PROBLEM No. 2

a. 1p

Consider an element dx of the rod, placed at distance x from the center of the rod. Its mass is dm = mdx/L each. Let $2l$ be the length of the rod at some moment, and let y be the corresponding length of the region x.

Since the object is homogenous at all times,

$$\frac{y}{x}=\frac{l}{L/2}$$

Let v be the velocity of the two ends of the rod at some moment.

$$v=\frac{\mathrm{d}l}{\mathrm{d}t}=\frac{\mathrm{d}(2l-L)}{2\,\mathrm{d}t}=\frac{\mathrm{d}\left(\frac{2l-L}{L}\right)}{\frac{2\,\mathrm{d}t}{L}}=\frac{L\,\mathrm{d}\varepsilon}{2\,\mathrm{d}t}=\frac{L}{2}\dot{\varepsilon}$$

The velocity of the element considered is

$$v(x)=\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}\left(\frac{lx}{L/2}\right)}{\mathrm{d}t}=\frac{x}{L/2}\frac{dl}{dt}=\frac{xv}{L/2}=\frac{2x}{L}\frac{L\dot{\varepsilon}}{2}=x\dot{\varepsilon}$$

The kinetic energy of the rod is

$$E_{\text{kin}}=2\int_0^{L/2}\frac{dmv^2(x)}{2}=\int_0^{L/2}x^2\dot{\varepsilon}^2\frac{mdx}{L}=\frac{m\dot{\varepsilon}^2}{L}\frac{x^3}{3}\Bigg|_0^{L/2}=\frac{mL^2\dot{\varepsilon}^2}{24}$$

b. 0.5p

Let S be the cross section of the rod, and V its volume. The elementary work done by the tensile force σS equals the increase in elastic potential energy.

$$\mathrm{d}E_{\text{pot}}=\mathrm{d}W=F\,\mathrm{d}(2l)=\sigma S\,\mathrm{d}(2l-L)=E\varepsilon\frac{V}{L}\mathrm{d}(2l-L)=\frac{mE}{\rho}\varepsilon\,\mathrm{d}\varepsilon=\mathrm{d}\left(\frac{mE}{2\rho}\varepsilon^2\right)\Rightarrow$$

$$E_{\text{pot}}=\frac{mE\varepsilon^2}{2\rho}$$

c. 0.5p

$$E_{\text{mech}}=E_{\text{kin}}+E_{\text{pot}}=\frac{mL^2\dot{\varepsilon}^2}{24}+\frac{mE\varepsilon^2}{2\rho}=\text{constant}\Rightarrow\dot{E}_{\text{mech}}=\frac{mL^2\dot{\varepsilon}\ddot{\varepsilon}}{12}+\frac{mE\varepsilon\dot{\varepsilon}}{\rho}=0\Rightarrow$$

$$\frac{mL^2\ddot{\varepsilon}}{12}+V\sigma=0$$

Dividing by L we get:

$$m\left(\frac{L\varepsilon}{12}\right)^{\cdot\cdot}=-\sigma S\Rightarrow\ddot{x}_{\text{equivalent}}=\left(\frac{L\varepsilon}{12}\right)^{\cdot\cdot}$$

d. 0.5p

Dividing also by m we get:

$$\frac{L^2}{12}\ddot{\varepsilon}+\frac{E}{\rho}\varepsilon=0\Rightarrow\omega^2=\frac{12E}{\rho L^2}\Rightarrow T_{\text{long}}=\pi L\sqrt{\frac{\rho}{3E}}$$

e. 0.5p

Consider very thin spherical layers of radius x and thickness dx. Their masses are:

$$\mathrm{d}m = m\frac{4\pi x^2 \mathrm{d}x}{\frac{4\pi R^3}{3}} = \frac{3m}{R^3}x^2\,\mathrm{d}\,x$$

Let r be the radius of the sphere and v the velocity of its surface at some moment. The argument goes similarly as in section A.

