and is none other than the kinetic energy per unit volume of a moving medium in which a stationary object is placed.) The proportionality coefficient C_x in equation (2) depends primarily on the object's shape and also on the nature of the flow and on the speed (when it varies widely). In aerodynamics this value is called the *coefficient of frontal resistance*, while in hydrodynamics it's referred to as the *coefficient of hydrodynamic resistance* or the *drag coefficient*.

Compare figures 1 and 2 and note an important difference between these two examples. A liquid flows around the object and doesn't leave a "shadow" of empty space behind it, unlike what happens in a cold rarefied medium. So a continuous medium presses against an object not only frontally (that is, against the prow, if the object were a ship) but elsewhere as well. The direction of this force acting on the object from the rear (from the stern) coincides with the direction of motion. Thus the flow of a liquid or a dense gas around an object results in a decrease in the total resistive force F_x. Flowing around the object, the liquid retains some of its momentum, not transferring it to the object. This feature is very important. In the next experiment we'll see how the coefficient of hydrodynamic resistance drops due to the flow around an object.

Case 3. Let's find the hydrodynamic resistance met by a ball placed in a stream of fluid (that is, the value of C_x for a sphere in a flowing liquid). This experiment is very simple and doesn't require any fancy equipment. All you need is a pail or a bathtub, a jar with a known volume, a meter stick, a watch with a second hand, and a light ball (a Ping-Pong ball or a small rubber ball).

Turn on the faucet and fill the pail or bathtub with water. Put the ball on the surface of the water—as you might expect, it skips away from the stream. However, if you place the ball under the stream (or push the ball close to it), it is caught by the stream and held where the stream hits the surface of the water.

What happens if you vary the flow? When the stream's speed is small, the ball floats on the surface (fig. 3a) and it can stay near the stream indefinitely. Sometimes the ball's center moves away from the stream's axis—when that happens, the ball spins around a horizontal axis like a small turbine.

When you increase the stream's flow, the ball sinks deeper, stops spinning, and positions itself at the stream's axis, oscillating ever so

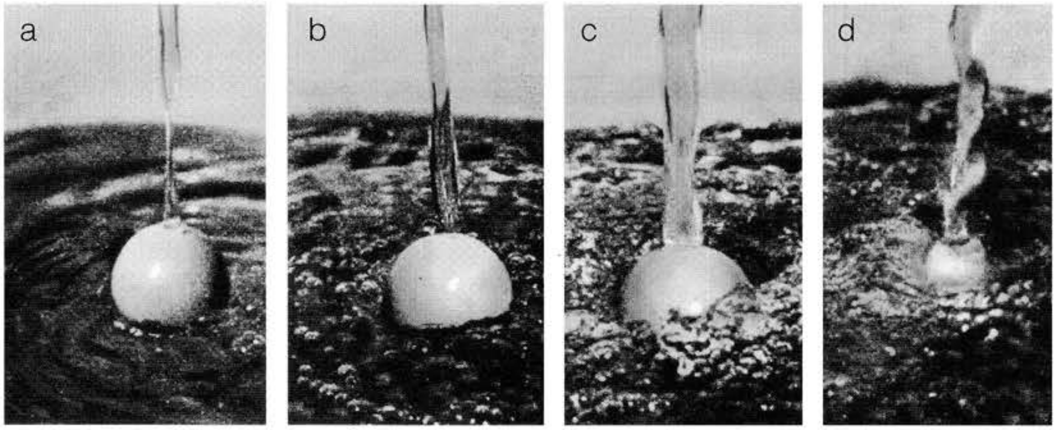


Figure 3
Photographs of a ball under streams with different volumetric flow rates.

slightly (fig. 3b and 3c). Finally, when you open the faucet even more, the ball becomes submerged entirely (fig. 3d).

The submergence of the light hollow ball is mainly due to hydrodynamic resistance—that is, the force exerted by the flow. This force can easily be measured by how deeply the ball is submerged.

Let's open the faucet in such a way the ball is submerged to a certain depth (equal to the ball's radius, for example) and maintained in this equilibrium state. You can control this submergence by eye. In the equilibrium state the buoyant force F_b counterbalances the ball's weight mg and the resistive force F_r:

$$mg + F_r = F_b.$$

In this equation $m = \rho_b {}^4/_3\pi R^3$, where R is the ball's radius and ρ_b is its average density. The radius of a Ping-Pong ball is $R \cong 1.9$ cm and its mass is $m \cong 2.5$ g—that is, $\rho_b \cong 0.09$ g/cm^3, which is approximately 1/11 that of water ($\rho_w = 1$ g/cm^3). This is why the ball floats with hardly any submergence. The buoyant force acting on the half-submerged ball is $F_b = {}^4/_3\pi R^3\rho_w g/2$. The value of F_r can be taken from Newton's equation (2) by inserting $S = \pi r^2$, where r is the radius of the stream:

$$\frac{4}{3}\pi R^3\rho_b g + C_x \pi r^2 \frac{\rho_w v^2}{2} = \frac{1}{2}\cdot\frac{4}{3}\pi R^3 \rho_w g,$$

which yields the coefficient of hydrodynamic resistance C_x:

$$C_x = \frac{4}{3}\left(\frac{R}{r}\right)^2 \frac{gR}{v^2}\left(1 - 2\frac{\rho_b}{\rho_w}\right). \quad (3)$$

To calculate C_x from this equation, we need to know the values of r and v, which are not easy to measure directly. However, these terms can be found indirectly (fig. 4) by measuring the volumetric flow of water in the stream Q ($Q = \pi r^2 v$, which is the volume of water passing through a unit cross section per unit time), its length L, and the initial radius r_0 (near the faucet), and by taking into account that the volumetric flow of water is constant in any part of the stream (the so-called *continuity equation*):

$$r_0^2 v_0 = r^2 v, \quad (4)$$

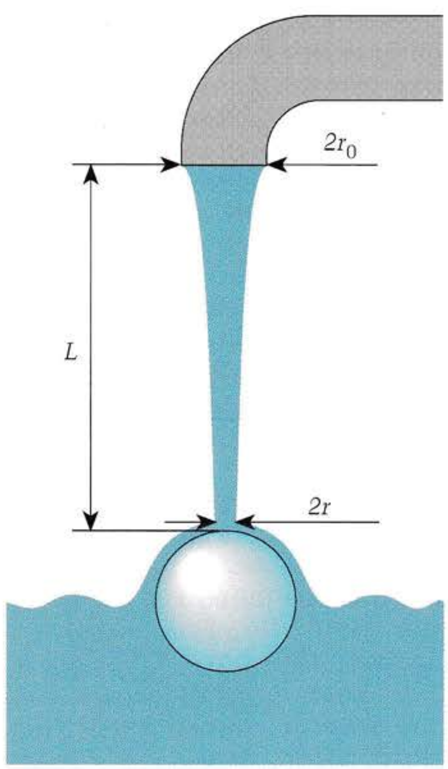


Figure 4
Calculating the coefficient of hydrodynamic resistance for a ball under a stream of water.

where v_0 is the stream's speed near the faucet. We can determine the volumetric flow of water by noting the time it takes to fill the jar. Knowing Q and r_0, we can find $v_0 = Q/\pi r_0^2$. The values for v_0 and L yield $v = \sqrt{v_0^2 + 2gL}$. Finally, we can determine r from equation (4). Before you do your experiment, think about the physical meaning of equation (4) and why a stream gets thinner as it travels downward, and what equation describes the decrease in its radius (see case 4).

Now make your measurements.

Here are my results. To submerge a Ping-Pong ball halfway under a stream of water whose length $L = 60$ cm and initial radius $r_0 = 0.8$ cm, the flow of water Q must be approximately equal to 130 cm^3/s. This gives $C_x \cong 0.5$. If we vary the stream's length L, we must simultaneously change the flow to keep the ball at the same depth. However, the experimental values of C_x were practically the same regardless of the flow. The experimental data obtained for different volumetric flows of water (corresponding to different lengths of the stream) are given in figure 5 (on the next page), where the broken line shows the theoretical values for the stream's diameter $D = 2r$ where the stream touches the ball. Do some experiments on your own with streams of different lengths to convince yourself that regardless of r and v, the values obtained for the coefficient of hydrodynamic resistance are

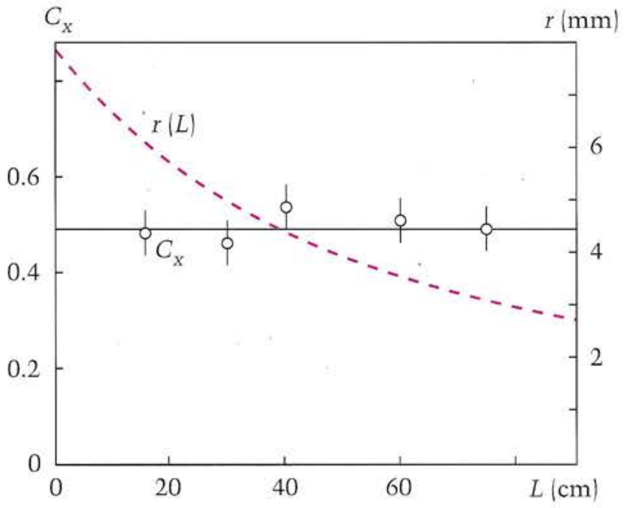


Figure 5

Experimental data on the hydrodynamic resistance coefficient C_x for a ball in a liquid stream as a function of stream length L. The broken line corresponds to the calculated dependence for a stream of radius r that submerges the ball halfway for various stream lengths L.

approximately the same—that is, $C_x = 0.5 \pm 0.1$.

Now let's try to make sense of our results from the physical viewpoint. We established two important facts. First, in calculating the resistive force F_x we used Newton's equation (2) and found that this equation correctly describes the experiment, because C_x was almost constant. So we verified this equation within the range of speeds $v \cong 1$–10 m/s. Second, in measuring C_x we found that, strangely enough, the ball takes only about 1/4 of the stream's momentum from it—the rest goes into the water where the ball is floating. Thus the coefficient of hydrodynamic resistance proved to be about 1/2 and not 2, as would be the case if the "collision" of the stream with the ball were absolutely inelastic (see equation (1)).

This experimental fact is not at all obvious. At first glance it seems that a narrow stream falling on the almost flat top of the ball ($r \ll R$) should transfer most of its momentum to it. In reality our experiments showed that the stream flows around the ball almost without loss of speed and drops down under the ball, preserving a significant portion (three quarters!) of its momentum. Thus the flow around an object indeed decreases the hydrodynamic resistance!

Now, some physics lovers will say that the factors 1/4 and 1 are of the same order of magnitude. True enough. But not every problem can be solved within an order of magnitude. Everybody agrees that there is a obvious difference between filling a fuel tank with 400 tons of oil and filling it with 100 tons. The lion's share of energy used by submarines, racing cars, and electric locomotives goes to overcoming the resistance of the media they pass through. So decreasing C_x even by a few percentage points can be considered a victory.

Case 4. Equation (2) is successfully used in practice when, say, one needs to evaluate the force of wind on a sail or a building, or the resistive force acting on a moving object, whether plane, bird, car, or submarine. This equation describes one of the fundamental laws in aerodynamics. However, one should keep the following nuances in mind. Since the pressure in a fluid depends on its speed, the resulting force of hydro- or aerodynamic resistance depends on how the fluid flows around the object—that is, on the speed of the fluid near the object's surface; whether the flow is smooth (laminar) or turbulent; where on the object's surface the medium separates from the object; and so on. It is this variety in the way continuous media flow near objects (see, for example, figure 6) that makes it difficult at times to compute F_x and C_x. For some objects like spheres, disks (athwart the flow), relatively short cylinders, and so on, the value of C_x is about 0.1–1 for a broad range of velocities. For a streamlined, elongated droplike object with a smooth curved front, the coefficient of dynamic resistance can be as low as 0.04–0.06.

Much smaller values for the coefficient of hydrodynamic resistance—down to about 0.01—can be found in the natural world. The dolphin is a classic example. However, to decrease resistance, fish and other aquatic animals have not only streamlined shapes but a whole "bag of tricks," including specially adapted types of skin and grease or mucus on their scales. In addition, they use their muscles to regulate the flow and prevent the formation of vortices that would drain all their energy. With these natural means the animals do scientists and engineers one better—many of their "technical solutions" have yet to be implemented in the devices invented by

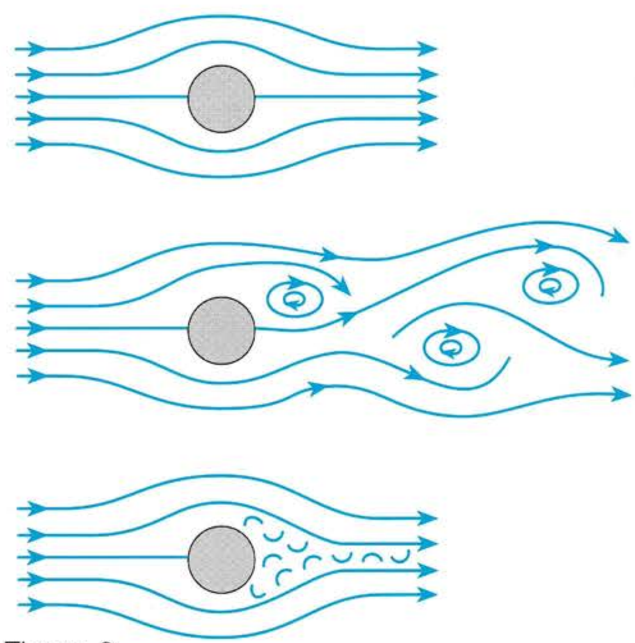


Figure 6

Flow lines of a liquid passing around a cylinder for different flow speeds—low, moderate, and comparatively large.

human beings. We still have a long way to go . . .

Case 5. At the dawn of aviation, an unexpected phenomenon was discovered in experiments aimed at measuring frontal resistance in aerodynamic tunnels. At comparatively high air speeds, a further increase in speed resulted in a drastic decrease (by a factor of two or more) in frontal resistance for spheres and certain other objects. The researchers were able to determine the conditions necessary for such a jump to occur, measure it, and explain it. They found that the jump was caused by a shift of the point where flow separation occurs at large velocities and by a corresponding decrease in the width of the vortex area behind the object. By improving the conditions of flow around the rear of the object, one could realize an immediate reduction of C_x.

This phenomenon has a curious consequence. Calculations have shown that if very large hailstones existed on Earth and their diameters increased as they fell to the critical value D_{cr} = 13 cm, the hail's speed would increase suddenly by a factor of 2.5, from 160 km/h to 400 km/h! (Other estimates of D_{cr} give a value just under 10 cm.) There is evidence that even nowadays hailstones are sometimes as large as a hen's egg or even an orange. But who knows, maybe larger hailstones fell in times past. Might they have approached the critical value D_{cr}? After all, in earlier ages the atmospheric "machine" on our planet worked harder than it does now and produced storms that were much more severe. Can the answer to one of science's biggest mysteries be hidden here? Maybe the dinosaurs were killed off by huge, speeding hailstones! Smaller animals could have found nooks to hide in and so avoid the pathetic fate of their towering cohabitants . . .

Case 6. Newton's equation for the force of resistance is by no means universal. For example, it correctly estimates the force acting on a spoon sinking in a jar of honey. But it doesn't describe the way a ball-bearing drops in a tall bottle of vegetable oil, and doesn't come close to explaining why a thick milky fog (that bane of drivers, pilots, and sailors) descends to Earth for such a long time.

In each of these cases, to find the resistance we need to know not only the shape and size of the object, its relative speed, and the density of the medium, but also another parameter of the medium—its viscosity, which is usually denoted by η. This coefficient shows how large the internal (or viscous) friction in a fluid is.

If viscous forces predominate as a sphere moves in a medium, the sphere is affected by a drag force that must be calculated not by Newton's equation (2) but by Stokes's equation:

$$F_x = 6\pi\eta R v.$$

But how are we to know whether a particular medium is viscous or not, and what equation we need to use to find the resistance? Let's consider a spherical water drop that falls freely in the air (or a hailstone, thus neglecting the difference between the densities of ice and water). We are interested in the drop's steady-state speed, which can be measured and then inserted into the equation for dynamic equilibrium, resulting in the dependence of the frontal resistance on the drop's size and speed as well as on the characteristics of the medium, its density, and the dynamic viscosity coefficient. The drop's mass is considered to be constant—that is, we neglect both condensation and evaporation.

The graph in figure 7 shows how the speed of the falling raindrops depends on their diameter. The data are taken from textbooks on meteorology. The notations near the curve show the type of precipitation corresponding to observed drops of a particular diameter.

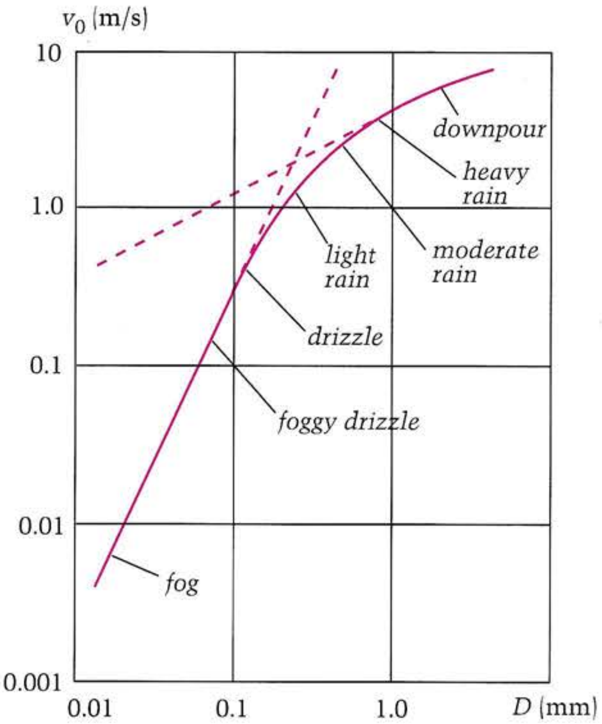


Figure 7

Logarithmic dependence of the speed of falling raindrops on their diameter.

This figure shows that the smallest drops (0.01–0.1 mm in diameter) fall with a speed $v_0 \sim D^2$. This is possible only if the resistive force obeys Stokes's law, which gives the equation for the rate of precipitation of the fog's drops:

$$v_0^{(S)} = \frac{1}{18} D^2 \frac{\rho}{\eta} g,$$

where $\eta \cong 1.8 \cdot 10^{-5}$ kg/m · s at T = 300 K. For air and other gases, $\eta \sim \sqrt{T}$ and hardly varies with the density of the gas.

An increase in the drop's size and speed results in a decrease in the relative contribution of the viscous friction forces (proportional to $\eta v R$) compared to the forces that are proportional to $R^2\,(\rho v^2/2)$ and described by Newton's equation. (Sometimes they are referred to as the *inertial response of the medium* in order not to confuse them with the forces of internal friction.)

Large raindrops ($D \gtrsim 1$ mm in

diameter) are hardly affected by the viscous friction of the air. Their steady-state speed in the atmosphere is proportional to $\sqrt{D}$:

$$v_0^{(\mathrm{N})} = \sqrt{\frac{4}{3C_x}\frac{\rho}{\rho_{\mathrm{air}}}Dg}, \qquad (5)$$

where $C_x \cong 0.5$. This equation is obtained from equation (2).

