PROBLEM No. 3

a. 0.5p

Deriving the Lorentz transformations two-fold, we get

$$a_x = a'_x \left(\frac{\sqrt{1-\frac{u^2}{c^2}}}{1+\frac{u}{c^2}v'_x} \right)^3$$

In our case $u = v_x$ and $v_x' = 0$.

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = a'\left(1-\frac{v_x^2}{c^2}\right)^{\frac{3}{2}}$$

$$F_x = \frac{ma_x}{1-\frac{v_x^2}{c^2}} = m_0 a' = \text{constant}$$

b. 0.5p

$$v_x = c\sin\alpha \Rightarrow \frac{\mathrm{d}(c\sin\alpha)}{\left(1-\sin^2\alpha\right)^{\frac{3}{2}}} = a'\mathrm{d}t \Rightarrow c\tan\alpha = a't + C$$

At $t = 0$, $v_x = 0$, so $\alpha = 0$ and $C = 0$.

$$\frac{\frac{v_x}{c}}{\sqrt{1-\frac{v_x^2}{c^2}}} = \frac{a't}{c} \Rightarrow v = c\frac{\frac{a't}{c}}{\sqrt{1+\left(\frac{a't}{c}\right)^2}}$$

c. 0.5p

$$\mathrm{d}t' = \mathrm{d}t\sqrt{1-\frac{v^2}{c^2}} = \frac{\mathrm{d}t}{\sqrt{1+\left(\frac{a't}{c}\right)^2}}\ ;\ \frac{a't}{c} = \sinh\tau \Rightarrow \mathrm{d}t' = \frac{c}{a'}\mathrm{d}\tau \Rightarrow t' = \frac{c}{a'}\tau + C$$

Again $C = 0$, so

$$t' = \frac{c}{a'}\operatorname{arcsinh}\left(\frac{a't}{c}\right)$$

d. 1p

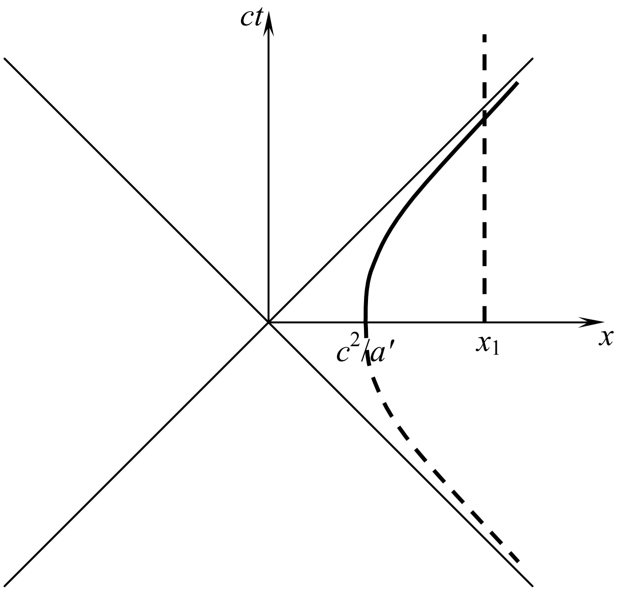
$$\left.\begin{array}{l} -c^2(\mathrm{d}t')^2 = -c^2(\mathrm{d}t)^2 + (\mathrm{d}x)^2 \\ \frac{a't}{c} = \sinh\tau \Rightarrow \mathrm{d}t = \frac{c}{a'}\cosh\tau\,\mathrm{d}\tau \end{array}\right\} \Rightarrow (\mathrm{d}x)^2 = \frac{c^4}{a'^2}\left(\cosh^2\tau - 1\right)(\mathrm{d}\tau)^2 \Rightarrow dx = \frac{c^2}{a'}\sinh\tau\,\mathrm{d}\tau \Rightarrow$$

$$x = \frac{c^2}{a'}\cosh\tau + C$$

At $t = t' = 0$, $x_0 = c^2/a'$, so again $C = 0$.

e. 1p

$$ct = \frac{c^2}{a'}\sinh\tau \Rightarrow x^2 - (ct)^2 = \left(\frac{c^2}{a'}\right)^2 \Rightarrow \frac{x^2}{\left(\frac{c^2}{a'}\right)^2} - \frac{(ct)^2}{\left(\frac{c^2}{a'}\right)^2} = 1$$



f. 0.5p

$$\rho_0 = \frac{c^2}{a'} \Rightarrow \begin{cases} x = \rho_0 \cosh\tau \\ ct = \rho_0 \sinh\tau \end{cases}$$

g. 0.5p

$$\begin{cases} x = \rho\cosh\tau \\ ct = \rho\sinh\tau \end{cases} \quad ; \quad \begin{cases} \rho = \sqrt{x^2 - (ct)^2} \\ \tau = \operatorname{arctanh}\left(\dfrac{ct}{x}\right) \end{cases}$$

