Barem de evaluare şi de notare

Se punctează în mod corespunzător oricare altă modalitate de rezolvare corectă a problemei

PROBLEM 3: BLACK HOLES PHYSICS

a. 1.0 point

For Minkovski (flat) spacetime $f(r) = g(r) = 1$.

$$(ds)^2 = -c^2(dt)^2 + (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2$$

b. 1.0 point

$$(ds)^2 = -c^2\left(1-\frac{r_S}{r}\right)(dt)^2 + \frac{(dr)^2}{1-\frac{r_S}{r}} = -c^2(dt')^2 \Rightarrow$$

$$\left(1-\frac{r_S}{r}\right)\left(\frac{dt}{dt'}\right)^2 - \frac{\left(\frac{dr}{dt'}\right)^2}{c^2\left(1-\frac{r_S}{r}\right)} = 1 \Rightarrow$$

$$\frac{d^2r}{dt'^2} + \frac{r_S c^2}{2r^2}\left[1+\frac{\left(\frac{dr}{dt'}\right)^2}{c^2\left(1-\frac{r_S}{r}\right)}\right] - \frac{r_S}{2r^2}\frac{1}{1-\frac{r_S}{r}}\left(\frac{dr}{dt'}\right)^2 = 0 \Rightarrow$$

$$a = \frac{d^2r}{dt'^2} = -\frac{r_S c^2}{2r^2}$$

c. 0.5 point

The observer will always experience the same acceleration as in the classical case.

$$a = -\frac{GM}{r^2} \Rightarrow r_S = \frac{2GM}{c^2}$$

d. 1.0 point

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$$a=\frac{dv}{dt'}=\frac{dv}{dr}\frac{dr}{dt'}=\frac{1}{2}\frac{d\left(v^2\right)}{dr}=-\frac{r_S c^2}{2r^2}\Rightarrow$$

$$d\left(v^2\right)=r_S c^2\frac{1}{r^2}dr\Rightarrow v^2=r_S c^2\frac{1}{r}+C$$

From the initial conditions, $C=-r_S c^2/r_0$, so

$$v=-c\sqrt{r_S\left(\frac{1}{r}-\frac{1}{r_0}\right)}=-\sqrt{\frac{2GM}{r_0}\left(\frac{r_0}{r}-1\right)}$$

Again, from the standpoint of the observer this is just the classical result.

e. **0.5 point**

$$dt'=\frac{dr}{v}=-\frac{1}{c}\sqrt{\frac{r_0 r}{r_S\left(r-r_0\right)}}dr\Rightarrow$$

$$t'=\int_{r_0}^{r_S}\left(-\frac{1}{c}\right)\sqrt{\frac{r_0}{r_S}\frac{\frac{r}{r_0}}{1-\frac{r}{r_0}}}dr=$$

$$=\frac{r_0}{c}\sqrt{\frac{r_0}{r_S}}\left[-\sqrt{x\left(1-x\right)}\Big|_{\frac{r_S}{r_0}}^{1}-\arccos\left(\sqrt{x}\right)\Big|_{\frac{r_S}{r_0}}^{1}\right]=$$

$$=\frac{r_0}{c}\sqrt{\frac{r_0}{r_S}}\left[\sqrt{\frac{r_S}{r_0}\left(1-\frac{r_S}{r_0}\right)}+\arccos\left(\sqrt{\frac{r_S}{r_0}}\right)\right]=$$

$$=\frac{r_0}{c}\left[\sqrt{\left(1-\frac{2GM}{c^2 r_0}\right)}+\sqrt{\frac{c^2 r_0}{2GM}}\arccos\left(\sqrt{\frac{2GM}{c^2 r_0}}\right)\right]$$

And once again this is just the classic result for a falling body.

f. **0.5 point**

$$\left(1-\frac{r_S}{r}\right)\left(\frac{dt}{dt'}\right)^2-\frac{v^2}{c^2\left(1-\frac{r_S}{r}\right)}=1\Rightarrow$$

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$$\frac{r-r_S}{r}\left(\frac{dt}{dt'}\right)^2 = 1+\frac{c^2\frac{r_S\left(r_0-r\right)}{r_0 r}}{c^2\frac{r-r_S}{r}} \Rightarrow$$

$$\left(\frac{dt}{dt'}\right)^2 = \frac{r}{r-r_S}\cdot\frac{r_0 r - r_S r}{r_0\left(r-r_S\right)} \Rightarrow$$

$$dt = dt'\cdot\frac{r}{r-r_S}\sqrt{\frac{r_0-r_S}{r_0}} = -\frac{dr}{c}\cdot\frac{r}{r-r_S}\sqrt{\frac{r_0-r_S}{r_0}}\sqrt{\frac{r_0 r}{r_S\left(r_0-r\right)}} =$$

$$= \frac{\left|dr\right|}{c}\cdot\frac{r}{r-r_S}\sqrt{\frac{r}{r_S}\cdot\frac{r_0-r_S}{r_0-r}}$$