It should be noted that equation (5) is only an approximation because, unlike drops of fog or hail, the shape of large raindrops deviates greatly from the spherical. The forces of surface tension are overpowered by the dynamic pressure of the frontal air flow, which deforms the drop. The largest raindrops ($D \leqslant 5$ mm), attaining a speed of 8 m/s, are flattened so much they break into smaller drops.

Drops of intermediate diameter—$0.1 \text{ mm} \leqslant D \leqslant 1$ mm—obey neither Stokes's nor Newton's equation. For these drops the inertial response of the medium and the forces of internal friction are comparable.

Case 8. There is yet another mechanism of deceleration. This third mechanism is also encountered in many situations and can predominate under certain conditions. Fill a pail or bathtub with water and submerge a pencil vertically to a certain depth. Move the pencil parallel to the water's surface. You won't see anything interesting if the speed of the pencil is small. However, when the pencil moves quickly, it leaves behind a group of diverging waves that carry off energy. Energy is needed to produce such waves, and in this case it comes from the hand that overcomes the water's resistance.

By experience we know that a moving object often produces waves, and these waves can be quite different. The pattern of waves generated by a ship or motor boat in deep water differs from shallow capillary waves that can be seen in a pail, a glass, or a saucer. The waves of a motor boat in shallow water differ from those in deep water. A supersonic plane produces a shock wave in a three-dimensional medium (air), as opposed to the two-dimensional waves made by a ship on the boundary surface between water and air. Also, unlike ordinary waves in water, shock waves in a gas are accompanied by a strong compression of the medium, which also heats up.

The amount of resistance from waves depends on how the waves are produced—that is, on the parameters of the medium bearing the waves, on how fast and in what manner they spread, and above all on the leading edge and size of the object. Usually the resistance from waves increases drastically with the speed v—provided, of course, that this increase doesn't cause the object to jump out of the water, as outboard-motor boats and hydrofoil craft do.

The calculation of wave resistance in any particular case is a difficult problem that can usually be solved only on the basis of experimental data. However, it's rather easy to estimate how large the wave resistance might be.

For example, consider the uniform straight-line motion of a motor boat on a deep lake. Two slanting waves originate at the bow and stern of the boat, which are symmetrical relative to the boat's course. These waves interfere and produce a pattern that is very familiar to fishermen and swimmers. The angle α formed by the waves' crests and the direction of the boat's motion does not depend on the boat's speed and is approximately equal to 20°. Two or three small crests can be seen on either side of the boat. The amplitudes of the other waves are far smaller and can be neglected, as their contribution to the final estimate is very small. The boat's waves run for hundreds of meters and die away slowly.

A wave crest can be approximated by a triangle with altitude H measured from the equilibrium level of the lake. The length of the triangle's base is $\lambda/2$, where λ is the wavelength of the wave produced by the boat. The length of the wave front generated per unit time is $v \cos \alpha$. The work performed to produce the crest is accumulated in the form of the wave's potential energy. Bearing in mind that in periodic processes the potential and kinetic energies are equal, we obtain an estimate of the power needed for a moving boat to generate a group of waves:

$$\begin{aligned} P &= \frac{1}{2}H\frac{\lambda}{2}v\cos\alpha\cdot\rho g\frac{H}{3}2\cdot 2n \\ &= \frac{1}{3}\rho g H^2 v\lambda n\cos\alpha, \end{aligned} \qquad (6)$$

where $2n$ is the total number of crests behind the boat. The factor $H/3$ is the average height to which the water is raised (with the distribution of mass within a crest taken into account).

Here's a typical example. At a speed of 18 km/h a boat produces waves with a height of 0.3 m and a wavelength of about 0.6 m, $n = 3$. Equation (6) yields the power $P \cong 3 \cdot 10^3$ W $\cong 4$ hp. Within a factor of 2, this value is equal to the minimum power required of a motor to accelerate a small boat to this speed.

So we see that a significant amount of power is needed just to make waves!

Problems and questions

1. While studying free fall, Galileo would simultaneously throw a cannon-ball with a mass of 80 kg and a musket ball with mass of about 200 g from a tall tower. Did the air have a significant effect on the objects as they fell? The tower was about 60 m high.

2. A new model of locomotive differs from the old one in its motor, which now has 1.5 times the power of its predecessor. How much faster is the new model?

3. Why are the data points scattered in figure 4? Find the value of the coefficient C_x in the experiment if a ball is submerged completely under a stream of water, as in case 3 described above. What conclusions can you draw from your experiment? What can you say about the precision of your data?

4. Show that a stream's radius at a distance L from a faucet is

determined by the equation $r(L) = r_0(1 + 2gL/v_0^2)^{-1/4}$, where v_0 is the speed of the water at the faucet, g is the acceleration due to gravity, and r_0 is the initial radius of the stream. The stream doesn't spray drops, and the effects of friction and surface tension can be neglected.

5. How might we visualize the flow of water under a sphere? Conduct an experiment with a stream of water from a pipe. What is the nature of the water flow under the sphere?

6. When a spoon is inserted into a stream of water, the stream reacts differently depending on which side of the spoon is turned toward the water. Why?

7. What is the steady-state speed v_0 of a Ping-Pong ball falling through the air?

ANSWERS, HINTS & SOLUTIONS ON PAGE 62

"SURPRISES OF THE CUBIC FORMULA" CONTINUED FROM PAGE 20

coefficients? Our formula tells us that it suffices for cubic equations—at least, if we allow complex, and not just real, expressions under the radicals.

It turns out that fourth-degree equations can also be solved in radicals.[5] But *equations of degree five and higher are unsolvable in radicals.* It's very likely that you've already heard about this last theorem.[6] Its proof is in fact much simpler than is generally supposed. But that's a subject for another article.

[5]This was explained in "The Great Art," as well as in "What You Add is What You Take" in the November/December 1994 issue of *Quantum*.—*Ed.*

[6]For instance, you might have read the *Quantum* article about Évariste Galois in the November/December 1991 issue, where this theorem is discussed.—*Ed.*

HOW DO YOU FIGURE?

Challenges in physics and math

Math

M171

Reproducing ones. Does there exist a quadratic polynomial $P(x)$ with integer coefficients such that for any positive integer n whose decimal notation consists only of ones the notation of $P(n)$ also consists only of ones? (A. Perlin)

M172

Recurrent unit fractions. Is it possible to select a subsequence of (a) 5, (b) n, (c) infinitely many terms from the sequence 1, 1/2, 1/3, 1/4, ... such that each term in it (except the first two) is the difference of the preceding two? (S. Tokarev)

M173

In search of similarity. Find all the points X on the side BC of a triangle ABC such that the triangle XPQ, where P and Q are the (a) circumcenters, (b) centroids, (c) orthocenters of the triangles AXB and AXC, is similar to ABC. (E. Turkevich)

M174

Estimating the derivative. A function f is differentiable on a segment $[a, b]$ of length 4. Prove that there is a point x inside the segment such that $f'(x) - (f(x))^2 < 1$ ($f'(x)$ is the derivative of f). (F. Vainshtein)

M175

Slicing a pyramid. A cross section of a regular tetrahedron is a quadrilateral. Prove that its perimeter is no less than twice the length of an edge of the tetrahedron, but less than three times this length. (V. Proizvolov, A. Savin)

Physics

P171

Ball in a glass of water. A small wooden sphere is attached by a nonstretchable cord of length l = 30 cm to the bottom of a cylindrical vessel filled with water. The distance from the bottom's center to the point where the cord is attached is r = 20 cm. The vessel is whirled about the vertical axis passing through the bottom's center. At what angular velocity does the cord make an angle $\alpha = 30°$ with the vertical? (V. Mozhayev).

P172

Sand on a membrane. A horizontal membrane is strewn with fine sand. The membrane oscillates with a frequency ν = 500 Hz in the vertical plane. What is the amplitude of the membrane's oscillations if the grains of sand are thrown to a height h = 3 mm relative to the membrane's equilibrium position? (B. Bukhovtsev)

P173

Gas processing. One kilomole of ideal monoatomic gas under standard conditions is processed from condition 1 to condition 2 in two different ways: $1 \to 3 \to 2$ and $1 \to 4 \to 2$ (fig. 1). Find the ratio of the thermal energies that must be transferred to the gas in these processes.

Figure 1

P174

Capacitor with faulty insulation. A flat plate with parallel layers has a thickness h and is made of a weakly conducting material with specific resistance ρ. This plate is placed inside a parallel plate capacitor, but not touching either capacitor plate. The capacitor is then charged to a potential V_0. Find the maximum current that flows across the conducting plate after the capacitor is short-circuited. The area of each plate of the capacitor and of the conducting plate is S; the distance d between the capacitor's plates is much smaller than the size of the plates and $h < d$. (V. Deryabkin)

P175

Lens and inclined mirror. A point source of light is placed at some distance under a convex lens (fig. 2). Where and how should a flat mirror be placed to produce a parallel beam of light coming out of the lens in the direction shown by the arrow? Draw the light rays. (V. Aleshkevich)

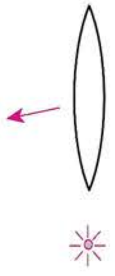


Figure 2

ANSWERS, HINTS & SOLUTIONS ON PAGE 57

How enlightene

"The rays emitted by the eye ... straight line."—Euclid

"A visual image is formed by ... visible objects and entering ...

"[E]ach small fraction of the ... along a straight line emergin... point. In this sense light bea... straight lines."—Christiaan H...

"As for light, there is no insta... along a curved path or benc... shadow."—Isaac Newton

REALLY, WHAT COULD BE more "evident" than light? As the displayed quotations show, however, sometimes the greatest minds in science "wandered around in the dark " when it came to light beams. The law of rectilinear propagation of light, known experimentally from ancient times, was like a litmus test for any theory of light, demanding a "straight answer," so to speak. The ideas of our distant ancestors, however naive they may seem now, paved the way to further developments in understanding the nature of light and its effects.

The concept of a "light beam" is a product of our everyday experience

Christiaan Huygens

and is very ancient indeed. It came from observing the heavenly bodies and shadows; from studying perspective, as artists and architects did; and from measuring plots of land. And even nowadays, aren't there problems that can't be solved without this understanding?

This installment of the Kaleidoscope offers you a chance to catch a glimpse of the law of rectilinear propagation concealed behind the play of light and shadow in many phenomena.

Questions and problems

1. A round pencil and a cylindrical fluorescent lamp are placed parallel to each other. Where is the region of complete shadow from the pencil?

2. Why aren't cylindrical fluorescent lamps used in film projectors?

3. How should a point source of light, a flat screen, and an object be positioned so that the shadow of the object on the screen is similar to the object's outline?

4. Under what circumstances will an opaque object cast a shadow without a penumbra (that is, a partial shadow)?

5. When will a body cast *only* a penumbra?

6. How can you tell if you're in the penumbra of some object?

7. Why is it harder to see the unevenness of a road's surface during the day than at night, when the road is illuminated by your car's headlights?

8. In a forest with tall trees and thick foliage, you can see circles of light on the ground on sunny days. What are these spots, and why are they round?

9. Is the shadow cast by a ball always circular?

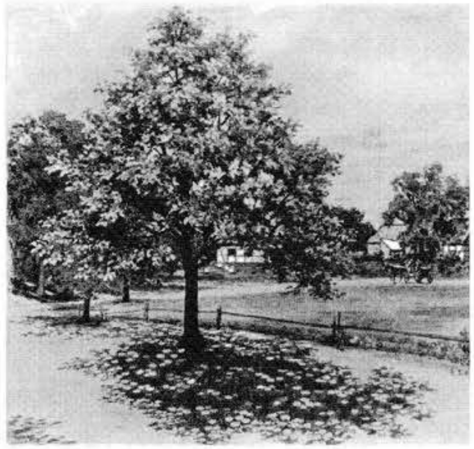
10. Why are the shadows formed by your legs on the ground clearly outlined, while the shadow of your head is blurry?

11. When a burning candle is illuminated by a bright electric lamp, shadows not only of the candle but of the flame itself are cast on a white screen. Can a source of light (that is, a flame) cast its own shadow?

12. Sometimes lighting fixtures are pointed at the ceiling rather than down. What's the point of this?

13. The shadows cast by the vertical posts of a soccer goal are longer in the morning and evening than at midday. Does the length of the shadow cast by the cross bar vary?

14. Using a pin, make a small hole in a piece of thick paper. Close one eye and hold the paper in front of the other at a distance of about 10 cm. Slowly raise a straight pin with its head up in such a way that it touches your eyelash. In the circular bright



ned are you?

eyes travel along a

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d by rays emitted by
ng the eye."—Alhazen

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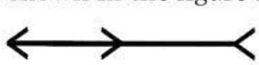
background of the hole in the paper, an image of the pin appears, with its head down and moving downward. Explain this phenomenon.

15. Can you cover up a star with a match? (Naturally you'll be making your observation with one eye closed.)

16. When we're sitting near a bonfire, why does it seem that the objects on the other side of it are swaying?

17. Light rays from the Sun are virtually parallel when they arrive at the Earth. Why do they look fan-shaped when they pass through clouds? (See the painting in Gallery Q.)

18. What can be said about the length of both parts of the horizontal line shown in the figure below?



Microexperiment

Place a screen at a small distance (up to 50 cm) from a burning candle. Put a pencil between them—first vertically, then horizontally. What do the shadows look like? Why?

It's interesting that . . .

. . . it was Euclid who proposed the model of the rectilinear light beam, which is the fundamental principle of geometric optics. In his *Optics* he studied such problems as shadow formation, how to create images with pinholes, and estimating the apparent size of objects and their distance from an observer.

. . . in the Middle Ages optics, perspective, and meteorology were parts of a single discipline. Confusion reigned in optical matters, and visual perception was not trusted. Eyesight was considered the most misleading of the senses.

. . . optics is, in its most literal sense, the science of vision. It was the camera obscura that laid to rest the notion of light-bearing rays emanating from the human eye. It turned optics into the science of light.

. . . the foundation of modern geometrical optics was laid by Johannes Kepler in 1604. At that time he wrote a manuscript entitled *Supplements to Vitellius* in which he explained the functioning of the eye and any other optical device by considering each point of a body as a source of diverging rays. The impetus behind the creation of this fundamental work came from the demands of astronomy.

. . . Huygens presented his famous principle to show that the wave theory of light could explain the known laws of optics, including the rectilinear propagation of light. However, it was Augustine Jean Fresnel who managed to do this, by making Huygens's principle more exact.

. . . the first optical (semaphore-based) telegraph connected Paris and Lille at the end of the 17th century. In the middle of the last century there were a number of optical telegraph lines in Russia, and the longest was the St. Petersburg–Warsaw line, which had 149 intermediate stations. It took only a few minutes for a signal to pass from one city to the other—alas, only in the daytime and when the visibility was good.

. . . the angle of vision of the human eye is much larger than one might think. Actions that occur at an angle of 90° on either side can be detected directly by our subconscious mind.

. . . only in the 20th century have experiments by physicists and physicians proved that the brain inverts the upside-down images sent to it by the eyes. To establish this, scientists wore special glasses that inverted what they were seeing. After a number of days the brain turned everything right-side up.

Isaac Newton

—Compiled by
Alexander Leonovich

ANSWERS, HINTS & SOLUTIONS
ON PAGE 61

PHYSICS CONTEST

Moving matter

"God . . . created matter with motion and rest in its parts, and . . . now conserves in the universe, by His ordinary operations, as much of motion and of rest as He originally created." —René Descartes, Principia Philosophiae *(1644)*

by Arthur Eisenkraft and Larry D. Kirkpatrick

HOW FAST CAN YOU THROW a baseball? How fast is a speeding bullet? Restricted to simple tools in the laboratory, both measurements can be completed with a clever approach and some elementary physics.

Although the baseball's speed would be difficult to measure directly, you can throw it into a box and measure the movement of the box across the table. Such an arrangement requires a box that will slow the ball down appreciably. The ball is thrown into the box. The box slides across the table and comes to a stop. Measuring the distance the box travels allows us to find the work done by friction. This then allows us to find the combined velocity of the box and ball when they began sliding. Since momentum is conserved in all collisions, we can back up one more step and determine the initial speed of the ball before it got embedded in the box.

Let's assume that a 0.5-kg ball is hurled into a 10-kg box that is resting on the table. If the box slides 2 m before coming to rest, we know that the work W done on the box is Fd, where F is the force of friction and d is the distance the box travels along the table. We can determine F independently by pulling the box at a constant speed along the table with a spring scale or by first determining the coefficient of friction μ and the normal force F_n. Let's assume that the force of friction is 20 N. The momentum of the ball before the collision with the box is equal to the combined momentum of the ball and box after the impact:

$$mv_0 = (m + M)V,$$

where m is the mass of the ball, M is the mass of the box, v_0 is the velocity of the ball, and V is the velocity of the ball and box. The kinetic energy of the ball and box is lost due to the work done by the frictional force:

$$\frac{1}{2}(m + M)V^2 = f \cdot d.$$

In our selected example, the ball's velocity is approximately 60 m/s, or 120 mph. (Does this agree with your calculated value?)

For bullets, it's preferable to suspend the box as a pendulum. This method was first used by Benjamin Robins in 1742. Professor A. P. French (MIT) shared with us a description by Robins in the *Philosophical Transactions of the Royal Society* for 1742–1743 (Vol. 42, pp. 437–56). In this journal, Robins described how he deliberately designed the pendulum to provide the first reliable way of measuring the speed of bullets.

With the bullet embedded in the pendulum, the pendulum rises and the initial kinetic energy of the pendulum becomes gravitational potential energy. It is then only necessary to measure the change in the vertical height of the pendulum to determine the speed of the bullet.

In both the baseball and bullet approaches, a common error is to assume that the initial kinetic energy of the object is equal to the kinetic energy after the collision. Since the object and the target stick together, we should recognize that this is an inelastic collision, in which kinetic energy is not conserved.

In some classroom demonstrations, a bullet is shot into the pendulum. In alternative demonstrations, an arrow is shot into the pendulum. The speed of the arrow in this latter experiment can be determined by a second, unrelated approach. The kinetic energy of the arrow is equal to the work performed by the bowstring, which can be determined by measuring the average force for

Art by Tomas Bunk


POETRY IN MOTION
TOM BUNK

every centimeter of pull. When two unrelated approaches provide an equivalent measure for the speed of the arrow, our confidence in the value and utility of the technique rises.

For this month's *Quantum* contest problem, we'll look at a fun Tarzan-and-Jane problem that we used on the first screening test for the US Physics Olympiad Team. Then we'll challenge you to find the speed of a bullet using a ballistic pendulum. Finally, we'll ask you to design an apparatus to be used in school physics laboratories.

A. Tarzan (mass = 80 kg) is standing on a small hill (height = 10 m) when he spots Jane (mass = 40 kg) in danger in the valley below. He grabs a vine and swings toward Jane. Grabbing her at the bottom of the arc, he hopes to make it to the 5-meter-high hill on the other side. Is Tarzan successful?

B. A bullet of mass 5 g hits a target of mass 5 kg that is free to swing as a pendulum. The cord holding the target is 5 m long. The target captures the bullet in a very short time and moves 30 cm horizontally. Calculate the velocity of the bullet.

C. You wish to design a laboratory apparatus that propels a ball by compressing a spring. The ball then travels a short distance into a target that is free to swing as a pendulum. What are the relative masses of the ball and target that produce the maximum movement of the target for a given initial compression of the spring?

D. A second variation of your laboratory apparatus has the ball stick (with a Velcro™-type fastener) to the bottom of a uniform rod. How is the final angle dependent on the mass of the ball and mass of the rod?