$$v = R\dot{\varepsilon} \Rightarrow v(x) = x\dot{\varepsilon} \Rightarrow \mathrm{d}\,E_{\mathrm{kin}} = \frac{\mathrm{d}\,mv^2(x)}{2} = \frac{3mx^2\,\mathrm{d}\,x}{R^3}\frac{x^2\dot{\varepsilon}^2}{2} \Rightarrow E_{\mathrm{kin}} = \frac{3mR^2\dot{\varepsilon}^2}{10}$$

$$\mathrm{d}\,E_{\mathrm{pot}} = \mathrm{d}\,W = \sigma S\,\mathrm{d}\,r = \varepsilon E 4\pi R^2 R\,\mathrm{d}\,\varepsilon = \frac{3EV\,\mathrm{d}\left(\varepsilon^2\right)}{2} = \mathrm{d}\left(\frac{3mE}{2\rho}\varepsilon^2\right)$$

$$E_{\mathrm{mech}} = \frac{3m}{2}\left(\frac{R^2\dot{\varepsilon}^2}{5} + \frac{E\varepsilon^2}{\rho}\right) = \text{constant}$$

f. 0.5p

$$\dot{E} = 0 \Rightarrow \frac{R^2\dot{\varepsilon}\ddot{\varepsilon}}{5} + \frac{E\varepsilon\dot{\varepsilon}}{\rho} = 0 \Rightarrow \omega^2 = \frac{5E}{\rho R^2} \Rightarrow T_{\mathrm{radial}} = 2\pi R\sqrt{\frac{\rho}{5E}}$$

g. 0.5p

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} - \mu\dfrac{\sigma_y}{E} \\ \varepsilon_y = \dfrac{\sigma_y}{E} - \mu\dfrac{\sigma_x}{E} \end{cases} \Rightarrow \begin{cases} \sigma_x = \dfrac{E\left(\varepsilon_x + \mu\varepsilon_y\right)}{1-\mu^2} \\ \sigma_y = \dfrac{E\left(\varepsilon_y + \mu\varepsilon_x\right)}{1-\mu^2} \end{cases}$$

h. 0.5p

$$\begin{cases} m\ddot{x}_{\mathrm{equivalent}} = -\sigma_x\dfrac{V}{L} \\ m\ddot{y}_{\mathrm{equivalent}} = -\sigma_y\dfrac{V}{l} \end{cases} \Rightarrow \begin{cases} \dfrac{mL\ddot{\varepsilon}_x}{12} + \dfrac{E\left(\varepsilon_x + \mu\varepsilon_y\right)}{1-\mu^2}\dfrac{V}{L} = 0 \\ \dfrac{ml\ddot{\varepsilon}_y}{12} + \dfrac{E\left(\varepsilon_y + \mu\varepsilon_x\right)}{1-\mu^2}\dfrac{V}{l} = 0 \end{cases}$$

i. 1.5p

By replacing the sought solutions into the system of equations we get

$$\begin{cases} -\dfrac{\omega^2 AL^2}{12} + \dfrac{E\left(A + \mu B\right)}{\rho\left(1-\mu^2\right)} = 0 \\ -\dfrac{\omega^2 Bl^2}{12} + \dfrac{E\left(B + \mu A\right)}{\rho\left(1-\mu^2\right)} = 0 \end{cases}$$

By dividing the two equations term by term we get a simpler one:

$$\frac{AL^2}{Bl^2} = \frac{A + \mu B}{B + \mu A}$$

Let us denote the ratio of the two amplitudes by r.

$$r\frac{L^2}{l^2}=\frac{r+\mu}{1+r\mu}\Rightarrow \mu L^2r^2+\left(L^2-l^2\right)r-\mu l^2=0\Rightarrow$$

$$r_{1;2}=\frac{-\left(L^2-l^2\right)\pm\sqrt{\left(L^2-l^2\right)^2+4\mu^2L^2l^2}}{2\mu L^2}$$

Returning r in the second equation we get:

$$\omega^2=\frac{12E}{\rho l^2\left(1-\mu^2\right)}\left[1+\mu\frac{-\left(L^2-l^2\right)\pm\sqrt{\left(L^2-l^2\right)^2+4\mu^2L^2l^2}}{2\mu L^2}\right]\Rightarrow$$

$$\omega_{1;2}=\sqrt{\frac{6E\left[L^2+l^2\pm\mu\sqrt{\left(L^2-l^2\right)^2+\left(2\mu Ll\right)^2}\right]}{\rho L^2l^2\left(1-\mu^2\right)}}$$

