2. OSCILLATIONS OF ELASTIC BODIES

In this problem gravitational effects are to be neglected. Unless otherwise stated, substances are to be considered homogenous and isotropic at all times.

A. Consider a very thin elastic rod, prevented from bending. The rod has length L, density ρ, and Young elasticity modulus E. The cross section of the rod is to be taken as constant. The rod is slightly stretched from both ends and then let to oscillate freely.
a. Write the kinetic energy of the rod at some moment in terms of its mass m, length L, and change rate of the tensile strain ε.
b. Write the potential elastic energy of the rod at some moment in terms of m, ρ, E, and ε.
c. Show that the conservation of mechanical energy for the rod leads to the characteristic differential equation for an undamped harmonic oscillation. Resorting to the analogy with a point mass m acted upon by the force σS (σ being the tensile stress and S being the cross section), specify the quantity that plays here the role of coordinate.
d. Write the expression for the period of small longitudinal oscillations of the rod in terms of L, ρ, and E.

B. Consider an elastic sphere of radius R, made of the same material as the rod before. Let ε be the tensile strain $\Delta R/R$. The sphere is slightly compressed uniformly and then let to oscillate freely.
e. Write the mechanical energy of the sphere at some moment in terms of m, R, ρ, E, ε and the change rate of ε.
f. Write down the expression for the period of small radial oscillations of the sphere in terms of R, ρ, and E.

C. Consider a very thin rectangular elastic plate, prevented from bending. The plate has the linear dimensions L and l respectively. The thickness of the plate is to be taken as constant.
The effects of the tensile stresses σ_x and σ_y acting on the plate are **NOT** independent, in the sense that a stretching on one of the directions leads to a shrinking on the other direction. In the limits of Hooke's Law, this can be written as:

$$\varepsilon_x = -\mu \frac{\sigma_y}{E}, \qquad \varepsilon_y = -\mu \frac{\sigma_x}{E}$$

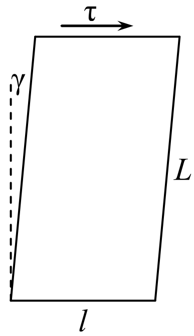
where the dimensionless factor μ (Poisson's ratio) is somewhere in the range of 0.3.
g. Express σ_x and σ_y in terms of ε_x, ε_y, E, and μ.
h. Write the system of differential equations for the movement of a point mass m on two orthogonal directions, using as coordinates the equivalent found at point **c**.
i. Find the possible values of ω for which the solutions of the above system are simple undamped harmonic oscillations (modes):

$$\varepsilon_x = A\sin\omega t, \qquad \varepsilon_y = B\sin\omega t.$$

Express the results in terms of L, l, ρ, E, and μ.

j. In general, the solution of the above system of equations is a superposition of the two modes found. For a square plate ($L = l$) and a weak Poisson ratio ($\mu^2 \ll 1$), express the beats period in terms of μ and the period T_{long} of longitudinal oscillations of a rod with the same length.

D. Now, instead of being squeezed, the plate is slightly slanted along one of the dimensions, by the action of a shear stress τ, as in the diagram alongside. The shear strain is defined to be $\tan\gamma \approx \gamma$, and Hooke's Law takes the form:



$$\gamma = \frac{1}{G}\tau \ ; [\tau]_{\text{SI}} = [G]_{\text{SI}} = \text{N/m}^2 ,$$

where G is the so-called *shear modulus*.

k. Express G in terms of E and μ.

l. Find the period of the small slanting oscillations of the plate in terms of L, ρ, and G. Express the same period in terms of μ and the period of a rod with the same length undergoing longitudinal oscillations, T_{long}.

E. A cylinder of radius R and length L, made of the same material as before, is slightly twisted and let to oscillate freely. The torsion strain is defined as the angle θ with which the cylinder is twisted, under the stress of a torque. Hooke's Law takes now the form:

$$\theta = \frac{1}{C}M \ ; [C]_{\text{SI}} = [M]_{\text{SI}} = \text{Nm} ,$$

where C is the so-called *elastic torsion constant*.

m. Find the period of the small twisting oscillations of the cylinder in terms of L, ρ, and G.

n. Express the elastic torsion constant in terms of R, L, E, and μ.