PROBLEM 1: MECHANICS

A. Determining the radius of curvature of a planar curve by means of Mechanics

a. **1.0 point**

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}$$

$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = f'(x)\cdot\dot{x}$$

b. **1.0 point**

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \dot{v}_x = \ddot{x}$$

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}\left[f'(x)\cdot\dot{x}\right] = f''(x)\cdot\dot{x}^2 + f'(x)\cdot\ddot{x}$$

$$\vec{a}_\mathrm{t} = \frac{\vec{a}\vec{v}}{v}\cdot\frac{\vec{v}}{v} = \frac{a_x v_x + a_y v_y}{\sqrt{v_x^2+v_y^2}}\cdot\frac{v_x\vec{i}+v_y\vec{j}}{\sqrt{v_x^2+v_y^2}} = \frac{v_x\left(a_x v_x + a_y v_y\right)\vec{i} + v_y\left(a_x v_x + a_y v_y\right)\vec{j}}{v_x^2+v_y^2}$$

c. **1.0 point**

$$\vec{a}_\mathrm{n} = \vec{a} - \vec{a}_\mathrm{t} =$$

$$= a_x\vec{i} + a_y\vec{j} - \frac{v_x\left(a_x v_x + a_y v_y\right)\vec{i} + v_y\left(a_x v_x + a_y v_y\right)\vec{j}}{v_x^2+v_y^2} =$$

$$= \frac{\left[a_x\left(v_x^2+v_y^2\right) - v_x\left(a_x v_x + a_y v_y\right)\right]\vec{i} + \left[a_y\left(v_x^2+v_y^2\right) - v_y\left(a_x v_x + a_y v_y\right)\right]\vec{j}}{v_x^2+v_y^2} =$$

$$= \frac{\left(a_x v_y^2 - a_y v_x v_y\right)\vec{i} + \left(a_y v_x^2 - a_x v_x v_y\right)\vec{j}}{v_x^2+v_y^2} =$$

$$= \frac{\left(a_x v_y - a_y v_x\right)\left(v_y\vec{i} - v_x\vec{j}\right)}{v_x^2+v_y^2}$$

$$\left|\vec{a}_\mathrm{n}\right| = \frac{\left|a_x v_y - a_y v_x\right|}{\sqrt{v_x^2+v_y^2}}$$

d. **0.5 point**

$$R=\frac{v^2}{a_n}=$$

$$=\frac{\sqrt{v_x^2+v_y^2}^3}{\left|a_x v_y-a_y v_x\right|}=$$

$$=\frac{\left[\dot{x}^2+\left(f'(x)\cdot\dot{x}\right)^2\right]^{3/2}}{\left|\ddot{x}\cdot f'(x)\cdot\dot{x}-\left(f''(x)\cdot\dot{x}^2+f'(x)\cdot\ddot{x}\right)\cdot\dot{x}\right|}=$$

$$=\frac{\left|\dot{x}\right|^3\left[1+f'^2(x)\right]^{3/2}}{\left|\dot{x}^3\cdot f''(x)\right|}=$$

$$=\frac{\left[1+f'^2(x)\right]^{3/2}}{\left|f''(x)\right|}$$

e. **0.5 point**

$$\left.\begin{aligned} f'(x_0)&=2Ax_0\\ f''(x_0)&=2A \end{aligned}\right\}\Rightarrow R=\frac{\left(1+4A^2x_0^2\right)^{3/2}}{2A}$$