E. After using your propulsion device for the ballistic pendulum, you decide to have some real fun by inventing a target game. The propulsion device is mounted on a table and propels a marble horizontally in order to hit a target on the floor below. In your first attempt you compress the spring 1.0 cm and the marble falls 30 cm short of the target, which is 3.0 m horizontally from the edge of the table. How much should you compress the spring in your second attempt to get a perfect hit?

Please send your solutions to *Quantum*, 1840 Wilson Boulevard, Arlington VA 22201-3000 within a month of receipt of this issue. The best solutions will be noted in this space and their authors will receive special certificates from *Quantum*.

Gravitational redshift

In the November/December 1995 issue of *Quantum*, we asked our readers to combine the effects of the gravitational redshift and the Doppler redshift to determine the mass and radius of a white dwarf star. Essentially correct solutions were submitted by Christopher Rybak, a senior at The Prairie School in Racine, Wisconsin; Lori Sonderegger, a senior at Amity Regional High School in Woodbridge, Connecticut; and Art Hovey, her AP physics teacher last year.

A. We use the conservation of energy to find the shift in the frequency of a photon emitted at the surface of the white dwarf:

$$hf - \frac{GMm}{R} = hf' - \frac{GMm}{r},$$

where h is Planck's constant, G is the gravitational constant, M is the mass of the star, m is the effective mass of the photon, and f and f' are the photon's frequencies at radii R (the surface) and $r > R$, respectively. Since $\Delta f \ll f$, the effective mass of the photon does not change appreciably and we have used a single value for m ($m = hf/c^2$). Therefore,

$$h\Delta f = \frac{GMhf}{c^2}\left(\frac{1}{r} - \frac{1}{R}\right),$$

or

$$\frac{\Delta f}{f} = \frac{GM}{c^2}\left(\frac{1}{r} - \frac{1}{R}\right). \qquad (1)$$

In the limit $r \gg R$, we get the desired result:

$$\frac{\Delta f}{f} = -\frac{GM}{Rc^2}.$$

The minus sign indicates that the frequency decreases (that is, it is redshifted) as the photon leaves the white dwarf.

B. In the text of the problem, we showed that $f'/f \cong 1 - \beta$, or

$$\frac{\Delta f}{f} \cong -\beta. \qquad (2)$$

Equating equations (1) and (2) and setting $r = R + d$, we find that

$$\beta = \frac{GM}{c^2}\left(\frac{1}{R} - \frac{1}{R+d}\right) = \frac{GMd}{c^2R(R+d)}.$$

Inverting both sides of this equation, we obtain

$$\frac{1}{\beta} = \frac{Rc^2}{GM}\left(\frac{R}{d} + 1\right).$$

Therefore, if we plot $1/\beta$ versus $1/d$, we should obtain a straight line with a slope equal to $R^2c^2/GM \equiv \alpha R$ and a y-intercept equal to $Rc^2/GM = \alpha$.

Graphing the data gives a slope of $3.2 \cdot 10^{12}$ m and an intercept of $0.29 \cdot 10^5$. These values yield $R = 1.1 \cdot 10^8$ m and $M = 5.1 \cdot 10^{30}$ kg, which are in the right ballpark for a white dwarf.

MATH INVESTIGATIONS

The conductor of a set

And some new problems arising from it

by George Berzsenyi

SIXTEEN YEARS AGO I POSED the following problem in the short-lived Texas Mathematical Olympiad (TMO), which served as a forerunner for the American Invitational Mathematics Examination (AIME): *What is the largest positive integer that cannot be represented in the form $7m + 3n$, where m and n are positive integers?* At that time I was aware of the fact that, more generally, if a and b are relatively prime positive integers, then every integer greater than or equal to $(a-1)(b-1)$ can be expressed in the form $xa + yb$, where x and y are nonnegative integers. But I was not familiar with the more general setting of the problem, recently called to my attention by my friend Béla Bajnok of Gettysburg College. Béla suggested that I consult Herbert S. Wilf's delightful book *Generatingfunctionalology*, published by Academic Press. There, on page 88, Wilf defines the *conductor* of a set $S = \{a_1, a_2, \ldots, a_M\}$ of relatively prime positive integers to be the smallest integer N such that every integer $n \ge N$ can be expressed in the form

$$n = x_1a_1 + x_2a_2 + \ldots + x_Ma_M,$$

where $x_1, x_2, \ldots, x_M$ are nonnegative integers. According to Wilf, while the case $M = 2$ is completely solved, there are no general "formulas" if $M \ge 3$ and no good algorithms for calculating the conductor for $M \ge 4$. Béla and two of the participants of his undergraduate research program made some progress on sets of the form $\{a, a + d, a + 2d\}$ and $\{a, a + d, a + 3d\}$, where a and d are positive integers. I hereby challenge my readers to parallel and extend their accomplishments.

Another of Béla's students, Charlie Ross, was instrumental in resolving the following problem of mine, thereby enabling me to pose it in the USA Mathematical Talent Search: **Rearrange the integers 1, 2, 3, ..., 97 into a sequence $a_1, a_2, a_3, \ldots, a_{97}$ so that the absolute value of the difference of a_{i+1} and a_i is either 7 or 9 for each $i = 1, 2, 3, \ldots, 96$.** Charlie, who was a member of one of this year's winning teams in the Mathematical Contest in Modeling, also managed to solve the following more general problem, which I hereby share with my readers: *Suppose that a, b, and k are positive integers and that a and b are relatively prime. Let $N = k(a + b) + 1$. Then the integers from 1 to N can be arranged in a list starting with 1 and ending with N so that each pair of neighbors in the list differ by either a or b.* As in the Conductor Problem, it should be possible to extend the above result from the set $\{a, b\}$ to sets of three or more elements. While the two problem areas are not closely related, I came up with the second problem by thinking about the first, and wondering what would happen if one allowed for subtraction in the Conductor Problem. This is why I am presenting them together at this time.

Feedback

I was glad to learn that Stan Wagon of Macalester College found the topic of my May/June 1995 *Quantum* column interesting enough to pose the following as Problem 805 in his "Problem of the Week" program on the Internet: True or false: If n is a positive integer, then $n^5 + 5$ and $(n+1)^5 + 5$ are relatively prime. He received several insightful responses to this problem. In particular, Ilan Vardi's investigations are the most far-reaching. I will try to report on them in a future issue. Alternately, my readers may wish to contact Ilan directly via e-mail (vardi@macalstr.edu) to learn about his results.

I also heard again from Les Reid of Southwest Missouri State University, who succeeded in determining $G(5, k)$ and conjecturing the corresponding formulas for $G(4, k)$ and $G(7, k)$. Again, more specifics will follow in a later column.

I am also happy to report that the problem area introduced in my March/April *Quantum* column was further popularized by Donald T. Piele in his "Mathematica Pearls" column in the Fall 1995 issue of *Mathematica in Education and Research*. I will report on the findings of his readers as soon as I learn about them.

Finally, it seems that the topic of my very first *Quantum* column, which appeared in the May 1990 issue, is still generating interest in the mathematical community. My friend and former colleague Brian Winkel (of West Point Military Academy) recently sent me a copy of Volume 28 Number 2 of *Mathematical Spectrum*, which featured an article on "The Roseberry Conjecture" by Filip Sajdak.

FOLLOW-UP

Queens on a cylinder

Further adventures on nonstandard chessboards

by Alexey Tolpygo

THE FAMOUS PROBLEM OF arranging eight chess queens on a chessboard so that none of them attacks another (two of its 96 solutions are shown in figure 1) generated heaps of extensions and modifications. The most obvious variation is to change the dimensions of the board. Or we can replace queens with other chess pieces. Sometimes the board itself is reshaped: we can "glue" a pair of its opposite sides together to make it into a cylinder (fig. 2a and 2b); and if the other pair of edges is also glued, we get a torus (fig. 2c). Some of these extensions have already appeared in *Quantum*: the Queen Problem on the torus was considered in "Torangles and Torboards" (March/April 1994), and the other chess royalty was the main character in "Signals, Graphs, and Kings on a Torus" and challenge M156 in the November/December 1995 issue.

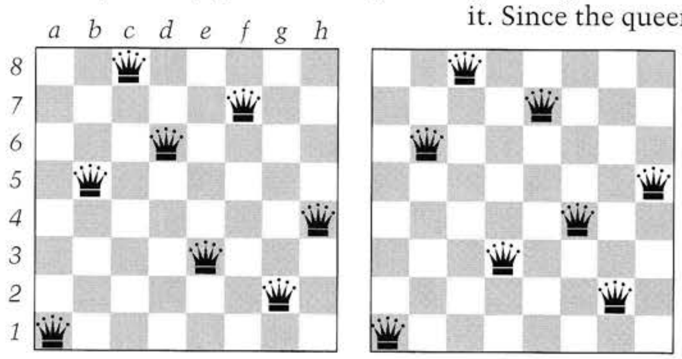


Figure 1

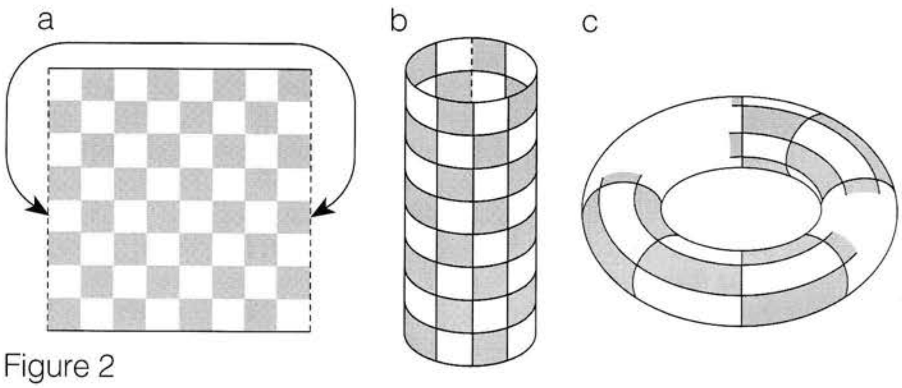


Figure 2

This article sheds new light on the cylindrical Queen Problem and gives a general method for tackling it. Since the queen moves like both the rook and the bishop, let's start with these pieces.

Rook, bishop ... monk?

For simplicity, we'll draw the cylindrical board just like the ordinary one—we'll simply mark the edges that are glued together (fig. 2a). So on such a board a rook attacks the same 14 squares as on the ordinary chessboard (fig. 3)—at least, as long as there are no other pieces. But if we put the rook on, say, b3 and a pawn on c3, the f3 square becomes inaccessible to the rook on the ordinary board (it is blocked by the pawn), while on the cylinder the rook can get there via a3 and h3. However, blocking won't be an issue for us, so we can be sure that on the cylinder and torus the rook is just as "powerful" as on the plane.

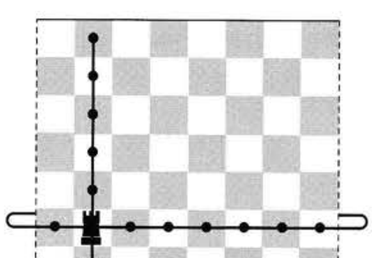


Figure 3

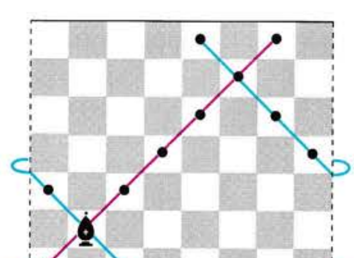


Figure 4

Things are quite different with the bishop. You know that it attacks 7 to 13 squares on an ordinary board, depending on its position. But on the 8×8 board glued into a cylinder, the bishop always attacks 13 squares (fig. 4). It would attack 14 squares, like the rook, but on the cylinder the two diagonals it attacks always have *two* squares in common (one of which is the square it stands on).

For boards of a different size this may not be the case. For instance, on the 7×7 board the two diagonals have only one intersection and the bishop attacks 12 squares, like the rook (fig. 5). Note that the bishop, on

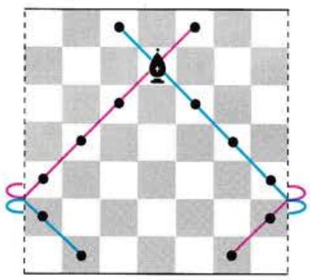


Figure 5

this board, is not restricted to a single color. But the situation gets much more interesting if the board is a rectangle with unequal sides.

Take, for example, the 6×8 board (fig. 6). A bishop on square a1 attacks only five squares on its diagonal. Glue together the vertical edges: now *two* diagonals, of five squares each, become accessible. But if, instead of the vertical edges, we glue together the horizontal rows 1 and 6, the two available diagonals will have seven squares each (not counting the bishop's location). In both cases the diagonals intersect, so the actual numbers of attacked squares are 9 and 12, respectively. Check that these numbers don't depend on the bishop's position.

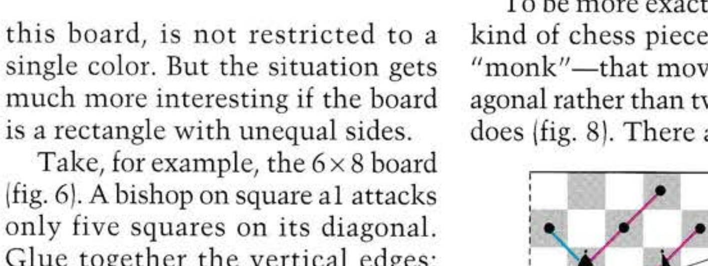


Figure 6

Problem 1. A rectangular $m \times n$ board is glued (a) along its vertical edges, (b) horizontally. How many squares on these cylindrical boards will be attacked by a bishop? For what numbers m and n do the two attacked diagonals intersect? Not intersect? Intersect for only one of the two methods of gluing?

But it's even more interesting to make a torus out of the board. As seen in figure 7, on the $n \times (n+1)$ board, the bishop attacks all squares at once (in the figure $n = 4$). Again,

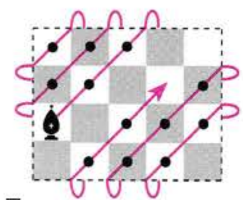


Figure 7

the bishop can access squares of both colors on this board. Not only that, it doesn't have to move along both diagonals—one of them suffices!

To be more exact, consider a new kind of chess piece—we'll call it a "monk"—that moves along one diagonal rather than two, as the bishop does (fig. 8). There are two kinds of

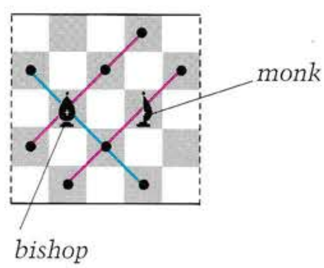


Figure 8

monks—"right-handed" and "left-handed," for want of better terms. Since for our problem it makes no difference, we'll assume that our monks are right-handed. Figure 7 also shows that a monk controls the entire $n \times (n+1)$ toroidal board from one square. This is true as well for a bishop.

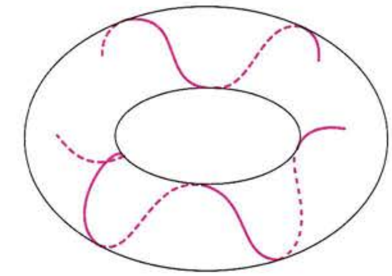


Figure 9

Figure 9 shows how a monk would move on an actual torus. This type of path is important in modern geometry—it's called the *winding* of the torus.

However, one piece attacking the entire board is, perhaps, too much of a good thing. So let's come back to the square board, but not necessarily measuring 8×8—let's examine the $n \times n$ board. It will suffice to make a cylinder out of it—figure 10 shows that the squares attacked by a monk (or a bishop) are the same for the cylinder and the torus.

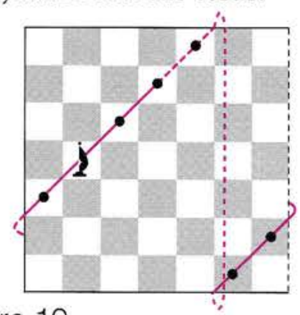


Figure 10

So, how many bishops can be arranged on this board without attacking one another? (This "professional courtesy" will be a continuing requirement that I won't repeat every time.) Clearly, this number is not greater than n, because there are only n diagonals of the same direction. And n bishops can always be arranged: simply put them in any horizontal row (fig. 11).

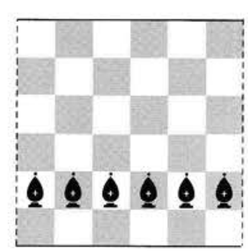


Figure 11

Problem 2. How many bishops can be arranged on an ordinary 8×8 board (square, not cylindrical)?

Problem 3. An 8×12 board is glued into a (a) torus, (b) cylinder of height 8, (c) cylinder of height 12. How many (right-handed) monks can be arranged on this board in each case? And what about bishops?

It's also easy to arrange n rooks on the $n \times n$ board, whether it's a square, cylinder, or torus. They can always be stood along a diagonal. (It's interesting that the solution for bishops is given by a rook move and for rooks by a bishop move.)

Now let's return to the queens. We saw that it was difficult even to arrange monks on a rectangular nonsquare board, to say nothing of bishops or queens, so we'll confine ourselves to a square $n \times n$ board. On the other hand, it follows from the considerations above that a queen can be replaced with a "nun," which moves like both a rook and monk (fig. 12).

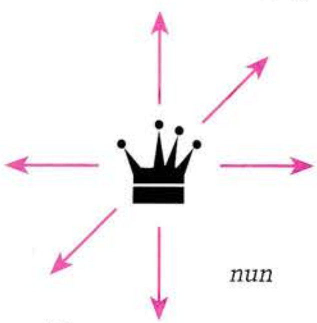


Figure 12

Nuns and vacuum tubes

Is it possible to arrange n nuns on the $n \times n$ board? Before we answer this question, consider the following rather old problem about a device that was replaced long ago by the transistor.

Plug problem. A vacuum tube has a round plug with n pins and its receptacle has n holes, so there are n ways for the tube to be plugged in. The holes are numbered $1, 2, \ldots, n$. Can we number the pins such that no matter how the tube is plugged in there is exactly one pin that fits into the hole with the same number?

This problem was proposed at Moscow Mathematical Olympiad for the cases $n = 7$ and $n = 8$. In the first case the answer is yes, while in the second case it's no. If $n = 7$ and the holes are numbered in the natural order $1, 2, \ldots, 7$, we can simply number the pins in the reverse order $7, 6, \ldots, 1$. Verify that this numbering solves the problem. There are other solutions as well—for instance, the order of pin numbers can be 1, 3, 5, 7, 2, 4, 6. It's clear what these numberings must look like for $n = 8$, but neither of them gives the desired result: for some turns of the tube there will be several numbers that coincide; for the other turns all corresponding numbers will be different.

To prove that for $n = 8$ other numberings are no good either, suppose the required numbering exists, plug in the tube, and denote by a_i the number of the pin that goes in the ith hole. The set of numbers $\{a_1, a_2, \ldots, a_8\}$ coincides with $\{1, 2, \ldots, 8\}$ (the braces in this notation show that we ignore the order of the elements), and $a_i = i$ for exactly one i. Turn the tube by one hole (in the direction of increasing hole numbers). Then the pin a_i will match the hole $i + 1$. (To be more exact, this is true for $i = 1, 2, \ldots, 7$; the pin a_8 matches the hole numbered $1 = 8 + 1 \pmod 8$. Similar problems will arise below, so let's agree from here on to consider all numbers modulo 8.) After the tube is turned there will be exactly one match again—that is, $a_i = i + 1 \pmod 8$ for one and only one i. The same will be true for other turns: for exactly one i we'll have $a_i = i + 2$; for another i, $a_i = i + 3$; and so on.