These equations require that $x > 0$ and $\rho > 0$, so using these new parameters one can cover only the quadrant of spacetime characterized by $x > |ct|$.

h. 1p

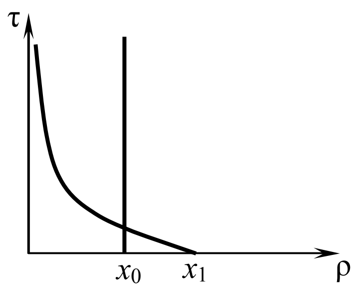
$$\left.\begin{array}{l} \mathrm{d}x = \mathrm{d}\rho\cosh\tau + \rho\sinh\tau\,\mathrm{d}\tau \\ \mathrm{d}(ct) = c\,\mathrm{d}t = \mathrm{d}\rho\sinh\tau + \rho\cosh\tau\,\mathrm{d}\tau \end{array}\right\} \Rightarrow$$

$$ds^2 = -c^2(\mathrm{d}t)^2 + (\mathrm{d}x)^2 = (\mathrm{d}\rho)^2 - \rho^2(\mathrm{d}\tau)^2 = -c^2\frac{\rho^2}{c^2}(\mathrm{d}\tau)^2 + (\mathrm{d}\rho)^2 \;;\; f = \frac{\rho^2}{c^2} \;;\; g = 1$$

i. 0.5p

$$\rho = \frac{x_1}{\cosh\tau} \Leftrightarrow \tau = \operatorname{arccosh}\left(\frac{x_1}{\rho}\right)$$

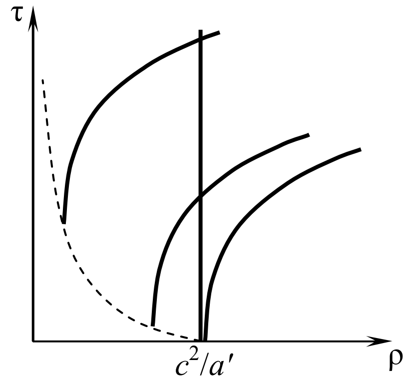
$$\Delta\rho = \frac{c^2}{a'}$$



j. 0.5p

The observer will receive only those signals emitted before the beacon exits the quadrant of spacetime described by the Rindler metric.

$$ct \le x = x_0 \Rightarrow t_{\lim} = \frac{x_0}{c} \Rightarrow N = \left[\frac{t_{\lim}}{T_0}\right] + 1 = \left[\frac{x_0}{cT_0}\right] + 1$$



In the case of the light,

$$ds^2 = 0 \Rightarrow (\mathrm{d}\rho)^2 = \rho^2 (\mathrm{d}\tau)^2 \Rightarrow \frac{\mathrm{d}\rho}{\rho} = \mathrm{d}\tau$$

At $\tau = 0$, $\rho_0 = x_0$, so

$$\ln\frac{\rho}{\rho_0} = \tau \Rightarrow \rho = x_0\,\mathrm{e}^{\tau}$$

k. 1.5p

Let ρ_e and τ_e be the spacetime coordinates for the emission of a pulse.

$$\rho_e = \sqrt{x_0^2 - c^2 t^2}\ ;\ \tau_e = \operatorname{arctanh}\left(\frac{ct}{x_0}\right)$$

$$\tanh\tau_e = \frac{\mathrm{e}^{\tau_e} - \mathrm{e}^{-\tau_e}}{\mathrm{e}^{\tau_e} + \mathrm{e}^{-\tau_e}} = \frac{ct}{x_0} \Rightarrow \mathrm{e}^{2\tau_e} - 1 = \frac{ct}{x_0}\left(\mathrm{e}^{2\tau_e} + 1\right) \Rightarrow \mathrm{e}^{2\tau_e} = \frac{1 + \frac{ct}{x_0}}{1 - \frac{ct}{x_0}} \Rightarrow \mathrm{e}^{\tau_e} = \sqrt{\frac{x_0 + ct}{x_0 - ct}}$$

$$\rho = \frac{\rho_e}{\mathrm{e}^{\tau_e}}\mathrm{e}^{\tau} = (x_0 - ct)\mathrm{e}^{\tau} = \rho_0 \Rightarrow \mathrm{e}^{\tau} = \frac{x_0}{x_0 - ct} \Rightarrow \tau = \ln\left(\frac{x_0}{x_0 - ct}\right)$$