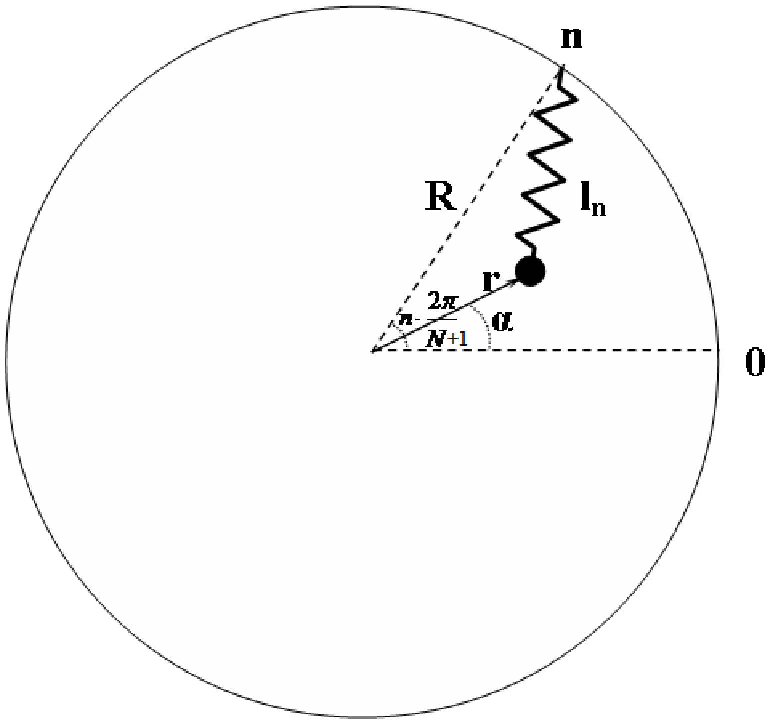
f. **0.5 point**

$$f(x_0)=-1\Rightarrow \sin 2x_0=-1\Rightarrow x_0=\frac{3}{4}\pi$$

$$\left.\begin{aligned} f'(x_0)&=2\cos\left(2\cdot\frac{3}{4}\pi\right)=0\\ f''(x_0)&=-4\sin\left(2\cdot\frac{3}{4}\pi\right)=4 \end{aligned}\right\}\Rightarrow R=\frac{1}{4}=0.25\,\mathrm{m}$$

$$T=2\pi\sqrt{\frac{R}{g}}\approx 1\,\mathrm{s}$$

B. Springs on a circle



g. 0.5 point

According to the cosine rule:

$$l_n^2 = R^2 + r^2 - 2Rr\cos\left(\frac{2n\pi}{N+1} - \alpha\right)$$

h. 0.25 point + 0.25 point

$$E_{\text{kin}} = \frac{m\left(\dot{r}^2 + r^2\dot{\alpha}^2\right)}{2}$$

$$E_{\text{pot}} = \sum_{n=0}^{N} \frac{kl_n^2}{2} = $$

$$= \frac{k}{2}\sum_{n=0}^{N}\left[R^2 + r^2 - 2Rr\cos\left(\frac{2n\pi}{N+1} - \alpha\right)\right]$$

i. 0.25 point + 0.25 point

One can use the sum of a geometric progression having complex terms.

$$\sum_{n=0}^{N}\left[\cos\left(\frac{2n\pi}{N+1}\right) + i\sin\left(\frac{2n\pi}{N+1}\right)\right] = \sum_{n=0}^{N} e^{i\frac{2n\pi}{N+1}} =$$

$$= \frac{\left(e^{i\frac{2\pi}{N+1}}\right)^{N+1} - 1}{e^{i\frac{2\pi}{N+1}} - 1} =$$

$$= \frac{\cos 2\pi + i\sin 2\pi - 1}{\cos\left(\frac{2\pi}{N+1}\right) + i\sin\left(\frac{2\pi}{N+1}\right) - 1} =$$