Add up these eight equations. After rearranging terms, we get

$$\begin{aligned} &a_1 + a_2 + \ldots + a_8 \\ &= (1+2+\ldots+8)+(1+2+\ldots+8) \pmod 8 \\ &= 0 \pmod 8. \end{aligned}$$

On the other hand, since $\{a_1, \ldots, a_8\} = \{1, \ldots, 8\}$, $a_1 + a_2 + \ldots + a_8 = 1 + 2 + \ldots + 8 = 36 \neq 0 \pmod 8$. This contradiction completes the proof.

Exercise. Try to figure out why the same argument doesn't lead to a contradiction for $n = 7$.

What does this problem have to do with our problem about nuns? It turns out that the two problems are very closely related.

Assertion. The "peaceful" arrangement of n nuns on the cylindrical $n \times n$ board is possible if and only if the plug problem is solvable for n pins.

This assertion isn't all that interesting in and of itself (it simply asserts that the answers to two different problems happen to coincide). But it's possible to assert much more. Namely, suppose the (right-handed) nuns are placed on the squares $(1, i_1), (2, i_2), \ldots, (n, i_n)$. Then they do not attack one another if and only if $(i_1, \ldots, i_n)$ is a solution to the plug problem. Check this yourself.

Peaceful nuns

From the discussion above it follows directly that eight nuns cannot be arranged on the 8×8 cylindrical board, let alone eight queens. On the other hand, an arrangement of nuns is always possible on a board with odd-numbered dimensions. It's easiest to set them along the "left" diagonal, as shown in figure 13, where you can also see the squares attacked diagonally by the first and third nuns.

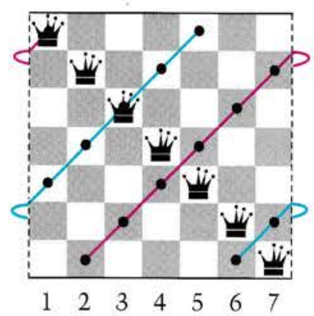


Figure 13

This solution corresponds to the reverse numbering of the pins in the plug problem. The second ("odd-before-even") numbering of the pins yields the arrangement of nuns by the knight's move (fig. 14). There are many other solutions.

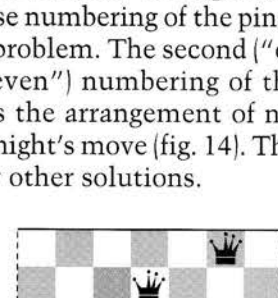


Figure 14

Art by Leonid Tishkov

In "fantasy chess" there is a piece called the "night knight." It moves like the usual knight, but by any number of ordinary moves at one time in the same diagonal direction. Strictly speaking, the arrangement of nuns in figure 14 is by the night knight's move, rather than the ordinary knight's.

One other thing: if the board is considered a torus, the arrangement in figure 14 defines one of its windings.

So, our discussion has led us to the following conclusion: on a cylindrical (or toroidal) $n \times n$ board, a peaceful arrangement of nuns is possible for odd n and impossible for even n.

As for queens, it's not possible to arrange n queens on these boards for even n. What about odd n? Our first solution (along the "left" diagonal) clearly doesn't work. As for the second solution (by the "night knight" move), you can verify on your own that it works if n is not divisible by 3. (See, for instance, figure 14—you can see that the nuns, even if endowed with full royal power, won't "quarrel."[1])

For $n = 3k$ the "night knight" arrangement is no good. Not only that, no "torus winding" solution will work here either. (By that I mean arrangements of the form $(1, 1)$, $(1 + k, 1 + l)$, $(1 + 2k, 1 + 2l)$, $\ldots$, $(1 + (n-1)k, 1 + (n-1)l)$, where the coordinates are taken modulo n. That is, we place the queens one by one, the first in the square $(1, 1)$, the second shifted k squares to the right and l squares up relative to the first, and so on; whenever a queen leaves the board, we shift it back n squares down, or left, or both—this corresponds to pasting together the opposite edges of the board.)

Fifteen queens? Too many!

Are solutions not based on windings possible (for $n = 3k$)? Clearly there are no solutions for $n = 3$. Let's prove that the case $n = 15$ is also unsolvable. Our proof below will make use of "labeled sets," or "multisets." Before we begin the proof, let's clarify what we mean by this term.

Labeled sets can be defined in several ways. For us, the simplest definition, connected with multiplicity, will suffice. A (finite) *labeled set M* consists of n arbitrary different elements $x_1, \ldots, x_n$; but unlike ordinary sets, each of the elements can enter M a number of times rather than just once. If k_i is this number of repetitions (that is, the multiplicity) of x_i, we say that the total number of elements in M is $N = k_1 + k_2 + \ldots + k_n$ rather than n. Equivalently, a labeled set can be described as containing the elements $y_1, \ldots, y_N$, once each, but some of these elements may be the same.

This notion is handy when we deal with, say, the roots of algebraic equations. For example, it's natural to think of the equation $x^3 - 3x + 2 = (x-1)^2(x+2) = 0$ as having three roots (1, 1, and -2) rather than two (1 and -2). Under this definition we can say that the set of roots of the equation $f(x)g(x) = 0$ is the union of the sets of the roots of $f(x) = 0$ and $g(x) = 0$, and no caveats about multiplicities are needed. In general, the number of elements in the union of any two labeled sets *always* equals the sum of the elements in both sets. With ordinary sets this is not necessarily true, because we must take their common elements into account.

Now, armed with this notion, we can proceed to the proof. Suppose we managed to arrange n queens on the cylindrical (or toroidal) $n \times n$ board—one in each horizontal row and one in each vertical file. Denote by i_1 the number of the row in which the queen stands in the first file, i_2 the number of the row with the queen in the second file, and so on. We see that

$$\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}. \quad (1)$$

You can verify that the queens will not attack one another diagonally if and only if the following two conditions hold:

$$\{i_1 - 1, i_2 - 2, \ldots, i_n - n\} = \{1, 2, \ldots, n\} \pmod{n} \quad (2)$$

and

$$\{i_1 + 1, i_2 + 2, \ldots, i_n + n\} = \{1, 2, \ldots, n\} \pmod{n}. \quad (3)$$

Notice that the first of these conditions is equivalent to the "nonaggression pact" between right-handed nuns in the same arrangement, and condition (3) plays the same role for left-handed nuns.

Now let's "add" the corresponding sides of equations (1) through (3) as labeled sets:

$$\{i_1, i_1 - 1, i_1 + 1, i_2, i_2 - 2, i_2 + 2, \ldots, i_n, i_n - n, i_n + n\} = 3\{1, 2, \ldots, n\} \pmod{n}.$$

Recall that we are investigating the case $n = 15$. Now the decisive step follows: we pass from an equation modulo 15 to an equation modulo 3 (this is possible because two numbers with the same remainders when divided by 15 will certainly yield the same remainders when divided by 3). Since $\{1, 2, \ldots, 15\} = 5\{1, 2, 3\} \pmod{3}$, we can write

$$\{i_1, i_1 - 1, i_1 + 1, \ldots, i_{15}, i_{15} - 15, i_{15} + 15\} = 15\{1, 2, 3\} \pmod{3}. \quad (4)$$

On the other hand, it's clear that $\{i_1, i_1 - 1, i_1 + 1\} = \{1, 2, 3\} \pmod{3}$. Similarly, $\{i_k, i_k - k, i_k + k\} = \{1, 2, 3\} \pmod{3}$ for all k except multiples of 3—that is, for $k = 1, 2, 4, 5, 7, 8, 10, 11, 13, 14$. Subtracting the corresponding equations from equation (4), we get

$$\{i_3, i_3 - 3, i_3 + 3, i_6, \ldots, i_{15} + 15\} = 5\{1, 2, 3\} \pmod{3}. \quad (5)$$

But $i = i - 3 = i + 3 \pmod{3}$, so the left side of this equation can be written as $3\{i_3, i_6, i_9, i_{12}, i_{15}\}$. This means that the multiplicity of any number on the left side of equation (5) is divisible by 3. But the multiplicity of any number on the right side of equation (5) is 5, which is not divisible by 3. This contradiction proves that the Queen Problem on the cylinder for $n = 15$ is unsolvable.

Problem 4. Prove that it is impossible to arrange 21 queens on a 21×21 cylindrical chessboard.

Problem 5. For what numbers m and n does a night knight (which moves in one direction) control all the squares of an $m \times n$ toroidal chessboard?

[1]See also the solution to problem 3 in "Torangles and Torboards" (March/April 1994, p. 58.—*Ed.*

LOOKING BACK

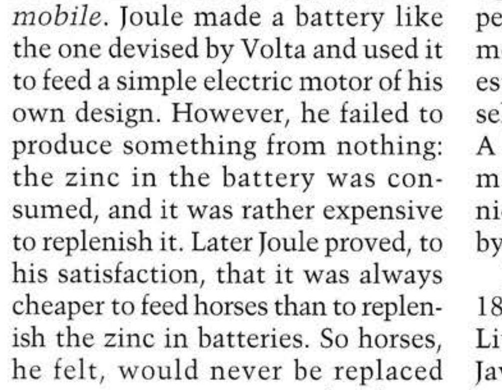
A brewer and two doctors

How Joule, von Mayer, and von Helmholtz worked their way to the law of conservation of energy

by Gennady Myakishev

IT'S BOTH INSTRUCTIVE AND interesting to learn how the discoverers of the law of conservation of energy made this fundamental step in physics.

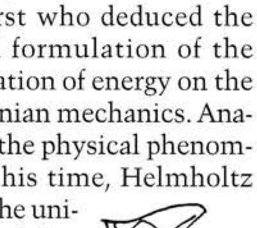
James Prescott Joule (1818–1889) was the owner of a brewery. His interest in this problem was first aroused when he saw the newly invented electric motor. Joule was quite a practical man, and no wonder he was tempted by the idea of creating an eternal and inexhaustible source of energy—a *perpetuum mobile*. Joule made a battery like the one devised by Volta and used it to feed a simple electric motor of his own design. However, he failed to produce something from nothing: the zinc in the battery was consumed, and it was rather expensive to replenish it. Later Joule proved, to his satisfaction, that it was always cheaper to feed horses than to replenish the zinc in batteries. So horses, he felt, would never be replaced completely in industry by electric motors.



This experience impelled him to investigate the relationships between heat and all other kinds of energy, and he decided to find out whether an exact quantitative relationship exists between heat and mechanical energy (G. Lipson, *Great Experiments in Physics*). The amount of energy transferred turned out to be equivalent to the mechanical work performed, and thus the value of the mechanical equivalent of heat was established: 4.187 J/cal. Joule himself obtained the value of 4.155 J/cal. A hundred years of experience in modernizing measurement techniques has improved Joule's result by less than 1%.

Julius Robert von Mayer (1814–1878) was a physician by education. Living in the tropics (the island of Java) as a ship's doctor, he treated sailors during epidemics of pulmonary disease according to the accepted practice of the time—by bleeding the patient from a vein in the arm. He noticed that the blood from these veins was much lighter than it was for sailors on ships in the northern climes. It could be even taken for arterial blood! There was an evident relationship between the temperature difference between the human body and its surroundings, on the one hand, and the degree of oxidation of the blood, on the other. Mayer came to the conclusion that there is a relationship in a living body between food consumption and heat production.

Herman Ludwig Ferdinand von Helmholtz (1821–1894) graduated from a military medical institute. He was the first who deduced the mathematical formulation of the law of conservation of energy on the basis of Newtonian mechanics. Analyzing most of the physical phenomena known in his time, Helmholtz demonstrated the universal nature of the law of conservation of energy. It's worth noting that all processes occurring in living organisms also obey this principle.



Isn't it curious that the originators of the law of conservation of energy law were not physicists by training?

Art by Sergey Ivanov

AT THE BLACKBOARD I

A pivotal approach

Applying rotation in problem solving

by Boris Pritsker

ONE KIND OF TRANSFORMATION that preserves distance is rotation. By the definition of rotation, the entire plane is turned about some point through a given angle clockwise or counterclockwise. Thus the size and shape of any figure are kept invariant, but its points all move along arcs of concentric circles. The center (which may or may not "belong" to the figure being rotated) is the only point that remains fixed. Because rotation preserves distance, it takes any figure into a congruent figure. The angles between corresponding lines are equal to each other and to the given angle of rotation. These very important properties of rotation can be widely used in problem solving. They simplify the solutions to many difficult problems, and make the solutions elegant and beautiful. In this article we'll look at some methods and techniques for problem solving that make use of rotation and its properties.

For example, let's perform a simple rotation of a given segment AB about a given point O (the center) through a given angle θ (fig. 1). AB is transformed into $A'B'$. The angle between the lines containing these segments equals θ, and by the properties of rotation $AB = A'B'$.

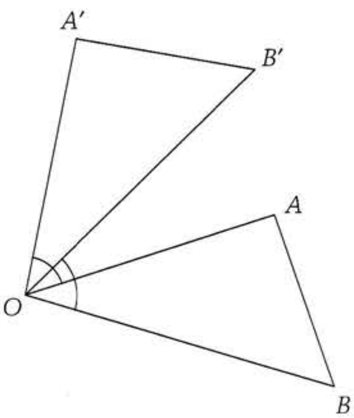


Figure 1

All the constructions that follow are based on these results, and all the proofs are based on the same properties of rotation.

Problem 1. Two squares $AMKB$ and $ACPT$ are drawn externally on the sides AB and AC of a given triangle ABC. Prove that the distance between points M and T is equal to twice the length of the median of triangle ABC drawn to the side BC.

Solution. Let's denote by O the point of intersection of the diagonals of the square $ACPT$ and consider the 90° rotation of the square about its center O, taking T into A and A into C (it's a clockwise rotation). Before going on, we have to make one auxiliary construction: draw a line through B parallel to AC and draw a line through C parallel to AB (fig. 2). Let A' be the point of their intersection and D the midpoint of BC. Thus we obtain a parallelogram $ABA'C$ whose center is the point D. Let's show that the rotation performed above takes the segment MT into AA'—this will prove the statement of the problem.

Since $MA = AB$ and $AB = A'C$, then $MA = A'C$. We know that $MA \perp AB$ and $AB \parallel A'C$; therefore, $MA \perp A'C$. This means that the segment MA is rotated into segment $A'C$, so point M is rotated into point A'. We have already noted that point T is rotated into point A. Thus the segment MT is rotated into segment AA' and therefore $MT = AA'$ by the properties of rotation. The segment AD is the median of triangle ABC and

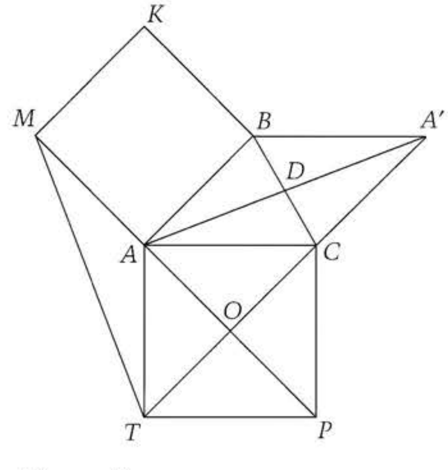


Figure 2

$AD = \frac{1}{2}AA'$ (by the property of the parallelogram), so $MT = 2AD$, and the proof is complete.

Problem 2. The points M and N are chosen on the sides BC and CD, respectively, of a given square $ABCD$ such that $BM : MC = 3:1$ and $CN:ND = 3:1$. Prove that $AM \perp BN$.

Proof. Let O be the center of the given square. A 90° rotation of the square about O takes point B into point C, C into D (fig. 3). Since our rotation takes segment BC into segment CD, and since $BM : MC = CN:ND$, then point M must be taken into a point that divides the segment CD in the same ratio as the point M divides BC—that is, M rotates into N. According to the properties of rotations, the angle between corresponding rays must be 90°; therefore, $AM \perp BN$, which was to be proved.

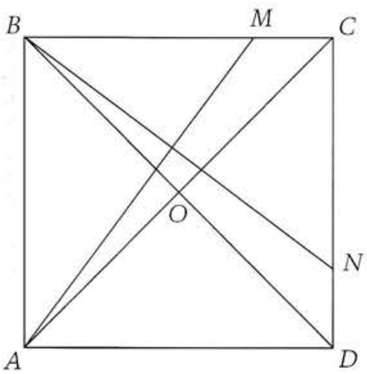


Figure 3

Problem 3. Construct an equilateral triangle such that its three vertices are located on three given concentric circles.

Solution. Choose any point A on the middle circle and rotate the smallest circle about A through an angle of 60° (fig. 4). The circle is transformed into a circle with center O' with the same radius. Let's denote the points of its intersection with the biggest circle by B and B'. (If there is only one point of intersection, the problem has a unique solution; if these circles do not intersect, the problem has no solution.) The last step is to draw the circle with center B and radius AB and the circle with center B' and radius AB'. The points of intersection of these circles with the smallest circle give us the third vertices of the triangles ABC and $AB'C'$—the desired equilateral triangles with vertices located on the three given concentric circles. It would be good practice to

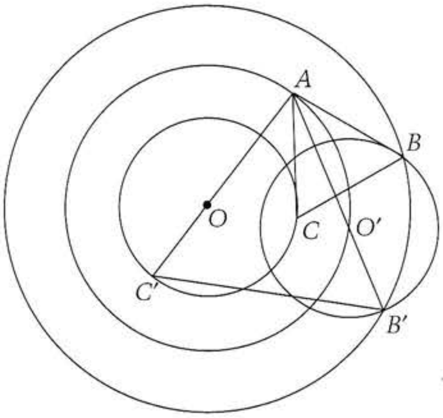


Figure 4

Art by Sergey Ivanov

prove that these triangles satisfy the conditions of the problem.

The rotation that has the most interesting properties of all is the one that transforms every line into a line parallel to it. This is the *half-turn*, or rotation through 180°, which transforms each ray into an oppositely directed ray. Clearly a half-turn is completely determined by its center. Another name for the half-turn is *central symmetry*. A point and its image are called *symmetric* with respect to the center of rotation.

Problem 4. Given an angle and a point M inside it, find a segment with its midpoint at M and its endpoints on the sides of the angle.

Solution. Let O be the vertex of the given angle. The half-turn about M takes O into point O' such that $OM = O'M$ and all three points lie on one line (fig. 5). Let us draw lines through point O' parallel to the given lines. Denote the points of intersection of these lines and given lines by A and B. Then $OAO'B$ is a parallelogram by construction. Since M is the midpoint of OO' (by the property of a half-turn), it must be the midpoint of the second diagonal of parallelogram $OAO'B$ as well. Therefore, AB is a solution. The reader can verify that there are no more solutions.

Problem 5. The midpoint M of the side AB of trapezoid $ABCD$ ($BC \parallel AD$) is connected to points C and D. Prove that the area of the triangle MCD is equal to one half the area of trapezoid $ABCD$.