But

$$\left.\begin{array}{l} \dfrac{r}{r_S}>1 \\ \dfrac{r_0-r_S}{r_0-r}>1 \\ \dfrac{r}{r-r_S}>\dfrac{r}{r-r_S} \end{array}\right\} \Rightarrow dt > \frac{\left|dr\right|}{c}\cdot\frac{r_S}{r-r_S} \Rightarrow$$

$$\int_{r=r_0}^{r=r_S} dt > \int_{r_S}^{r_0}\frac{r_S}{c}\cdot\frac{1}{r-r_S}dr = \infty$$

So from the point of view of someone outside the black hole, the falling observer remains frozen just outside the black hole's rim for an eternity, never crossing the event horizon!! Of course, this result is just an approximation, valid in the idealized regime in which the falling observer's effect on the gravitational field (spacetime) can be ignored.

g. **1.0 point**

$$dS = \frac{dU}{T} = \frac{d\left(Mc^2\right)}{\dfrac{\hbar c^3}{8\pi G k_B M}} = \frac{8\pi G k_B}{\hbar c}MdM$$

By definition and common sense, no black hole, no entropy. So

$$S = \frac{4\pi G k_B M^2}{\hbar c} = \frac{4\pi G k_B}{\hbar c}\cdot\frac{r_S^2 c^4}{4G^2} = \frac{k_B c^3}{4G\hbar}A$$

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h. **0.5 point**

$$E_{\text{radiated}} = (M_1 + M_2 - M_{1+2})c^2$$
$$S_{1+2} \geq S_1 + S_2 \Rightarrow A_{1+2} \geq A_1 + A_2 \Rightarrow M_{1+2}^2 \geq M_1^2 + M_2^2 \Rightarrow$$
$$E_{\max} = \left(M_1 + M_2 - \sqrt{M_1^2 + M_2^2}\right)c^2$$

i. **0.5 point**

$$ds = \frac{dr}{\sqrt{G(r)}} \approx \frac{dr}{\sqrt{G'(r_h)\cdot(r - r_h)}} \Rightarrow$$
$$R = \int_{r_h}^{r_h+\varepsilon} ds = \frac{1}{\sqrt{G'(r_h)}} \cdot 2\sqrt{r - r_h}\Big|_{r_h}^{r_h+\varepsilon} = \frac{2\sqrt{\varepsilon}}{\sqrt{G'(r_h)}}$$

j. **0.5 point**

$$F(r) \approx F'(r_h)\cdot(r - r_h)$$
$$ds = c\sqrt{F(r_h + \varepsilon)}d\tau \approx \sqrt{F'(r_h)\cdot\varepsilon}\cdot d(c\tau)$$
$$L = \int_0^P ds = \sqrt{F'(r_h)\cdot\varepsilon}\cdot P$$

k. **1.0 point**

$$F(r_h) = G(r_h) = 0 \Rightarrow 1 - \frac{2GM}{c^2 r_h} = 0 \Rightarrow r_h = r_S$$
$$F'(r_h) = G'(r_h) = \frac{1}{r_S}$$
$$\sqrt{F'(r_h)\cdot\varepsilon}\cdot P = 2\pi\frac{2\sqrt{\varepsilon}}{\sqrt{G'(r_h)}} \Rightarrow P = \frac{4\pi}{\sqrt{F'(r_h)\cdot G'(r_h)}} = 4\pi r_S$$
$$4\pi\frac{2GM}{c^2} = \frac{\hbar c}{k_B T_H} \Rightarrow T_H = \frac{\hbar c^3}{8\pi G k_B M}$$

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l. **1.0 point**

$$C = \frac{dU}{dT_{\mathrm{H}}} = \frac{d\left(Mc^2\right)}{dT_{\mathrm{H}}} = c^2 \frac{d\left(\frac{\hbar c^3}{8\pi G k_{\mathrm{B}} T_{\mathrm{H}}}\right)}{dT_{\mathrm{H}}} = -\frac{\hbar c^5}{8\pi G k_{\mathrm{B}} T_{\mathrm{H}}^2}$$

m. **0.5 point**

$$W = \sigma T_{\mathrm{H}}^4 A = \frac{\pi^2 k_{\mathrm{B}}^4}{60\hbar^3 c^2} \cdot \frac{\hbar^4 c^{12}}{8^4 \pi^4 G^4 k_{\mathrm{B}}^4 M^4} \cdot 4\pi \cdot \frac{4G^2 M^2}{c^4} = \frac{\hbar c^6}{15360\pi G^2 M^2}$$

n. **0.5 point**

$$W = -\frac{dU}{dt} \Rightarrow \frac{\hbar c^6}{15360\pi G^2 M^2} = -\frac{c^2 dM}{dt} \Rightarrow$$

$$\tau = \int_0^\tau dt = -\frac{15360\pi G^2}{\hbar c^4} \int_M^0 M^2 dM = \frac{5120\pi G^2 M^3}{\hbar c^4}$$

For the given values, τ is of the order of 10^{77} s, which is approximately 10^{60} times greater than the age of the universe!!