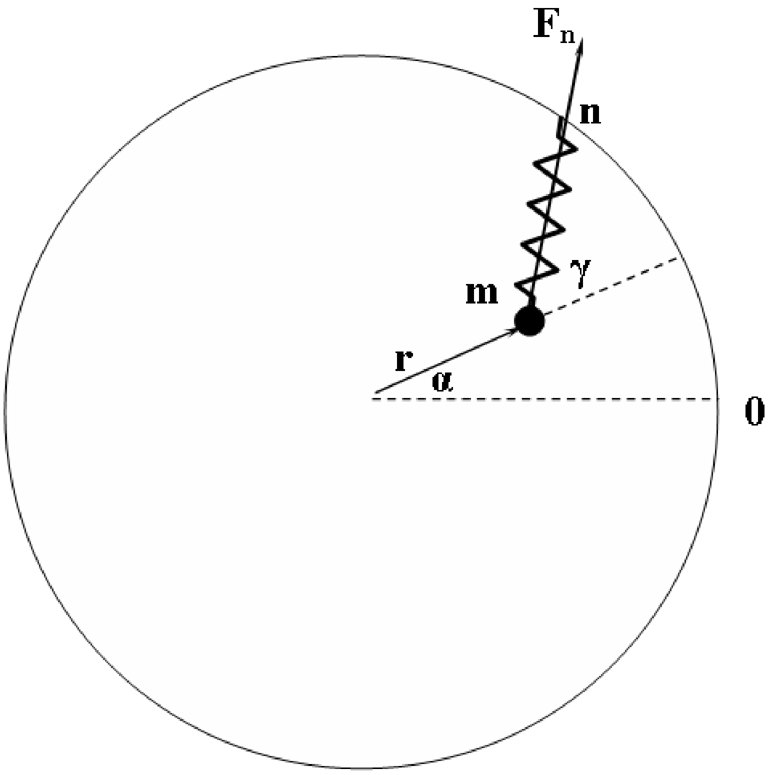
$$= 0$$

So

$$\sum_{n=0}^{N}\cos\left(\frac{2n\pi}{N+1}\right) = \sum_{n=0}^{N}\sin\left(\frac{2n\pi}{N+1}\right) = 0$$

$$E_{\text{pot}} = \frac{k}{2}\left[\sum_{n=0}^{N}\left(R^2 + r^2\right) - 2Rr\sum_{n=0}^{N}\cos\left(\frac{2n\pi}{N+1} - \alpha\right)\right] = \frac{N+1}{2}k\left(R^2 + r^2\right)$$

j. 0.5 point



The force with which the n-th spring is acting on the object is

$$F_n = kl_n$$

According to the sine rule,

$$\frac{\sin(\pi-\gamma)}{R} = \frac{\sin\left(\frac{2n\pi}{N+1}-\alpha\right)}{l_n}$$

The torque of F_n equals

$$M_n = r\cdot F_n \sin\gamma =$$

$$= r\cdot kl_n \cdot \frac{R\sin\left(\frac{2n\pi}{N+1}-\alpha\right)}{l_n} =$$

$$= kRr\sin\left(\frac{2n\pi}{N+1}-\alpha\right)$$

The total torque equals

$$M = \sum_{n=0}^{N} M_n = \sum_{n=0}^{N} kRr\sin\left(\frac{2n\pi}{N+1}-\alpha\right) = 0$$

So the angular momentum is indeed constant.

$$L = mr^2\dot\alpha$$

k. 0.5 point

$$E = E_{\text{kin}} + E_{\text{pot}} = \frac{1}{2}m\left(\dot r^2 + \frac{L^2}{m^2r^2}\right) + \frac{1}{2}k(N+1)\left(R^2+r^2\right) = \text{constant} \Rightarrow$$

$$\dot E = m\left(\dot r\ddot r - \frac{L^2}{m^2}r^{-3}\dot r\right) + k(N+1)r\dot r = 0 \Rightarrow$$

$$\ddot r - \mathcal{L}^2 r^{-3} + \omega^2 r = 0$$

l. 0.5 point

$$\dot r = \frac{2z\dot z}{2\sqrt{z^2+K}} = \frac{z\dot z}{r} \Rightarrow$$

$$\ddot r = \frac{\left(\dot z^2 + z\ddot z\right)r - z\dot z\dot r}{r^2} = \frac{\dot z^2 + z\ddot z}{r} - \frac{z\dot z}{r^2}\cdot\frac{z\dot z}{r} \Rightarrow$$