Solution. To solve the problem, we first find the point K that is symmetric to point C with respect to point M (fig. 6). Now we'll show that point K necessarily lies on the extension of the side AD of the given trapezoid. Since $BM = MA$ (by the conditions of the problem), $MC = MK$,

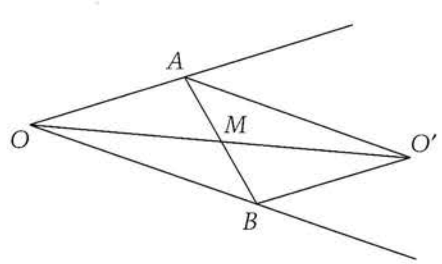


Figure 5

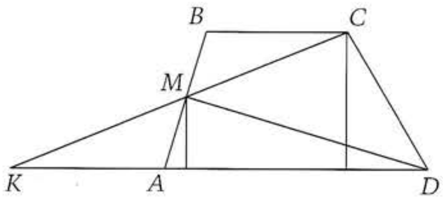


Figure 6

and $\angle BMC = \angle AMK$ (by construction), triangles BMC and AMK are congruent. Therefore, $\angle MKA = \angle MCB$. But these angles are alternate interior angles, so KA must be parallel to BC, which means that K belongs to the extension of AD. It's clear that triangles AMK and BMC have equal areas. So the area of the trapezoid $ABCD$ is equal to the area of the triangle KCD. To complete the proof, it's enough to show that the area of KCD is equal to twice the area of CMD. But this is easy—DM is a median in triangle KCD, and a median always divides a triangle into two triangles of equal area (the proof of this standard fact is left to the reader).

Problem 6. Given three points O, M, and N, construct a square such that O is its center and points M and N are located on opposite sides of the square (or their extensions).

Solution. Construct point M' symmetric to point M and N' symmetric to point N with the respect to point O (fig. 7). Then lines MN' and $M'N$ are parallel by the properties of central symmetry. Next draw the perpendicular FF' through point O to lines MN' and $M'N$ (point F lies on MN', point F' lies on $M'N$), and the line l passing through O parallel to MN' and $M'N$.

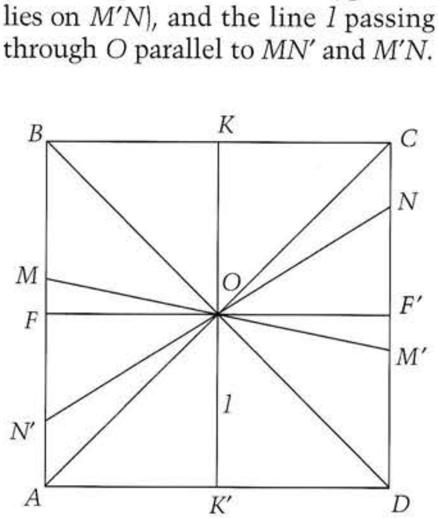


Figure 7

Locate points K and K' on line l such that $OK = OK' = OF = OF'$. To complete the construction draw lines through points K and K' parallel to FF'. Denote the points of intersection of these lines with MN' and $M'N$ as B, A, D, and C, respectively. $ABCD$ is a rectangle, and $AB = BC = CD = DA$ (by construction). Therefore, $ABCD$ is the desired square.

Sometimes M and N are on the sides of the square, and sometimes on their extensions. The reader can investigate the different cases. What happens if M and N are reflections of each other in point O?

In conclusion, I'd like to offer several problems for you to solve on your own so that you become more familiar with these properties of rotation.

Problem 7. Point P is chosen outside square $ABCD$, and segments PA, PB, PC, PD are drawn. Prove that the perpendicluar lines drawn from point A to BP, point B to CP, point C to DP, and point D to AP all have a common point. Is the statement true if point P is chosen inside the square?

Problem 8. Point M is an arbitrary point on the side AB of square $ABCD$. Construct the square inscribed in the given square such that M is one of its vertices.

Problem 9. Construct an equilateral triangle such that its three vertices are located on three given parallel lines.

Problem 10. Construct a square such that three of its vertices are located on three given parallel lines.

Problem 11. Given two circles and a point P that does not lie on any of them, draw a line through P such that the two given circles cut off a segment whose midpoint is P.

Problem 12 (math challenge M109 in the March/April 1994 issue of *Quantum*). From the vertex A of a square $ABCD$ two rays are drawn inside the square. From vertices B and D, perpendiculars are dropped to the two rays: BK and DM are dropped to one of them, and BL and DN are dropped to the other. Prove that the segments KL and MN are congruent and perpendicular. (D. Nyamsuren). ◘

Boris Pritsker *taught mathematics in Kiev, Ukraine. He currently resides in New York City.*

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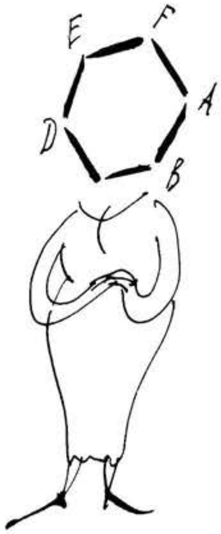


The mathematician, the physicist, and the engineer

A trove of science jokes on the Internet

AN INTREPID DUTCHMAN by the name of Joachim Verhagen has done us all a great service. He has collected hundreds of science jokes from dozens of newsgroups on the Internet, and from individual e-mail correspondents, and has made them available on the World Wide Web. We offer here a small sample of his growing anthology, culled from the section in which the characteristic foibles and small but satisfying victories of various scientific professions are featured.

Verhagen has detected an intriguing mathematical relationship among the various disciplines. Noting that he visits more biology newsgroups than others in his joke hunting, he writes: "All those biology groups do not mean that I have a lot of biology humor. As a matter of fact, the more exact a science the more humor I find. The relation is close to: mathematics : physics : chemistry : biology = 8 : 4 : 2 : 1. It seems that more exact sciences are more humorous."

The entire collection can be found at http://www.fys.ruu.nl/~nienhuys/scijokes. (Be forewarned that some sensibilities may be offended by a few of the anecdotes, though the overwhelming bulk of the material there is quite innocuous.)

Red rubber ball

A mathematician, a physicist, and an engineer were all given a red rubber ball and told to find the volume. The mathematician carefully measured the diameter and evaluated a triple integral. The physicist filled a beaker with water, put the ball in the water, and measured the total displacement. The engineer looked up the model and serial numbers in his red-rubber-ball table. (Anonymous addendum: "If it were my company: the engineer tries to look up the model and serial numbers, can't find them, and tells his manager, 'It's just not going to work.'")

Desert breakdown

A computer science student, an engineering student, and a meteorology student are going through the desert in a jeep. Suddenly the jeep stops and they're left sitting there, wondering what happened.

The engineering student pipes up, "It must be the fan belt that's broken down. The engine has overheated, so we'll just have to wait until it cools down, bodge the fan belt,[1] and we'll be fine."

The meteorology student replies, "Naw, it's not that. It's just the ambient heat in this place. It's not allowing the engine to breathe correctly. We just have to wait till night."

The computer science student thinks about this for a minute, then says, "Yeah, you may be right, but I've got an idea. What say we all get out, then get back in again?"

Murky research

Three persons with degrees in mathematics, physics, and biology are locked up in dark rooms for research purposes.

A week later the researchers open the first door. The biologist steps out and reports: "Well, I sat around until I started to get bored, then I searched the room and found a can, which I smashed on the floor. There was food in it, which I ate when I got hungry. That's it."

Then they free the physicist, who says: "I walked along the walls to get an image of the room's geometry,

[1]This joke was posted from the United Kingdom, where "bodge" is apparently a term of art.—*Ed.*

then I searched it. There was a metal cylinder at five feet into the room and two feet to the left of the door. If felt like a can, and I threw it at the left wall at the right angle and velocity for it to crack open."

Finally, the researchers open the third door and hear a faint voice out of the darkness: "Let *C* be an open can . . ."

Defining moment

What is "pi"?

Mathematician: "Pi is the number expressing the relationship between the circumference of a circle and its diameter."

Physicist: "Pi is 3.1415927 plus or minus 0.000000005."

Engineer: "Pi is about 3."

The doctor knows

A mathematician, a physicist, and a physician were asked: "What is 2 times 2?"

The physicist takes out a notebook and starts scribbling. After three days of the most complex calculations, he finds, by using the Earth's radius and the gravitational constant, that "it lies somewhere between pi and two times the square root of 3."

The mathematician comes back after a week with dark rings under his eyes and declares: "Friends and colleagues, there is a solution."

The physician says simply: "Four."

The others look at him sharply. "Oh, well," they say, "you *memorized* it."

Einstein in Elysium

Einstein dies and goes to heaven, only to be informed that his room is not yet ready. "I hope you will not mind waiting in a dormitory," he is told by the doorman (we'll call him Pete). "We are very sorry, but it's the best we can do, and you will have to share the room with others."

Einstein says this is no problem at all, there's no need to make such a big fuss. So Pete leads him to the dorm.

They enter, and Albert is introduced to everyone present.

"Here is your first roommate," says Pete. "He has an IQ of 180!"

"Why, that's wonderful!" says Albert. "We can discuss mathematics."

"And here is your second roommate. His IQ is 150."

"That's wonderful!" says Albert. "We can discuss physics."

"And here is your third roommate. His IQ is 100."

"Wonderful!" says Albert. "We can discuss the latest plays at the theater."

Just then another roommate reaches out to shake Albert's hand.

"I'm your last roommate," he says. "I'm sorry, but my IQ is only 80."

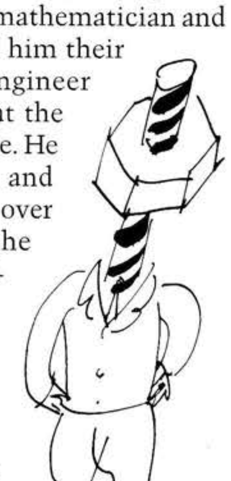
Albert smiles back at him and says, "So, where do you think interest rates are headed?"

Take your pick

A mathematician and a physicist are trying to measure the height of a flagpole using a long tape measure. The mathematician takes the tape measure, walks up to the flagpole, and begins to shinny up the pole. A short way up, he slips and falls down.

The physicist notices a ladder lying nearby in the bushes. He leans the ladder against the pole, but it reaches only halfway up. He climbs the ladder and tries to shinny up from there, but he also slips and falls.

While they sit near the pole, scratching their heads, an engineer walks by, so the mathematician and the physicist tell him their problem. The engineer notices a crank at the base of the flagpole. He turns the crank, and the flagpole tilts over until it lies on the ground. The engineer stretches out the tape measure, cranks the pole back up, and tells the mathematician and the physicist: "It's 15 meters."



As the engineer walks off into the distance, the mathematician looks at the physicist and says: "Isn't that just like an engineer? You ask him for the height, and he gives you the length."

But some people believe the story goes like this:

A team of engineers were required to measure the height of a flagpole. They had only a tape measure, and they were getting quite frustrated trying to keep the tape against the pole. It kept falling down, and so on.

A mathematician comes along, learns of their problem, and proceeds to remove the pole from the ground and measure it easily.

When he leaves, one engineer says to the other: "Just like a mathematician! We need to know the height, and he gives us the length."[2]

Too clever by a half

There is a glass half full of water.

Mathematician: "The glass is half full."

Physicist: "The glass is half empty."

Engineer: "The glass is too big."

That's not funny!

An engineer, a physicist, and a mathematician find themselves in an anecdote—indeed, an anecdote quite similar to many that you have no doubt already heard. After some observations and rough calculations, the engineer realizes the situation and starts laughing. A few minutes later the physicist understands too and chuckles to himself happily, as he now has enough experimental evidence to publish a paper.

This leaves the mathematician somewhat perplexed, as he had observed right away that he was the subject of an anecdote, and deduced quite rapidly the presence of humor from similar anecdotes, but considered this anecdote to be too trivial a corollary to be significant, let alone funny. ◘

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[2]Some of the jokes in Verhagen's collection are actually "variations on a theme," leaving it to readers to pick their favorite version.—*Ed.*

Physics in the news

Calculus, "obesity," and the laws of scaling

by Albert A. Bartlett

THIS STORY IS BUILT AROUND a quotation from the widely read international news magazine *The Economist*. A short news story told of the increased intake of dietary fat by young prople in Japan, and it stated that "the average 16-year-old Japanese girl has grown 4% heavier since 1975, although she is only 1% taller."[1]

This makes it sound as if Japanese girls are experiencing a "trend toward obesity." In order to understand and evaluate this, we must define what we mean by a trend toward obesity. We will say that there is no trend toward obesity if, in a given period of time, all linear dimensions of a person increase by the same fractional amount. If a person's height, width, and breadth all increase by 1%, we would say there was no trend toward obesity. However, if in a given time one's lateral dimensions—width and breadth—increase by a larger fractional amount than one's increase in height, then we'll say that there is a trend toward obesity. The 1% and 4% data certainly seem to indicate a trend toward obesity until we look carefully at the elementary arithmetic of scaling.

First, let's look at large changes in size. Suppose I have similar objects whose different sizes are all indicated by different values of a length L. The surface areas of the objects will then vary as L^2 and the volumes will vary as L^3. If the objects are all made of the same homogeneous material, the masses and weights of the objects will also vary as L^3.

The simple example of a cube can be used to illustrate this. A cube has an edge length L, a surface area $6L^2$, and a volume L^3. For a sphere, the radius R is the measure of size. The area is $4\pi R^2$ and the area is $^4/_3\pi R^3$. This general dependence of area and volume on linear dimensions is true for all similar objects, even though we may not know the formula for the area or volume of the objects. Thus we can say that if, by magic, we could create a human who was twice my height, breadth, and width, then the following would be true:

1. Since all linear dimensions have been doubled, the length of a belt to go around the waist of the large person would be twice the length of my belt. The same would be true of the circumference of the neck, the length of the arms, and so on.
2. The area of the cloth required to make clothes for the large person would be $2^2 = 4$ times the area required to make the same style of clothes for me.
3. The volume, mass, and weight of the large person would be $2^3 = 8$ times my volume, mass, and weight.

This has some interesting consequences. What would be the pressure in the knee joints of the large person, compared to the pressure in my knee joints? The area of the knee joints has increased by a factor of 4, while the load or weight they are to carry has increased by a factor of 8. Since pressure is force per unit area, we can see that the pressure in the knee joints has increased by a factor of $8/4 = 2$. When standing, the large person would have twice the pressure in the knee joints that I have in my knee joints when I'm standing!

This opens up a whole realm of

[1]*The Economist*, September 16, 1995, p. 74.

understanding. Nature has worked out mechanisms for knee joints that operate best in one range of pressures. If nature built big people, it would arrange for the big people to have about the same pressure in the knee joints as the small people have. Since the large person's knee joint has to handle eight times the load, the area of the knee joint would be approximately eight times as large as in the smaller people. So the person who was twice my height would have a knee joint diameter that is $\sqrt{8} = 2.8$ times the diameter of my knee joints. The larger person's leg diameter would have to be larger with respect to his other body dimensions than would be the case with me.

Remember *Gulliver's Travels* by Jonathan Swift: the Liliputians were only about 15 cm high, and the Brobdingnagians were many times as tall as Gulliver, yet Swift portrays them all as having the same proportions as Gulliver. It won't work. The Liliputians would need legs of only a very small diameter relative to their height, and the Brobdingnagians would need legs with an enormous diameter compared to their body height. On a more general scale, this tells us why the diameter of the leg of an insect may be about 1% of the length of the insect's body, while the diameter of the leg of an elephant is about 10% of the length of the elephant's body. In the extreme case, it tells us why a whale can't walk. There may not be room enough under a whale's body to put four normal flesh-and-bone legs of sufficient diameter to hold up the large weight of the whale's enormous body. The only way whales can move is to be in water and take advantage of the Archimedean principle.

But what if the increase in height is only 1%, which is far less than doubling? We need to know what happens to body area and volume when all body dimensions are increased by the same small fractional amount. To examine this, we must derive what I believe to be the most important relation in differential calculus.

Suppose I have two variables x and y that are related by the equation

$$y = Ax^n, \tag{1}$$

where A and n are constants. If we differentiate both sides of this equation, we have

$$dy = nAx^{n-1}dx. \tag{2}$$

If we divide equation (2) by equation (1), we get

$$\frac{dy}{y} = n\frac{dx}{x}. \tag{3}$$

Equation (3) is my candidate for Most Useful Equation in Differential Calculus.

To illustrate this utility, let's take the expression for the period T of a simple pendulum:

$$T = 2\pi\sqrt{\frac{L}{g}}. \tag{4}$$

We seek to answer the question: If we increase the length L of the pendulum by 1%, how much does the period change? In this case A is $2\pi/\sqrt{g}$ and n is 1/2. So equation (3) becomes

$$\frac{dT}{T} = \frac{1}{2}\frac{dL}{L}. \tag{5}$$

We can read this to say that a 1% increase in L ($dL/L = 0.01$) gives an *increase* in the period T of 1/2% ($dT/T = 0.005$).

We can now inquire about the consequences of increasing g by 1%. Equation (3) becomes

$$\frac{dT}{T} = -\frac{1}{2}\frac{dg}{g}. \tag{6}$$

This can be read to say that an increase of 1% in the free-fall acceleration g results in a *decrease* in the period of the pendulum by 1/2%. The decrease is indicated by the minus sign. The utility of equation (3) is enormous. If you have a simple power relationship such as equation (1) between two variables that is either exact or approximate, then you can, by inspection, tell the fractional change in one variable when a given fractional change is made in the other variable. For example, if you heat an object so that thermal expansion causes its length to increase by 1%, then you know at once that its area increases by approximately 2% and its volume by 3%.

And now, let's return to people. If the height and all other body dimensions increase by 1%, then the larger person would have the same proportions as a smaller person, and one would say that there was no trend toward obesity. If all linear dimensions increase by 1%, then the belt size would increase by 1%, the area of cloth required to make clothes would increase by 2%, and the body volume, mass, and weight would all increase by 3%. Thus we can see that 3% of the 4% increase in weight comes from the larger size and is not an indication of a trend toward obesity. The remaining (4% – 3%) increase in weight is an indication that the lateral dimensions have increased slightly more than 1%, which would indicate a small trend toward obesity.

What about the pressure in the knee joints? If all the linear dimensions are increased by 1%, the larger person will have a 1% increase in the pressure in the knee joints. The 1% increase due to the obesity—that is, the difference (4% – 3%)—will also add 1% to the pressure in the knee joints, so the data in the story can be used to conclude that the pressure in the knee joints is probably about 2% higher.

So, you see how misinterpretations, spoken or implied, in news stories can be rooted out by simple analysis. Now it's up to you to keep a vigilant eye on the media! ◘

Related *Quantum* articles

Anatoly Mineyev, "From Mouse to Elephant," March/April 1996, p. 18.

———, "Trees Worthy of Paul Bunyan," January/February 1994, p. 4.

A. Zherdev, "Horseflies and Flying Horses," May/June 1994, p. 32.

Albert A. Bartlett *is an emeritus professor of physics at the University of Colorado, Boulder CO 80309-0390.*

Duracell awards $100,000 to young inventors

Virginia student takes first place in Duracell/NSTA Scholarship Competition

WHAT'S MISSING IN HOMES and businesses that the creativity of high school students can supply? Try the *Portable Visual Field Analyzer*, *Early Warning Safety Helmet*, *Motion-Activated Electronic Note Poster*, *Talking Color Identifier for the Blind*, *RipeRanger*, and the *FM Transmitter/Receiver Auto-Train Safety Device*. These battery-powered health and safety, time- and dollar-saving inventions were the top winners in the 14th annual Duracell/NSTA Scholarship Competition. The 1996 competition awarded over $100,000 in savings bonds.