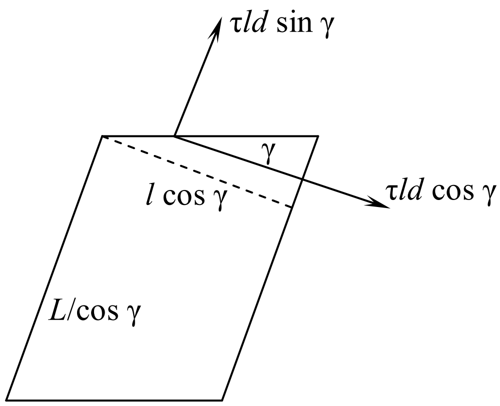
j. 0.5p

$$L=l\Rightarrow\omega_{1;2}=\sqrt{\frac{6E\left(2L^2\pm2\mu^2L^2\right)}{\rho L^4\left(1-\mu^2\right)}}=\sqrt{\frac{12E\left(1\pm\mu^2\right)}{\rho L^2\left(1-\mu^2\right)}}$$

$$\Delta\omega=\sqrt{\frac{12E}{\rho L^2}}\left(\sqrt{\frac{1+\mu^2}{1-\mu^2}}-1\right)\approx\mu^2\sqrt{\frac{12E}{\rho L^2}}\Rightarrow T_{\text{beats}}=\frac{T_{\text{long}}}{\mu^2}$$

k. 1.5p

Let d be the thickness of the plate. The shear force τld can be decomposed into a stretching component along L (x-axis) and a shrinking component along l (y-axis).



$$\sigma_x=\frac{\tau ld\sin\gamma}{ld}\;;\;\sigma_y=\frac{\tau ld\cos\gamma}{\left(L/\cos\gamma\right)d}\Rightarrow$$

$$\varepsilon_x=\frac{\tau\sin\gamma}{E}-\mu\left(-\frac{\tau l\cos^2\gamma}{LE}\right)$$

$$\varepsilon_y=-\frac{\tau l\cos^2\gamma}{LE}-\mu\frac{\tau\sin\gamma}{E}$$

But

$$\varepsilon_x=\frac{\frac{L}{\cos\gamma}-L}{L}=\frac{1-\cos\gamma}{\cos\gamma}\;;\;\varepsilon_y=\frac{l\cos\gamma-l}{l}=-\left(1-\cos\gamma\right)\Rightarrow$$

$$\begin{cases}\dfrac{E}{\tau}\dfrac{1-\cos\gamma}{\cos\gamma}=\sin\gamma+\mu\dfrac{l}{L}\cos^2\gamma\\[2ex]\dfrac{E}{\tau}\left(1-\cos\gamma\right)=\dfrac{l}{L}\cos^2\gamma+\mu\sin\gamma\end{cases}$$

Multiplying the second equation by μ and subtracting it from the first one we get:

$$\frac{E}{\tau}\left(1-\cos\gamma\right)\left(\frac{1}{\cos\gamma}-\mu\right)=\sin\gamma\left(1-\mu^2\right)\Rightarrow\frac{E\gamma^2}{2\tau}\left(1-\mu\right)\approx\gamma\left(1-\mu^2\right)\Rightarrow\gamma=\frac{2\tau\left(1+\mu\right)}{E}$$

$$G = \frac{E}{2(1+\mu)}$$

l. 0.5p

The quantities involved in the shear deformation are absolutely analogous to those describing the longitudinal deformation.

$$T_{\text{slant}} = \pi L\sqrt{\frac{\rho}{3G}} = T_{\text{long}}\sqrt{2(1+\mu)}$$

m. 0.5p

Consider very thin cylindrical layers of radius x and thickness dx. When the cylinder is twisting, each one of them is subject to a very small shear.

$$T_{\text{twist}} = \pi L\sqrt{\frac{\rho}{3G}}$$

n. 1p

Let α be a very small angle with witch one cap of the cylinder rotates with respect to the other. Then the slanting angle of a cylindrical layer is:

$$x\alpha = L\gamma \Rightarrow \gamma = \frac{x}{L}\alpha$$

The corresponding shear stress is

$$\tau = G\frac{x}{L}\alpha$$

The elementary shear force acting on the cap is

$$\mathrm{d}F = \tau\,\mathrm{d}S = G\frac{x}{L}\alpha 2\pi x\,\mathrm{d}x$$

The corresponding elementary torque is

$$\mathrm{d}M = \mathrm{d}F \cdot x = \frac{2\pi G\alpha x^3 \mathrm{d}x}{L}$$

$$M = \frac{2\pi G\alpha}{L}\int_0^R x^3\,\mathrm{d}x = \frac{2\pi G R^4\alpha}{4L} \Rightarrow C = \frac{\pi G R^4}{2L} = \frac{\pi E R^4}{4L(1+\mu)}$$