$$\frac{\dot{z}^2+z\ddot{z}}{r}-\frac{z^2\dot{z}^2}{r^3}-\frac{\mathcal{L}^2}{r^3}+\omega^2 r=0$$
$$\left(\dot{z}^2+z\ddot{z}\right)\left(z^2+K\right)-z^2\dot{z}^2-\mathcal{L}^2+\omega^2\left(z^2+K\right)^2=0$$
$$\left(\ddot{z}+\omega^2 z\right)z^3+K\ddot{z}z+K\dot{z}^2+2K\omega^2 z^2+K^2\omega^2-\mathcal{L}^2=0$$

m. **0.5 point**

The first term of the equation cancels anyway.

$$-K\omega^2A^2\cos^2\left(\omega t+\phi_0\right)+K\omega^2A^2\sin^2\left(\omega t+\phi_0\right)+2K\omega^2A^2\cos^2\left(\omega t+\phi_0\right)+K^2\omega^2-\mathcal{L}^2=0$$
$$K\omega^2A^2+K^2\omega^2-\mathcal{L}^2=0\Rightarrow A^2=\frac{\mathcal{L}^2}{K\omega^2}-K$$

n. **0.5 point**

$$r(t)=\sqrt{\frac{\mathcal{L}^2-K^2\omega^2}{K\omega^2}\cos^2\left(\omega t+\phi_0\right)+K}=$$
$$=\frac{1}{\omega}\sqrt{\frac{\mathcal{L}^2\cos^2\left(\omega t+\phi_0\right)+K^2\omega^2\sin^2\left(\omega t+\phi_0\right)}{K}}=$$
$$=\frac{1}{\omega\sqrt{2K}}\sqrt{\mathcal{L}^2\left[1+\cos\left(2\omega t+2\phi_0\right)\right]+K^2\omega^2\left[1-\cos\left(2\omega t+2\phi_0\right)\right]}=$$
$$=\frac{1}{\omega\sqrt{2K}}\sqrt{\left(\mathcal{L}^2+K^2\omega^2\right)+\left(\mathcal{L}^2-K^2\omega^2\right)\cos\left(2\omega t+2\phi_0\right)}$$
$$T_r=\frac{2\pi}{2\omega}$$

But since the values of r repeat after every rotation by π, it means that the period of the motion is

$$T=2\pi\sqrt{\frac{1}{N+1}\frac{m}{k}}$$

o. **0.5 point**

$$L=0\Rightarrow\mathcal{L}=0\Rightarrow r(t)=\sqrt{K}\left|\sin\left(\omega t+\phi_0\right)\right|$$
$$L=0\Rightarrow\dot{\alpha}=0\Rightarrow\alpha=\text{constant}$$

According to the previous part, except for the moments when $r = 0$ and α is not defined, there will be two constant values for α, corresponding to two opposing directions. So in this case the object will perform simple harmonic motion with amplitude $\sqrt{K}$.

p. **0.5 point**

$$r = \text{constant} = \sqrt{K} \Rightarrow L^2 - K^2\omega^2 = 0 \Rightarrow \left|\frac{L}{m}\right| = K\sqrt{\frac{k}{m}(N+1)} \Rightarrow r = \sqrt{\frac{|L|}{\sqrt{(N+1)mk}}}$$

q. **0.5 point**

We can think of this situation as having N + 1 springs of positive constant k and another (N + 1)/d springs of "negative" stiffness –k. The potential energy E_{pot} derived in **i.** holds for both ensembles (with appropriate N and k) and angular momentum is still conserved. Thus all the results from part **j.** onwards continue to hold if we replace (N + 1) by (N + 1) + [– (N + 1)/d]. Thus

$$\omega'^2 = \left[N + 1 - \frac{N+1}{d}\right]\frac{k}{m} \Rightarrow \omega' = \omega\sqrt{1 - \frac{1}{d}}$$