From 100 finalist entries assembled at Duracell headquarters, the judges tested and selected 41 devices for further awards, and the six top inventions were recognized at the National Science Teachers Association convention in St. Louis at the end of March. The first-place winner is Ian Hagemann of Great Falls, Virginia, a senior at Thomas Jefferson High School for Science and Technology. Second-place winners are Eugene Agresta of Closter, New Jersey, a junior at Northern Valley Regional High School; Kyle Beucke, of Stony Brook, New York, a junior at Ward Melville High School; Sean Breheny of Scranton, Pennsylvania, a junior at the Scranton Preparatory School; Daniel Durand of Shoreham, New York, a junior at Shoreham-Wading River High School; and Jeremy Kiser of Akron, Ohio, who is a junior at Springfield High School.

Looking for blind spots

The first-place winner, Ian Hagemann, was awarded a $20,000 bond for developing a portable and inexpensive optical testing system. His invention, the Portable Visual Field Analyzer, or PVFA, is an automated mechanism for testing eyesight—specifically for detecting "blind spots" on the retina. The energy-efficient PVFA uses a microcomputer unit and runs on one 9-volt battery that is regulated to 5 V. "Drugstores and pharmacies," Hagemann says, "may consider adding a PVFA to their array of self-diagnostic equipment." The PVFA and its inventor were featured in a front-page article in the *Washington Post*.

Hagemann wants to study cognitive science in college and plans to become a physician. He speaks French and Spanish, plays blues guitar and is a licensed model aircraft pilot. Hagemann's sponsor for the competition is David Bell, a teacher at the Thomas Jefferson High School for Science and Technology in Alexandria, Virginia.

From helmets to cantaloupes

Eugene Agresta is a second-place winner of a $10,000 bond for the Early Warning Safety Helmet, a bicycle helmet that vibrates and alerts riders by detecting vehicles approaching from behind. This safety device is especially helpful for young children and people who are hearing impaired. Agresta says, "As far as I know, no commercially available product exists like my Early Warning Safety Helmet. It might prevent some injuries or even fatal accidents." After high school, Agresta plans to attend a technical school.

Kyle Beucke is a second-place winner of a $10,000 bond for inventing a talking Post-it™-type note that can't blow away or get lost. The Motion-Activated Electronic Note Poster always delivers its message because it is electronically activated and verbally alerts anyone who walks past it. "It's difficult to make sure someone is aware that a message has been left for them," says Beucke. "My device records a brief message and senses when someone walks by, activating the message playback." Beucke collects beetles and is a member of his school's varsity Science Olympiad team.

Sean Hugh Breheny invented an aid for visually impaired persons that recognizes and reads colors—the Talking Color Identifier for the Blind. Also a second-place $10,000 winner, this handy device audibly identifies the colors of objects from a library of 16 basic colors. When pressed against an object, a microprocessor begins a short computer program that uses a photocell to measure reflected light and determine the color of the object. Although his primary interest is electronics, Breheny represented the United States playing the tin whistle in an international music competition in Ireland last year. Breheny wants to study electrical engineering or electronics, but has not yet decided on a college.

Daniel James Durand invented a small device with both household and commercial applications that tells if fruit is ripe without squeezing or cutting into it. Durand says the RipeRanger "provides a previously unavailable solution to the woes of fruit shoppers everywhere by giving consumers reassurance that the fruit they buy will be ripe. It's also the first hand-held device that has field-use capabilities." RipeRanger can give readings of the levels of ripeness in various pieces of fruit by comparing the sugar content with a ripeness standard. For his efforts, Durand received a $10,000 second-place bond. Durand volunteers at a local hospital on weekends and writes a weekly sports column for his community newspaper.

After learning about a serious train accident at a railroad crossing, Jeremy James Kiser developed a mechanism for school buses and other vehicles that warns of approaching trains. The FM Transmitter/Receiver Auto-Train Safety Device sends out a signal alerting those within a half-mile range. "I feel that this should be a mandatory piece of equipment found on school buses around the nation," says Kiser. The safety signal is designed to jam radio signals. An alternate design has a separate receiver that sends warning signals when the radio is off. Kiser received a second-place $10,000 bond. He would like to attend an art school after high school.

The six first- and second-place winners, their parents, and their sponsoring teachers were guests of Duracell at an awards ceremony in St. Louis on March 28 moderated by NASA Teacher in Space Designee Barbara Morgan. The winners demonstrated their inventions for an audience of teachers and scientists and exhibited them for thousands attending the National Science Teachers Association's 44th annual convention.

Other winners

The following third-place winners received $1,000 savings bonds (their teacher-advisors are noted in parentheses): Michael Goelzer, St. Albans School, Washington, D.C. (Robert Morse); Joe Wolin, Eagle High School, Eagle, Idaho (Robert Beckwith); Matthew Smith, Lane Technical School, Chicago, Illinois (Michael Robles); Maria Vornbrock, Watkins Mill High School, Gaithersburg, Maryland (Jean Maloney); John Verde, Arundel Senior High School, Gambrills, Maryland (Don Higdon); Stefan Kazachki, North Carolina School of Science and Math, Durham, North Carolina (Chuck Britton); Matthew Hicks, Batavia High School, Batavia, New York (Gary Heim); Judy Dong, Midwood High School at Brooklyn College, Brooklyn, New York (Stanley Shapiro); Mary Gates, Springfield High School, Akron, Ohio (Nicholas Frankovits); Chris Vondrachek, Newberg High School, Newberg, Oregon (Terry Coss).

The following fourth-place winners received $500 savings bonds (their teacher-advisors are noted in parentheses): Rahul Athavale, Boca Raton High School, Boca Raton, Florida (JoAnne Weise); Chama Cascade, W. Hawaii Exploration Academy, Kailua-Kona, Hawaii (Bill Woerner); Matthew Schultz, Harrison High School, Farmington Hills, Michigan (Dennis King); Brian Siegrist, Northern Valley Regional High School, Demarest, New Jersey (Javier Rabelo); Alexis Astorga, Northern Valley Regional High School, Demarest, New Jersey (Javier Rabelo); Gary Fischer, Kittatinny Regional High School, Newton, New Jersey (Ronald Garbarini); Bruce Noll, Ocean City High School, Ocean City, New Jersey (Michael Wilbraham); David Nussbaum, Half Hollow Hills School West, Dix Hills, New York (Harold Shaver); Carlyn Scheinfeld, Jericho High School, Jericho, New York (Allen Sachs); David Chiang, Manhasset High School, Manhasset, New York (Peter Guastella); Arti Anand, Jericho High School, Jericho, New York (Allen Sachs); Thomas Sapienza, Shoreham–Wading River High School, Shoreham, New York (John Holzapfel); Shayna Lustig, Shoreham–Wading River High School, Shoreham, New York (Jim Baglivi); Matthew Hoimes, Ward Melville High School, East Setauket, New York (Melanie Krieger); Robert Long Jr., Lakota High School, West Chester, Ohio (Linda Noble); William Fyock, Warwick High School, Lititz, Pennsylvania (Laurel Hess); Jeffrey Huber, Hempfield High School, Landisville, Pennsylvania (Glenn Shaffer); Nick Berg, B. Reed Henderson High School, West Chester, Pennsylvania (Charles Wood); Brian Ground, J. L. Mann High School, Greenville, South Carolina (Hugh Gilchrist); Mark McGrath, Hilton Head High School, Hilton Head Island, South Carolina (Richard Winger); Michael McTaggart, South Carolina Governor's School for Science and Mathematics, Hartsville, South Carolina (Kurt Wagner); Brian Rosenthal, The Kinkaid School, Houston, Texas (Herman Keith); Matthew Rodgers, Academy of Science and Technology, Conroe, Texas (Scott Rippetoe); John Claus, Academy of Science and Technology, Conroe, Texas (Scott Rippetoe); William Gould, Lake Braddock Secondary School, Burke, Virginia (Don Ehrenberger).

A recent survey of past winners and their teachers indicates that the competition influences career choices and fields of study. It appeals to a wide range of students with interests in industrial arts, vocational/technical studies, and the sciences. Teachers reported that the competition is useful for motivating students to learn more science and technology as they solve the problems associated with developing their inventions.

Every student who entered the 1996 competition received a long-distance calling card and a certificate of participation. Many finalists will have their devices displayed at conventions and exhibits throughout the country. Administered by the National Science Teachers Association, the Duracell/NSTA Scholarship Competition has awarded over $600,000 in scholarships, savings bonds, and cash awards to nearly 700 students over the last fourteen years.

To enter the Duracell/NSTA Scholarship Competition, ninth through twelfth grade students design and build a device that is educational, useful, or entertaining and is powered by one or more Duracell batteries. Judging is based on the creativity, practicality, and energy efficiency of the device as well as the clarity of the written description. Proposals for entries are due at NSTA each January.

Duracell U.S.A., headquartered in Bethel, Connecticut, is a division of Duracell Inc., the world's leading manufacturer of high-performance alkaline batteries.

Bulletin Board

CyberTeaser truth-tellers

Just about everyone who entered the CyberTeaser contest—a continuing feature of *Quantum*'s World Wide Web site—submitted a correct answer. This was very gratifying indeed. Now if we could just get all our entrants to supply their *mailing* address when they submit their solution!

These folks were the first to submit a correct answer to the CyberTeaser (brainteaser B172 in this issue):

Oleg Shpyrko (Rochester, New York)
Xi-an Li (Middlebury, Vermont)
Sabrina Sowers (Baltimore, Maryland)
Matthew Wong (Edmonton, Alberta)
Maeline Krig (Minnesota)
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Marc Oliver Rieger (Konstanz, Germany)
Jeff Canter (Nashville, Tennessee)
May T. Lim (Philippines)

With this edition of the CyberTeaser, we added a new twist: every person who submitted a correct answer was eligible to win a copy of *Quantum Quandaries*, a collection of the first 100 brainteasers from *Quantum* magazine. Check our Web page to find out who won the book. And while you're there, take a crack at the new CyberTeaser. Our address is **http://www.nsta.org/quantum**.

Free curriculum materials on the Web

Microsoft Corporation and the National Science Teachers Association (NSTA) have teamed up to provide a field-tested, comprehensive curriculum for high school science as part of the Global Schoolhouse (GSH), a popular resource area on the Internet. The new GSH Web site, launched at the NSTA convention in St. Louis in March, gives science educators in-depth descriptions of the content components of the new National Science Education Standards and science teaching materials designed to achieve those standards.

At the GSH Web site, teachers have free access to learning resources developed by NSTA, called "Micro-Units," for more than 80 topics in biology, chemistry, Earth and space science, and physics. The materials are linked to the National Science Education Standards recently developed by the National Research Council of the National Academy of Sciences. For teachers, Micro-Units include materials for helping students learn the content of the unit, along with assessment tools. Student materials provide basic activities, hands-on investigations, and readings for each unit.

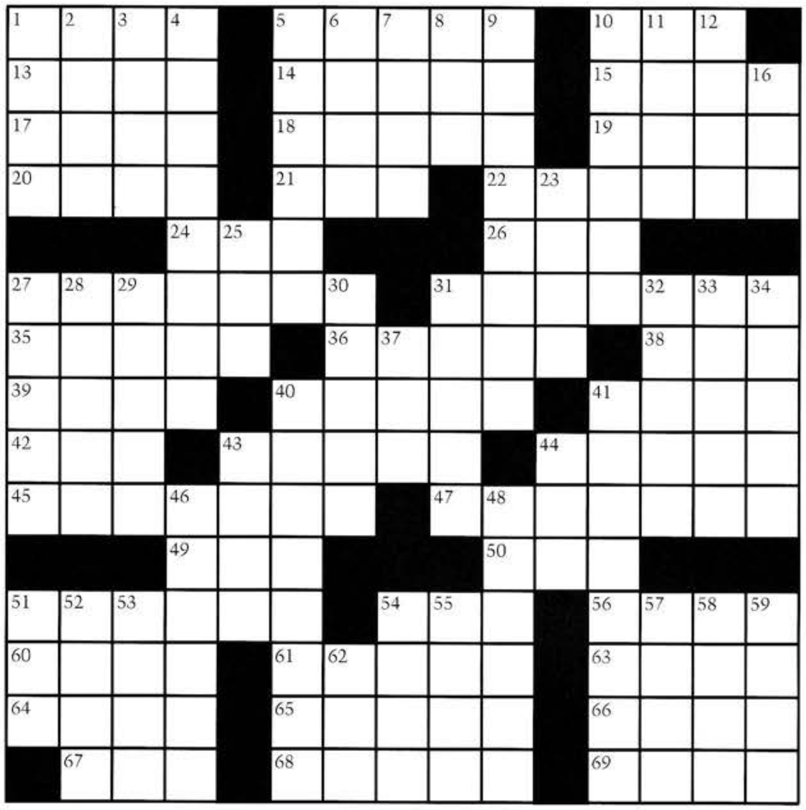
The overall curriculum framework and the Micro-Units are the work of the Scope, Sequence and Coordination of High School Science (SS&C) project, an NSTA initiative funded by the National Science Foundation. In addition to supporting the new science education standards, the curriculum's integrated and coordinated design reflects the best research on how students learn. High school students study all four natural sciences every year across all grade levels, instead of the traditional "layer cake" pattern of a single science subject each school year.

Before being posted to the Web, materials are reviewed by experienced science teachers, teaching scientists, and research scientists and are field tested in schools. Curriculum materials for the ninth-grade level are available now; materials for 10th through 12th grades will follow. Science teachers who download and use the nearly 2,000 pages of free materials can also provide feedback to help improve the curriculum.

The Scope, Sequence and Coordination home page is located at **http://www.gsh.org/NSTA_SSandC**. The site is hosted on the Global Schoolhouse, developed jointly by the Global SchoolNet Foundation and Microsoft.

Criss × cross science

by David R. Martin



Across

1 Arab robes
5 699,646 (in base 16)
10 Unified field theory: abbr.
13 57,018 (in base 16)
14 Kennedy's Sec'y of the Interior Stewart Lee ___
15 Shady mountain side
17 Afresh
18 Change the color of
19 Po tributary
20 Shed skin
21 Unrefined metal
22 Element in diamonds
24 Row
26 Make lace
27 Learned person
31 ___ diffraction
35 Blows a horn
36 Perch
38 Hydrogen and oxygen
39 Heedless
40 One ft^3/s
41 ___ function (piecewise const. function)
42 Expert
43 10^{-15}: pref.
44 Large group
45 Daughter of Minos
47 ___ reactor
49 JFK's predecessor
50 Poise reciprocal
51 Like Jupiter or Saturn
54 Pie ___ mode
56 Publisher Conde ___ (1874–1942)
60 Anthropologist ___ Hrdlicka (1869–1943)
61 Capillary permeability increaser
63 Theater award
64 10^{-9}: pref.
65 Japanese novelist ___ Dazai
66 Construction beam (2 wds.)
67 ___ Allen belts
68 Ballots
69 British writer Walter ___ ___ Mare (1873–1956)

Down

1 Swedish botanist ___ Afzelius (1750–1837)
2 Seismologist ___ Gutenberg (1889–1960)
3 English chemist Frederick Augustus ___ (1827–1902)
4 ___ waveform
5 ___ borealis
6 ___ wax (ozocerite)
7 ___ oil (Juniper tar oil)
8 Insect
9 ___ field
10 Volume units
11 52,187 (in base 16)
12 Pedestal part
16 Container
23 43,758 (in base 16)
25 Hirt and Pacino
27 Band
28 Vehicle
29 Old Testament Prophet
30 Finnish port
31 Refrigerant
32 Saltpetre
33 970,458 (in base 16)
34 Person to be shunned
37 Superlative: suff.
40 ___ radiation
41 Electric switch
43 64,986 (in base 16)
44 Coolidge's successor
46 Inventor Thomas ___
48 It's also like 51A
51 Astronomer ___ Oort
52 Kingly Norse name
53 ___ cava (vein)
54 Med. school subj.
55 Calcium oxide
57 German physicist Ernst ___ (1840–1905)
58 Continental crust rocks
59 10^{12} pref.
62 Same: pref.

SOLUTION IN THE NEXT ISSUE

SOLUTION TO THE MARCH/APRIL PUZZLE

E	A	R	S		N	T	Y	P	E		E	G	A	S
G	E	O	N		O	H	A	I	N		S	I	N	E
A	B	B	E		H	O	L	E	S		A	R	N	E
D	I	E	L	S		R	O	T		E	U	L	E	R
			L	E	D		W	A	T	T				
C	H	I		C	O	S			H	A	L	L	E	Y
H	O	L	T		P	A	E	D	O		A	E	A	A
E	Y	E	S		P	R	O	E	M		K	N	A	R
A	L	F	A		L	I	N	E	S		E	Y	E	D
P	E	T	R	I	E			D	O	T		A	B	S
				O	R	E	S		N	O	S			
F	R	E	O	N		N	O	T		N	O	D	E	S
A	I	D	S		B	O	L	A	S		L	A	L	O
B	O	I	L		C	L	I	N	E		A	L	A	N
A	T	T	O		S	A	D	H	E		R	E	N	E

ANSWERS, HINTS & SOLUTIONS

Math

M171

The answer is yes. Consider the polynomial $P(x) = x(9x + 2)$. If

$$n = \underbrace{11\ldots11}_{k},$$

then

$$9n + 2 = 1\underbrace{00\ldots00}_{k-1}1.$$

Therefore,

$$P(n) = \underbrace{11\ldots11}_{k} \cdot 1\underbrace{00\ldots00}_{k-1}1 = \underbrace{11\ldots11}_{2k}.$$

M172

The answer is positive in cases (a) and (b) and negative in case (c). To construct an n-term sequence with the desired properties, we can take the first n Fibonacci numbers 1, 2, 3, 5, … , f_n (where $f_{k+1} = f_k + f_{k-1}$), reverse their order, and divide them all by their least common multiple (LCM) N. Any number f_k/N thus obtained will be of the form $1/m$, and for any $k = 1, \ldots, n - 2$,

$$\frac{f_k}{N} = \frac{f_{k+2}}{N} - \frac{f_{k+1}}{N}.$$

For instance, for $n = 5$ we have LCM(1, 2, 3, 5, 8) = 120, so one of many possible answers is the sequence 1/15, 1/24, 1/40, 1/60, 1/120.

An infinite sequence of this sort is impossible, because if we reduce the first two terms of such a sequence to their common denominator N and compute all subsequent terms without simplifying the emerging fractions, we'll represent them all in the form b_k/N, $k = 1, 2, \ldots$, where $N \geq b_1 > b_2 > \ldots$ (and $b_{k+1} = b_{k-1} - b_k$). So there can be no more than N terms. (N. Vasilyev)

M173

Let's start with part (b). Here two positions of point X are possible: the midpoint M of BC and the base H of the altitude dropped from A on BC (fig. 1). These two points merge into one if $AB = AC$. Let L and N be the midpoints of AC and AB, respectively. Then XQ lies on the median XL of triangle ABC, and $XQ : XL = 2 : 3$. Similarly, $XP : XN = 2 : 3$. It follows that triangles XPQ, XLN are similar, with ratio of similarity 2:3. So we can look for points X such that $\triangle XNL$ is similar to $\triangle ABC$. Since for any X the area of $\triangle XNL$ is 1/4 that of $\triangle ABC$, the ratio of similarity of these triangles must be 1/2. So, if $\triangle XNL \sim \triangle ABC$, we can assume that the side of $\triangle ABC$ corresponding to $NL = {}^1/_2 BC$ is BC. (Otherwise both triangles are isosceles or equilateral and our assumption holds anyway.) Now we have two possibilities: $\angle XNL = \angle BCA = \angle NLA$, which means that NX is parallel to AC and $X = M$ (fig. 1a), or $\angle XNL = \angle ABC = \angle ANL$, which means that point X is symmetric to A with respect to NL—that is, $X = H$ (fig. 1b). Both points M and H clearly satisfy the condition.