Let t^* be the moment the observer receives the last signal.

$$v(t^*) = c\frac{\frac{a't^*}{c}}{\sqrt{1 + \left(\frac{a't^*}{c}\right)^2}}$$

The frequency received is

$$\nu = \nu_0\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} = \nu_0\sqrt{\frac{1 - \frac{\frac{a't^*}{c}}{\sqrt{1 + \left(\frac{a't^*}{c}\right)^2}}}{1 + \frac{\frac{a't^*}{c}}{\sqrt{1 + \left(\frac{a't^*}{c}\right)^2}}}} = \nu_0\sqrt{\frac{\sqrt{1 + \left(\frac{a't^*}{c}\right)^2} - \frac{a't^*}{c}}{\sqrt{1 + \left(\frac{a't^*}{c}\right)^2} + \frac{a't^*}{c}}} = \nu_0\left[\sqrt{1 + \left(\frac{a't^*}{c}\right)^2} - \frac{a't^*}{c}\right]$$

But

$$ct^* = \frac{c^2}{a'}\sinh\tau \Rightarrow \frac{a't^*}{c} = \sinh\tau \Rightarrow \nu = \nu_0(\cosh\tau - \sinh\tau) = \nu_0\,\mathrm{e}^{-\tau} = \nu_0\left(1 - \frac{ct}{x_0}\right)$$

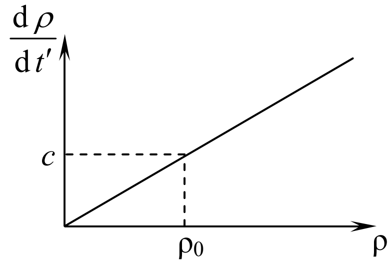
$$t = NT_0 \Rightarrow \nu = \nu_0\left\{1-\frac{cT_0}{x_0}\left(\left[\frac{x_0}{cT_0}\right]+1\right)\right\}$$

l. 0.5p

$$\frac{\mathrm{d}\rho}{\mathrm{d}t'}=\frac{\mathrm{d}\rho}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t'}=\left(x_0-ct\right)\mathrm{e}^{\tau}\frac{a'}{c}=\frac{a'}{c}\rho$$

Upon reception,

$$\mathrm{e}^{\tau}=\frac{x_0}{x_0-ct}\Rightarrow\frac{d\rho}{dt'}=\left(x_0-ct\right)\frac{x_0}{x_0-ct}\frac{a'}{c}=\frac{c^2}{a'}\frac{a'}{c}=c$$



m. 1p

$$\tanh\tau=\frac{ct}{x_0}\Rightarrow\frac{1}{\cosh^2\tau}\mathrm{d}\tau=\frac{c}{x_0}\mathrm{d}t\Rightarrow\mathrm{d}t=\frac{x_0}{c}\left(1-\tanh^2\tau\right)\mathrm{d}\tau=\frac{x_0}{c}\left(1-\frac{c^2t^2}{x_0^2}\right)\mathrm{d}\tau$$

$$\frac{\mathrm{d}\left(\mathrm{d}t\right)}{\mathrm{d}x_0}=\frac{\mathrm{d}\tau}{c}\frac{\mathrm{d}\left(\frac{x_0^2-c^2t^2}{x_0}\right)}{\mathrm{d}x_0}=\frac{\mathrm{d}\tau}{c}\frac{x_0^2+c^2t^2}{x_0^2}$$

$$\varepsilon=\frac{\frac{\mathrm{d}\tau}{c}\frac{x_0^2+c^2t^2}{x_0^2}\Delta x_0}{\frac{\mathrm{d}\tau}{c}\frac{x_0^2-c^2t^2}{x_0}}=\frac{x_0^2+c^2t^2}{x_0^2-c^2t^2}\frac{\Delta x_0}{x_0}$$

n. 0.5p

$$\varepsilon=\frac{\Delta x_0}{x_0}=\frac{h}{\frac{c^2}{a'}}=\frac{gh}{c^2}\approx\frac{10\,\mathrm{m/s^2}\cdot 360\cdot 10^3\,\mathrm{km}}{9\cdot 10^{16}\,\mathrm{m^2/s^2}}=4\cdot 10^{-11}$$

$$\Delta t=4\cdot 10^{-11}\cdot 365\cdot 24\cdot 3600\approx 1.26\cdot 10^{-3}\,\mathrm{s}$$