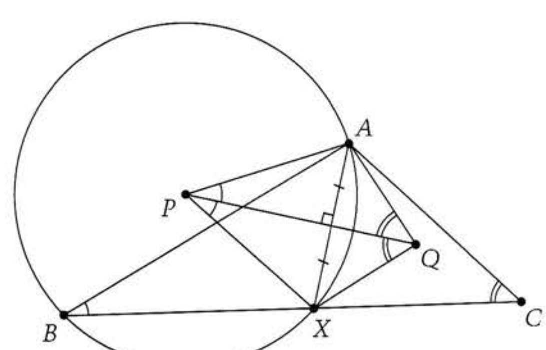


Figure 2

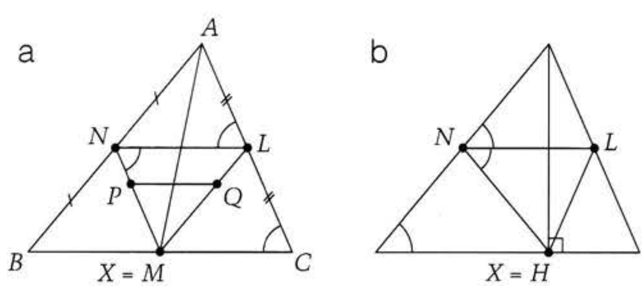


Figure 1

Now let's turn to part (a). Here the two triangles in question are similar for all points X on BC. We can skip the case where $AX \perp BC$: it is the situation considered above, because in this case, in the notation of figure 1b, $X = H$, $P = N$, $Q = L$. So suppose that, say, $\angle AXB > 90°$. Then angle ABX is acute (fig. 2); its vertex B lies on the same side of AX as the circumcenter P of $\triangle ABX$; and, by the Inscribed Angle Theorem, $\angle XPA = 2\angle XBA$. If angle ACX is also acute, a similar argument shows that $\angle XQA = 2\angle XCA$. It remains to notice that PQ is the perpendicular bisector of AX and, in the case we consider, P and Q are on different sides of AX. So $\angle XPQ = {}^1/_2\angle XPA = \angle CBA$ and $\angle XPQ = {}^1/_2\angle XQA = \angle BCA$ and, by the AAA condition for similarity, $\triangle XPQ \sim \triangle ABC$. This argument has to be slightly modified if

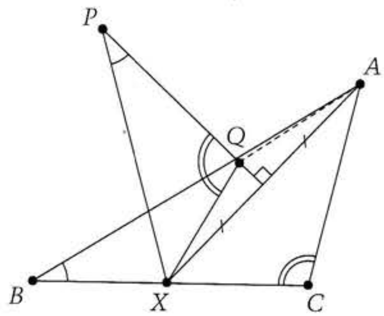


Figure 3

$\angle ACX > 90°$ (fig. 3)—here both angles XQP and XCA equal $180° - {}^1/_2\angle XQA$.

Finally, part (c). In this case also any point X can be taken. This follows from the perpendicularity of the sides PQ, QX, XP of $\triangle XPQ$ to the sides BC, CA, AB of $\triangle ABC$, respectively (fig. 4): rotating one triangle by 90°, we make all its sides parallel to the corresponding sides of the other, and the similarity becomes clear. (V. Dubrovsky, E. Turkevich)

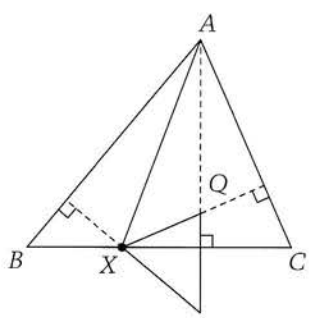


Figure 4

M174

Suppose the statement of the problem is wrong. Then

$$f'(x) - (f(x))^2 \geq 1$$

for any point x on the given segment. Rewrite this inequality as

$$\frac{f'(x)}{1+\left(f(x)\right)^2} \geq 1,$$

or

$$\Phi'(x) \geq 0,$$

where $\Phi(x) = \tan^{-1} f(x) - x$ (this follows from the fact that $(\tan^{-1} y)' = 1/(1 + y^2)$ and the formula for the derivative of a composite function). So $\Phi(x)$ is a nondecreasing function, and $\Phi(b) - \Phi(a) \geq 0$—that is,

$$\tan^{-1} f(b) - \tan^{-1} f(a) \geq b - a = 4.$$

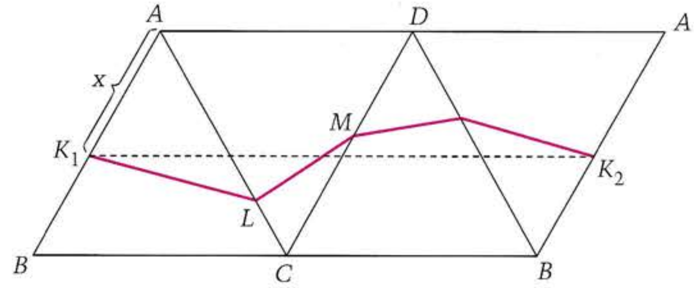


Figure 6

But $\tan^{-1} y$ ranges in the interval $(-\pi/2, \pi/2)$, so the difference of its values in any two points cannot exceed $\pi < 4$. This contradiction completes the solution. It is clear now that the number 4 in the statement can be replaced with π.

M175

First we'll establish the lower bound for the perimeter. If we cut the tetrahedron shown in figure 5 along its edges DA, AB, and BC and unfold it so that it lies flat, we'll get a parallelogram. The perimeter P of the cross section turns into a polygonal path K_1K_2 such that the segment K_1K_2 is parallel to two sides of the parallelogram. It is clear from the figure that $K_1K_2 = 2a$, where a is the length of an edge, so we have $P \geq 2a$. Notice that $P = 2a$ if and only if the plane of the section is parallel to the two edges (AD and BC) that it does not cross.

To prove the second inequality, imagine that the plane cutting the pyramid moves parallel to itself (fig. 6) so that the cross section remains a quadrilateral. The two extreme positions are those where the plane reaches the endpoints of one of the pyramid's edges. For each of these the quadrilateral degenerates into a triangle (or even a double segment). But the perimeter of any triangular cross section is smaller than $3a$, because the distance between any two points of an equilateral triangle (except a pair of its vertices) is smaller than its side length.

It remains to show that any "intermediate" cross section will have a perimeter no greater than one of the two extremes (for the triangles). If $x = AK$, where K is the vertex of the section on the edge AB (fig. 7), then the perimeter is a linear function of x (that is, $P = kx + b$).

For instance, in the case shown in figure 7, KL is proportional to $AK = x$ (that is, $KL = c_1x$); the same is true for AL (that is, $AL = c_2x$); so $CL = a - c_2x$. Then LM and CM are proportional to CL, so they also depend linearly on x, and so on.

However, any linear function on a segment takes its maximum at one of the segment's endpoints. The value $3a$ is never achieved, because the "extremal" triangular cross section in this case would have to

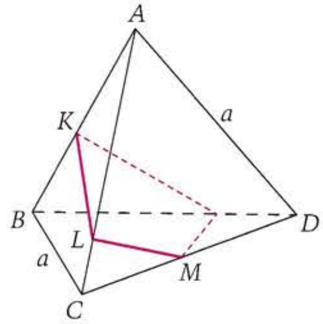


Figure 5

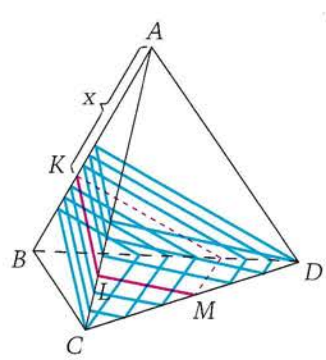


Figure 7

coincide with the pyramid's face, and any parallel cross section would be a triangle in this case. But it's possible to obtain a perimeter arbitrarily close to $3a$ by drawing a plane close to a face. (N. Vasilyev, V. Proizvolov)

Physics

P171

The cord with the sphere will be deflected toward the axis of revolution (see figure 8). In the inertial reference frame the sphere is affected by three forces: the buoyant force $\mathbf{F}_b$, the tension of the cord $\mathbf{T}$, and the force of gravity $\mathbf{F}$ ($F = \rho_s Vg$, where ρ_s is the sphere's density and V is its volume).[1] The sum of the projections of these forces on the vertical axis is zero:

$$\rho_w Vg - \rho_s Vg - T\cos\alpha = 0,$$

where ρ_w is the density of water. The sphere's revolution along a circle of radius $r - l\sin\alpha$ induced by the glass's rotation is described by the following equation:

$$\rho_s V\omega^2(r - l\sin\alpha) = \rho_w V\omega^2(r - l\sin\alpha) - T\sin\alpha.$$

A simultaneous solution of these two equations gives us ω:

$$\omega = \sqrt{\frac{g\tan\alpha}{r - l\sin\alpha}} = 10.6\ \text{s}^{-1}.$$

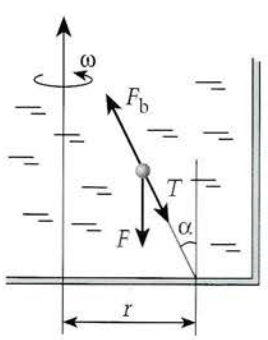


Figure 8

[1]For details about the buoyant force in a moving liquid, see "Meandering Down to the Sea" in the July/August 1992 issue of *Quantum*.

P172

Consider the membrane's oscillations to be harmonic—that is,

$$x = A\sin\omega t,$$

where x is the deviation from the equilibrium position, A is the amplitude, and $\omega = 2\pi\nu$ is the angular frequency of the oscillations. At $t = 0$, we have chosen the membrane to be at its equilibrium position and moving in the upward direction.

A grain of sand moving with the membrane is affected by two forces: gravity $m\mathbf{g}$ and the normal force $\mathbf{N}$ due to the membrane (fig. 9). Thus the motion of a grain is described by

$$\mathbf{N} - m\mathbf{g} = m\mathbf{a}.$$

At the moment the grain loses contact with the membrane, the supporting force is zero, so the grain's acceleration is $\mathbf{a} = -\mathbf{g}$. Then the grain ascends to a height h as a body thrown vertically from a height h_0 equal to the membrane's deflection from the equilibrium position at moment t_0, and with an initial velocity v_0 equal to that of the membrane at the same moment t_0 (see figure 9). According to the law of conservation of mechanical energy,

$$mgh_0 + \frac{mv_0^2}{2} = mgh. \qquad (1)$$

Now let's consider the motion of the membrane in more detail and connect the values h_0 and v_0

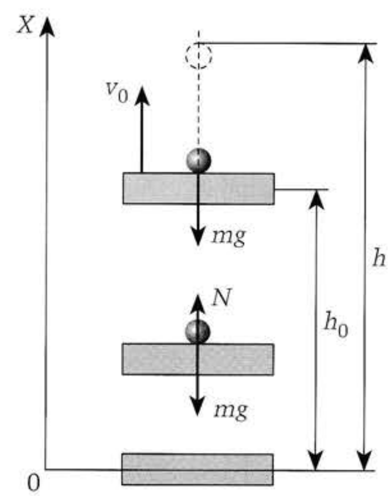


Figure 9

with the amplitude of the membrane's oscillation. At any moment the membrane's velocity is $v = \omega A\sin(\omega t + \pi/2) = \omega A\cos\omega t$, and its acceleration $a = -\omega^2 x = -A\sin\omega t$. At the moment of separation t_0,

$$v = v_0 = \omega A\cos\omega t_0,$$
$$a = -g = -\omega^2 A\sin\omega t_0,$$
$$x = h_0 = A\sin\omega t_0.$$

From the equation for the acceleration,

$$\sin\omega t_0 = g/\omega^2 A,$$

so

$$h_0 = g/\omega^2,$$
$$v_0 = \omega A\cos\omega t_0 = \sqrt{\omega^2 A^2 - g^2\ \omega^2}.$$

Inserting the values of h_0 and v_0 into equation (1) yields

$$A = \frac{\sqrt{2gh\omega^2 - g^2}}{\omega^2} \cong 77\cdot 10^{-6}\ \text{m}$$
$$\cong 0.077\ \text{mm}.$$

P173

According to the first law of thermodynamics, the thermal energy transferred to the gas is expended on changing its internal energy ΔU and performing work W:

$$Q_\text{I} = \Delta U_\text{I} + W_\text{I},\ Q_\text{II} = \Delta U_\text{II} + W_\text{II}.$$

Here the subscript I refers to the process $1 \to 3 \to 2$, and the subscript II refers to the process $1 \to 4 \to 2$. Since the gas is monatomic, the following equations are true for one mole of the gas:

$$U = \frac{3}{2}RT, \quad \Delta U = \frac{3}{2}R\Delta T.$$

It follows that the change in the gas's internal energy corresponding to the transition from state 1 to state 2 depends only on the change in the gas's temperature $\Delta T = T_2 - T_1$ and not on the way the gas moves from one state to another. Thus

$$\Delta U_\text{I} = \Delta U_\text{II} = \Delta U = \frac{3}{2}R(T_2 - T_1).$$

To find the temperatures T_1 and

T_2, let's write down the equations of state for an ideal gas for states 1 and 2 (see figure 1 on page 31):

$$P_0V_0 = RT_1,\ 2P_0 \cdot 2V_0 = RT_2,$$

from which we get

$$T_2 - T_1 = \frac{3P_0V_0}{R}, \quad \Delta U = \frac{9}{2}P_0V_0.$$

Now we get the values for the work W_{I} and W_{II} performed by the gas. In the first case (the process $1 \to 3 \to 2$) the gas doesn't perform work during the $1 \to 3$ phase, but during the isobaric expansion (the $3 \to 2$ phase) the work performed is

$$W_{\text{I}} = P\Delta V = 2P_0(2V_0 - V_0) = 2P_0V_0.$$

In the second case (the process $1 \to 4 \to 2$) the gas performs work only during the phase $1 \to 4$:

$$W_{\text{II}} = P_0(2V_0 - V_0) = P_0V_0.$$

Thus

$$Q_{\text{I}} = \Delta U + W_{\text{I}} = \frac{13}{2}P_0V_0,$$

$$Q_{\text{II}} = \Delta U + W_{\text{II}} = \frac{11}{2}P_0V_0.$$

The ratio we seek is

$$\frac{Q_{\text{I}}}{Q_{\text{II}}} = \frac{13}{11}.$$

P174

Any conducting material between the plates will come to equilibrium with surface charges matching the surface charges on the capacitor plates. Thus there is no E-field inside the conductor (even a non-ideal one). When the capacitor is short circuited, the charges on the capacitor plates neutralize each other almost instantaneously, as they would in a perfect conductor. Therefore, the potential difference across the conducting plate is V_0. Because the resistance of the plate is $R = \rho h/S$, the current due to the internal migration of charges across the plate is maximum at the initial instant and given by

$$I = \frac{V_0}{R} = \frac{V_0 S}{\rho h}.$$

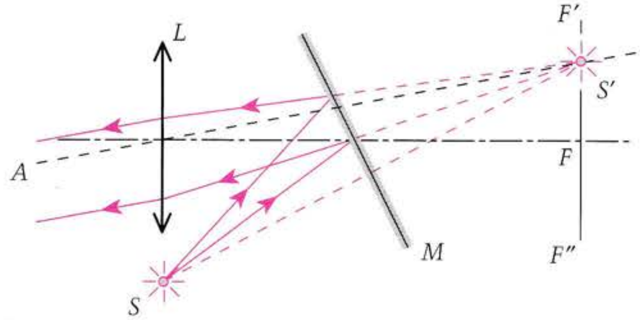


Figure 10

P175

Let $F'F''$ be the focal plane of the lens (fig. 10). Draw an auxiliary optical axis AA' parallel to the arrow. If the light source is located in the focal plane at point S' where it crosses the axis AA', the lens collects the rays into a parallel beam directed along the axis AA'. This parallel beam emerges from the lens in the direction shown by the arrow.

In this case the light source for the lens will be the image of the source S in the flat mirror. To place this image at the point S', the mirror must be placed at the middle of the segment SS' and perpendicular to it. The light rays are shown in figure 10.

Brainteasers

B171

Divide the factors in the initial product into groups of ten factors: $1 \cdot 2 \cdot 3 \cdot \ldots \cdot 10$; $11 \cdot 12 \cdot 13 \cdot \ldots \cdot 20, \ldots$; the last group will be incomplete. After we "thin out" these products, all of them (except the last) will end in the same digit as $1 \cdot 3 \cdot 7 \cdot 9$—that is, they'll all end in 9. Since $9 \cdot 9 = 81$ ends in one and the number of complete products, 199, is odd, the last digit of their product is 9. The remaining product is $1991 \cdot 1993$ and ends in 3. So the answer is 7 ($9 \cdot 3 = 27$).

B172

Every honest inhabitant of the island answered "yes" to one question; every liar answered "yes" to two questions. So the total number of positive answers $60 + 40 + 30 = 130$ equals the sum of the number of honest persons plus twice the number of liars. If each liar were counted once, we'd simply get the entire population, 100. So the number of liars is $130 - 100 = 30$.

B173

The bubbles envelop the tablet with a layer of practically constant thickness. Since the pill dissolves uniformly over its surface, its diameter shrinks very slowly, while its thickness diminishes quite noticeably (we're speaking of a *relative* decrease, of course). So the surface area of the tablet and the volume of the bubbles remain virtually constant, while its mass decreases rapidly. At a certain point the buoyant force exceeds the tablet's weight and lifts it to the surface.

B174

The answer is shown in figure 11.

B175

The answer is any number greater than 3 except 5. Notice first that no two vertices can belong to the same triangle (we exclude the "partition into one triangle"), so the vertices belong to three different triangles. After cutting them off, we are left with a convex polygon, which can be a triangle (four parts—fig. 12a); but it can never be the

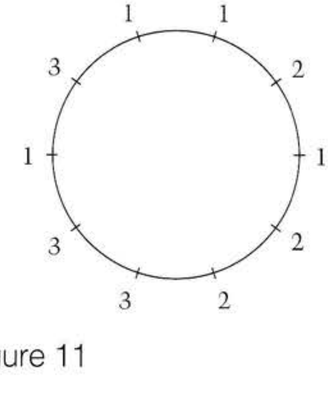


Figure 11

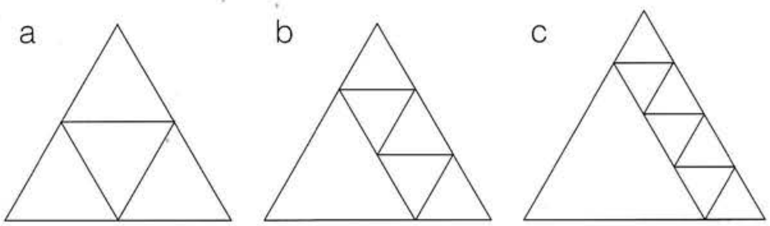


Figure 12

union of two (equilateral) triangles, because such a union is convex only if it is a rhombus, which is impossible in our case. Figures 12b and 12c show partitions into six and eight parts. It remains to say that subdividing a triangle of an n-piece partition into four triangles as in figure 14a increases the number of pieces by three. This trick applied to the partitions in the figures yields $4 + 3k$, $6 + 3k$, and $8 + 3k$ triangular pieces for any $k \geq 0$ (these numbers give the remainders 1, 0, 2, respectively, when divided by 3). (V. Dubrovsky)

Kaleidoscope

1. See figure 13.

2. Projection lamps must produce conical light beams. These can be formed only by point sources of light, which fluorescent lamps are not.

3. They should be located on the same line, which must be perpendicular to the screen and object.

4. When the light comes from a point source.

5. When the light source is larger than the body and the screen is located farther from the body than the apex of the cone of complete shadow (see figure 13).

6. If you can see part of the light source from where you are.

7. When illuminated by car headlights, the raised areas of the pavement cast shadows that can be seen from a distance.

8. These bright spots are images of the Sun formed by many camera obscuras (gaps in the foliage) on the screen (ground). When the gap's size is larger than the Sun's image on the ground, the shape of the spots varies.

9. No (see, for example, figure 14).

10. The individual parts of an extended light source produce overlapping shadows. The resulting shadow will have a sharper edge when the object is closer to the screen.

11. The heated opaque particles in the flame block the passage of light from the lamp, and at the same time they emit light of weaker intensity. As a result the part of the screen behind the flame is illuminated less and is perceived as shadow.

12. This type of illumination produces no clear-cut shadows.

13. No, it doesn't.

14. The pin casts a shadow on the retina that is orientated in the same way as the pin itself. As usual, the brain flips the image, so the shadow is perceived as being upside down.

15. No, because the light arrives from a star as parallel rays, and the shadow of the match doesn't cover the pupil of the eye completely, because at night it is fully dilated (fig. 15).

16. The air over a bonfire is heated to varying temperatures at different places, so its density isn't constant. Light beams passing through a heterogeneous medium do not propagate along straight lines, and so they distort the images of objects.

17. This is the result of perspective. A similar impression is produced when you look down a stretch of railroad tracks.

18. By direct measurement you can verify that both segments are equal, although your first impression probably says otherwise—it's an optical illusion.

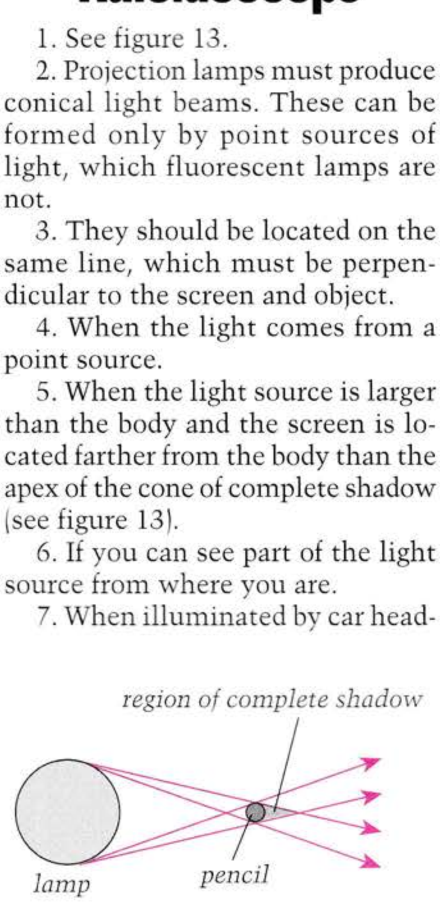


Figure 13

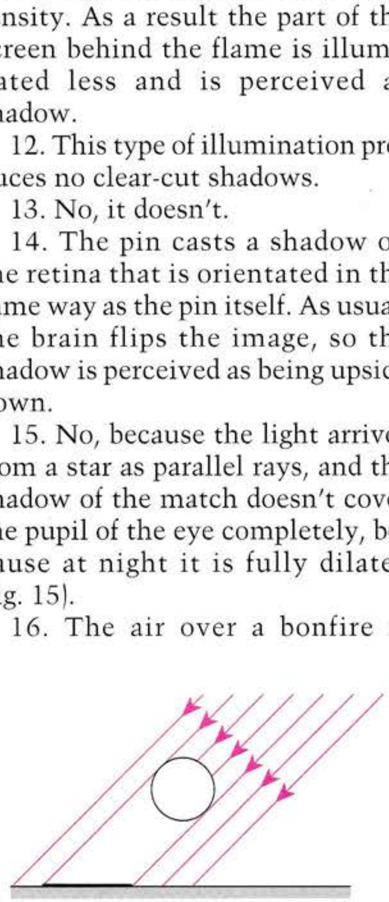


Figure 14

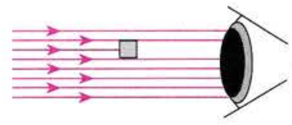


Figure 15

Microexperiment

In the first case the pencil's shadow is more clear-cut than in the second case, because the width of the candle's flame is less than its height. See also the answer to problem 10.

Cutting facets

1. Points A and B, P and Q are symmetric about the diameter of the given circle perpendicular to AB. It follows that $AQ = BP = CK$ and $BQ = PA = DM$. In addition, all these segments are parallel, so $AQKC$ and $BQMD$ are parallelograms.

2. Let M be the midpoint of AB, and let Q be the second intersection point of the circles. Since the triangles AQB are all similar to one another, the same is true for triangles AQM: they have a constant angle AQM and a constant ratio QM/AM. Therefore, point M is the image of A under the spiral similarity about Q with this constant angle and this constant ratio. So the locus of M is the image of the circle QPA—that is, a certain circle passing through Q and P.

3. The spiral similarity about Q that takes A into B maps one circle

onto the other and so maps the tangent AC to the first circle onto BC. Therefore, the angle of rotation $\angle AQB$ is equal to one of the angles formed by lines AC and BC at point C. If we inspect all possible situations more carefully (or use oriented angles instead), we'll see that $\angle AQB + \angle ACB = \pi$ if points Q and C are on different sides of AB, and $\angle AQB = \angle ACB$ if Q and C are on the same side of AB. In either case, the points A, B, Q, and C lie on the same circle.

4. The theorem about the Gauss line follows from our statement about concurrent diagonals of parallelograms. To see why, let the given quadrilateral be $NBJM$ (using the notation from the article). We will reconstruct figure 12 (in the article) around this quadrilateral. We extend NB and MJ to meet at K, and NM and BJ to meet at D. Then we find points A and C so that $BNMA$ and $KNDC$ are parallelograms (fig. 16). Consider the dilation with factor 2 and center N. It takes the midpoint of diagonal BM into A, and the midpoint of KD into C. Now we have a figure like figure 12, and it was shown that lines AC, BD, and KM are concurrent—in other words, that points A, C, and J are collinear. So the given midpoints (their pre-images under the initial dilation) lie on the same line (a midline in triangle NJC).

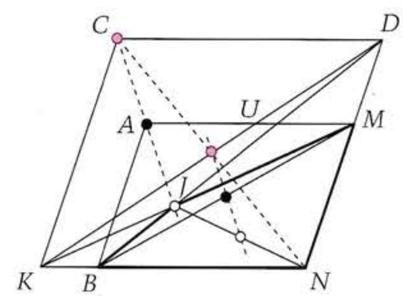


Figure 16

5. In the notation of figure 14 in the article, both mutually dual statements amount to the same fact: the line joining the intersection of $a = AB'$ and $b' = A'B$ and the intersection of $a' = AC'$ and $b = A'C$ passes through the intersection point of the lines BC' and $B'C$. We can transform the dual theorem of Pappus's theorem into the theorem about concurrent diagonals (and vice versa) by a central projection that sends points A and A' in figure 14 to infinity, thus creating two triples of parallel lines (lines a, b, c turn into, say, KC, BP, ND in figure 12, and lines a', b', c' into LM, CD, KN).

Against the current

(Answers to selected items)

1. In Galileo's experiment the musket ball and cannonball fell to the ground practically simultaneously. However, if Galileo could have measured short intervals of the order of 10^{-2} s, he would have observed that the musket ball was slightly behind the cannonball. This is because the relative contribution of the air resistance to the total forces affecting the falling musket ball is greater than that affecting the cannonball.

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2. By a factor of $\sqrt[3]{1.5} \cong 1.15$.

5. If you add a small amount of dye to the stream of water without disturbing it (you can use ink or potassium permanganate), you can see that the flow under the ball is turbulent and chaotic.

6. The coefficient C_x depends strongly on the orientation of the spoon relative to the stream's direction.

7. If a Ping-Pong ball is dropped from a high place, the steady-state speed of about 8 m/s will occur in a time $\tau \cong 2$–3 s.

Corrections

March/April 1996

On page 59 several overbars failed to print as expected in column 2. Beginning at line 16, the text should read: "... $2(k + 1) - 1$ end in $\overline{1}$ and is obtained by adding $\overline{1}$ to the odd representation of $k + 1$. The number one has two representations: 1 and $1\overline{1}$."

* * * * *

The Publisher's Page contains a misprint due to a copyediting error. In the second-last paragraph, line 7, the vector should read "∇A" (not "$\nabla \mathbf{A}$").

* * * * *

The solution to brainteaser B166 (p. 58) contained a misprint. The equation in the last line of the first paragraph should read: "$5(n + 1) = 3[(n + 1) + 2]$." As stated, n is the number of days before Friday that Ciliegia worked. But he also worked Friday, so in the equation "n" should have been "$n + 1$" (as it is in the next paragraph). The revised equation produces $n = 2$, as printed.

* * * * *

Several readers wrote to tell us they found shorter solutions for brainteaser B167:

$$1(562)437 \rightarrow 1(456)237 \rightarrow 1234567.$$

We notice that both the brainteaser author (p. 58) and the boy in the picture (p. 9) take three books from the middle, leaving two books on each end. If one must take three books from the middle, leaving a pair of books at each end, and then insert the three tomes either at one end or between the volumes at one end, the author's solution is valid. However, this restriction is not articulated in the statement of the problem, and as it stands, the solution sent in by our correspondents is both valid and shorter.

January/February 1996

Due to an editing error, the solution to challenge P162 contains an incorrect value for one of the parameters. On page 46 in the third column, 13 lines from the bottom, for "$\mu = 14.5$ g/mole" read "$\mu = 29$ g/mole." A tip of the foolscap to Nick Secor, a graduate student in physics at the University of New Mexico, for alerting us to this.

November/December 1995

In the Kaleidoscope the following sentence appears (p. 33):

> The Polish mathematician Kazimierz Kuratowsky [sic] and the Russian mathematician Lev Pontryagin independently proved that a graph is planar if and only if it contains neither of these two graphs (the complete 5-node graph and "houses-and-wells") as a subgraph.

In addition to misspelling Kuratowski's name, the sentence perpetuates a misattribution that apparently dates from 1962. In that year A. A. Zykov's Russian translation of C. Berge's *Théorie des graphes et ses applications* (1958) was published. Zykov added the following footnote to Berge's treatment of the Theorem on Planar Graphs: "This theorem was introduced (but not published) by L. S. Pontryagin in 1927 and in 1930, and independently of him, proved again by Kuratowski. As a result we call it the Pontryagin–Kuratowski Theorem."

A note by Kennedy, Quintas, and Syslo in *Historia Mathematica* (12 [1985], pp. 356–68) traces the history of the theorem. Briefly, in 1929 Kuratowski announced his discovery, and in 1930 he published it. In his published paper he planted the seed for the later confusion. In a footnote, Kuratowski wrote: "I have learned from Mr. Alexandroff, that a theorem for graphs, analogous to my theorem, had been found by Mr. Pontrjagin several years ago, but has not been published so far."

We will never know if Pontryagin proved the Theorem on Planar Graphs in 1927. There is evidence (some of it from Pontryagin himself) that he was in fact working through an early and incomplete version of the Kuratowski Theorem, transmitted to him via Alexandrov, and so his purported proof would have stemmed as much from Kuratowski as from his own inner resources. In the end, however, "there can be no dispute that Kuratowski published the first written and correct proof of the Theorem" (Kennedy et al., p. 363). It is also telling that "[b]efore Zykov's translation of [Berge's book], Soviet mathematicians universally referred to the Theorem as the Kuratowski Theorem" (p. 364).

Our thanks to *Quantum* advisory board member Alexander Soifer for bringing this to our attention.

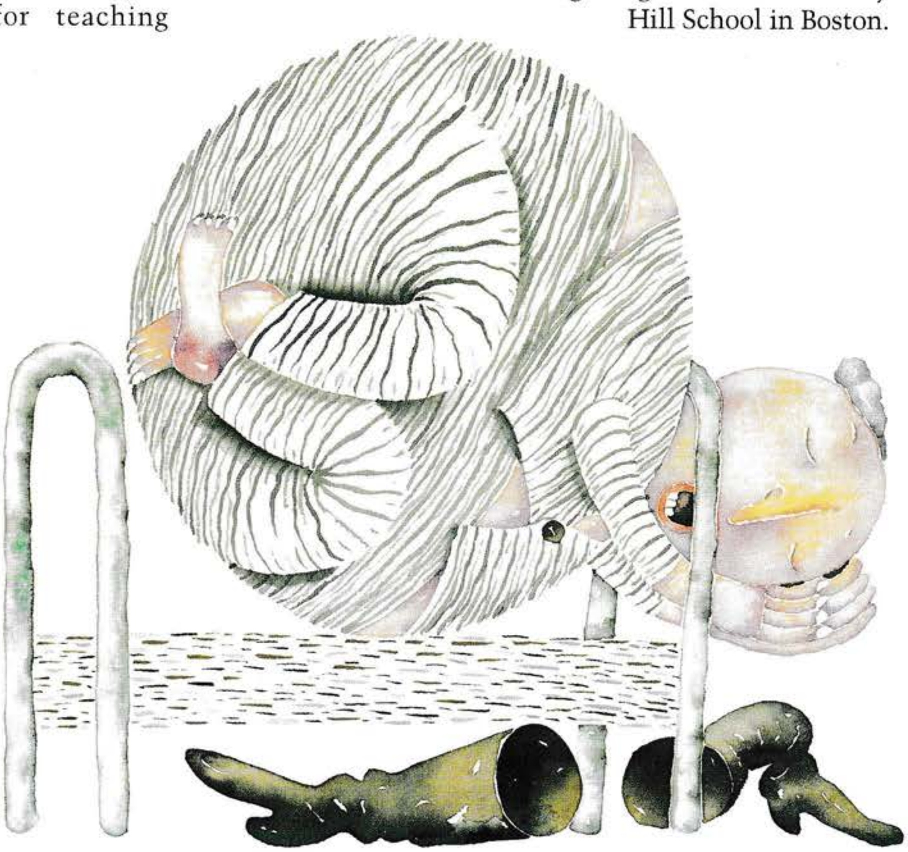
Why won't Weeble Wobbly go to bed?

Is the problem real, or just in his head?

by L. Borovinsky

IF YOU STUDY THE RUSSIAN language, sooner or later you'll encounter the great Samuel Marshak. He was a literary scholar who translated Shakespeare into Russian, but he is best known for his poetry for children. There is probably not a Russian alive who doesn't know of him, and many of his poems are perfectly suitable for elementary textbooks for teaching Russian to foreigners.

Here's a poem Marshak wrote about a popular toy that seems to have no nationality. The Russians call it Vanka-Vstanka. In English this might be rendered as "Johnny Jump-Up." At any rate, it's about a doll you can't knock over—it always pops right back up. The translation below was done by Dasha Bolotina, an eighth-grader at the Shady Hill School in Boston.



Art by Sergey Ivanov

Dasha calls the toy Weeble Wobbly, which is as good a name as any.

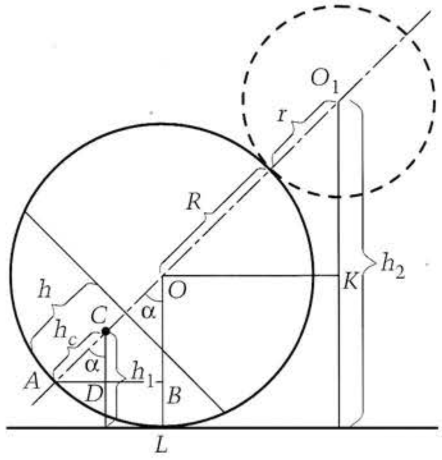
To sleep went the horses, to sleep went the chicks,
You can't hear the chatter of birds in their nests.
But one naughty boy by the name of Weeble,
And nicknamed Wobbly, cannot sleep or rest.

That Weeble, that Wobbly has unhappy nannies,
Each night they try putting Weeble to bed.
But Weeble's resisting, for he is not tired,
He jumps up and wobbles, and makes nannies mad.

The nannies try covering him with a blanket,
But Weeble just tosses the blanket away.
Again as before he stands up and wobbles,
And so he keeps standing all night and all day.

Weeble was treated by a very wise doctor,
The diagnosis he made sounded perfectly right:
"Weeble," he said, "you can't sleep like other people,
Because it's your head that is just much too light."

So, the doctor from the children's hospital found the reason for Weeble



Wobbly's strange behavior. Now we'll try to explain it with the help of physical laws and also find a cure for his insomnia. We'll figure out what it would take for Weeble Wobbly to lie down—or rather, to *stay* lying down—and get a good night's rest.

How is this invariably upright doll constructed? Imagine two adjoining spheres with radii R and r ($R > r$—see the figure above). The bigger sphere is the "body" and the smaller is the "head." (Sometimes the head is shaped like a cylinder with a horizontal axis, but for our purposes the shape of the head is of no importance.) The body contains a massive object of height h shaped like a slice from a sphere. The slice is defined partly by the lower surface of the toy's body and partly by a plane perpendicular to the axis passing through the centers of the spheres and the point where they touch.

If we try to lay Weeble Wobbly down on a horizontal surface and leave it to its own devices, it immediately stands up. Why? It's obvious that the vertical position corresponds to the state of stable equilibrium. There is a law of mechanics that says: "In a state of stable equilibrium, the center of gravity must be in the lowest of all possible positions." This means that the value of the potential energy caused by the force of gravity must be the smallest possible.

Let's find what conditions minimize the Weeble Wobbly's potential energy when it assumes the vertical position. To do this we disturb the equilibrium of the doll, inclining its axis to an angle α from the vertical. Let h_c be the height of the center of gravity of the massive object in the toy's body when the axis is vertical, M the mass of this object, and m mass of the toy's head. The mass of the body's shell doesn't matter, because the height of its center of gravity and thus its potential energy don't change when Weeble Wobbly tilts.

As you can see from the figure, the height of the massive object's center of gravity in the tilted position is

$$\begin{aligned} h_1 &= |CD| + |BL| \\ &= h_c \cos\alpha + R - R\cos\alpha, \end{aligned}$$

and the height of the head's center of gravity is

$$\begin{aligned} h_2 &= |LO| + |KO_1| \\ &= R + (R + r)\cos\alpha. \end{aligned}$$

The total potential energy of the massive object and the head is

$$\begin{aligned} E_p &= Mgh_1 + mgh_2 \\ &= (M + m)gR \\ &\quad + [m(R + r) - M(R - h_c)]g\cos\alpha. \end{aligned}$$

From this it follows that the potential energy of Weeble Wobbly is minimal when its axis is vertical (that is, when $\alpha = 0$) if the expression by which we multiply $g\cos\alpha$ is negative:

$$m(R + r) - M(R - h_c) < 0,$$

or

$$m < M\frac{R - h_c}{R + r}.$$

True, h_c is still unknown, but it can be expressed by the known parameters R and h (we'll accept this without proof):

$$h_c = h\frac{8R - 3h}{12R - 4h}.$$

So, we have found the exact mathematical condition showing how "light-headed" Weeble must be for him to resume the vertical position whenever he is pushed over—or put to bed.